Bayesian inference for reliability of systems and networks using the survival signature

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System reliability: structure function

System with $K \geq 2$ types of components

m_k components of type $k \in \{1, 2, \ldots, K\}$, with $\sum_{k=1}^{K} m_k = m$

State vector $\underline{x} = (\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^K)$, with $\underline{x}^k = (x_1^k, x_2^k, \ldots, x_{m_k}^k)$ the sub-vector representing the states of the components of type k with $x_i^k = 1$ if the ith component of type k functions and $x_i^k = 0$ if not.

Structure function $\phi(\underline{x}) = 1$ if system functions with state $\underline{x}$ and $\phi(\underline{x}) = 0$ if not.

Assume: $\phi(\underline{x})$ is not decreasing in any of the components of $\underline{x}$ ('coherent system') and $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$.
(This assumption can be deleted)

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Survival signature

The Survival Signature $\Phi(l_1, l_2, \ldots, l_K)$, with $l_k = 0, 1, \ldots, m_k$, is the probability that a system functions given that *precisely* l_k of its components of type k function, for each $k \in \{1, 2, \ldots, K\}$.

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with precisely l_k of their m_k components $x_i^k = 1$, so with $\sum_{i=1}^{m_k} x_i^k = l_k$.

Let $S_{l_1,\ldots,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, $k = 1, 2, \ldots, K$. Assuming exchangeability of the failure times of the m_k components of type k

$$\Phi(l_1, \ldots, l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \times \sum_{\underline{x} \in S_{l_1,\ldots,l_K}} \phi(\underline{x})$$

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Probability system functions at time t

Let $C_t^k \in \{0, 1, \ldots, m_k\}$ denote the number of components of type k in the system that function at time $t > 0$. If the probability distribution for the failure time of components of type k is known and has CDF $F_k(t)$, and we assume failure times of components of different types to be independent, then the probability that the system functions at time $t > 0$ is

$$P(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) P(\bigcap_{k=1}^{K} \{C_t^k = l_k\}) =$$

$$\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right]$$

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Example

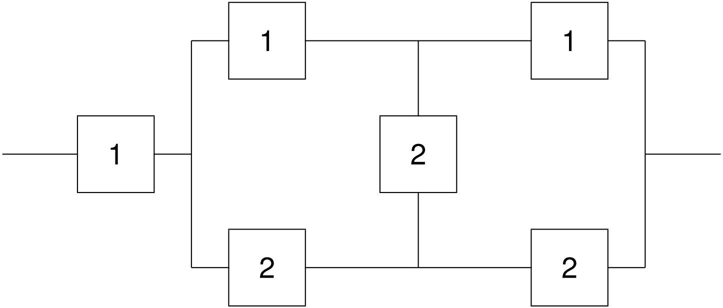

Figure: System with 2 types of components

l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_2	$\Phi(l_1, l_2)$
0	0	0	2	0	0
0	1	0	2	1	0
0	2	0	2	2	4/9
0	3	0	2	3	6/9
1	0	0	3	0	1
1	1	0	3	1	1
1	2	1/9	3	2	1
1	3	3/9	3	3	1

Table: Survival signature of this system

About survival signature:

Coolen FPA, Coolen-Maturi T (2012) On generalizing the signature to systems with multiple types of components. *Complex Systems and Dependability*, W Zamojski et al. (Eds.), Springer 115-130.

Coolen FPA, Coolen-Maturi T, Al-nefaiee AH (2014) Nonparametric predictive inference for system reliability using the survival signature. *Journal of Risk and Reliability*, to appear.

Coolen FPA, Coolen-Maturi T (2014) Modelling uncertain aspects of system dependability with survival signatures. *Edited Volume, Springer*, to appear.

Networks

We consider basic networks with one type of nodes and one type of links; both nodes and links can be unreliable.

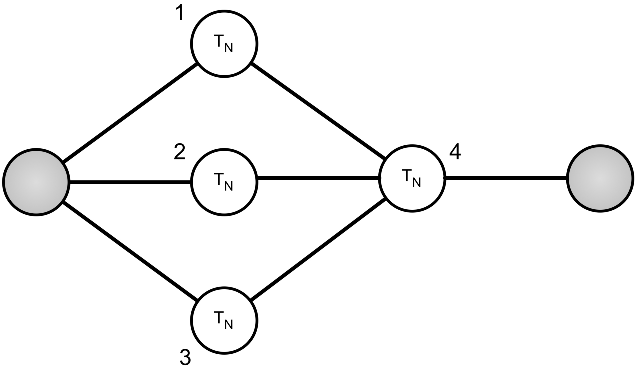

Figure: Topology of a simple network of 4 nodes and 7 links. Start and end nodes are shaded and assumed to function always.

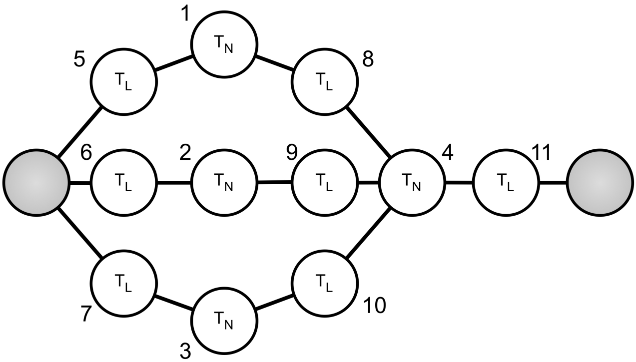

Figure: The same network with links represented as nodes so that nodes are the only unreliable components.

In this simple case, assume that there are m_N nodes and m_L links in the network, and let $m = m_N + m_L$. The state vector is then $\underline{x} = (\underline{x}_N, \underline{x}_L) \in \{0,1\}^m$, with $\underline{x}_N = (x_{N1}, \ldots, x_{Nm_N})$ representing the state of the nodes and $\underline{x}_L = (x_{L1}, \ldots, x_{Lm_L})$ representing the state of the links. The system state is $\phi(\underline{x})$.

The survival signature for l_N nodes and l_L links is

$$\Phi(l_N, l_L) \;=\; \binom{m_N}{l_N}^{-1} \binom{m_L}{l_L}^{-1} \sum_{\underline{x} \in S_{l_N l_L}} \phi(\underline{x}),$$

where $S_{l_N l_L} = \{\underline{x} \mid \sum_{i=1}^{m_N} x_{Ni} = l_N, \sum_{i=1}^{m_L} x_{Li} = l_L\}$ is the set of node and link states where exactly l_N nodes and l_L links are working.

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The reliability function of the network, where nodes and links have failure time distribution functions $F_N(t)$ and $F_L(t)$ respectively, is

$$
\begin{aligned}
P(T_S > t) &= \sum_{l_N=0}^{m_N} \sum_{l_L=0}^{m_L} \Phi(l_N, l_L) \times \\
&\binom{m_N}{l_N} [F_N(t)]^{m_N - l_N} [1 - F_N(t)]^{l_N} \times \\
&\binom{m_L}{l_L} [F_L(t)]^{m_L - l_L} [1 - F_L(t)]^{l_L}
\end{aligned}
$$

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l_N	l_L	$\Phi(l_N, l_L)$	l_N	l_L	$\Phi(l_N, l_L)$	l_N	l_L	$\Phi(l_N, l_L)$
2	3	0.014	3	3	0.043	4	3	0.086
2	4	0.057	3	4	0.171	4	4	0.343
2	5	0.143	3	5	0.393	4	5	0.714
2	6	0.286	3	6	0.643	4	6	0.857
2	7	0.500	3	7	0.750	4	7	1

Table: Survival signature for network ($\Phi(l_N, l_L) = 0$ when not shown)

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