Parametric and Topological Inference for Masked System Lifetime Data

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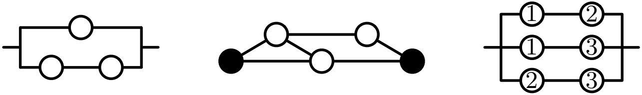
Structural Reliability Theory

- Interest lies in the reliability of 'systems' composed of numerous 'components'.



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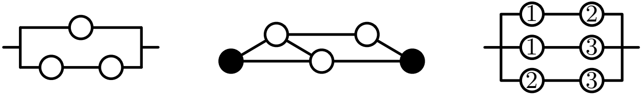

- Lifetime of the system, T, is determined by:
  - the lifetime of the components, $Y_i \sim F_Y(\,\cdot\,;\psi_i)$
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via either the *structure function* or *signature*.

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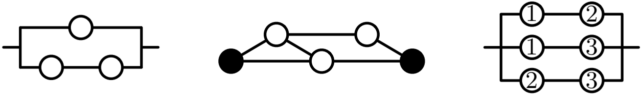

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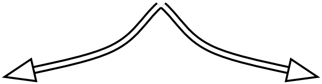

Probabilistic Analysis

Statistical Inference

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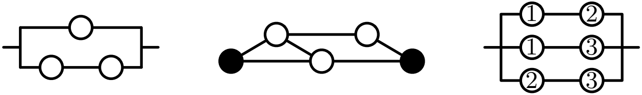

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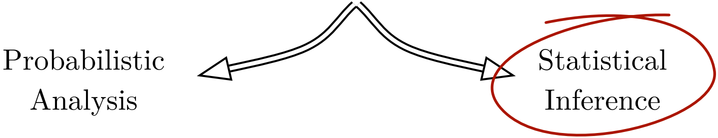
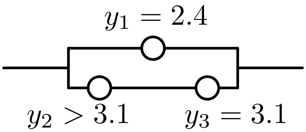


Masked System Lifetime Data

Traditionally, one may have failure time data on components and then infer the parameters ψ of the lifetime distribution.



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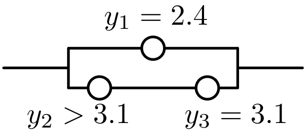

Trivial Bayesian inference:

$$f_{\Psi \,|\, Y}(\psi \,|\, \mathbf{y}) \propto \Big\{ f_Y(y_1; \psi)\, f_Y(y_3; \psi)\, (1 - F_Y(y_2; \psi)) \Big\}\, f_\Psi(\psi)$$

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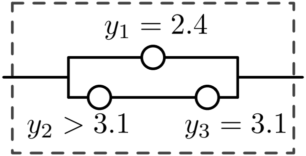

$$t = y_3 = y_{2:3} = 3.1$$

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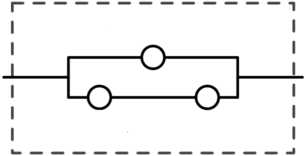

$$t = y_? = y_{?:3} = 3.1$$

Masked system lifetime data means only the failure time of the system as a whole is known, not the component failure times or indeed which components had failed.

The literature on inference for masked system lifetime data is extensive, but:

- heavily focused on specific structures (e.g. series/competing risk systems, see Reiser *et al.* (1995) or Kuo and Yang (2000))
- or focused on specific lifetime distributions (e.g. Exponential, see Gåsemyr and Natvig (2001))
- or does not focus on inferring the parameters of the model (e.g. infer hazard, see Ng *et al.* (2012)).

Why?

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Why? Nasty likelihood!

$$
\begin{aligned}
L(\psi; \mathbf{y}) &= \prod_{i=1}^{m} \frac{\partial}{\partial t} F_T(t; \psi) \bigg|_{t=t_i} \\
&= \prod_{i=1}^{m} \frac{\partial}{\partial t} \Big[1 - \{1 - F_{Y_2}(t)\} \{1 - F_{Y_3}(t)\} \Big] F_{Y_1}(t) \bigg|_{t=t_i}
\end{aligned}
$$

Missing Data

The missing data is what makes the inference hard. Tanner and Wong (1987) is a classic solution to this in a Bayesian framework assuming the missing data can be simulated.

Simulate *missing data* given *parameter*

$$f_{Y\,|\,\Psi,T}(\mathbf{y}_{1\cdot},\dots,\mathbf{y}_{m\cdot}\,|\,\psi,\mathbf{t})$$

$$f_{\Psi\,|\,Y,T}(\psi\,|\,\mathbf{y}_{1\cdot},\dots,\mathbf{y}_{m\cdot},\mathbf{t})$$

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Simulate *parameter* given *missing data*

Then, in the usual way the marginal samples from the second Gibbs step can be used for inference about ψ.

System Signatures

The signature (Samaniego, 1985) is less widely used than the structure function, but in some ways more elegant.

Definition (Signature)

The *signature* of a system is the n-dimensional probability vector $\mathbf{s} = (s_1, \dots, s_n)$ with elements:

$$s_i = \mathbb{P}(T = Y_{i:n})$$

where T is the failure time of the system and $Y_{i:n}$ is the ith order statistic of the n component failure times.

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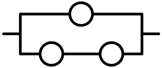
e.g.



$$\Longrightarrow \mathbf{s} = \left(0, \frac{2}{3}, \frac{1}{3}\right)$$

Sampling Latent Failure Times

It can be shown:

$$
\begin{aligned}
&f_{Y|\,T}(y_{i1}, \ldots, y_{in}; \psi \mid t) \\
&\quad \propto \sum_{j=1}^{n} \Big[f_{Y|\,Y<t}(y_{i(1)}, \ldots, y_{i(j-1)}; \psi) \\
&\qquad\qquad \times f_{Y|\,Y>t}(y_{i(j+1)}, \ldots, y_{i(n)}; \psi) \\
&\qquad\qquad \times \mathbb{I}_{\{t\}}(y_{i(j)}) \\
&\qquad\qquad \times \binom{n-1}{j-1} F_Y(t; \psi)^j \, \bar{F}_Y(t; \psi)^{n-j+1} \, s_j \Big]
\end{aligned}
$$

Signature based data augmentation

1. With probability

$$\mathbb{P}(j) \propto \binom{n-1}{j-1} F_Y(t_i;\psi)^j \, \bar{F}_Y(t_i;\psi)^{n-j+1} \, s_j$$

it was the jth failure that caused system failure.

2. Having drawn a random j, sample
   - $j-1$ values, $y_{i1}, \dots, y_{i(j-1)}$, from $F_{Y \mid Y<t_i}(\,\cdot\,;\psi)$, the distribution of the component lifetime conditional on failure before t_i
   - $n-j$ values, $y_{i(j+1)}, \dots, y_{in}$, from $F_{Y \mid Y>t_i}(\,\cdot\,;\psi)$, the distribution of the component lifetime conditional on failure after t_i

and set $y_{ij} = t_i$.

Prerequisites

This is a very general method. The prerequisites for use are,

1. The signature of the system;
2. The ability to perform standard Bayesian inference with the full data;
3. The ability to sample from $F_{Y|\,Y<t_i}(\,\cdot\,;\psi)$ and $F_{Y|\,Y>t_i}(\,\cdot\,;\psi)$.

Prerequisites

This is a very general method. The prerequisites for use are,

1. The signature of the system;

   Easy for systems that are not huge

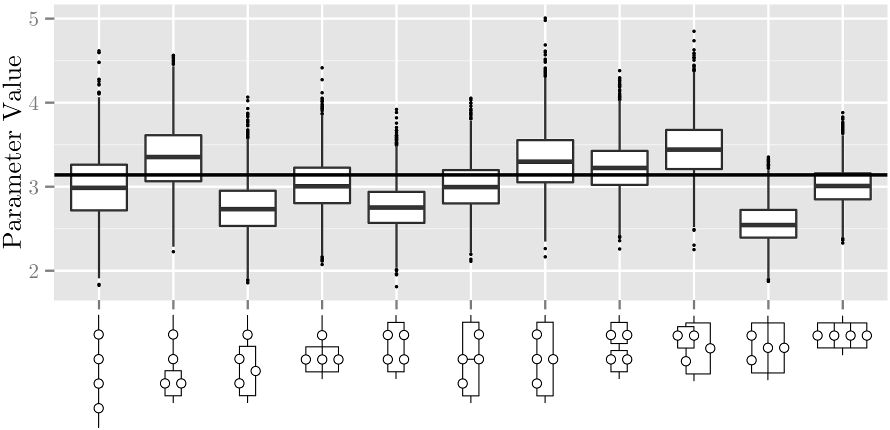
2. The ability to perform standard Bayesian inference with the full data;

   Easy for common lifetime distributions

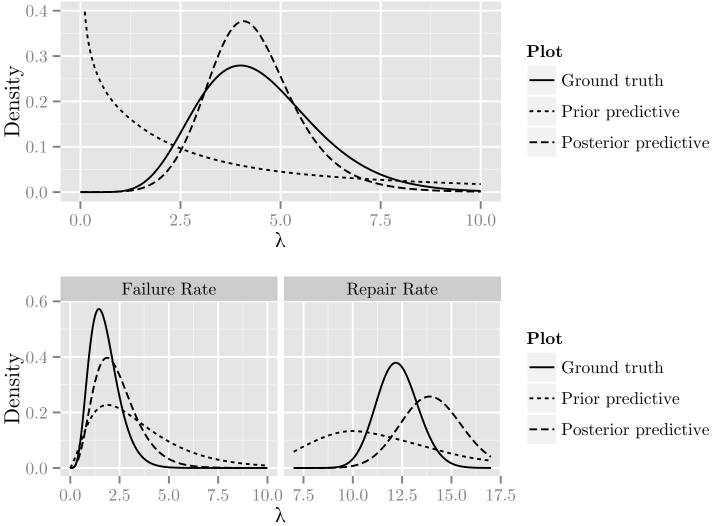
3. The ability to sample from $F_{Y|\,Y<t_i}(\,\cdot\,;\psi)$ and $F_{Y|\,Y>t_i}(\,\cdot\,;\psi)$.

   Depends!

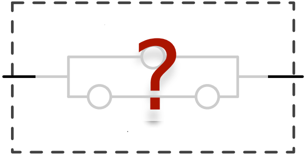
Canonical Exponential Component Lifetime Example



Exchangeable Systems



Unknown Topologies



$$t = y_? = y_{?:3} = 3.1$$

Uniqueness of the Signature

Type	Order	Signature repetition: Unique	2	3	4	5	6	7	Total
Coherent systems	2	2	0	0	0	0	0	0	2
	3	5	0	0	0	0	0	0	5
	4	14	3	0	0	0	0	0	20
	5	43	15	2	6	2	10	1	180
Coherent systems /w graph	2	2	0	0	0	0	0	0	2
	3	4	0	0	0	0	0	0	4
	4	11	0	0	0	0	0	0	11
	5	27	4	0	0	0	0	0	35

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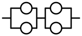
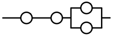
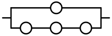
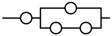
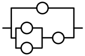
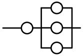
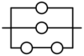
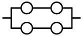
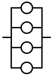
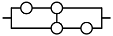
Signature & Topology

Order 4 coherent systems with graph representation.

System Topology	Signature	System Topology	Signature
	$(1, 0, 0, 0)$		$\left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$
	$\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$		$\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$
	$\left(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0\right)$		$\left(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4}\right)$
	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)$		$\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$
	$\left(0, \frac{2}{3}, \frac{1}{3}, 0\right)$		$(0, 0, 0, 1)$
	$\left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$		

Jointly Inferring the Topology

Omitting some rather technical MCMC details, if one uses block marginals:

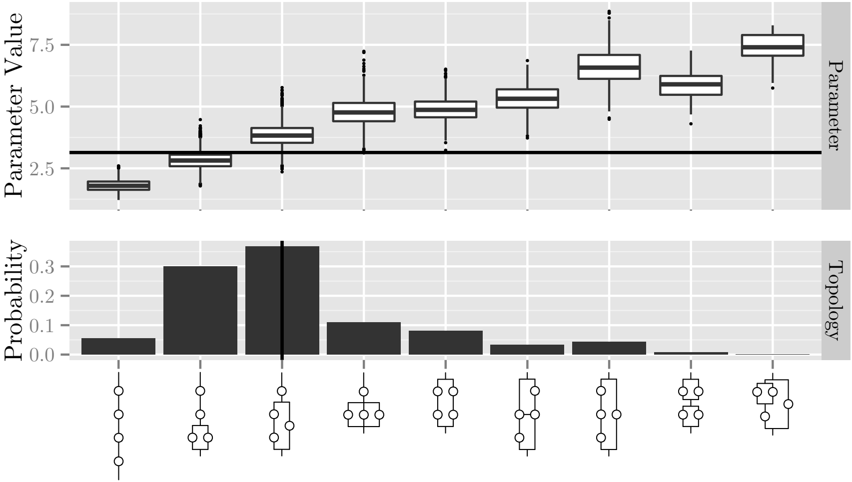
$$f_{\mathcal{M},Y\,|\,\Psi,T}(\mathcal{M}_j, \mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot}\,|\,\psi, \mathbf{t})$$
$$f_{\Psi\,|\,\mathcal{M},Y,T}(\psi\,|\,\mathcal{M}_j, \mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot}, \mathbf{t})$$

by sampling the former sequentially:

$$f_{\mathcal{M}\,|\,\Psi,T}(\mathcal{M}_j\,|\,\psi, \mathbf{t})$$
$$f_{Y\,|\,\mathcal{M},\Psi,T}(\mathbf{y}_{1\cdot}, \ldots, \mathbf{y}_{m\cdot}\,|\,\mathcal{M}_j, \psi, \mathbf{t})$$

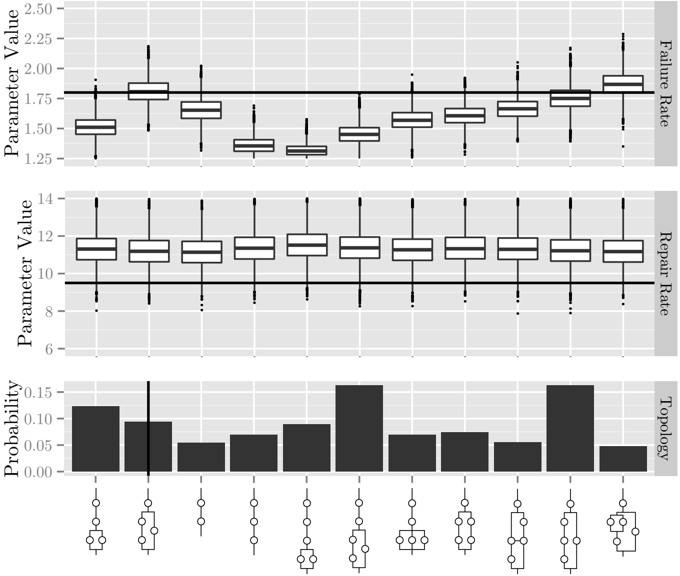
then the topology $\mathcal{M}_j$ can be jointly inferred with the parameters.

Canonical Exponential Component Lifetime Example



52 candidate topologies, low probabilities suppressed

Phase-type Component Lifetime Example



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