Introduction to Reliability Theory

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University of Oxford

Graduate Lecture Series
22nd May 2014

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Taster of ~~Introduction to~~ Reliability Theory

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Reliability theory

Reliability theory is concerned with quantification of the uncertainty in the lifetime of components and systems of components using probability theory.

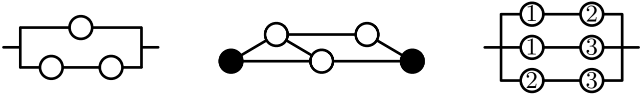
Leads to questions of:

- optimal system design; *
- inference given test data; **
- extreme value modelling;
- maintenance schemes;
- renewal theory & stochastic models of repair;
- shock models;
- ...

An interesting mixture of probability, statistics and applied work. (* = today)

Structural Reliability Theory

- Interest lies in the reliability of 'systems' composed of numerous 'components'.



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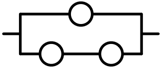
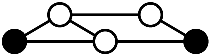
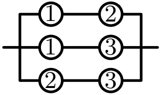

- Lifetime of the system, τ, is determined by:
  - the lifetime of the components, $T_i \sim F_{T_i}(\,\cdot\,;\psi_i)$
  - the structure of the system.
  - the possible presence of a repair process.

via the *structure function* or *signature* or *survival signature*.

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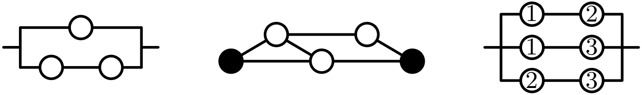

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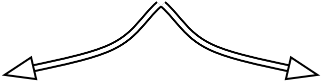
Probabilistic Analysis



Statistical Inference

Structural Reliability Theory

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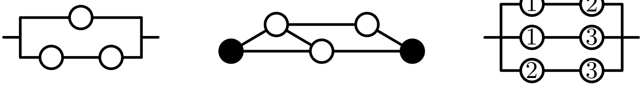

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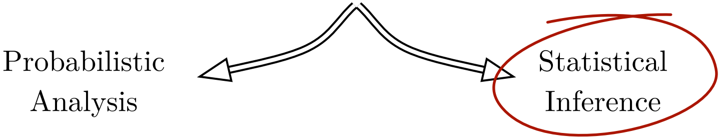
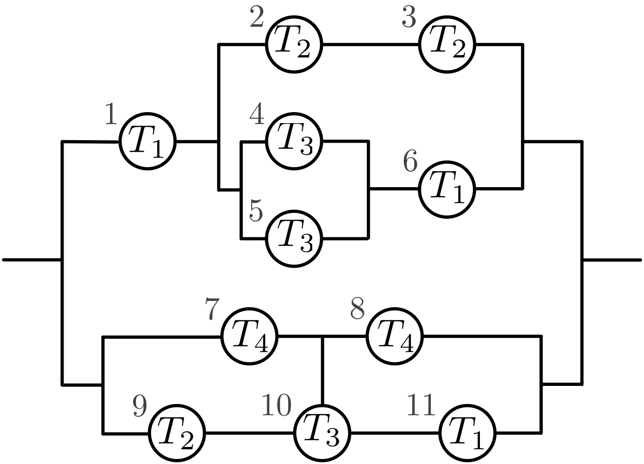
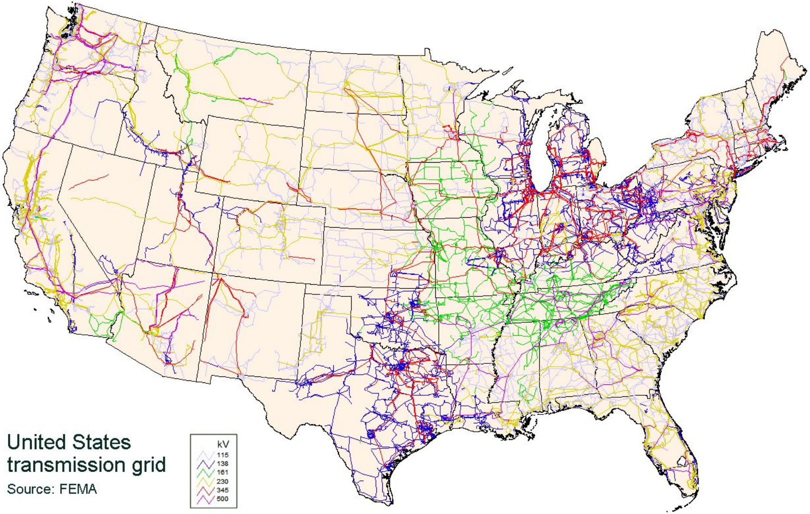
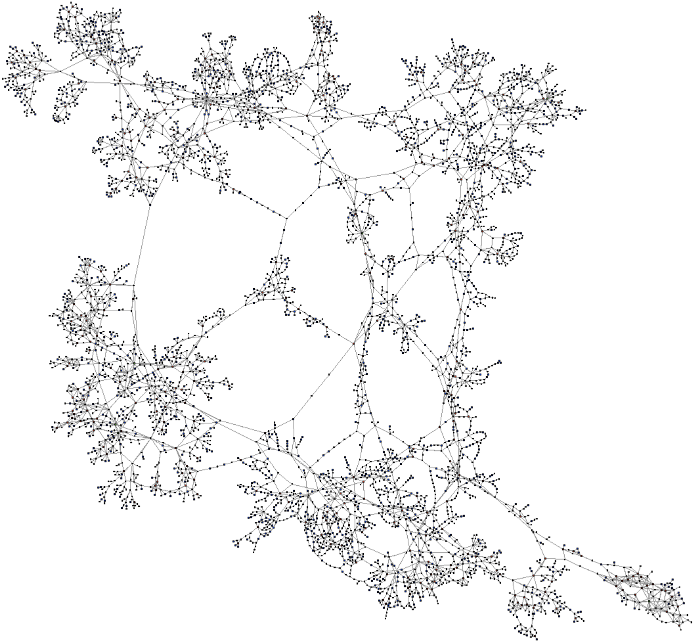


Example Toy System




United States
transmission grid
Source: FEMA
kV
115
138
161
230
345
500



Outline of talk

We will necessarily take a narrowly curated path through a small subset of reliability theory from absolute basics up to a simple example of recent research, but should give a flavour of the field.

1. Failure models for components (1930s);
   - survival function;
   - hazard & failure rate;
   - IFR/DFR.
2. Tools to analyse system structure
   - structure function (1960s);
   - signature (1980s);
   - survival signature (2010s).
3. Topological inference and Bayesian posterior predictive system lifetime (2012/3).

Component lifetimes

A component is defined to be any part of a system which is atomic from the perspective of a reliability analysis, meaning no constituent parts of the unit are modelled directly, only the unit as a whole. Note that a ‘component’ may itself be a system.

Thus, the lifetime T of a component is modelled directly by some lifetime (probability) distribution, typically with support $[0, \infty)$.

- Exponential
- Gamma
- Weibull
- Gompertz
- Coxian
- Phase-type
- ...

Survival function & hazard

In reliability, rather than focusing on the lifetime (probability) density or distribution directly, interest is typically in the *survival function*:

$$\bar{F}_T(t) := \mathbb{P}(T > t) = 1 - F_T(t)$$

and the *hazard rate*:

$$h(t) := \lim_{\delta \to 0^+} \frac{\mathbb{P}(t < T < t + \delta \mid T > t)}{\delta} = \frac{1}{\bar{F}_T(t)} \frac{\partial F_T(t)}{\partial t} = \frac{f_T(t)}{\bar{F}_T(t)}$$

The hazard rate is the instantaneous risk of failure and encapsulates the changing failure characteristics with time.

During expert elicitation the hazard rate can be crucial to ensure that the correct lifetime attributes can be captured by the chosen distribution.

Hazard rates

- IFR (increasing failure rate)
  $\iff \bar{F}_T(t)$ is log-concave
  $\implies \bar{F}_T(t)$ is a Pólya frequency function of order 2.
- IFRA (increasing failure rate average)
  $\implies$ integrated hazard is a star-shaped function.
- NBU (new better than used)
  $\iff F_T(t+\delta) \geq F_T(t)\, F_T(\delta) \quad \forall\, \delta \geq 0$

$$\text{IFR} \subset \text{IFRA} \subset \text{NBU}$$

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For example, the Exponential distribution is very special, having constant hazard. This is a direct relation to the memoryless property. It corresponds to a component which isn't subject to any burn-in or wear and tear and sits at the boundary of all the classes of distribution: it is IFR, IFRA, NBU, DFR, DFRA and NWU!

The structure function

Initially, we consider component and system operation at only a snapshot in time.
$\therefore$ consider a system of components $x_1, \ldots, x_n$ where $x_i \in \{0, 1\}$ denotes operation or failure of component i.
Birnbaum proposed the structure function for system analysis.

Definition (structure function)

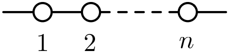
When the state of a system is dependent only on the state of the constituent components, then the binary random variable, ϕ, denoting operation of the system is a functional of the component states.

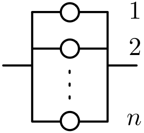
$$\phi := \varphi(X_1, \ldots, X_n)$$

The mapping $\varphi : \{0, 1\}^n \to \{0, 1\}$ is called the *structure function* of the system.

Simple systems

The structure function for series and parallel systems is trivial.





For the series structure:

$$\varphi(\underline{x}) = \min(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i$$

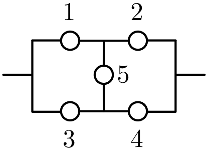
and the parallel system has structure function:

$$\varphi(\underline{x}) = \max(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$$

and combinations thereof . . .

Bridge system

Not hard to construct examples for which decomposing into series and parallel subsystems doesn't work.



$$\varphi(\underline{x}) = x_1x_2 + x_3x_4 + x_1x_4x_5 + x_2x_3x_5 - x_1x_2x_3x_5 - x_1x_3x_4x_5 - x_1x_2x_4x_5 - x_1x_2x_3x_4 - x_2x_3x_4x_5 + 2x_1x_2x_3x_4x_5$$

Path/cut sets

Definition ((Minimal) path set)

A set of components, P, of a system is said to be a *path set* if the system functions correctly whenever all the components in P function correctly.
If no proper subset of P is a path set, then P is said to be a *minimal path set*.

Definition ((Minimal) cut set)

A set of components, C, of a system is said to be a *cut set* if the system is failed whenever all the components in C have failed.
If no proper subset of C is a cut set, then C is said to be a *minimal cut set*.

Path/cut sets

For bridge system:

$$\mathcal{P} = \{\{1,2\},\{3,4\},\{1,4,5\},\{2,3,5\}\}$$
$$\mathcal{C} = \{\{1,3\},\{2,4\},\{1,4,5\},\{2,3,5\}\}$$

Path/cut sets

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$$\mathcal{P} = \{\{1,2\},\{3,4\},\{1,4,5\},\{2,3,5\}\}$$
$$\mathcal{C} = \{\{1,3\},\{2,4\},\{1,4,5\},\{2,3,5\}\}$$

Lemma

Let $\mathcal{P}$ be the collection of all minimal path sets of a system. If X is the set of currently operational components, then the system as a whole is operational if and only if $\exists\ P_i \in \mathcal{P}$ s.t. $P_i \subseteq X$

Similarly, let $\mathcal{C}$ be the collection of all minimal cut sets of a system. If X is the set of currently failed components, then the system as a whole is failed if and only if $\exists\ C_i \in \mathcal{C}$ s.t. $C_i \subseteq X$

Path/cut sets & the structure function

Theorem

The structure function of a system with collection of all minimal path sets $P_1, \dots P_r$ and collection of all minimal cut sets $C_1, \dots, C_s$ can be expressed in terms of the components of either:

$$\begin{aligned}\varphi(\underline{x}) &= 1 - \prod_{j=1}^{r} \left(1 - \prod_{i \in P_j} x_i \right) \\ &= \prod_{j=1}^{s} \left(1 - \prod_{i \in C_j} (1 - x_i) \right)\end{aligned}$$

Pause to ask some obvious questions

1. Does every possible structure function mapping $\varphi : \{0,1\}^n \to \{0,1\}$ correspond to some real-world system?

2. Does the structure function posses any probabilistic properties?

3. If so, can we answer all the interesting questions like the closure of systems of IFR/DFR/IFRA components?

4. In what sense do path/cut sets characterise systems?

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4. In what sense do path/cut sets characterise systems?

   All possible cut/path sets $\equiv$ all *coherent* systems

Coherency

Definition (Relevant component)

Consider a system of order n with state vector of components $(x_1, \ldots, x_{i-1}, Y, x_{i+1}, \ldots, x_n)$. The ith component Y is said to be *irrelevant* if:

$$\varphi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \varphi(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

for all possible realisations of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \{0, 1\}^{n-1}$.
If a component is not irrelevant, it is defined to be a *relevant* component.

Coherency

Definition (Monotone structure function)

The structure function $\varphi(\cdot)$ of an order n system is said to be *monotone* if

$$\underline{x} \leq \underline{y} \implies \varphi(\underline{x}) \leq \varphi(\underline{y})$$

where $\underline{x}, \underline{y} \in \{0,1\}^n$ and the inequality on the left is taken element-wise.

Definition (Coherent system)

A system is *coherent* if and only if the structure function representing the system is monotone and every component is relevant.

Coherency restricts the number of systems significantly.
$\exists\ 2^{2^3} = 256$ mappings $\varphi : \{0,1\}^3 \to \{0,1\}$, but there are only 5 coherent systems order 3.

Butterworth's set theoretic treatment

It's possible to completely characterise coherent systems purely by cut and path sets.

Theorem (Minimal cut-sets ≡ coherent systems)

If a coherent system ϕ has collection of all minimal cut sets $\mathcal{C}$, then $\bigcup_{C_i \in \mathcal{C}} C_i$ is the set of all components in ϕ.

Conversely, any collection of sets $\mathcal{C}$ such that $C_i \not\subset C_j,\ \forall\, C_i, C_j \in \mathcal{C}$ defines a collection of minimal cut sets of some coherent system comprising the components $\bigcup_{C_i \in \mathcal{C}} C_i$.

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Replace 'cut set' with 'path set' and above statement still true.

Definition (Duality)

Coherent system A is the dual of coherent system B if the minimal path sets of A are the minimal cut sets of B (and vice-versa).

Do we need anything more?

Job done? :)

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Job done? :(

Not really. On the face of it, the reliability function enables all statements of probability about the reliability of a system in a particular structure, but theoretical analysis of general system structures by this route is very difficult. Hence the answer 'no' to question 3 earlier.

Moreover, even for concrete structures it gets messy very fast . . . > 5 components in a non-trivial arrangement and the algebra is plain nasty.

On top of that, there is something mathematically unsatisfying about our main tool for systems not being invariant to relabelling of components!

Fast-forward to 1982 and enter the signature . . .

System signature

Samaniego was trying to understand closure properties of systems more deeply when he came up with an improved approach, the *signature*.

Definition (System signature)

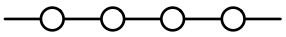
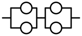
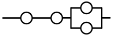
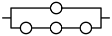
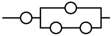
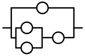
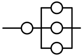
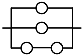
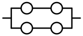
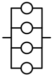
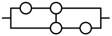
Consider a coherent system of order n, with independent and identically distributed component lifetimes. The *signature* of the system is the n-dimensional probability vector $\underline{s} = (s_1, \ldots, s_n)$ with elements:

$$s_i := \mathbb{P}(\tau = T_{i:n})$$

where τ is the failure time of the system and $T_{i:n}$ is the ith order statistic of the n component failure times.

Signature examples

All order 4 coherent systems with graph representation.

System Topology	Signature	System Topology	Signature
	$(1, 0, 0, 0)$		$\left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$
	$\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$		$\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$
	$\left(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0\right)$		$\left(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4}\right)$
	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)$		$\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$
	$\left(0, \frac{2}{3}, \frac{1}{3}, 0\right)$		$(0, 0, 0, 1)$
	$\left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$		

An important property

Signatures lead to many useful results. One is:

Theorem

Let $T_1, \dots, T_n \sim T$ be the iid component lifetimes of an order n coherent system with signature $\underline{s}$. Let τ be the system lifetime. Then,

$$\bar{F}_\tau(t) := \mathbb{P}(\tau > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} F_T(t)^j \, \bar{F}_T(t)^{n-j}$$

Corollary

If additionally $F_T(\cdot)$ is absolutely continuous then,

$$f_\tau(t) := -(\partial/\partial t)\mathbb{P}(\tau > t) = \sum_{i=1}^{n} i \, s_i \binom{n}{i} F_T(t)^{i-1} \, \bar{F}_T(t)^{n-i} f_T(t)$$

Multiple components?

Most systems have more than one type of component! So can we extend the signature?

Let there be K component types, $k \in \{1, \ldots, K\}$, with m_k components of type k in the system.

Define $q_k(j_k) := \mathbb{P}(\tau = T^k_{j_k:m_k})$, and:

$$q := (q_1(1), \ldots, q_1(m_1), q_2(1), \ldots, q_2(m_2), \ldots, q_K(1), \ldots, q_K(m_K))$$

$$\implies \mathbb{P}(\tau > T) = \sum_{k=1}^{K} \sum_{j_k=1}^{m_k} q_k(j_k) \mathbb{P}(T^k_{j_k:m_k} > t)$$

Seems to achieve the same structure/lifetime separation, just need to be able to compute everything.

Not so fast ... computing $q_k(j_k)$ painful

Let $K = 2$,

$$\begin{aligned} q_1(j_1) &= \mathbb{P}(\tau = T^1_{j_1:m_1}) \\ &= \sum_{j_2=0}^{m_2} \Big[\mathbb{P}(\tau = T^1_{j_1:m_1} \mid T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2}) \\ &\qquad \times \mathbb{P}(T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2}) \Big] \end{aligned}$$

Non-trivial to compare order statistics of two different distributions. For a general K component system, need to derive comparisons of

$$\prod_{\substack{l=1 \\ l \neq j}}^{K} (m_l + 1)$$

for order statistics from different probability distributions.

Notation change

In a system with K different component types, it becomes useful to change (without loss of generality) the vector of components notation so as to group components of the same type:

$$\underline{x} := (\underline{x}^1, \dots, \underline{x}^K)$$

where $\underline{x}^k$ is the vector of components of type $k \in \{1, \dots, K\}$,

$$\underline{x}^k := (x^k_1, \dots, x^k_{m_k})$$

Thus there are m_k components of type k in the system.

Simplifying signatures

Coolen rethought signatures with the objective of separating structure and component lifetimes for multiple components.

Definition (System survival signature)

Consider a system comprising K component types, with m_k components of type $k \in \{1, \dots, K\}$. Then the *system survival signature* $\Phi(l_1, \dots, l_K)$, with $l_k \in \{0, 1, \dots, m_k\}$, is the probability that the system functions given precisely l_k of its components of type k function.

$$\Phi(l_1, \dots, l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1} \right] \sum_{\underline{x} \in S_{l_1, \dots, l_K}} \varphi(\underline{x})$$

where $S_{l_1, \dots, l_K} = \{\underline{x} : \sum_{i=1}^{m_k} x_i^k = l_k \quad \forall\, k\}$

System lifetime

Now, letting $C_t^k \in \{0, 1, \dots, m_k\}$ be the RV denoting the number of components of type k in the system which function at time $t > 0$, it is possible to write the system lifetime in terms of the distributions of component type lifetimes. If $F_k(t)$ is the distribution of component k's lifetime, then

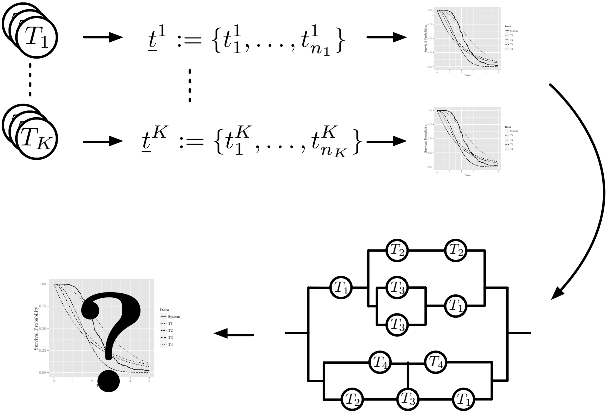
$$\mathbb{P}(\tau > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) \mathbb{P}\left(\bigcap_{k=1}^{K} \{ C_t^k = l_k \} \right)$$

$$\stackrel{iid}{=} \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi(l_1, \dots, l_K) \prod_{k=1}^{K} \binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [\bar{F}_k(t)]^{l_k} \right]$$

2013 : Bayesian inference with survival signature

Test data available on components considered exchangeable with those to be used in a system.

Objective: inference on system/network reliability.



A nonparametric model of components

At a fixed time t, probability component of type k functions is Bernoulli(p_t^k) for some unknown p_t^k.

$\implies$ number functioning at time t in iid batch of n_k is Binomial(n_k, p_t^k).

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Let $S_t^k \in \{0, 1, \dots, n_k\}$ be number of working components in test batch of n_k components of type k. Then,

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Given test data $\underline{t}^k = \{t_1^k, \dots, t_{n_k}^k\}$, for each t we can form corresponding observation from Binomial model

$$s_t^k = \sum_{i=1}^{n_k} \mathbb{I}(t_i^k > t)$$

Bayesian inference for nonparametric model

Taking prior $p_t^k \sim \text{Beta}(\alpha_t^k, \beta_t^k)$, exploit conjugacy result

$$p_t^k \,|\, s_t^k \sim \text{Beta}(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

Then, posterior predicitive for number of components surviving in a new batch of m_k components is

$$C_t^k \,|\, s_t^k \sim \text{Beta-binomial}(m_k, \alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

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Summary: for any fixed t, s_t^k provides a minimal sufficient statistic for computing posterior predictive distribution of the number of components surviving to t in a new batch, without any parametric model for component lifetime being assumed.

Propagating uncertainty to the system

Now take collection of component types $k \in \{1, \ldots, K\}$, each with test data $\underline{t} = \{\underline{t}^1, \ldots, \underline{t}^k\}$, and corresponding collection of minimal sufficient statistics for a fixed t, $\{s_t^1, \ldots s_t^K\}$.

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Survival probability for a new system τ^* comprising these component types follows naturally via posterior predictive and survival signature:

$$P(\tau^* > t \,|\, s_t^1, \ldots s_t^K)$$
$$= \int \cdots \int P(\tau^* > t \,|\, p_t^1, \ldots p_t^K) P(p_t^1 \mid s_t^1) \ldots P(p_t^K \mid s_t^K) \, dp_t^1 \ldots dp_t^K$$

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\begin{aligned}
&P(\tau^* > t \,|\, s_t^1, \dots s_t^K) \\
&= \int \cdots \int P(\tau^* > t \,|\, p_t^1, \dots p_t^K) P(p_t^1 \,|\, s_t^1) \dots P(p_t^K \,|\, s_t^K)\, dp_t^1 \dots dp_t^K \\
&= \int \cdots \int \left[\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) P\left(\bigcap_{k=1}^{K} \{ C_t^k = l_k \,|\, p_t^k \} \right) \right] \\
&\qquad\qquad \times P(p_t^1 \,|\, s_t^1) \dots P(p_t^K \,|\, s_t^K)\, dp_t^1 \dots dp_t^K
\end{aligned}
$$

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$$= \int \cdots \int \left[\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) P\left(\bigcap_{k=1}^{K} \{ C_t^k = l_k \,|\, p_t^k \} \right) \right] \times P(p_t^1 \mid s_t^1) \dots P(p_t^K \mid s_t^K) \, dp_t^1 \dots dp_t^K$$

$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) \prod_{k=1}^{K} \int P(C_t^k = l_k \,|\, p_t^k) P(p_t^k \mid s_t^k) \, dp_t^k$$

Final integral is simply the posterior predictive (Beta-binomial).

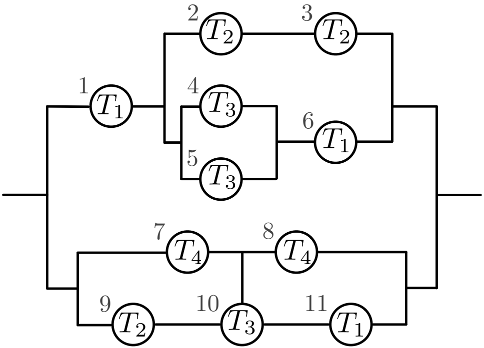
System survival probability

$$
\begin{aligned}
&P(\tau^* > t \,|\, s_t^1, \dots s_t^K) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) \\
&\qquad \times \prod_{k=1}^{K} \binom{m_k}{l_k} \frac{\mathrm{B}(l_k + \alpha_t^k + s_t^k, m_k - l_k + \beta_t^k + n_k - s_t^k)}{\mathrm{B}(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)}
\end{aligned}
$$

Incredibly easy to implement this algorithmically since survival signature has factorised the survival function by component type. Exercise for the viewer to convince themselves the same is not practical with the structure function!

Example system layout, $K = 4$, $n = 11$

Example system:



$T_1 \sim \text{Exp}(\lambda_1 = 0.55)$

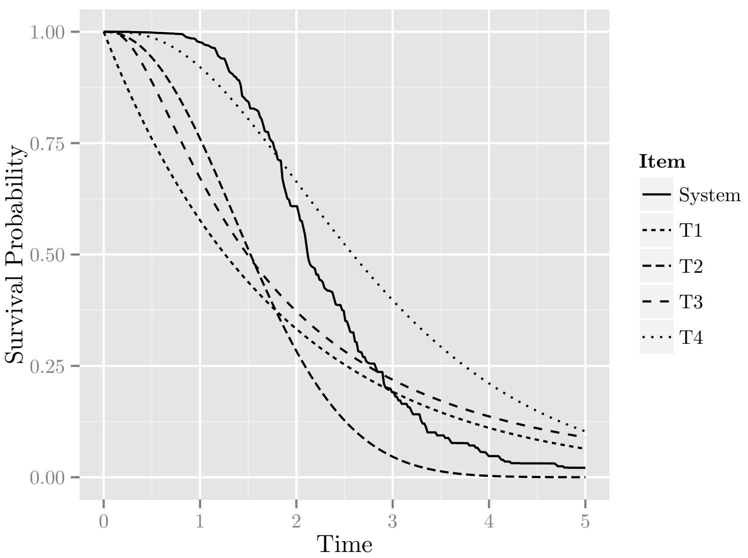
$T_2 \sim \text{Wei}(\lambda_2 = 1.8, \gamma_1 = 2.2)$

$T_3 \sim \text{Log-N}(\mu = 0.4, \tau = 1.234)$

$T_4 \sim \text{Gam}(\lambda_3 = 0.9, \gamma_2 = 3.2)$

Simulated test data with $n_k = 100 \; \forall \, k$

Posterior predictive survival curves



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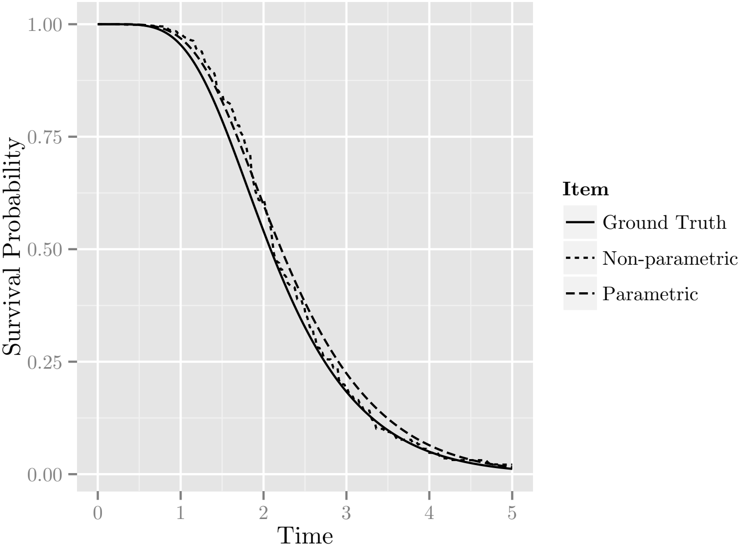
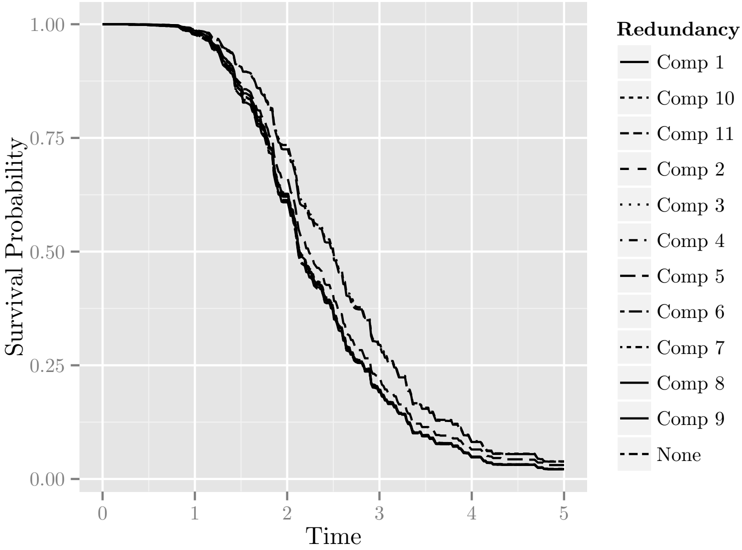


Optimal redundancy?



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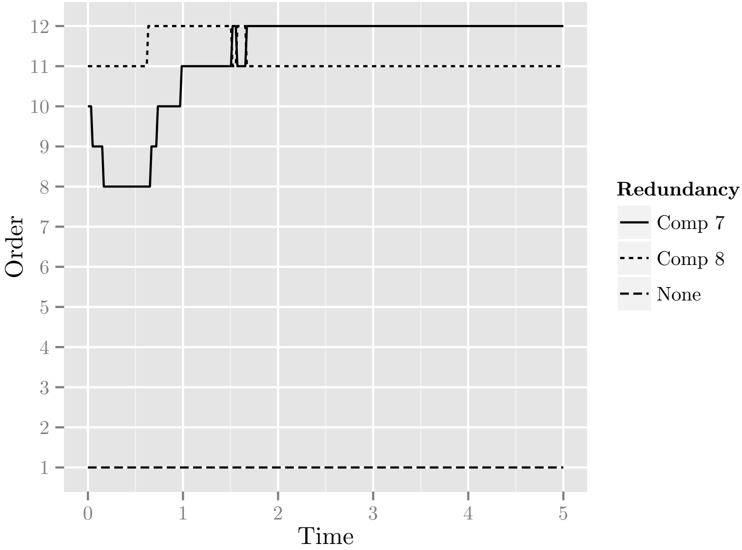
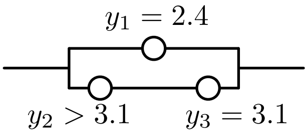


2012 : Topological inference with masked lifetime data



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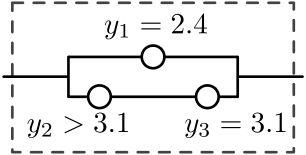
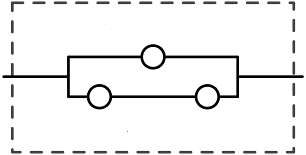


$$t = y_3 = y_{2:3} = 3.1$$

2012 : Topological inference with masked lifetime data



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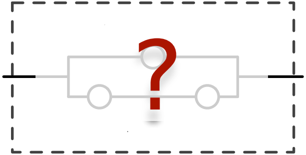

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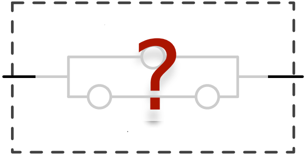

$$t = y_? = y_{?:3} = 3.1$$

$$L(\psi; \underline{t}) = \prod_{j=1}^{n} \sum_{i=1}^{n} i \, s_i \binom{n}{i} F_T(t_j; \psi)^{i-1} \, \bar{F}_T(t_j; \psi)^{n-i} f_T(t_j; \psi)$$

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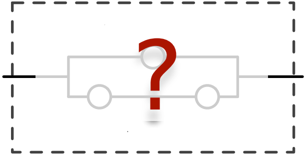

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$$L(\psi; \underline{t}, \underline{s}) = \prod_{j=1}^{n} \sum_{i=1}^{n} i\, s_i \binom{n}{i} F_T(t_j;\psi)^{i-1}\, \bar{F}_T(t_j;\psi)^{n-i} f_T(t_j;\psi)$$

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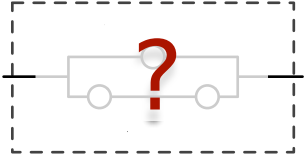

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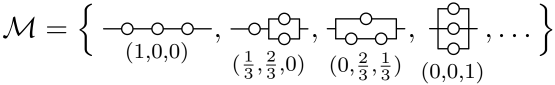
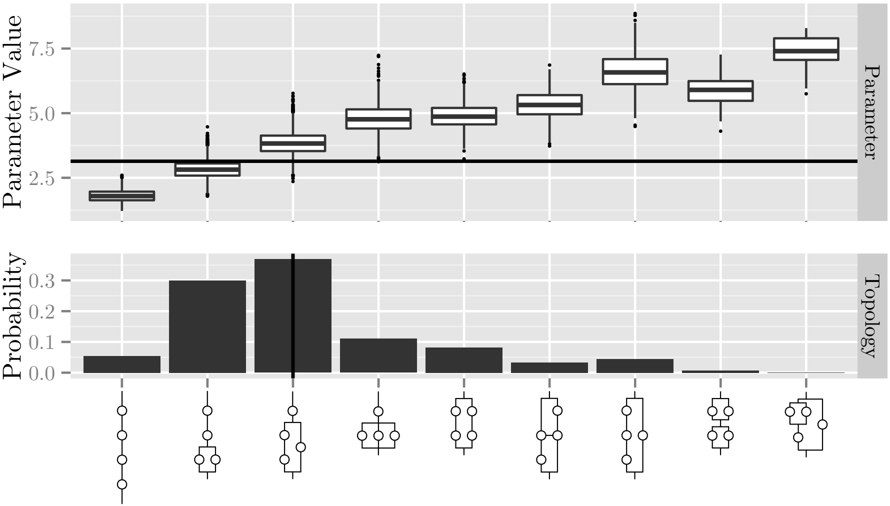


It is possible to then compute the necessary block/full conditionals to make a data augmentation scheme which explores some set of proposed topologies, $\mathcal{M}$.

Example posterior over topologies



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