Multi-level Monte Carlo for Reliability Theory

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1. Introduction

Analysing the reliability of large and complex engineered systems can be computationally challenging, especially in the context of many differing types of component.

In this poster we demonstrate a natural mapping of MLMC onto the simplest possible reliability problem of estimating a functional of expected system lifetime, providing orders of magnitude speedup compared to textbook brute-force approaches.

This points to the potential for wider use of multi-level methods throughout reliability theory.

3. MLMC (see Giles, 2015)

Consider a sequence of estimators $T_0, T_1, \dots,$ which approximates T_L with increasing accuracy, but also increasing cost. By linearity of expectation,

$$\mathbb{E}[T_L] = \mathbb{E}[T_0] + \sum_{\ell=1}^{L} \mathbb{E}[T_\ell - T_{\ell-1}],$$

and therefore we can use the following unbiased estimator for $\mathbb{E}[T_L]$,

$$\frac{1}{N_0}\sum_{n=1}^{N_0} T_0^{(0,n)} + \sum_{\ell=1}^{L}\left\{\frac{1}{N_\ell}\sum_{n=1}^{N_\ell}\left(T_\ell^{(\ell,n)} - T_{\ell-1}^{(\ell,n)}\right)\right\}$$

Level ℓ in the superscript (ℓ, n) indicates that samples used at each level are independent, but crucially the differences use common samples. Note, 'correction' since each T_ℓ generally *not* unbiased estimate.

Let $\sigma_0^2, \sigma_\ell^2$ and $\text{cost}_0, \text{cost}_\ell$ be the variance and expected cost of one sample of $T_0, T_\ell - T_{\ell-1}$ respectively. Then, overall for multi-level:

$$\text{cost}_{\text{MLMC}} = \sum_{\ell=0}^{L} N_\ell \cdot \text{cost}_\ell$$

$$\sigma^2_{\text{MLMC}} = \sum_{\ell=0}^{L} N_\ell^{-1} \cdot \sigma_\ell^2$$

$\therefore$ for accuracy $\varepsilon > 0$, $\text{cost}_{\text{MLMC}}$ minimised when $N_\ell \propto \sigma_\ell / \sqrt{\text{cost}_\ell}$

$$\implies \text{cost}_{\text{MLMC}} = \varepsilon^{-2}\left(\sum_{\ell=0}^{L} \sigma_\ell \sqrt{\text{cost}_\ell}\right)$$

Provided cost increases slower than variance decreases, can achieve savings.

4. mlmc R package

The `mlmc` R package (Aslett *et al.*, 2016) provides an easy to use interface which automates much of the MLMC estimation process.

User simply needs to define a level sampler which is provided to the `mlmc()` function.

All standard graphical diagnostics easily plotted via overloaded `plot()` function on MLMC result object.

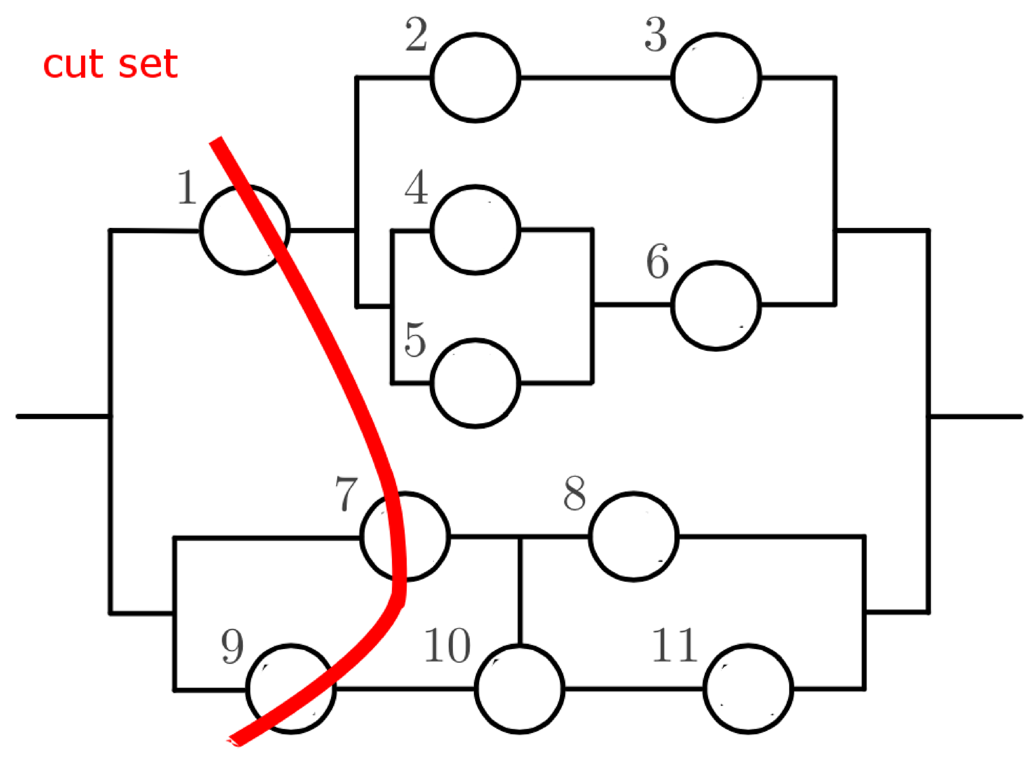
References

Aslett, L. J. M., Giles, M. B., Nagapetyan, T. and Vollmer, S. J. (2016), *mlmc: Tools for Multilevel Monte Carlo*. R package.
URL: *https://github.com/louisaslett/mlmc*

Barlow, R. E. and Proschan, F. (1981), *Statistical Theory of Reliability and Life Testing*, To Begin With Press.

Giles, M. (2015), `Multilevel Monte Carlo methods', *Acta Numerica* **24**, 259--328.

2. System lifetime simulation



Cut sets: A set of components, C, is a *cut set* of the system if the system is failed whenever all the components in C are failed.

A cut set is said to be a *minimal cut set* if no subset of it is also a cut set.

Set of all minimal cut sets, $\mathcal{C}$, characterises the operational state of a system completely.

System lifetime: (Barlow and Proschan, 1981)

$$T_S = \min_{C\in\mathcal{C}}\left\{\max_{c\in C}\{T_c\}\right\}.$$

Thus, the failure time for the system depends on the system structure (via $\mathcal{C}$) and the failure time distributions for each node.

Standard Monte Carlo:

$$\mathbb{E}[f(T_S)] \approx \hat{I}_n := \frac{1}{n}\sum_{i=1}^{n} f\left(\min_{C\in\mathcal{C}}\left\{\max_{c\in C}\{t_c^{(i)}\}\right\}\right)$$

where $t_c^{(i)} \sim F_c(\cdot)$. $\hat{I}_n \sim \text{N}(\mu, \sigma/\sqrt{n})$
$\therefore$ accuracy $\varepsilon > 0$ with α% confidence requires $n = z^2_{\alpha/2}\sigma^2\varepsilon^{-2} \implies \text{cost}_{\text{MC}} = \sigma^2 \cdot \varepsilon^{-2} \cdot |\mathcal{C}|$

$\therefore$ runtime depends on:

1. variance of the estimator;
2. target accuracy of the estimate;
3. number of cut sets.

Cheat? Use subset $\mathcal{C}' \subset \mathcal{C} \implies$

$$\min_{C\in\mathcal{C}'}\left\{\max_{i\in C}\{t_i\}\right\} = T'_S \geq T_S = \min_{C\in\mathcal{C}}\left\{\max_{i\in C}\{t_i\}\right\}.$$

But, $\hat{I}_n \to \eta \neq \mu$, and can only control variance

$$\mathbb{E}[(\hat{I}'_n - \mu)^2] = \frac{\sigma^2}{n} + (\eta-\mu)^2$$
$$= \text{var} + \text{bias}^2$$

5. Level selection in reliability problems

Can use coarse estimate idea in (2) with MLMC?

Sequence of estimators $T_0, \dots, T_L$ based on a nested sequence of minimal cutsets,

$$\mathcal{C}_0 \subset \cdots \subset \mathcal{C}_L = \mathcal{C}.$$

Note $T_L \equiv T_S$, which is not typically true in a general MLMC setting.

Need

1. geometric increase in cost;
2. geometric decrease in variance;
3. geometric decay in differences;
4. $\ell = 0$ should be cheap.

Cost: aim for doubling of min cutset collections ($\text{cost}_\ell = |\mathcal{C}_\ell|$), e.g.

$$|\mathcal{C}_0| = 8, \dots, |\mathcal{C}_5| = 250, |\mathcal{C}_6| = 500, |\mathcal{C}_7| = 1000$$

Prespecify these target sizes and select cutsets.

Level 0: Presimulate 100 component failures, take expectation and sort cutsets. Choose the $|\mathcal{C}_0|$ smallest.

Other levels: Continued selection based on sorted expectation works poorly.
We really want

$$\mathbb{E}[T_\ell - T_{\ell-1}] > \mathbb{E}[T_{\ell+1} - T_\ell]$$

i.e. given $\mathcal{C}_0, \dots, \mathcal{C}_{\ell-1}$ want level ℓ st $\mathbb{E}[T_{\ell-1} - T_\ell]$ is maximal. Note,

$$\mathbb{E}[T_{\ell-1} - T_\ell] \leq \mathbb{E}\left[T_{\ell-1} - \min\left\{T_{\ell-1}, \max_{C\in\mathcal{C}\setminus\mathcal{C}_{\ell-1}} C(\underline{T})\right\}\right]$$

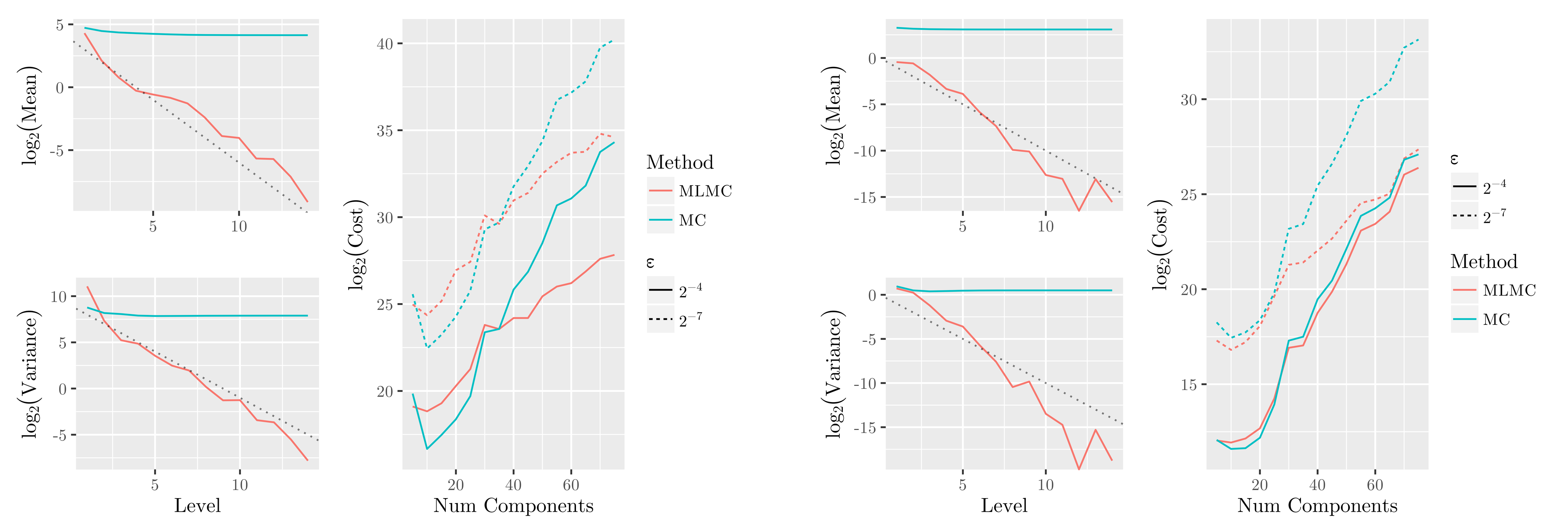
where $C(\underline{T})$ is cutset failure time.
$\therefore$ use the 100 presimulations to Monte Carlo estimate:

$$\mathbb{E}\left[T_{\ell-1} - \min\{T_{\ell-1}, C_i(\underline{T})\}\right] \quad \forall i \text{ st } C_i \in \mathcal{C}\setminus\mathcal{C}_{\ell-1}$$

and sort using this measure. Select smallest $|\mathcal{C}_\ell| - |\mathcal{C}_{\ell-1}|$.

6. Results

Left pairs: Diagnostic tests for largest system; Right: cost gains for nested randomly grown systems. Components Weibull with shape $\beta = 0.5$ (left), $\beta = 3$ (right) and uniformly distributed scale.



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