ON COMPUTING GAUSSIAN CURVATURE OF SOME WELL KNOWN DISTRIBUTIONS

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The Gaussian curvature of the surface at the point p is the product of the maximum and minimum curvatures in the family. The objective of this paper is to provide a deeper and broader understanding of the meaning of Gaussian curvature, using some more general alternative computational methods. We define the coefficients of the expected Fisher Information Matrix as the coefficients of the first fundamental form. Four different formulas, found in Struik (1961), are used, although we do not intend to compare the superiority of these formulas in computing the Gaussian curvature. We found that all four formulas can compute the Gaussian curvature effectively and successfully. This is demonstrated with three common examples.

1. INTRODUCTION

The Gaussian curvature, K, of the surface, at the point p, is the product of the extreme values of curvatures in the family. If p is a point on a regular surface in R^3 and K(p) is positive, then the two curvatures have the same sign and the point p is called an elliptic point of the surface. If K(p) is negative, then the two curvatures have opposite signs and the point p is called the hyperbolic point of the surface. Examples in this paper demonstrate these cases. If exactly one curvature equals zero, then the point p is a parabolic point of the surface. If the Gaussian curvature equals zero, then the surface is either planar or developer. Computing the Gaussian curvature plays a central role in determining the shape of the surface. It is also a well known fact that two surfaces which have the same Gaussian curvature are always isometric and bending invariant. For instance, Struik D.J. (1961, p120) provided an excellent example that demonstrated a correspondence between the points of a catenoid and that of a right helicoid, such that at corresponding points the coefficients of the first fundamental form and the Gaussian curvatures are identical. In fact, one surface can pass into the other by a continuous bending. This has been demonstrated by the deformation of six different stages. However, if the Gaussian curvature is different, then the two surfaces will not be isometric. For example, a sphere and plane are not locally isometric because the Gaussian curvature of a sphere is nonzero while the Gaussian curvature of a plane is zero. This is why any map of a portion of the earth must distort distances. In this paper, we define the coefficients of the expected Fisher Information Matrix as equal to the coefficients of the first fundamental form. There are numerous authors who have used this concept, including Barndorff-Nielsen O.E.,et.al.(1986, p87 equation (3.1) or (4.1)), and Kass R.E. (1997, p189). Using these defined metric tensors, we can then adopt the same notation and apply the formulas listed in Struik D.J. (1961). The Gaussian curvature then becomes a function of the coefficients of the first fundamental form and their first and second derivatives. In this paper, we suggest the following four systematic steps to compute the Gaussian curvature: Step 1-compute the coefficients of the expected Fisher Information Matrix or coefficients of the first fundamental form, namely, E,F and G; Step 2-compute the needed first or second derivative of E,F and G, and thus the six Christoffel symbols; Step 3-apply formula (D), which necessitates in the computation of the mixed Riemann curvature tensors $\Re^1_{121}$ and $\Re^2_{121}$; the subsequent computation of the inner product of this tensor with the metric tensor, F or G, results in the covariant Riemann curvature tensor $\Re_{1212}$; Step 4-observe that the Gaussian curvature has a very simple relation to Riemann symbols of the second kind. By adhearing to this procedure, the correct Gaussian curvature will be calculated. In the case where $F \neq 0$ or the parametric lines on the surface are not orthogonal, the computational procedure can be extremely tedious. It is always prudent to find a proper transformation to form an orthogonal system of parametric lines in order to simplify the computational procedures.

2. NOTATION AND TERMINOLOGY

In this section, we define the basic notations and terminologies that will apply in the next two sections. These notations and symbols can also be found in Struik, D.J. (1961) or Gray, A. (1993). First and foremost, we define the coefficients of the first fundamental form as;

$$E=-E\left(\frac{\partial^2 \ln f}{\partial u^2}\right),\quad F=-E\left(\frac{\partial^2 \ln f}{\partial v \partial u}\right),$$

$$\text{and } G=-E\left(\frac{\partial^2 \ln f}{\partial v^2}\right).$$

where f(u,v) are two parameters of the probability density functions. It is clear that E,F and G are functions of the parameters u and v. The expectations apply to the whole sample space where the random variables are defined. We also assume that the regular conditions of the information metrics are all satisfied. The details of these five conditions are summarized in Kass R.E. (1997, page 185, section 7.4.1). Next, we define the six well known Christoffel symbols (see Struik D.J. 1961, p107, equation (2-7) or Gray A p398)as follows:

$$\Gamma^1_{11}=\frac{GE_u-2FF_u+FE_v}{2(EG-F^2)},\quad \Gamma^2_{12}=\frac{EG_u-FE_v}{2(EG-F^2)},$$

$$\Gamma^2_{11}=\frac{2EF_u-EE_v-FE_u}{2(EG-F^2)},\quad \Gamma^1_{22}=\frac{2GF_v-GG_u-FG_v}{2(EG-F^2)},$$

$$\Gamma^1_{12}=\frac{GE_v-FG_u}{2(EG-F^2)},\quad \Gamma^2_{22}=\frac{EG_v-2FF_v+FG_u}{2(EG-F^2)}.$$

Since E, F and G are functions of parameters (u,v) and are continuously twice differentiable, E_u, E_v, F_u, F_v, G_u and G_v all exists and are all well defined. Because F=0, formula (A) turns out to be a simplified form of Gauss’ Equation. In 1997, Kass R.E. used formula (A) to compute the Gaussian curvature of trinomial and t families. In the next section, we will demonstrate that formulas (C) and (D) are heavily dependant on the six Christoffel symbols. Additionally, no assumption is made regarding F=0, and so the parametric lines are not necessarily orthogonal. However, if F=0, the six Christoffel symbols can be greatly simplified. The three distributions discussed here belong to this case.

3. THE FORMULA

In this section, we select four formulas that can be used to compute the Gaussian curvature.

(A) $$-\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right)$$

(B) $$-\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & F & G \\ E_U & F_U & G_U \\ E_V & F_V & G_V \end{vmatrix} -\frac{1}{2\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\frac{G_u-F_v}{\sqrt{EG-F^2}}-\frac{\partial}{\partial v}\frac{F_u-E_v}{\sqrt{EG-F^2}}\right]$$

(C) $$\frac{1}{D}\left[\frac{\partial}{\partial v}\left(\frac{D}{E}\Gamma^2_{11}\right)-\frac{\partial}{\partial u}\left(\frac{D}{E}\Gamma^2_{12}\right)\right] =\frac{1}{D}\left[\frac{\partial}{\partial u}\left(\frac{D}{G}\Gamma^1_{22}\right)-\frac{\partial}{\partial v}\left(\frac{D}{G}\Gamma^1_{12}\right)\right]$$

where $D^2 = EG - F^2$.

(D) $$\frac{\Re_{1212}}{EG-F^2}=\frac{(12,12)}{EG-F^2},$$

where $(12,12)=\Re_{1212}=\sum_{m=1}^{2}\Re^m_{121}g_{m2},$

$$\Re^{l}_{ijk} = \frac{\partial}{\partial u_j}\Gamma^{l}_{ik} - \frac{\partial}{\partial u_i}\Gamma^{l}_{jk} + \Gamma^{m}_{ik}\Gamma^{l}_{mj} - \Gamma^{m}_{jk}\Gamma^{l}_{mi},$$

sum on m,

where the quantities of $\Re^{l}_{ijk}$ are components of a tensor of the fourth order. This tensor is called the mixed Riemann curvature tensor. Notice that g_{11}, g_{12} and g_{22} are simply tensor notation for E,F and G. Formula (B) was developed by G. Frobenius while formula (C) was derived by J. Liouville. Clearly, formula (A) is a special case that is valid only when the parametric lines are orthogonal. Formula (D) is a general form represented in Riemann symbols of the first and second kind, respectively. In formula (D), $\Re_{1212}$, the inner product of the mixed Riemann curvature tensor and the metric tensor, is called the covariant Riemann curvature tensor; it is a covariant tensor of the fourth order. The components $\Re^{l}_{ijk}$ and $\Re_{1212}$ are also known as Riemann symbols of the first and second kind, respectively. Notice that Riemann symbols of the second kind will satisfy the relation $\Re_{1212} = -\Re_{1221} = -\Re_{2112} = \Re_{2121}$, the well-known property of skew-symmetry with respect to the last two indices. It is useful to be aware of the fact that the Christoffel symbols depend only on the coefficients of the first fundamental form and their derivatives. The same holds true for the mixed Riemann curvature tensor. From this point of view, as long as we can find the coefficients of the first fundamental form of a given distribution and their first and second derivatives, we can uniquely define the corresponding Christoffel symbols and hence mixed Riemann curvature tensors. Thus, the process of computing the covariant Riemann curvature tensor and Gaussian curvature is simplified. From a different perspective, we know that the mixed Riemann curvature tensor will link with the coefficient of the second fundamental form; namely e,f,and g, by $\Re^{n}_{121} = g^{n2}(eg - f^2)$, where

$$g^{11} = \frac{G}{EG - F^2}, \quad g^{12} = \frac{-F}{EG - F^2}, \quad g^{22} = \frac{E}{EG - F^2}.$$

The above relation can then be easily used to derive $\Re_{1212} = eg - f^2$, and the result will coincide with equation (7-3) of Struik D.J.(1961, p83), the original fundamental definition of Gaussian curvature. These points convince us that formulas (A) and (D) basically define the same quantity, but only in different forms. The reason why only formula (D) was selected for presentation is due to the following two facts: 1. to avoid repetition of Kass R.E (1997, p189); 2. when F=0, formulas (B) and (C) are trivially similar to formula (A). For example, in formula (C), we may substitute the following equation on the left hand side:

$$\frac{D}{E}\Gamma^{2}_{11} = \sqrt{\frac{G}{E}}\left(\frac{-E_v}{2G}\right) = \frac{-E_v}{2\sqrt{EG}}. \quad \text{or}$$

$$\frac{D}{E}\Gamma^{2}_{12} = \sqrt{\frac{G}{E}}\frac{G_u}{2G} = \frac{G_u}{2\sqrt{EG}}.$$

We can immediately calculate the same results as found from formula (A) while formula (D) results in a Riemann representation. In this way, we have supplied some more general alternative methods to compute the Gaussian curvature, including the case when $F \neq 0$.

4. THREE EXAMPLES

In this section, we give the details of three examples and demonstrate how we could apply formula (D) to compute our Gaussian curvature. The three examples will deal with the location-scale family of densities and the methods of finding those with negative Gaussian curvature. Kass R.E.(1997,p192 theorem 7.4.6) gave the general form of a location-scale manifold of density:

$$\left\{ p(x) = \frac{1}{v} f\left(\frac{x-u}{v}\right) \mid (u,v) \in R \times R_{+} \right\}$$

for some density function f. Then, the information metric of the

Riemannian geometry space has constant negative curvature. We provide the derivation of the formula for the Gaussian curvature of normal distribution in example 1, Cauchy distribution in example 2 and t family distribution in example 3.

Example 1: Let Ω_1 be a location scale manifold of density that has the following general form:

$$\Omega_1 = \left\{ f(x) = \frac{1}{\sqrt{2\pi v^2}} \exp(-\frac{(x-u)^2}{2v^2}) \,|\, (u,v) \in R \times R_+ \right\}$$

where u is the location parameter and v is the scale parameter. We also assume that the regular conditions of the information metric are satisfied. The first and second partial derivative, with respect to parametric lines u and v, are given as:

(4.1) $\frac{\partial^2 \ln f}{\partial u^2} = \frac{-1}{v^2}$, $\frac{\partial^2 \ln f}{\partial v^2} = \frac{1}{v^2} - \frac{3(x-u)^2}{v^4}$.

It is commonly known that the expected value and variance of the random variable x are u and v^2, respectively. From this, we could easily derive the coefficient of the first fundamental form

$$E = \frac{1}{v^2},\ F = 0,\ G = \frac{2}{v^2}.,$$

as well as their corresponding derivatives with respect to the parametric lines u and v :

(4.2)
$$E_u = 0,\ E_v = \frac{-2}{v^3},\ G_u = 0,\ G_v = \frac{-4}{v^3},$$
$$EG = \frac{2}{v^4},\ \sqrt{EG} = \frac{\sqrt{2}}{v^2} \text{ and } \frac{1}{\sqrt{EG}} = \frac{v^2}{\sqrt{2}}.$$

Substituting the listed results into formula (A), (B) or (C), it should be easy to compute the Gaussian curvature, obtaining $\frac{-1}{2}$. Again, we present the details for formula (D) only. We can derive $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0.$

$$\Re_{121}^2 = \frac{\partial}{\partial v}\Gamma_{11}^2 - \Gamma_{21}^1\Gamma_{11}^2 + \Gamma_{11}^2\Gamma_{22}^2 = \frac{-1}{2v^2}$$

$$\Re_{1212} = \Re_{121}^2 G = \frac{-1}{v^4}$$

$$K = \frac{\Re_{1212}}{EG} = \frac{-1}{v^4}\frac{v^4}{2} = \frac{-1}{2}.$$

Example 2: Let Ω_2 be the location scale manifold of density which has the following general form:

$$\Omega_2 = \left\{ f(x) = \frac{v}{\pi}\frac{1}{v^2 + (x-u)^2} \,|\, x \in R,\ (u,v) \in R \times R_+ \right\}.$$

where u is the location parameter and v is the scale parameter. The logarithm of the likelihood function of Cauchy density with one observation can be written as

(4.3) $\ln f = \ln\frac{v}{\pi} - \ln(v^2 + (x-u)^2)$.

As before, we can derive the first two partial derivatives with respect to the parametric lines u and v.

(4.4)
$$\frac{\partial^2 \ln f}{\partial u^2} = \frac{2((x-u)^2 - v^2)}{((x-u)^2 + v^2)},$$
$$\frac{\partial^2 \ln f}{\partial v^2} = \frac{-1}{v^2} + \frac{2(v^2 - (x-u)^2)}{(v^2 + (x-u)^2)},$$
$$\frac{\partial^2 \ln f}{\partial v \partial u} = \frac{-4v(x-u)}{(v^2 + (x-u)^2)^2}.$$

Taking the expected values of equations (4.4), we finally get the following results:

(4.5)
$$E = -E(\frac{\partial^2 \ln f}{\partial u^2}) = \frac{1}{2v^2},\ F = -E(\frac{\partial^2 \ln f}{\partial u \partial v}) = 0, \text{ and}$$
$$G = -E(\frac{\partial^2 \ln f}{\partial v^2}) = \frac{1}{2v^2}.$$

The derivatives of the coefficients of the first fundamental form and six Christoffel symbols are all straightforward computations. Due to the fact that the Cauchy distribution is the same as the normal distribution, that is, $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0.$, we use formula (D) to derive the Gaussian curvature.

$$\Re^2_{121} = \frac{\partial}{\partial v}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{22} = \frac{-1}{v^2}$$

$$EG = \frac{1}{4v^4}.$$

$$\Re_{1212} = \sum_{m=1}^{2} \Re^m_{121} g_{m2} = \Re^2_{121} G = \frac{-1}{2v^4}$$

$$K = \frac{\Re_{1212}}{EG} = 4v^4(\frac{-1}{2v^4}) = -2$$

Example 3: Let Ω_3 be the location-scale manifold of density that has the student t distribution and generally has the form:

$$\Omega_3 = \left\{ f(x) = \frac{1}{v}\frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\Gamma(\frac{r}{2})}(1+\frac{1}{r}(\frac{x-u}{v})^{-\frac{r+1}{2}}) \;\middle|\; x \in R, \quad (u,v) \in R \times R_+ \right\}.$$

where u is location parameter and v is scale parameter. Let us define the following variables to simplify the notation:

$$a = \frac{1}{r}, \qquad b = \frac{r+1}{2}, \qquad c_r = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\Gamma(\frac{r}{2})}.$$

Then the logarithm of likelihood function of family t, can be written as follows:

(4.6) $\ln f(x) = \ln c_r - b\ln(1 + a(\frac{x-u}{v})^2) - \ln v.$

From equation (4.6), we can derive the first and second partial derivatives :

$$\frac{\partial^2 \ln f}{\partial u^2} = \frac{2ab(a(x-u)^2 - v^2)}{(a(x-u)^2 + v^2)^2}.$$

(4.7)

$$\frac{\partial^2 \ln f}{\partial v^2} = v^{-2}\left[1 + \frac{-6ab(x-u)^2 v^{-2}(1 + a(x-u)^2 v^{-2}) + 4a^2 b(x-u)^4 v^{-4}}{(1 + a(x-u)^2 v^{-2})^2}\right]$$

$$\frac{\partial^2 \ln f}{\partial v \partial u} = \frac{4abv(x-u)}{(a(x-u)^2 + v^2)^2}$$

We can now take the expected values of (4.7), and have the following results.

$$E = -E(\frac{\partial^2 \ln f}{\partial u^2}) = -2ab(\frac{-r}{v^2(r+3)}) = \frac{r+1}{v^2(r+3)},$$

$$F = -E(\frac{\partial^2 \ln f}{\partial v \partial u}) = 0,$$

$$G = -E(\frac{\partial^2 \ln f}{\partial v^2}) = \frac{-1}{v^2}(1 - 3\frac{r+1}{r+3}).$$

It now becomes a routine procedure to compute the derivative of the coefficient of the first fundamental form and six Christoffel symbols. Compute the Riemann symbols of the first and second kind, respectively. Thus, the Gaussian curvature is calculated.

$$\Re^2_{121} = \frac{\partial}{\partial v}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{22} = \frac{-(r+1)}{2rv^2}$$

$$\Re_{1212} = \sum_{m=1}^{2} \Re^m_{121} = \Re^2_{121} G = \frac{-(r+1)}{v^4(r+3)}$$

$$K = \frac{\Re_{1212}}{EG} = \frac{-(r+3)}{2r}$$

5. CONCLUDING REMARK

One of the most important theorems of the 19th century is 'Theorema Egregium'. Many mathematicians at the end of the 18th century, including Euler and Monge, had used the Gaussian curvature, but only when defined as the product of the principal curvatures. Since each principal curvature of a surface depends on the particular way the surface is defined in R^3, there is no obvious reason for the product of the principal curvatures to be intrinsic to that particular surface. Gauss published in 1827 that the product of the principal curvatures depends only on the intrinsic geometry of the surface revolutionized differential geometry. Gauss wrote "'The Gaussian curvature of a surface is a bending invariant' ,'a most excellent theorem', This is a Theorema egregium'". In this theorem, Gauss proved that the Gaussian curvature ,K, of a surface, depends only on the coefficient of the first fundamental form and their first and second derivatives. This important geometric fact will link the concepts of bending and isometric mapping. By bending invariant, we mean that it is unchanged by such deformations of the surface when restricted to a limited region that does not involve stretching, shrinking, or tearing. When measured along a curve on the surface, the distance between two points on the surface is unchanged. The angle of the two tangent directions at the point is also unchanged. This property of surfaces expressible as bending invariant is called the intrinsic property. We would like to conclude this study by repeating Kass' (1989,1997) favorite and most interesting piece of trivia: "Suppose we ask which distribution in the t family is half way between Normal and Cauchy on the statistical curvature scale, the scale of sufficiency loss of the Maximum Likelihood Estimator. For Normal, $\gamma=0$ and for Cauchy, $\gamma=(\frac{5}{2})^{\frac{1}{2}}$. Thus, we seek γ such that $\gamma=\frac{1}{2}(\frac{5}{2})^{\frac{1}{2}}$. There is no reason why γ should turn out to be an integer; it merely has to be a number greater than 1. Since $\gamma=1$ for Cauchy and $\gamma=\infty$ for Normal, the answer is $\gamma=3$. Thus, in the sense of the insufficiency of the MLE, as measured by statistical curvature , the t, on 3 degrees of freedom, is halfway between Normal and Cauchy. This means that the statistical curvature of the t_3 distribution is the arithmetic mean of the statistical curvatures for the Cauchy and Normal distribution. From the Gaussian curvature that we derived in this paper, we showed that in Normal distribution we obtain $K=-\frac{1}{2}$,and in Cauchy distribution we obtain $K=-2,$ while in t family distribution with r degrees of freedom, we get $K=-\frac{r+3}{2r}$. In other words, the Gaussian curvature of the t_3 distribution is the geometric mean of the curvatures for the Cauchy and Normal distribution. Thus, we conclude that whether one uses statistical or geometric mean curvature, the t_3 may be considered half way in between a Normal and Cauchy distribution."

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