Parametric and Topological Inference for Masked System Lifetime Data

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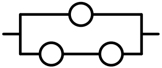
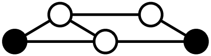
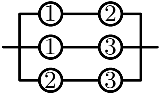
Structural Reliability Theory

- Interest lies in the reliability of ‘systems’ composed of numerous ‘components’.







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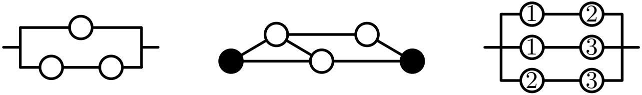

- Lifetime of the system, T, is determined by:
  - the lifetime of the components, $Y_i \sim F_Y(\,\cdot\,;\psi_i)$
  - the structure of the system.

via either the *structure function* or *signature*.

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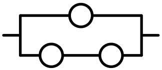
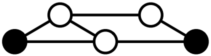
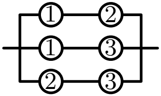

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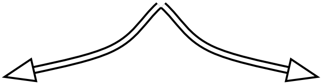
Probabilistic Analysis



Statistical Inference

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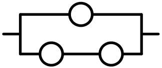
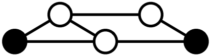
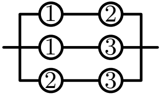

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Probabilistic Analysis

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Structure Functions & Signatures

The structure function (Birnbaum *et al.*, 1961) is a mapping $\varphi(\cdot) : \{0,1\}^n \to \{0,1\}$ which determines operation of the system given the state of the n components.

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The signature (Samaniego, 1985) is less widely used, but in some ways more elegant.

Definition (Signature)

The *signature* of a system is the n-dimensional probability vector $\mathbf{s} = (s_1, \dots, s_n)$ with elements:

$$s_i = \mathbb{P}(T = Y_{i:n})$$

where T is the failure time of the system and $Y_{i:n}$ is the ith order statistic of the n component failure times.

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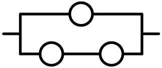
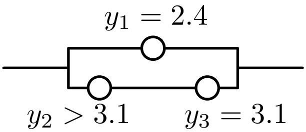
e.g.



$$\implies \mathbf{s} = \left(0, \frac{2}{3}, \frac{1}{3}\right)$$

Masked System Lifetime Data

Traditionally, one may have failure time data on components and then infer the parameters ψ of the lifetime distribution.



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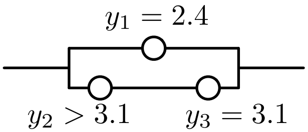

Trivial Bayesian inference:

$$f_{\Psi \mid Y}(\psi \mid \mathbf{y}) \propto \left\{ f_Y(y_1; \psi) \, f_Y(y_3; \psi) \, (1 - F_Y(y_2; \psi)) \right\} f_\Psi(\psi)$$

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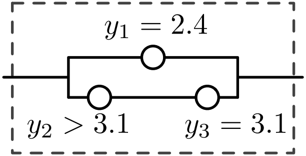

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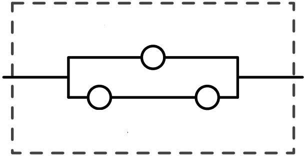

$$t = y_{?} = y_{?:3} = 3.1$$

Masked system lifetime data means only the failure time of the system as a whole is known, not the component failure times or indeed which components had failed.

The literature on inference for masked system lifetime data is extensive, but:

- heavily focused on specific structures (e.g. series/competing risk systems, see Reiser *et al.* (1995) or Kuo and Yang (2000))
- or focused on specific lifetime distributions (e.g. Exponential, see Gåsemyr and Natvig (2001))
- or does not focus on inferring the parameters of the model (e.g. infer hazard, see Ng *et al.* (2012)).

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Why? Nasty likelihood, even in i.i.d. case!

$$
\begin{aligned}
L(\psi; \mathbf{y}) &= \prod_{i=1}^{m} \frac{\partial}{\partial t} F_T(t; \psi) \Big|_{t=t_i} \\
&= \prod_{i=1}^{m} \frac{\partial}{\partial t} \Big[1 - \{1 - F_{Y_2}(t)\} \{1 - F_{Y_3}(t)\} \Big] F_{Y_1}(t) \Big|_{t=t_i}
\end{aligned}
$$

Missing Data

The missing data is what makes the inference hard. Tanner and Wong (1987) is a classic solution to this in a Bayesian framework assuming the missing data can be simulated.

$$f_{Y \,|\, \Psi, T}(\mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot} \,|\, \psi, \mathbf{t})$$

$$f_{\Psi \,|\, Y, T}(\psi \,|\, \mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot}, \mathbf{t})$$

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?

$$f_{\Psi\,|\,Y,T}(\psi \,|\, \mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{m\cdot}, \mathbf{t})$$

✓

Then, in the usual way the marginal samples from the Gibbs step are the required estimates:

$$f_{\Psi\,|\,T}(\psi \,|\, \mathbf{t}) = \int \cdots \int_{\mathbb{R}^+} f_{\Psi,Y|\,T}(\psi, \mathbf{y} \,|\, \mathbf{t}) \, d\mathbf{y}$$

Sampling Latent Failure Times

It can be shown:

$$
\begin{aligned}
f_{Y \mid T}&(y_{i1}, \dots, y_{in}; \psi \mid t) \\
&\propto \sum_{j=1}^{n} \Bigg[f_{Y \mid Y<t}(y_{i(1)}, \dots, y_{i(j-1)}; \psi) \\
&\qquad\qquad \times f_{Y \mid Y>t}(y_{i(j+1)}, \dots, y_{i(n)}; \psi) \\
&\qquad\qquad \times \mathbb{I}_{\{t\}}(y_{i(j)}) \\
&\qquad\qquad \times \binom{n-1}{j-1} F_Y(t;\psi)^j \, \bar{F}_Y(t;\psi)^{n-j+1} \, s_j \Bigg]
\end{aligned}
$$

Signature based data augmentation

1. With probability

$$\mathbb{P}(j) \propto \binom{n-1}{j-1} F_Y(t_i;\psi)^j \, \bar{F}_Y(t_i;\psi)^{n-j+1} \, s_j$$

it was the jth failure that caused system failure.

2. Having drawn a random j, sample
   - $j-1$ values, $y_{i1}, \dots, y_{i(j-1)}$, from $F_{Y \mid Y<t_i}(\,\cdot\,;\psi)$, the distribution of the component lifetime conditional on failure before t_i
   - $n-j$ values, $y_{i(j+1)}, \dots, y_{in}$, from $F_{Y \mid Y>t_i}(\,\cdot\,;\psi)$, the distribution of the component lifetime conditional on failure after t_i

and set $y_{ij} = t_i$.

Prerequisites

This is a very general method. The prerequisites for use are,

1. The signature of the system;
2. The ability to perform standard Bayesian inference with the full data;
3. The ability to sample from $F_{Y|\,Y<t_i}(\,\cdot\,;\psi)$ and $F_{Y|\,Y>t_i}(\,\cdot\,;\psi)$.

Prerequisites

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1. The signature of the system;

   Easy for systems that are not huge

2. The ability to perform standard Bayesian inference with the full data;

   Easy for common lifetime distributions

3. The ability to sample from $F_{Y|\, Y<t_i}(\,\cdot\,;\psi)$ and $F_{Y|\, Y>t_i}(\,\cdot\,;\psi)$.

   Depends!

Prerequisites

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2. The ability to perform standard Bayesian inference with the full data;
3. The ability to sample from $F_{Y|\,Y<t_i}(\,\cdot\,;\psi)$ and $F_{Y|\,Y>t_i}(\,\cdot\,;\psi)$.
   e.g. for Exponential one may generate random realisations of $Y|\,Y<t$ *as:*

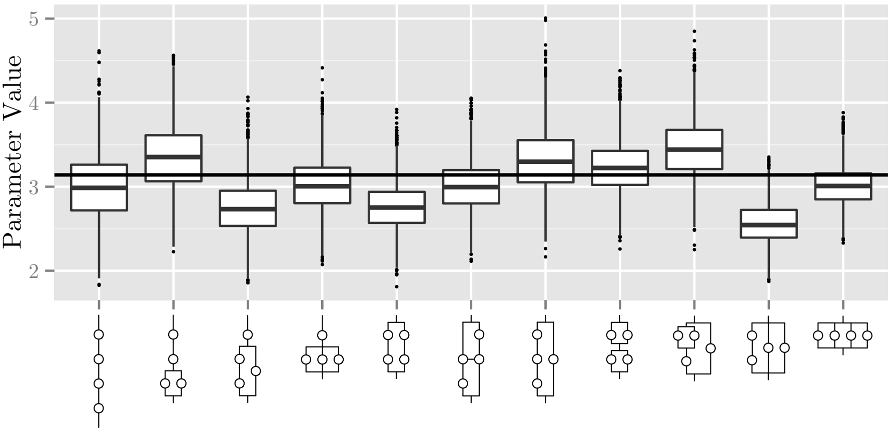
   $$y = -\lambda^{-1}\log[U(e^{-\lambda t}-1)+1] \qquad \textit{where} \qquad U\sim\text{Uniform}(0,1)$$

   and, random realisations of $Y|\,Y>t$ *as:*

   $$y = L + t \qquad \textit{where} \qquad L\sim\text{Exponential}(\lambda)$$

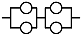
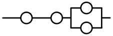
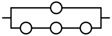
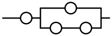
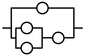
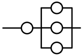
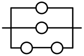
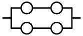
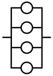
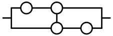
   by the memoryless nature of the Exponential.

Canonical Exponential Component Lifetime Example



Signature & Topology

Order 4 coherent systems with graph representation.

System Topology	Signature	System Topology	Signature
	$(1,0,0,0)$		$\left(0,\frac{1}{3},\frac{2}{3},0\right)$
	$\left(\frac{1}{2},\frac{1}{2},0,0\right)$		$\left(0,\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$
	$\left(\frac{1}{4},\frac{7}{12},\frac{1}{6},0\right)$		$\left(0,\frac{1}{6},\frac{7}{12},\frac{1}{4}\right)$
	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{2},0\right)$		$\left(0,0,\frac{1}{2},\frac{1}{2}\right)$
	$\left(0,\frac{2}{3},\frac{1}{3},0\right)$		$(0,0,0,1)$
	$\left(0,\frac{1}{2},\frac{1}{2},0\right)$		

Jointly Inferring the Topology

$$f_{Y\,|\,\Psi,T}(\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot}\,|\,\psi,\mathbf{t})$$

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$$f_{Y\,|\,\Psi,T}(\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot}\,|\,\psi,\mathbf{t},\mathbf{s})$$

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Jointly Inferring the Topology

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$$f_{\Psi\,|\,Y,T}(\psi\,|\,\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot},\mathbf{t},\cancel{\mathbf{s}})$$

Let $\mathcal{M}$ be a collection of signatures, then naïvely we might presume random scan Gibbs between:

$$f_{Y|\,\mathcal{M},\Psi,T}(\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot}\,|\,\mathcal{M}_j,\psi,\mathbf{t})$$
$$f_{\Psi\,|\,\mathcal{M},Y,T}(\psi\,|\,\mathcal{M}_j,\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot},\mathbf{t})$$
$$f_{\mathcal{M}\,|\,\Psi,Y,T}(\mathcal{M}_j\,|\,\psi,\mathbf{y}_{1\cdot},\ldots,\mathbf{y}_{m\cdot},\mathbf{t})$$

explores the posterior of:

$$f_{\mathcal{M},\Psi,Y|\,T}(\mathcal{M}_j,\psi,\mathbf{y}\,|\,\mathbf{t})$$

But, Harris ergodicity & positivity concerns

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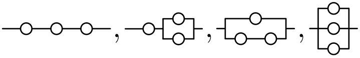
Toy Example of Problem

Assume $\mathbf{t}$ comprises 100 masked system lifetimes and $\mathcal{M}$ is all order 3 coherent systems with graph representation.

$\mathcal{M} = \Bigg\{$  $\Bigg\}$

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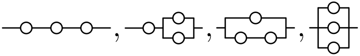
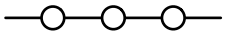
$\mathcal{M} = \{$  $\}$

Iteration 1

Let starting topology be $\mathcal{M}_1 =$  $\implies \mathbf{s} = (1, 0, 0)$.

Let ψ have sensible starting value.

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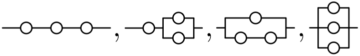

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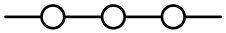
Let starting topology be $\mathcal{M}_1 =$  $\implies \mathbf{s} = (1, 0, 0)$.

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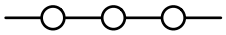
Then $f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{100\cdot} \mid$  $, \psi, \mathbf{t})$ will produce simulations st $t_i = y_{i(1:3)}\ \forall\, i$.

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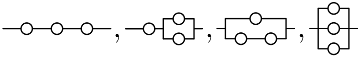
$\mathcal{M} = \{$  $\}$

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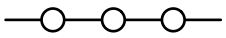
Let starting topology be $\mathcal{M}_1 =$  $\implies \mathbf{s} = (1, 0, 0)$.
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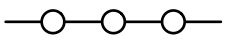
Then $f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{100\cdot} \mid$  $, \psi, \mathbf{t})$ will produce simulations st $t_i = y_{i(1:3)}\ \forall\, i$.

Thus, a move to $\mathcal{M}_2$ is harder. Moreover, moves to $\mathcal{M}_3$ or $\mathcal{M}_4$ are impossible.

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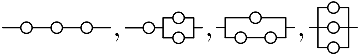
$$\mathcal{M} = \{ \dots \}$$



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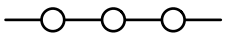
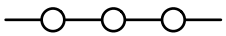
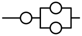
Let starting topology be $\mathcal{M}_1 =$  $\implies \mathbf{s} = (1, 0, 0)$.
Let ψ have sensible starting value.

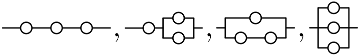
Then $f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{100\cdot} \mid$  $, \psi, \mathbf{t})$ will produce simulations st $t_i = y_{i(1:3)}\ \forall\, i$.

Thus, a move to $\mathcal{M}_2$ is harder. Moreover, moves to $\mathcal{M}_3$ or $\mathcal{M}_4$ are impossible.

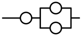
Assume a move to $\mathcal{M}_2 =$  is made though.

Toy Example of Problem

Assume $\mathbf{t}$ comprises 100 masked system lifetimes and $\mathcal{M}$ is all order 3 coherent systems with graph representation.

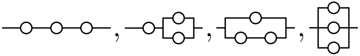
$\mathcal{M} = \Big\{$  $\Big\}$

Iteration 2

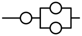
Topology is $\mathcal{M}_2 =$  $\implies \mathbf{s} = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

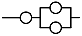
Toy Example of Problem

Assume $\mathbf{t}$ comprises 100 masked system lifetimes and $\mathcal{M}$ is all order 3 coherent systems with graph representation.

$\mathcal{M} = \Big\{$  $\Big\}$

Iteration 2

Topology is $\mathcal{M}_2 =$  $\implies \mathbf{s} = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

Now $f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{100\cdot} \mid$  $, \psi, \mathbf{t})$ will produce simulations where t_i is either $y_{i(1:3)}$ or $y_{i(2:3)}$. However, very low probability that $t_i = y_{i(1:3)}\ \forall\, i$ or $t_i = y_{i(2:3)}\ \forall\, i$.

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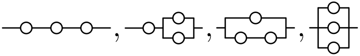
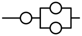
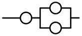
Toy Example of Problem

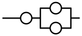
Assume $\mathbf{t}$ comprises 100 masked system lifetimes and $\mathcal{M}$ is all order 3 coherent systems with graph representation.

$\mathcal{M} = \{$  $\}$

Iteration 2

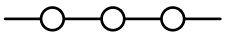
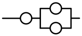
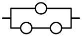
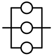
Topology is $\mathcal{M}_2 =$  $\implies \mathbf{s} = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

Now $f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{100\cdot} \mid$  $, \psi, \mathbf{t})$ will produce simulations where t_i is either $y_{i(1:3)}$ or $y_{i(2:3)}$. However, very low probability that $t_i = y_{i(1:3)}\ \forall\, i$ or $t_i = y_{i(2:3)}\ \forall\, i$.

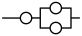
$\implies f_{\mathcal{M} \mid \Psi, Y, T}($  $\mid \psi, \mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{m\cdot}, \mathbf{t}) = 1$ is likely.

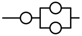
Toy Example of Problem

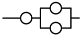
Assume $\mathbf{t}$ comprises 100 masked system lifetimes and $\mathcal{M}$ is all order 3 coherent systems with graph representation.

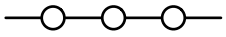
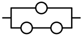
$$\mathcal{M} = \Big\{$$  ,  ,  ,  $$\Big\}$$

Iteration 2

Topology is $\mathcal{M}_2 =$  $\implies \mathbf{s} = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

Now $f_{Y \,|\, \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{100\cdot} \,|$  $, \psi, \mathbf{t})$ will produce simulations where t_i is either $y_{i(1:3)}$ or $y_{i(2:3)}$. However, very low probability that $t_i = y_{i(1:3)}\ \forall\, i$ or $t_i = y_{i(2:3)}\ \forall\, i$.

$\implies f_{\mathcal{M} \,|\, \Psi, Y, T}($  $| \,\psi, \mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{m\cdot}, \mathbf{t}) = 1$ is likely. Indeed:

$$f_{\mathcal{M} \,|\, \Psi, Y, T}\Big(\Big\{$$  ,  $$\Big\} \,\Big|\, \psi, \mathbf{y}_{1\cdot}, \dots, \mathbf{y}_{m\cdot}, \mathbf{t}\Big) > 0$$

$$\iff t_i = y_{i(1:3)}\ \forall\, i \text{ or } t_i = y_{i(2:3)}\ \forall\, i$$

The problem can be avoided by using the following full conditionals instead:

$$f_{\mathcal{M}, Y \mid \Psi, T}(\mathcal{M}_j, \mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot} \mid \psi, \mathbf{t})$$
$$f_{\Psi \mid \mathcal{M}, Y, T}(\psi \mid \mathcal{M}_j, \mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot}, \mathbf{t})$$

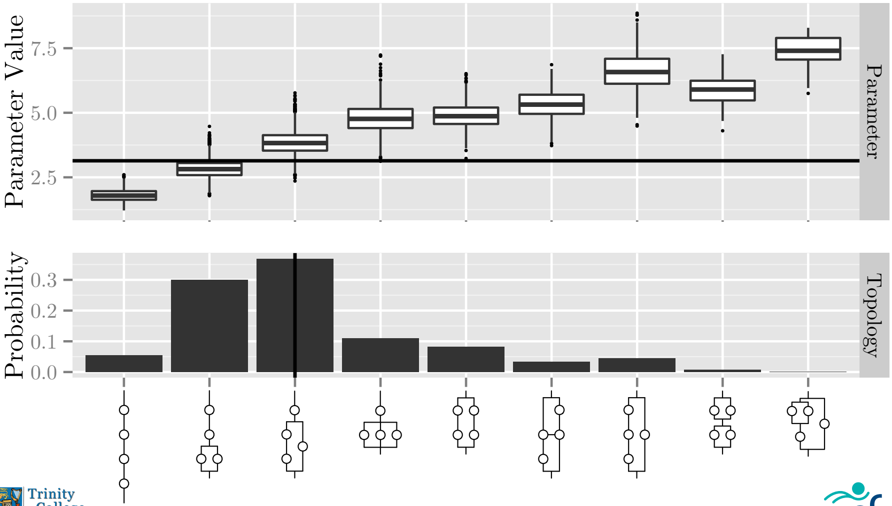
since the block marginals are concordant with positivity and ensure Harris ergodicity. Sampling the former sequentially:

$$f_{\mathcal{M} \mid \Psi, T}(\mathcal{M}_j \mid \psi, \mathbf{t})$$
$$f_{Y \mid \mathcal{M}, \Psi, T}(\mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot} \mid \mathcal{M}_j, \psi, \mathbf{t})$$

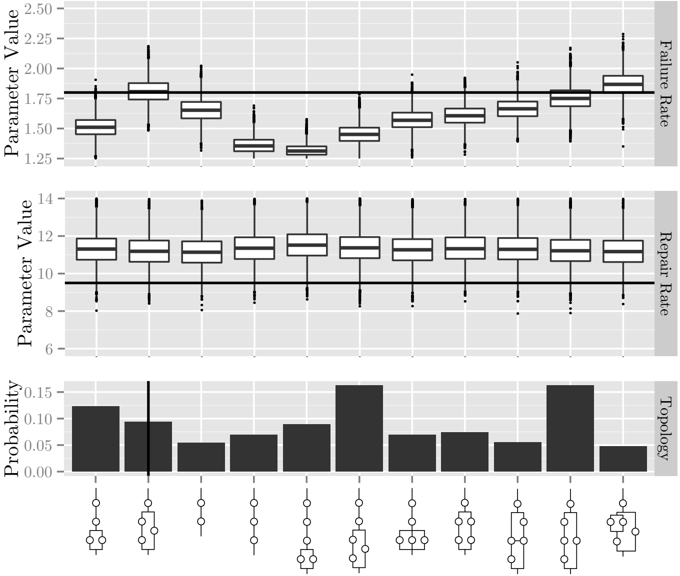
The latter is unchanged from before. For the former:

$$f_{\mathcal{M} \mid \Psi, T}(\mathcal{M}_j \mid \psi, \mathbf{t}) \propto \left\{ \prod_{i=1}^{m} f_{T \mid \Psi, \mathcal{M}}(t_i \mid \psi, \mathcal{M}_j) \right\} f_{\mathcal{M}}(\mathcal{M}_j)$$

Canonical Exponential Component Lifetime Example



Phase-type Component Lifetime Example



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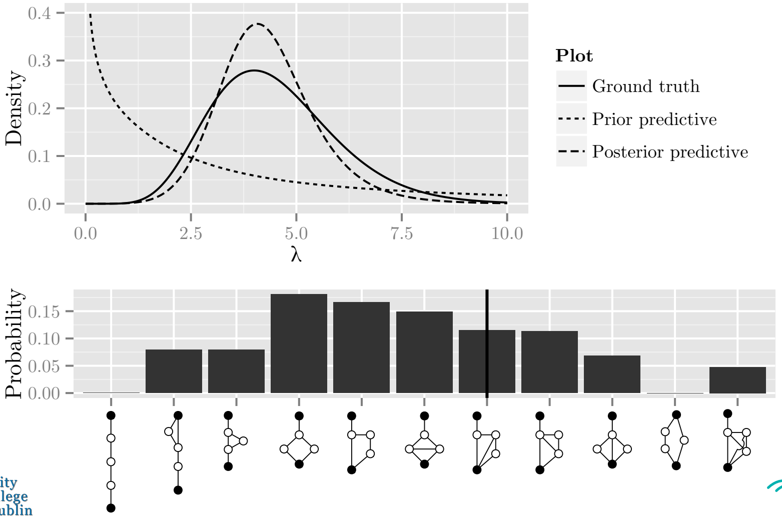
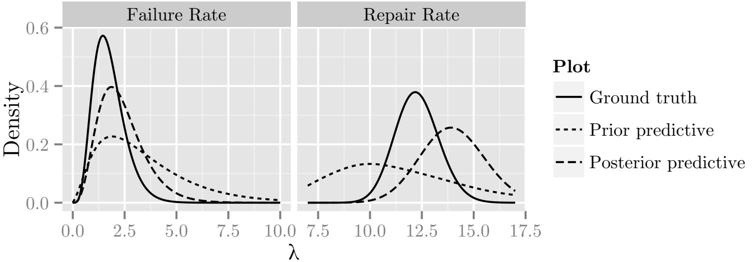
Exchangeable Systems

The i.i.d. systems assumption easily relaxed to exchangeability.



Phase-type Component Lifetimes

Extreme generality of the solution allows wide variety of component lifetime distributions. Solutions to the prerequisites have been derived for Phase-type distributed components.



May interpret as:

- Repairable redundant subsystems;
- Theoretically dense in function space of all positively supported continuous distributions.

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