Statistical methods for system reliability

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UTOPIAE Training School
22 November 2017

UTOPIÆ Uncertainty Treatment and Optimisation in Aerospace Engineering

MARIE CURIE ACTIONS

Durham University

Introduction

About me

- Cryptography and statistics — privacy preserving methodology
- Reliability theory
- Computational statistics, HPC for:
    - Markov-chain Monte Carlo on MPI clusters and GPUs
    - Hidden Markov Model acceleration for statistical genetics
    - machine learning
- Stats software
    - `ReliabilityTheory`, `HomomorphicEncryption`, …
    - RStudio AMIs for Amazon Web Services (incl Julia)

Bayesian inference

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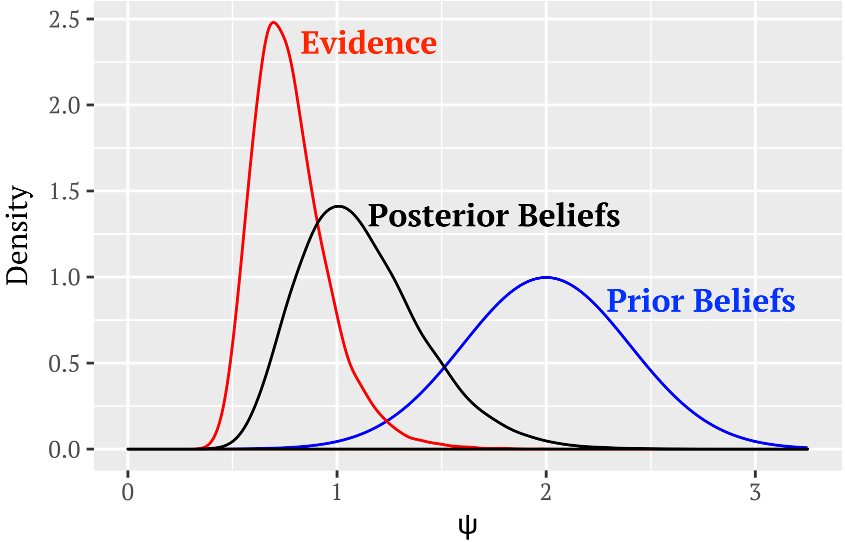
- Prior: all knowledge about ψ which is not contained in $\underline{t}$ is expressed via *prior* density $f_\Psi(\psi)$.
- Posterior: Bayes' Theorem enables us to rationally update the prior to our *posterior* belief in light of the new evidence (data).

Bayes' Theorem

$$f_{\Psi\,|\,T}(\psi\,|\,\underline{t}) = \frac{f_{T\,|\,\Psi}(\underline{t}\,|\,\psi)\; f_\Psi(\psi)}{\int_\Omega f_{T\,|\,\Psi}(\underline{t}\,|\,\psi)\; f_\Psi(d\psi)} \quad \propto f_{T\,|\,\Psi}(\underline{t}\,|\,\psi)\; f_\Psi(\psi)$$

Bayesian inference



Bayesian inference: prediction

Bayesian inference provides a natural method to perform prediction for a new observation $t^\star$, of the random variable T, in light of the observed data. All parameter uncertainty is integrated out to give a *posterior predictive density*:

$$f_{T \,|\, \underline{T}}(t^\star \,|\, \underline{t}) = \int f_{T \,|\, \Psi}(t^\star \,|\, \psi)\; f_{\Psi \,|\, \underline{T}}(d\psi \,|\, \underline{t})$$

Reminder of some probability densities

$X \sim \text{Beta}(\alpha, \beta)$ $\qquad \alpha > 0, \beta > 0$

$$\implies f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \qquad x \in [0,1]$$

$X \sim \text{Binomial}(n, p)$ $\qquad n \in \mathbb{N}, p \in [0,1]$

$$\implies f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x \in \{0, 1, \ldots, n\}$$

$X \sim \text{Beta-binomial}(n, \alpha, \beta)$ $\qquad n \in \mathbb{N}, \alpha > 0, \beta > 0$

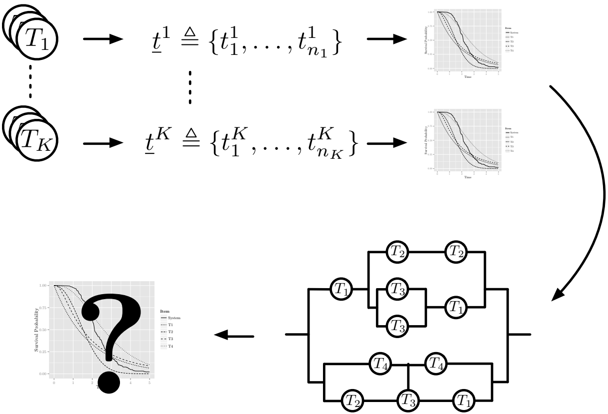
$$\implies f_X(x) = \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)}, \qquad x \in \{0, 1, \ldots, n\}$$

with $B(\alpha, \beta) := \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \left(= \dfrac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!} \text{ if } \alpha, \beta \in \mathbb{N}^+ \right)$

Problem Setting

Test data available on components to be used in a system.

Objective: Bayesian inference on system/network reliability given component test data.



Nonparametric method

A nonparametric model for components

At a fixed time t, probability component of type k functions is Bernoulli(p_t^k) for some unknown p_t^k.

$\implies$ number functioning at time t in iid batch of n_k is Binomial(n_k, p_t^k).

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Let $S_t^k \in \{0, 1, \ldots, n_k\}$ be number of working components in test batch of n_k components of type k. Then,

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Given test data $\underline{t}^k = \{t_1^k, \dots, t_{n_k}^k\}$, for each t we can form corresponding observation from Binomial model

$$s_t^k \triangleq \sum_{i=1}^{n_k} \mathbb{I}(t_i^k > t)$$

Bayesian inference for nonparametric model

Taking prior $p_t^k \sim \text{Beta}(\alpha_t^k, \beta_t^k)$, exploit conjugacy result

$$p_t^k \mid s_t^k \sim \text{Beta}(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

Then, posterior predicitive for number of components surviving in a new batch of m_k components is

$$C_t^k \mid s_t^k \sim \text{Beta-binomial}(m_k, \alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

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Summary: for any fixed t, s_t^k provides a minimal sufficient statistic for computing posterior predictive distribution of the number of components surviving to t in a new batch, without any parametric model for component lifetime being assumed.

Adding imprecision (see Frank's talk)

Imprecision can be added via prior sets for the Beta distribution. Hard to specify on α, β, so best to reparameterise:

$$n = \alpha + \beta \qquad y = \frac{\alpha}{\alpha + \beta}$$

- y is prior expectation of the probability, p, that the component functions;
- n is prior strength, since

$$\text{Var}(p) = \frac{y(1-y)}{n+1}$$

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Then we specify a prior set

$$\Pi = [\underline{n}, \overline{n}] \times [\underline{y}, \overline{y}]$$

Propagating uncertainty to the system

Now take collection of component types $k \in \{1, \ldots, K\}$, each with test data $\underline{t} = \{\underline{t}^1, \ldots, \underline{t}^k\}$, and corresponding collection of minimal sufficient statistics for a fixed t, $\{s_t^1, \ldots s_t^K\}$.

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Survival probability for a new system $S^\star$ comprising these component types follows naturally via posterior predictive and surival signature:

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\begin{aligned}
&P(T_{S^\star} > t \,|\, s_t^1, \dots s_t^K) \\
&= \int \cdots \int P(T_{S^\star} > t \,|\, p_t^1, \dots p_t^K) P(dp_t^1 \,|\, s_t^1) \dots P(dp_t^K \,|\, s_t^K)
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&= \int \cdots \int \left[\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) P\left(\bigcap_{k=1}^{K} \{C_t^k = l_k \mid p_t^k\} \right) \right] \\
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$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) \prod_{k=1}^{K} \int P(C_t^k = l_k \mid p_t^k) P(dp_t^k \mid s_t^k)$$

Final integral is simply the posterior predictive (Beta-binomial).

System survival probability

$$P(T_{S^*} > t \mid s_t^1, \dots s_t^K)$$
$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K)$$
$$\times \prod_{k=1}^{K} \binom{m_k}{l_k} \frac{B(l_k + \alpha_t^k + s_t^k, m_k - l_k + \beta_t^k + n_k - s_t^k)}{B(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)}$$

Incredibly easy to implement this algorithmically since survival signature has factorised the survival function by component type!

Why not structure function?

$$\phi(\underline{x}) = \prod_{j=1}^{s} \left(1 - \prod_{i \in C_j} (1 - x_i) \right)$$

where $\{C_1, \ldots, C_s\}$ is the collection of minimal cut sets of the system. Recall don't need $x \in \{0, 1\}$ — we can plug in probabilities. So why not?

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$$\begin{aligned} &P(T_{S^\star} > t \mid s_t^1, \dots s_t^K) \\ &\quad = \int \cdots \int \phi(p_t^{x_1}, \dots, p_t^{x_n}) P(dp_t^1 \mid s_t^1) \dots P(dp_t^K \mid s_t^K) \end{aligned}$$

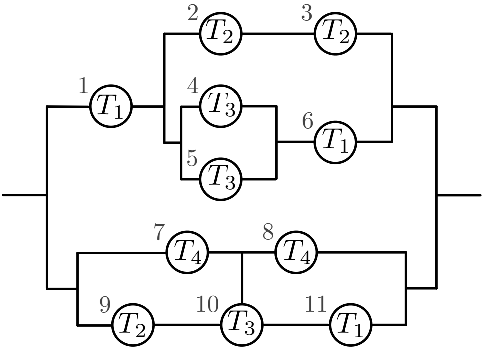
where $p_t^{x_i}$ is the element of $\{p_t^1, \dots, p_t^K\}$ corresponding to component i (i.e. component i is of type x_i)

Have fun with that integral for large K … !

Example

Example system layout, $K = 4$, $n = 11$

Example system:



$T_1 \sim \text{Exp}(\lambda_1 = 0.55)$

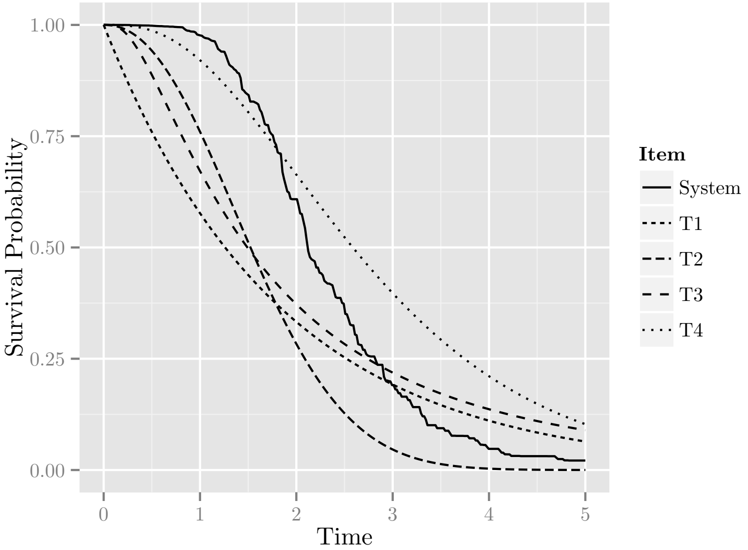
$T_2 \sim \text{Wei}(\lambda_2 = 1.8, \gamma_1 = 2.2)$

$T_3 \sim \text{Log-N}(\mu = 0.4, \tau = 1.234)$

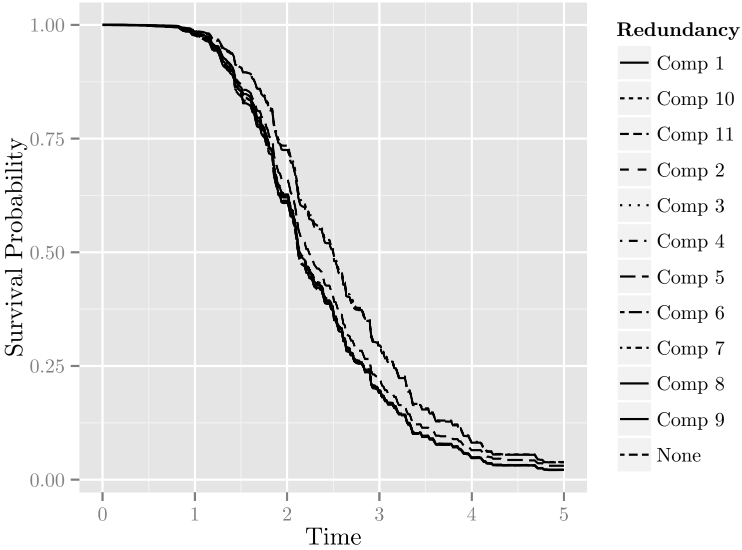
$T_4 \sim \text{Gam}(\lambda_3 = 0.9, \gamma_2 = 3.2)$

Simulated test data with $n_k = 100 \; \forall \, k$

Posterior predictive survival curves



Optimal redundancy?



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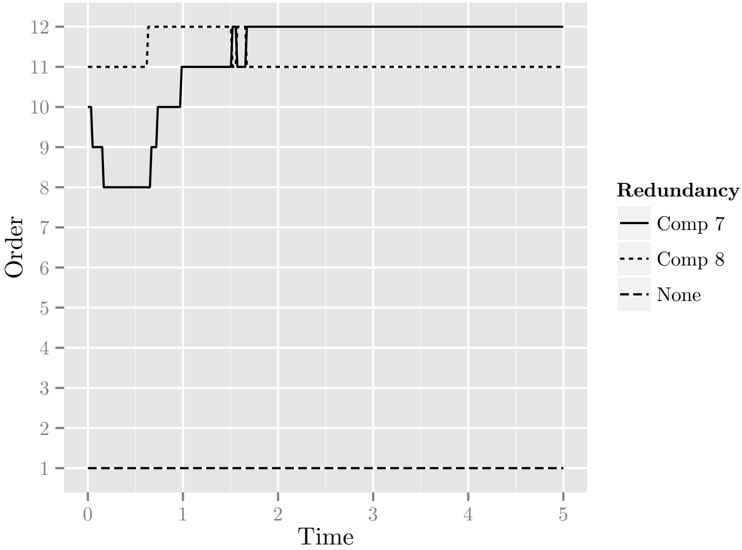

Parametric Method

Parametric models of components

The survival signature achieves the same factorisation of system lifetime when using parametric models for the components.

Model the lifetime of component k directly via likelihood function f_k

$$T_k \sim f_k(t; \psi_k)$$

As before, given test data $\underline{t}^k = \{t_1^k, \dots, t_{n_k}^k\}$ for component k, posterior density is:

$$f_{\Psi_k \mid \underline{T}^k}(\psi_k \mid \underline{t}^k) \propto f_{\Psi_k}(\psi_k) \prod_{i=1}^{n_k} f_k(t_i^k; \psi_k)$$

$$P(T_{S^\star} > t \,|\, \underline{t}^1, \ldots \underline{t}^K)$$
$$= \int \cdots \int P(T_{S^\star} > t \,|\, \psi_1, \ldots \psi_K) f_{\Psi_1 \,|\, \underline{T}^1}(d\psi_1 \,|\, \underline{t}^1) \ldots f_{\Psi_K \,|\, \underline{T}^K}(d\psi_K \,|\, \underline{t}^K)$$

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\end{aligned}
$$

Final term posterior predictive of l_k components of type k surviving to t.

Computing the integral for arbitrary models

Four possibilities. The posterior, $f_{\Psi_k \,|\, \underline{T}^k}(d\psi_k \,|\, \underline{t}^k)$, is:

1. in closed form and integral tractable;

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Thus, for samples $\psi_k^{(1)}, \dots, \psi_k^{(N)} \sim \Psi_k \,|\, \underline{T}^k$ we can always fall back to evaluating:

$$\frac{1}{N} \sum_{i=1}^{N} [F_k(t;\psi_k^{(i)})]^{m_k - l_k} [1 - F_k(t;\psi_k^{(i)})]^{l_k}$$

$$\xrightarrow{N\to\infty} \mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

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3. known upto normalising constant; $\rightarrow$ **Markov-chain MC**
4. unknown, due to black-box component model.

Note the integral is just:

$$\mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

Thus, for samples $\psi_k^{(1)}, \dots, \psi_k^{(N)} \sim \Psi_k \,|\, \underline{T}^k$ we can always fall back to evaluating:

$$\frac{1}{N} \sum_{i=1}^{N} [F_k(t;\psi_k^{(i)})]^{m_k - l_k} [1 - F_k(t;\psi_k^{(i)})]^{l_k}$$

$$\xrightarrow{N \to \infty} \mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

Computing the integral for arbitrary models

Four possibilities. The posterior, $f_{\Psi_k \,|\, \underline{T}^k}(d\psi_k \,|\, \underline{t}^k)$, is:

1. in closed form and integral tractable; $\rightarrow$ **easy**
2. known distribution, intractable integral; $\rightarrow$ **Monte Carlo**
3. known upto normalising constant; $\rightarrow$ **Markov-chain MC**
4. unknown, due to black-box component model. $\rightarrow$ **ABC**

Note the integral is just:

$$\mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

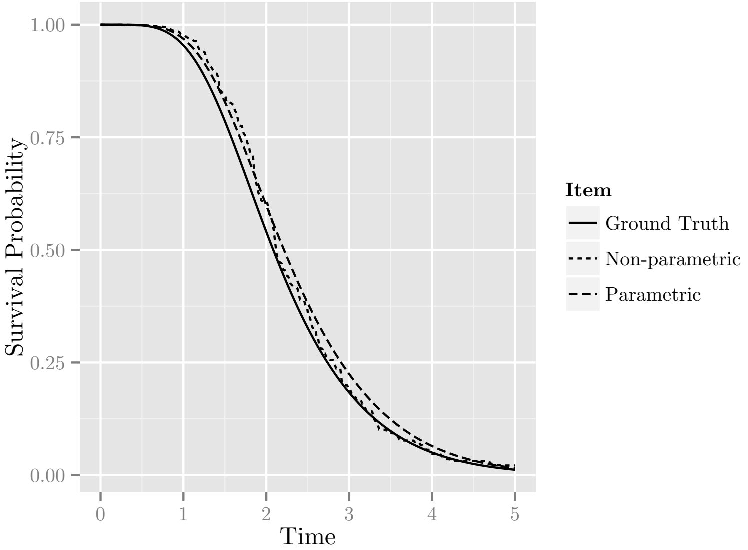
Thus, for samples $\psi_k^{(1)}, \dots, \psi_k^{(N)} \sim \Psi_k \,|\, \underline{T}^k$ we can always fall back to evaluating:

$$\frac{1}{N} \sum_{i=1}^{N} [F_k(t;\psi_k^{(i)})]^{m_k - l_k} [1 - F_k(t;\psi_k^{(i)})]^{l_k}$$

$$\xrightarrow{N \to \infty} \mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

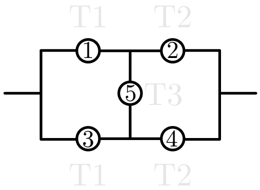
Example

Posterior predictive survival curves for both methods



Software

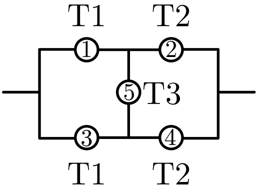
R package: ReliabilityTheory




```
library("ReliabilityTheory")

sys <- graph.formula(s -- 1 -- 2 -- t, s -- 3 -- 4 -- t,
                     1 -- 5 -- 2, 3 -- 5 -- 4)
```

R package: ReliabilityTheory




```
sys <- graph.formula(s -- 1 -- 2 -- t, s -- 3 -- 4 -- t,
                     1 -- 5 -- 2, 3 -- 5 -- 4)
sys <- setCompTypes(sys, list("T1" = c(1, 3),
                              "T2" = c(2, 4),
                              "T3" = c(5)))
sig <- computeSystemSurvivalSignature(sys)
```

References

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