Imperial College London

Imperial College London

Department of Mathematics

Harvesting Volatility Risk Premium

Author: Shibo Lu (CID: 01210524)

A thesis submitted for the degree of

MSc in Mathematics and Finance, 2019-2020

Declaration

The work contained in this thesis is my own work unless otherwise stated.

Signature:

Acknowledgements

I would like to express my deepest gratitude to my thesis supervisor, Vladimir Lucic, for his brilliant advice and inspiration during the period of my master's thesis. This work could not have been completed without his passionate guidance and support. Thanks are also due to my parents for their everlasting help and encouragement. I would like to thank specially my girlfriend Jiaqi Gong for her great support during my difficult time. Finally, I wish to thank all my classmates who made my days more special.

Abstract

Volatility risk premia refers to the fact that on average implied volatility in many asset classes is above realized volatility. This phenomenon can be consistently modeled withing incomplete markets framework, for example jump-diffusion or stochastic volatility models. The situation is particularly clear in the equities space, where the risk premia harvested in "normal" times, via, say selling a portfolio of Delta-hedged options (e.g. in [Bakshi and Kapadia, 2015]), is partially paid back in the stressed regimes during which the Delta-hedged options portfolio, which is short Gamma, suffers losses. In this project, we do simulations on harvesting volatility risk premia by short selling delta hedged options under different model assumptions and compare with the theoretical gains by theoretical derivation. We found that both stochastic volatility and jumps will cause loss in our delta hedged option portfolio and there could be extremely large potential loss due to jumps. Finally, we also tests when volatility risk premia are harvested via volatility swap, how the delta hedged option's value will change with different parameters.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Introduction to volatility risk premium

The volatility risk premium (VRP) refers to the phenomenon that option-implied volatility tends to exceed realized volatility of the same underlying asset over time. This creates a profit opportunity for volatility sellers. This difference is most apparent in broad market equity indices such as the SP 500 Index. Very limited disclosure in the industry can be found on harvesting VRP. Ge [Ge, 2016] listed three ways of harvesting VRP.

The first one is based on option strategies, i.e. by selling index options and delta hedging with index. This method is usually applied in production portfolios and empirical research has been conducted by industrial traders in Ge [Ge, 2016] and Gmbh [GMBH, 2017]. However, a very limited disadvantage of this strategy is the vulnerability to financial crisis. The hedged portfolio will be highly likely to have very large negative loss when there are sudden jumps in the underlying stock price. Studies show that using options to collect equity VRP was profitable in most historical periods in Bakshi and Kapadia[Bakshi and Kapadia, 2015], Ge and Bouchey [Wei and Bouchey, 2015])

Another strategy used in practice is through another financial instrument variance swap. A variance swap is different from a traditional swap in many ways. It does not have the periodic cash exchanges, and it is a structured contract that stipulates a strike level at the initiation date and pays out only at the expiration date, based on the difference between the realized variance (volatility squared) of a given asset (usually an equity index) and the strike level which is chosen so that the expected value of the swap contract is zero at the initiation of the contract. A variance swap, such like a traditional swap, has a theoretical notional that is used to compute gains or losses but is not exchanged. The notional of a variance swap, however, can be tricky to compute or understand. It is derived from another theoretical value called the vega notional. [Ge, 2016] In this project, we always use unit notional for variance swap strategy. The variance swap will make steady profits mostly when realized volatility is below the strike level. When realized volatility surpasses variance strikes, however, losses occur and the losses have the potential to be significant.

The third way is via VIX futures. VIX Index futures are the newest class of instruments that can be used to harvest the volatility risk premium. The CBOE introduced the VIX in 1993 as a benchmark for equity market risk, computed from the implied volatility of near-term, at-the-money SP 100 Index options. On the other side of the trade, shorting VIX futures may be a straightforward way to harvest the volatility risk premium. However, construction of portfolio including VIX futures tends to be complex due to the change of term structure in highly volatile markets and selection of VIX futures need to be managed with care and precision in order to maintain a consistent and profitable trading strategy, which makes it hard to study mathematically. Therefore, we will not cover this strategy in this project.

Since the VRP harvesting requires implied volatility to be higher than the realised volatility of the underlying asset, in highly volatile markets, there is highly likely to be great loss in all three strategies. This project tries to simulate such market conditions with stochastic volatility, focusing on option strategies and variance swap strategies. Using different models will give different results on the two strategies. We conducted research on selling delta hedged option strategy in four different cases. The first case assumes that the stock price has simple geometric Brownian motion followed by a modified Heston model with stochastic volatility assumption. Then we consider Merton jump diffusion model with constant volatility and the final case concentrates on the stochastic volatility Merton Jump diffusion model.

In the meantime, we also considered a combination of the first two strategies. An important topic covered in this project is the way the delta hedged option position performs with different parameters, keeping VRP harvested from variance swap.

1.2 Parameters and Notations

For convenience, all basic notations and parameters are introduced in this section

- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution
- $T-t$ is the time to maturity (expressed in years)
- S_t is the spot price of the underlying asset
- K is the strike price
- r is the risk free rate (annual rate, expressed in terms of continuous compounding)
- σ is the volatility of returns of the underlying asset
- $\hat{\sigma}$ is the implied volatility of an option i.e. the volatility that makes theoretical value of the option equal to its price under some model assumptions (e.g. Black Scholes).
- σ_v is the volatility of stochastic volatility V_t
- q is the continuously paid dividend
- $n(\cdot)$ is the derivative of the cumulative distribution function of the standard normal distribution
- W_t and B_t are both Wiener processes or Brownian motions
- N_t is Poisson process with jump intensity λ
- P_t is compound Poisson process defined by

$$P_t = \sum_{i=1}^{N_t} J_i$$

  where U_i has lognormal distribution $\ln(J_i + 1) \sim N\left(\gamma, \delta^2\right)$
- $C(\cdot)$ is the call price of European style option with payoff $\max(S_T - t, 0)$.

- $\mathcal{C}(\cdot)$ is the Black-Scholes call price and $\mathcal{C}^{\mathrm{MJ}}(\cdot)$ is the call price under Merton jump diffusion process
- Δ_t, $\frac{\partial C_t}{\partial S_t}$ and $\partial_S C_t$ are used as sensitivity of call option price to underlying asset which is also known as delta
- Γ_t or $\frac{\partial^2 C_t}{\partial S_t^2}$ are used as second order sensitivity of call option price to underlying asset which is also known as gamma

1.3 Project structure

The project is formed of three different parts. The first one focuses on harvesting volatility risk premium with Black Scholes models and stochastic volatility models such as Heston model. The second part incorporates models with jumps, for example, Merton jump diffusion model and stochastic jump diffusion model. The final section mainly concentrates on harvesting VRP via variance swap. We are particularly interested in how tuning parameters to keep VRP harvested from variance swap constant will influence the performance of short selling delta hedged options portfolio.

1.4 Black Scholes model and delta hedging

The Black–Scholes model is a mathematical model for the dynamics of a financial market containing derivative investment instruments. In the derivation of this model, the stock price was assumed to be geometric Brownian motion (GBM) with SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{1.4.1}$$

From the partial differential equation in the model, known as the Black–Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{1.4.2}$$

one can deduce the Black–Scholes formula

$$C(S,t) = N(d_1)S - e^{-r(T-t)}KN(d_2) \tag{1.4.3}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

The price of a corresponding put option based on put-call parity is:

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)} - S_t + C(S,t) \\ &= e^{-r(T-t)}KN(-d_2) - S_tN(-d_1). \end{aligned}$$

The key idea behind the model is to hedge the option by buying and selling the underlying asset in just the right way and, as a consequence, to eliminate risk. This type of hedging is called 'continuously revised delta hedging' and is the basis of more complicated hedging strategies such as those engaged in by investment banks and hedge funds.

It is worth noticing the strong assumption of dynamics of underlying asset. When asset price does not follow geometric Brownian motion, the pricing formula will not give a fair price of the option. The main reason is due to the inaccuracy of delta hedge strategy in this case since delta hedge under Black Scholes model only hedges the uncertainty associated with the Brownian motion. However, it is still worth testing how Black Scholes formula performs under different model assumptions.

We can consider the scenario of selling at time 0 an European call option at implied volatility $\hat{\sigma}$, i.e. for the price $p = \mathcal{C}(T, S_0, K, r, \hat{\sigma})$ and then following a Black-Scholes delta-hedging trading strategy based on constant volatility $\hat{\sigma}$ until the option expires at time T. We denote $C(t,s) = \mathcal{C}(T-t, s, K, r, \hat{\sigma})$, so that the hedged portfolio, with value process Π_t, is constructed by holding $\Delta_t := \partial_S C(t, S_{t-})$ units of the risky asset S, and the remainder $\beta_t := \frac{1}{B_t}(\Pi_t - \Delta_t S_t)$ units in the risk free asset B (a unit notional zero coupon bond with YTM r). This portfolio, initially funded by the option sale ($\Pi_0 = p$) defines a self-financing trading strategy. Hence the portfolio value process Π satisfies the SDE

$$\Pi_t = p + \int_0^t \partial_S C(u, S_u)\,\mathrm{d}S_u + \int_0^t r\left(\Pi_u - \partial_S C(u, S_{u-})\, S_u\right)\mathrm{d}u \tag{1.4.4}$$

or equivalently, [Bakshi and Kapadia, 2015] defines $\Pi_{t,t+\tau}$ as the hedging error between time t and $t+\tau$

$$\Pi_{t,t+\tau} = C_{t+\tau} - C_t + \int_t^{t+\tau} \Delta_u dS_u + \int_t^{t+\tau} r\left(C_u - \Delta_u S_u\right) du \tag{1.4.5}$$

Note that 1.4.4 and 1.4.5 are not only valid when Black Scholes (1.4.1) PDE holds. The equivalence between these two definitions should be clear with Ito's formula on C_t. In this project, we will use either of the two definition depending on the situation.

However, delta hedging does not work well for jump diffusion processes and will give a relatively big loss when jumps occur. However, jumps will create value for variance swap. In this project, we also compare the effectiveness of VRP harvesting using both variance swap and delta hedged options.

1.5 Change of measure

VRP appears when implied volatility is greater than realised volatility. The concept can also be interpreted as when risk neutral volatility is greater than realised volatility. Therefore, in order to find opportunities to harvest VRP we shall use Girsanov theorem to change to risk neutral probability measure.

Theorem 1.5.1 (Girsanov Theorem). *Let $\{W_t\}$ be a Wiener process on the Wiener probability space $\{\Omega, \mathcal{F}, P\}$. Let $\{X_t\}$ be a measurable process adapted to the natural filtration of the Wiener process $\{\mathcal{F}_t^W\}$ with $X_0 = 0$. Define the Doléans-Dade exponential $\mathcal{E}(X)_t$ of X with respect to W*

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$$

where $[X]_t$ is the quadratic variation of X_t. If $\mathcal{E}(X)_t$ is a strictly positive martingale, a probability *measure $\mathcal{Q}$ can be defined on $\{\Omega, \mathcal{F}\}$ such that we have Radon-Nikodym derivative*

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{F_t} = \mathcal{E}(X)_t$$

Then for each t the measure $\mathcal{Q}$ restricted to the unaugmented sigma fields $\mathcal{F}_t^W$ is equivalent to P restricted to $\mathcal{F}_t^W$. Furthermore, if Y is a local martingale under P, then the process

$$\tilde{Y}_t = Y_t - [Y, X]_t$$

is a Q local martingale on the filtered probability space $\left\{\Omega, \mathcal{F}, \mathcal{Q}, \left\{\mathcal{F}_t^W\right\}\right\}$.

Girsanov Theorem provides a convenient way of finding the equivalent martingale. As a simple example, we can find risk neutral measure $\mathcal{Q}$ which is done in Black-Scholes model via Radon-Nkodym derivative:

$$\frac{dQ}{dP} = \mathcal{E}\left(\int_0^t \frac{r-\mu}{\sigma} dW_s\right)$$

where r denotes the instantaneous risk free rate, μ the asset drift and σ volatility. Then we can say that under risk neutral measure $\mathcal{Q}$

$$dS_t^* = \sigma S_t^* dW_t^*$$

where $W_t^* = W_t - \frac{r-\mu}{\sigma}t$. By Girsanov's Theorem, $(W_t^*)_{0\le t\le T^*}$ is a standard Brownian motion with respect to $\mathcal{Q}$ and hence $\mathrm{d}\log S_t^*$ is a martingale under $\mathcal{Q}$.

Chapter 2

Delta hedged options under Black Scholes based models

2.1 No dividend paying geometric Brownian motion

In this section, we use simplest delta hedging strategy with call option price obtained from Black Scholes option pricing formula as 2.2.2 and delta being defined as Black-Scholes delta

$$\Delta^{\mathrm{BS}}(t, S_t) = N'(d_1)$$

where $N'(\cdot)$ is the first order derivative of standard normal distribution CDF and d_1 is defined as in Section 1.4. We will give both theoretical derivation and simulation results to demonstrate the effectiveness of Black Scholes delta hedging strategy in different models and make comparison between different models.

2.1.1 VRP harvesting via delta hedged option

In this section, we stick to the geometric Brownian motion assumption where stock price follows SDE in 2.2.1. Further, it can be easily proved by Ito's formula, and we obtain

$$C_t = C_0 + \int_0^t \Delta_u dS_u + \int_0^t \left(\frac{\partial C_u}{\partial u} + \frac{1}{2}\sigma^2 S_u^2 \frac{\partial^2 C_u}{\partial S_u^2} \right) du \tag{2.1.1}$$

$$= C_0 + \int_0^t \Delta_u dS_u + \int_0^t r \left(C_u - \Delta_u S_u \right) du \tag{2.1.2}$$

which is the statement that the call option can be replicated by trading a stock and a bond. Combining equation 2.1.2 with the definition of delta-hedged gains in equation 1.4.5, it is apparent that, with continuous trading, $\Pi_{t,t+\tau} = 0$ over every horizon τ for any starting time t. More generally, it can verified that $\Pi_{t,t+\tau} = 0$ is a property common to all one-dimensional Markov Ito price processes:

$$\frac{dS_t}{S_t} = \mu_t \left[S_t \right] dt + \sigma_t \left[S_t \right] dW_t^1$$

for any deterministic finite difference process $\mu_t[S_t]$ and $\sigma_t[S_t]$.

Theoretically, it is impossible to do continuous delta hedging in real world. Therefore, one needs to rebalance hedging positions discretely, in which case $\Pi_{t,t+\tau}$ will not necessarily be zero. Over the life of the option, where the hedge is rebalanced at each of the dates $t_n, n = 0, 1, \dots N-1$ (where we define $t_0 = t, t_N = t + \tau$) and $t_{n+1} - t_n = \tau/N$. Define the discrete delta-hedged gains

$\pi_{t,t+\tau}$ as

$$\pi_{t,t+\tau} \equiv C_{t+\tau} - C_t - \sum_{n=0}^{N-1} \Delta_{t_n} \left(S_{t_{n+1}} - S_{t_n}\right) - \sum_{n=0}^{N-1} r \left(C_t - \Delta_{t_n} S_{t_n}\right) \frac{\tau}{N}$$

In research conducted in [Bertsimas et al., 1997] as well as those in [Figlewski, 1989], the asymptotic distribution of the discretely hedged option portfolio has a mean of zero, and is symmetric and it is influenced by a variety of parameters. Simulation results also show a convergence to Dirac delta function with steps being smaller. From Bertsimas, Kogan and Lo(2000), $\sqrt{N}\pi \Rightarrow \frac{1}{\sqrt{2}} \int_t^{t+\tau} \sigma^2 S_u^2 \frac{\partial^2 C_u}{\partial S_u^2} dW_u$ where W_u is a Wiener process, independent of W_u^1. Thus, the asymptotic distribution of the discretely hedged option portfolio has a mean of zero, and is symmetric. Simulation results can also verify such convergence.

Here we are more interested in how selling options with different implied volatility will influence the hedging error. In this case, Black Scholes PDE becomes

$$\frac{\partial C}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

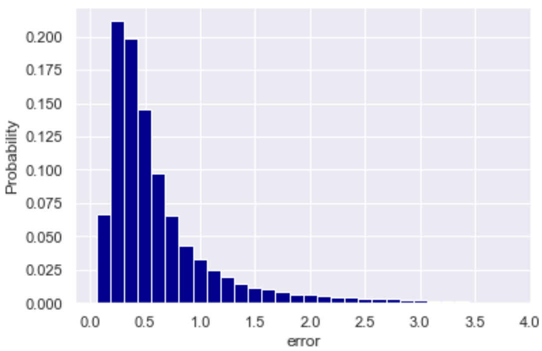
Substituting the new PDE to 1.4.5 and applying Ito's formula give us the volatility risk premium we can harvest

$$\mathrm{VRP}_t = \frac{1}{2} \int_0^t \Gamma\left(u, S_{u-}\right) S_u^2 \left(\hat{\sigma}^2 - \sigma^2\right) \mathrm{d}u.$$

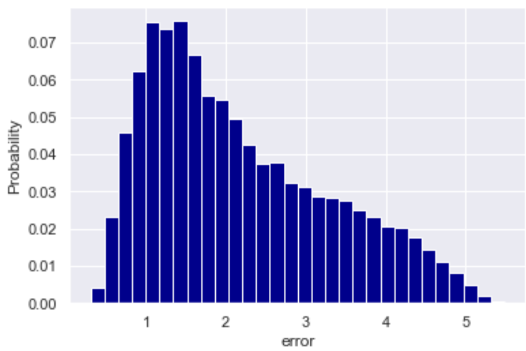
This formula shows that the nature of VRP harvesting via delta hedged options is by a short position of gamma which is positive almost surely in Black Scholes model.

2.1.2 Simulation results

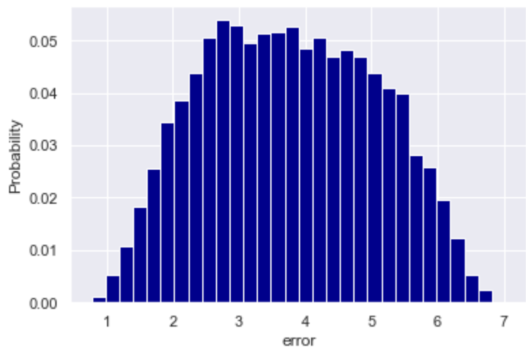
We use the following parameters throughout simulations in this section: $T = 1$, $\mu = 0.2$, $r = 0.05$ and $\sigma = 0.3$. The simulations are based on 252 trading days every year and daily steps refers to how many simulations we will do for each day. The more steps for each day, the closer our simulation is to continuous trading. We can indeed see the convergence clearly from the simulation results. See appendix for convergence of delta hedged option as number of steps increases. The following plots show when we can harvest VRP via selling delta hedged options with higher implied volatility than realised volatility, the distribution of the gains we can obtain.



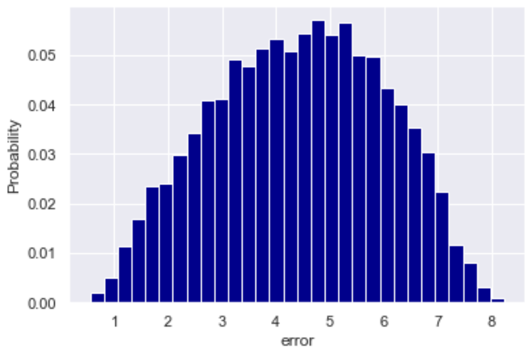
(a) ITM, K = 60



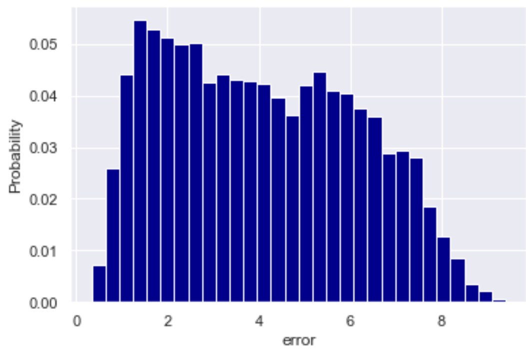
(b) ITM, K = 80



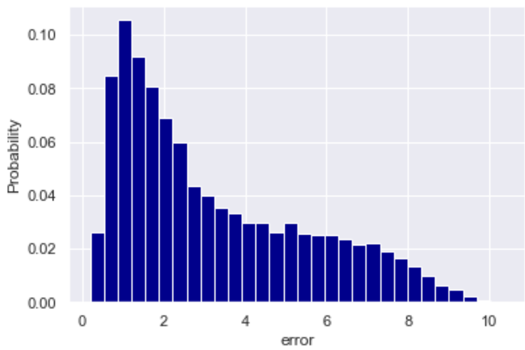
(c) ATM, K = 100



(d) OTM, K = 120



(e) OTM, K = 140



(f) OTM, K = 160

Figure 2.1: BS Delta Hedging Error under GBM without Dividend

Here is also a table that summarises the mean, variance and skewness of gains distribution with different strikes.

K	mean	variance	skewness
60	-1.5572	0.7829	0.7829
80	0.2179	2.6304	0.3955
90	3.0582	1.5552	0.2579
100	2.1476	2.9574	-0.3302
110	4.2569	1.9409	0.02627
120	4.3844	2.5021	-0.08733
130	4.3383	3.4840	0.006903
140	4.0577	4.3520	0.2259
160	3.2332	5.3100	0.7968

Table 2.1: Distribution of VRP harvested from selling options with higher implied volatility under BS model without dividends

We can see that it is the most profitable to see options that are out of the money. Selling in the money options are in general not as profitable as out of the money options. However, if the option is deep out of money, it is not very likely to be exercised so the initial price of selling such option is very low which lowers the final profit of our strategy. However, when the strike price increases, the variance is increasing as K rises, which exerts uncertainty on our short selling option strategy.

2.2 Dividend paying geometric Brownian motion

In this section, we add continuously paid dividends to GBM in the simulation of underlying share price which gives the SDE that describes the dynamics of S as

$$\frac{dS_t}{S_t} = (\mu + q)dt + \sigma dW_t. \tag{2.2.1}$$

In the mean time, the pricing formula becomes

$$C\left(S,t\right) = e^{-q(T-t)}N\left(d_1\right)S - e^{-r(T-t)}KN\left(d_2\right), \tag{2.2.2}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

In the meantime, it can be easily proved that the Black Scholes PDE extends to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \tag{2.2.3}$$

Also, delta could be obtained easily by taking derivative of C with respect to S

$$\Delta_t = e^{-q(T-t)}N'(d_1)$$

2.2.1 VRP harvesting with delta hedged option

We would like to consider a simple extension of Black Scholes model by adding continuously paid dividends to the model. Under this model, by Ito's formula, we get

$$\begin{aligned} C_t =& C_0 + \int_0^t \Delta_u dS_u + \int_0^t \left(\frac{\partial C_u}{\partial u} + \frac{1}{2}\sigma^2 S_u^2 \frac{\partial^2 C_u}{\partial S_u^2}\right) du \qquad (2.2.4) \\ =& C_0 + \int_0^t \Delta_u dS_u + \int_0^t \left(rC_u - (r-q)\Delta_u S_u\right) du. \qquad (2.2.5) \end{aligned}$$

Combining 2.2.5 with formula 1.4.5, one can obtain the theoretical delta hedge error when a unit of option at market volatility is sold.

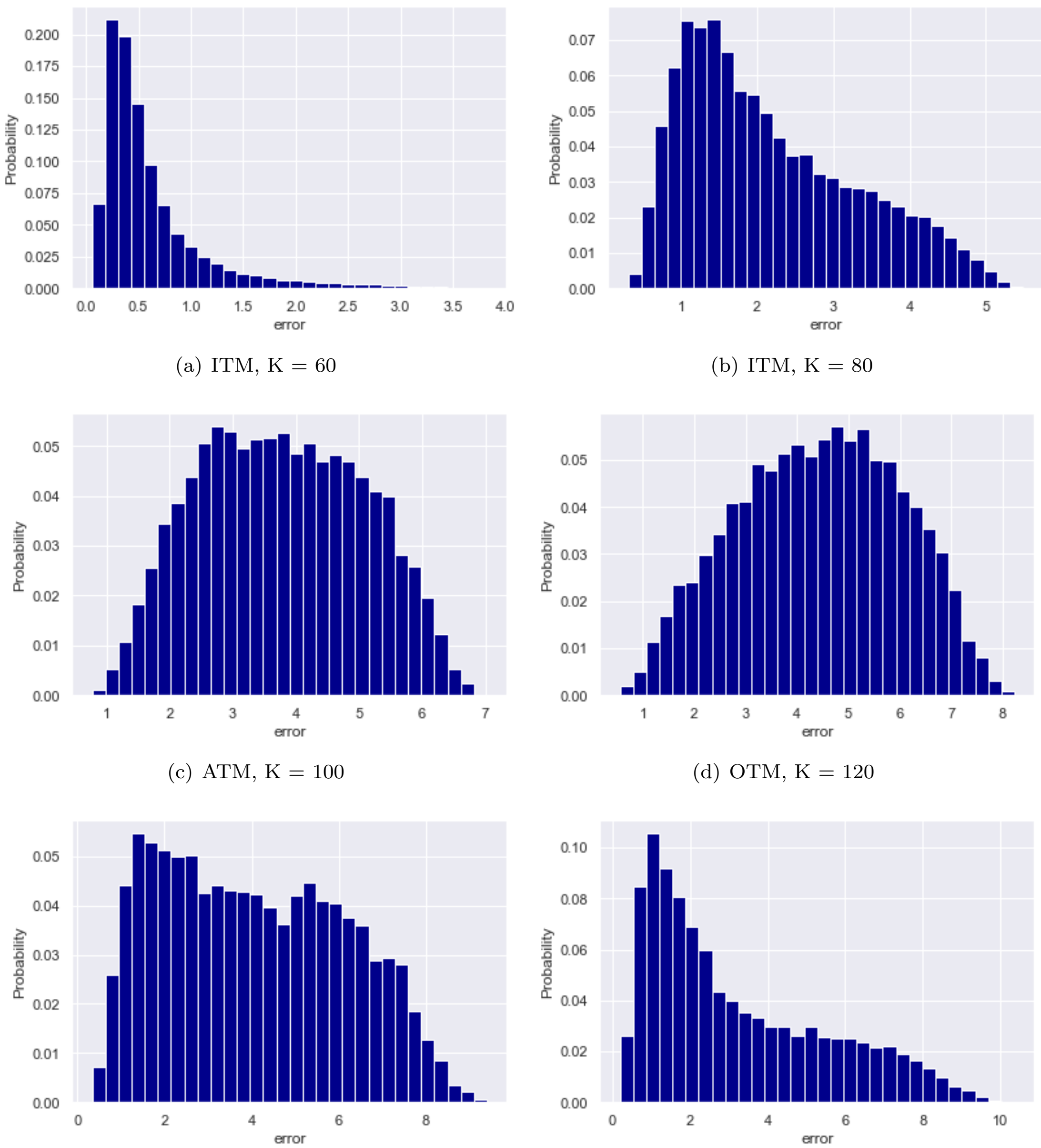
$$\Pi_{t,t+\tau} = -\int_t^{t+\tau} q\Delta(u, S_u)S_u du \tag{2.2.6}$$

Thus, in dividend paying case, without discrete hedging error, the simple delta hedging strategy already gives a natural hedging loss. This will give potential loss for simple delta hedging strategy even after we adjust the delta hedging strategy. Theoretical VRP that we can harvest in this case is

$$\text{VRP}_t = \int_0^t \left(\frac{1}{2}\Gamma\left(u, S_u\right)S_u^2\left(\hat{\sigma}^2 - \sigma^2\right) - q\Delta(u, S_u)S_u\right) \mathrm{d}u \tag{2.2.7}$$

2.2.2 Simulation results

We use the following parameters throughout simulations in this section: $T = 1$, $\mu = 0.2$, $r = 0.05$, $\sigma = 0.3$ and $q = 0.02$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 12, which means that the total steps is 3024. Also, when harvesting VRP, the implied volatility is set $\hat{\sigma} = 0.4$.



(a) ITM, K = 60

(b) ITM, K = 80

(c) ATM, K = 100

(d) OTM, K = 120

(e) OTM, K = 140

(f) OTM, K = 160

Figure 2.2: BS Delta Hedging Error under GBM with Dividend

From figures, we can see that due to the negative term associated with dividends in 2.2.6, VRP will be significantly influenced by this term because of the high delta of an in-the-money option. However, we can see that when selling deep out-of-money options, due to the very low delta which eliminates the negative element in 2.2.7, we may be able to make considerable profits.

Here is also a table that summarises the distribution of delta hedged gains at different strikes.

K	mean	variance	skewness
60	0.6223	0.28817	2.3510
80	0.2179	2.6303	0.3955
90	1.2883	3.0695	-0.04439
100	2.1476	2.9574	-0.3301
110	2.8502	2.6910	0.02627
120	3.2458	2.5769	-0.1936
140	3.3066	3.1681	0.2931
150	3.0738	3.5765	0.5249
160	2.7930	3.9680	0.7819

Table 2.2: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

2.3 Stochastic volatility model without dividend

In this section, we will explore the effect of stochastic volatility on the hedging error of delta hedged options. We use the following model assumption that is consistent with [Bakshi and Kapadia, 2015].

$$\frac{dS_t}{S_t} = \mu_t\left[S_t, \sigma_t\right]dt + \sigma_t dW_t^1 \tag{2.3.1}$$

$$d\sigma_t = \theta_t\left[\sigma_t\right]dt + \eta_t\left[\sigma_t\right]dW_t^2 \tag{2.3.2}$$

where the correlation between the two Weiner processes, W_t^1 and W_t^2, is ρ. It may be noted that volatility, σ_t, follows an autonomous stochastic process; the drift coefficient, $\theta_t\left[\sigma_t\right]$,and the diffusion coefficient, $\eta_t\left[\sigma_t\right]$, are functionally independent of S_t. In particular, here we only focus on the Heston model whose SDE is

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t \tag{2.3.3}$$

$$dV_t = \kappa\left(\theta - V_t\right)dt + \nu\sqrt{V_t}dB_t \tag{2.3.4}$$

where the correlation between the two Weiner processes, W_t and B_t, is ρ.

Theorem 2.3.1 (Theoretical solution to Heston model)**.** *With Heston model defined by 2.3.3 and 2.3.4, The theoretical solution to this model can be written as*

$$C^H(S, v, t) = SP_1 - KP(t, T)P_2$$

where

$$P_j(x, v, T; \ln[K]) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\log K} f(x, v, T; \phi)}{i\phi}\right]d\phi$$

with

$$f_f(x, v, t; \phi) = e^{c(T - r_i\phi) + D(T - t_i\phi)v + Hx}$$

where

$$C(\tau; \phi) = n\phi i\tau + \tfrac{a}{\sigma^2}\left\{\left(b_j - \rho\sigma\phi i + d\right)\tau - 2\ln\left[\tfrac{1 - ge^{dr}}{1 - g}\right]\right\}$$
$$D(\tau; \phi) = \tfrac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\tfrac{1 - e^{\omega r}}{1 - ge^{dt}}\right]$$

and

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}$$

$$d = \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}$$

For detailed proof of the theoretical solution, see [Heston, 1993].

We can see that the theoretical solution to Heston model is very complicated and requires Fast Fourier Transform method while finding the solutions. Hence, it is not convenient to simulate the delta hedging strategy with its own delta. Therefore, due to the time limit of this project, we will only perform delta hedging based on Black Scholes delta. The simulation of stochastic volatility model can be a future direction of research.

Though Monte Carlo simulation is not very doable for delta hedging in Heston model, theoretical expression of hedging error can still be obtained as shown in the following proposition.

Proposition 2.3.2. *Let the stock price process follow the dynamics given in equations 2.3.3 and 2.3.4. Moreover, suppose the volatility risk premium is of the general form where* $\delta_t^v[\sigma_t] \equiv -\operatorname{Cov}_t\left(\frac{\mathrm{d}S_t}{S_t}, d_t\right)$. *Then,*
1. The delta-hedged gains with Black Scholes call option delta formula at implied volatility $\hat{\sigma}$, $\Pi^{BS}_{t,t+\tau}$, *is given by*

$$\Pi^{BS}_{t,t+\tau} = \int_t^{t+\tau} \frac{\partial C_u}{\partial V_u} dV_u + \int_t^{t+\tau} b(\hat{\sigma}) du$$

where

$$b(\hat{\sigma}) = \frac{1}{2}(\hat{\sigma}^2 - V_u)S_u^2 \frac{\partial^2 C_u}{\partial S_u^2} + \frac{1}{2}\nu^2 V_u \frac{\partial^2 C_u}{\partial V_u^2} + \rho\nu\sqrt{V_u}\sigma_u S_u \frac{\partial^2 C_u}{\partial S_u \partial V_u}$$

2. The delta hedged error with its intrinsic delta formula $\Pi_{t,t+\tau}$

$$\Pi_{t,t+\tau} = \int_t^{t+\tau} \delta_u^v[\sigma_u] \frac{\partial C_u}{\partial \sigma_u} du + \int_t^{t+\tau} \nu\sqrt{V_t} \frac{\partial C_u}{\partial \sigma_u} dB_u$$

Proof. For 1, we use Ito's formula to get

$$C_{t+\tau} = C_t + \int_t^{t+\tau} \frac{\partial C_u}{\partial S_u} dS_u + \int_t^{t+\tau} \frac{\partial C_u}{\partial V_u} dV_u + \int_t^{t+\tau} b_u du \tag{2.3.5}$$

where $b_u \equiv \frac{\partial C_u}{\partial u} + \frac{1}{2}V_u^2 S_u^2 \frac{\partial^2 C_u}{\partial S_u^2} + \frac{1}{2}\nu^2 V_u \frac{\partial^2 C_u}{\partial V_u^2} + \rho\nu\sqrt{V_u}\sigma_u S_u \frac{\partial^2 C_u}{\partial S_u \partial V_u}$. Then we substitute 2.3.5 into 1.4.5 and wrap up to get

$$\Pi^{BS}_{t,t+\tau} = \int_t^{t+\tau} \frac{\partial C_u}{\partial V_u} dV_u + \int_t^{t+\tau} b_u du + \int_t^{t+\tau} r\left(C_u - \Delta_u S_u\right) du \tag{2.3.6}$$

Then substitute Black Scholes PDE into 2.3.6 to get

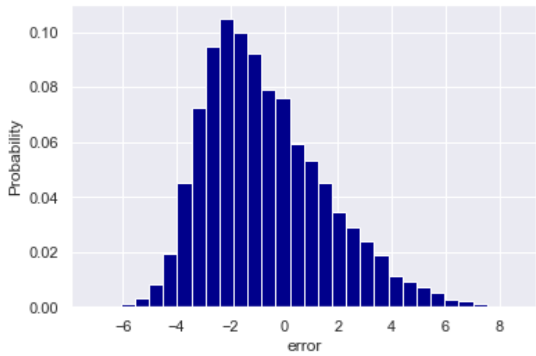
$$\Pi^{BS}_{t,t+\tau} = \int_t^{t+\tau} \frac{\partial C_u}{\partial V_u} dV_u + \int_t^{t+\tau} \frac{1}{2}(\hat{\sigma}^2 - V_u)S_u^2 \frac{\partial^2 C_u}{\partial S_u^2} + \frac{1}{2}\nu^2 V_u \frac{\partial^2 C_u}{\partial V_u^2} + \rho\nu\sqrt{V_u}\sigma_u S_u \frac{\partial^2 C_u}{\partial S_u \partial V_u} du \tag{2.3.7}$$

For 2, See [Bakshi and Kapadia, 2015] □

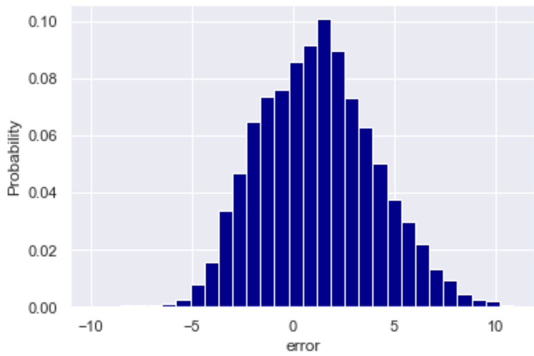
Theoretical results in Proposition 2.3.2 indicate that using Black Scholes delta to hedge the stochastic volatility will give relatively large error in Heston model. Theoretical solution to Heston model is very complicated, so it is very difficult to apply delta hedging with Heston model. In stochastic volatility model, harvesting VRP via variance swaps tends to be a better choice. However, it is still possible to harvest VRP through Black Scholes delta hedging strategy.

2.3.1 Simulation results

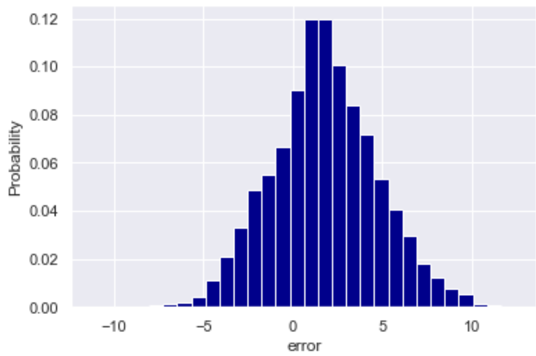
We use the following parameters throughout simulations in this section: $T = 1$, $\mu = 0.2$, $r = 0.05$, $V_0 = 0.3$, $\kappa = 0.2$, $\theta = 0.2$, $\nu = 0.2$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 12, which means that the total steps is 3024. Also, when harvesting VRP, the implied volatility is set $\hat{\sigma} = 0.4$.



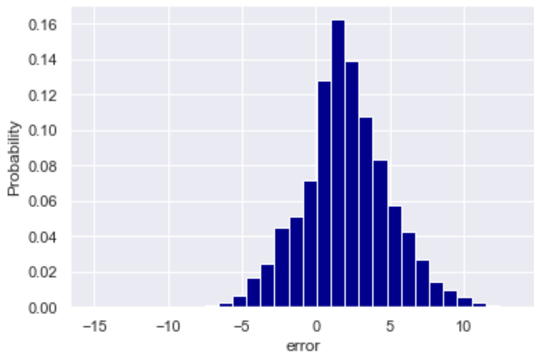
(a) ITM, K = 80



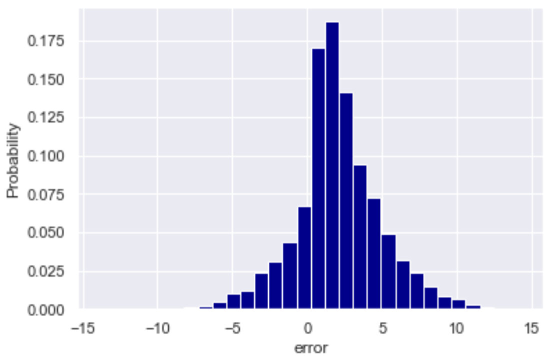
(b) ATM, K = 100



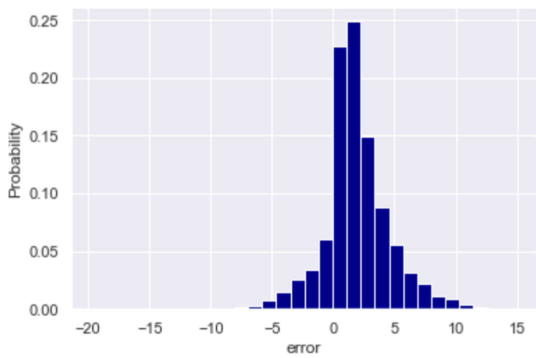
(c) OTM, K = 110



(d) OTM, K = 120



(e) OTM, K = 130



(f) OTM, K = 140

Figure 2.3: BS Delta Hedging Error under GBM with Dividend

Also, statistical data on delta hedged option portfolio with different strikes is shown below

It is worth noticing that compared with previous results the variance is not decreasing as the option goes deep out of money. This is partly because as option goes deep out of money, the stochastic volatility will have very little influence on our delta hedged option portfolio.

K	mean	variance	skewness
80	-0.7054	5.0996	0.6992
100	1.2137	8.2229	0.1939
110	1.7726	8.7551	0.08096
120	2.0226	8.6247	0.05738
130	2.0421	8.1509	0.08784
140	1.8943	7.4413	0.1692
150	1.7097	6.6926	0.1915

Table 2.3: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

Chapter 3

Delta hedged options under models with jumps

3.1 Merton jump diffusion model

3.1.1 Model specification

In Merton jump diffusion model, changes in the asset price consist of normal (continuous diffusion) component that is modeled by a Brownian motion with drift process and abnormal (discontinuous, i.e. jump) component that is modeled by a compound Poisson process. Asset price jumps are assumed to occur independently and identically. The probability that an asset price jumps during a small time interval dt can be written using a Poisson process $\mathrm{d}N_t$ as

$$\frac{dS_t}{S_{t-}} = (\alpha - \lambda\mu_J)\mathrm{d}t + \sigma \mathrm{d}W_t + (J_t - 1)\, dN_t. \tag{3.1.1}$$

Equivalently, the model can also be written in form of compound Poisson process.

$$dS_t/S_{t-} = (\alpha - \lambda\mu_J)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}\tilde{P}_t \tag{3.1.2}$$

Under this model, the analytical solution to the stochastic differential equation can be written as

$$S_t = S_0 \exp\left[\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right) t + \sigma B_t + \sum_{k=1}^{N_h} Y_k\right] \tag{3.1.3}$$

MJD model is an example of an incomplete model because there are many equivalent martingale risk-neutral measures $\mathcal{Q} \sim \mathcal{P}$ under which the discounted asset price process $\{e^{-rt}S_i; 0 \leq t \leq T\}$ becomes a martingale. Merton finds his equivalent martingale risk-neutral measure $\mathcal{Q}_M \sim \mathcal{P}$ by changing the drift of the Brownian motion process while keeping the other parts (most important is the jump measure, i.e. the distribution of jump times and jump sizes) unchanged:

$$S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda k\right) t + \sigma W_t^{Q_u} + \sum_{k=1}^{N_t} Y_k\right] \text{ under } \mathbb{Q}_M$$

Note that $W_t^{Q_M}$ is a standard Brownian motion process on $(\Omega, \mathcal{F}, \mathbb{Q}_M)$ and the process $\{e^{-rt}S_t; 0 \leq t \leq T\}$ is a martingale under $\mathbb{Q}_M$. Then, a European option price $\mathcal{C}^{MJ}(t, S_t)$ with payoff function $H(S_T)$

is calculated as:

$$\mathcal{C}^{\mathrm{MJ}}(t, S_t) = e^{-r(T-t)} E^{\mathrm{Q}_4}\left[H(S_T) \mid \mathcal{F}_t\right]$$

After complicated calculations, the analytical solution to the option value under MJD model can be expressed as two equivalent forms in [Matsuda, 2004]

$$\mathcal{C}^{\mathrm{MJ}}(t, S_t) = e^{-r\tau} \sum_{i\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^i}{i!} E^{Q_u}\left[H\left\{S_t \exp\left\{\left(r - \lambda\left(e^{\mu+\delta^2/2} - 1\right) + \frac{i\mu + i\delta^2/2}{\tau} - \frac{1}{2}\sigma_i^2\right)\tau + \sigma_i B_\tau^{Q_M}\right\}\right\}\right] \tag{3.1.4}$$

$$= \sum_{i\geq 0} \frac{e^{-\bar{\lambda}\tau}(\bar{\lambda}\tau)^i}{i!} \mathcal{C}^{BS}\left(\tau, S_i; \sigma_i \equiv \sqrt{\sigma^2 + \frac{i\delta^2}{\tau}}, r_i \equiv r - \lambda\left(e^{\mu+\delta^2/2} - 1\right) + \frac{i\mu + i\delta^2/2}{\tau}\right) \tag{3.1.5}$$

3.1.2 Black Scholes delta hedging error

Under MJD frame, we can test how Black Scholes hedging strategy performs when jumps occur. We can derive the hedging error with Black Scholes delta under this model according to the derivation in [Davis, 2010]. With portfolio construction process described in Section 1.4, the portfolio value process X satisfies the SDE

$$\begin{aligned} X_t =& p + \int_0^t \partial_S C(u, S_{u-}) \mu S_{u-} \mathrm{d}u + \int_0^t \partial_S C(u, S_{u-}) \sigma S_{u-} \mathrm{d}W_u \\ &+ \int_0^t \partial_S C(u, S_{u-}) S_{u-}(z) J_u \mathrm{d}u + \int_0^t (X_u - \partial_S C(u, S_{u-}) S_u) r \mathrm{d}u \end{aligned}$$

Now define $Y_t = C(t, S_t)$, so that in particular $Y_0 = p$. Applying the Itô formula gives

$$\begin{aligned} Y_t =& p + \int_0^t \partial_t C(u, S_{u-}) \mathrm{d}u + \int_0^t \partial_S C(u, S_{u-}) \mu S_{u-} \mathrm{d}u \\ &+ \int_0^t \partial_S C(u, S_{u-}) \sigma S_{u-} \mathrm{d}W_u + \frac{1}{2}\int_0^t \partial_{SS}^2 C(u, S_{u-}) \sigma^2 S_{u-}^2 \mathrm{d}u \\ &+ \int_0^t (C(u, S_{u-}(1 + J_u)) - C(u, S_{u-})) \mathrm{d}t \end{aligned}$$

Note that this expression is independent of V_t since it is Black Scholes Option pricing formula. Thus the 'hedging error' process defined by $Z_t := X_t - Y_t$ satisfies the SDE

$$\begin{aligned} Z_t =& \int_0^t r X_u \mathrm{d}u - \int_0^t \left(r S_{u-} \partial_S C(u, S_{u-}) + \partial_t C(u, S_{u-}) + \frac{1}{2}\sigma^2 S_{u-}^2 \partial_{SS}^2 C(u, S_{u-}) \mathrm{d}u\right. \\ &- \int_0^t (C(u, S_{u-}(1 + J_u(z))) - C(u, S_{u-}) - \partial_S C(u, S_{u-}) S_{u-} J_u(z)) \mathrm{d}u \end{aligned} \tag{3.1.6}$$

Since in this expression, we use the Black Scholes Option Price. We would need to substitute Black Scholes PDE

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

Therefore, with implied volatility $\hat{\sigma}$, the payoff of selling a delta hedged option would be

$$Z_t = \int_0^t r Z_u \mathrm{d}u + \frac{1}{2}\int_0^t \Gamma(u, S_{u-}) S_{u-}^2 \left(\hat{\sigma}^2 - \sigma^2\right) \mathrm{d}u$$

$$-\int_0^t \left(C\left(u, S_{u-}\left(1+J_u(z)\right)\right)-C\left(u, S_{u-}\right)-\partial_S C\left(u, S_{u-}\right) S_{u-} J_u(z)\right) \mathrm{d}u \tag{3.1.7}$$

From 3.1.7, it can be seen that apart from the first two terms, the final term is the effect of jumps. However, we are unable to hedge this part of the jumps. This jump part tends to give a relatively large hedging error in extreme cases.

3.1.3 Merton delta hedging error

Instead of using Black Scholes delta hedging strategy, since theoretical value of the option delta can be obtained from the theoretical expression of option pricing formula of MJD model, we can use a more accurate way of hedging especially to reduce the theoretical hedging error especially at jumps. First of all, we need to introduce the PDE derived from MJD model

$$\frac{\partial V}{\partial t}+\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 V}{\partial S_t^2}+rS_t\frac{\partial V}{\partial S_t}-rV+\lambda E\left[V\left((1+J_t)S_t, t\right)-V\left(S_t, t\right)\right]-\lambda S_t\frac{\partial V}{\partial S_t}\mu_J=0 \tag{3.1.8}$$

Hence, we substitute 3.1.8 (with σ replaced by $\hat{\sigma}$) into 3.1.6 to get the hedging error by Merton delta

$$Z_t=\int_0^t rZ_u \mathrm{d}u+\frac{1}{2}\int_0^t \Gamma\left(u, S_{u-}\right) S_{u-}^2\left(\hat{\sigma}^2-\sigma^2\right)\mathrm{d}u-\int_0^t \partial_S C\left(u, S_{u-}\right) S_{u-} J_u \mathrm{d}u \tag{3.1.9}$$

We can see that, continuously, Merton delta hedging can hedge the jumps to some extent under continuous trading assumptions, which, in practice, is impossible. This is because we are unable to know when jumps will happen and adjust our delta instantaneously based on the occurrence of jumps under discrete hedging settings.

Moreover, in practical simulations, we can also spot that the Merton delta hedging strategy is sometimes an over-hedged strategy which means that there will be positive gains when tradings trades are not frequent enough. This gives us a clue to guess that Merton delta gives us a compensation for the potential loss at the occurrence of jumps. However, for both Black Scholes delta and Merton delta, the model is unable to converge to Dirac delta function as shown in Appendix A.1.2.

3.1.4 Simulation results

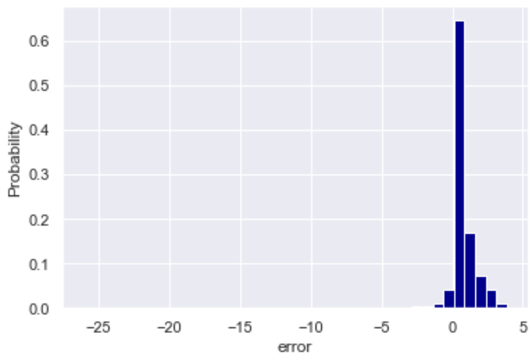
In this section, We use the following parameters throughout simulations in this section: $T = 1$, $\mu = 0.2$, $r = 0.05$, $\lambda = 0.25$. The jump size parameters are set as $\gamma = 0$ and $\delta = 0.2$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 12, which means that the total steps is 3024. Also, when harvesting VRP, the implied volatility is set $\hat{\sigma} = 0.4$.

Black Scholes delta VRP harvesting

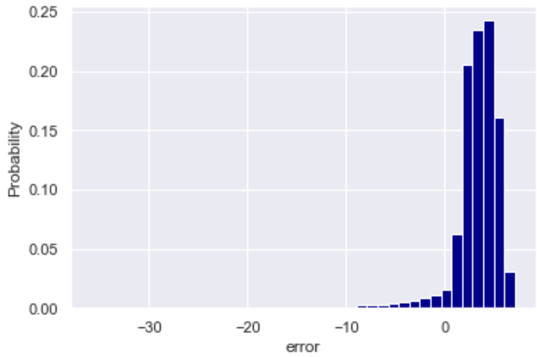
First of all, we will test how Black Scholes delta performs in this case. Due to the unpredictability of jumps in MJD model, we are unable to hedge the jumps which will give big loss due to jump effect as shown in 3.1.7. Simulation results are shown in Fig 3.1.

We can see from the plots that there is highly likely to be a huge jump loss for delta hedged options at all strikes although the majority of cases yield a profit for the portfolio. In order to better monitor the risk, we also look at 95% Value at Risk (VaR) of the portfolio with different strikes as shown in Table 3.1.

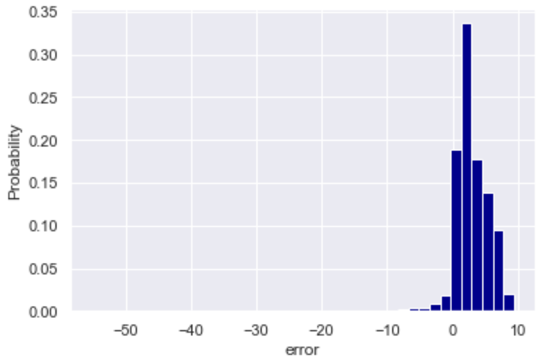
From the table, although the most profitable choice is to sell out of the money option with



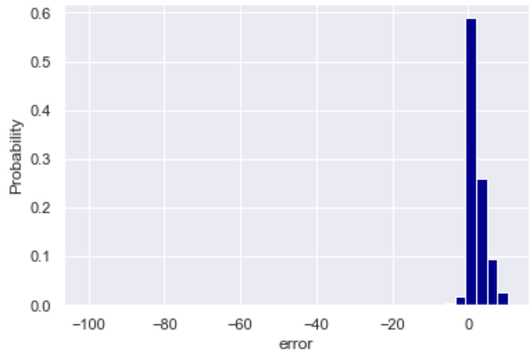
(a) ITM, K = 60



(b) ATM, K = 100



(c) OTM, K = 140



(d) OTM, K = 160

Figure 3.1: BS Delta Hedging Error under Merton Jump Diffusion Model without Dividend

K	mean	variance	skewness	95% VaR
60	0.8851	0.9028	-1.8275	0.08344
80	2.5305	3.6000	-3.2656	0.0053
100	3.7544	7.4331	-5.9030	-0.1836
120	3.9093	8.9042	-3.9253	-0.0912
140	3.2544	9.7621	-3.8346	0.2594
160	2.4693	7.8993	-3.2053	0.2082

Table 3.1: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

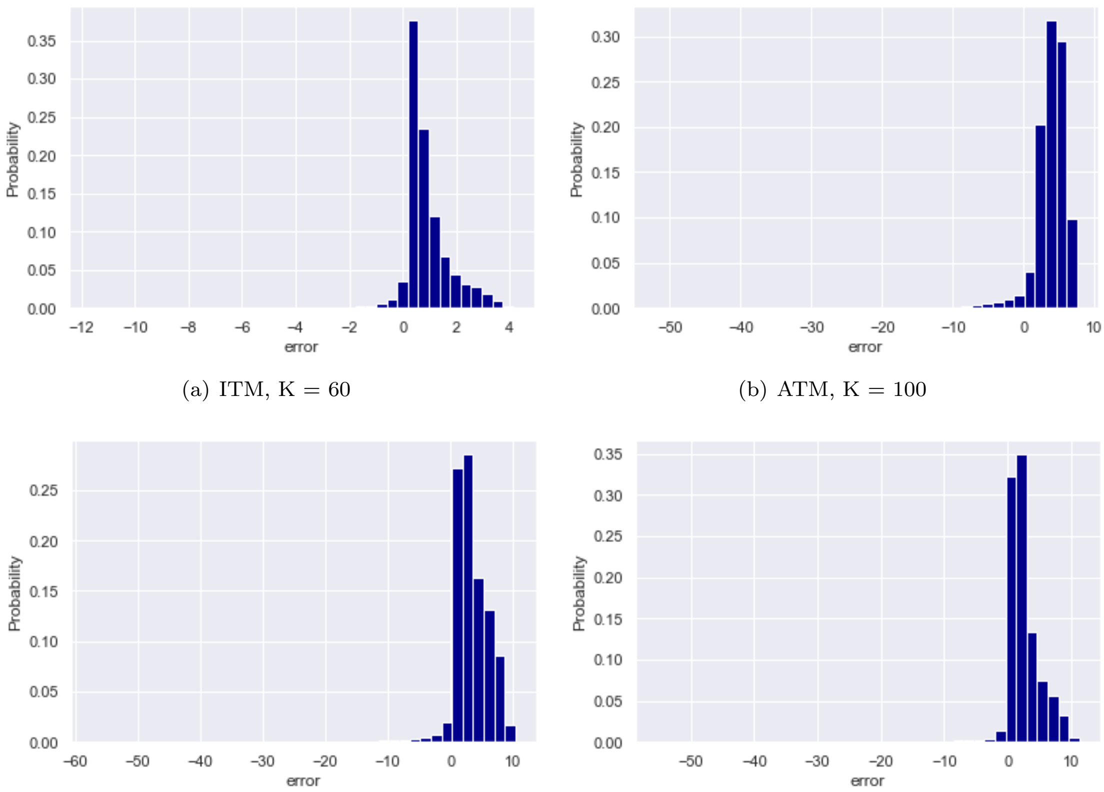
strike between 120 and 140, the 95% VaR will be negative in some cases, which means that there could potentially be significant loss in the portfolio and the variance in this interval is also the biggest. Extra attention needs to be paid on extreme loss

Merton delta VRP harvesting

More precise hedging requires delta hedging for Merton delta. Delta hedging strategy will not give a convergent histogram to Dirac delta function. This is mainly because of discrete adjustments of the trading position. In continuous trading assumptions, there will be a sudden change in delta, which makes it possible to adjust the position immediately before jumps. However, this is impossible for discrete hedging adjustments. For convergence plots, check Appendix A for details.

We would like to test VRP harvesting by Merton delta as well. The simulation results are shown in Fig 3.1 and Table 3.2 shows the distribution of delta hedged option portfolio's pnl.

Compared with Black Scholes delta hedging strategy, Merton delta gives relatively smaller mean profit. However, the variance and 95%VaR are lower than Black Scholes strategy, which, from the



(a) ITM, K = 60

(b) ATM, K = 100

(c) OTM, K = 140

(d) OTM, K = 160

Figure 3.1: Merton Delta Hedging Error under Merton Jump Diffusion Model without Dividend

K	mean	variance	skewness	95%VaR
60	0.7210	0.8040	-4.6063	0.1584
80	2.1785	3.2375	-2.9949	0.4047
100	3.2691	6.3111	-4.0654	0.3213
120	3.3930	8.1484	-3.7174	0.5651
140	2.8469	8.6138	-3.9091	0.5051
160	2.0364	8.5434	-7.8268	0.3812

Table 3.2: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

perspective of risk management, gives a more steady VRP than BS delta hedging.

3.2 Stochastic volatility jump diffusion model

3.2.1 Model specification

This is the most complicated model that we are considering. It is a combination of Merton jump diffusion model and stochastic volatility model. The full model is consistent with the model in asset price is assumed to follow a jump-diffusion model and the asset volatility is allowed to be stochastic. Specifically, the dynamics under the physical probability measure P are

$$\mathrm{d}\ln S_t = \left[r - q + \delta_s V_t + \frac{V_t}{2}\right]\mathrm{d}t + \sqrt{V_t}dW_t + J_t dN_t - \lambda\mu_J dt \tag{3.2.1}$$

$$\mathrm{d}V_t = \kappa\left(\theta - V_t\right)\mathrm{d}t + \sigma_v V_t^{\nu}\mathrm{d}B_t \tag{3.2.2}$$

where W_t and B_t are two correlated Wiener processes with the correlation coefficient equal to ρ. Denote the mean and variance of J_t by μ_J and σ_J. Note that dW_t and $J_t dN_t$ have respective variances equal to dt and $\lambda\left(\mu_j^2+\sigma_j^2\right) dt$. Thus, $V_t+\lambda\left(\mu_j^2+\sigma_j^2\right)$ is the variance rate of the asset price process. As in price and volatility processes are dependent through two correlated diffusive terms W_t and B_t.

3.2.2 Change of Measure for compound Poisson process

Due to the compound Poisson process in the model, the risk neutral measure is not unique. We need to adapt Girsanov theorem to a version for Poisson process as in [Privault, 09].

Theorem 3.2.1 (Change of measure for Poisson process). *Let $(N_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda>0$ and jump size distribution $\nu(dx)$. In our case, the jump size is log-normally distributed with mean $e^\gamma-1$. Consider another intensity parameter $\tilde{\lambda}>0$ and jump size distribution $\tilde{\nu}(dx)$, and let*

$$\psi(x):=\frac{\tilde{\lambda}}{\lambda} \frac{\tilde{\nu}(dx)}{\nu(dx)}-1, \quad x \in \mathbb{R}$$

Then, under the probability measure$\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ defined by the Radon-Nikodym density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{\mathrm{d}\widetilde{\mathbb{P}}_{\lambda,\nu}}:=\mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}\left(1+\psi\left(Z_k\right)\right)$$

the process

$$Y_t:=\sum_{k=1}^{N_t} Z_k, \quad t \in \mathbb{R}_+$$

is a compound Poisson process with modified intensity $\tilde{\lambda}>0$, and modified jump size distribution $\tilde{\nu}(dx)$

For detailed proof of Theorem 3.2.1, check [Privault, 09].

Corollary 3.2.2 (Girsanov theorem for compound Poisson process). *The compensated process*

$$Y_t-\tilde{\lambda} t \mathbb{E}_{\tilde{\nu}}[Z]$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ defined by the RadonNikodym density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{\mathrm{d}\widetilde{\mathbb{P}}_{\lambda,\nu}}=\mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}\left(1+\psi\left(Z_k\right)\right)$$

Proof of this corollary follows from Proposition 3.2.1 and by applying Ito's formula for compound Poisson process. Check [Privault, 09].

Lemma 3.2.3 (Risk neutral price dynamics). *Under a chosen risk neutral measure $\mathcal{Q}$ with stock price being described as*

$$\mathrm{d}\log S_t=[r-q]\,\mathrm{d}t+\sqrt{V_t}dW_t^Q+J_t^* dN_t-\lambda^*\mu_J^* V_t dt \tag{3.2.3}$$

$$\mathrm{d}V_t=\kappa\left(\theta-V_t+\delta_v V_t\right)\mathrm{d}t+\sigma_v V_t^\nu \mathrm{d}B_t^Q \tag{3.2.4}$$

where $\mathrm{d}B_t^Q \mathrm{d}W_t^Q = \rho \mathrm{d}t$*, define density*

$$\pi_t = \exp(-r(T-t))\varepsilon\left(-\int_0^t \zeta_t dW_\mathrm{t}\right)\exp\left(\sum_{i,\tau_i\leq t} J_i^*\right)$$

where $\varepsilon(\cdot)$ *denotes the stochastic exponential, and where* ζ *are the market prices of the Brownian shocks in the price and volatility defined by*

$$\zeta_t^{(1)} = \delta^s\sqrt{V_t}, \quad \zeta_i^{(2)} = -\frac{1}{\sqrt{1-\rho^2}}\left(\rho\delta^s + \frac{\delta^r}{\sigma_v}\right)\sqrt{V_t}$$

Then $\pi_t\exp(rt)$ *and* $\pi_t\exp(qt)S_t$ *are both local martingale*

To prove Lemma 3.2.3, Appendix A of [Pan, 2002] has detailed explanations. This gives us a bases for theoretical option pricing under this model and also showed that a valid risk neutral measure does exist. In the meantime, we can also find the pricing PDE that the process follows.

Proposition 3.2.4 (Option pricing PDE of stochastic jump diffusion model)**.** *Assume* C_t *is the solution to the call price under stochastic jump diffusion model, then* C_t *satisfies the following PDE*

$$\frac{1}{2}V_tS^2\frac{\partial^2 C}{\partial S^2} + \frac{\sigma_v^2V_t^{2\nu}}{2}\frac{\partial^2 C}{\partial V_t^2} + \rho\sigma_vV_t^{\nu+1/2}\frac{\partial^2 C}{\partial S_t\partial V_t} + (r-q-\mu_J^*\lambda^*V_t)\,S\frac{\partial C}{\partial S} + [\kappa_v(\theta - V_t) + \delta_vV_t]\frac{\partial C}{\partial V_t}$$
$$+\frac{\partial C}{\partial t} - rC + E\left[C\left(S_{u-}(1+J_u^*)\right) - C\left(S_{u-}\right)\right] = 0$$

where J_t^* *is the jump size under risk neutral measure* $\mathcal{Q}$

Proof. Apply Ito's formula on $\exp(-rt)C_t$

$$\begin{aligned}
\exp(-rt)C_t =& C_0 + \int_0^t \partial_tC\left(u,S_{u-}\right)\mathrm{d}u + \int_0^t \big((r-q-\mu_J^*\lambda^*V_t)\,\partial_SC\left(u,S_{u-}\right)S_{u-} \\
&+ (\kappa(\theta - V_t) + \delta_vV_t)\,\partial_VC\left(u,S_{u-}\right)\big)\mathrm{d}u + \int_0^t \partial_SC\left(u,S_{u-}\right)\sqrt{V_u}S_{u-}\mathrm{d}W_u \\
&+ \frac{1}{2}\int_0^t \partial_{SS}^2C\left(u,S_{u-}\right)V_uS_{u-}^2\mathrm{d}u + \int_0^t \partial_VC\left(u,S_{u-}\right)\sigma_vV_t^\nu\mathrm{d}B_t \\
&+ \frac{1}{2}\int_0^t \partial_{VV}^2C\left(u,S_{u-}\right)\sigma_v^2V_t^{2\nu}\mathrm{d}u + \int_0^t \rho\sigma_v\partial_{SV}^2C\left(u,S_{u-}\right)\mathrm{d}u \\
&+ \int_0^t \left(C\left(u,S_{u-}\left(1+J_u^*\right)\right) - C\left(u,S_{u-}\right)\right)\mathrm{d}u - \int_0^t rC(u,S_{u-})\mathrm{d}u
\end{aligned} \tag{3.2.5}$$

By Corollary 3.2.3, due to the local martingale property of π_t and hence $\exp(-rt)C_t$, forcing the drift to be 0 gives the PDE. □

3.2.3 Delta hedging error

Black Scholes delta

With portfolio constructed as in Section 1.4, we derive the expression of portfolio value as before

$$\begin{aligned}
X_t =& p + \int_0^t \partial_SC\left(u,S_{u-}\right)\mu S_{u-}\mathrm{d}u + \int_0^t \partial_SC\left(u,S_{u-}\right)\sigma S_{u-}\mathrm{d}W_u \\
&+ \int_0^t \partial_SC\left(u,S_{u-}\right)S_{u-}(z)J_u\mathrm{d}u + \int_0^t \left(X_u - \partial_SC\left(u,S_{u-}\right)S_u\right)r\mathrm{d}u
\end{aligned}$$

Hence we can get the dynamics of call option's value by Ito's formula

$$
\begin{aligned}
C_t =& C_0 + \int_0^t \partial_t C\left(u, S_{u-}\right) \mathrm{d}u + \int_0^t \left(r - q - \mu_J^* \lambda^* V_t\right) \partial_S C\left(u, S_{u-}\right) \mu_u S_{u-} + \kappa(\theta - V_t)\partial_V C\left(u, S_{u-}\right)\mathrm{d}u \\
&+ \int_0^t \partial_S C\left(u, S_{u-}\right) \sqrt{V_u} S_{u-} \mathrm{d}W_u + \frac{1}{2}\int_0^t \partial_{SS}^2 C\left(u, S_{u-}\right) V_u S_{u-}^2 \mathrm{d}u \\
&+ \int_0^t \partial_V C\left(u, S_{u-}\right) \sigma V_t^{\nu} \mathrm{d}B_t + \frac{1}{2}\int_0^t \partial_{VV}^2 C\left(u, S_{u-}\right) \sigma^2 V_t^{2\nu} \mathrm{d}u + \int_0^t \rho \sigma_v V_u^{\nu+1/2} \partial_{SV}^2 C(u, S_{u-}) \mathrm{d}u \\
&+ \int_0^t \left(C\left(u, S_{u-}\left(1 + J_u\right)\right) - C\left(u, S_{u-}\right)\right) \mathrm{d}u
\end{aligned}
$$

Therefore, under stochastic volatility jump diffusion model, with Black Scholes PDE substituted in to $Z_t = X_t - C_t$ the final hedging error with Black Scholes delta with implied volatility $\hat{\sigma}$ is

$$
\begin{aligned}
Z_t =& \int_0^t r Z_u \mathrm{d}u + \int_0^t \Big(\frac{1}{2}(\hat{\sigma}^2 - V_u) S_{u-}^2 \partial_{SS}^2 C(u, S_{u-}) - \rho \sigma_v V_u^{\nu+1/2} \partial_{SV}^2 C(u, S_{u-}) + (q + \mu_J^* \lambda^* V_t) S_{u-} \partial_S C\left(u, S_{u-}\right) \\
&- \kappa(\theta - V_t)\partial_V C\left(u, S_{u-}\right)\Big)\mathrm{d}u + \int_0^t \partial_V C\left(u, S_{u-}\right) \sigma_v V_t^{\nu} \mathrm{d}B_t + \int_0^t \partial_S C\left(u, S_{u-}\right) \sqrt{V_u} S_{u-} \mathrm{d}W_u \\
&- \int_0^t \left(C\left(u, S_{u-}\left(1 + J_u\right)\right) - C\left(u, S_{u-}\right) - \partial_S C\left(u, S_{u-}\right) S_{u-} J_u\right) \mathrm{d}u
\end{aligned}
$$

Since the Black Scholes delta only hedges the uncertainty associated with W_t and could not hedge stochastic volatility and jumps, the error follows a complicated stochastic differential equation. It is difficult to see whether there is still any opportunity to harvest volatility risk premium. Therefore, we will simulate how this delta hedge performs.

Stochastic volatility jump diffusion delta

As before, we are interested in the delta hedging portfolio perform in stochastic volatility jump diffusion model.

$$
\begin{aligned}
C_{t+\tau} - C_t =& \int_t^{t+\tau} \frac{\partial C_u}{\partial u} \mathrm{d}u + \int_t^{t+\tau} \frac{\partial C_u}{\partial S_u} dS_u + \int_t^{t+\tau} \frac{\partial C_u}{\partial V_u} \mathrm{d}V_u + \int_t^{t+\tau} \frac{V_u S_u^2}{2} \frac{\partial^2 C_u}{\partial S_u^2} \mathrm{d}u \\
&+ \int_t^{t+\tau} \frac{\sigma_v^2 V_u^{2\gamma}}{2} \frac{\partial^2 C_u}{\partial V_u^2} du + \int_t^{t+\tau} \rho \sigma_v V_u^{\gamma+1/2} \frac{\partial^2 C_u}{\partial S_u \partial V_u} \mathrm{d}u \\
&+ \int_t^{t+\tau} \Big(C\left(S_{u-}(1 + J_u)\right) - C\left(S_{u-}\right)\Big) \mathrm{d}u
\end{aligned}
$$

where J_u is the jump under phisical measure $\mathcal{P}$.

Then substituting the PDE in 3.2.4 and $C_{t+\tau} - C_t$ into the definition of $\Pi(t, t+\tau)$ as in 1.4.5 and let $\Delta_t = \frac{\partial C_t}{\partial S_t}$. We would get

$$
\begin{aligned}
\Pi_{t,t+\tau} =& \int_t^{t+\tau} \left(\mu_J^* \lambda^* V_u \frac{\partial C_u}{\partial S_u} \sigma_u S_u - \delta_v V_u \frac{\partial C_u}{\partial \sigma_u}\right) du + \int_0^t \partial_S C\left(u, S_{u-}\right) \sqrt{V_u} S_{u-} \mathrm{d}W_u \\
&+ \int_0^t \partial_V C\left(u, S_{u-}\right) \sigma_v V_t^{\nu} \mathrm{d}B_t - \int_t^{t+\tau} C_u\left(S_u(1 + J_u^*)\right) - C_u\left(S_u(1 + J_u)\right) \mathrm{d}u
\end{aligned}
$$

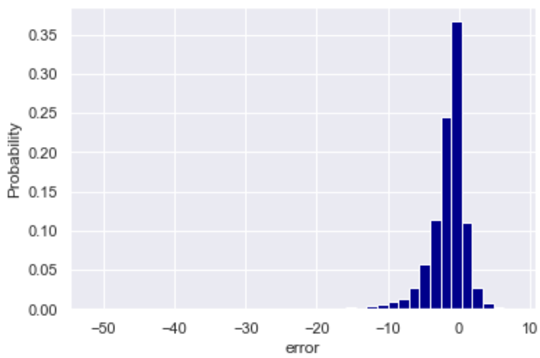
Then we can also find the expectation of this gain by

$$
\begin{aligned}
\mathbf{E}_t\left[\Pi_{t,t+\tau}\right] =& \mathbf{E}_t\Big[\int_t^{t+\tau} \Big(\mu_J^* \lambda^* S_u V_u \frac{\partial C_u}{\partial S_u} - \delta_v V_u \frac{\partial C_u}{\partial V_u}\Big)\mathrm{d}u + \int_t^{t+\tau} \Big(C\left(S_u(1 + J_t)\right) - C\left(S_u(1 + J_t^*)\right)\Big)\mathrm{d}u\Big] \\
=& \int_t^{t+\tau} \mathbf{E}_t\Big(\mu_J^* \lambda^* S \frac{\partial C_u}{\partial S_u} - \delta_v V_t \frac{\partial C_u}{\partial V_u}\Big)\mathrm{d}u + \int_t^{t+\tau} E\big[C\left(S_u(1 + J_t)\right) - C\left(S_u(1 + J_t^*)\right)\big]\mathrm{d}u
\end{aligned}
$$

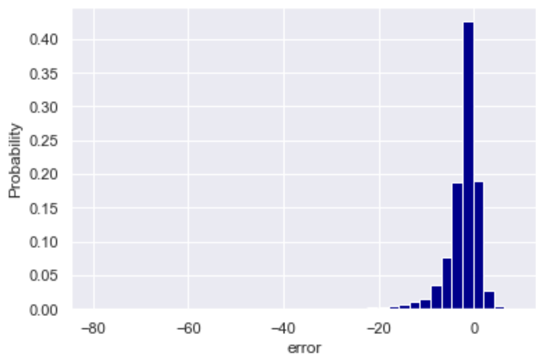
3.2.4 VRP harvesting via Black Scholes and Merton delta

In this section, We use the following parameters throughout simulations: $T = 1$, $\mu = 0.2$, $r = 0.05$, $\lambda = 0.25$, $v_0 = 0.09$, $\sigma_v = 0.3$ and $\nu = 0.5$. The jump size parameters are set as $\gamma = 0$ and $\delta = 0.2$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 12, which means that the total steps is 3024. Also, when harvesting VRP, the implied volatility for Black Scholes and Merton option pricing is set $\hat{\sigma} = 0.4$.

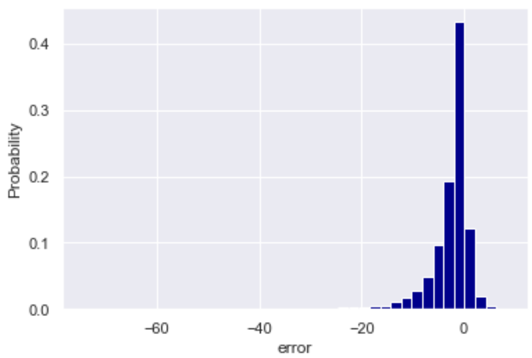
First of all, we try to harvest VRP via selling Black Scholes delta hedged option as before. Simulation results are shown in **Fig** 3.2 and **Table** 3.3.



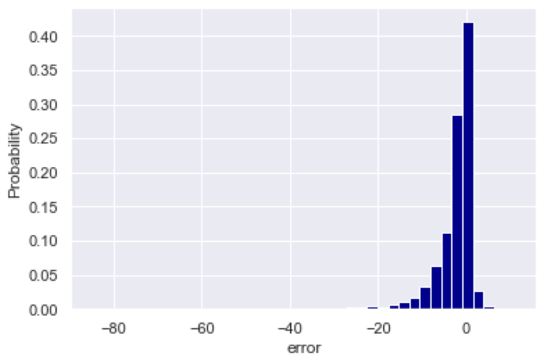
(a) ITM, K = 80



(b) ATM, K = 100



(c) OTM, K = 120



(d) OTM, K = 140

Figure 3.2: BS Delta Hedging Error under Stochastic Volatility Jump Diffusion Model with Dividend

K	mean	variance	skewness
80	-1.6358	3.2375	-2.9949
100	-2.3239	6.3111	-4.0654
120	-2.6802	6.3111	-4.0654
140	-2.7324	8.1484	-3.7174

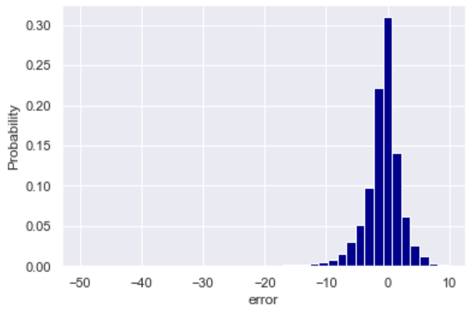
Table 3.3: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

However, it is not profitable to sell option with higher implied volatility in this case although we take higher risk. This is because the effect of stochastic volatility and jumps accumulates and causes more loss than simple Heston model and MJD model.

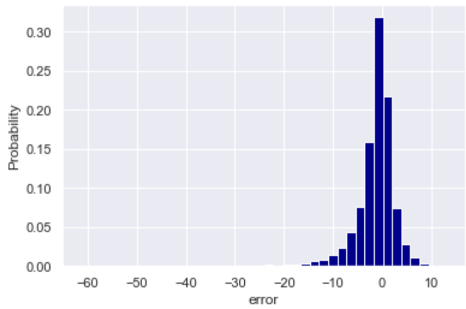
With the failure of Black Scholes delta, we try to sell delta hedged options via Merton delta defined by

$$\Delta^{MJ}(\tau = T - t, S_t) = \sum_{i \geq 0} \frac{e^{-\bar{\lambda}\tau}(\bar{\lambda}\tau)^i}{i!} \Delta^{BS}\left(\tau, S_i; \sigma_i \equiv \sqrt{\sigma^2 + \frac{i\delta^2}{\tau}}, r_i \equiv r - \lambda\left(e^{\mu+\delta^2/2} - 1\right) + \frac{i\mu + i\delta^2/2}{\tau}\right)$$

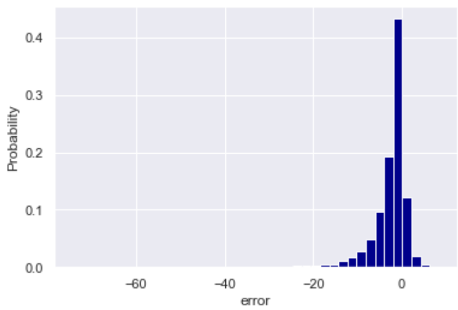
By hedging with Merton delta, we will get the following results as shown in **Fig** 3.3 and **Table** 3.4.



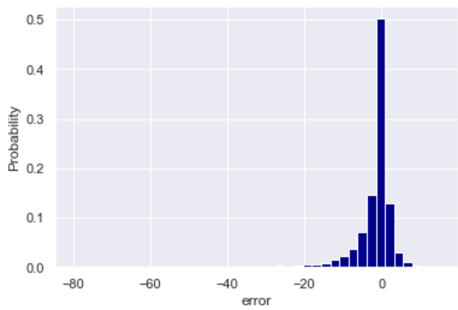
(a) ITM, K = 80



(b) ATM, K = 100



(c) OTM, K = 120



(d) OTM, K = 140

Figure 3.3: Merton Delta Hedging Error under Stochastic Volatility Jump Diffusion Model with Dividend

K	mean	variance	skewness
80	-0.9010	11.0551	-2.4789
100	-1.3416	18.6023	-2.85350
120	-1.6156	25.7214	-4.5929
140	-1.6977	26.6449	-3.8480

Table 3.4: Distribution of VRP harvested from selling options with higher implied volatility under BS model with continuous dividends

It can be seen that although it is better to use Merton delta to hedge, it is still not profitable to see delta hedged options in this model. Therefore, we should not use delta hedged option

Chapter 4

Variance swaps and delta hedged options

4.1 Variance swaps under different models

a variance swap is a forward contract on future realized price variance, variance being the square of volatility. At expiry the receiver of the "floating leg" pays (or owes) the difference between the realized variance (or volatility) and the agreed upon strike. At inception the strike K is generally chosen such that the fair value of the swap is zero. This strike is referred to as fair variance (or fair volatility). Like other swaps, the payoff is determined based on a notional amount that is never exchanged. However, in the case of a variance swap, the notional amount is specified in terms of vega, which is normally called vega notional. The payoff of a variance swap is given as follows:

$$\frac{N_{\text{vega}}}{2K}\left(\sigma^2_{\text{realised}} - K^2\right)$$

where N_{var} is vega notional, $\sigma^2_{\text{realised}}$ is annualised realised variance, and K is variance strike.

Variance swaps will have different forms of payoff in different models. The most commonly used model is stochastic volatility model and jump diffusion.

4.1.1 Black Scholes and Heston model

Black Scholes model

Mathematically, assuming stock price to be geometric Brownian motion with constant volatility and apply Black Scholes model for option pricing will not result in any value for variance swap as the implied volatility can be calculated and there is no meanings to use variance swap. However, in practice, volatility skew is often spotted, which is the difference in implied volatility between out-of-the-money options, at-the-money options, and in-the-money options. Therefore, variance swaps can also be used to capture the difference between implied volatility and realised volatility. We can relate the payoff of a variance swap and underlying stock price using lto's Lemma. We first assume that the underlying stock is described by geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

Applying Ito's formula, we get:

$$d\left(\log S_t\right) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t \quad \text{and} \quad \frac{dS_t}{S_t} - d\left(\log S_t\right) = \frac{\sigma^2}{2}dt$$

Taking integrals, the total variance is:

$$\text{variance} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \ln\left(\frac{S_T}{S_0}\right)\right)$$

Therefore, an estimate of the variance is

$$\sigma^2 = \frac{252}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{S_i}{S_{i-1}}\right)\right]^2$$

The fair strike can be found in [Tsoukalas and Zeng, 2009]

$$K^2 = \frac{2e^{rT}}{T}\left[\int_0^{S_0}\frac{P_0(K)}{K^2}dK + \int_{S_0}^{\infty}\frac{C_0(K)}{K^2}dK\right]$$

where P_0 and C_0 are put option price and call option price at time $t = 0$

Heston model

With model defined in 2.3.3 and 2.3.4, the fair value (strike) of this contract is obtained as expectation under the risk-neutral measure $\mathbb{Q}$ of the variance process,

$$\text{VarSwap} = \mathbb{E}^Q\left[\frac{1}{T}\int_0^T v_t dt\right] \tag{4.1.1}$$

With

$$\Lambda_t := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t$$

the usual assumption is to postulate existence of a function $\Lambda t(\cdot)$ such that

$$\delta_t^v[v_t] = \left[\ln(\Lambda_t), W^{(2)}\right]_t$$

such that under $\mathbb{Q}$ the asset dynamics become (we assume zero risk-free rate)

$$\begin{aligned} dS_t/S_t &= \sqrt{v_t}dW_t^Q \\ dV_t &= (\theta_t[V_t] - \delta_v[V_t])\,dt + \eta_t[V_t]\,dB_t^Q \end{aligned}$$

where $\mathrm{d}W_t^Q\mathrm{d}B_t^Q = \rho\mathrm{d}t$. This in turn yields the fair strike of the Variance Swap as

$$SW_{0,T}^Q = V_0 + \frac{1}{T}\int_0^T\int_0^t \mathbb{E}^Q\left[\theta_u[v_u] - \lambda[v_u]\,du\right]dt$$

The realized variance of the asset between 0 and T is

$$\overline{RV}_{0,T}^{\mathcal{P}} = V_0 + \frac{1}{T}\int_0^T\left(\int_0^t \theta_u[V_u]\,du + \int_0^t \eta_u[V_u]\,dB_u\right)dt$$

so the total realized PnL from selling short the Variance Swap is

$$-\frac{1}{T}\int_0^T\left(\int_0^t \lambda_u[V_u]\,du + \int_0^t \eta_u[V_u]\,dB_u\right)dt$$

with the $\mathbb{P}$ -expected value

$$-\frac{1}{T}\int_0^T\int_0^t \lambda_u[V_u]\,\mathrm{d}u\mathrm{d}t.$$

Therefore positive "implied minus realized" spread corresponds to negative value of the market price of volatility risk.

Under Heston model, we can derived the variance s Since solution to Heston stochastic volatility equation 2.3.4 is $E_0^{\mathcal{P}}[V_t] = \theta + e^{-\kappa t}(V_0 - \theta)$, and the solution under risk neutral measure for equation is $E_0^{\mathbb{Q}}[V_t] = \frac{\kappa\theta}{\kappa+\delta^V} + e^{-(\kappa+\delta^v)t}\left(V_0 - \frac{\kappa\theta}{\kappa+\delta^v}\right)$

$$
\begin{aligned}
&\overline{RV}^{\mathcal{P}}_{0,T} - SW^{Q}_{0,T} \\
=&\frac{1}{T}E_0^{\mathcal{P}}\left[\int_0^T v_I dt\right] - \frac{1}{T}E_0^{Q}\left[\int_0^T v_I dt\right] \\
=&\frac{1}{T}\int_0^T \left(\theta + e^{-\kappa t}(v_0 - \theta)\right)dt - \frac{1}{T}\int_0^T \left(\frac{\kappa\theta}{\kappa - \delta^v} + e^{-(\kappa-\delta^v)t}\left(v_0 - \frac{\kappa\theta}{\kappa - \delta^v}\right)\right)dt \\
=&\theta + \frac{1 - e^{-\kappa T}}{\kappa T}(v_0 - \theta) - \left(\frac{\kappa\theta}{\kappa - \delta^v} + \left(\frac{1 - e^{-(\kappa-\delta_v)t}}{(\kappa - \delta_v)T}\right)\left(v_0 - \frac{\kappa\theta}{\kappa - \delta_v}\right)\right) \qquad (4.1.2)
\end{aligned}
$$

4.1.2 Merton jump diffusion model

Under MJD model in 3.1.6, we can calculate the difference between realised volatility due to jump rate,

$$\overline{RV}^{\mathcal{P}}_{0,T} = E_0^{\mathcal{P}}\left[\int_0^T \left(\frac{dS_t}{S_t}\right)^2\right] = E_0^{\mathcal{P}}\left[\int_0^T \left(\sigma^2 \mathrm{d}t + J_t^2 dN_t\right)\right]$$

$$SW^{Q}_{0,T} = E_0^{Q}\left[\int_0^T \left(\frac{dS_t}{S_t}\right)^2\right] = E_0^{Q}\left[\int_0^T \left(\sigma^2 dt + J_t^2 dN_t\right)\right]$$

The difference can be derived as follows:

$$\overline{RV}^{\mathcal{P}}_{0,T} - SW^{Q}_{0,T} = E_0^{\mathcal{P}}\left[\int_0^T J_t^2 dN_t\right] - E_0^{Q}\left[\int_0^T J_t^2 dN_t\right]$$

In order to derived the quadratic variation of jumps, we need to use the following lemma from [Gatheral, 2006].

Lemma 4.1.1. *With probability measure* $\mathbb{P}$ *and jump size is log normally distributed with where* J_i *has lognormal distribution* $\log(J_i + 1) \sim N(\gamma, \delta^2)$

$$E_0^{\mathcal{P}}\left[\int_0^T J_t^2 dN_t\right] = \lambda^2 T^2 \mu_J^2 + \lambda T \sigma_J^2$$

where $\mu_J = e^{\gamma + \frac{\delta^2}{2}} - 1$ *and* $\sigma_J = e^{2\gamma+\delta^2}\left(e^{\delta^2} - 1\right)$

Proof. Let P_t denote the return of a compound Poisson process so that

$$P_T = \sum_{i}^{N_T} J_i$$

with the J_i i.i.d. with distribution density $\mu(\mathrm{d}x)$ and N_T a Poisson process with mean λT. Define the quadratic variation as

$$\langle P \rangle_T = \sum_{i=1}^{N_T} |J_i|^2$$

Then the quadratic variation has expectation

$$\mathbb{E}\left[\langle P\rangle_T\right] = \mathbb{E}\left[N_T\right]\mathbb{E}\left[|J_i|^2\right] = \lambda T\int_0^T y^2\mu(y)dy$$

Also

$$\mathbb{E}\left[P_T\right] = \lambda T\int_0^T y\mu(y)dy$$

and

$$\mathbb{E}\left[P_T^2\right] = \lambda T\int_0^T y^2\mu(y)dy + (\lambda T)^2\left(\int_0^T y\mu(y)dy\right)^2 = \lambda T\mathrm{Var}(J_t) + \lambda^2T^2\left(E\left[J_t\right]\right)^2$$

or

$$\mathbb{E}\left[\langle P\rangle_T\right] = \mathbb{E}\left[P_T^2\right] - \mathbb{E}\left[P_T\right]^2 = \mathrm{Var}\left[P_T\right]$$

□

However, in order to find the theoretical solution to MJD model, we need to define a risk neutral measure for MJD model. As discussed in Section 3.1, MJD is an incomplete model which means that there are more than one risk neutral measure.

Choice 1: Keeping λ and μ_J both unchanged

This change of measure identity was initially proposed by Merton. [Merton, 1976]. By using an equivalent martingale that keeps both λ and μ_J constant, we would be able to find $\mathrm{d}W_t^Q$ such that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \varepsilon\left(-\int_0^T \zeta_\tau dW_\tau\right)$ for some ζ_t that can satisfy the drift change that makes $\mathrm{e}^{(r-q)t}S_t$ a martingale. Merton also showed that the solution under this model is

$$S_t = S_0\exp\left[\left(r-q-\lambda\mu_J\right)t + \sigma M_t^{\mathbb{Q}} + \sum_{k=1}^{N_i}\ln(1+J_k)\right]$$

under $\mathbb{Q}$ The advantage of this model is that this model has the fewest parameters that we need to take care of as $\mu_J^* = \mu_J$ and $\lambda^* = \lambda$ are fixed, which will significantly simplify our calculation at later stage. However, it also eliminates a free parameter that we can use to calibrate our model.

In this case, there will be no difference between realised volatility and implied volatility. Hence

$$\overline{RV}_{0,T}^{\mathcal{P}} - SW_{0,T}^{Q} = 0$$

Choice 2: Keeping λ unchanged and free the jump size

This change of measure identity was from Pan's paper [Pan, 2002]. Here we set $\lambda^* = \lambda$ and let the new jump size J_t^* has a new distribution $\nu(\mathrm{d}x)$ with mean μ_J^* and variance σ_J^2 unchanged. $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \varepsilon\left(-\int_0^T \zeta_\tau dW_\tau\right)\prod_{k=1}^{N_T}\left(\frac{\tilde{\nu}(\mathrm{d}x)}{\nu(\mathrm{d}x)}\right)$ where $\zeta_t = \delta^s\sqrt{V}$ with some constant δ^s. It can be proved that this identity also makes $\mathrm{e}^{(r-q)t}S_t$ a martingale. This model freezes $\lambda^* = \lambda$ and let μ_J change by changing the distribution of J_t under risk neutral measure. We then can choose μ_J^* freely and it will influence our choice of δ^s in order to keep the martingale condition. The solution to this model now becomes

$\mathrm{S}_t = S_0 \exp\left[(r-q-\lambda\mu_J^*)\,t + \sigma M_t^{\mathbb{Q}} + \sum_{k=1}^{N_i} \ln(1+J_k^*)\right]$

In this case, the payoff of variance swap will be

$$\overline{RV}_{0,T}^{\mathcal{P}} - SW_{0,T}^{Q} = \lambda^2 T^2 \big(\mu_J^2 - (\mu_J^*)^2\big) \tag{4.1.3}$$

Choice 3: Vary λ and μ under some constraints

This change of measure identity was from Duan et al.[Duan and Yeh, 2010]. In this model, we change both μ_J and λ. This time we use the Randon Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \varepsilon\left(-W_t\right) \mathrm{e}^{-(\lambda^*-\lambda)T} \prod_{k=1}^{N_T} \left(\frac{\tilde{\nu}(\mathrm{d}x)}{\nu(\mathrm{d}x)}\right).$$

We let N_t^* be a Poisson process with intensity λ^* and independent of $W_t^{\mathbb{Q}}$ under $\mathbb{Q}$ and B_t^*; J_t^* is an independent normal random variable under measure $\mathbb{Q}$ with a new mean μ_J^* but its standard deviation remains unchanged at σ_J. This model gives us more freedom to control the whole process. As long as the dynamics are like described in equation (3) and (4), we would be able to ensure that $\mathrm{e}^{(r-q)t}S_t$ a martingale. We would be able to see that choosing different equivalent martingale gives different expressions on the value of derivatives such as vairance swaps and delta hedged options that we will discuss in the following section.

$$\overline{RV}_{0,T}^{\mathcal{P}} - SW_{0,T}^{Q} = (\lambda T \mu_J)^2 + \lambda T \sigma^2 - (\lambda^* T \mu_J^*)^2 - \lambda^* T \sigma^2$$

4.1.3 Stochastic Volatility jump diffusion model

For stochastic volatility jump diffusion model, we fix the risk neutral measure as in Lemma 3.2.3. The model can be treated as a combination of Heston model and MJD model with choice 2. Thus, the payoff of variance swap is

$$\overline{RV}_{0,T}^{\mathcal{P}} - SW_{0,T}^{Q} = \theta + \frac{1-e^{-\kappa T}}{\kappa T}(v_0-\theta) - \left(\frac{\kappa\theta}{\kappa-\delta^v} + \left(\frac{1-e^{-(\kappa-\delta_v)t}}{(\kappa-\delta_v)T}\right)\left(v_0 - \frac{\kappa\theta}{\kappa-\delta_v}\right)\right) + \lambda^2T^2(\mu_J^2 - (\mu_J^*)^2) \tag{4.1.4}$$

4.2 Variance Swap and delta hedged options

In this section, we are interested in for a fixed VRP in terms of variance swap, the distribution of delta hedging error changes for options with varying strikes. As in Section 4.1, we will consider to fix the variance swap under different model assumptions.

Heston model

Under Heston model defined as 2.3.3 and 2.3.4, we can change κ and θ which can influence the VRP harvested from variance swap as in 4.1.2. In order to fix the VRP, we let $v_0 = \frac{k\theta}{k-\delta_v}$. Then k, θ and δ_v should have the following relationship

$$(v_0-\theta)\left(1-\frac{1-e^{-kT}}{kT}\right) = VRP$$

Thus, we will have

$$\theta = v_0 - VRP\left(1-\frac{1-e^{-kT}}{kT}\right)^{-1} \quad \text{and} \quad \delta_v = k\left(1-\frac{\theta}{v_0}\right)$$

We use the following parameters throughout simulations in the Heston model section: $T = 1$, $\mu = 0.2$, $r = 0.05$, $\sigma = 0.3$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 12, which means that the total steps is 3024. Also, when calculating delta hedged gains, the implied volatility is set $\hat{\sigma} = 0.3$. Under Heston model assumption, we change κ, θ and δ_v to keep VRP from total variance swap unchanged. In the mean time, we adjust strike price K to see the effect of different parameters for options with different strikes

Three rounds of simulations were run based on Black Scholes delta hedging when $K = 80$, $K = 100$ and $K = 120$. See Appendix A.2.1 for detailed simulation results and graphs.

From simulation results, when we increase κ, θ and δ_v while fixing the VRP from variance swap unchanged, there are a few statistical property that we can spot.

1. The delta hedged gains' distribution tends have more negative expectation value and tends to be more negatively skewed. However, it has lower variance in the meantime.
2. The variance and skewness decrease as parameters increase.
3. Out of the money option in general gives the highest expected payoff for the delta hedged portfolio and with small κ, θ and δ, the expected payoff is even positive while for in the money and deep out of money options, profits are hard to make.

Stochastic jump diffusion model

The effect of jumps only influence the extreme loss in MJD model. Therefore, the distribution will not have significant change when we change the parameters. Thus, we can directly look at the stochastic jump diffusion model where we can adjust both the jump and stochastic volatility parameters.

We use the following parameters throughout simulations in the stochastic jump diffusion model section: $T = 1$, $\mu = 0.2$, $r = 0.05$, $v_0 = 0.09$. The number of simulations is set to be 10000. The simulations are based on 252 trading days and we set the steps on each day to be 8, which means that the total steps is 2017. Also, when calculating delta hedged gains, the implied volatility is set $\hat{\sigma} = 0.3$. We change κ, θ, δ_v, λ and μ_J^* to keep VRP from total variance swap unchanged. In the mean time, we adjust strike price K to see the effect of different parameters for options with different strikes.

The stochastic volatility can be controlled exactly as Heston model and for the jump part, by 4.1.3, we can fix μ_J and vary λ and μ_J^* (γ^* in particular) by

$$\gamma^* = \log\left(\mu_J^2 - \frac{VRP}{\lambda^2 T^2}\right) + \frac{b^2}{2}$$

Three rounds of simulations were run based on Merton delta hedging when $K = 80$, $K = 100$ and $K = 120$. See Appendix A.2.2 for detailed simulation results and graphs.

A few observations can be made based on simulation results:

1. Compared to Heston model, for stochastic volatility jump diffusion model, as parameters increase, the expected payoff is less negative.
2. The variance peaks at certain parameter. 3. The large negative loss dominates the expectation, but increasing λ and

Conclusion

By simulation and theoretical derivation, we prove and observe that there exists volatility risk premia under Black Scholes model. Adding continuous paid dividend will lower the volatility risk premia, but it is still profitable to sell delta hedged strategy. Also, stochastic volatility will pull down volatility risk premia. Black Scholes delta hedged option can still make positive expected payoff, but by theoretical derivation the Heston model's own delta should be more suitable for harvesting VRP.

Adding jumps to the model will cause the delta hedged option portfolio to have a very large negative loss in most cases. Black Scholes delta hedged portfolio and Merton delta hedged portfolio both give positive expected payoff, but Merton delta will better control risks. For stochastic volatility jump diffusion models, it is no longer profitable to sell either Black Scholes delta hedged option or Merton delta hedged options due to the significant loss caused by both stochastic volatility and jumps.

In the stochastic volatility jump diffusion model, we use another instrument variance swap to harvest volatility risk premium. We simulated the delta hedged options as we change parameters while keeping VRP harvested from var swap fixed. Overall, delta hedged options are highly likely to end up with loss and losses are sometimes significant. However, as parameters become larger, the delta hedged option will have lower loss.

Future research may include

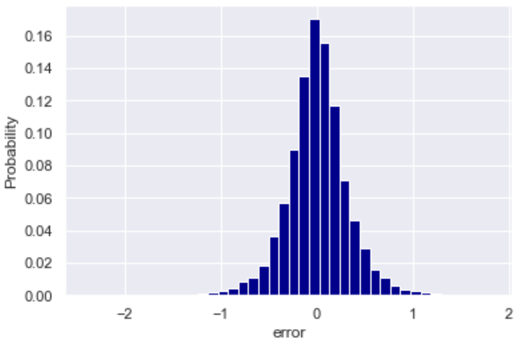
1. Find the delta hedging error for stochastic volatility model with theoretical delta and compare with Black Scholes delta hedging. It can be expected that with model's own delta, the hedging effect should be much better.
2. Though pure delta hedging is not as stable as variance swap in volatility premium harvesting. However, research has shown that for empirical $S\&P$ 500 data, volatility risk premium does exist. It may be possible to find a model that can better estimate the index (e.g. GARCH) and make hedging a dynamic programming problem or we can even apply reinforcement learning to harvest VRP.

Appendix A

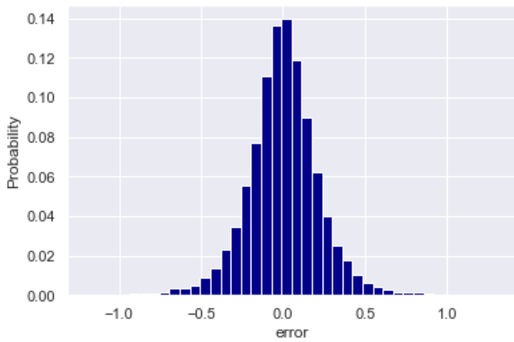
Simulation results

A.1 Convergence of Delta hedging

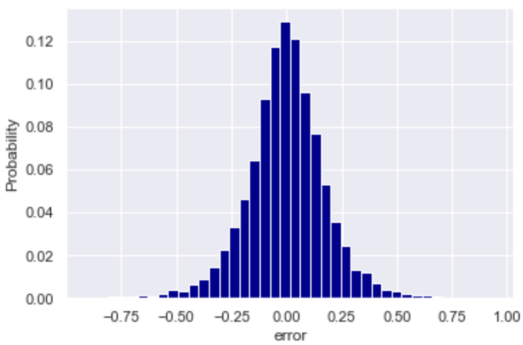
A.1.1 Black Scholes model



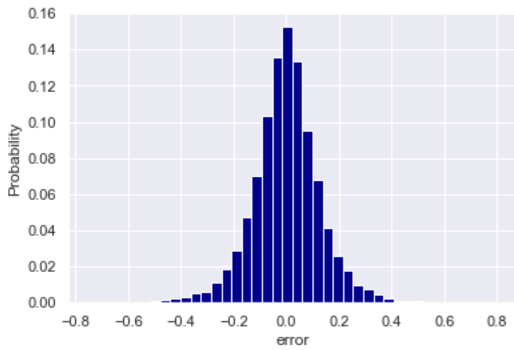
(a) daily steps = 5



(b) daily steps = 8



(c) daily steps = 12



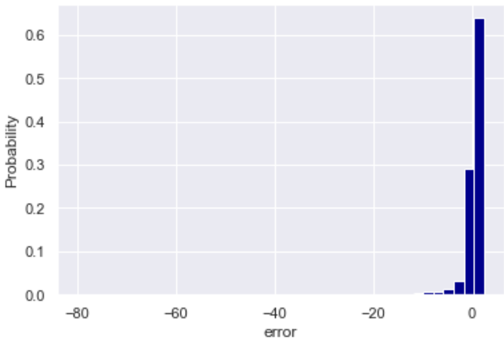
(d) daily steps = 24

Figure A.1: Convergence of pnl of delta hedged option portfolio under Black Scholes model

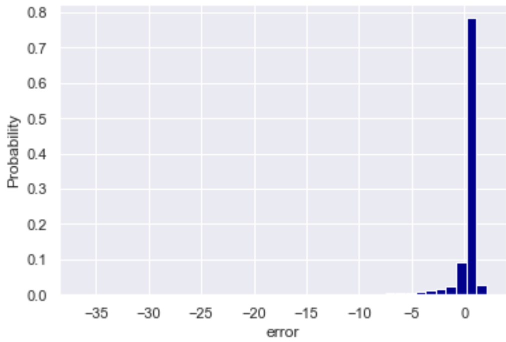
A.1.2 Merton Jump diffusion model

A.2 Change in delta hedged option's gains with VRP from variance swap fixed

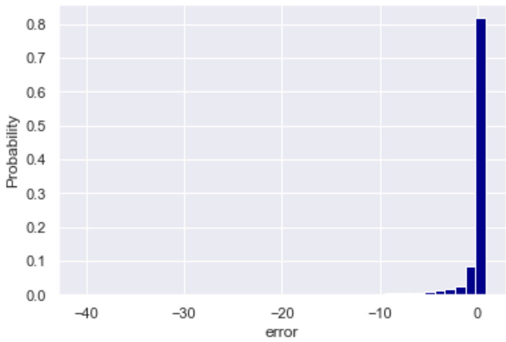
A.2.1 Heston model



(a) daily steps = 5

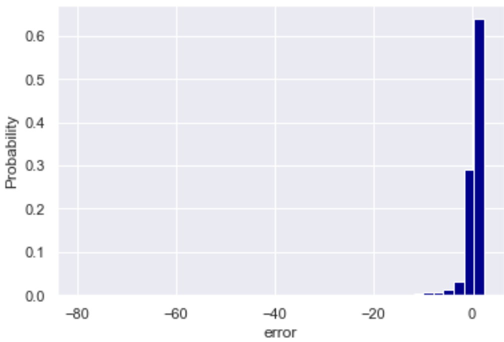


(b) daily steps = 8

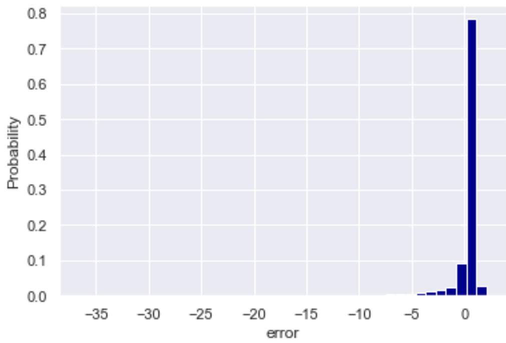


(c) daily steps = 12

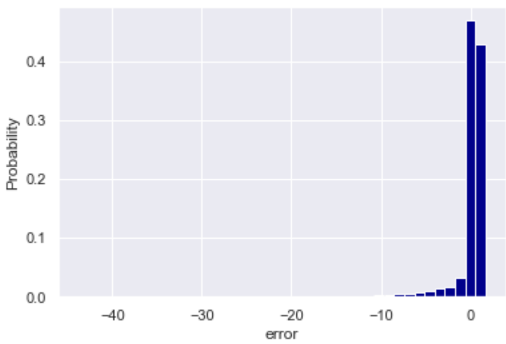
Figure A.2: Non-convergence of pnl of Black Scholes delta hedged option portfolio under MJD model



(a) daily steps = 5

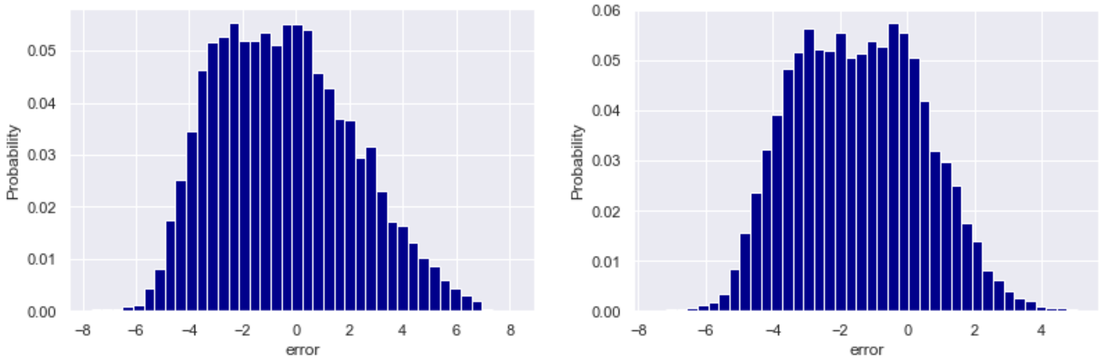


(b) daily steps = 8



(c) daily steps = 12

Figure A.3: Non-convergence of pnl of Merton delta hedged option portfolio under MJD model

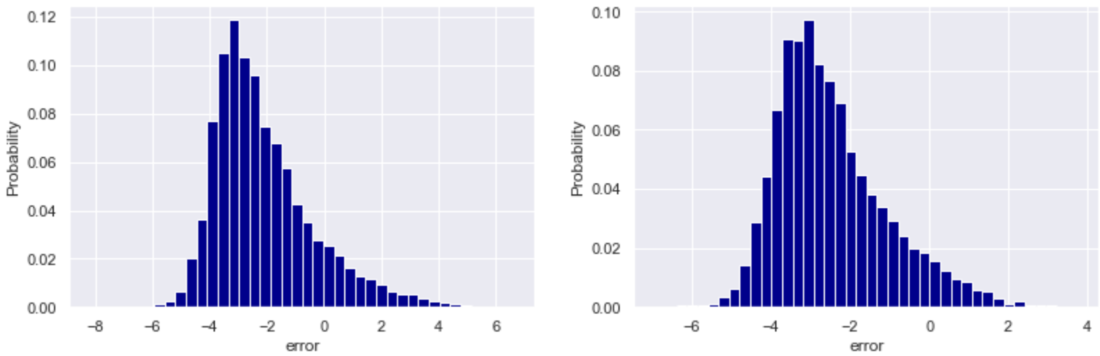


(a) k = 1, θ = 0.02408, δ_v = 0.8494 (b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.4: Distribution of OTM Black Scholes delta hedged options under Heston model at K = 120

κ	θ	δ_v	mean	variance	skewness
1	0.02408	0.8494	-0.4569	6.6475	0.3454
1.2	0.04028	0.8978	-0.6987	5.6341	0.2672
1.4	0.05174	0.9473	-0.9712	5.1371	0.2517
1.6	0.06023	0.9976	-1.1491	4.5611	0.2069
1.8	0.06676	1.04889	-1.3653	4.0983	0.2269
2.0	0.07074	1.08789	-1.4480	3.6972	0.1585

Table A.1: Statistics of distribution of ATM BS delta hedged options with $K = 100$ under Heston model

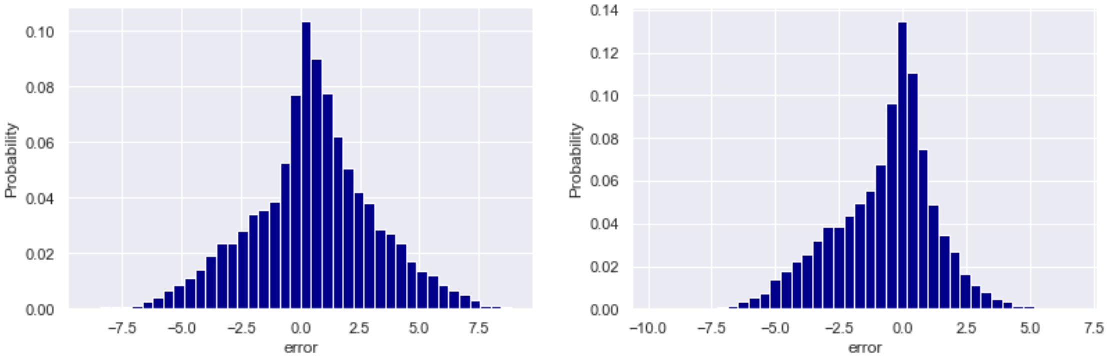


(a) k = 1, θ = 0.02408, δ_v = 0.8494 (b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.5: Distribution of ITM Black Scholes delta hedged options under Heston model at K = 80

κ	θ	δ_v	mean	variance	skewness
1	0.02408	0.8494	-2.1193	3.0732	1.1100
1.2	0.04028	0.8978	-2.1935	2.7780	1.0277
1.4	0.05174	0.9473	-2.2778	2.4544	1.0543
1.6	0.06023	0.9976	-2.3808	2.2436	0.9431
1.8	0.06676	1.04889	-2.4529	2.0853	0.9172
2.0	0.07074	1.08789	-2.4811	1.9562	0.8316

Table A.2: Statistics of distribution of ITM BS delta hedged options with $K = 80$ under Heston model



(a) k = 1, θ = 0.02408, δ_v = 0.8494

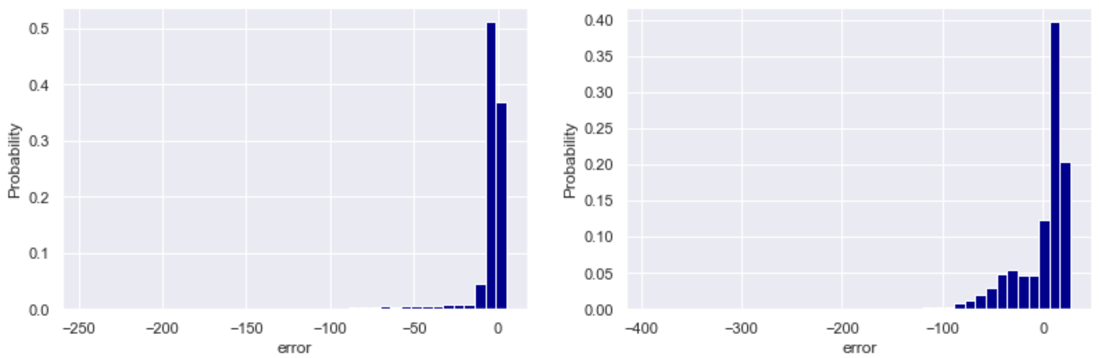
(b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.6: Distribution of OTM Black Scholes delta hedged options under Heston model at K = 120

κ	θ	δ_v	mean	variance	skewness
1	0.02408	0.8494	0.5129	6.8146	-0.05614
1.2	0.04028	0.8978	0.1282	5.9253	-0.2093
1.4	0.05174	0.9473	-0.1557	5.2065	-0.2107
1.6	0.06023	0.9976	-0.3680	4.67707	-0.3319
1.8	0.06676	1.04889	-0.6039	4.2707	-0.3836
2.0	0.07074	1.08789	-0.7289	3.9714	-0.4810

Table A.3: Statistics of distribution of OTM Merton delta hedged options with $K = 120$ under Heston model

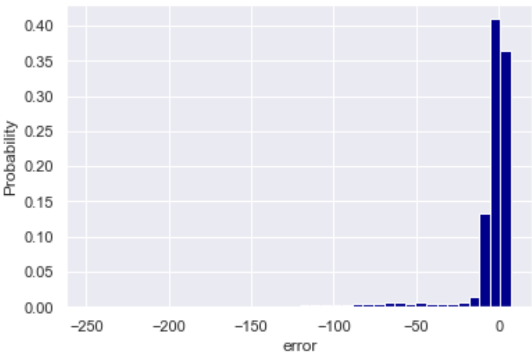
A.2.2 Stochastic volatility Model with Merton delta hedging



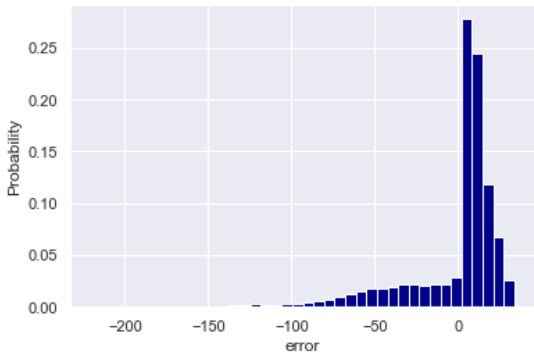
(a) k = 1, θ = 0.02408, δ_v = 0.8494

(b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.7: Distribution of ITM Merton delta hedged options under stochastic volatility jump diffusion model at K = 80

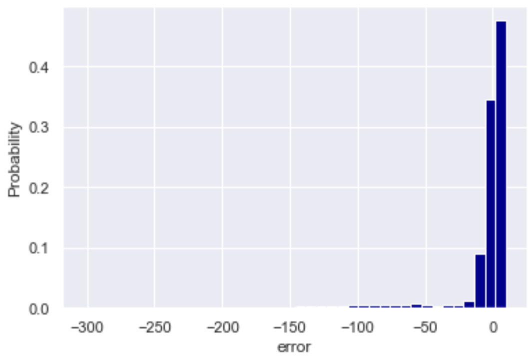


(a) k = 1, θ = 0.02408, δ_v = 0.8494

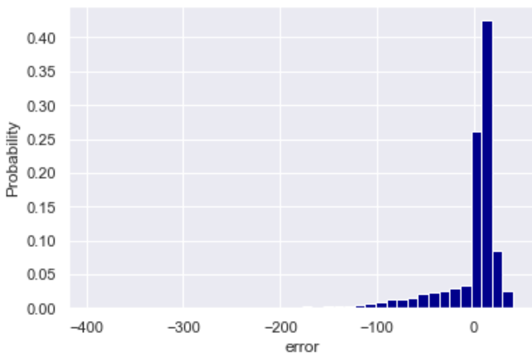


(b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.8: Distribution of ATM Merton delta hedged options under stochastic jump diffusion model at K = 100



(a) k = 1, θ = 0.02408, δ_v = 0.8494



(b) k = 2, θ = 0.07074, δ_v = 1.08789

Figure A.9: Distribution of OTM Merton delta hedged options under stochastic jump diffusion model at K = 120

κ	θ	δ_v	λ	a_Q	mean	variance	skewness
1	0.02408	0.8494	0.1	-0.2146	-6.6762	480.8092	-4.3054
1.2	0.04028	0.8978	0.18	-0.1668	-5.0400	772.8338	-3.1239
1.4	0.05174	0.9473	0.26	-0.1375	-3.1912	880.4824	-2.5114
1.6	0.06023	0.9976	0.34	-0.1157	-2.6099	904.7619	-2.2371
1.8	0.06676	1.04889	0.42	-0.09788	-1.90727	891.4981	-2.2964
2.0	0.07074	1.08789	0.46	-0.08221	-1.9788	861.5896	-2.2430

Table A.4: Statistics of distribution of ITM Merton delta hedged options at $K = 80$ under stochastic volatility jump diffusion model

κ	θ	δ_v	λ	a_Q	mean	variance	skewness
1	0.02408	0.8494	0.1	-0.2146	-5.9146	660.5863	-4.3084
1.2	0.04028	0.8978	0.18	-0.1668	-4.3507	974.9351	-3.3307
1.4	0.05174	0.9473	0.26	-0.1375	-3.1227	1063.4742	-2.8865
1.6	0.06023	0.9976	0.34	-0.1157	-2.2331	1029.1765	-2.7997
1.8	0.06676	1.04889	0.42	-0.09788	-2.4183	1087.5049	-2.8771
2.0	0.07074	1.08789	0.46	-0.08221	-1.8359	998.7658	-2.8265

Table A.5: Statistics of distribution of ATM Merton delta hedged options at $K = 100$ under stochastic volatility jump diffusion model

κ	θ	δ_v	λ	a_Q	mean	variance	skewness
1	0.02408	0.8494	0.1	-0.2146	-6.0048	257.4232	-5.0621
1.2	0.04028	0.8978	0.18	-0.1668	-4.8614	452.3301	-3.3307
1.4	0.05174	0.9473	0.26	-0.1375	-3.1575	639.3861	-2.6005
1.6	0.06023	0.9976	0.34	-0.1157	-2.4859	733.420	-2.0758
1.8	0.06676	1.04889	0.42	-0.09788	-2.4183	709.0029	-1.7817
2.0	0.07074	1.08789	0.46	-0.08221	-1.4416	725.5316	-2.0685

Table A.6: Statistics of distribution of OTM Merton delta hedged options at $K = 120$ under stochastic volatility jump diffusion model

Bibliography

[Bakshi and Kapadia, 2015] Bakshi, G. and Kapadia, N. (2015). Delta-Hedged Gains and the Negative Market Volatility Risk Premium. *The Review of Financial Studies*, 16(2):527–566.

[Bertsimas et al., 1997] Bertsimas, D., Kogan, L., and Lo, A. W. (1997). Pricing and hedging derivative securities in incomplete markets: An e-aritrage model. (6250).

[Davis, 2010] Davis, M. (2010). *Black–Scholes Formula.*

[Duan and Yeh, 2010] Duan, J.-C. and Yeh, C.-Y. (2010). Jump and volatility risk premiums implied by vix. *Journal of Economic Dynamics and Control*, 34(11):2232 – 2244. Special Issue: 2008 Annual Risk Management Conference held in Singapore during June 30 - July 2, 2008.

[Figlewski, 1989] Figlewski, S. (1989). Options arbitrage in imperfect markets. *The Journal of Finance*, pages pp. 1289–1311.

[Gatheral, 2006] Gatheral, J. (2006). The volatility surface: A practitioner's guide.

[Ge, 2016] Ge (2016). A survey of three derivative based methods to harvest the volatility risk premium in equity markets. *Journal of Investing.*

[GMBH, 2017] GMBH, F. T. (2017). Harvesting the sp500 volatility risk premium. *The Hedge Fund Journal*, page Issue 121.

[Heston, 1993] Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6(2):327–343.

[Matsuda, 2004] Matsuda, K. (2004). Introduction to merton jump diffusion model.

[Merton, 1976] Merton, R. (1976). Option prices when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144.

[Pan, 2002] Pan, J. (2002). The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics*, 63(1):3 – 50.

[Privault, 09] Privault, N. (09). Lecture notes in financial mathematics.

[Tsoukalas and Zeng, 2009] Tsoukalas, G. and Zeng, X. (2009). Lecture notes in advanced topics in financial engineering.

[Wei and Bouchey, 2015] Wei, G. and Bouchey, P. (2015). Delta-neutral performance attri- bution for historical option strategies. *Working Paper, Parametric Portfolio Associates LLC.*