Chapter 6

Linear small-amplitude waves

6.1 Introduction

One of the main difficulties of fluid mechanics is its intrinsic non-linearity, explicitly visible in the $(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}$ term in the equation of motion. This makes it difficult to find exact solutions, except in those cases where there is a lot of symmetry[1]. An example of such a situation is the Solar Wind model treated in Chapter 4: it is a steady flow $(\partial/\partial t = 0)$ and in addition the flow is spherically symmetric so that the direction of the flow lines is known in advance: the radial direction. This is in fact the maximum symmetry that is non-trivial for a flow in three dimensions, the trivial case being a flow that is steady with a globally constant speed, density and pressure.

Another way to simplify the equations is to look at small perturbations around an equilibrium where there is a force balance. This equilibrium state that is a solution of the fluid equations. One then looks at small deviations from that equilibrium, assuming that the changes in velocity, density and pressure remain small. If that is the case, nonlinear terms can be neglected when describing the evolution of these small perturbations, for instance: all variations in fluid quantities such as velocity, density and pressure can be expressed as *linear* functions of the *displacement field* $\boldsymbol{\xi}(\boldsymbol{x}\ ,\ t)$ that decribes how far individual fluid elements are displaced from their equilibrium position. This vector field (simply called the *displacement vector* from now on) plays a pivotal role in the theory. The linearization technique works well if the amplitude of the displacement vector remains sufficiently small.

As an illustration of this technique, often referred to as *perturbation analysis*, I will look at an analogous situation in classical mechanics.

[1] I use the term *symmetry* here in a general mathematical sense. As such, it includes not only spatial symmetries, such as spherical symmetry with $\partial/\partial\theta = \partial/\partial\phi = 0$, but also the case of a steady flow the the symmetry $\partial/\partial t = 0$.

6.1.1 Perturbation analysis of particle motion in a potential

Consider a particle of mass m moving in one dimension x in a potential $V(x)$, which leads to a force $F(x) = -\mathrm{d}V/\mathrm{d}x$. The equation of motion for this particle reads:

$$m\,\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}\;. \tag{6.1.1}$$

Now let's assume that there is an equilibrium position x_0 where the force $F(x)$ vanishes. This implies that the potential satisfies

$$\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0} = 0\;. \tag{6.1.2}$$

Consider a particle at rest at the equilibrium position $x = x_0$. We now perturb the particle, shifting its position from $x = x_0$ to $x = x_0 + \xi$. How will the particle move?

In the immediate vicinity of x_0 (i.e. for *small* ξ) the potential can be expanded in powers of $\xi = x - x_0$ as:

$$V(x_0+\xi) \approx V_0 + \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0} \xi + \frac{1}{2}\left(\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right)_{x=x_0} \xi^2 + \cdots \tag{6.1.3}$$

Here $V_0 \equiv V(x_0)$. If we break off the expansion for the potential at the quadratic term, and use the equilibrium condition (6.1.2) we get

$$V(x_0+\xi) \approx V_0 + \frac{1}{2}k\,\xi^2\;, \tag{6.1.4}$$

where $k \equiv (\mathrm{d}^2 V/\mathrm{d}x^2)_{x=x_0}$. Now substituting

$$x(t) = x_0 + \xi(t) \tag{6.1.5}$$

into the equation of motion (6.1.1), and using $\xi = x - x_0$ so that for constant x_0 one has

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\,\frac{\mathrm{d}V}{\mathrm{d}\xi} = \frac{\mathrm{d}V}{\mathrm{d}\xi}\;, \tag{6.1.6}$$

one finds:

$$m\,\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\frac{\mathrm{d}V}{\mathrm{d}\xi} = -k\xi\ . \tag{6.1.7}$$

By breaking off the expansion (6.1.3) of the potential at the quadratic term in ξ, we get a *linear* equation of motion for the dispacement $\xi(t)$ of the particle. Had we included terms proportional to ξ^3, there would be a corresponding **nonlinear** term $\propto \xi^2$ in the equation of motion for ξ. By making this choice we have *linearized* the problem. We must therefore assume that $|\xi|$ remains sufficiently small so that our approximation for $V(x_0 + \xi)$ remains valid.

The equation of motion (6.1.7) looks like the equation of motion for a linear oscillator if $k > 0$. In that case the force is directed back towards the equilibrium position x_0, and the solution is a harmonic oscillation around the equilibrium position:

$$\xi(t) = \xi_0\ \cos(\omega t + \alpha)\ , \tag{6.1.8}$$

where ξ_0 is the amplitude of the oscillation and the oscillation frequency equals

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{m}\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0}}\ . \tag{6.1.9}$$

The amplitude ξ_0 and phase angle α follow directly from initial conditions: the displacement $\xi(0) = \xi_0\ \cos\alpha$ and the velocity $(\mathrm{d}\xi/\mathrm{d}t)_0 = -\omega\xi_0\ \sin\alpha$ at $t = 0$.

The condition $k > 0$ corresponds to:

$$\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0} > 0\ . \tag{6.1.10}$$

Condition (6.1.10) is simply that the position x_0 must correspond with a *minimum* in the potential. In that case the equilibrium is *stable* since a small perturbation from the equilibrium position leads to a harmonic oscillation of the particle around that position. The stable case is illustrated below.

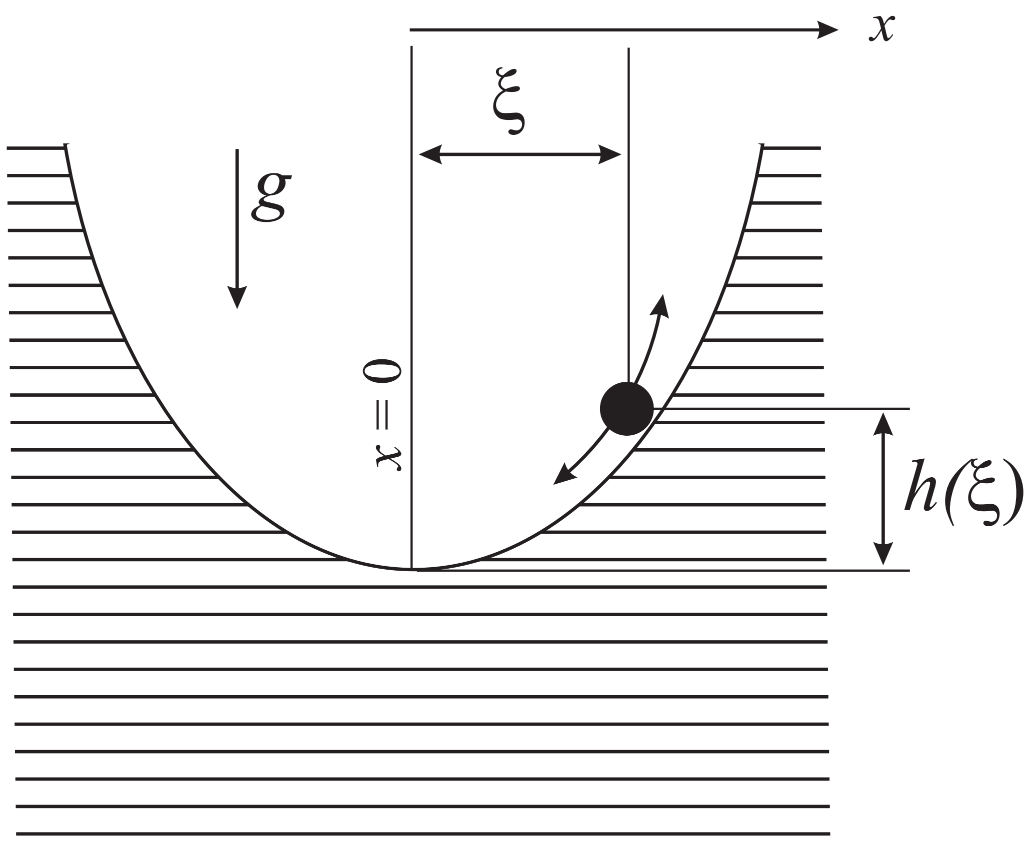


Figure 6.1: *A simple example of a stable oscillation is the motion of a spherical ball in a bowl under the influence of gravity. The gravitational potential energy equals* $V(\xi) = mg\,h(\xi)$*, with* g *acceleration of gravity and where* ξ *is the horizontal distance to the point where the bottom of the bowl reaches its lower level. This point coincides with* $x = 0$*. Also,* $h(\xi)$ *is the height above the lowest point at distance* ξ*. The minimum of the potential occurs in this example at* $x = 0$*, and the constant* k *in this case equals* $k = mg\,(\mathrm{d}^2h/\mathrm{d}x^2)_{x=0}$.

If on the other hand the equilibrium position is at a *maximum*, so that $k < 0$ and

$$\left(\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right)_{x=x_0} < 0 \; , \tag{6.1.11}$$

the force is always directed *away* from equilibrium position x_0. In that case the solution of the equation of motion for ξ reads

$$\xi(t) = \xi_+ \exp(\sigma t) + \xi_- \exp(-\sigma t) \; . \tag{6.1.12}$$

The term proportional to ξ_+ grows exponentially in time, and will dominate the solution when $\sigma t \gg 1$. The *growth rate* σ is

$$\sigma \equiv \sqrt{\frac{|k|}{m}} = \sqrt{\frac{1}{m} \left|\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right|_{x=x_0}} \; . \tag{6.1.13}$$

The amplitude of the term $\propto \xi_+$ doubles in a time interval $\Delta t = \tau = \ln 2/\sigma = 0.693/\sigma$, and grows without bound. This exponential growth of $\xi(t)$ (in the linear approximation) implies that the equilibrium is *linearly unstable*: the particle will move further and further away from the equilibrium position. This means that our assumption that linearization is allowed must inevitably break down when the displacement becomes sufficiently large. It is still possible that the exact motion (without the linearization assumption) is stable, so that the equilibrium is *linearly unstable*, but *nonlinearly stable*. In what follows, we will not consider that case and assume that the presence of a linear instability signals a true instability of the system.

This example of perturbation anaysis illustrates the main features of an approach that is also valid in fluid mechanics. There we will also perturb an equilibrium, and derive a linear equation of motion for a small displacement $\Delta \boldsymbol{x} = \boldsymbol{\xi}$ from that equilibrium. If the equilibrium is stable we will find the linear waves (oscillations) the fluid is able to support. If the equilibrium turns out to be unstable, we will find the linear growth rate of the instability. Like in the case of ordinary mechanics, the perturbation approach allows us to determine the stability of an equilibrium state.

6.2 What constitutes a wave?

In an ideal fluid in a *stable* equilibrium, small pertubations in pressure, density and temperature propagate as waves. The qualification 'small' in this context means that a number of conditions must be satisfied:

- The amplitude of the pressure perturbation ΔP, density perturbation $\Delta\rho$ and the temperature perturbation ΔT are all small compared with the average pressure, density, and temperature:

$$|\Delta P| \ll P \quad , \quad |\Delta\rho| \ll \rho \quad , \quad |\Delta T| \ll T \; . \tag{6.2.1}$$

- The displacement $\Delta\boldsymbol{x} \equiv \boldsymbol{\xi}$ of a fluid element must be small compared with the wavelength λ of the wave, and the wavelength is small compared with the scale length L on which the average pressure, density or temperature of the fluid change:

$$|\boldsymbol{\xi}| \ll \lambda \ll L \; . \tag{6.2.2}$$

If these conditions are not fulfilled, a description in terms of simple linear and purely harmonic waves is not possible.

We will mostly deal with the case of *plane waves* where it is assumed that ΔP, $\Delta\rho$, ΔT and $\boldsymbol{\xi}$ all vary harmonically in space and time, with a well-defined wavelength and wave frequency[2].

Such harmonic behaviour is to be expected. Consider for example what happens in a sound wave, which is simply a periodic train of alternating regions of slightly higher and slightly lower pressure than the average pressure. When the gas is locally compressed so that the density increases, the associated local pressure increase will lead to a pressure force directed away from the compression region. This presure force induces a motion of the gas away from the compression which, by virtue of mass conservation, decreases the density. This density decrease can not stop instantaneously due to the inertia of the material. Therefore it continues until the region becomes *less* dense than its surroundings. The region is now under-pressurized and the direction of the pressure force reverses. As a result, the material flows back into the region. Without some form of friction, this cycle will continue indefinitely.

[2] In the case where we are dealing with cylindrical or spherical waves, as opposed to plane waves, the situation is more complicated.

6.3 The plane wave representation

The displacement of a fluid element in a harmonic plane wave can be represented in terms of complex functions as

$$\boldsymbol{\xi}(\boldsymbol{x}\ ,\ t) = \boldsymbol{a} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc}\ . \tag{6.3.1}$$

Here $\boldsymbol{a}$ is a complex amplitude, $\boldsymbol{k}$ the *wave vector*, which is related to the wavelength λ by

$$\boldsymbol{k} = \frac{2\pi}{\lambda}\,\hat{\boldsymbol{n}} \tag{6.3.2}$$

with $\hat{\boldsymbol{n}}$ a unit vector perpendicular to the wave front, ω is the wave frequency and the notation 'cc' denotes the *complex conjugate*. The complex conjugate must be included to keep $\boldsymbol{\xi}$ (which is an observable quantity!) real-valued. Such a representation is equivalent (but much more convenient, as we will see) to a representation in terms of sines and cosines. In fact it is equivalent with

$$\boldsymbol{\xi}(\boldsymbol{x}\ ,\ t) = 2|a|\ \hat{\boldsymbol{e}}_a\ \cos\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t + \alpha\right)\ , \tag{6.3.3}$$

if we write $\boldsymbol{a} = a\hat{\boldsymbol{e}}_a$ where $\hat{\boldsymbol{e}}_a$ is a (real) unit vector. The phase angle α is related to the real and imaginary parts of the complex amplitude $\boldsymbol{a}$:

$$\alpha = \tan^{-1}\left(\mathrm{Im}(a)/\mathrm{Re}(a)\right) \equiv \tan^{-1}\left(a_{\rm i}/a_{\rm r}\right)\ , \tag{6.3.4}$$

and $|a|$ is

$$|a| = \sqrt{a_{\rm r}^2 + a_{\rm i}^2}\ . \tag{6.3.5}$$

Note that the displacement $\boldsymbol{\xi}(\boldsymbol{x}\ ,\ t)$ is a *field* on space-time, just as the fluid velocity. The velocity perturbation associated with this displacement is

$$\Delta\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}\ . \tag{6.3.6}$$

For the other quantities that vary as a result of the presence of the waves a similar expressions can be written down. For instance, one can write for the density and pressure variations

$$\begin{aligned} \Delta\rho &= \tilde{\rho}\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc}\ , \\ \Delta P &= \tilde{P}\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc}\ . \end{aligned} \tag{6.3.7}$$

The fundamental equations of the flow will (after linearization) provide the relation between $\Delta\rho$ or ΔP and the displacement $\boldsymbol{\xi}$, and between the complex amplitudes $\boldsymbol{a}$, $\tilde{\rho}$ and $\tilde{P}$.

This description will be valid provided the wave number and wave frequency satisfy

$$\omega \mathrm{T} \gg 1 \quad , \quad |\boldsymbol{k}|L \gg 1\ . \tag{6.3.8}$$

Here L is the lengthscale of the spatial variation of the properties of the fluid, and T the timescale on which the fluid changes its properties. The wave period $P = 2\pi/\Omega$ must be much shorter than the time scale on which the fluid changes its global properties, and the wavelength $\lambda = 2\pi/k$ must be much smaller than the scale on which inhomogeneities occur in the fluid or gas. If these equalities are marginally satisfied, there are advanced methods, such as the WJKB method[3] that is also used to construct approxiate solutions to the wave equations of quantum mechanics.

[3]See for instance: P.M. Morse& H. Feshbach, 1953: *Methods of Theoretical Physics* Vol. II, p. 1095, McGraw-Hill, New York.

6.4 Lagrangian and Eulerian perturbations

In Chapter 2.1 we already noted the two different time derivatives that play a role in fluid mechanics: the partial (or *Eulerian*) time derivative $\partial/\partial t$ which gives the change at a fixed coordinate position, and the total (or *Lagrangian*) time derivative $\mathrm{d}/\mathrm{d}t$ which is the derivative following the flow. We also pointed out the difference between the Eulerian perturbation δQ of some quantity (field) $Q(\boldsymbol{x}\ ,\ t)$ as measured at some fixed position, and the Lagrangian perturbation ΔQ, given a small change in position $\Delta\boldsymbol{x}$:

$$\Delta Q = \delta Q + (\Delta\boldsymbol{x} \cdot \boldsymbol{\nabla})\, Q\ . \tag{6.4.1}$$

These definitions for Eulerian and Lagrangian derivatives and variations can be given a precise mathematical meaning. If the flow field is well-behaved, it is possible to assign to each fluid element a label that will identify it unambiguously. A simple choice for such a label is the position the fluid element has at some arbitrary reference time t_0:

Lagrangian label: the position $\boldsymbol{x}(t_0) \equiv \boldsymbol{x}_0$ of each fluid element at $t = t_0$.

One can think of the position of a fluid element as a function of time t *and* of the label $\boldsymbol{x}_0$, which marks its position at time t_0:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_0\ ,\ t)\ . \tag{6.4.2}$$

This is equivalent with an 'initial condition' $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Evaluating this function $\boldsymbol{x}(\boldsymbol{x}_0, t)$ at *fixed* $\boldsymbol{x}_0$ as a function of t gives you the trajectory of a given fluid element: a flow line. Changing the value of $\boldsymbol{x}_0$ at fixed t takes you to a different fluid element, and you are moving (in a continuous fashion) from flowline to flowline. The label $\boldsymbol{x}_0$ is carried along by a flow element, is constant along a given flowline and must therefore satisfy

$$\frac{\mathrm{d}\boldsymbol{x}_0}{\mathrm{d}t} = 0\ . \tag{6.4.3}$$

The Lagrangian time derivative can be re-interpreted in these terms as

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}_0}\ . \tag{6.4.4}$$

In contrast, the partial (Eulerian) time derivative is taken with the coordinate position $\boldsymbol{x}$ kept fixed:

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} \ . \tag{6.4.5}$$

In the same manner one can define the Lagrangian perturbation ΔQ and its Eulerian counterpart δQ for any fluid quantity $Q(\boldsymbol{x}\ ,\ t)$ as

$$\begin{aligned} \Delta Q &= \text{perturbation of } Q \text{ with } \boldsymbol{x}_0 \text{ fixed ,} \\ \delta Q &= \text{perturbation of } Q \text{ with } \boldsymbol{x} \text{ fixed .} \end{aligned} \tag{6.4.6}$$

This definition ensures that ΔQ is the change as seen by an observer moving with the flow.

There is an important set of relations between these variations, spatial derivatives and the Eulerian and Lagrangian time derivatives, which follow directly from the formal definitions (6.4.4), (6.4.5) and (6.4.6):

$$\boxed{\delta\left(\frac{\partial Q}{\partial t}\right) = \frac{\partial\,\delta Q}{\partial t} \quad , \quad \delta(\boldsymbol{\nabla} Q) = \boldsymbol{\nabla}(\delta Q) \quad , \quad \Delta\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right) = \frac{\mathrm{d}\,\Delta Q}{\mathrm{d}t} \ .} \tag{6.4.7}$$

These results will prove useful when we derive the wave properties below.

6.4.1 Velocity, density and pressure perturbations in a wave

The displacement field (wave amplitude) $\boldsymbol{\xi}(\boldsymbol{x}\,,\,t)$ as defined above corresponds to the change of the coordinates (associated with a fixed coordinate grid) as seen by a hypothetical observer who is moving with the oscillating motion of the fluid in the wave: the sloshing motion. An observer fixed to the grid on the other hand is by definition always at the same coordinate position. This implies for a small-amplitude wave that the following relations must be valid:

$$\Delta\boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{x}\,,\,t) \quad , \quad \delta\boldsymbol{x} = \boldsymbol{0}\;. \tag{6.4.8}$$

We can use the unperturbed position $\boldsymbol{x}$ of the fluid as Lagrangian labels to identify different fluid elements[4]. Each fluid element is then displaced according to the simple prescription

$$\boldsymbol{x} \longrightarrow \overline{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}\,,\,t)\;. \tag{6.4.9}$$

If we use the definition (6.4.6) and relation (6.4.1), which give the relation between the Lagrangian and Eulerian variations in some quantity Q, one finds:

$$\boxed{\Delta Q = \delta Q + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})Q\;.} \tag{6.4.10}$$

This is the connection between the Lagrangian and Eulerian variation in a small-amplitude wave, neglecting terms of order $|\boldsymbol{\xi}|^2$ and higher. The quantity Q can be a scalar, vector or tensor. We will now use these relations to systematically calculate the velocity, density and pressure perturbations that induced by the wave motion. There are other methods to do this, but they tend to be *ad hoc* and not generally valid.

[4]From this point onwards, I will write $\boldsymbol{x}$ rather than $\boldsymbol{x}_0$ for the unperturbed position of a fluid element.

The velocity perturbation

We can apply the relations derived in the previous Section immediately to calculate the velocity perturbation induced by the wave. The Lagrangian velocity perturbation equals

$$\begin{aligned} \Delta \boldsymbol{V} &\equiv \Delta\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right) = \frac{\mathrm{d}\,\Delta\boldsymbol{x}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{\xi}\ . \end{aligned} \tag{6.4.11}$$

The Eulerian velocity variation seen by a fixed observer now follows from (6.4.1) as:

$$\begin{aligned} \delta \boldsymbol{V} &= \Delta\boldsymbol{V} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{V} \\ &= \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{V}\ . \end{aligned} \tag{6.4.12}$$

These relations simplify considerably in the case where the fluid is globally at rest so that $\boldsymbol{V} = \boldsymbol{0}$. In that case one has $\Delta\boldsymbol{V} = \delta\boldsymbol{V} = \partial\boldsymbol{\xi}/\partial t$. Note that we consistently neglect all higher order terms $\propto |\boldsymbol{\xi}|^2\ ,\ |\boldsymbol{\xi}|^3\ \cdots$.

The density perturbation

The density change follows from a simple argument of mass conservation quite similar to the one used to derive the continuity equation in Chapter 2.6. Consider different fluid elements, their unperturbed position separated by an infinitesimal vector $\mathrm{d}\boldsymbol{x}$, which we write in component form as

$$\mathrm{d}\boldsymbol{x} \equiv (\mathrm{d}x_1\ ,\ \mathrm{d}x_2\ ,\ \mathrm{d}x_3)\ . \tag{6.4.13}$$

The wave motion (6.4.9) transports each fluid element to a new position according to

$$x_i \longrightarrow \overline{x}_i = x_i + \xi_i(\boldsymbol{x}\ ,\ t) \quad \text{for } i = 1\ ,\ 2\ ,\ 3\ . \tag{6.4.14}$$

This means that the vector $\mathrm{d}\boldsymbol{x}$ is stretched and tilted according to the prescription

$$\mathrm{d}x_i \longrightarrow \mathrm{d}\overline{x}_i = \frac{\partial \overline{x}_i}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial \overline{x}_i}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial \overline{x}_i}{\partial x_3}\,\mathrm{d}x_3 \qquad (6.4.15)$$

By using the *Einstein summation convention*, where a summation is implied whenever an index is repeated, we can write $\mathrm{d}\overline{x}_i$ as

$$\mathrm{d}\overline{x}_i = \frac{\partial \overline{x}_i}{\partial x_j}\,\mathrm{d}x_j \equiv D_{ij}\,\mathrm{d}x_j \ . \qquad (6.4.16)$$

In this expression the summation is over the index j for $j = 1\ ,\ 2\ ,\ 3$. The quantity $D_{ij} \equiv \partial\overline{x}_i/\partial x_j$ is a tensor, the so-called *deformation tensor*. This tensor contains in principle all the information needed to calculate how the vector $\mathrm{d}\boldsymbol{x}$ connecting two neighbouring points is changed as a result of the fluid motion. Using (6.4.14) one can calculate the components of this tensor:

$$D_{ij} = \frac{\partial \overline{x}_i}{\partial x_j} = \delta_{ij} + \frac{\partial \xi_i}{\partial x_j} \ . \qquad (6.4.17)$$

In matrix form this corresponds to

$$\boldsymbol{D} = \begin{pmatrix} 1 + \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_3} \\ \dfrac{\partial \xi_2}{\partial x_1} & 1 + \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_3} \\ \dfrac{\partial \xi_3}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_2} & 1 + \dfrac{\partial \xi_3}{\partial x_3} \end{pmatrix} . \qquad (6.4.18)$$

This tensor generally is a function of position and time.

Now consider the infinitesimal volume $\mathrm{d}\mathcal{V}$ defined by the three infinitesimal vectors $\mathrm{d}\boldsymbol{X} \equiv (\mathrm{d}X\ ,\ 0\ ,\ 0)$, $\mathrm{d}\boldsymbol{Y} \equiv (0\ ,\ \mathrm{d}Y\ ,\ 0)$ and $\mathrm{d}\boldsymbol{Z} \equiv (0\ ,\ 0\ ,\ \mathrm{d}Z)$ that all connect to neighbouring fluid elements. The infinitesimal volume enclosed by these three vectors is given by the general rule (2.6.6):

$$\mathrm{d}\mathcal{V} = \mathrm{d}\boldsymbol{X} \cdot (\mathrm{d}\boldsymbol{Y} \times \mathrm{d}\boldsymbol{Z}) = \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \ . \qquad (6.4.19)$$

Each of these three vectors changes as a result of the wave motion, in a manner described by recipe (6.4.16).

For instance, the infinitesimal vector $\mathrm{d}\boldsymbol{X} = (\mathrm{d}X\ ,\ 0\ ,\ 0)$ becomes:

$$\mathrm{d}\overline{\boldsymbol{X}} = \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{X} = \left(1 + \frac{\partial \xi_1}{\partial x_1}\ ,\ \frac{\partial \xi_2}{\partial x_1}\ ,\ \frac{\partial \xi_3}{\partial x_1}\right)\ \mathrm{d}X\ . \tag{6.4.20}$$

The first component is along the unperturbed vector, and corresponds to a change of length of the vector, which increases when $\partial\xi_1/\partial x_1 > 0$ or decreases when $\partial\xi_1/\partial x_1 < 0$. The other two components are in the direction perpendicular to the unperturbed vector. For that reason they correspond to a *rotation* of the vector that changes the orientation $\mathrm{d}\overline{\boldsymbol{X}}$ with respect to $\mathrm{d}\boldsymbol{X}$. This is illustrated in the figure below. Similar expressions can be written down for $\mathrm{d}\overline{\boldsymbol{Y}}$ and $\mathrm{d}\overline{\boldsymbol{Z}}$:

$$\mathrm{d}\overline{\boldsymbol{Y}} = \left(\frac{\partial \xi_1}{\partial x_2}\ ,\ 1 + \frac{\partial \xi_2}{\partial x_2}\ ,\ \frac{\partial \xi_3}{\partial x_2}\right)\ \mathrm{d}Y\quad ,\quad \mathrm{d}\overline{\boldsymbol{Z}} = \left(\frac{\partial \xi_1}{\partial x_3}\ ,\ \frac{\partial \xi_2}{\partial x_3}\ ,\ 1 + \frac{\partial \xi_3}{\partial x_3}\right)\ \mathrm{d}Z\ . \tag{6.4.21}$$

The volume enclosed by the new separation vectors $\mathrm{d}\overline{\boldsymbol{X}}$, $\mathrm{d}\overline{\boldsymbol{Y}}$ and $\mathrm{d}\overline{\boldsymbol{Z}}$ is

$$\mathrm{d}\overline{\mathcal{V}} = \mathrm{d}\overline{\boldsymbol{X}} \cdot \left(\mathrm{d}\overline{\boldsymbol{Y}} \times \mathrm{d}\overline{\boldsymbol{Z}}\right)\ . \tag{6.4.22}$$

Let us write this in component form, using the totally anti-symmetric *Levi-Cevita tensor* ϵ_{ijk} which is defined by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for } i\ j\ k \text{ an even permutation of } 1\ 2\ 3; \\ -1 & \text{for } i\ j\ k \text{ an uneven permutation of } 1\ 2\ 3; \\ 0 & \text{if any of the } i\ j\ k \text{ have the same value} \end{cases}\ . \tag{6.4.23}$$

This definition implies $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$, and all other components vanish. In terms of this tensor, the components of the cross product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be written as (remember the summation convention!)

$$(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk}\ A_j B_k\ . \tag{6.4.24}$$

The volume-element (6.4.22) expressed in component notation is

$$\mathrm{d}\overline{\mathcal{V}} = \epsilon_{ijk}\ \mathrm{d}\overline{X}_i\ \mathrm{d}\overline{Y}_j\ \mathrm{d}\overline{Z}_k\ . \tag{6.4.25}$$

Using (6.4.20) for $\mathrm{d}\overline{\boldsymbol{X}}$ in component form, $\mathrm{d}\overline{X}_i = D_{i1}\,\mathrm{d}X$, and the corresponding expressions $\mathrm{d}\overline{Y}_i = D_{i2}\,\mathrm{d}Y$, $\mathrm{d}\overline{Z}_i = D_{i3}\,\mathrm{d}Z$, one finds:

$$\mathrm{d}\overline{\mathcal{V}} = \epsilon_{ijk}\, D_{i1}\, D_{j2}\, D_{k3}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\ . \tag{6.4.26}$$

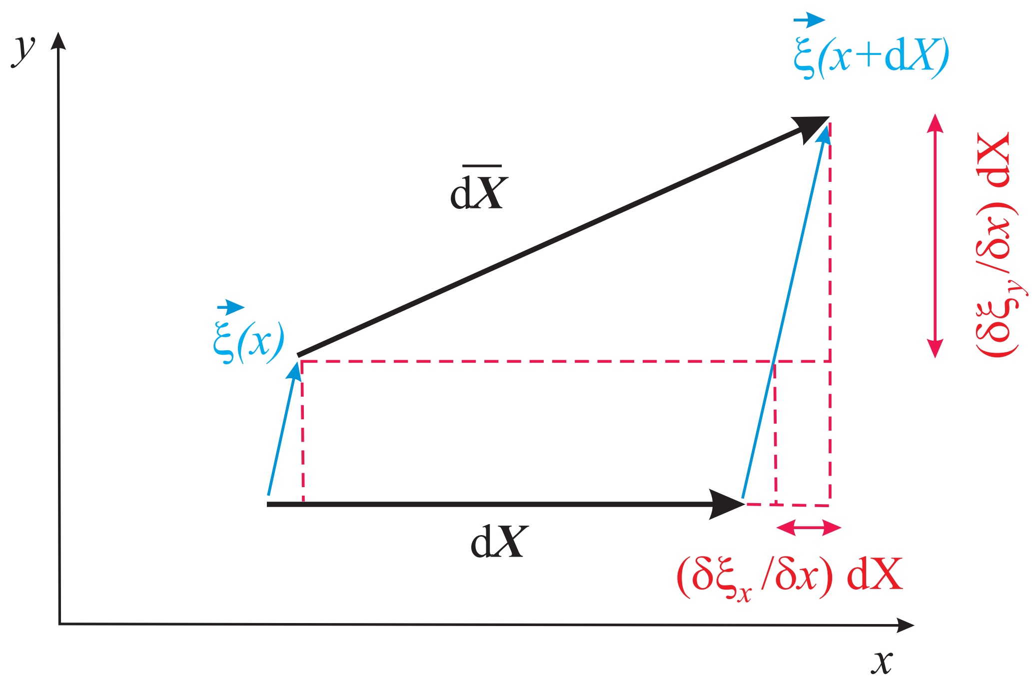


Figure 6.2: *The stretching and rotation of the infinitesimal vector* $\mathrm{d}\boldsymbol{X} = (\mathrm{d}X\,,\,0\,,\,0)$*, illustrated for two dimensions in the* x-y *plane. The change of the vector is characterized by a displacement vector* $\boldsymbol{\xi}(x\,,\,y\,,\,t)$ *at its root, and by a displacement vector* $\boldsymbol{\xi}(x + \mathrm{d}X\,,\,y, t)$ *at its tip. The difference between the* x*-components of these two displacement vectors leads to stretching of the vector* $\mathrm{d}\boldsymbol{X}$ *by an amount* $\propto (\partial\xi_x/\partial x)\,\mathrm{d}X$*, while the difference between the* y*-components,* $\xi_y(x + \mathrm{d}X) - \xi_y(x) \approx (\partial\xi_y/\partial x)\,\mathrm{d}X$*, rotates the vector away from its original orientation parallel to the* x*-axis, with a rotation angle* $\propto (\partial\xi_y/\partial x)\,\mathrm{d}X$*, all to first order in* $|\boldsymbol{\xi}|$.

The product involving the Levi-Cevita tensor and the three factors of D_{ij} is actually the determinant of the deformation tensor[5]:

$$\epsilon_{ijk}\, D_{i1}\, D_{j2}\, D_{k3} \equiv \det(\boldsymbol{D}) \equiv D(\boldsymbol{x}\,,\, t) = \left\| \begin{array}{ccc} 1 + \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_3} \\ \dfrac{\partial \xi_2}{\partial x_1} & 1 + \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_3} \\ \dfrac{\partial \xi_3}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_2} & 1 + \dfrac{\partial \xi_3}{\partial x_3} \end{array} \right\| \; . \tag{6.4.27}$$

This means that expression (6.4.26) for the volume $\mathrm{d}\overline{\mathcal{V}}$ is simply

$$\mathrm{d}\overline{\mathcal{V}} = D(\boldsymbol{x}\,,\, t)\, \mathrm{d}\mathcal{V} \; . \tag{6.4.28}$$

Writing out the determinant of the deformation tensor one finds:

$$D(\boldsymbol{x}\,,\, t) = 1 + \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3} + \text{terms of order } |\boldsymbol{\xi}|^2 \text{ and } |\boldsymbol{\xi}|^3. \tag{6.4.29}$$

Using the definition[6]

$$\frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3} = \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \; , \tag{6.4.30}$$

one has

$$D(\boldsymbol{x}\,,\, t) = 1 + \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \text{terms of order } |\boldsymbol{\xi}|^2 \text{ and } |\boldsymbol{\xi}|^3. \tag{6.4.31}$$

The perturbed volume (6.4.28) therefore equals, neglecting terms of order $|\boldsymbol{\xi}|^2$ and $|\boldsymbol{\xi}|^3$:

$$\boxed{\mathrm{d}\overline{\mathcal{V}} = (1 + \boldsymbol{\nabla} \cdot \boldsymbol{\xi})\; \mathrm{d}\mathcal{V} \; .} \tag{6.4.32}$$

[5] This is easily checked by fully writing out the product.

[6] Simply associate x_1 with the x-coordinate, x_2 with the y-coordinate and x_3 with the z-coordinate.

The density change follows from the conservation of the mass contained in the volume,

$$\mathrm{d}m = \rho\,\mathrm{d}\mathcal{V} = \overline{\rho}\,\mathrm{d}\overline{\mathcal{V}} = \text{constant} . \tag{6.4.33}$$

This implies:

$$\overline{\rho} = \rho \left(\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\overline{\mathcal{V}}}\right) . \tag{6.4.34}$$

Using (6.4.32) one can express the new density in terms of the old and $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$:

$$\overline{\rho} = \frac{\rho}{(1+\boldsymbol{\nabla}\cdot\boldsymbol{\xi})} \approx \rho\ (1-\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) . \tag{6.4.35}$$

Here I have used the approximation $(1+\eta)^{-1} \approx 1-\eta+\mathcal{O}(\eta^2)$ that is valid for $|\eta| \ll 1$.

The Lagrangian variation of the density is by definition

$$\Delta\rho = \overline{\rho} - \rho = -\rho\ (\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) . \tag{6.4.36}$$

The Eulerian density perturbation follows from Eqn. (6.4.10): [7]

$$\begin{aligned} \delta\rho &= -\rho\ (\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\,\rho \\ &= -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) . \end{aligned} \tag{6.4.37}$$

[7] Here I use the vector identity $f(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})f = \boldsymbol{\nabla}\cdot(f\boldsymbol{A})$.

The pressure variations ΔP and δP

We consider an adiabatic gas without external heat sources or heat sinks. This means that the pressure must obey the adiabatic gas law $P \propto \rho^{\gamma}$ for a given fluid element. Then the pressure depends only on the density, and we can calculate the pressure change following a fluid element from the density change. The Lagrangian pressure perturbation, $\Delta P = \overline{P}(\boldsymbol{x} + \boldsymbol{\xi} \, , \, t) - P(\boldsymbol{x} \, , \, t)$, therefore equals

$$\Delta P = \left(\frac{\partial P}{\partial \rho} \right) \, \Delta \rho = -\gamma P \, \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) \, . \tag{6.4.38}$$

The Eulerian pressure perturbation $\delta P = \overline{P}(\boldsymbol{x}) - P(\boldsymbol{x})$ follows in the now familiar fashion:

$$\delta P = -\gamma P \, \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) - \left(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} \right) P \, . \tag{6.4.39}$$

The table on the following page collects all the results we have derived in this Section for the perturbations that are associated with a small-amplitude wave with displacement vector $\boldsymbol{\xi}(\boldsymbol{x} \, , \, t)$. In the Box below, the principles behind this derivation are illustrated for the much simpler case of a one-dimensional flow, where one can (temporarily) forget about the vector character of the displacement $\boldsymbol{\xi}$.

Perturbed quantities in a linear adiabatic wave

Quantity	**Lagrangian perturbation**	**Eulerian perturbation**
Position $\boldsymbol{x}$	$\Delta \boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{x}\,,\,t)$	$\delta \boldsymbol{x} = \mathbf{0}$ (by definition!)
Velocity $\boldsymbol{V}(\boldsymbol{x}\,,\,t)$	$\Delta \boldsymbol{V} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\xi}$	$\delta \boldsymbol{V} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{V}$
Density $\rho(\boldsymbol{x}\,,\,t)$	$\Delta \rho = -\rho\ (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})$	$\delta \rho = -\rho\ (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\,\rho$
		$= -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{\xi})$
Pressure $P(\boldsymbol{x}\,,\,t)$	$\Delta P = -\gamma P\ (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})$	$\delta P = -\gamma P\ (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\,P$

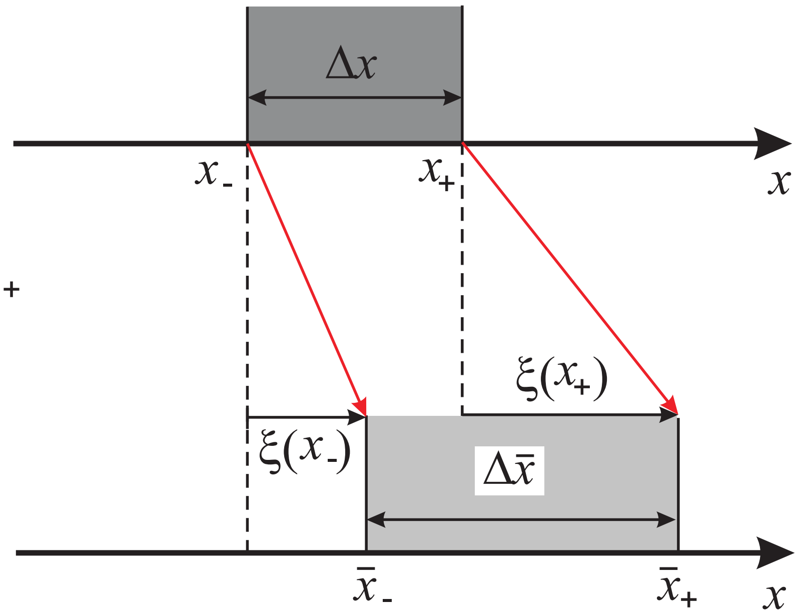


Figure 6.3: *A volume-element with width Δx is stretched as a result of the difference between the displacement $\xi(x_-)$ at the trailing edge, and the displacement $\xi(x_+)$ at the leading edge. Due to these displacements, the new width equals $\Delta \bar{x}$. The example shown is for the case of expansion where $\partial\xi/\partial x > 0$ so that $\Delta\bar{x} > \Delta x$. The opposite case (where $\partial\xi/\partial x < 0$, not shown) would compress the volume-elements so that $\Delta\bar{x} < \Delta x$.*

The one-dimensional case

The derivation of the Lagrangian density change $\Delta\rho$ and the pressure change ΔP (and their Eulerian counterparts $\delta\rho$ and δP) given above is quite general, but also rather complicated. Some insight can be gained from the one-dimensional case, where one does not have to worry about the vector-character of the displacement. Consider a one-dimensional fluid with density $\rho(x\,,\,t)$ and pressure $P(x\,,\,t)$. The position of all fluid elements changes as a result of a perturbation (sound wave). If we label this position with an x-coordinate, we can represent the effect of the perturbation by:

$$x \longrightarrow \bar{x} \equiv x + \xi(x\,,\,t)\ . \qquad (6.4.40)$$

This defines the displacement $\xi(x\,,\,t)$ for the one-dimensional case. The role of the small 'volume' is now played by the interval Δx, see the figure above.

Consider the fluid element with its trailing edge at $x_- \equiv x$ and the leading edge at $x_+ = x_- + \Delta x$. The mass of the fluid element is

$$\Delta m = \rho \, \Delta x \ . \tag{6.4.41}$$

Due to the perturbation (6.4.40) the trailing edge of the volume changes its position from x_- to $\bar{x}_- = x_- + \xi(x_- \, , \, t)$, whereas the leading edge changes its position from x_+ to $\bar{x}_+ = x_+ + \xi(x_+ \, , \, t)$. The width of the fluid element is now equal to:

$$\begin{aligned} \Delta\bar{x} &= \bar{x}_+ - \bar{x}_- \\ &= x_+ + \xi(x_+ \, , \, t) - (x_- + \xi(x_- \, , \, t)) \ . \end{aligned} \tag{6.4.42}$$

Now using $x_- = x$ and $x_+ = x + \Delta x$ one finds:

$$\begin{aligned} \Delta\bar{x} &= \Delta x + \xi(x + \Delta x \, , \, t) - \xi(x \, , \, t) \\ &\approx \Delta x + \frac{\partial \xi}{\partial x} \, \Delta x \ . \end{aligned} \tag{6.4.43}$$

Here I have used the fact that Δx is infinitesimally small. One concludes that the new and the old 'volume' are related by

$$\Delta\bar{x} = \left(1 + \frac{\partial \xi}{\partial x}\right) \Delta x \ . \tag{6.4.44}$$

This is the one-dimensional analogue of relation (6.4.32). Note that the fluid element is compressed (so that $\Delta\bar{x} < \Delta x$) when $\partial\xi/\partial x < 0$, and expands (so that $\Delta\bar{x} > \Delta x$) in the case $\partial\xi/\partial x > 0$.

Mass conservation (Δm = constant) now reads $\rho \, \Delta x = \bar{\rho} \, \Delta\bar{x}$, so the new density is

$$\bar{\rho} = \rho \, \frac{\Delta x}{\Delta\bar{x}} \ . \tag{6.4.45}$$

Using (6.4.44) one has

$$\bar{\rho} = \frac{\rho}{1 + \dfrac{\partial \xi}{\partial x}} \approx \rho \left(1 - \frac{\partial \xi}{\partial x} \right) , \tag{6.4.46}$$

where I have assumed that $|\xi|$ is small compared with the wavelength λ of the perturbation, which implies that $|\partial\xi/\partial x| \sim |\xi|/\lambda$ is much smaller than unity.

The new density $\bar{\rho}$ is the density in the displaced fluid element, which is now at a position $\bar{x} = x + \xi$. So we should write relation (6.4.46) more precisely as:

$$\bar{\rho}(x + \xi \ , \ t) = \rho(x \ , \ t) \left(1 - \frac{\partial \xi}{\partial x} \right) . \tag{6.4.47}$$

This defines the *Lagrangian* density perturbation as

$$\Delta\rho = \bar{\rho}(x + \xi \ , \ t) - \rho(x \ , \ t) = -\rho(x \ , \ t) \left(\frac{\partial \xi}{\partial x} \right) . \tag{6.4.48}$$

This is the one-dimensional version of relation (6.4.36).

The density at the old (unperturbed) position follows from using (for small ξ)

$$\bar{\rho}(x + \xi \ , \ t) \approx \bar{\rho}(x \ , \ t) + \xi \left(\frac{\partial \rho}{\partial x} \right) . \tag{6.4.49}$$

Note that I have replaced $\partial\bar{\rho}/\partial x$ by $\partial\rho/\partial x$, which is allowed since the difference between ρ and $\bar{\rho}$ (and the two density derivatives) is of order $|\xi|$, and can be neglected since we are only considering terms *linear* in ξ in relation (6.4.49). Substituting this into relation (6.4.47) and re-ordering terms one finds:

$$\bar{\rho}(x \ , \ t) = \rho(x \ , \ t) \left(1 - \frac{\partial \xi}{\partial x} \right) - \xi \left(\frac{\partial \rho}{\partial x} \right) . \tag{6.4.50}$$

The *Eulerian* density perturbation is (by definition) the difference between the new and the old density at the old (unperturbed) position. It follows from the previous relation as

$$\delta\rho = \bar{\rho}(x\;,\;t) - \rho(x\;,\;t) = -\rho\,\left(\frac{\partial \xi}{\partial x}\right) - \xi\,\left(\frac{\partial \rho}{\partial x}\right)\;. \tag{6.4.51}$$

This result for $\delta\rho$ can be written more compactly as

$$\delta\rho = -\frac{\partial}{\partial x}\left(\rho\,\xi\right)\;. \tag{6.4.52}$$

This is the one-dimensional version of Eqn. (6.4.37).

In the special case of a *uniform* mass density, where $\partial\rho/\partial x = 0$ everywhere in the unperturbed fluid, there is no difference between the Eulerian and Lagrangian density perturbations:

$$\delta\rho = \Delta\rho = -\rho\,\left(\frac{\partial \xi}{\partial x}\right) \quad \textbf{(uniform fluid only!)} \tag{6.4.53}$$

In the general case $\Delta\rho$ and $\delta\rho$ do not coincide.

The pressure perturbation due to the displacement can be calculated in much the same manner. For an adiabatic gas, where

$$P(\rho) \propto \rho^{\gamma}\;, \tag{6.4.54}$$

we can use (6.4.45) to write:

$$\bar{P}(x+\xi\;,\;t) = P(x\;,\;t)\,\left(\frac{\bar{\rho}}{\rho}\right)^{\gamma} = P(x\;,\;t)\,\left(\frac{\Delta x}{\Delta\bar{x}}\right)^{\gamma}\;. \tag{6.4.55}$$

Using (6.4.44) we have

$$\bar{P}(x+\xi\ ,\ t) = P(x\ ,\ t)\ \left(1 + \frac{\partial \xi}{\partial x}\right)^{-\gamma}\ . \tag{6.4.56}$$

Using $|\partial\xi/\partial x| \ll 1$ we can approximate this by:

$$\bar{P}(x+\xi\ ,\ t) = P(x\ ,\ t)\ \left(1 - \gamma\frac{\partial \xi}{\partial x}\right)\ . \tag{6.4.57}$$

The Lagrangian perturbation of the pressure follows immediately:

$$\Delta P \equiv \bar{P}(x+\xi\ ,\ t) - P(x\ ,\ t) = -\gamma P\ \left(\frac{\partial \xi}{\partial x}\right)\ . \tag{6.4.58}$$

The Eulerian perturbation can be found using (compare Eqn. 6.4.49)

$$\bar{P}(x+\xi\ ,\ t) \approx \bar{P}(x\ ,\ t) + \xi\ \left(\frac{\partial P}{\partial x}\right)\ . \tag{6.4.59}$$

Upon substitution of this relation into (6.4.57), and after a re-arrangement of terms, one finds:

$$\delta P \equiv \bar{P}(x\ ,\ t) - P(x\ ,\ t) = -\gamma P\ \left(\frac{\partial \xi}{\partial x}\right) - \xi\ \left(\frac{\partial P}{\partial x}\right)\ . \tag{6.4.60}$$

Only if the pressure gradient vanishes in the unperturbed fluid, so that $\partial P/\partial x = 0$ everywhere, do the Lagrangian and the Eulerian pressure perturbations coincide:

$$\Delta P = \delta P = -\gamma P\ \left(\frac{\partial \xi}{\partial x}\right) \quad \text{(\textbf{uniform} fluid only!)} \tag{6.4.61}$$

In three dimensions Eqn. (6.4.58) becomes Eqn. (6.4.38), and (6.4.60) becomes (6.4.39).

6.5 Sound waves

The results derived in the previous Section allow us to calculate the properties of an adiabatic sound wave propagating in a stationary, homogeneous fluid. We assume that $\boldsymbol{V} = 0$ everywhere and that avarage density ρ and average pressure P are are independent of position. Because of that assumption, and the fact that the unperturbed fluid is stationary, there is no difference between the linear Lagrangian variations and the Eulerian variations:

$$\text{homogeneous fluid:} \Longleftrightarrow \delta Q = \Delta Q, \tag{6.5.1}$$

a relation that is valid for any quantity $Q(\boldsymbol{x}\ ,\ t)$ in the fluid.

We introduce the small displacement $\Delta \boldsymbol{x} \equiv \boldsymbol{\xi}(\boldsymbol{x}\ ,\ t)$ of a fluid element, due to the presence of a sound wave, that takes the form (6.3.1),

$$\boldsymbol{\xi}(\boldsymbol{x}\ ,\ t) = \boldsymbol{a} \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t\right) + \mathrm{cc}\ . \tag{6.5.2}$$

Pressure and density fluctuations induced by this wave satisfy

$$\Delta\rho = \delta\rho = -\rho\left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}\right) \quad , \quad \Delta P = \delta P = -\gamma P\left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}\right)\ . \tag{6.5.3}$$

The velocity induced by the wave equals

$$\delta \boldsymbol{V} = \Delta \boldsymbol{V} = \frac{\partial \boldsymbol{\xi}}{\partial t}\ . \tag{6.5.4}$$

From the properties of the exponential function,

$$\begin{aligned} \frac{\partial}{\partial t}\left[\exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t\right)\right] &= -i\omega \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t\right)\ , \\ \frac{\partial}{\partial x_i}\left[\exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t\right)\right] &= ik_i \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t\right)\ , \end{aligned} \tag{6.5.5}$$

we can calculate the velocity perturbation and the density- and pressure perturbations in terms of $\boldsymbol{\xi}$ by using (6.5.3) and (6.5.2):

$$\begin{pmatrix} \delta \boldsymbol{V}(\boldsymbol{x}\,,\,t) \\ \\ \delta\rho(\boldsymbol{x}\,,\,t) \\ \\ \delta P(\boldsymbol{x}\,,\,t) \end{pmatrix} = \begin{pmatrix} -i\omega\boldsymbol{a} \\ \\ -\rho\, i(\boldsymbol{k}\cdot\boldsymbol{a}) \\ \\ -\gamma P\, i(\boldsymbol{k}\cdot\boldsymbol{a}) \end{pmatrix} \times\ \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc}\ . \tag{6.5.6}$$

This incidently shows that $\tilde{\rho} = -i\rho\,(\boldsymbol{k}\cdot\boldsymbol{a})$ and that $\tilde{P} = -i\gamma P\,(\boldsymbol{k}\cdot\boldsymbol{a})$. The only missing ingredient is an equation of motion that links the velocity $\delta\boldsymbol{V} = \partial\boldsymbol{\xi}/\partial t$ to the density and pressure perturbations. Consider the equation of motion for the gas:

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\frac{1}{\rho}\,\boldsymbol{\nabla}P\ . \tag{6.5.7}$$

From the Lagrangian perturbation of the left-hand-side of this equation we obtain the acceleration of the fluid elements due to the wave. For this acceleration term we can use the fact that taking the Lagrangian variation Δ and the comoving time derivative $\mathrm{d}/\mathrm{d}t$ commute. Using (6.5.4) one finds:

$$\Delta\left(\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right) = \frac{\mathrm{d}\,\Delta\boldsymbol{V}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} = \frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\ . \tag{6.5.8}$$

In the last equality I have used that the *unperturbed* velocity vanishes: $\boldsymbol{V} = 0$.

The Lagrangian perturbation of the right-hand-side gives the pressure force per unit mass due to the waves. This term can be evaluated using [1] the fact that we have assumed that both the unperturbed pressure P and the unperturbed density ρ are constant everywhere, and [2] by applying (6.5.1) and the properties listed in Eqn. (6.4.7):

$$\begin{aligned} \Delta\left(\frac{1}{\rho}\,\boldsymbol{\nabla}P\right) &= \frac{1}{\rho}\,\Delta\left(\boldsymbol{\nabla}P\right) \quad \text{(as } \boldsymbol{\nabla}P = 0 \text{ in the } \textit{unperturbed} \text{ fluid)} \\ &= \frac{1}{\rho}\,\delta\left(\boldsymbol{\nabla}P\right) \quad \text{(as } \Delta = \delta \text{ in a homogeneous fluid)} \\ &= \frac{1}{\rho}\,\boldsymbol{\nabla}\,\delta P \qquad \text{(as } \delta(\boldsymbol{\nabla}P) = \boldsymbol{\nabla}\delta P.\text{)} \end{aligned} \tag{6.5.9}$$

The steps taken in this last derivation are only true for the *linear* perturbations. The perturbed version of the equation of motion obtained in this fashion is the equation that governs the perturbations due to sound waves:

$$\begin{aligned} \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\frac{1}{\rho}\, \boldsymbol{\nabla}\, \delta P \\ &= \frac{\gamma P}{\rho}\, \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) \; . \end{aligned} \tag{6.5.10}$$

Here I have substituted expression (6.5.3) for δP. The relation

$$\frac{\gamma P}{\rho} \equiv C_{\rm s}^2 \tag{6.5.11}$$

defines the *adiabatic sound speed* $C_{\rm s}$. One can write (6.5.10) as a *wave equation* in three dimensions:

$$\boxed{\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - C_{\rm s}^2\, \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) = 0 \; .} \tag{6.5.12}$$

In conclusion: in order to find the equation of motion for the displacement vector $\boldsymbol{\xi}(\boldsymbol{x}\, , \, t)$ one has to perturb and linearize the equation of motion, expressing all quantities (such as the velocity and pressure perturbations) in terms of $\boldsymbol{\xi}$ and its derivatives and ruthlessly dropping all terms that are quadratic (or higher order) in $\boldsymbol{\xi}$.

If we now substitute the plane wave assumption (6.5.2) for $\boldsymbol{\xi}$ and make use of the properties of the exponential factor, this equation is converted into a set of linear *algebraic* equations for the amplitude $\boldsymbol{a}$ [8], given ω and $\boldsymbol{k}$:

$$\boxed{\omega^2\, \boldsymbol{a} - C_{\rm s}^2\, (\boldsymbol{k} \cdot \boldsymbol{a})\, \boldsymbol{k} = 0 \; .} \tag{6.5.13}$$

[8] There is a similar equation for the complex conjugate $\boldsymbol{a}^*$, but that equation does not contain any new information: it is simply the complex conjugate of the equation for $\boldsymbol{a}$. We can therefore safely ignore it in what follows, as I show in more detail below.

In order to simplify the algebra, assume that the sound wave propagates in the $x - y$ plane so that $\boldsymbol{k} = (k_x \; , \; k_y \; , \; 0)$. In that case we have

$$\boldsymbol{k} \cdot \boldsymbol{a} = k_x a_x + k_y a_y \; .$$

It is always possible to define your coordinate system in such a way that this choice is valid, as long as one is dealing with *plane* waves.

By writing out the three spatial components of equation (6.5.13) explicitly we get three coupled, linear algebraic equations for a_x, a_y and a_z that can be represented in matrix form:

$$\begin{pmatrix} \omega^2 - k_x^2 C_{\rm s}^2 & -k_x k_y \, C_{\rm s}^2 & 0 \\ -k_y k_x \, C_{\rm s}^2 & \omega^2 - k_y^2 C_{\rm s}^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = 0 \; . \tag{6.5.14}$$

Matrix algebra[9] tells us that there are only non-trivial solutions, i.e. solutions where the a_i do not all vanish, when the determinant of the 3×3 matrix M_{ij} in (6.5.14),

$$M_{ij} \equiv \omega^2 \, \delta_{ij} - C_{\rm s}^2 k_i k_j \; , \tag{6.5.15}$$

vanishes. This determinant equals

$$\det \left(M_{ij} \right) = \omega^2 \, \left\{ \left(\omega^2 - k_x^2 C_{\rm s}^2 \right) \left(\omega^2 - k_y^2 C_{\rm s}^2 \right) - \left(k_x k_y \, C_{\rm s}^2 \right)^2 \right\} \; . \tag{6.5.16}$$

Re-ordering terms, and putting the determinant equal to zero, yields a relation between wave frequency ω and the wave number $\boldsymbol{k}$, the so-called *dispersion relation*. For sound waves in a stationary fluid or gas this dispersion relation is

$$\boxed{\omega^4 \, \left(\omega^2 - k^2 \, C_{\rm s}^2 \right) = 0 \; ,} \tag{6.5.17}$$

with $k^2 = k_x^2 + k_y^2$.

[9] e.g. G.B. Arfken & H.J. Weber, 2005: *Mathematical Methods for Physicists*, Sixth Edition, Elsevier Academic Press, Chapter 3.

There are two types of solutions: the solution $\omega = 0$ does not really correspond with a wave: the corresponding amplitude does not vary in time. Strictly speaking, this solution should be discarded for this reason.

The remaining two solutions correspond to a positive- and a negative frequency sound wave:

$$\omega(\boldsymbol{k}) = +kC_{\rm s} \quad , \quad \omega(\boldsymbol{k}) = -kC_{\rm s} \; , \tag{6.5.18}$$

with $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2}$. The frequency of the sound waves depends only on the sound speed and the magnitude of the wave vector, but **not** on the direction of $\boldsymbol{k}$! This means that sound waves in a stationary fluid propagate with equal velocity in all directions. There is no preferred direction. We will see below that this is no longer true for sound waves in a *moving* fluid. In that case, the direction of the fluid velocity $\boldsymbol{V}$ introduces a preferred direction.

Using the three possible solutions for ω in the original equations one can determine the corresponding *eigenvectors*. It is easily checked that the solution $\omega = 0$ must have $a_x = a_y = 0$ and $a_z \neq 0$ or $a_x/a_y = -k_y/k_x$ and $a_z = 0$. In both cases $\boldsymbol{a} \perp \boldsymbol{k}$. This can also be seen directly from (6.5.13): if we substitute $\omega = 0$ it reduces to $C_{\rm s}^2(\boldsymbol{k} \cdot \boldsymbol{a})\boldsymbol{k} = 0$, which has the solution $\boldsymbol{k} \cdot \boldsymbol{a} = 0$.

Sound waves on the other hand must have

$$a_x/a_y = k_x/k_y \quad , \quad a_z = 0 \; . \tag{6.5.19}$$

This implies that the sound wave amplitude and the wave vector must be parallel:

$$\boldsymbol{a}_{\rm sound} \parallel \boldsymbol{k} \; . \tag{6.5.20}$$

Sound waves are compressive *longitudinal waves*. The main properties of a sound wave are illustrated in the figure below.

Now that we know the frequency and the polarization of the sound wave, we can immediately write down the relation between the amplitude $|\boldsymbol{\xi}| = \sqrt{2\boldsymbol{a} \cdot \boldsymbol{a}^*}$ of the wave, and the velocity, density and pressure perturbations. From (6.5.6) and (6.5.20) one finds:

$$
\begin{aligned}
|\delta \boldsymbol{V}| &= C_{\rm s}\, k|\boldsymbol{\xi}| \; , \\
|\delta\rho| &= \rho\, k|\boldsymbol{\xi}| = \rho\, \frac{|\delta \boldsymbol{V}|}{C_{\rm s}} \; , \\
|\delta P| &= \gamma P\, k|\boldsymbol{\xi}| = \gamma P\, \frac{|\delta\rho|}{\rho} \; .
\end{aligned}
\tag{6.5.21}
$$

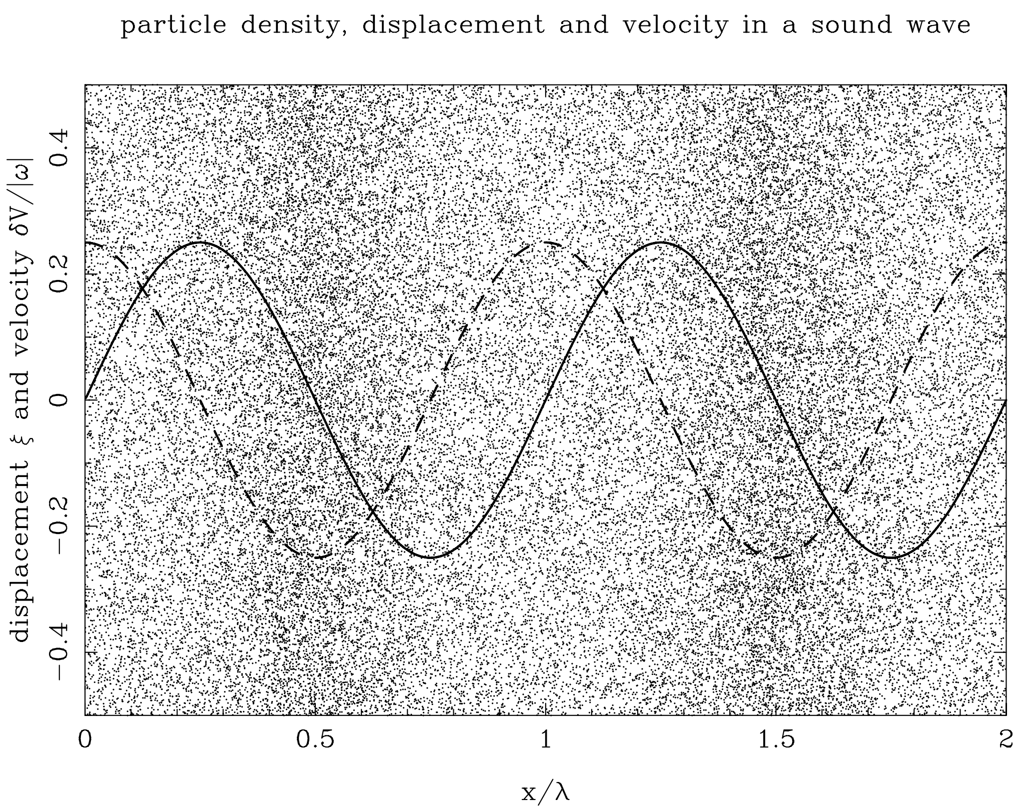


Figure 6.4: *The density ρ, displacement ξ and velocity δV in a sound wave of wavelength λ and frequency ω propagating in the x-direction. This figure shows a 'snapshot' of the wave, the density represented by the position of a large number of 'test-particles' carried passively along by the flow, the displacement by a solid sinusoidal curve, and the velocity is represented by $\delta V/|\omega|$: the dashed curve. Note that with this scaling, the velocity curve has the same amplitude as the displacement curve, (see Eqn. 6.5.6) but is shifted by $\lambda/2$, i.e. the velocity curve is 90^o out of phase. Note that the density is largest at those locations where where the displacement derivative satisfies $\partial\xi/\partial x < 0$ and simultaneously $\xi = 0$.*

What about the complex conjugate?

This derivation treats the algebra resulting from the plane wave assumption,

$$\boldsymbol{\xi}(\boldsymbol{x}\,,\,t) = \boldsymbol{a}\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc}\,,$$

in a rather cavalier fashion. To justify the approach taken, i.e. converting differential equations for $\boldsymbol{\xi}$ to an algebraic equation for the amplitude $\boldsymbol{a}$, I will look at this approach in more detail, taking the case of sound waves as an example.
The partial differential equation (wave equation) for sound waves reads

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} - C_{\mathrm{s}}^2\,\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right) = 0\,.$$

Now writing the plane-wave assumption as

$$\boldsymbol{\xi}(\boldsymbol{x}\,,\,t) = \boldsymbol{a}\,e^{+iS} + \boldsymbol{a}^*e^{-iS}$$

with

$$S(\boldsymbol{x}\,,\,t) \equiv \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$$

the phase of the wave and $\boldsymbol{a}^*$ the complex conjugate of the (complex) wave amplitude, substitution of this expression into the wave equation yields:

$$\left[\omega^2\boldsymbol{a} - C_{\mathrm{s}}^2\,\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{a}\right)\right]\,e^{+iS} + \left[\omega^2\boldsymbol{a}^* - C_{\mathrm{s}}^2\,\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{a}^*\right)\right]\,e^{-iS} = 0\,.$$

This equation should be satisfied for *all* values of $\boldsymbol{x}$ and t, meaning for all values of the phase $S(\boldsymbol{x}\,,\,t)$. Since

$$e^{\pm iS} = \cos S \pm i\sin S\,,$$

the above equation can only be satisfied for all $\boldsymbol{x}$ and t if the two factors in the square brackets are **both** zero:

$$\omega^2\boldsymbol{a} - C_{\mathrm{s}}^2\,\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{a}\right) \;=\; 0\,,$$

and

$$\omega^2\boldsymbol{a}^* - C_{\mathrm{s}}^2\,\boldsymbol{k}\left(\boldsymbol{k}\cdot\boldsymbol{a}^*\right) \;=\; 0\,.$$

However, the second equation is simply the complex conjugate of the first equation (assuming that ω and $\boldsymbol{k}$ are real quantities), so it contains no new information (as $0^* = 0$). Therefore it is sufficient to solve only one of then, the equation for $\boldsymbol{a}$. If the wave frequency becomes complex, the story is a bit more complicated, but the final conclusion is the same: **in the algebraic equations resulting from the plane wave assumption we can forget the phase factor $e^{\pm iS}$ after differentiation, and the complex conjugate. In effect you only need to solve a set of equations for the components of the amplitude vector $\boldsymbol{a}$.**

6.5.1 Wave kinematics: phase- and group velocity

The propagation of the waves is characterized by two velocities: the *phase velocity* $\boldsymbol{v}_{\rm ph}$ and the *group velocity* $\boldsymbol{v}_{\rm gr}$. The phase velocity is the velocity at which points or surfaces of *constant* phase move. This phase is defined by writing Eqn. (6.5.2) as

$$\boldsymbol{\xi}(\boldsymbol{x}\,,\,t) = \boldsymbol{a}\exp\left[iS(\boldsymbol{x}\,,\,t)\,\right] + {\rm cc}\;, \tag{6.5.22}$$

where, for waves in a uniform steady fluid, the phase S is simply

$$S(\boldsymbol{x}\,,\,t) \equiv \boldsymbol{k}\cdot\boldsymbol{x} - \omega t\;.$$

The phase velocity $\boldsymbol{v}_{\rm ph}$ is defined by the requirement that an observer moving with this velocity stays on a surface of constant wave phase:

$$\left(\frac{{\rm d}S}{{\rm d}t}\right)_{\rm ph} = \frac{\partial S}{\partial t} + (\boldsymbol{v}_{\rm ph}\cdot\boldsymbol{\nabla})S = 0\;. \tag{6.5.23}$$

Since we have

$$\frac{\partial S}{\partial t} = -\omega \quad , \quad \frac{\partial S}{\partial x_i} = k_i\;, \tag{6.5.24}$$

this condition means that the phase velocity must satisfy

$$\omega(\boldsymbol{k}) - \boldsymbol{k}\cdot\boldsymbol{v}_{\rm ph} = 0\;. \tag{6.5.25}$$

The obvious choice is[10]

$$\boxed{\boldsymbol{v}_{\rm ph} = \frac{\omega(\boldsymbol{k})}{k}\,\hat{\boldsymbol{\kappa}}\;,} \tag{6.5.26}$$

with $\hat{\boldsymbol{\kappa}} \equiv \boldsymbol{k}/k$ the unit vector along the wave vector.

[10] One can always add an arbitrary velocity $\boldsymbol{v}_\perp \perp \boldsymbol{k}$ to $\boldsymbol{v}_{\rm ph}$ and still satisfy this condition. The only sensible and non-arbitrary choice however is to put this perpendicular velocity to zero.

The group velocity $\boldsymbol{v}_{\rm gr}$ is defined as the velocity with which the *wave amplitude* propagates. Its value can be determined by the following argument. For simplicity, I use a one-dimensional example.

Consider a *wave packet*, containing waves of different wavelengths, centered in a small bandwidth $\Delta k \ll k$ around some central wave number k. In that case, the displacement can be represented as an integral counting all wave numbers present in the packet[11]

$$\xi(x\ ,\ t) = \int_{-\infty}^{+\infty} \frac{{\rm d}k'}{2\pi}\,\mathcal{A}(k')\,e^{ik'x - i\omega(k')t}\ . \tag{6.5.27}$$

An example of such a superposition of waves is shown below. The typical spatial extent of the wavepacket equals $\Delta x \approx 1/\Delta k$. The differential wave amplitude (the so-called Fourier amplitude) $\mathcal{A}(k)$ satisfies

$$\mathcal{A}(k') = 0 \ \text{ for } |k' - k| \gg \Delta k\ , \tag{6.5.28}$$

i.e. $\mathcal{A}(k')$ is strongly peaked around wave number k.

The wave packet will evolve in time as the waves propagate. Everywhere along the path of the wavepacket (and at each wave number) the local dispersion relation $\omega = \omega(k)$ must be satisfied. This determines the wave frequency at some wave number $k + \Delta k$ near k as

$$\omega(k + \Delta k) \approx \omega(k) + \Delta k\,\left(\frac{\partial\omega}{\partial k}\right)\ . \tag{6.5.29}$$

Using this expansion, together with the fact that the Fourier amplitude is strongly peaked around wave number k, one can write:

$$\xi(x, t) \approx e^{ikx - i\omega(k)t} \times \underbrace{\int_{-\infty}^{+\infty} \frac{{\rm d}k'}{2\pi}\,\mathcal{A}(k')\,e^{i\Delta k\,[x - (\partial\omega/\partial k)t\,]}}_{\text{effective amplitude}}\ . \tag{6.5.30}$$

Here $\Delta k \equiv k' - k$. The integral over k' defines what can be considered as the effective amplitude of the wave packet.

[11] This is an example of a so-called *Fourier representation*. It is needed to represent a wave packet with a finite spatial size $L \sim 1/\Delta k$. In contrast, a monochromatic wave ($\Delta k = 0$) always has an **infinite** spatial extent.

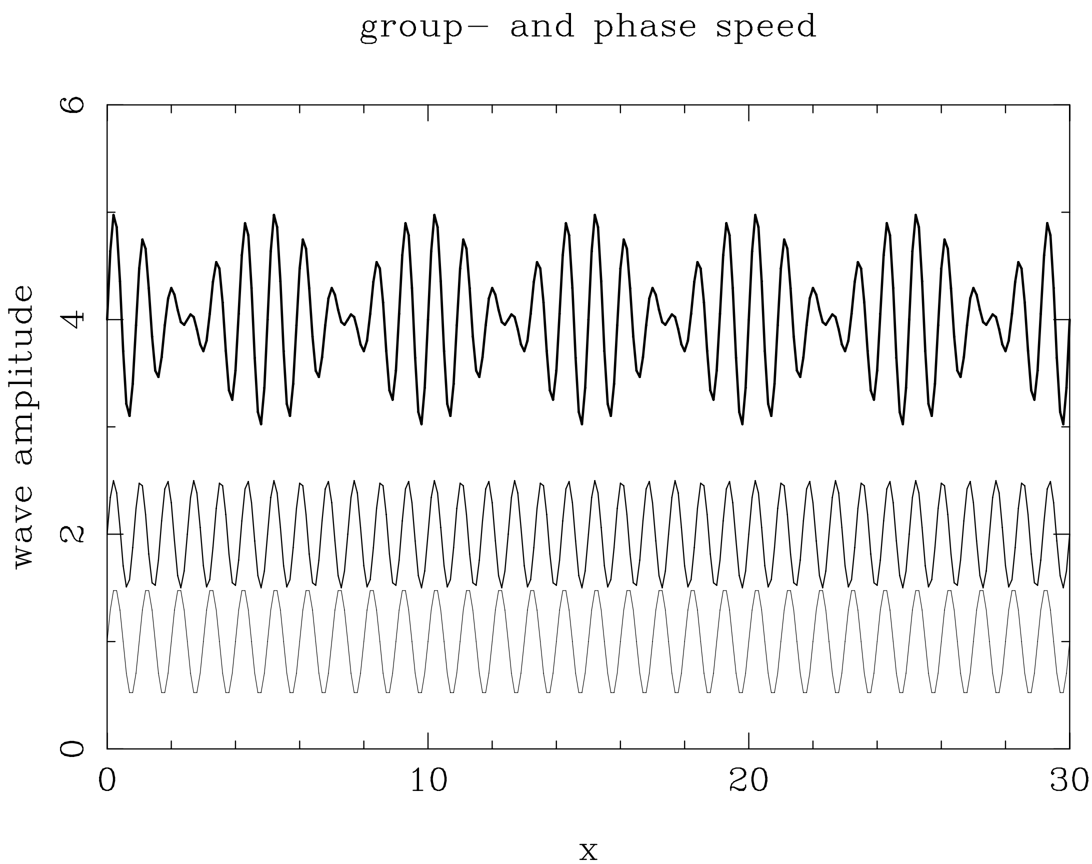


Figure 6.5: *The wave pattern that results from adding two sinusiodal waves, with a slightly different wave number k and frequency ω. These two waves are the two sinus-like curves at the bottom of the figure. Here the relation between the frequency and wave number is chosen to be of the form $\omega(k) = \sqrt{k^2c^2 + \omega_0^2}$. The two waves together interfere to form the wave shown at top. The resulting amplitude modulation in this wave travels at the group velocity. The rapid sinusoidal variation on the other hand travels at the phase speed.*

This effective amplitude will be vanishingly small due to the sinusoidal behaviour of the exponential factor in the integrand, the result of **destructive interference**, **except** at those positions where the phase factor in that exponential term vanishes:

$$x - \left(\frac{\partial \omega}{\partial k}\right) t = 0 . \tag{6.5.31}$$

At those points the different Fourier amplitudes add up, a case of **constructive interference**. Condition (6.5.31) therefore determines the position of the wave packet, and defines the group velocity in this one-dimensional example as

$$v_{\rm gr} = \left(\frac{{\rm d}x}{{\rm d}t}\right)_{\rm packet} = \frac{\partial \omega}{\partial k} . \tag{6.5.32}$$

In three dimensions this generalizes to:

$$\boxed{\boldsymbol{v}_{\rm gr} = \frac{\partial \omega(\boldsymbol{k})}{\partial \boldsymbol{k}} = \left(\frac{\partial \omega}{\partial k_x} , \frac{\partial \omega}{\partial k_y} , \frac{\partial \omega}{\partial k_z}\right) .} \tag{6.5.33}$$

For sound waves, the phase and group velocity are equal:

$$\boldsymbol{v}_{\rm ph} = \boldsymbol{v}_{\rm gr} = C_{\rm s} \hat{\boldsymbol{\kappa}} . \tag{6.5.34}$$

Such waves are said to show *no dispersion*: the amplitude and phase propagate with the same velocity, regardless the wavelength or frequency. If the sound waves in our atmosphere were not almost dispersionless, human hearing would have to be much more sophisticated to discern intelligible signals from human speech, which covers a frequency range of ~ 100 Hz to ~ 1 kHz, or to enjoy music which covers a range ~ 10 Hz to ~ 20 kHz.

6.6 Sound waves in a moving fluid

Now consider sound waves propagating in a moving medium with velocity $\boldsymbol{V}$. If we assume that the wavelength of the waves concerned is much smaller than the scale on which this velocity changes, and that the wave period is much shorter than the timescale on which the temporal variation of $\boldsymbol{V}$ occurs, we may treat this situation (to lowest order) as a case where the fluid velocity is constant and uniform. In that approximation, the relation between the displacement vector and Lagrangian velocity perturbation is

$$\Delta \boldsymbol{V} = \left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \right) \boldsymbol{\xi} \; ,$$

while the Lagrangian perturbation of the acceleration is

$$\frac{\mathrm{d}\Delta \boldsymbol{V}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \right)^2 \boldsymbol{\xi} \; .$$

The only difference with the case treated above, where the fluid was at rest, is a consistent replacement of the time derivatives:

$$\frac{\partial}{\partial t} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}), \tag{6.6.1}$$

The ordinary time derivative is replaced by the comoving derivative in the unperturbed flow. The density- and pressure variations depend only on the spatial derivatives of $\boldsymbol{\xi}$, and remain unchanged, e.g.

$$\Delta P = \delta P = -\gamma P \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) \; .$$

Therefore, we can immediately write down the wave equation for sound waves in a moving fluid that corresponds to Eqn. (6.5.12), which is valid in a stationary fluid:

$$\left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \right)^2 \boldsymbol{\xi} - C_{\mathrm{s}}^2 \, \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) = 0 \; . \tag{6.6.2}$$

If we now again assume a plane wave solution,

$$\boldsymbol{\xi}(\boldsymbol{x} \, , \, t) = \boldsymbol{a} \exp \left(i \boldsymbol{k} \cdot \boldsymbol{x} - i \omega t \right) + \mathrm{cc} \; , \tag{6.6.3}$$

it is easily checked that we find essentially the same dispersion relation as before,

$$\left(\omega - \boldsymbol{k} \cdot \boldsymbol{V} \right)^2 \, \boldsymbol{a} - C_{\mathrm{s}}^2 \left(\boldsymbol{k} \cdot \boldsymbol{a} \right) \boldsymbol{k} = 0 \; , \tag{6.6.4}$$

except for the replacement

$$\omega \implies \omega - \boldsymbol{k} \cdot \boldsymbol{V} \equiv \tilde{\omega} , \tag{6.6.5}$$

i.e. the wave frequency ω is replaced by the *Doppler-shifted* frequency $\tilde{\omega}$, which corresponds to the frequency of the wave seen by an observer moving with the fluid, i.e. the frequency in the *fluid rest frame*. This is a simple consequence of replacement rule (6.6.1), which implies that the time derivative of the displacement vector $\boldsymbol{\xi}(\boldsymbol{x} \, , \, t)$ is:

$$\begin{gathered} \left(\frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla}) \right) \boldsymbol{a} \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t \right) = \\ -i \left(\omega - \boldsymbol{k} \cdot \boldsymbol{V} \right) \, \boldsymbol{a} \exp\left(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t \right) \end{gathered} \tag{6.6.6}$$

We find the following dispersion relation for sound waves in a moving fluid:

$$\tilde{\omega} = \omega - \boldsymbol{k} \cdot \boldsymbol{V} = \pm |\boldsymbol{k}| C_{\rm s} \, , \tag{6.6.7}$$

or equivalently:

$$\boxed{\omega(\boldsymbol{k}) = \boldsymbol{k} \cdot \boldsymbol{V} \pm |\boldsymbol{k}| C_{\rm s} \, .} \tag{6.6.8}$$

If we now calculate the *group* velocity, the velocity with which signals can propagate, we find in this case:

$$\boxed{\boldsymbol{v}_{\rm gr} = \frac{\partial \omega}{\partial \boldsymbol{k}} = \boldsymbol{V} \pm C_{\rm s} \, \hat{\boldsymbol{\kappa}} \, ,} \tag{6.6.9}$$

with $\hat{\boldsymbol{\kappa}} = \boldsymbol{k}/|\boldsymbol{k}|$ as before. This result simply says that sound waves are *dragged along* by the moving fluid at velocity $\boldsymbol{V}$, and propagate *with respect to the fluid* at the (local) sound speed in the direction of $\boldsymbol{k}$.

6.7 Non-planar sound waves

The previous two Sections could leave the impression that all sound waves are plane waves, where the displacement vector $\boldsymbol{\xi}(\boldsymbol{x} , t)$ is given by relation (6.5.2). This is not the case. As a counter-example I will briefly consider the case of spherical sound waves.

Let us assume that we place a spherical membrane of radius R at the origin. This membrane acts as a loudspeaker: its radius varies harmonically in time with a prescribed frequency ω:

$$R(t) = R_0 + \delta R \, \cos(\omega t) \; . \tag{6.7.1}$$

We will assume that the amplitude of this vibration is small: $\delta R \ll R_0$.

The gas surrounding this vibrating sphere must respond: at the surface of the vibrating sphere the gas must move in concert. There the displacement vector of the fluid is given by:

$$\boldsymbol{\xi}(r = R_0 , \, t) = \delta R \, \cos(\omega t) \, \hat{\boldsymbol{e}}_r \; . \tag{6.7.2}$$

These radial motions induce density- and pressure fluctuations, which in turn lead to the emission of spherical sound waves.

In order to derive the equation for these spherical waves I will use a different form of the wave equation for sound waves. The original equation (6.5.12) reads

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - C_{\rm s}^2 \, \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) = 0 \; . \tag{6.7.3}$$

Sound waves are the compressive solutions of the wave equation, with

$$\Delta \equiv \frac{\delta \rho}{\rho} = - \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) \neq 0 \; . \tag{6.7.4}$$

We can therefore 'isolate' the sound waves by taking the divergence on both sides of the wave equation and demanding that $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \neq 0$. Using the relations

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \right) = \frac{\partial^2 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})}{\partial t^2} = - \frac{\partial^2 \Delta}{\partial t^2} \quad , \quad \boldsymbol{\nabla} \cdot \left[\boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) \right] = - \boldsymbol{\nabla}^2 \Delta \tag{6.7.5}$$

one finds the following equation that *exclusively* describes sound waves:

$$\left\{ \frac{\partial^2}{\partial t^2} - C_{\rm s}^2 \, \boldsymbol{\nabla}^2 \right\} \Delta(\boldsymbol{x} \, , \, t) = 0 \; . \tag{6.7.6}$$

If the waves are spherical, meaning that the surfaces of constant phase are expanding spheres since the waves are outgoing waves that propagate away from the origin, the function $\Delta(\boldsymbol{x} \, , \, t)$ can only depend on the distance r to the origin, and on time. The resulting wave equation reads in spherical coordinates:

$$\left\{ \frac{\partial^2}{\partial t^2} - \frac{C_{\rm s}^2}{r^2} \, \frac{\partial}{\partial r} \left(r^2 \, \frac{\partial}{\partial r} \right) \right\} \Delta(r \, , \, t) = 0 \; . \tag{6.7.7}$$

The frequency ω is fixed: it must equal the vibration frequency of the sphere. This means we can put

$$\Delta(r \, , \, t) = \tilde{\Delta}(r) \, e^{-i\omega t} + {\rm cc} \; , \tag{6.7.8}$$

and use

$$\frac{\partial^2 \Delta}{\partial t^2} = -\omega^2 \, \Delta \; . \tag{6.7.9}$$

The wave equation (6.7.7) can be then written as an equation for $\tilde{\Delta}(r)$[12]:

$$\frac{1}{r^2} \, \frac{{\rm d}}{{\rm d}r} \left(r^2 \, \frac{{\rm d}\tilde{\Delta}}{{\rm d}r} \right) = -\frac{\omega^2}{C_{\rm s}^2} \, \tilde{\Delta} \; . \tag{6.7.10}$$

Note that the partial derivatives with respect to r have been replaced by ordinary derivatives. This is allowed since $\tilde{\Delta}$ only depends on r.

This equation must be solved subject to sensible boundary conditions. At the surface of the vibrating sphere one has condition (6.7.2). We will deal with this condition later. At large distances the wave should die out, meaning that $\Delta \downarrow 0$ when $r \to \infty$. Equation (6.7.10) can be cast in the form

$$r^2 \, \frac{{\rm d}^2 \tilde{\Delta}}{{\rm d}r^2} + 2r \, \frac{{\rm d}\tilde{\Delta}}{{\rm d}r} + k^2 r^2 \, \tilde{\Delta} = 0 \; , \tag{6.7.11}$$

[12] Once again, you can forget about the complex conjugate for the time being.

Here k is defined as

$$k \equiv \frac{\omega}{C_{\rm s}} \; , \tag{6.7.12}$$

and essentially plays the role of the wave number. The solution to this equation is

$$\tilde{\Delta}(r) = A \, j_0(kr) + B \, n_0(kr) \; . \tag{6.7.13}$$

Here $j_0(x)$ and $n_0(x)$ are the *spherical Bessel functions* of order zero[13], and A and B are arbitrary constants, to be determined from the boundary conditions. The two functions j_0 and h_0 can be given in closed form:

$$j_0(x) = \frac{\sin x}{x} \quad , \quad n_0(x) = -\frac{\cos x}{x} \; . \tag{6.7.14}$$

Both functions satisfly the boundary condition $\tilde{\Delta} \rightarrow 0$ as $r \rightarrow \infty$. We are forced to use another argument to decide which Bessel function (or what combination of the two functions) to use.

The form of these spherical Bessel functions shows that we are truly dealing with waves: the density $\delta\rho(r \, , \, t) = \rho \, \Delta(r \, , \, t)$ varies harmonically in time, and the spherical ripples due to $\sin(kr)$ and $\cos(kr)$ have a wavelength $\Delta r = \lambda = 2\pi/k = 2\pi C_{\rm s}/\omega$. This is exactly the same wavelength one would assign to a plane sound wave with the same frequency. There is a major difference however: the typical amplitude of the spherical waves scales as $1/r$, as both $j_0(kr)$ and $n_0(kr)$ have a factor $1/kr$ in front of the sine and cosine term. In a plane wave (without damping) on the other hand the amplitude is constant. This reflects the different geometry of a spherical wave.

We can decide which combination of the two spherical Bessel functions to use by employing the following argument. Far from the vibrating sphere, at a distance that is large compared to both R_0 and λ, the wave should look like an *outgoing* (almost) plane wave to a local observer. Assuming without loss of generality that $\omega > 0$, this means that $\Delta(r \, , \, t)$ should vary like:

$$\Delta_{\rm out}(r \, , \, t) \sim \cos(kr - \omega t + \alpha) \; . \tag{6.7.15}$$

In contrast, an *ingoing* spherical wave propagating towards the origin would behave like

$$\Delta_{\rm in}(r \, , \, t) \sim \cos(-kr - \omega t + \alpha) \; . \tag{6.7.16}$$

[13] see for instance: Arfken & Weber, *Mathematical Methods for Physicists*, Ch. 11.7.

This argument implies that the solution should be chosen to be equal to:

$$\tilde{\Delta}(r) = A \, \left(j_0(kr) + i n_0(kr)\right) \equiv A \, h_0^{(1)}(kr) \; . \tag{6.7.17}$$

The function

$$h_0^{(1)}(x) = j_0(x) + i n_0(x) = \frac{1}{ix} \exp(ix) \tag{6.7.18}$$

is the zero-th order spherical Hankel function of the first kind.

The remaining constant A can only be found by considering the boundary condition at the surface of the vibrating sphere (i.e. at $r = R_0$). Because of the spherical symmetry of the problem, the motion of the fluid induced by the sphere will be in the radial direction:

$$\boldsymbol{\xi}(\boldsymbol{x} \, , \, t) = \xi(r \, , \, t) \, \hat{\boldsymbol{e}}_r \; . \tag{6.7.19}$$

Consequently, the linearized equation of motion only has a radial component,

$$\rho \, \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial \delta P}{\partial r} \; , \tag{6.7.20}$$

compare Eqn. (6.5.10). The associated pressure perturbation is

$$\delta P = \gamma P \, \Delta(r \, , \, t) \; . \tag{6.7.21}$$

and because of the harmonic time dependence we can rewrite (6.7.20) as:

$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 \, \xi = -C_{\rm s}^2 \, \frac{\partial \Delta}{\partial r} \; . \tag{6.7.22}$$

Here I have used $C_{\rm s}^2 = \gamma P/\rho$. This gives the amplitude of the motion as

$$\xi(r \, , \, t) = \left(\frac{C_{\rm s}^2}{\omega^2}\right) \frac{\partial \Delta}{\partial r} = \frac{1}{k^2} \, \frac{\partial \Delta}{\partial r} \; . \tag{6.7.23}$$

Substituting the solution

$$\Delta(r\;,\;t) = A h_0^{(1)}(kr)\exp(-i\omega t) + \mathrm{cc} \tag{6.7.24}$$

One finds:

$$\xi(r\;,\;t) = \frac{A}{k}\,h'^{(1)}_0(kr)\exp(-i\omega t) + \mathrm{cc} \tag{6.7.25}$$

Here $h'^{(1)}_0(x) \equiv \mathrm{d}h_0^{(1)}/\mathrm{d}x$ is the first derivative of the Hankel function. At the surface of the sphere we must satisfy the condition that the velocity, and therefore the amplitude ξ matches the motion of the surface, $\xi(R_0\;,\;t) = \delta R\,\cos(\omega t)$. This condition fixes A:

$$A = \frac{k\,\delta R}{2h'^{(1)}_0(kR_0)}\;. \tag{6.7.26}$$

This solves the problem of spherical sound waves generated by a vibrating sphere. Note that the coefficient A is proportional to the amplitude δR of the vibration on the spherical membrane.

The full solution of the sound waves emitted by a vibrating spherical membrane now reads, for $r > R_0$:

$$\xi(r\;,\;t) = \tfrac{1}{2}\delta R\,\left(\frac{h'^{(1)}_0(kr)}{h'^{(1)}_0(kR_0)}\right)\,\exp(-i\omega t) + \mathrm{cc.} \tag{6.7.27}$$

6.8 Some astrophysical applications of waves

6.8.1 The Jeans instability

Around 1902, Sir James Jeans investigated the stability of a self-gravitating fluid. This calculation considers the fate of small-amplitude waves ('sound waves') in a fluid which generates its own gravity. This means one one has to solve the equation of motion and the continuity equation in concert with Poisson's equation for the gravitational potential:

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} &= -\frac{1}{\rho}\,\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi \; , \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\,(\boldsymbol{\nabla}\cdot\boldsymbol{V}) \; , \\ \boldsymbol{\nabla}^2\Phi &= 4\pi G\,\rho \; , \end{aligned} \tag{6.8.1}$$

together with the adiabatic gas law $P(\rho) \propto \rho^{\gamma}$. The unperturbed state on which these waves are superposed is sometimes referred to as *Jeans' swindle*: a fluid with uniform density ρ, pressure P and no gravity: $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = 0$. There can be no gravitational acceleration in a uniform fluid: the gravitational acceleration $\boldsymbol{g}$ is a vector. Its direction would introduce a *preferred* direction, which can not be present in an infinite homogeneous medium that looks the same everywhere and in every direction. One must therefore conclude that $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = 0$, which implies $\Phi =$ constant. However, according to Poisson's equation one has $\boldsymbol{\nabla}^2\Phi = 4\pi G\,\rho$. This will only give a constant Φ if $\rho = 0$. This inconsistency is glossed over by assuming that Poisson's equation only applies to the density *fluctuations* induced by the waves.

The results derived for the velocity, density and pressure perturbations in sound waves remain valid in this case:

$$\delta\boldsymbol{V} = \frac{\partial\boldsymbol{\xi}}{\partial t} \quad , \quad \delta\rho = -\rho\,(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \quad , \quad \delta P = -\gamma P\,(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) = C_{\mathrm{s}}^2\,\delta\rho \; . \tag{6.8.2}$$

The equation of motion for the perturbations must be modified in order to take the effect of gravity into account. It now reads:

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\frac{1}{\rho}\,\boldsymbol{\nabla}\delta P - \boldsymbol{\nabla}\delta\Phi \; . \tag{6.8.3}$$

Here I have used that, according to Jean's Swindle, the gravitational acceleration acting on a fluid element is

$$\delta \boldsymbol{g} = -\boldsymbol{\nabla} \delta\Phi \ . \tag{6.8.4}$$

This acceleration is caused by the gravitational action of the density fluctuations: density enhancements in the waves tend to attract the surrounding matter. Poisson's equation links the potential perturbations to the fluctuations in the density:

$$\boldsymbol{\nabla}^2 \, \delta\Phi = 4\pi G \, \delta\rho \ . \tag{6.8.5}$$

Let us define the relative density perturbation:

$$\Delta \equiv \frac{\delta\rho}{\rho} = -(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) \ . \tag{6.8.6}$$

Substituting for the pressure perturbation δP from (6.8.2), the equation of motion becomes:

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = C_{\rm s}^2 \, \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - \boldsymbol{\nabla} \delta\Phi \ . \tag{6.8.7}$$

Using the fact that $C_{\rm s} = \sqrt{\gamma P/\rho}$ is constant, we can take the divergence of both sides of the equation, effectively isolating the compressive ($\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \neq 0$) 'sound-like' solutions:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) &= C_{\rm s}^2 \, \boldsymbol{\nabla}^2(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - \boldsymbol{\nabla}^2 \delta\Phi \\ &= C_{\rm s}^2 \, \boldsymbol{\nabla}^2(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + 4\pi G\rho \, (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) \ . \end{aligned} \tag{6.8.8}$$

Here I have used $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}(\cdots) = \boldsymbol{\nabla}^2 \cdots$, and I have employed Poisson's equation (6.8.5) to eliminate $\boldsymbol{\nabla}^2 \, \delta\Phi$ in terms of the density perturbation:

$$\boldsymbol{\nabla}^2 \delta\Phi = 4\pi G \, \delta\rho = 4\pi G\rho \, \Delta \ . \tag{6.8.9}$$

Equation (6.8.8) is a linear equation for $\Delta = \delta\rho/\rho$:

$$\left[\frac{\partial^2}{\partial t^2} - C_{\rm s}^2\,\boldsymbol{\nabla}^2 - 4\pi G\,\rho\right]\,\Delta = 0\ . \tag{6.8.10}$$

The rest of the analysis proceeds along the same lines as for sound waves. Consider a plane wave solution, where the relative density perturbation $\Delta = \delta\rho/\rho$ takes the form[14]

$$\Delta(\boldsymbol{x}\,,\,t) = \tilde{\Delta}\,\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + {\rm cc}\ . \tag{6.8.11}$$

A substitution of this assumption for $\Delta(\boldsymbol{x}\,,\,t)$ into (6.8.10) yields the dispersion relation for compressive (sound) waves in a self-gravitating fluid:

$$\omega^2 = k^2 C_{\rm s}^2 - 4\pi G\,\rho\ . \tag{6.8.12}$$

The last term on the right-hand-side gives the modification of sound waves due to gravity. The solution of this equation,

$$\omega(\boldsymbol{k}) = \pm\sqrt{k^2 C_{\rm s}^2 - 4\pi G\,\rho}\ , \tag{6.8.13}$$

describes fundamentally different behaviour at short and long wavelengths.

The dividing line between these two types of behaviour is at the wavelength $\lambda_{\rm J}$, the so-called *Jeans length*, where the wave frequency $\omega(\boldsymbol{k})$ vanishes. Defining $k_{\rm J} = 2\pi/\lambda_{\rm J}$ one must have $k_{\rm J}^2 C_{\rm s}^2 = 4\pi G\,\rho$, and one finds:

$$\lambda_{\rm J}^2 = \left(\frac{2\pi}{k_{\rm J}}\right)^2 = \frac{\pi C_{\rm s}^2}{G\rho}\ . \tag{6.8.14}$$

For waves with a wavelength $\lambda < \lambda_{\rm J}$ the argument of the square root in (6.8.13) is positive, and the wave frequency is real.

[14] In terms of the plane-wave expression (6.5.2) for $\boldsymbol{\xi}(\boldsymbol{x}\,,\,t)$ the amplitude $\tilde{\Delta}$ is related to the displacement amplitude $\boldsymbol{a}$ by $\tilde{\Delta} = -i(\boldsymbol{k}\cdot\boldsymbol{a})$, see Eqn. (6.5.6).

However, for wavelengths $\lambda > \lambda_{\rm J}$ the argument of the square root is *negative*, and the wave frequency becomes purely imaginary. The solution (6.8.13) for $\lambda > \lambda_{\rm J}$ can be written in terms of the Jeans length:

$$\omega = \pm ikC_{\rm s}\sqrt{\frac{\lambda^2}{\lambda_{\rm J}^2} - 1} \equiv i\sigma \; . \tag{6.8.15}$$

Imaginary frequencies, where $\omega = i\sigma$, lead to exponentially growing or decaying perturbations. Solution (6.8.15) always has one exponentially growing mode and one decaying mode. The decaying mode is not very important as it dies away. The assumed time-dependence means that the relative density perturbation behaves as

$$\Delta(\boldsymbol{x} \, , \, t) \propto e^{-i\omega t} = e^{\sigma t} \; . \tag{6.8.16}$$

If $\mathrm{Im}(\omega) = \sigma > 0$ the perturbation grows exponentially in time. It decays if $\sigma < 0$. Here there is always a solution with $\sigma > 0$, which implies that the wave amplitude gets larger and larger. Our assumption that the pressure, density and velocity perturbations associated with the wave all remain small will ultimately break down. When such a situation arises, the equilibrium state used to calculate the wave properties is said to be **linearly unstable** against suitable perturbations:

> If there is a solution with $\mathrm{Im}\,\omega(\boldsymbol{k}) \equiv \sigma(\boldsymbol{k}) > 0$ a linear instability arises

(6.8.17)

The importance of the Jeans length $\lambda_{\rm J}$ as the wavelength that separates stable from unstable oscillations can be illustrated in another way. The pressure force and the gravitational force due to the perturbation are

$$\begin{aligned} \boldsymbol{F}_{\rm p} &= -\boldsymbol{\nabla}\delta P = -\gamma P \, k^2 \boldsymbol{a} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc} \; , \\ \boldsymbol{F}_{\rm g} &= -\rho \, \boldsymbol{\nabla}\delta\Phi = 4\pi G \, \rho^2 \, \boldsymbol{a} \exp\left(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\right) + \mathrm{cc} \; . \end{aligned} \tag{6.8.18}$$

Here I have used the plane wave assumption, and the fact that $\boldsymbol{a} \parallel \boldsymbol{k}$. One sees that the pressure force and the gravitational force are 180 degrees out of phase: they work in opposite directions, physically obvious as gravity promotes mass concentrations while pressure forces try to negate them.

Comparing the amplitude of these two forces one has:

$$\frac{|\boldsymbol{F}_{\rm g}|}{|\boldsymbol{F}_{\rm p}|} = \frac{4\pi G\,\rho}{k^2 C_{\rm s}^2} = \left(\frac{\lambda}{\lambda_{\rm J}}\right)^2 \; . \tag{6.8.19}$$

In the stable case ($\lambda < \lambda_{\rm J}$) the amplitude of pressure force is larger than the amplitude of the gravitational force, and the system is stable. In the case $\lambda > \lambda_{\rm J}$ the amplitude of the gravitational force is larger than the amplitude of the pressure force. In that case the system is unstable, and the density enhancements in the wave will continue to grow.

This is illustrated in the two figures above. It shows the displacement ξ, the velocity $\delta v = \partial\xi/\partial t$, the pressure force and the gravitational force in a plane wave propagating in the x-direction. The first figure considers the stable case $\lambda = \lambda_{\rm J}/\sqrt{2}$, the second figure considers the unstable case with $\lambda = \sqrt{2}\,\lambda_{\rm J}$.

We encountered a similar unstable situation in our simple perturbation analysis of a single particle moving in a potential well. In that case, it turned out that an equilibrium is unstable if ${\rm d}^2V/{\rm d}x^2 < 0$ at the equilibrium point. The example of the Jeans' instability shows that in fluid dynamics you can have a situation where there are stable as well as unstable solutions to the equations of motion. However, if there is an unstable solution, the system is unstable and can not persist.

6.8.2 The zero-frequency mode

For completeness sake, I mention the fact that the zero-frequency waves present in our discussion of sound waves are also present in Jeans' problem. This can be seen by taking

$$\boldsymbol{\nabla}\times\;\text{(Equation of motion 6.8.7)}\;.$$

Using the vector identity

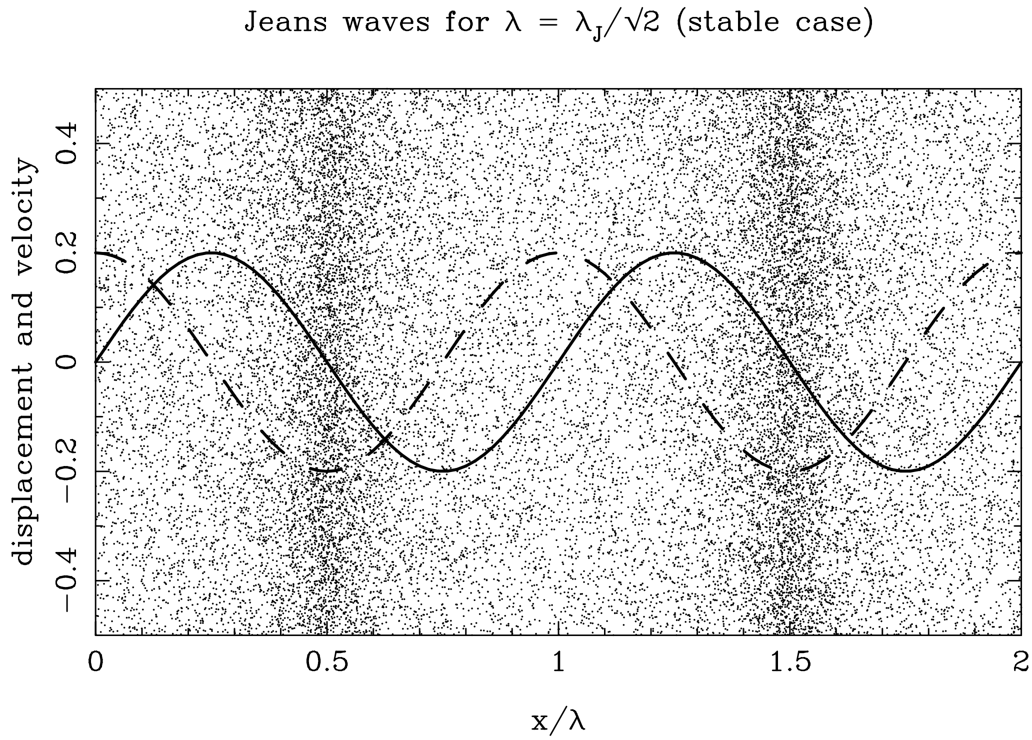
$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}f = 0\;, \tag{6.8.20}$$

valid for an arbitrary function $f(\boldsymbol{x}\;,\;t)$, this leads to

$$\frac{\partial^2}{\partial t^2}\left(\boldsymbol{\nabla}\times\boldsymbol{\xi}\right) = 0\;. \tag{6.8.21}$$

Note that this equation does not show any coupling to gravity since $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\delta\Phi = 0$. Substituting a plane wave solution for $\boldsymbol{\xi}(\boldsymbol{x}\;,\;t)$ (c.f. 6.5.2) one immediately finds

$$\omega^2\;(\boldsymbol{k}\times\boldsymbol{a}) = 0\;. \tag{6.8.22}$$



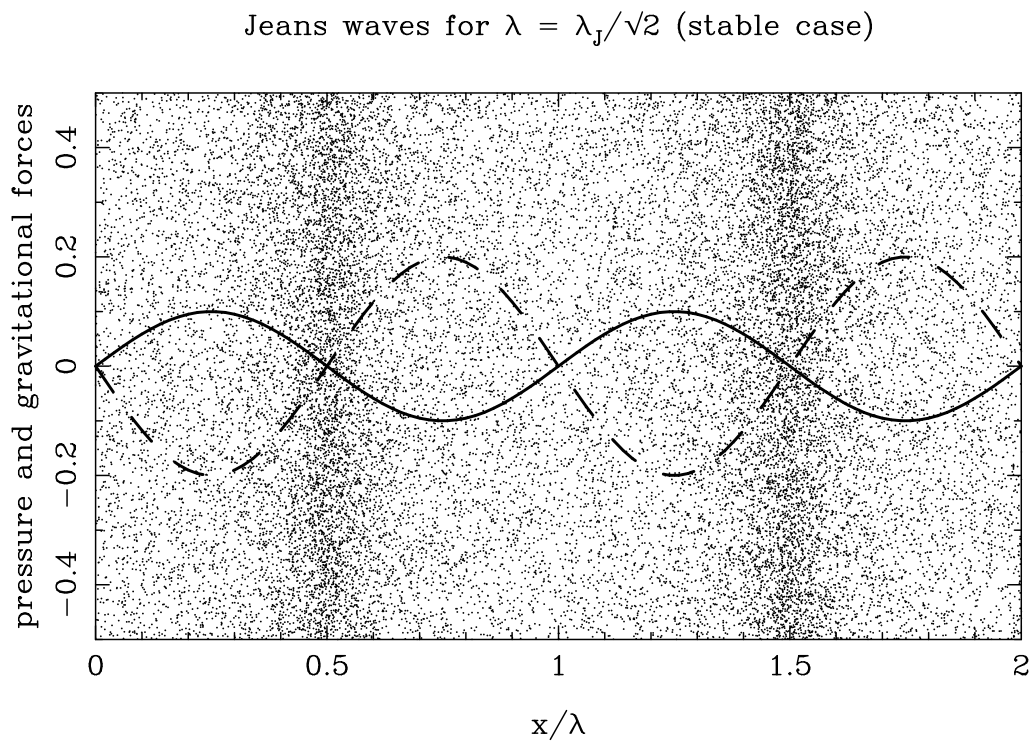


Figure 6.6: *The displacement (top panel, solid curve), velocity (top panel, dashed curve) gravitational force (bottom panel, solid curve) and pressure force(bottom panel, dashed curve) in a linear sound wave in a self-gravitating fluid. Shown is the stable case with wavelength* $\lambda = \lambda_{\rm J}/\sqrt{2}$*. In this case the amplitude of the pressure force is twice that of the gravitational force. The small dots are test particles moving with the fluid, and show where the compressions and rarefactions are located.*

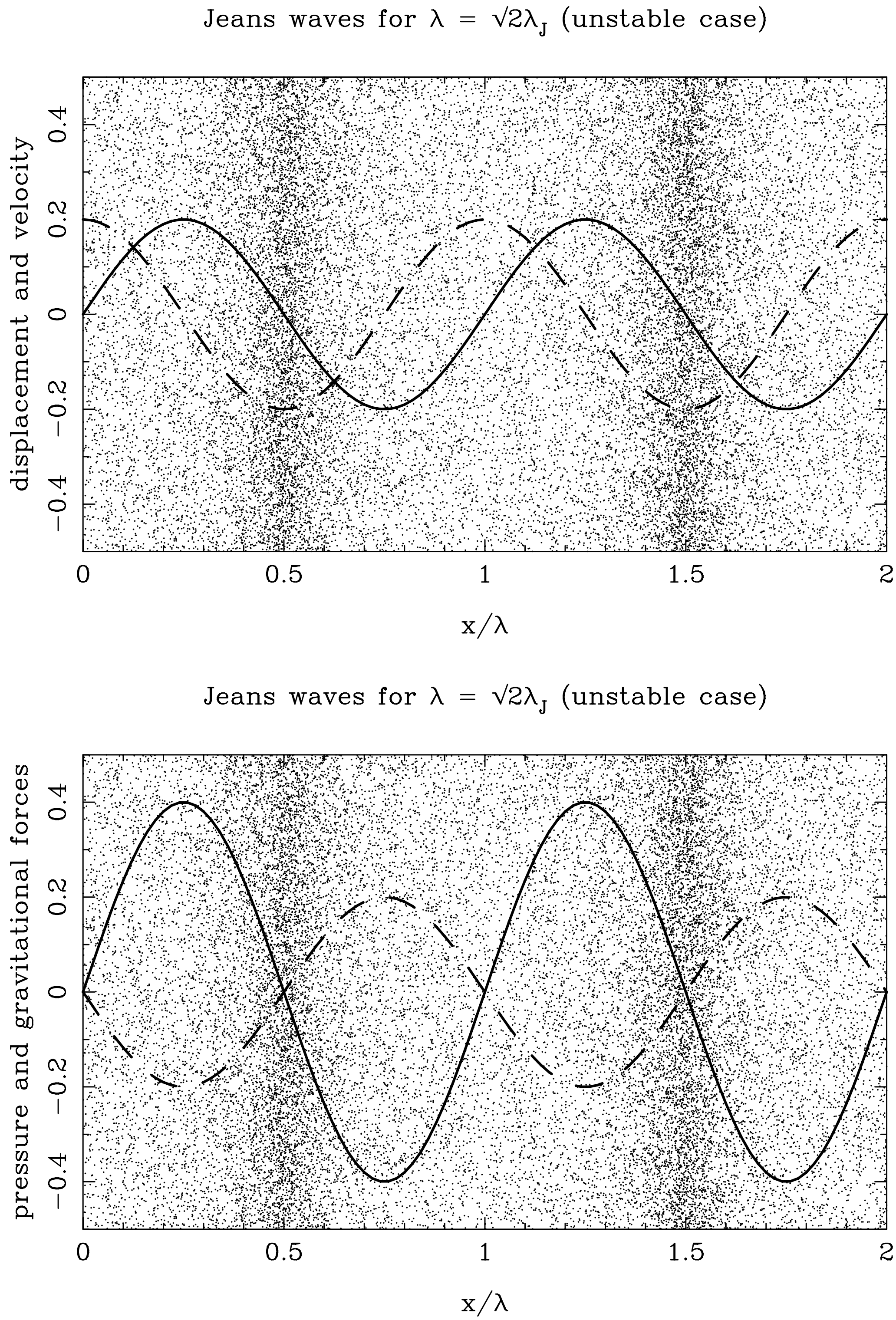


Figure 6.7: *The displacement (top panel, solid curve), velocity (top panel, dashed curve) gravitational force (bottom panel, solid curve) and pressure force(bottom panel, dashed curve) in a linear sound wave in a self-gravitating fluid. Shown is the unstable case with wavelength* $\lambda = \sqrt{2}\,\lambda_{\rm J}$. *In this case the amplitude of the pressure force is half that of the gravitational force.*

The only non-trivial solution where $\boldsymbol{k} \times \boldsymbol{a} \neq 0$ must have $\omega = 0$. The compressive (longitudinal) waves which play a role in the Jeans Instability have $\boldsymbol{k} \parallel \boldsymbol{a}$, just like ordinary sound waves.

6.8.3 A simple physical explanation of the Jeans Instability

The physics behind the Jeans Instability can be understood in a different manner without referring to waves and their stability. This alternative approach uses a stability criterion based on an energy argument. Consider a spherical cloud of hydrogen gas ($\mu \approx 1$) with radius a, uniform density ρ, temperature T and pressure $P = \rho\mathcal{R}T$. The total energy $W(a)$ of this cloud is

$$W(a) = \int_0^M \mathrm{d}m(r) \left[\frac{3}{2}\,\mathcal{R}T - \frac{Gm(r)}{r}\right] \equiv U_{\rm th} + U_{\rm gr} \; . \tag{6.8.23}$$

Here

$$\mathrm{d}m(r) = 4\pi r^2 \,\rho\, \mathrm{d}r \quad , \quad m(r) = \frac{4\pi}{3}\,\rho\, r^3 \tag{6.8.24}$$

are the mass contained in a spherical shell between r and $r + \mathrm{d}r$, and the mass contained within a radius r respectively. The total mass of the cloud is

$$M = \frac{4\pi}{3}\,\rho\, a^3 \; . \tag{6.8.25}$$

The term $3\mathcal{R}T/2$ in integral (6.8.23) is the thermal energy per unit mass in an ideal gas with adiabatic index $\gamma = 5/3$. The term involving $\Phi(r) = -Gm(r)/r$ is the gravitational binding energy per unit mass at radius r. Integrating these quantities over all mass elements in the cloud yields the total cloud energy W.

The integration of this expression is relatively straightforward. One finds:

$$W(a) = \tfrac{3}{2}\, M\mathcal{R}T - \tfrac{3}{5}\,\frac{GM^2}{a} \; . \tag{6.8.26}$$

I now consider the effect of a change $-\Delta a$ (with $\Delta a > 0$) in the radius of the cloud, so that the radius decreases from a to $a - \Delta a$. Let us assume that this change occurs adiabatically, so that no heat is added to, or extracted from the gas.

In that case, the thermodynamical equations of Chapter 2.5 tell us that the thermal energy changes according to $\mathrm{d}U_{\rm th} = -P\,\mathrm{d}\mathcal{V}$. The volume change is $\Delta\mathcal{V} = -4\pi a^2\,\Delta a$. This means that the thermal energy of the cloud changes by an amount

$$\Delta U_{\rm th} = -P\,\Delta\mathcal{V} \approx \rho\mathcal{R}T\,4\pi a^2\,\Delta a\;. \tag{6.8.27}$$

The change of the gravitational binding energy due to the change in radius from a to $a - \Delta a$ is

$$\Delta U_{\rm gr} \approx \left(\frac{\partial U_{\rm gr}}{\partial a}\right) \times (-\Delta a) = -\frac{3}{5}\left(\frac{GM^2}{a^2}\right)\,\Delta a\;. \tag{6.8.28}$$

Here I have used that the total mass M of the cloud is conserved. Adding these two contributions yields the change of the total energy, $\Delta W = \Delta U_{\rm th} + \Delta U_{\rm gr}$, of the cloud:

$$\Delta W \approx \left(3M\mathcal{R}T - \frac{3}{5}\,\frac{GM^2}{a}\right) \times \left(\frac{\Delta a}{a}\right)\;. \tag{6.8.29}$$

Now there are two possibilites:

- If $\Delta W > 0$ the change *costs* energy since the increase in the inward gravitational force is smaller than the increase of the outward pressure force that resists the volume change. In this case the cloud is **stable**.
- If $\Delta W < 0$, the change *liberates* energy! The inward gravitational force increases faster than the outward pressure force. This implies that, once started, the contraction of the cloud will continue, leading to *gravitational collapse*. The cloud is **unstable**, which can be interpreted as a consequence of the fact that physical systems tend to evolve towards a minimum-energy state.

Using $M = 4\pi\rho a^3/3$ expression (6.8.29) can be rewritten as

$$\Delta W = 3M\mathcal{R}T\left(1 - \frac{a^2}{\overline{\lambda}_{\rm J}^2}\right) \times \left(\frac{\Delta a}{a}\right)\;. \tag{6.8.30}$$

The characteristic length $\overline{\lambda}_{\rm J}$ in this expression is defined by:

$$\overline{\lambda}_{\rm J} = \sqrt{\frac{15}{4\pi}} \left(\frac{\mathcal{R}T}{G\rho} \right)^{1/2} . \tag{6.8.31}$$

This characteristic length is almost the same as the Jeans length introduced in the previous Section when we discussed the Jeans Instability. Using $C_{\rm s} = (5\mathcal{R}T/3)^{1/2}$ one finds

$$\overline{\lambda}_{\rm J} = \left(\frac{3}{2\pi} \right) \lambda_{\rm J} \approx 0.5 \, \lambda_{\rm J} \; .$$

The above criterion for (in)stability leads to the following conclusion: if the cloud has a radius $a > \overline{\lambda}_{\rm J}$ (and therefore the cloud diameter is larger than $2\overline{\lambda}_{\rm J} \sim \lambda_{\rm J}$) it will be unstable against gravitational collapse since $\Delta W < 0$. Smaller clouds, with $a < \overline{\lambda}_{\rm J}$, are stable as $\Delta W > 0$.