Privacy and Security in Bayesian Inference

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Introduction

Joint work with ...

This project is now joint work with Sam Livingstone, UCL.

Motivation

Security in statistics applications is a growing concern:

- computing in a ‘hostile’ environment (e.g. cloud computing);
- donation of sensitive/personal data (e.g. medical/genetic studies);
- complex models on constrained devices (e.g. smart watches)
- running confidential algorithms on confidential data (e.g. engineering reliability)

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- Differential privacy
  - on outcomes of ‘statistical queries’
  - guarantees of privacy for individual observations

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- Data privacy
    - at rest
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    - data pooling
- Model privacy
    - prior distributions
    - model formulation

The standard problem …

- Three parties have private data of the same type.
- There is a Bayesian model of mutual interest.
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Agreed model

$\pi(\cdot \mid \psi)$

$\pi(\psi)$

Private data

$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=1}^{n_1}$

$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=n_1+1}^{n_1+n_2}$

$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=n_1+n_2+1}^{N}$

Work with Murray Pollock, Hongsheng Dai & Gareth Roberts.

The perspective for today …

- **Eve** has a private model, including prior information which may itself be private.
- **Cain** and **Abel** have private data which is relevant to the fitting of Eve's model.

Can Eve fit a model, pooling data from Cain and Abel without observing their raw data and without revealing her model and prior information? Abel also doesn't trust Cain …

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$$\pi(\psi)$$

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Cryptography the solution?

Encryption can provide security guarantees …

Easy

$$\mathtt{Enc}(k_p, m) \rightleftharpoons c \qquad\qquad \mathtt{Dec}(k_s, c) = m$$

Hard without k_s

… but is typically 'brittle'.

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Arbitrary addition and multiplication is possible with **fully homomorphic encryption** schemes (Gentry, 2009).

$$\begin{tikzcd}
m_1 \arrow[d, "\mathtt{Enc}(k_p, \cdot)"] & m_2 \arrow[d] \arrow[r, "+"] & m_1 + m_2 \\
c_1 & c_2 \arrow[r, "\oplus"] & c_1 \oplus c_2 \arrow[u, "\mathtt{Dec}(k_s, \cdot)"']
\end{tikzcd}$$

Back to the problem ...

$\pi(\cdot \,|\, \psi)$

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$$\downarrow$$

$$\mathbf{x}_i^\star = \mathrm{Enc}(k_p, \mathbf{x}_i)$$

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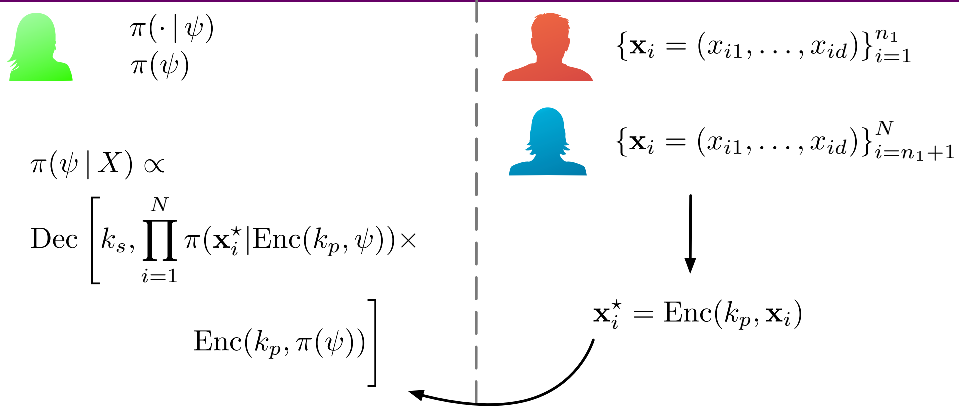
$$\mathbf{x}_i^\star = \mathrm{Enc}(k_p, \mathbf{x}_i)$$

$$\pi(\psi \mid X) \propto$$
$$\mathrm{Dec}\left[k_s, \prod_{i=1}^{N} \pi(\mathbf{x}_i^\star | \mathrm{Enc}(k_p, \psi)) \times \mathrm{Enc}(k_p, \pi(\psi))\right]$$

Back to the problem ...



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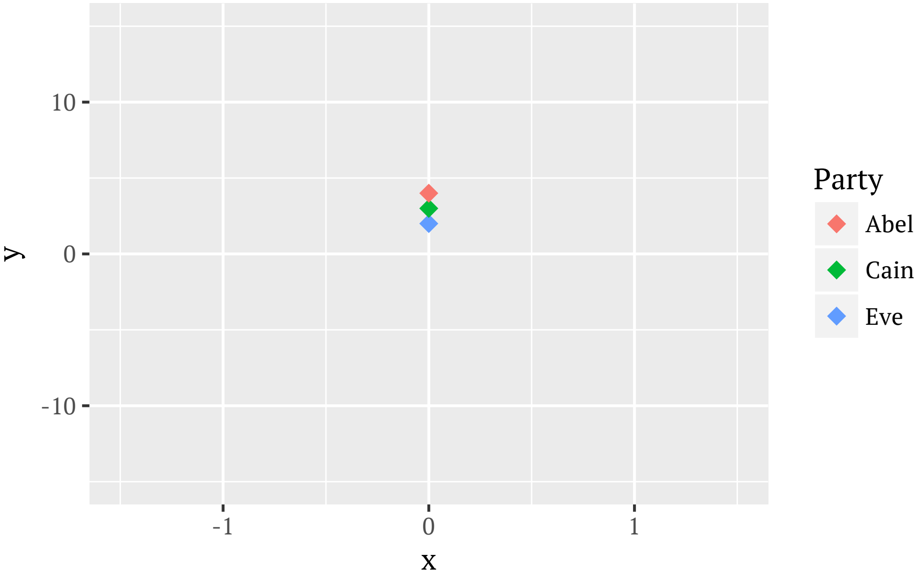
✗ Likelihood restricted to low degree polynomials

✗ Can only handle very small *N* due to multiplicative depth

✗ MAP/posterior? How? MCMC?

✗ Who holds secret key?

(Simplified) look at Homomorphic Secret Sharing



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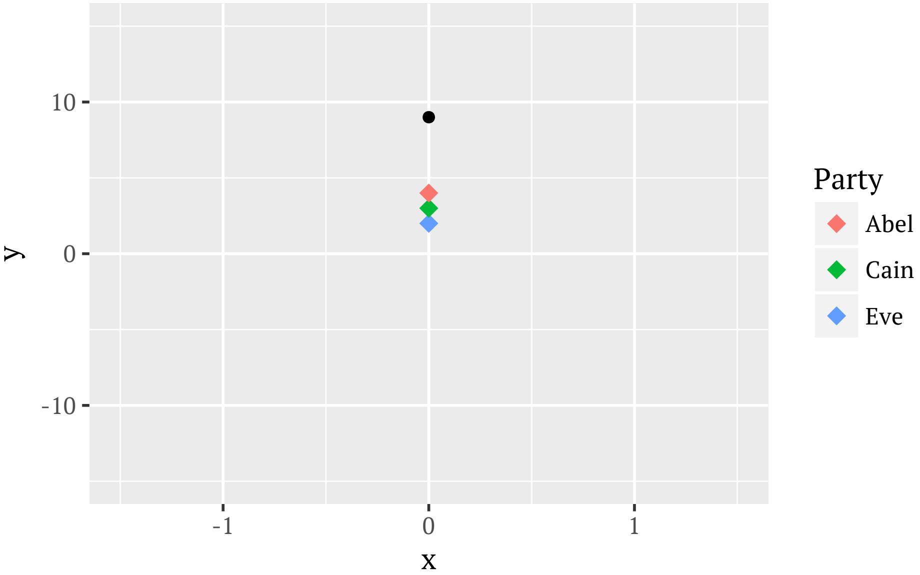

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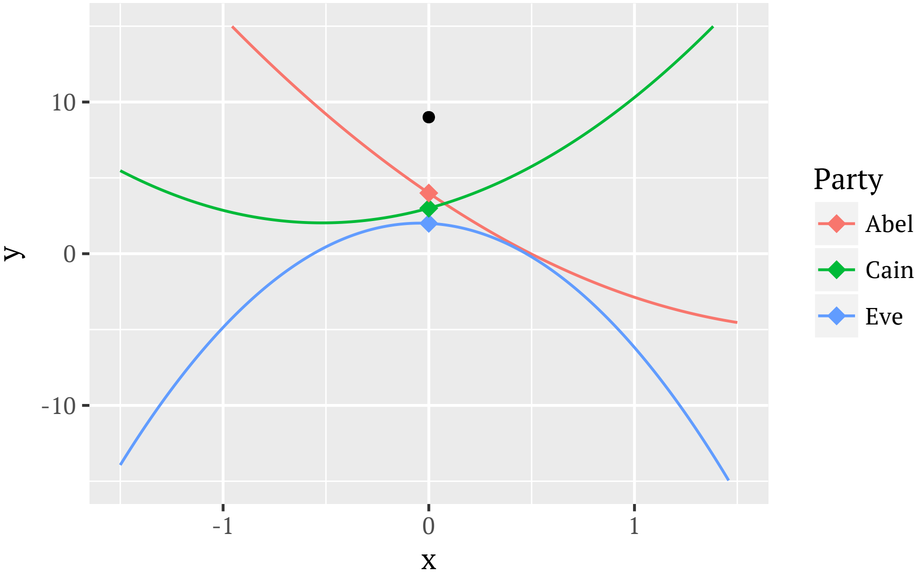

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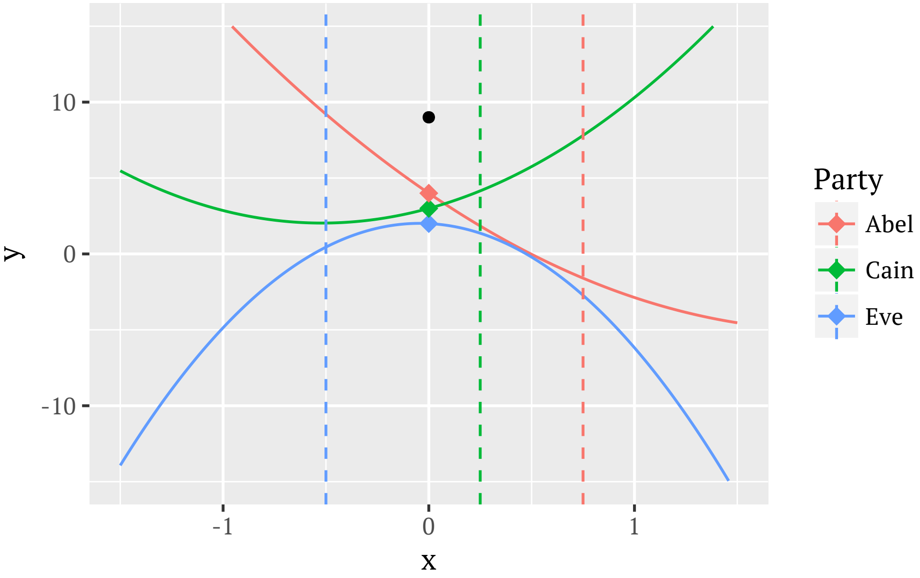

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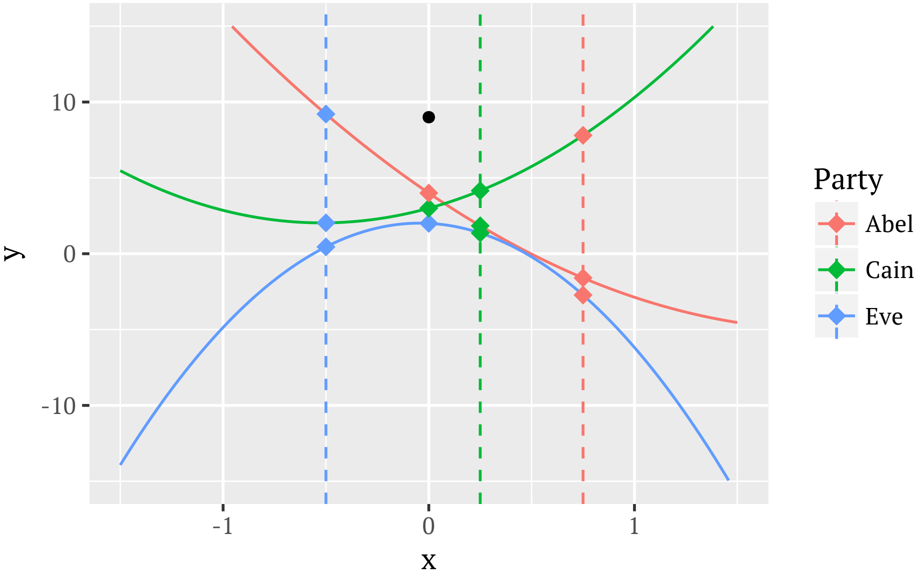

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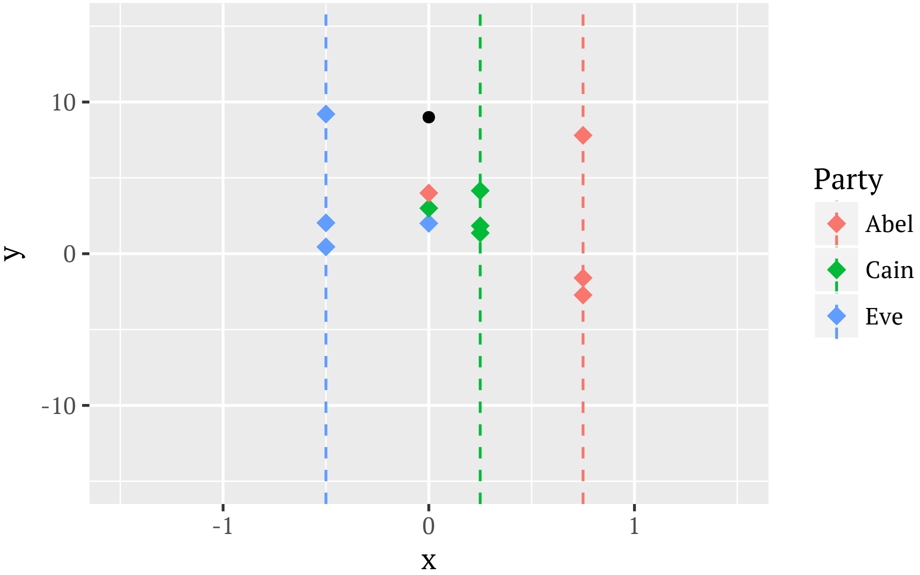

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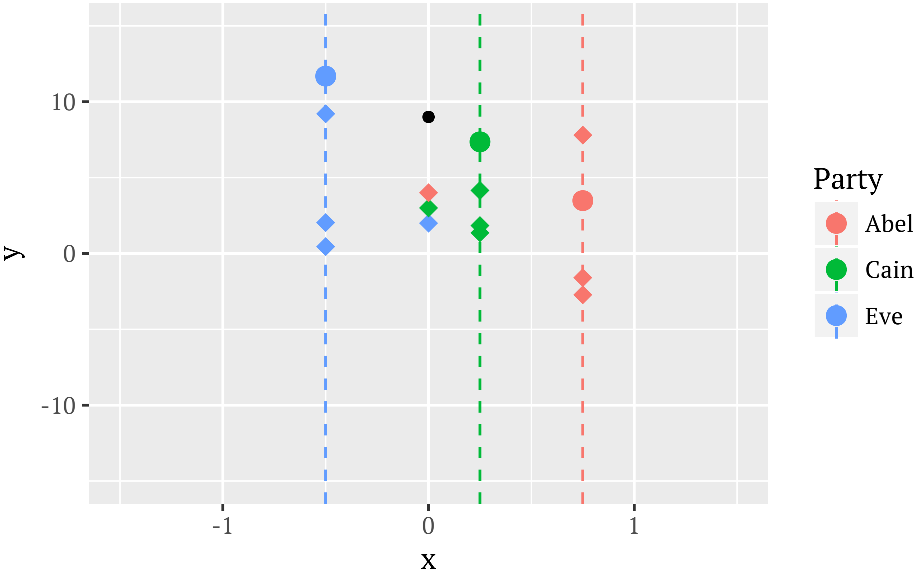

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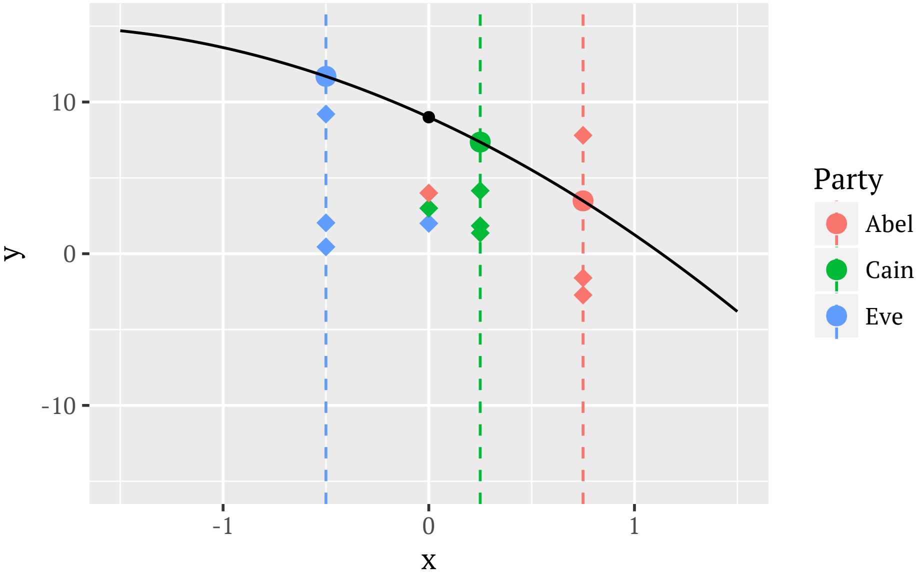

Eve, Cain & Abel

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Approximate Bayesian Computation

Approximate Bayesian Computation

1. Sample $\psi_j \sim \pi(\psi),\ j \in \{1, \ldots, m\}$
2. For each ψ_j, simulate a dataset Y_j from $\pi(\cdot \mid \psi_j)$ of the same size, N, as X.
3. Accept ψ_j if $d(S(X), S(Y_j)) < \varepsilon$.

Where $S(\cdot)$ is some (vector) of summary statistics; $d(\cdot, \cdot)$ is a distance metric; and ε is a user defined threshold.

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Problems: $d(\cdot,\cdot)$ can only be low degree polynomials;
Must compute $S(\cdot)$ secretly for Cain and Abel's pooled data;
Naïve ABC performs poorly & choosing ε blindfolded.

Naïve encrypted ABC (I) – Eve & data owners $1, \dots, P$

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5. All compute $d_j^{\star p} = d\left(S^{\star p}(X), S^{\star p}(Y_j)\right)$, where $d(\cdot)$ is a **homomorphically computable distance metric.**

Naïve encrypted ABC (II) – Eve & data owners $1, \ldots, P$

6. All send their shares, $d_j^{\star p}$, to a randomly chosen data owner $k \in 1, \ldots, P$

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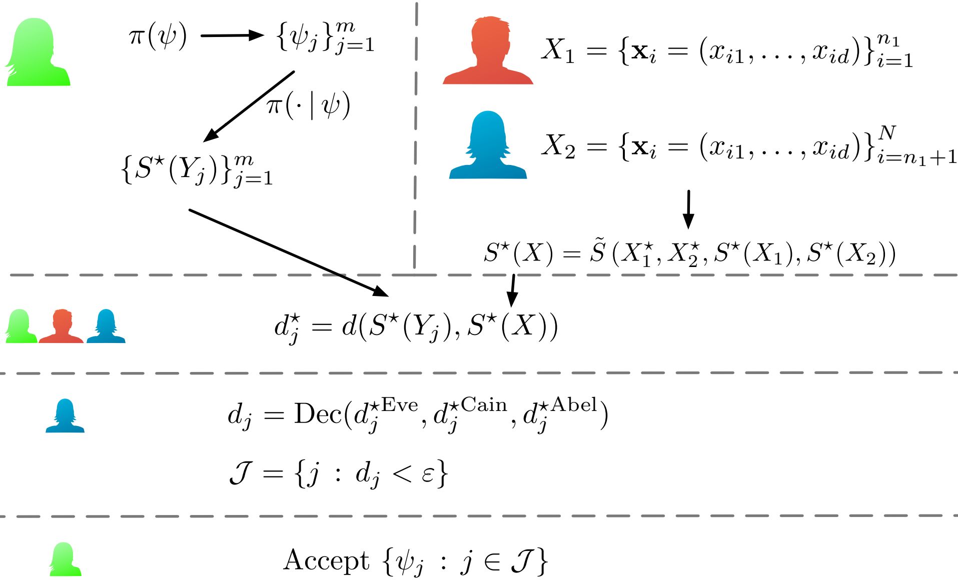
8. Data owner k sends to Eve a list of those indices j such that $d_j < \varepsilon$.

Naïve encrypted ABC (III) – in pictures



$$\pi(\psi) \longrightarrow \{\psi_j\}_{j=1}^m$$

$$\pi(\cdot \mid \psi)$$

$$\{S^\star(Y_j)\}_{j=1}^m$$

$$X_1 = \{\mathbf{x}_i = (x_{i1}, \ldots, x_{id})\}_{i=1}^{n_1}$$

$$X_2 = \{\mathbf{x}_i = (x_{i1}, \ldots, x_{id})\}_{i=n_1+1}^{N}$$

$$S^\star(X) = \tilde{S}\left(X_1^\star, X_2^\star, S^\star(X_1), S^\star(X_2)\right)$$

$$d_j^\star = d(S^\star(Y_j), S^\star(X))$$

$$d_j = \mathrm{Dec}(d_j^{\star\mathrm{Eve}}, d_j^{\star\mathrm{Cain}}, d_j^{\star\mathrm{Abel}})$$

$$\mathcal{J} = \{j \,:\, d_j < \varepsilon\}$$

$$\text{Accept } \{\psi_j \,:\, j \in \mathcal{J}\}$$

Points to note

- Samples ψ_j are never seen by Cain and Abel
- Eve learns only an accept/reject
    - Final distances between summary statistics decrypted by Cain or Abel
- Cain and Abel do not learn about each other's data
    - only see composite distance between pooled summary stats and Eve's simulation
    - can make distances information theoretically secure by adding random values generated by Cain, Abel and Eve
- **BUT**, Cain and Abel do have to know $S(\cdot)$, which in most ABC settings is model dependent $\implies$ risk to Eve

Obstacles to cryptographic ABC

- Homomorphically computable pooling of summary statistics
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- Homomorphically computable pooling of summary statistics
- Summary statistics that don't reveal model
- Homomorphically computable distance metric
- Blindfold selection of ε
    - Propose using ABC-PMC/SMC, with distance chosen to retain $\alpha\%$ of samples instead. Eve then uses accepted ψ_j on step t to propose step $t+1$ and repeat algorithm.
    - Standard idea — details omited.

Cryptographically Secure Inference

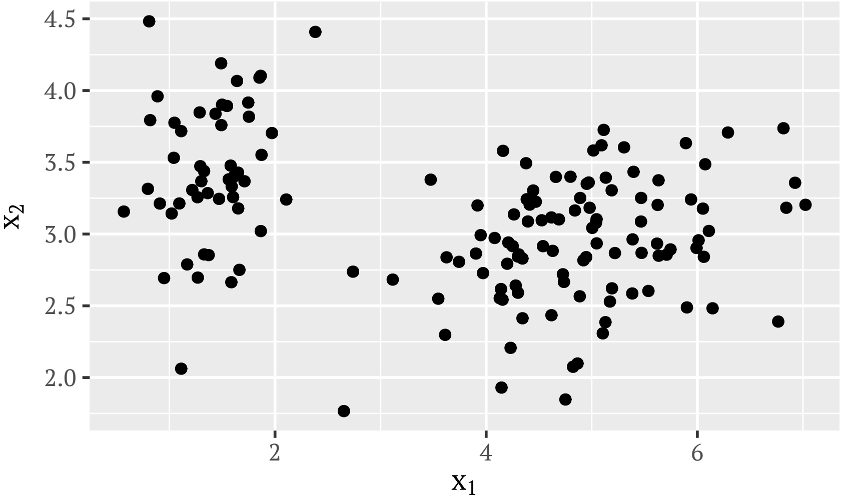
Collection of Coarse Random Marginals (CCRM)

Construct in the manner of a decision forest:

- Grow T trees, each to predetermined fixed depth L
- Choose variable $v \in \{1, \ldots, d\}$ uniformly at random
- Each split point uniformly at random in range of $x_{\cdot v}$
    - Thus Cain and Abel must provide range of each variable in the data, though this range need not be tight
    - e.g. release $(\min_i x_{iv} + \eta, \max_i x_{iv} + \eta)$ for $\eta \sim N(0, \sigma^2)$ with σ^2 chosen not to exclude too large a range
- $\mathbf{s} = S(\cdot)$ is then the counts of observations in each terminal leaf
    - vector of $T2^L$ counts
    - $\tilde{S}(\cdot)$ is then simply vector addition
- Define

$$d(S(X), S(Y_j)) = \sum_{i=1}^{T2^L} \left(s_i^X - s_i^{Y_j} \right)^2$$

Collection of Coarse Random Marginals (CCRM)



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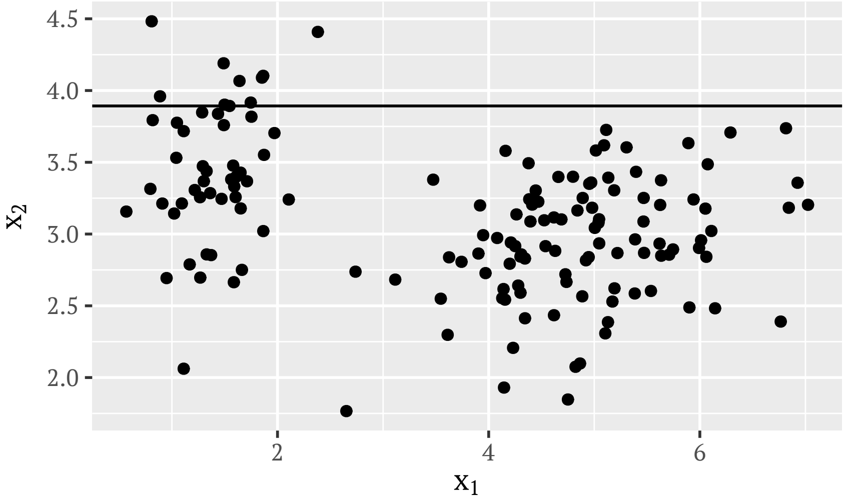

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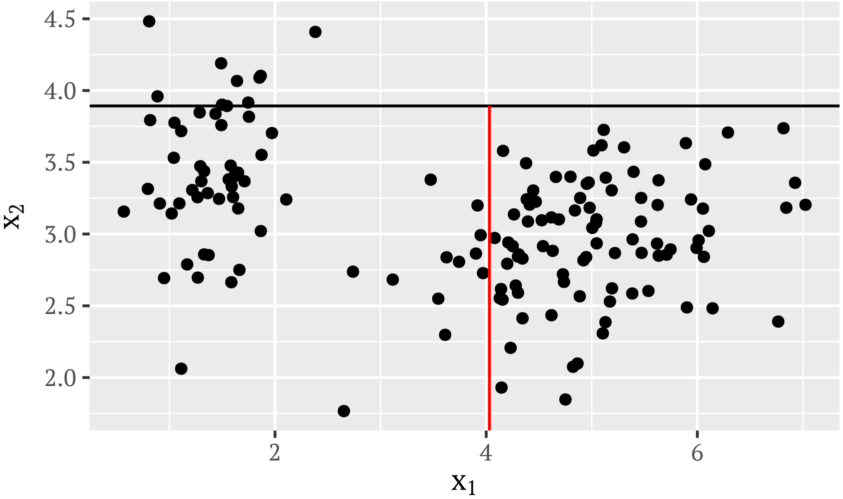

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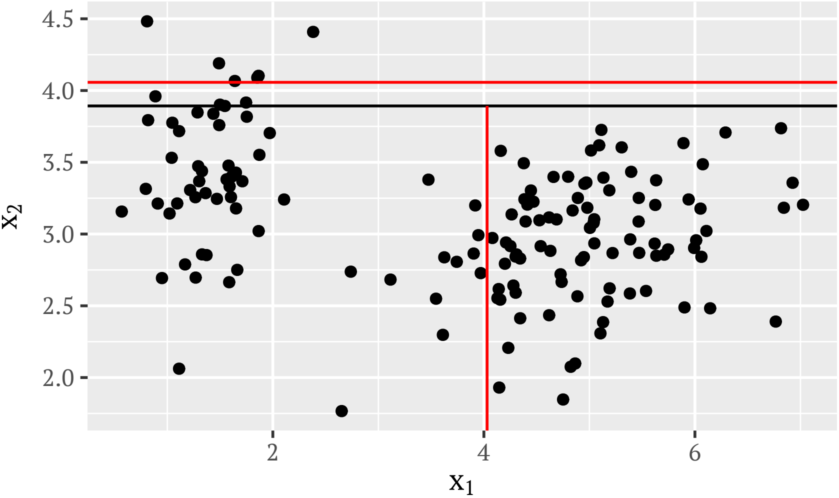

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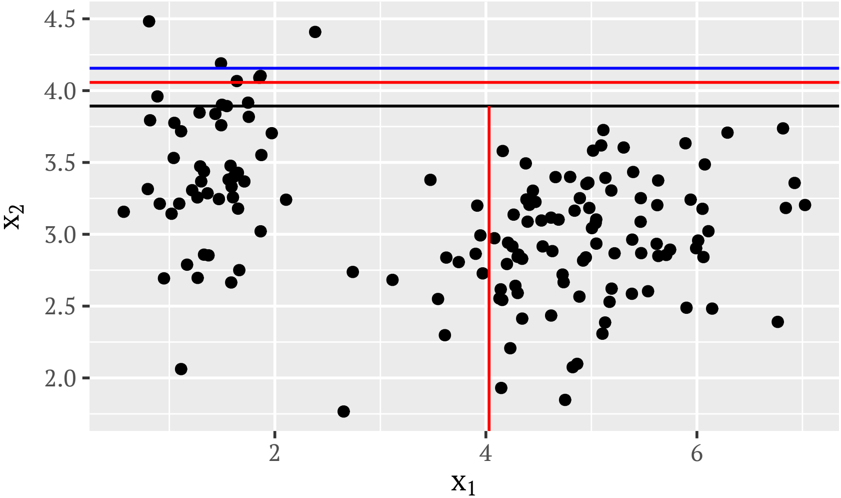

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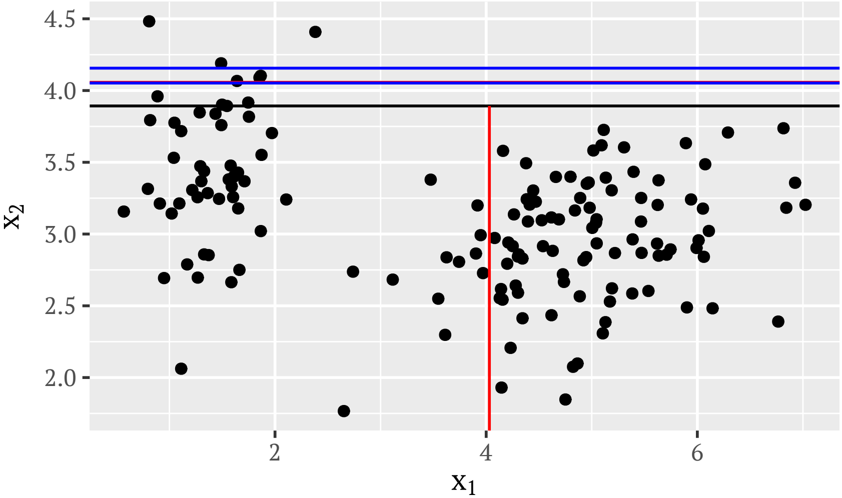

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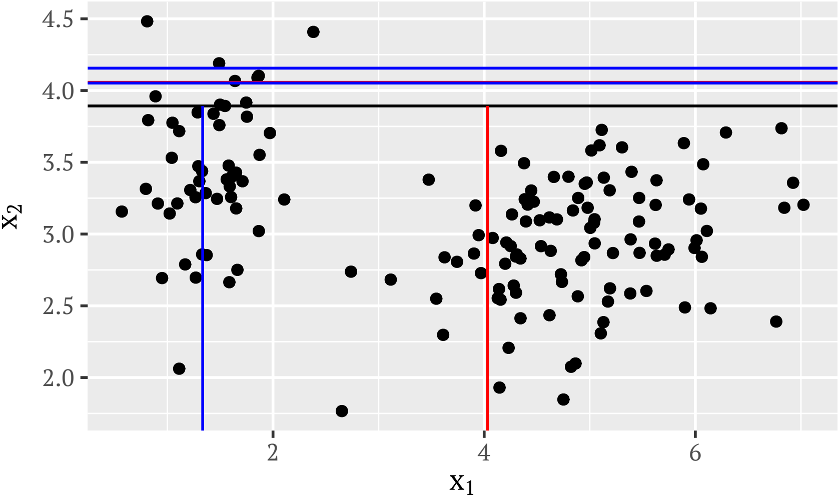

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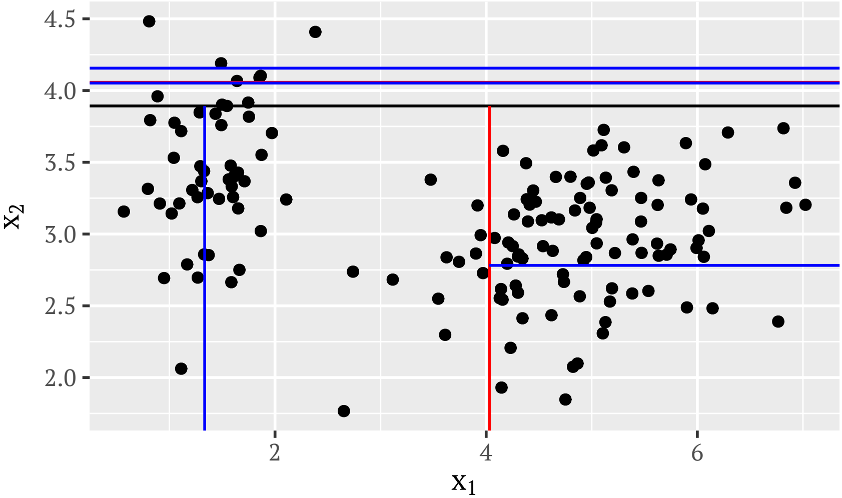

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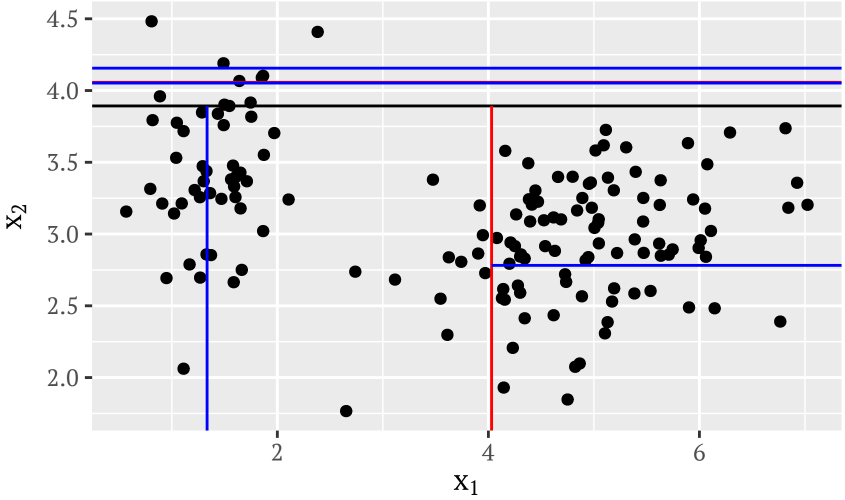


$$S(X) = (\dots, 3, 3, 0, 3, 43, 33, 64, 24, \dots)$$

CCRM solutions

- Homomorphically computable pooling of summary statistics
  - **simple vector addition**
- Summary statistics that don't reveal model
  - **CCRM is completely random, grown the same way for all models and data sets. Only weak information about range of each variable leaked.**
- Homomorphically computable distance metric
  - **sum of squared differences**

Variance of distance metric per CRM

Lemma *Let the random variable V be multinomially distributed with success probabilities $p = (p_1, \ldots, p_k)$ for n trials. Then,*

$$
\begin{aligned}
&\mathrm{Var}\left(\sum_{i=1}^{k}(V_i - c_i)^2\right) \\
&= \sum_{i=1}^{k}\Bigg[\left({}^{n}C_{n-4} - n^2(n-1)^2\right)p_i^4 + \left(6\,{}^{n}C_{n-3} + 2n(n-1)(4c_i - n)\right)p_i^3 \\
&\quad + \left(7n(n-1) - n^2 - 4c_i n(2n-3)(1+c_i)\right)p_i^2 + \left(n + 4c_i n(c_i - 1)\right)p_i \\
&\quad + \sum_{\substack{j=1 \\ i\neq j}}^{k}\Big[- n(2c_i - 1)(2c_j - 1)p_i p_j + 2n(n-1)(2c_j - 1)p_i^2 p_j \\
&\quad + 2n(n-1)(2c_i - 1)p_i p_j^2 - 2n(n-1)(2n-3)p_i^2 p_j^2\Big]\Bigg]
\end{aligned}
$$

$\Longrightarrow$ can be used to weight random marginals differently.

ABCDE: Approximate Bayesian Computation Done Encrypted

Tying it all together:

- ABC-PMC/SMC
- Homomorphic Secret Sharing with data pooling
- CCRM summary statistic protecting model/prior privacy
- Pooled $S(\cdot)$ computable encrypted from multiple data owners
- Distance computable encrypted and not learned by modeller
- Variance of each CRM computable encrypted for weighting

Selected connections in ABC literature

- Bernton, E., Jacob, P. E., Gerber, M., & Robert, C. P. (2019). Approximate Bayesian computation with the Wasserstein distance. *Journal of the Royal Statistical Society: Series B*, 81(**2**), 235-269.
- Gutmann, M. U., Dutta, R., Kaski, S., & Corander, J. (2017). Likelihood-free inference via classification. *Statistics and Computing*, 1-15.
- Fearnhead, P., & Prangle, D. (2012). Constructing summary statistics for approximate Bayesian computation: semi-automatic approximate Bayesian computation. *Journal of the Royal Statistical Society: Series B*, 74(**3**), 419-474.

Examples

Toy example

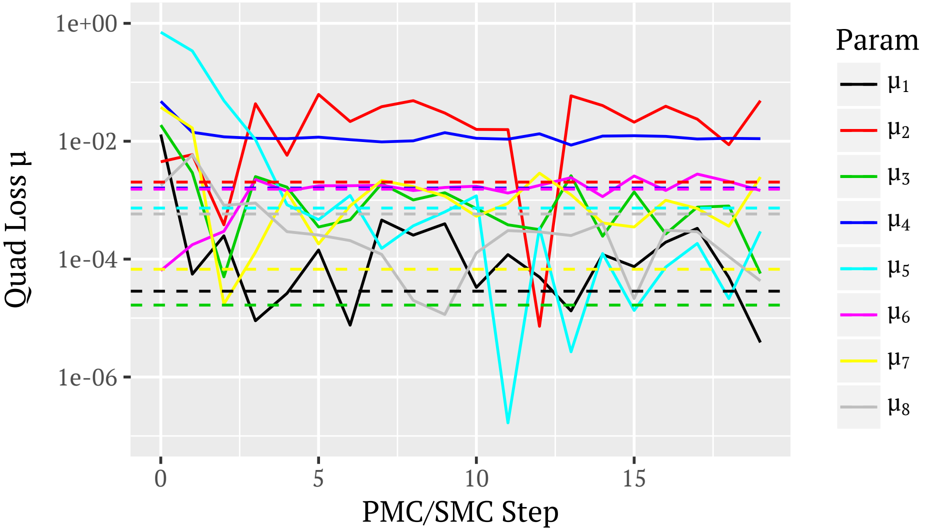
Super simple first example, 8-dimensional multivariate Normal.

$$X \sim \mathrm{N}(\boldsymbol{\mu} = \mathbf{0}, \Sigma = I)$$
$$\mu_i \sim \mathrm{N}(\eta_i, \sigma = 2)$$

where η_i chosen independently uniformly at random on the interval $[-1, 1]$ for repeated experiments.

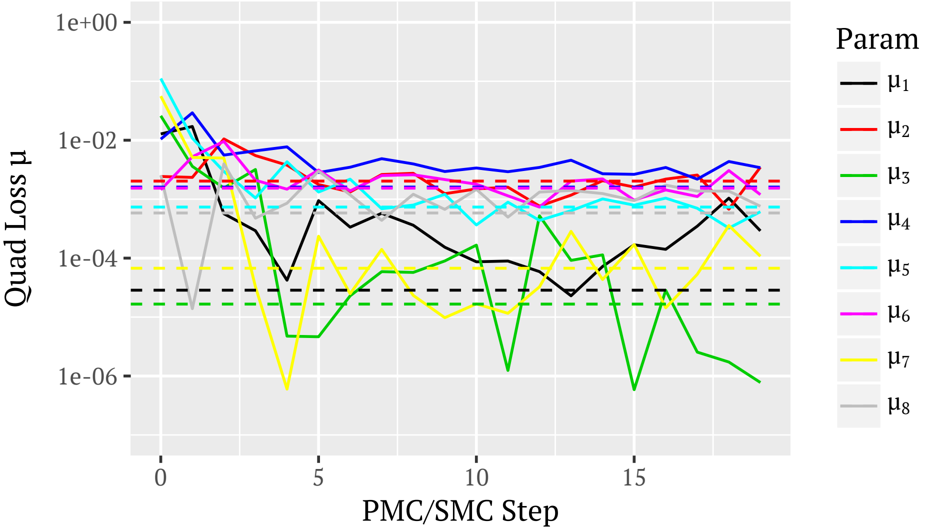
- Simulate $n = 1000$ observations
- Range of all dimensions taken to be $[-4, 4]$ for construction of CCRM, without checking true range of X
- Standard ABC used $S(X) = (\bar{x}_1, \dots, \bar{x}_8)$

Toy example: 8D Normal, marginal quadratic loss



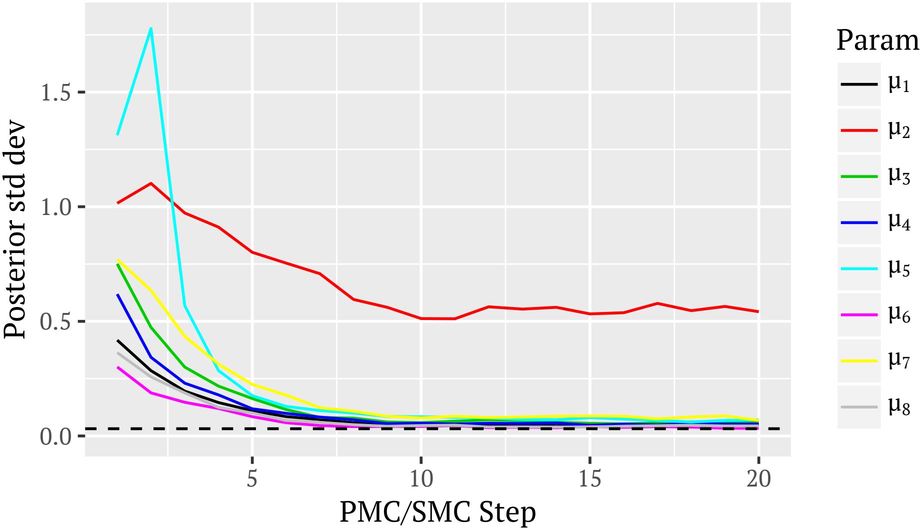
$$n = 10^3, T = 20, L = 2, m = 10^4, \alpha = 0.01$$

Toy example: 8D Normal, marginal quadratic loss



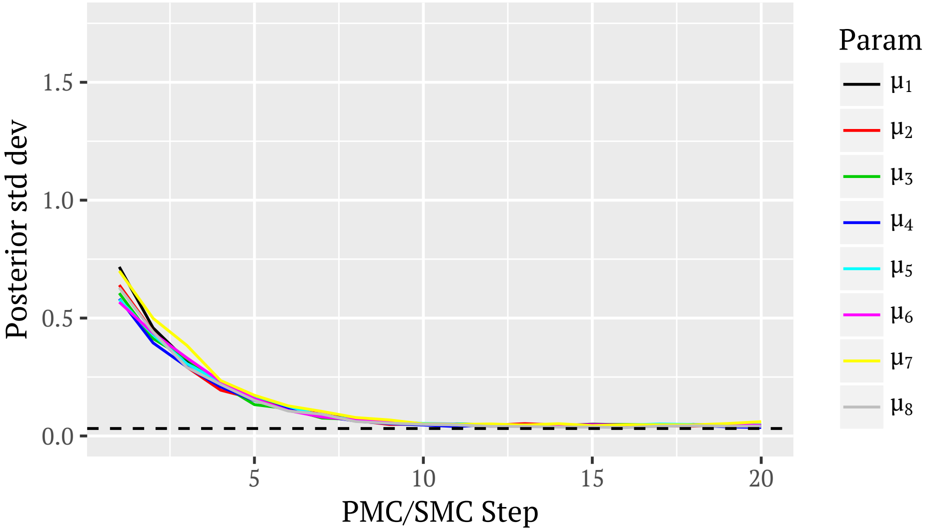
$$n = 10^3, T = 1000, L = 2, m = 10^4, \alpha = 0.01$$

Toy example: 8D Normal, marginal posterior σ



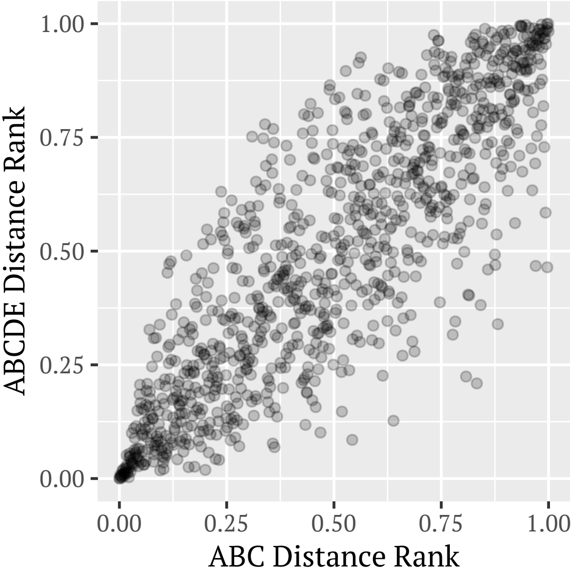
$$n = 10^3, T = 20, L = 2, m = 10^4, \alpha = 0.01$$

Toy example: 8D Normal, marginal posterior σ



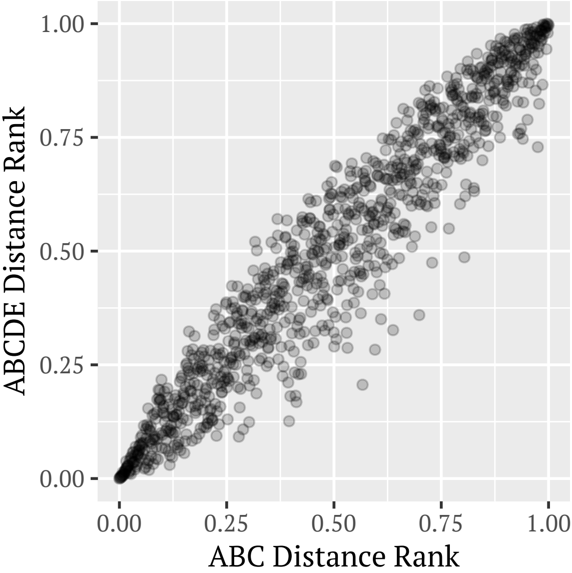
$$n = 10^3, T = 1000, L = 2, m = 10^4, \alpha = 0.01$$

Toy example: distance concordance



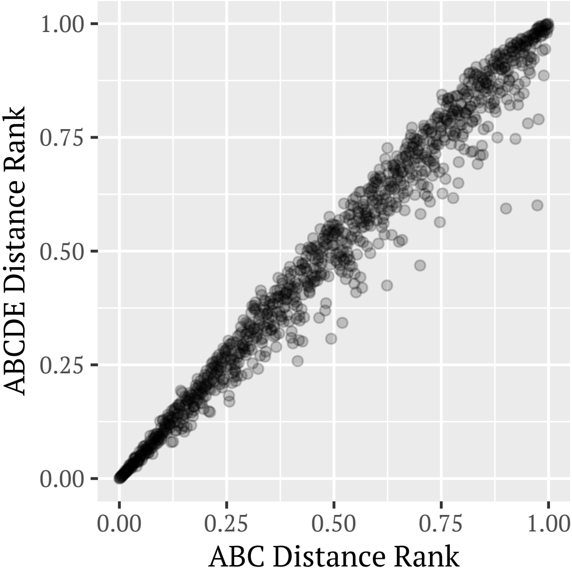
$T = 20$

Toy example: distance concordance



$T = 100$

Toy example: distance concordance



$T = 1000$

Expected quadratic loss

Can understand lowest ABC error achievable without Monte Carlo error:

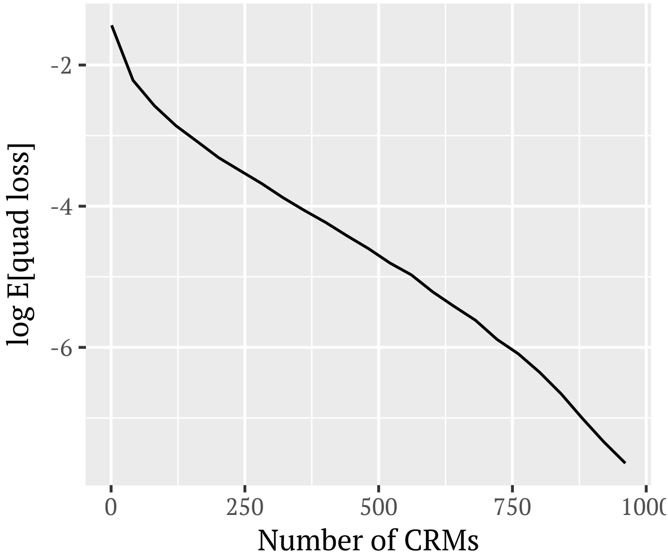
$$\mathbb{E}\left[(\mu-\hat{\mu})^2 \,|\, T=t\right]$$
$$=\frac{1}{|\mathcal{A}^t|}\int_{\mathcal{A}^t}\left(\mu-\int_{-\infty}^{\infty}\theta\,\mathbb{P}\left(S(x)=S(x^{\text{obs}})\,|\,da_1,\dots,da_t\right)\,\pi(d\theta)\right)^2$$

because for 1-level CRMs:

$$\mathbb{P}\left(S(x)=S(x^{\text{obs}})\,|\,da_1,\dots,da_t\right)$$
$$=\prod_{k=1}^{t}\binom{n}{m_k}F_{v_k}(X<a_k)^{m_k}(1-F_{v_k}(X<a_k))^{n-m_k}$$

where $m_k=\#\{i : x_i^{\text{obs}}<a_k\}$.

Expected quadratic loss



g-and-k distribution (Haynes et al. 1997)

Defined via inverse distribution function

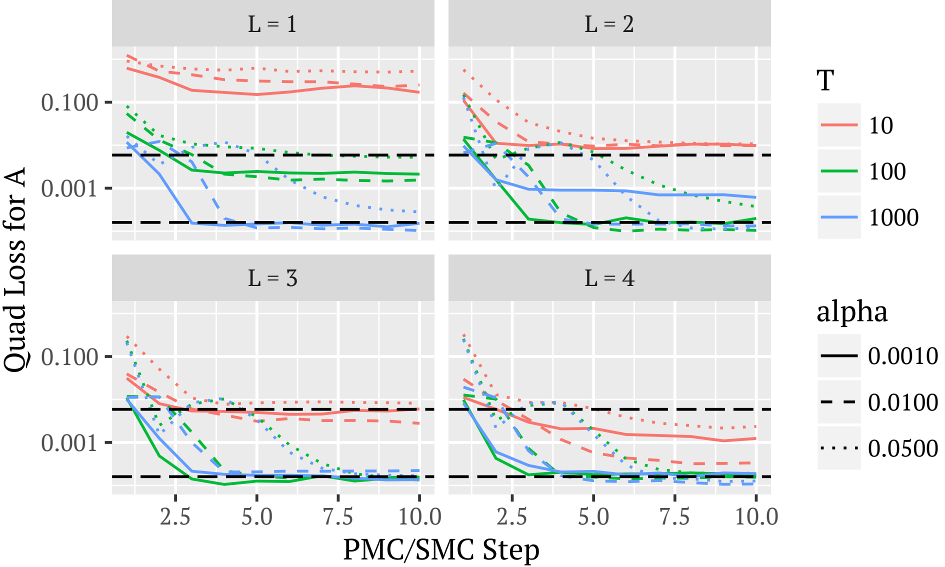
$F^{-1}(x \mid A, B, g, k) =$

$$A + B \left[1 + 0.8 \frac{1 - \exp\big(-g\Phi^{-1}(x)\big)}{1 + \exp\big(-g\Phi^{-1}(x)\big)} \right] \big(1 + \Phi^{-1}(x)^2\big)^k \Phi^{-1}(x)$$

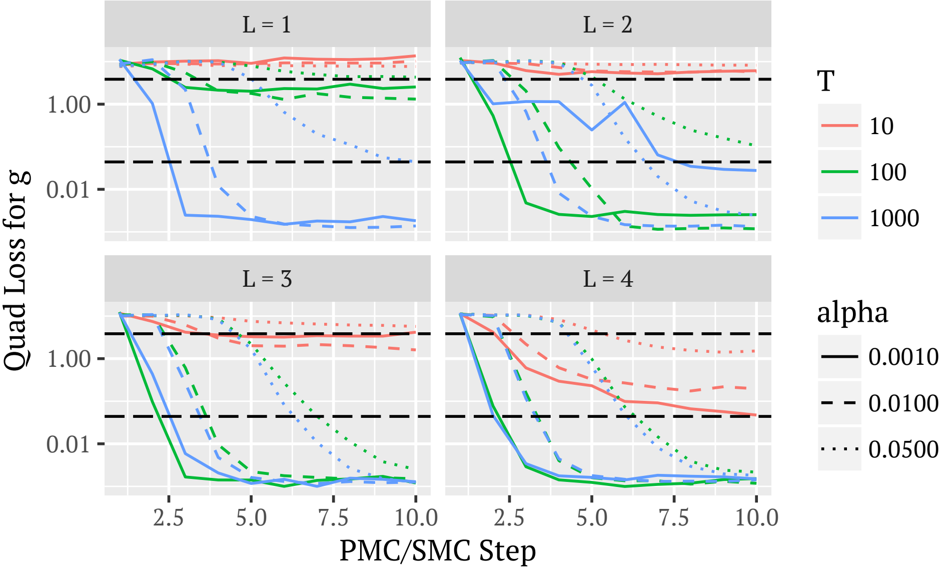
Following Allingham et al. (2009) and Fearnhead & Prangle (2012), take:

- $A = 3, B = 1, g = 2, k = \frac{1}{2}$
- simulate $n = 10000$ observations
- standard ABC uses the order statistics, $S(X) = (x_{(1)}, \dots, x_{(n)})$

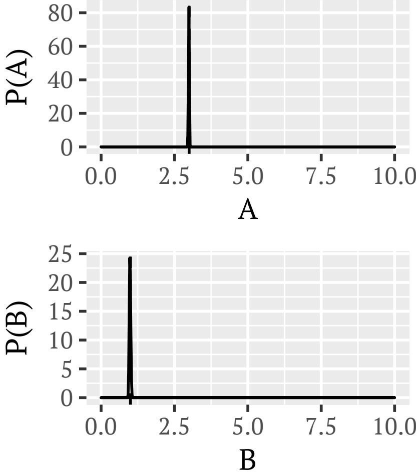
g-and-k: quadratic loss



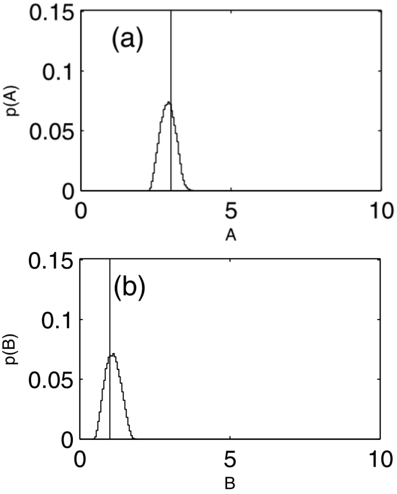
g-and-k: quadratic loss



g-and-k: density plots

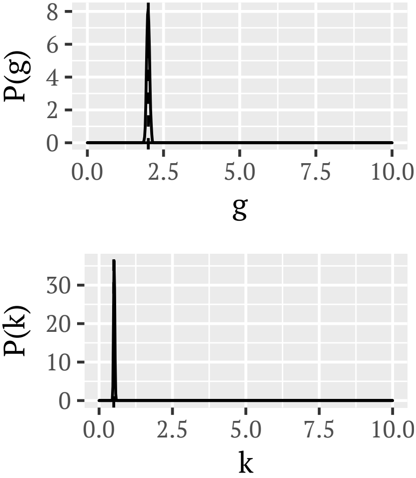


$T = 1000, L = 3, m = 10^5, \alpha = 0.01$

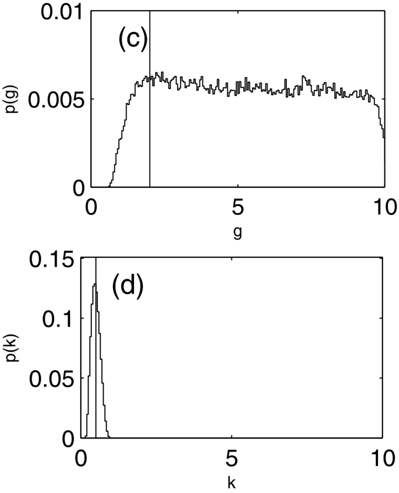


Allingham et al (2009)

g-and-k: density plots



$T = 1000, L = 3, m = 10^5, \alpha = 0.01$



Allingham et al (2009)

Tuberculosis Transmission (Tanaka et al. 2006)

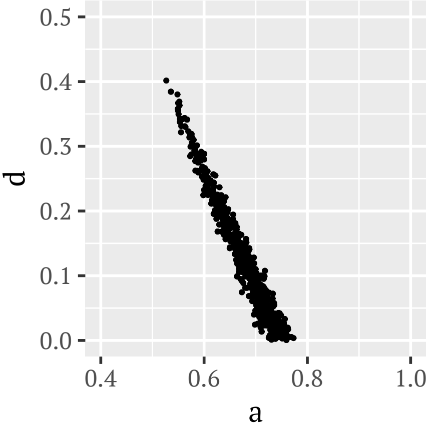
Model of transmission of disease,

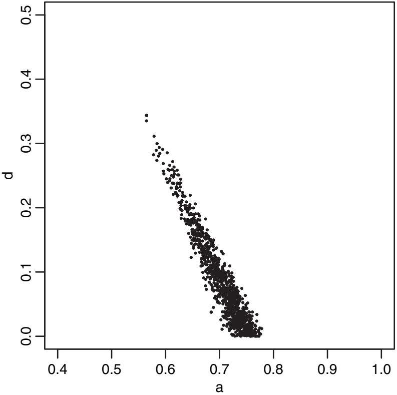
- ‘birth’ of new infections, rate α
- ‘death’ recovery or mortality of carrier, rate δ
- ‘mutation’ genotype of bacterium mutates within carrier, rate θ (infinite-alleles assumption)

$X_i(t)$ num infections type i at time t; $G(t)$ num unique genotypes.

- San Francisco tuberculosis data 1991/2, 473 samples (no time)
- Fearnhead & Prangle (2012) transform $(\alpha/(\alpha+\delta+\theta), \delta/(\alpha+\delta+\theta))$
- $S(X) = (G(t_{\text{end}})/473, 1 - \sum_i (X(t_{\text{end}})/473)^2)$

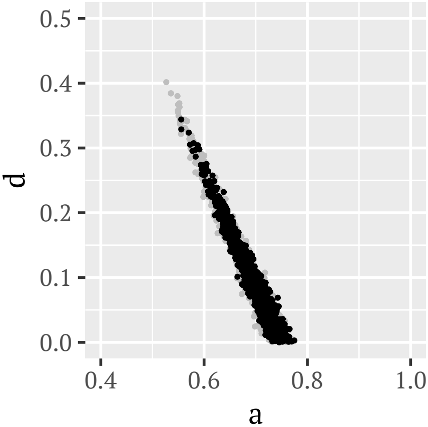
Posterior samples

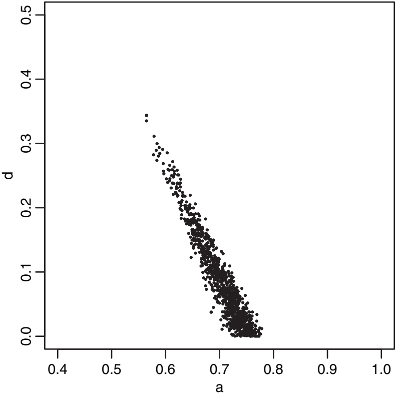




Semi-automatic ABC

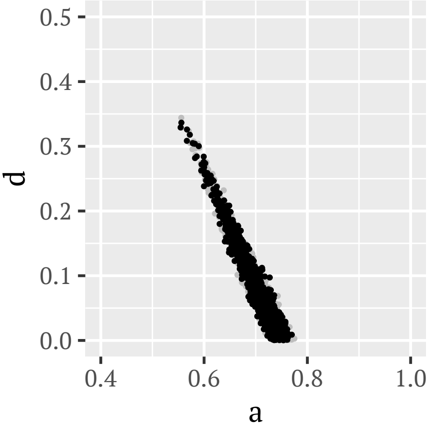
Posterior samples

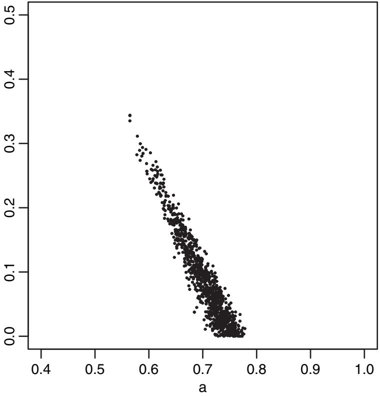




Semi-automatic ABC

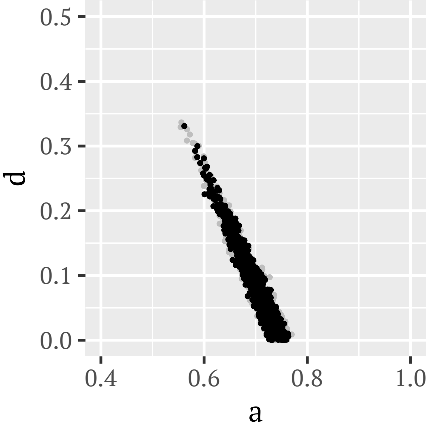
Posterior samples

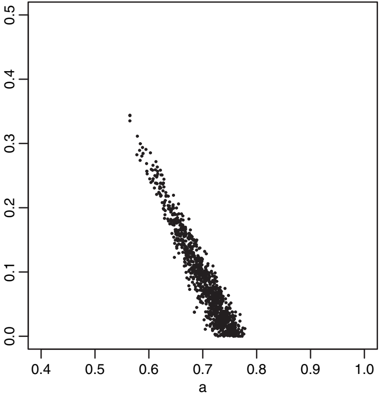




Semi-automatic ABC

Posterior samples





Semi-automatic ABC

Theory

One dimensional asymptotics (I)

Proposition: When $d = 1$, if $\rho_T(S(x), S(y)) := \sum_{k=1}^{T} \rho(S_k(x), S_k(y))$ for some discrepancy $\rho : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ then as $T \to \infty$

$$\lim_{T \to \infty} \frac{\rho_T(S(x), S(y))}{T} \xrightarrow{a.s.} \int_{-\infty}^{\infty} \rho(F_X(z), F_Y(z))dz,$$

where F_X and F_Y are the empirical cumulative distribution functions for the data sets $x_{1:n}$ and $y_{1:n}$ respectively. In particular

1. If $\rho_T(S(x), S(y)) := \|S(x) - S(y)\|_1$, then $T^{-1}\rho_T(S(x), S(y)) \xrightarrow{a.s.} W_1(x_{1:n}, y_{1:n})$
2. If $\rho_T(S(x), S(y)) := \|S(x) - S(y)\|_2^2$, then $T^{-1}\rho_T(S(x), S(y)) \xrightarrow{a.s.} \int_{-\infty}^{\infty}(F_X(z) - F_Y(z))^2 dz.$

One dimensional asymptotics (II)

Corollary: As $T \to \infty$ the following Central Limit Theorem holds:

$$\frac{T^{-1}\rho_T(S(x),S(y)) - \int \rho(F_X(z),F_Y(z))dz}{\sqrt{T}} \Rightarrow N(0,\sigma^2),$$

where $\sigma^2 := \mathrm{Var}_u[\rho(F_X(u),F_Y(u))]$.

$\Longrightarrow$

- convergence of the distance is $O(\sqrt{T})$
- for large enough T estimates of uncertainty can be made using the Gaussian approximation.

Higher dimensions

- Currently hard to see that it matches known distances
- Can get non-asymptotic bounds on uncertainty of CCRM estimator
- Asymptotics in L
- Some very early work on benefits of $L > 1$ with correlation structure

Conclusions

- So far, this …
  - Provides encrypted inference whilst preserving model, prior and data privacy
  - Enables pooling of multiple data owners
  - Theoretically arbitrary low-dimensional models
  - Some theoretical justification in 1D case
- … but this is work-in-progress! Currently in progress:
  - Method of ensuring differential privacy
  - Encrypted software implementation of this scheme
  - Best use of weights
  - Fuller understanding of accuracy for CCRM choices
  - Data as a service
- Perhaps also useful as a model independent summary statistic for unencrypted ABC too?
- Questions, comments and discussion welcome!