

Privacy and Security in Bayesian Inference

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Introduction

Motivation

Security in statistics applications is a growing concern:

- computing in a ‘hostile’ environment (e.g. cloud computing);
- donation of sensitive/personal data (e.g. medical/genetic studies);
- complex models on constrained devices (e.g. smart watches)
- running confidential algorithms on confidential data (e.g. engineering reliability)

Perspectives on “privacy”

- Differential privacy
 - on outcomes of ‘statistical queries’
 - guarantees of privacy for individual observations

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- Data privacy
 - at rest
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 - data pooling
- Model privacy (see other work with Sam Livingstone, UCL)
 - prior distributions
 - model formulation

The perspective for today ...

- **Eve, Cain** and **Abel** have private data of the same type.
- There is a model (including prior) of mutual interest.
- Inference would be improved by pooling the data, but privacy constraints (eg GDPR) prevent this.

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- **Eve, Cain** and **Abel** have private data of the same type.
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Can Eve, Cain and Abel pool their data in order to fit a Bayesian model without revealing the raw data?

Agreed model

$$\pi(\cdot | \psi)$$

$$\pi(\psi)$$

Private data



$$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=1}^{n_1}$$



$$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=n_1+1}^{n_1+n_2}$$



$$\{\mathbf{x}_i = (x_{i1}, \dots, x_{id})\}_{i=n_1+n_2+1}^N$$

Differential Privacy

Differential privacy quantifies the privacy level of ‘statistical queries’.

Informally, for today this is: “how much can be learned about the original data when we learn about the Bayes posterior from MCMC samples?”

Strong statement: assume adversary has access to arbitrary auxiliary information.

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Definition (*Differential Privacy*)

We say that a randomised algorithm \mathcal{M} is (ϵ, δ) -differentially private if for all $\mathcal{S} \subseteq \text{Range}(\mathcal{M})$ and for all x, y such that $\|x - y\|_1 \leq 1$:

$$\mathbb{P}(\mathcal{M}(x) \in \mathcal{S}) \leq \exp(\epsilon)\mathbb{P}(\mathcal{M}(y) \in \mathcal{S}) + \delta$$

Previous work

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- Post process these with accept/reject step.
- View as penalty MCMC algorithm.
- Final posterior samples shown to be differentially private.

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Today: Can we produce a method with better efficiency properties than penalty MCMC by leveraging additional security and privacy methods? This is work in progress ...

Cryptography the solution?

Encryption can provide security guarantees ...

$$\text{Enc}(k_p, m) \rightleftharpoons c$$

Easy

Hard without k_s

$$\text{Dec}(k_s, c) = m$$

... but is typically 'brittle'.

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Arbitrary addition and multiplication is possible with **fully homomorphic encryption** schemes (Gentry, 2009).

$$\begin{array}{ccccc}
 m_1 & & m_2 & \xrightarrow{+} & m_1 + m_2 \\
 \downarrow \text{Enc}(k_p, \cdot) & & \downarrow & & \uparrow \text{Dec}(k_s, \cdot) \\
 c_1 & & c_2 & \xrightarrow{\oplus} & c_1 \oplus c_2
 \end{array}$$

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$$\pi(\psi | X) \propto$$

$$\text{Dec} \left[k_s, \prod_{i=1}^N \pi(\mathbf{x}_i^* | \text{Enc}(k_p, \psi)) \text{Enc}(k_p, \pi(\psi)) \right]$$



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- ✗ Likelihood restricted to low degree polynomials
- ✗ Can only handle very small N due to multiplicative depth
- ✗ MAP/posterior? How?
- MCMC?
- ✗ Who holds secret key?

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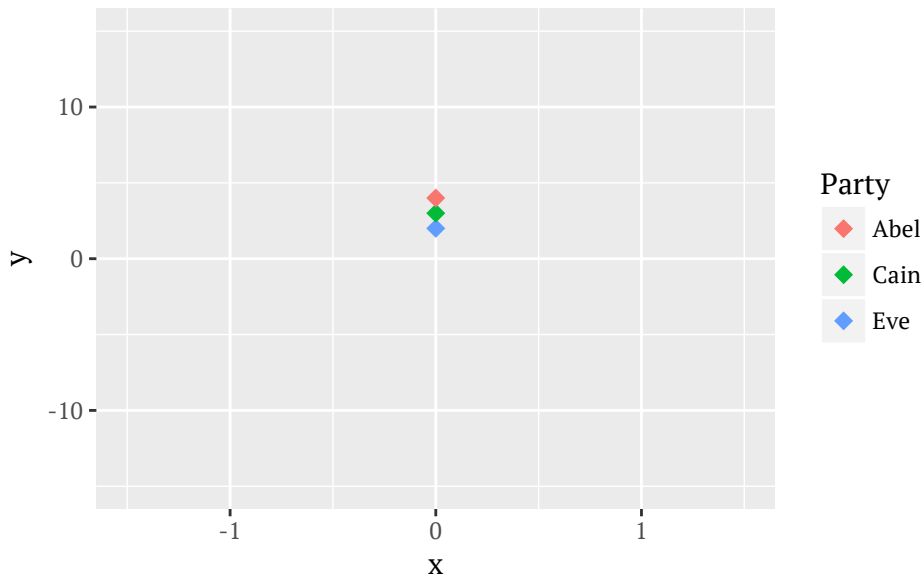
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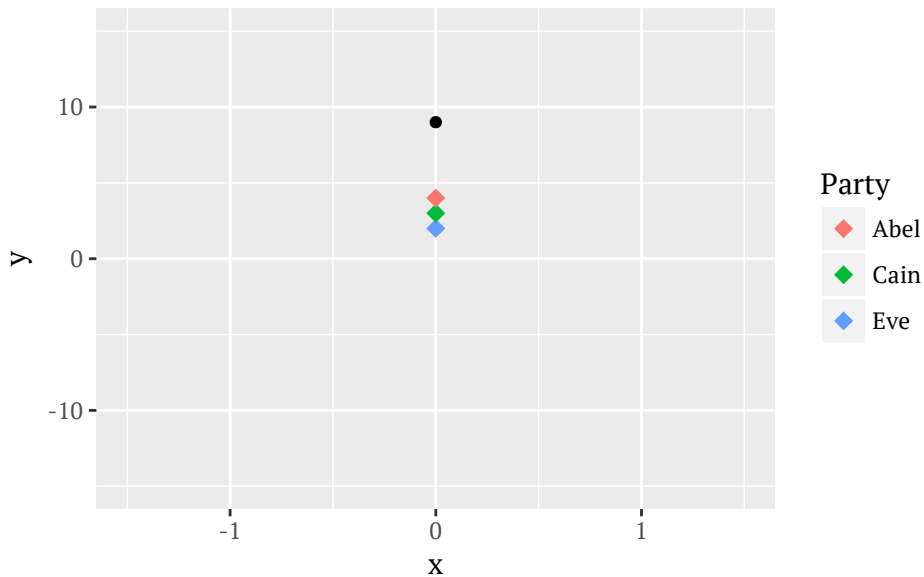
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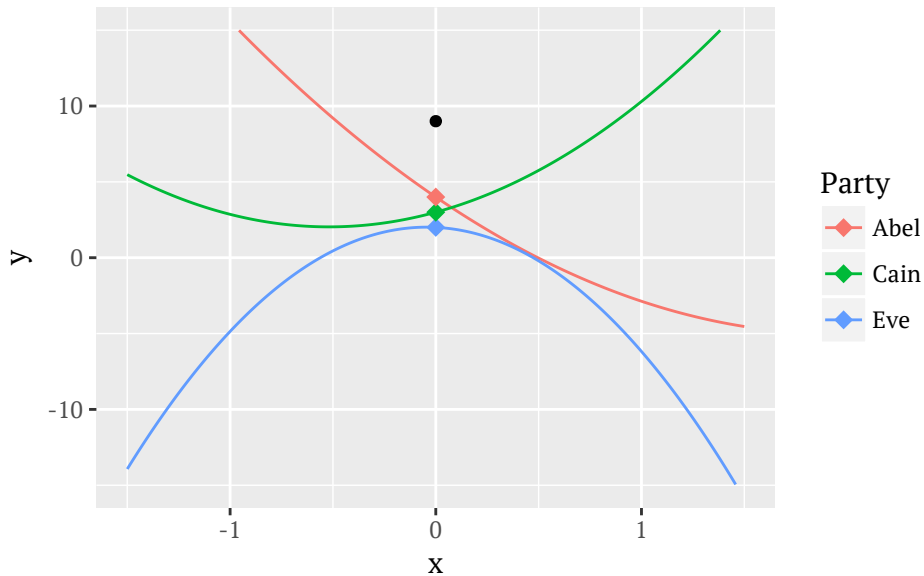
(Simplified) look at Homomorphic Secret Sharing



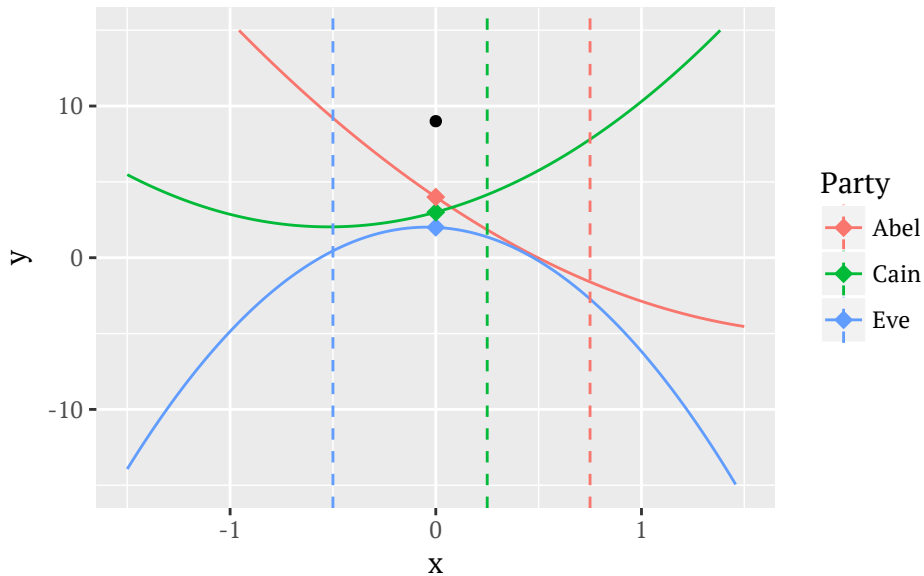
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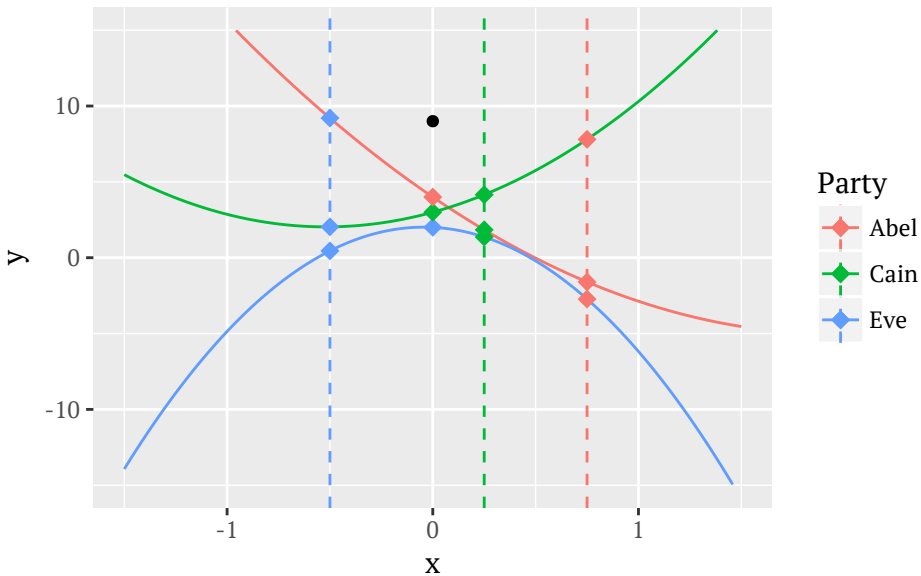
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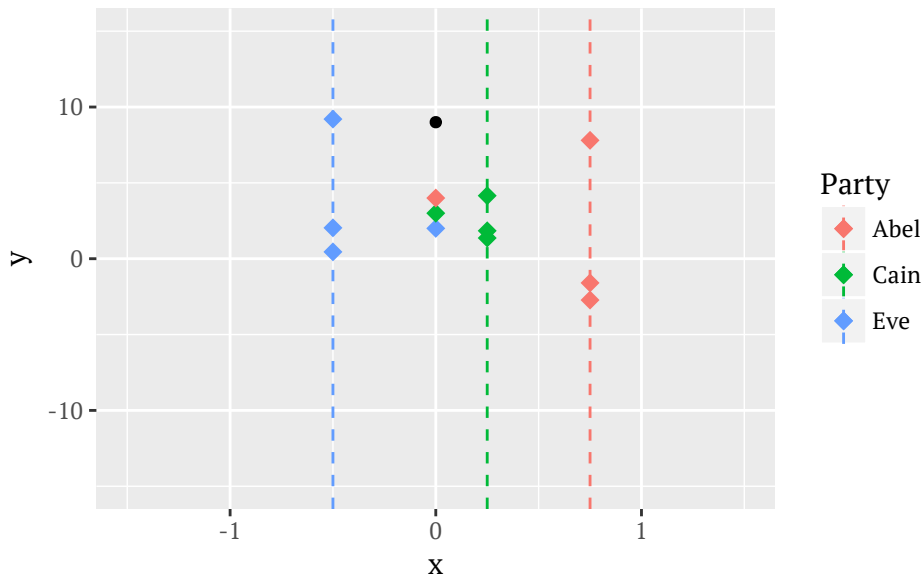
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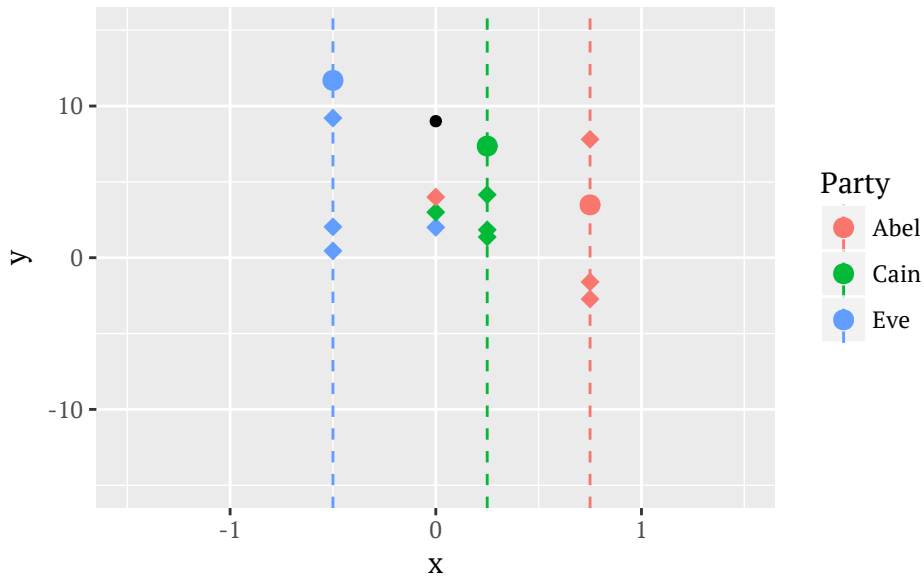
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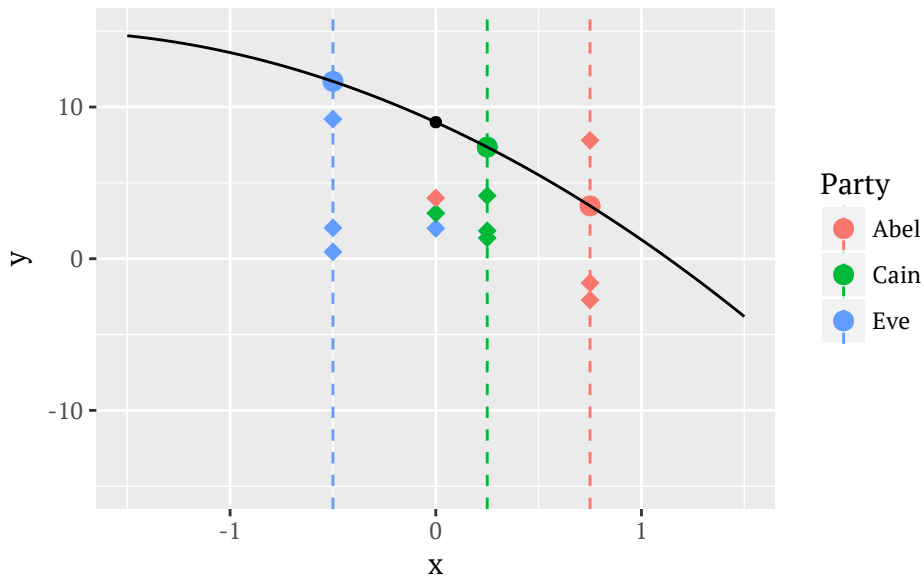
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Confidential MCMC

Metropolis-Hastings

To sample from a target (unnormalised) density of interest, $\pi(\theta)$.

- 1 Initialise with a sample θ_0 .
- 2 Given a sample θ_i , propose a new sample $\theta' \sim q(\cdot | \theta_i)$.
- 3 Compute $\alpha(\theta_i, \theta') = \min \{1, r(\theta_i, \theta')\}$ where

$$r(\theta_i, \theta') := \frac{\pi(\theta')q(\theta_i | \theta')}{\pi(\theta_i)q(\theta' | \theta_i)} \quad (1)$$

- 4 With probability $\alpha(\theta_i, \theta')$ set $\theta_{i+1} = \theta'$, else set $\theta_{i+1} = \theta_i$.
- 5 Repeat steps 2–4 for a fixed number of iterations.

Bayesian inference

Often assume independence so that

$$\pi(\theta) \equiv \pi(\theta | \mathbf{y}) \propto p(\theta) \prod_{i=1}^N p(y_i | \theta)$$

In privacy setting, consider partition of observation indices, $\{\mathcal{J}_i\}_{i=1}^m$, st

$$\bigcup_{i=1}^m \mathcal{J}_i = \{1, \dots, N\} \text{ and } \mathcal{J}_i \cap \mathcal{J}_j = \emptyset \quad \forall i \neq j$$

where participant j only has access to $\{y_i\}_{i \in \mathcal{J}_j}$. Then write Bayesian posterior:

$$\pi(\theta | \mathbf{y}) \propto p(\theta) \prod_{j=1}^m \prod_{i \in \mathcal{J}_j} p(y_i | \theta)$$

Log-likelihood shares

Define portion of likelihood computable by participant j ,

$$p_j^*(\theta) := \prod_{i \in \mathcal{I}_j} p(y_i | \theta)$$

Then,

$$\log \pi(\theta) = \log p(\theta) + \sum_{j=1}^m \log p_j^*(\theta)$$

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and acceptance ratio becomes,

$$\begin{aligned} \log r(\theta_i, \theta') &= \log p(\theta') - \log p(\theta_i) \\ &\quad + \sum_{j=1}^m (\log p_j^*(\theta') - \log p_j^*(\theta_i)) \\ &\quad + \log q(\theta_i | \theta') - \log q(\theta' | \theta_i) \end{aligned}$$

All done?

So, are we finished? Simply compute the acceptance ratio using homomorphic secret shares?

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Not so fast ... completely deterministic so no differential privacy guarantee can be provided when parties observe value of acceptance ratio!

Achieving differential privacy

Rewrite Metropolis-Hastings in an exactly equivalent way:

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- 3 Sample $U \sim \text{Unif}(0, 1)$ and compute
 $\eta = \log r(\theta_i, \theta') - \log U$

- 4 Set

$$\theta_{i+1} = \begin{cases} \theta_i & \text{if } \eta < 0 \\ \theta' & \text{if } \eta \geq 0 \end{cases}$$

- 5 Repeat steps 2–4 for a fixed number of iterations.

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If we can compute η and establish $\eta \geq 0$, then the HSS step is a randomised algorithm.

Requirements

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But, this requires also hiding uniform random sample $U \sim \text{Unif}(0, 1)$. If a participant observes U , they can:

- learn $\log r(\theta_i, \theta')$ if they observe η .
- learn a bound on $\log r(\theta_i, \theta')$ if they observe $\eta \gtrless 0$.

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Note:

$$U \sim \text{Unif}(0, 1) \implies -\log U \sim \text{Exp}(1)$$

From Devroye (1986),

$$T, V, W \sim \text{Unif}(0, 1) \implies W(-\log TV) \sim \text{Exp}(1)$$

\therefore collaboratively compute with two participants, one secret shares W , the other $-\log TV$.

Confidential MCMC algorithm

- 1 Initialise with a sample θ_0 .
- 2 Given a sample θ_i , propose a new sample $\theta' \sim q(\cdot | \theta_i)$.
- 3 Participant 1 samples $U, V \sim \text{Unif}(0, 1)$
- 4 Participant 2 samples $W \sim \text{Unif}(0, 1)$
- 5 Compute $\eta := \log r(\theta_i, \theta') + W \log UV$ via homomorphic secret shares

- 6 Set

$$\theta_{i+1} = \begin{cases} \theta_i & \text{if } \eta < 0 \\ \theta' & \text{if } \eta \geq 0 \end{cases}$$

- 7 Repeat steps 2–5 for a fixed number of iterations.

Level of Differential Privacy

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In one iteration, we achieve same level of differential privacy as when observing a single iid draw from posterior:

Lemma (single iteration DP)

A single iteration of the confidential MCMC algorithm has differential privacy,

$$\frac{\mathbb{P}(\eta < 0 \mid \mathbf{y})}{\mathbb{P}(\eta < 0 \mid \mathbf{y}_{-i})} \leq e^{2C}$$

where $C = \sup_{y, y', \theta} |\log \pi(y \mid \theta) - \log \pi(y' \mid \theta)|$.

Level of Differential Privacy

Under repeated sampling to form a full MCMC output, differential privacy can still be achieved:

Theorem (MCMC trace DP)

Let d_θ be the dimension of parameter θ and let

$$\sup_{\mathbf{y}, \theta} \left| \frac{\partial \log \pi(\mathbf{y} | \theta)}{\partial \theta_i} \right| \leq M$$

Then, k iterations is differentially private with

$$\left(\varepsilon = (4d_\theta n^{-1/2} M) \left(\sqrt{2k \log(1/\delta)} + k e^{4d_\theta n^{-1/2} M} - k \right), \delta \right)$$

Conclusion

Work in progress ...

- 1 Characterising how much of an improvement this provides vs not using cryptographic methods
- 2 Implementation is in development with
 - Shamir's secret sharing extended to including multiplication
 - fully secure network communication built in
 - automatic parsing and evaluation of a provided function circuits
- 3 Performance of the technique:
 - minimising circuit size?
 - optimal ordering of operations (accomodate latency)?
 - preemptive computation?
- 4 Important extensions:
 - beyond honest-but-curious security
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