Privacy and Security in Bayesian Inference

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Introduction

Motivation

Security in statistics applications is a growing concern:

- computing in a 'hostile' environment (e.g. cloud computing);
- donation of sensitive/personal data (e.g. medical/genetic studies);
- complex models on constrained devices (e.g. smart watches)
- running confidential algorithms on confidential data (e.g. engineering reliability)
- big(ger) data (e.g. pooling data sources)

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- Differential privacy
  - on outcomes of ‘statistical queries’
  - guarantees of privacy for individual observations

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- Data privacy
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- Model privacy (see other work with Sam Livingstone, UCL)
  - prior distributions
  - model formulation

The perspective for today …

- **Eve**, **Cain** and **Abel** have private data of the same type.
- There is a Bayesian model of mutual interest.
- Inference would be improved by pooling the data, but privacy constraints (eg GDPR) prevent this.

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Can Eve, Cain and Abel pool their data in order to fit a Bayesian model without revealing the raw data?

Agreed model

$$\pi(\cdot \mid \psi)$$

$$\pi(\psi)$$

Private data

$$\{\mathbf{x}_i = (x_{i1}, \ldots, x_{id})\}_{i=1}^{n_1}$$

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Strong statement: we assume an adversary has access to arbitrary auxilliary information ... data being ‘big’ not a protection.

Definition (*Differential Privacy*)

We say that a randomised algorithm $\mathcal{M}$ is (ε, δ)-differentially private if for all $\mathcal{S} \subseteq \text{Range}(\mathcal{M})$ and for all x, y such that $\|x - y\|_1 \leq 1$:

$$\mathbb{P}(\mathcal{M}(x) \in \mathcal{S}) \leq \exp(\varepsilon)\mathbb{P}(\mathcal{M}(y) \in \mathcal{S}) + \delta$$

Previous work

Everyone sees fitted model parameters, differential privacy of output important. Previous perspectives applied at the combination step.

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- Post process these with accept/reject step.
- View as penalty MCMC algorithm.
- Final posterior samples shown to be differentially private.

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Today: Can we produce a method with better efficiency properties than penalty MCMC by leveraging cryptographic methods?

Cryptography the solution?

Encryption can provide security guarantees …

Easy

$$\texttt{Enc}(k_p, m) \rightleftharpoons c \qquad\qquad \texttt{Dec}(k_s, c) = m$$

Hard without k_s

… but is typically 'brittle'.

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Arbitrary addition and multiplication is possible with **fully homomorphic encryption** schemes (Gentry, 2009).

$$\begin{tikzcd}
m_1 \arrow[d, "\texttt{Enc}(k_p, \cdot)"] & m_2 \arrow[r, "+"] \arrow[d] & m_1 + m_2 \\
c_1 & c_2 \arrow[r, "\oplus"] & c_1 \oplus c_2 \arrow[u, "\texttt{Dec}(k_s, \cdot)"']
\end{tikzcd}$$

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$$\pi(\psi \mid X) \propto$$

$$\text{Dec}\left[k_s, \prod_{i=1}^{N} \pi(\mathbf{x}_i^\star | \text{Enc}(k_p, \psi)) \text{Enc}(k_p, \pi(\psi))\right]$$

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✗ Likelihood restricted to low degree polynomials
✗ Can only handle very small N due to multiplicative depth
✗ MAP/posterior? How? MCMC?
✗ Who holds secret key?

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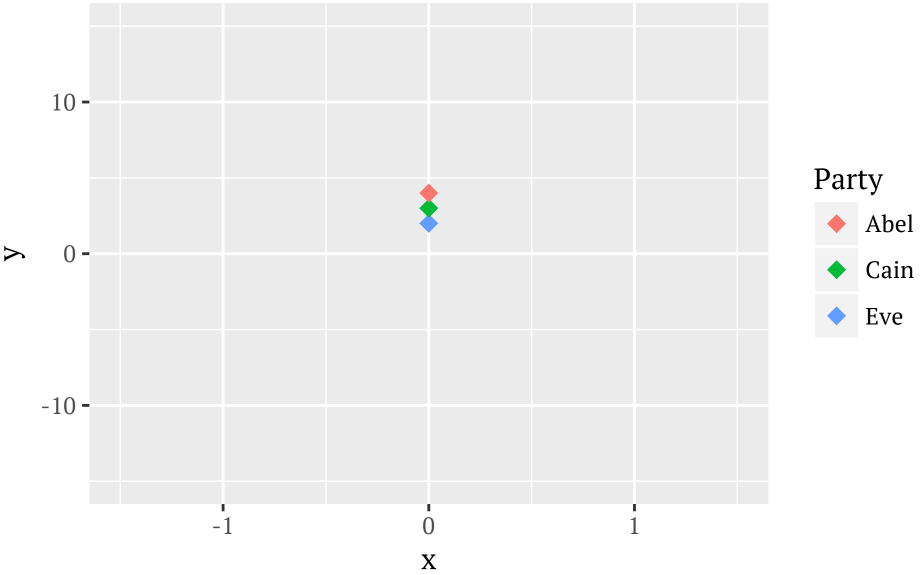
(Simplified) look at Homomorphic Secret Sharing



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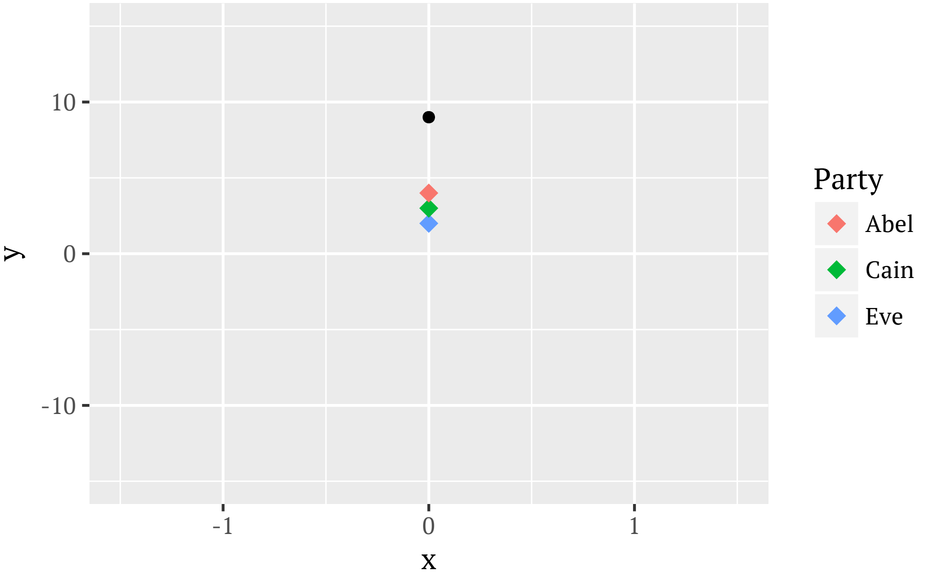

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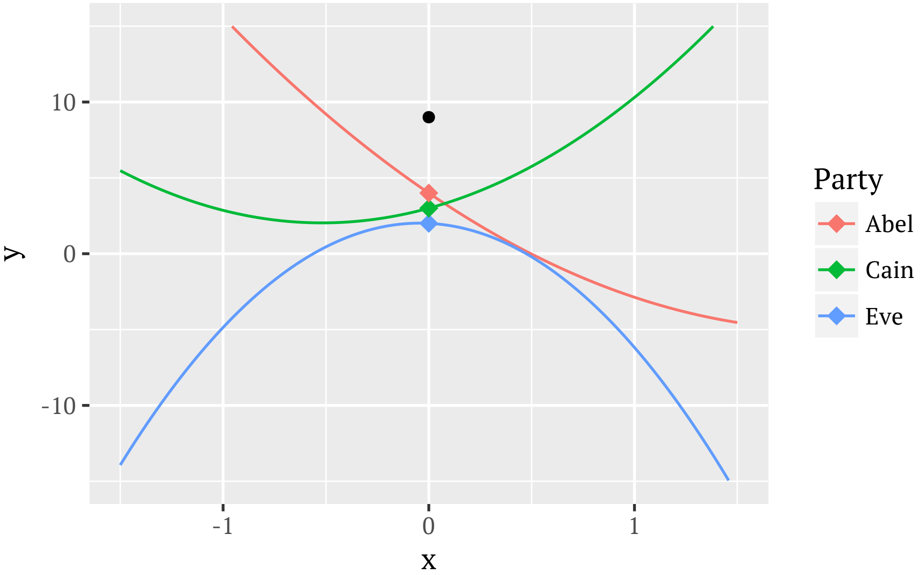

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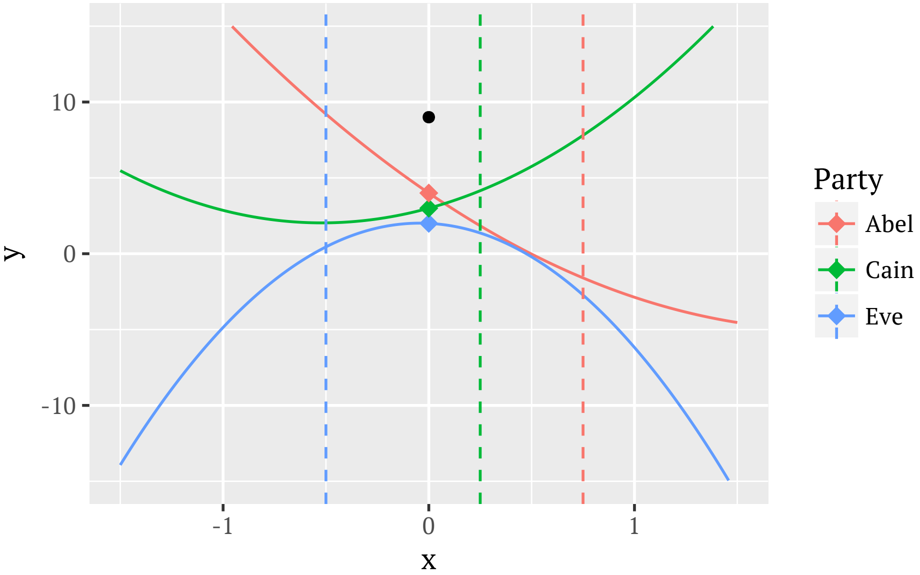

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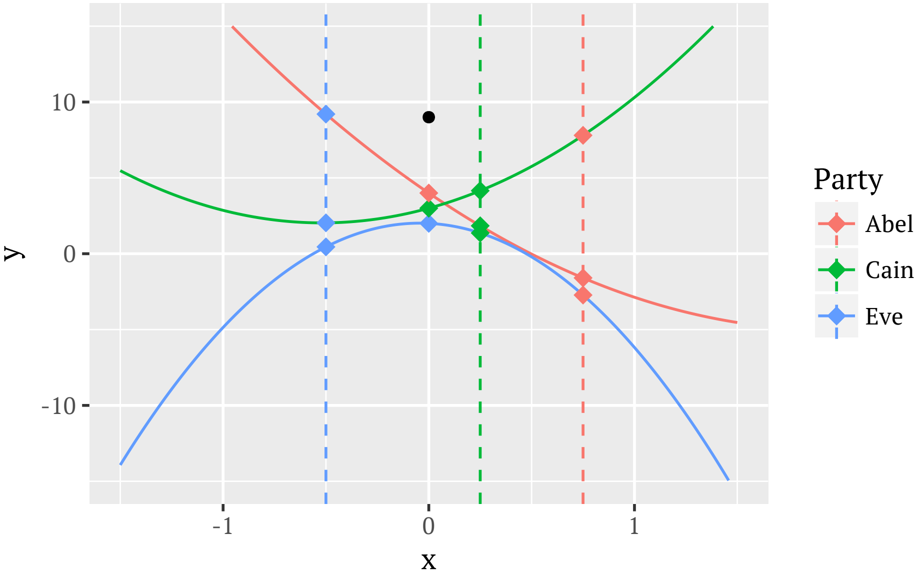

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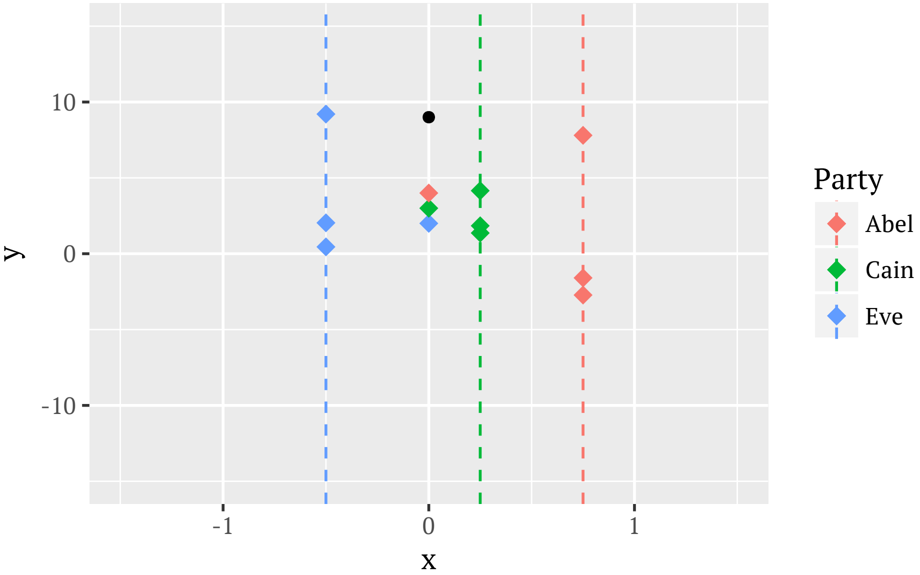

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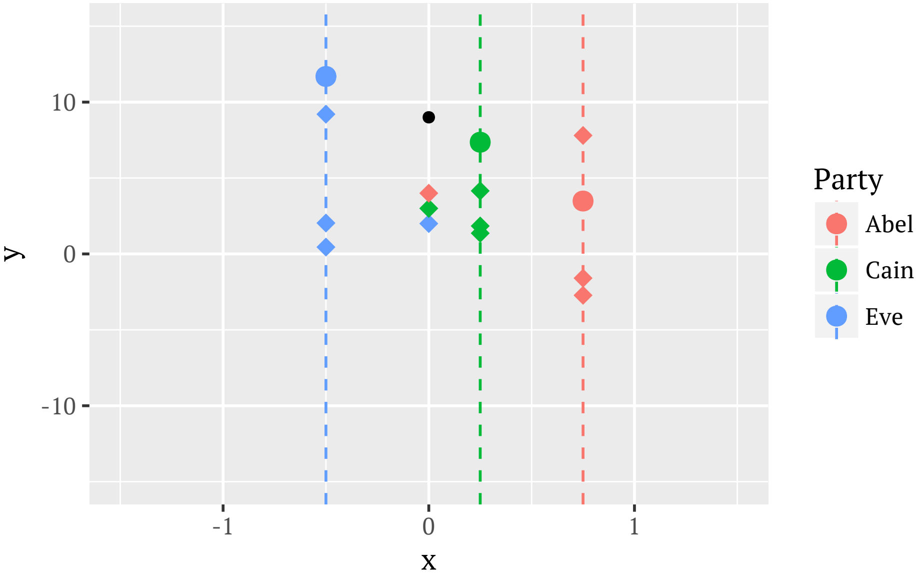

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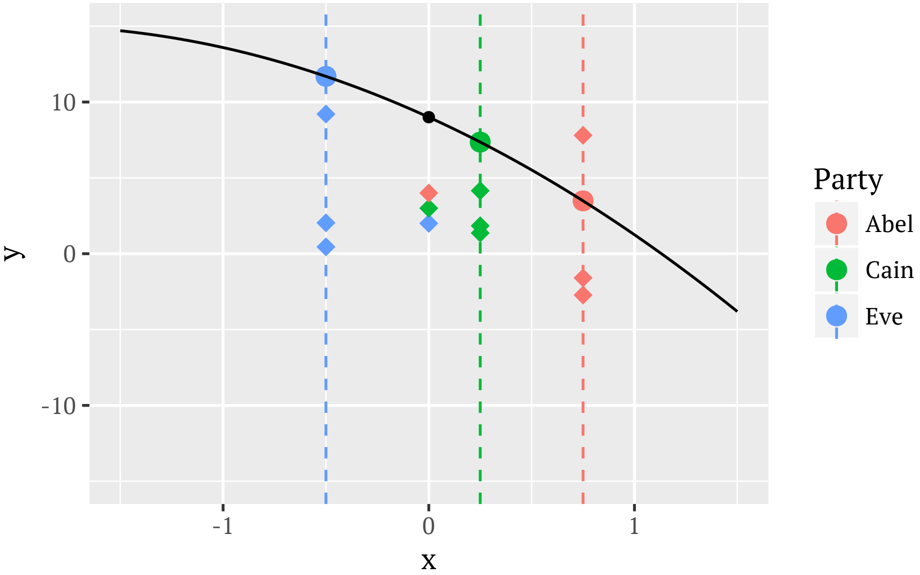

Confidential MCMC

Metropolis-Hastings

To sample from a target (unnormalised) density of interest, $\pi(\theta)$.

1. Initialise with a sample θ_0.
2. Given a sample θ_i, propose a new sample $\theta' \sim q(\cdot \mid \theta_i)$.
3. Compute $\alpha(\theta_i, \theta') = \min\{1, r(\theta_i, \theta')\}$ where

$$r(\theta_i, \theta') := \frac{\pi(\theta')q(\theta_i \mid \theta')}{\pi(\theta_i)q(\theta' \mid \theta_i)} \tag{1}$$

4. With probability $\alpha(\theta_i, \theta')$ set $\theta_{i+1} = \theta'$, else set $\theta_{i+1} = \theta_i$.
5. Repeat steps 2–4 for a fixed number of iterations.

Bayesian inference

Often assume independence so that

$$\pi(\theta) \equiv \pi(\theta \,|\, \mathbf{y}) \propto p(\theta) \prod_{i=1}^{N} p(y_i \,|\, \theta)$$

In privacy setting, consider partition of observation indices, $\{\mathcal{I}_i\}_{i=1}^{m}$, st

$$\bigcup_{i=1}^{m} \mathcal{I}_i = \{1, \dots, N\} \text{ and } \mathcal{I}_i \cap \mathcal{I}_j = \emptyset \;\; \forall\, i \neq j$$

where participant j only has access to $\{y_i\}_{i \in \mathcal{I}_j}$. Then write Bayesian posterior:

$$\pi(\theta \,|\, \mathbf{y}) \propto p(\theta) \prod_{j=1}^{m} \prod_{i \in \mathcal{I}_j} p(y_i \,|\, \theta)$$

Log-likelihood shares

Define portion of likelihood computable by participant j,

$$p_j^\star(\theta) := \prod_{i \in \mathcal{I}_j} p(y_i \,|\, \theta)$$

Then,

$$\log \pi(\theta) = \log p(\theta) + \sum_{j=1}^{m} \log p_j^\star(\theta)$$

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and acceptance ratio becomes,

$$\begin{aligned}\log r(\theta_i, \theta') = \log p(\theta') - \log p(\theta_i) \\ + \sum_{j=1}^{m} \left(\log p_j^\star(\theta') - \log p_j^\star(\theta_i) \right) \\ + \log q(\theta_i \,|\, \theta') - \log q(\theta' \,|\, \theta_i)\end{aligned}$$

All done?

So, are we finished? Simply compute the acceptance ratio using homomorphic secret shares?

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Not so fast … completely deterministic so no differential privacy guarantee can be provided when parties observe value of acceptance ratio!

Achieving differential privacy

Rewrite Metropolis-Hastings in an exactly equivalent way:

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Achieving differential privacy

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3. Sample $U \sim \text{Unif}(0, 1)$ and compute $\eta = \log r(\theta_i, \theta') - \log U$
4. Set
$$\theta_{i+1} = \begin{cases} \theta_i & \text{if } \eta < 0 \\ \theta' & \text{if } \eta \geq 0 \end{cases}$$
5. Repeat steps 2–4 for a fixed number of iterations.

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If we can compute η *and* establish $\eta \gtreqless 0$, then the HSS step is a randomised algorithm.

Requirements

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But, this requires also hiding uniform random sample $U \sim \text{Unif}(0, 1)$. If a participant observes U, they can:

- learn $\log r(\theta_i, \theta')$ if they observe η.
- learn a bound on $\log r(\theta_i, \theta')$ if they observe $\eta \gtrless 0$.

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Note:

$$U \sim \text{Unif}(0, 1) \implies -\log U \sim \text{Exp}(1)$$

From Devroye (1986),

$$T, V, W \sim \text{Unif}(0, 1) \implies W(-\log TV) \sim \text{Exp}(1)$$

$\therefore$ collaboratively compute with two participants, one secret shares W, the other $-\log TV$.

Confidential MCMC algorithm

1. Initialise with a sample θ_0.
2. Given a sample θ_i, propose a new sample $\theta' \sim q(\cdot \mid \theta_i)$.
3. Participant 1 samples $U, V \sim \text{Unif}(0, 1)$
4. Participant 2 samples $W \sim \text{Unif}(0, 1)$
5. Compute $\eta := \log r(\theta_i, \theta') + W \log UV$ via homomorphic secret shares
6. Set

$$\theta_{i+1} = \begin{cases} \theta_i & \text{if } \eta < 0 \\ \theta' & \text{if } \eta \geq 0 \end{cases}$$

7. Repeat steps 2–5 for a fixed number of iterations.

Level of Differential Privacy

Can entirely hide the value of η by taking product with random positive value, so can assume we just observe accept/reject decision.

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In one iteration, we achieve same level of differential privacy as when observing a single iid draw from posterior:

Lemma (single iteration DP)

A single iteration of the confidential MCMC algorithm has differential privacy,

$$\frac{\mathbb{P}(\eta < 0 \,|\, \mathbf{y})}{\mathbb{P}(\eta < 0 \,|\, \mathbf{y}_{-i})} \leq e^{2C}$$

where $C = \sup_{y, y', \theta} |\log \pi(y \,|\, \theta) - \log \pi(y' \,|\, \theta)|$.

Level of Differential Privacy

Under repeated sampling to form a full MCMC output, differential privacy can still be achieved:

Theorem (MCMC trace DP)

Let d_θ be the dimension of parameter θ and let

$$\sup_{\mathbf{y},\theta} \left| \frac{\partial \log \pi(\mathbf{y} \mid \theta)}{\partial \theta_i} \right| \leq M$$

Then, k iterations is differentially private with

$$\left(\varepsilon = (4 d_\theta n^{-1/2} M) \left(\sqrt{2k \log(1/\delta)} + k e^{4 d_\theta n^{-1/2} M} - k \right), \delta \right)$$

Conclusion

Work in progress ...

1. Characterising how much of an improvement this provides vs not using cryptographic methods
2. Implementation is in development with
   - Shamir's secret sharing extended to including multiplication
   - fully secure network communication built in
   - automatic parsing and evaluation of a provided function circuits
3. Performance of the technique:
   - minimising circuit size?
   - optimal ordering of operations (accomodate latency)?
   - preemptive computation?
4. Important extensions:
   - beyond honest-but-curious security
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