Canonical Transformations

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We want to find coordinate transformations $\{q,p\} \rightarrow \{Q,P\}$ and Hamiltonian functions $H(q,p,t), H'(Q,P,t)$ such that the form of the canonical equations of motion are preserved. Namely:

$$\dot{q} = \frac{\partial H}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q}$$
$$\dot{Q} = \frac{\partial H'}{\partial Q} \quad ; \quad \dot{P} = -\frac{\partial H}{\partial Q}$$

These transformations are called "canonical transformations". Not every coordinate transformation is so special; we will now derive the conditions for transformations to be "canonical" (i.e.. preserve the form of the canonical equations of motion), and we will find out the differences between the "old" and "new" Hamiltonian functions.

We know that we can derive canonical equations of motion for $\{q,p\}$ from an action principle of the form $S = \int(p\dot{q} - H)dt$. We can also derive canonical equations of motion for $\{Q,P\}$ from the action $S' = \int(P\dot{Q} - H')dt$. The solutions to the action principle for are unchanged if $S - S' = \int(df/dt)dt$, for $f = f(q,t)$ a function of coordinates and time. If we use a restricted version of the action principle, keeping both q and p fixed at the initial and final times, then the functions f can be a function of coordinates and momenta: $F = F(q,p,t)$. Each such choice of function F will "generate" a change of coordinates if

$$\begin{aligned} \int \frac{dF}{dt}dt &= \int(p\dot{q} - H)dt - \int(P\dot{Q} - H')dt \\ \int dF &= int(pdq - Hdt - (PdQ - H'dt) \\ dF &= pdq - PdQ + (H' - H)dt \end{aligned}$$

This is a restricting condition on the coordinates, since the combination $pdq - PdQ + (H' - H)dt$ will not in general be an exact differential (i.e., we would not be be able to find an exact integral). When integrated, this expression gives us a function $F(q, Q, t)$ such that

$$\frac{\partial F}{\partial q} = p; \quad \frac{\partial F}{\partial Q} = -P; \quad \frac{\partial F}{\partial t} = H' - H$$

A different, and more useful, way to view these equations is to think that *any* function $F(qQ, t)$ will generate a coordinate transformation $\{q, p\} \to \{Q, P\}$, where the change of coordinates are given by the equations

$$\frac{\partial F}{\partial q} = p; \quad \frac{\partial F}{\partial Q} = P$$

The equations of motion in either coordinate set will have the form of canonical equations of motion, if $H'(Q, P, t) = H(q, p, t) + \partial F/\partial t$, where in the right hand side we use $q = q(Q, P, t), F = F(q(Q, P, t), p(Q, P, t), t)$.

The formalism can be derived in a straightforward manner for n generalized coordinates.

For example, consider the generating function $F = qQ$. The coordinate change will be

$$\begin{aligned} \frac{\partial F}{\partial q} = Q &= p \\ \frac{\partial F}{\partial Q} = q &= -P \end{aligned}$$

The new coordinate q is the old momentum p, and the new momentum P is minus the old coordinate q! Just like in Lagrangian mechanics the generalized coordinates need not have units of length or be position coordinates, in Hamiltonian mechanics the canonical momenta need not be related to linear momenta or velocities; in fact the roles of coordinates and momenta are interchangeable (up to a sign!).

The solutions to the new equations of motion should of course be the same physical solutions as for the "old" problem. T illustrate this point, let's consider the Hamiltonian of a simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

The canonical equations of motion are $\dot{q} = \partial H/\partial p = p/m$ and $\dot{p} = -\partial H/\partial q = -kq$. Using the q equation into the p equation, we obtain the SHO equation

$$\ddot{q} = -\frac{k}{m}q$$

with the usual solution $q(t) = q_0 \cos(\omega t + \phi)$, with $\omega^2 = k/m$.

If we perform the change of coordinates generated by $F = qQ$, the new Hamiltonian is the same as the old one because $\partial F \partial t = 0$. The change of coordinates is $q = -P, p = Q$, so the new Hamiltonian is

$$H' = \frac{Q^2}{2m} + \frac{1}{2}k(-P)^2 = \frac{Q^2}{2m} + \frac{1}{2}kP^2$$

The new equations of motion are

$$\dot{Q} = \frac{\partial H'}{\partial P} = kP$$
$$\dot{P} = -\frac{\partial H'}{\partial Q} = Q/m$$

Again using the $\dot{Q}$ equation into the $\dot{P}$ equation, we obtain a SHO equation

$$\frac{\ddot{Q}}{k} = -\frac{Q}{m}$$

with the usual solution $Q(t) = Q_0 \cos(\omega t + \psi_0)$, and $P(t) = \dot{Q}/k = -(Q_0\omega/k)\sin(\omega t + \psi_0)$, and $\omega^2 = k/m$. We see that the "old solution"

$$q(t) = q_0 \cos(\omega t + \phi)$$

is the same as the "new" solution

$$-P(t) = (Q_0\omega/k)\sin(\omega t + \psi_0)$$

if $\psi_0 = \phi_0 + \pi/2$ and $Q_0 = q_0\omega/k$. The initial values of the coordinates are indeed the ones that tell us about differences in the choices for coordinates. If the oscillator is a mass on a spring, a physical statement of initial conditions could be "start from rest with the spring stretched by a length Δ from its rest length. From that physical description, we have to come up with values for our constants of integration, be those q_0, ϕ_0 or Q_0, ψ_0. In this case, the condition "at rest" is described by $\phi_0 = 0$, but $\psi_0 = \pi/2$; while the initial spring length tells us that $q_0 = \Delta$ and $Q_0 = \Delta\omega/k$.

We know the Hamiltonian functions are the same under the transformation generated by $F = qQ$ because $\partial F/\partial t = 0$, but the Lagrangian functions will not be the same:

$$L' - L = (P\dot{Q} - H') - (p\dot{q} - H) = P\dot{Q} - p\dot{q} = (-q)\dot{p} - p\dot{q} = -d(qp)/dt$$

The Lagrangian functions differ by a total time derivative of the function $-qp = -mq\dot{q}$. This is *not* the generic transformation we learned about in Goldstein's problem 1-8, since it is a function of $q, \dot{q}$, not just q: canonical transformations are more general.

We can use Legendre transformations to find generating functions of different pairs of variables other than q, Q associated with $F(q, Q, t)$. For example, if we want generating functions of q, P (that is, using P instead of Q), we know that $\Phi(q, P) = F(q, Q) + PQ$, and

$$d\Phi = d(F + PQ) = pdq - PdQ + (H' - H)dt + PdQ + QdP = pdq + QdP + (H' - H)dt$$

so the equations for the coordinate transformations and the change in Hamiltonian is

$$\frac{\partial \Phi}{\partial q} = p; \quad \frac{\partial \Phi}{\partial P} = Q; \quad \frac{\partial \Phi}{\partial t} = H' - H$$

Let us consider, for example, a change of coordinates generated by $\Phi(q, P, t) = qP - f(q, t)$. The new and old Hamiltonian functions are not the same now, but

$$H' = H + \frac{\partial \Phi}{\partial t} = H - \frac{\partial f}{\partial t}$$

The coordinate transformation is

$$p = \frac{\partial \Phi}{\partial q} = P - \frac{\partial f}{\partial q}$$

$$Q = \frac{\partial \Phi}{\partial P} = q$$

Only momenta are transformed, while coordinates are the same.

The change in Lagrangian will be

$$L' = P\dot{Q} - H' = \left(p + \frac{\partial f}{\partial q}\right)\dot{q} - \left(H - \frac{\partial f}{\partial t}\right) = (p\dot{q} - H) + \left(\frac{\partial f}{\partial q} + -\frac{\partial f}{\partial t}\right) = L + \frac{df}{dt}$$

This is the case we studied in Goldstein's problem 1-8 and revisited many times through the course. These transformation includes the case of the electromagnetic gauge transformation in Problem 1-9, for example.

We have seen then that *all* of the transformations generated by arbitrary functions of coordinates $f(q, t)$, are canonical transformations generated by functions of the form $\Phi(q, P, t) = qP - f(q, t)$, a small set of the generic functions $\Phi(q, P, t)$ we could use.