The Condition Number of the PageRank Problem

Sepandar D. Kamvar and Taher H. Haveliwala

Stanford University
{sdkamvar,taherh}@cs.stanford.edu

Abstract. We determine analytically the condition number of the PageRank problem. Specifically, we prove the following statement:
"Let P be an $n \times n$ row-stochastic matrix whose diagonal elements $P_{ii} = 0$. Let c be a real number such that $0 \leq c < 1$. Let E be the $n \times n$ rank-one row-stochastic matrix $E = \boldsymbol{e}\boldsymbol{v}^T$, where $\boldsymbol{e}$ is the n-vector whose elements are all $e_i = 1$, and $\boldsymbol{v}$ is an n-vector that represents a probability distribution.
Define the matrix $A = [cP + (1-c)E]^T$. The problem $A\boldsymbol{x} = \boldsymbol{x}$ has condition number $\kappa = (1+c)/(1-c)$."
This statement has implications for the accuracy to which PageRank can be computed, currently and as the web scales. Furthermore, it provides a simple proof that, for values of c that are used by Google, small changes in the link structure of the web do not cause large changes in the PageRanks of pages of the web.

1 Theorem

Theorem 1. *Let P be an $n \times n$ row-stochastic matrix whose diagonal elements $P_{ii} = 0$.. Let c be a real number such that $0 \leq c \leq 1$. Let E be the $n \times n$ rank-one row-stochastic matrix $E = \boldsymbol{e}\boldsymbol{v}^T$, where $\boldsymbol{e}$ is the n-vector whose elements are all $e_i = 1$, and $\boldsymbol{v}$ is an n-vector that represents a probability distribution*[1].
Define the matrix $A = [cP + (1-c)E]^T$. The problem $A\boldsymbol{x} = \boldsymbol{x}$ has condition number $\kappa = (1+c)/(1-c)$.

2 Notation and Preliminaries

P is an $n \times n$ row-stochastic matrix whose diagonal elements $P_{ii} = 0$. E is the $n \times n$ rank-one row-stochastic matrix $E = \boldsymbol{e}\boldsymbol{v}^T$, where $\boldsymbol{e}$ is the n-vector whose elements are all $e_i = 1$ and $\boldsymbol{v}$ is an n-vector whose elements are all non-negative and sum to 1. A is the $n \times n$ column-stochastic matrix:

$$A = [cP + (1-c)E]^T \tag{1}$$

We let $\boldsymbol{x}$ be the dominant eigenvector of A. By convention, we choose eigenvectors $\boldsymbol{x}$ such that $||\boldsymbol{x}||_1 = 1$. Since A is a non-negative matrix, the dominant eigenvector $\boldsymbol{x}$ is also non-negative. Therefore,

$$\boldsymbol{e}^T\boldsymbol{x} = ||\boldsymbol{x}||_1 = 1 \tag{2}$$

Since A is column-stochastic, it's dominant eigenvalue $\lambda_1 = 1$, $1 \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq 0$. That is,

$$A\boldsymbol{x} = \boldsymbol{x} \tag{3}$$

[1] i.e., a vector whose elements are nonnegative and whose L_1 norm is 1.

3 Proof of Theorem 1

We prove this case via a series of lemmas.

Lemma 1. $E^T \boldsymbol{x} = \boldsymbol{v}$.

Proof. By definition, $E = \boldsymbol{e}\boldsymbol{v}^T$. Therefore, $E^T \boldsymbol{x} = \boldsymbol{v}\boldsymbol{e}^T \boldsymbol{x}$. From equation 2, $\boldsymbol{e}^T \boldsymbol{x} = 1$. Therefore, $E^T \boldsymbol{x} = \boldsymbol{v}$, and Lemma 1 is proved.

Lemma 2. The eigenvalue problem $A\boldsymbol{x} = \boldsymbol{x}$ can be rewritten as the nonsingular system of equations $(I - cP^T)\boldsymbol{x} = (1-c)\boldsymbol{v}$.

Proof. From $A\boldsymbol{x} = \boldsymbol{x}$, we can rearrange terms to get

$$(I - A)\boldsymbol{x} = 0.$$

By the definition of A (equation 1):

$$[I - (cP + (1-c)E)^T]\boldsymbol{x} = 0.$$

From Lemma 1, $E^T \boldsymbol{x} = \boldsymbol{v}$. Therefore, $(I - cP^T)\boldsymbol{x} - (1-c)\boldsymbol{v} = 0$. Rearranging terms, we get $(I - cP^T)\boldsymbol{x} = (1-c)\boldsymbol{v}$, and Lemma 2 is proved.

Lemma 3. $\boldsymbol{x} = (I - cP^T)^{-1}\boldsymbol{v}$.

Proof. Let $M = I - cP^T$. Then $M^T = I - cP$. Since P has zeros on the diagonals and is row-stochastic, and since $c < 1$, $I - cP$ is strictly diagonally dominant and therefore invertible. Since M^T is invertible, M is also invertible. Therefore, we may write $\boldsymbol{x} = (I - cP^T)^{-1}\boldsymbol{v}$ and Lemma 3 is proved.

Lemma 4. $||I - cP^T||_1 = 1 + c$.

Proof. Since the diagonal elements of cP^T are all zero,

$$||I - cP^T||_1 = ||I||_1 + c||P^T||_1 = 1 + c||P^T||_1.$$

Since P^T is a column-stochastic matrix, $||P^T||_1 = 1$. Thus, $||I - cP^T||_1 = 1 + c$ and Lemma 4 is proved.

Lemma 5. $||(I - cP^T)^{-1}||_1 = 1/(1-c)$.

Proof. Recall from equation 1 that $A = [cP + (1-c)E]^T$, where $E = \boldsymbol{e}\boldsymbol{v}^T$ and v is some n-vector whose elements are non-negative and sum to 1. Let $\boldsymbol{x}(\boldsymbol{e_i})$ be the n-vector that satisfies the following equations:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{e_i} \\ A\boldsymbol{x}(\boldsymbol{e_i}) &= \boldsymbol{x}(\boldsymbol{e_i}) \\ ||\boldsymbol{x}(\boldsymbol{e_i})||_1 &= 1. \end{aligned}$$

From Lemma 2, $\boldsymbol{x} = (1-c)(I-cP^T)^{-1}\boldsymbol{v}$. Therefore, $\boldsymbol{x}(\boldsymbol{e_i}) = (1-c)(I-cP^T)^{-1}\boldsymbol{e_i}$. Taking the norm of both sides, $||\boldsymbol{x}(\boldsymbol{e_i})||_1 = (1-c)||(I-cP^T)^{-1}\boldsymbol{e_i}||_1$. Since $||\boldsymbol{x}(\boldsymbol{e_i})||_1 = 1$, we have

$$||(I-cP^T)^{-1}\boldsymbol{e_i}||_1 = 1/(1-c). \tag{4}$$

Notice that $(I-cP^T)^{-1}\boldsymbol{e_i}$ gives the ith column of $(I-cP^T)^{-1}$. Thus, from equation 4, the L1 norm of the matrix $(I-cP^T)^{-1}$ is $||(I-cP^T)^{-1}|| = 1/(1-c)$.

Lemma 6. The 1-norm condition number of $\boldsymbol{x} = (I-cP^T)^{-1}\boldsymbol{v}$ is $\kappa = (1+c)/(1-c)$.

Proof. By definition, the 1-norm condition number κ of the problem $\boldsymbol{y} = M^{-1}\boldsymbol{b}$ is given by $\kappa = ||M||_1||M^{-1}||_1$. From Lemmas 4 and 5, this is $\kappa = (1+c)/(1-c)$.

4 Implications

The matrix A is used by Google to compute PageRank, an estimate of web-page importance used for ranking search results [3]. PageRank is defined as the stationary distribution of the Markov chain corresponding to the $n \times n$ stochastic transition matrix A^T. The matrix P corresponds to the web link graph; in making P stochastic, there are standard techniques for dealing with web pages with no outgoing links [1].

The strongest implication of this result has to do with the stability of PageRank. A proof of stability of PageRank is given in [2], but we show a tighter stability bound here. Imagine that the Google matrix A is perturbed slightly, either by modifying the link structure of the web (by adding or taking away links), or by changing the value of c. Let us call this perturbed matrix $\tilde{A} = A + \epsilon B$, where ϵB is the "error matrix" describing the change to the web matrix A. Let $\boldsymbol{x}$ be the PageRank vector corresponding to the web matrix A, and let $\tilde{\boldsymbol{x}}$ be the vector corresponding to the web matrix $\tilde{A}$. It is known that, for a linear system of equations,

$$||\boldsymbol{x} - \tilde{\boldsymbol{x}}||_1 \leq \kappa\epsilon||B||$$

From Theorem 1, we can rewrite this as:

$$||\boldsymbol{x} - \tilde{\boldsymbol{x}}||_1 \leq \epsilon\frac{1+c}{1-c}||B||$$

What this means is, for values of c near to 1, PageRank is not stable, and a small change in the link structure may cause a large change in PageRank. However, for smaller values of c such as those likely used by Google ($.8 < c < .9$), PageRank is stable, and a small change in the link structure will cause only a small change in PageRank.

Another implication of this is the accuracy to which PageRank may be computed. Again, for values of c likely used by Google, PageRank is a *well-conditioned* problem meaning that it may be computed accurately by a stable algorithm. However, for values of c close to 1, PageRank is an *ill-conditioned* problem, and it cannot be computed to great accuracy by any algorithm.

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