The Classification of Quasithin Groups
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ABSTRACT. Around 1980, G. Mason announced the classification of the quasithin finite simple groups of characteristic 2-type in which all proper simple sections are known; but he neither completed nor published his work. We provide a proof of a stronger theorem classifying those groups, which is independent of Mason. In particular we close this gap in the proof of the classification of the finite simple groups. We also prove a corollary classifying quasithin groups of even type: providing a bridge to the program of Gorenstein, Lyons, and Solomon; their program seeks to produce a new, simplified proof of the classification of the finite simple groups.

To Pam and Judy

Contents

Preface

The classification of the quasithin simple groups of even characteristic can be thought of as roughly one fourth of the classification of the finite simple groups. The two volumes in this series provide the first proof that each group in this class is a known simple group. This result closes a gap in the classification of the finite simple groups which has existed for over twenty years.

In addition the series is part of an ongoing effort to reorganize and simplify the original proof of the classification of the finite simple groups, and to write the proof down carefully in a relatively short number of pages (e.g., less than ten thousand). The effort includes the "GLS" series of Gorenstein, Lyons, and Solomon, which at the moment consists of the five volumes [**GLS94**]–[**GLS02**], but it also includes smaller projects such as [**Asc94**] and [**BG94**].

A detailed discussion of these matters appears in the introductions to each of the two volumes in our series. Roughly speaking, the first volume consists of fairly general results on finite groups (with emphasis on quasithin groups) which serve as the foundation for the classification of the quasithin groups. The second volume consists of a proof that the groups listed in our Main Theorem are the simple quasithin groups of even characteristic, all of whose proper simple sections are known simple groups.

We would be remiss if we did not acknowledge the assistence of a number of people:

During the many years we have worked on this project, each of us visited and benefited from the hospitality of many universities and faculties, whose assistance we gratefully acknowledge.

In particular, we would like to thank Ulrich Meierfrankenfeld for calling our attention to Stellmacher's *qrc*-Lemma, and stating it in the form we use heavily as our Theorem D.1.5. Ulrich also read portions of the manuscript and suggested various simplifications.

We thank Robert Guralnick and Gunter Malle, whose work in [**GM02**] and [**GM04**] establishes important results on representations of finite simple groups related to failure of factorization, some of which have been unpublished for years. They relieved us of the need to prove those results; we thank them for providing us with prepublication copies of their work, and also for reading over the parts of our work which apply their work.

Similarly we would like to thank Richard Lyons and Ronald Solomon, who read over and helped improve our final chapter, which proceeds under the hypothesis of the GLS series.

We also thank the University of Florida group theory seminar (including Chat Ho and Peter Sin) for reading other parts of the manuscript, correcting various errors, and suggesting simplifications.

Most importantly, we would like to thank John Thompson for reading large portions of the two volumes and suggesting numerous improvements and simplifications. The authors, and indeed the finite group theory community, owe him a great debt of gratitute for his selfless work benefiting us all.

Volume I: Structure of strongly quasithin $\mathcal{K}$-groups

The proof of our Main Theorem classifying QTKE-groups appears in Volume II of this series. In Volume I we collect results which are used in that proof, but which are not explicitly about QTKE-groups. We have chosen to place such results here, so as to not interupt the flow of the proof of the main theorems. Some of the results are known and fairly general. Others are more specialized and are proved here for the first time.

Introduction to Volume I

The treatment of the "quasithin groups of even characteristic" is one of the major steps in the Classification of the Finite Simple Groups.   (for short, the Classification). As a part of the original Classification program, Geoff Mason announced a classification of a subclass of the quasithin groups in about 1980, but he never published his work, and the preprint he distributed [**Mas**] is incomplete in various ways. In two lengthy volumes, we now treat the quasithin groups of even characteristic; in particular, we close that gap in the proof of the Classification.

In this first volume, we record and develop the machinery necessary to prove our Main Theorem, which classifies the simple quasithin $\mathcal{K}$-groups of even characteristic. The second volume implements that proof. Each volume contains an Introduction discussing its contents, and each contains a statement of the Main Theorem with some definitions and discussion to help the reader understand the statement of the theorem and place it in context.

Section 0.1 of this Introduction to Volume I contains the statement of the Main Theorem. Section 0.2 consists of a brief overview of the most important general topics in Volume I. Then the subsequent sections give more details about what we prove in each of the general categories. The final section 0.13 of the Introduction describes the references we appeal to during the course of the proof; these "background references" consist of texts, and a small number of papers in the literature which have the same small controlled set of background references.

The Introduction to Volume II contains an extended discussion of the proof of the Main Theorem and some discussion of the history of the quasithin group problem.

0.1. Statement of Main Results

We begin by defining the class of groups considered in our Main Theorem; since the definitions are somewhat technical, we also supply some motivation. For definitions of more basic group-theoretic notation and terminology, the reader is directed to [**Asc86a**].

The quasithin groups are the "small" groups in that part of the Classification where the actual examples are primarily the groups of Lie type defined over a field of characteristic 2. We first translate the notion of the "characteristic" of a linear group into the setting of abstract groups: Let G be a finite group, $T \in Syl_2(G)$, and let $\mathcal{M}$ denote the set of maximal 2-local subgroups of G. [1] We define G to be of *even characteristic* if

$$C_M(O_2(M)) \leq O_2(M) \text{ for all } M \in \mathcal{M} \text{ containing } T.$$

[1]A *2-local subgroup* is the normalizer of a nonidentity 2-subgroup.

The class of simple groups of even characteristic contains some families in addition to the groups of Lie type in characteristic 2. In particular it is larger than the class of simple groups of *characteristic 2-type* ($C_M(O_2(M)) \leq O_2(M)$ for *all* $M \in \mathcal{M}$), which played the analogous role in the original proof of the Classification.

The Classification proceeds by induction on the group order. Thus if G is a minimal counterexample to the Classification, then each proper subgroup H of G is a $\mathcal{K}$-*group*: that is, all composition factors of each subgroup of H lie in the set $\mathcal{K}$ of known finite simple groups.

Finally quasithin groups are "small" by a measure of size introduced by Thompson in the N-group paper [**Tho68**]: Define

$$e(G) := \max\{m_p(M) : M \in \mathcal{M} \text{ and } p \text{ is an odd prime}\}$$

where $m_p(M)$ is the *p-rank* of M (the maximum rank of an elementary abelian p-subgroup of M). When G is of Lie type in characteristic 2, $e(G)$ is a good abstract approximation of the Lie rank of G. Janko called the groups with $e(G) \leq 1$ "thin" groups, leading Gorenstein to define G to be *quasithin* if $e(G) \leq 2$. The groups of Lie type of characteristic 2 and Lie rank 1 or 2 are the "generic" simple quasithin groups of even characteristic.

Define a finite group H to be *strongly quasithin* if $m_p(H) \leq 2$ for all odd primes p. Thus the 2-locals of quasithin groups are strongly quasithin.

We combine the three principal conditions into a single hypothesis:

Hypothesis 0.1.1 (Main Hypothesis). *Define G to be a* QTKE-group *if*

(QT) G is quasithin,
(K) all proper subgroups of G are $\mathcal{K}$-groups, and
(E) G is of even characteristic.

We prove:

Theorem 0.1.2 (Main Theorem). *The finite nonabelian simple QTKE-groups consist of:*

(1) (Generic case) Groups of Lie type of characteristic 2 *and Lie rank at most* 2, *but $U_5(q)$ only for $q = 4$.*
(2) (Certain groups of rank 3 *or* 4*)* $L_4(2)$, $L_5(2)$, $Sp_6(2)$.
(3) (Alternating groups) A_5, A_6, A_8, A_9.
(4) (Lie type of odd characteristic) $L_2(p)$, *p a Mersenne or Fermat prime,* $L_3^\epsilon(3)$, $L_4^\epsilon(3)$, $G_2(3)$.
(5) (Sporadic) M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, He, Ru.

We recall that there is an "original" or "first generation" proof of the Classification made up by and large of work done before 1980, and a "second generation" program in progress, whose aim is to produce a somewhat different and simpler proof of the Classification. The two programs take the same general approach, but often differ in detail. Our paper is a part of both efforts.

In particular Gorenstein, Lyons, and Solomon (GLS) are in the midst of a major program to revise and simplify the proof of part of the Classification. We also prove a corollary to our Main Theorem, which supplies a bridge between that result and the GLS program. The corollary is discussed in detail in Volume II.

0.2. An overview of Volume I

In this section, we briefly describe in general terms the main topics covered in Volume I. Then in the remaining sections of the Introduction, we give more details about each topic.

We have made some attempt to collect subtopics of the same flavor together in chapters. However, linear exposition sometimes makes this difficult. Also the complexity and duration of the project has contributed to less than optimal organization.

Many of the results in this volume are known or even "well known". Sometimes proofs appear in the literature, but just as often the results are part of the "folklore". One important contribution of this volume is to collect these basic facts and lemmas about finite groups in one place, and in many cases write down the first published proof of a lemma. We also prove a number of new results which are of interest outside of the context of quasithin groups. At the end of this section, we indicate some of the results in Volume I which we regard as most interesting in their own right.

Volume I records and proves results in the following general categories:

We record, for the convenience of the reader, the statements of some basic lemmas from texts appearing in our list of background references. We also establish more specialized corollaries which are fairly immediate consequences of these basic results.

We record and prove many facts about quasithin $\mathcal{K}$-groups. The quasithin hypothesis is not inherited by all relevant sections of a quasithin group, but the $\mathcal{K}$-group hypothesis is inherited. Thus an important early step is to determine all quasithin and strongly quasithin semisimple $\mathcal{K}$-groups. Then we list and establish many facts about the automorphism groups, Schur multipliers, subgroup structure, and $\mathbf{F}_2$-representations of such groups, which are crucial to the proof of the Main Theorem, and in many cases of independent interest.

Another crucial early step is to give a qualitative description of the structure of strongly quasithin $\mathcal{K}$-groups. This work builds on the earlier results on the semisimple groups.

An important tool in the proof of the Main Theorem is Thompson factorization (see section 0.6) and various related notions. There are lemmas about Thompson factorization scattered through the literature, and other results exist in the folklore. We collect and reprove some of these lemmas, and also establish new results related to Thompson factorization.

An (abstract) "minimal parabolic" is a finite group G such that a Sylow 2-subgroup of G is not normal in G and is contained in a unique maximal subgroup of G. The notion was first considered by P. McBride. We give proofs of known facts about minimal parabolics, and prove many new results about strongly quasithin minimal parabolics.

The notion of "pushing up" is not well defined in the literature. Roughly speaking, one kind of pushing up involves finding a common nontrivial normal 2-subgroup U of some pair G_1, G_2 of 2-locals in a finite group G, given that some large 2-subgroup R is embedded in G_1 and G_2 in some suitable way; if we can produce U then we have "pushed up" the pair G_1, G_2 by embedding it in the 2-local $N_G(U)$. For example, R might be normal in G_1 and Sylow in G_2 with $F^*(G_2) = O_2(G_2)$, in which case it suffices to find a nontrivial characteristic subgroup of R normal in G_2.

Another type of pushing up considers the following setup: Let R be a 2-subgroup of G and define

$$C(G,R) := \langle N_G(C) : 1 \neq C \text{ char } R\rangle.$$

In this pushing up problem, the object is to determine those groups G with $F^*(G) = O_2(G)$ and $C(G,R) < G$ for suitable "'large" 2-subgroups R of G.

There are many pushing up results in the literature. We record some of these, give proofs of others, and prove some new pushing up theorems; see section 0.8.

Weak closure is another important tool for analyzing groups of even characteristic. Weak closure originated in the work of Thompson, and was used by him to good effect in the N-group paper. We give a treatment of weak closure, and include proofs of a few new lemmas; see section 0.9.

Yet another important tool in the theory of groups of even characteristic is the so-called "amalgam method." These techniques go back to Tutte [**Tut47**] and Sims [**Sim67**], and were first used in the context of groups of even characteristic by Thompson in the N-group paper. The modern amalgam method was begun by Goldschmidt in [**Gol80**], and was extended by various authors such as Delgado, Meierfrankenfeld, and Stellmacher. Our approach is a bit different from the standard approach, and is described briefly in section 0.10.

Finally to prove the Main Theorem, or almost any classification theorem in finite simple group theory, one needs "recognition theorems": Results which say that if $\bar{G} \in \mathcal{K}$ and G is a finite group such that some family of subgroups of G "closely resembles" a suitable family in $\bar{G}$, then G is isomorphic to $\bar{G}$. Sometimes we prove our own recognition theorems, and sometimes we appeal to a theorem already in the literature. In particular, Volume I contains proofs of several recognition theorems, including results on groups of Lie type of Lie rank 2, and on the Rudvalis sporadic group Ru. These results are of independent interest.

As promised, we close this initial section with a short list of results from Volume I which be believe are of particular interest in contexts other than that of the quasithin groups:

(1) Theorem F.4.8 and its corollaries F.4.24, F.4.26, and F.4.31, which supply recognition theorems for extensions of groups of Lie type of Lie rank 2. We believe that the approach in Theorem F.4.8, of characterizing completions of amalgams via the existence of short cycles in the coset complex, deserves more attention.

(2) Theorem J.1.1 which provides a 2-local recognition theorem for the Rudvalis group Ru. Chapter J also contains a proof that the standard list of subgroups of Ru of prime order, and their normalizers, is correct.

(3) Our extension of Stellmacher's *qrc*-Lemma for QTKE-groups. Formally the result appears as Theorem 3.1.6 in Volume II; however, all but a few details of the proof appear in Volume I. The result states:

THEOREM 0.2.1. *Let G be a simple QTKE-group, $T \in Syl_2(G)$, $T \leq M_0 \leq M \in \mathcal{M}$, and H a subgroup of G minimal subject to $T \leq H \not\leq M$ and $O_2(H) \neq 1$. Assume that V is a normal elementary abelian 2-subgroup of M_0 which satisfies $O_2(M_0/C_{M_0}(V)) = 1$, $O_2(M_0) = C_T(V)$, and $H \cap M$ normalizes V or $O^2(M_0)$. Then either $O_2(\langle M_0, H\rangle) \neq 1$ or $\hat{q}(M_0/C_{M_0}(V), V) \leq 2$.*

The parameter $\hat{q}$ is defined in section 0.6.3. It should be possible to remove the quasithin hypothesis from the Theorem.

(4) In section C.1, we supply a proof of the Local $C(G,T)$-Theorem for $\mathcal{K}$-groups. The $\mathcal{K}$-group hypothesis makes possible a proof which is much simpler than the original.

(5) The statement and proof of "Baumann's Argument" in lemma B.2.18, and of "Glauberman's Argument" in lemma C.1.21. To our knowledge, this is the first time these important results in local group theory have been written down explicitly.

(6) The basic inheritance properties of "offenders in 2F-modules" in section D.2.

(7) Theorem G.11.2 in section G.11, which lists the possible actions of the normalizer of a large extraspecial 2-subgroup of a QTKE-group on the Frattini quotient of the 2-group. It should be possible to extend this result to arbitary $\mathcal{K}$-groups.

0.3. Basic results on finite groups

Section A.1.2 contains basic results on finite groups. Some results are listed without proof, with references to texts in our background references where proofs can be found. Many of these results are known to finite group theorists by standard names such as the Three Subgroup Lemma, the Baer-Suzuki Theorem, etc. When we refer to such results, we sometimes do so via the standard name and the lemma number, or sometimes just via the standard name.

Section A.1.2 also contains some more specialized corollaries to the basic lemmas; for example, corollaries which hold only in strongly quasithin groups. We supply the easy proofs of such corollaries.

Lemma A.1.21 gives properties of supercritical subgroups of p-groups of odd order. The term "supercritical subgroup" is not used in the literature; it is a term we have coined. However, the notion certainly already exists in the folklore, extending the notion of "critical subgroup" due to Thompson.

0.4. Semisimple quasithin and strongly quasithin $\mathcal{K}$-groups

In section A.2, we determine the simple quasithin and strongly quasithin $\mathcal{K}$-groups in Theorems B (A.2.2)and C (A.2.3), respectively. Since the quasithin hypothesis is not inherited by enough proper sections, it is these results which give us control of the composition factors of proper subgroups of QTKE-groups. Moreover Theorem A (A.2.1) essentially extends Theorems B and C to semisimple groups, so we also have control over semisimple sections of proper subgroups of QTKE-groups.

Much of the rest of Volume I is devoted to establishing various properties of the groups appearing in Theorem C; we will say more about this later.

0.5. The structure of SQTK-groups

An *SQTK-group* is a strongly quasithin $\mathcal{K}$-group. In particular the 2-local subgroups of QTKE-groups are SQTK-groups.

In section A.3, we establish qualitative results about the structure of SQTK-groups. The description is in terms of $\mathcal{C}$-components of the group, a notion introduced and defined in that section; here is a summary:

A $\mathcal{C}$*-component* of a finite group G is a subgroup of G minimal subject to being nontrivial, perfect, and subnormal in G. We write $O_\infty(G)$ for the largest solvable normal subgroup of G. From A.3.3, for each $\mathcal{C}$-component L of G, $L/O_\infty(L)$ is a nonabelian simple group; and the map $L \mapsto LO_\infty(G)/O_\infty(G)$ is a bijection between the set of $\mathcal{C}$-components of G and the set of components of $G/O_\infty(G)$.

Assume G is an SQTK-group and write G^∞ for the last term in the derived series of G. In section A.3, we establish four important facts about G: First, G^∞ is the product of the $\mathcal{C}$-components of G. Second, if L is a $\mathcal{C}$-component of G, then aside from a special class of exceptions described in proposition A.3.6, $L/O_2(L)$ is quasisimple. Third, if K is a $\mathcal{C}$-component of G distinct from L, then $[L,K] \leq O_2(G)$; that is, distinct $\mathcal{C}$-components of G commute modulo $O_2(G)$. Fourth, either L is normal in G, or L is one of a number of special exceptions listed in lemma A.3.8.3, and L^G is of order 2.

Section A.1.2 also contains more specialized results about the structure of SQTK-groups.

0.6. Thompson factorization and related notions

As is the case with so many notions crucial to the study of groups of even characteristic, Thompson factorization was introduced by Thompson. We begin our discussion in subsection 0.6.1, with the most basic setup for Thompson factorization and its translation into the language of $\mathbf{F}_2$-representation theory. We also discuss lemmas which extend the basic setup. Then in subsections 0.6.2 and 0.6.3, we move on to generalizations of Thompson factorization and corresponding problems in representation theory.

0.6.1. Thompson factorization. Let G be a finite group. The *Thompson subgroup* of G (at the prime 2) is the subgroup $J(G)$ of G generated by the elementary abelian 2-subgroups of G of maximal rank. We will be most interested in the Thompson subgroups of 2-groups, and in particular in $J(T)$ for T a Sylow 2-group of a finite group G. Assume that $F^*(G) = O_2(G)$ and let $Z := \Omega_1(Z(T))$, $V := \langle Z^G \rangle$, and $G^* := G/C_G(V)$. Then $O_2(G^*) = 1$, and the theory of Thompson factorization (cf. B.2.7) tells us that either

(a) We have Thompson factorization: $G = N_G(J(T))C_G(V) = N_G(J(T))C_G(Z)$, or

(b) V is an *FF-module* for G^*: that is, there is a nontrivial elementary abelian 2-subgroup A^* of G^* such that

$$r_{A^*,V} := \frac{m(V/C_V(A^*)))}{m(A^*)} \leq 1.$$

In case (a), Thompson factorization gives us information about G in terms of $N_G(J(T))$ and $C_G(Z)$ or $C_G(V)$, and can allow us to "push up" G inside some larger group: For example $G = C(G,T)$ since Z and $J(T)$ are characteristic in T; and if $C_G(V)$ is 2-closed, then $J(T) \trianglelefteq G$.

In case (b), we are led to the study of faithful FF-modules for finite groups G^* with $O_2(G^*) = 1$. It develops (cf. the discussion in subsection 0.11.2) that such modules are rare, and hence we also have strong information about G in case (b), which can also be used to push up G.

Indeed it is not necessary to restrict attention to Sylow 2-groups of G; similar results hold for 2-subgroups R such that $C_R(V) \in Syl_2(C_G(V))$. Another extension was initiated by Baumann: Define the *Baumann* subgroup of a 2-group R to be

$$\mathrm{Baum}(R) := C_R(\Omega_1(Z(J(R)))).$$

Under suitable restrictions on R and G, Baumann's analysis shows that if Thompson factorization fails, and we set $B := \mathrm{Baum}(R)$ and $L := \langle B^G \rangle$, then $C_B(V) \in Syl_2(C_L(V))$, so we can apply the theory of Thompson factorization and pushing up to the pair B, L.

In sections B.1 and B.2, we record and prove basic facts from the theory of Thompson factorization and FF-modules. In addition we state and prove "Baumann's Argument" in B.2.18. This is the argument Baumann used in [**Bau76**] to prove "Baumann's Lemma" (see B.6.10). It is part of the folklore that this argument applies in much greater generality; but to our knowledge, lemma B.2.18 constitutes the first instance in the literature where an extension is stated explictly, much less proved.

0.6.2. Extensions of Thompson factorization. One can also define higher Thompson subgroups as follows: Let $T \in Syl_2(G)$, $m := m(T)$, and for i a nonnegative integer, define $J_i(T)$ to be the subgroup generated by all elementary abelian 2-subgroups of T of rank at least $m - i$. Thus $J_0(T) = J(T)$.

Under suitable hypotheses on G and nonnegative integers i and j, one can establish analogues of Thompson factorization of the form

$$G = N_G(J_i(T))C_G(\Omega_1(Z(J_{i+j}(T))));$$

see for example 5.53 in Thompson [**Tho68**].

In his preprint on quasithin groups, Mason establishes some extensions of this lemma of Thompson. Using our theory involving the parameter $n(G)$ discussed in section 0.9, in section E.5 we establish other extensions of Thompson's lemma, suggested by Mason's generalization. This theory is again useful in pushing up, and also in conjunction with weak closure.

0.6.3. An extension of Stellmacher's *qrc*-Lemma. In [**Ste92**], Stellmacher introduces an innovative and very useful variation on Thompson factorization: the *qrc*-Lemma. However the *qrc*-Lemma as stated in [**Ste92**] is probably a bit opaque to anyone who has not already seen it applied. The version we state and prove in section D.1 as D.1.5 was communicated to us by Meierfrankenfeld; its value is also probably unclear to the uninitiated, so here is an overview: In D.1.5 we have a pair of subgroups G_1 and G_2 of a group G with a common Sylow 2-subgroup, and an elementary abelian normal 2-subgroup V of G_1 such that $O_2(G_1^*) = 1$, where $G^* := G_1/C_{G_1}(V_1))$. Then, subject to some extra constraints, D.1.5 says that one of five restrictive conclusions holds, which are discussed briefly below.

We go on to prove an extension of the *qrc*-Lemma which appears as Theorem 3.1.6 in Volume II, (and is reproduced as Theorem 0.2.1 in section 0.2); but is really proved in Volume I, as we now indicate: In cases (2) and (4) of D.1.5, $q(G_1^*, V) \leq 2$, where $q(G_1^*, V)$ is the minimum value of $r_{A^*,V}$, as A^* ranges over the nontrivial elementary abelian 2-subgroups of G_1^* with $[V, A^*, A^*] = 0$. By Thompson Replacement (cf. B.1.4.5), V is an FF-module iff $q(G_1^*, V) \leq 1$, so the condition that $q(G_1^*, V) \leq 2$ can be viewed as a generalization of the condition that V is an FF-module. In case (3) of D.1.5, the dual of V is an FF-module for G_1^*; we show in B.5.13 that if G_1 is an SQTK-group, then in this case again $q(G_1^*, V) \leq 2$. In case (1), lemma E.2.15 tells us that $\hat{q}(G_1^*, V) < 2$, where $\hat{q}(G_1^*, V)$ is the minimum value of $r_{A^*,V}$, as A^* ranges over the nontrivial elementary abelian 2-subgroups of G_1^* with $[V, A^*, A^*, A^*] = 0$. Finally in case (5) in the proper context, Theorem C.5.8 shows that if G is a QTKE-group, then $\langle G_1, G_2 \rangle$ is contained in a 2-local subgroup of G, so we have pushed up the pair G_1, G_2. Now visibly $\hat{q}(G_1^*, V) \leq q(G^*, V)$, so that ultimately the *qrc*-Lemma says that (in the proper context in a QTKE-group) we can push up the pair G_1, G_2 unless $\hat{q}(G_1^*, V) \leq 2$.

Meierfrankenfeld and Stellmacher refer to $\mathbf{F}_2 G_1^*$-modules V with $\hat{q}(G_1^*, V) \leq 2$ as *2F-modules*. Thus 2F-modules are generalizations of FF-modules, and our extension of the *qrc*-Lemma says that such modules are an obstruction to pushing up in QTKE-groups. It is not unreasonable to expect that this extension of the *qrc*-Lemma holds in general without our QTKE-hypotheses.

0.7. Minimal parabolics

An (abstract) "minimal parabolic" is a finite group G such that a Sylow 2-subgroup of G is not normal in G and is contained in a unique maximal subgroup of G. The notion was first considered by P. McBride. The general structure of a minimal parabolic is well known; cf. B.6.8. In section E.2, we go on to establish more detailed results about minimal parabolics which are SQTK-groups. In one such result (see E.2.15) we show that in case (1) of the *qrc*-Lemma, when G_2 is an SQTK-group then $\hat{q}(G_1^*, V) < 2$, as discussed in the previous section.

0.8. Pushing up

Having developed the basics of Thompson factorization, the most important tool in pushing up, we begin our formal study of pushing up in chapter C.

The obstructions to pushing up in a finite group G with $F^*(G) = O_2(G)$ are the *blocks* of G: The subnormal subgroups $L = O^2(L)$ of G such that $L/O_2(L)$ is quasisimple or $\mathbf{Z}_3$, $U(L) := [O_2(L), L] \leq Z(O_2(L))$, and L is irreducible on $\tilde{U}(L) := U(L)/C_{U(L)}(L)$.

A block L is of *type* $L_2(2^n)$ for $n > 1$ an integer if $L/O_2(L) \cong L_2(2^n)$ and $\tilde{U}(L)$ is the natural module for $L/O_2(L)$. Similarly for $m \geq 3$ an integer, the block L is of *type* A_m if $L/O_2(L) \cong A_m$ and $\tilde{U}(L)$ is the natural module for $L/O_2(L)$. Thus for example a block of type A_3 is isomorphic to A_4; by convention we also refer to such blocks as blocks of *type* $L_2(2)$, since $A_3 \cong \mathbf{Z}_3 \cong L_2(2)'$.

Blocks of type $L_2(2^n)$ and blocks of type A_m with m odd are the most important obstruction to pushing up. This is already evident in the case of solvable 2-locals in the N-group paper, where Thompson produces blocks of type $L_2(2)$ via an appeal to a theorem of Sims. In his work on thin groups, Aschbacher used early work of Glauberman to produce $L_2(2^n)$-blocks, and "Glauberman's Argument" to produce A_5-blocks (cf. 5.3 in Aschbacher [**Asc78a**]). Glauberman's work eventually led to the Glauberman-Niles/Campbell Theorem, which we record as Theorem C.1.18, with a reference to the paper [**GN83**] of Glauberman and Niles for a proof.

The argument of Glauberman used to produce A_5-blocks has been used in the literature in other pushing up problems. In lemma C.1.21 we formalize and prove a version of "Glauberman's Argument"; as far as we know this is the first time such a generalization has been written down explictly and proved.

In Theorem C.1.29 we give a new proof of the "Local $C(G,T)$-Theorem" for $\mathcal{K}$-groups. The Local $C(G,T)$-Theorem says that if $F^*(G) = O_2(G)$, $T \in Syl_2(G)$, and $C(G,T) \leq M \leq G$, then $G = ML_1 \cdots L_r$, where L_i is an $L_2(2^{n_i})$ or $A_{2^{n_i}+1}$-block. The $\mathcal{K}$-group assumption makes possible a proof which is much simpler than Aschbacher's original proof in [**Asc81a**].

We also record in Theorems C.1.32 and C.1.34, a pushing up theorem of Meierfrankenfeld and Stellmacher (with a reference to their original proof in [**MS93**]) which applies to certain 2-radical subgroups of a group G such that $G/O_2(G)$ is essentially a group of Lie type in characteristic 2 of Lie rank 2.

Recall a 2-subgroup R of G is *radical* if $R = O_2(N_G(R))$. In sections C.2 through C.4, we develop a theory of pushing up for radical subgroups of SQTK-groups satisfying some extra properties. This theory is a special case of work in [**Asc81b**], although we supply our own proofs except in the case of some easy lemmas.

0.9. Weak closure

Weak closure originated in the work of Thompson, particularly in the N-group paper. Chapter E contains a discussion of weak closure, much of which comes from [**Asc81c**]. Theorem E.6.3 is the deepest result; in a sense it extends Theorem 11.1 in [**Asc81c**] from groups of characteristic 2-type to QTKE-groups, in that it relaxes the hypothesis in [**Asc81c**] that G is of characteristic 2-type to the hypothesis that G is of even characteristic, although the quasithin hypothesis is added.

Most of the remaining theory of weak closure in chapter E is formal and fairly easy to establish; see section E.3.

Section E.1 contains a discussion of the parameter $n(G)$ of a finite group G, which roughly speaking measures the maximum of the 2-ranks $m_2(F/O_2(F))$ of subgroups F in a certain family of subgroups associated to some fixed Sylow 2-subgroup T of G. This family is critical in weak closure and other pushing up situations. The family appeared first in [**Asc81c**], and the lemmas in section E.1 come from that reference.

0.10. The amalgam method

In the amalgam method, one focuses attention on a pair G_1, G_2 of finite subgroups of a group G sharing a common Sylow 2-subgroup T, such that no nontrivial subgroup of T is normal in $G_0 := \langle G_1, G_2 \rangle$. Assume also that $F^*(G_i) = O_2(G_i)$ for $i = 1, 2$. The amalgam method concentrates on the coset graph Γ with vertices $G_0/G_1 \cup G_0/G_2$, and two cosets adjacent if their intersection is nonempty. Write γ_i, $i = 0, 1$, for G_{i+1} regarded as a vertex of Γ.

Our version of the method is not standard: we also give ourselves a normal elementary abelian 2-subgroup V of G_1 such that $C_T(V) = O_2(C_{G_1}(V))$ and $O_2(\bar{G}_1) = 1$, where $\bar{G}_1 := G_1/C_{G_1}(V)$. For $\gamma_0 g \in \Gamma$, let $V_{\gamma_0 g} := V^g$. In our version of the amalgam method, we consider the parameter b, which is the greatest positive integer such that V fixes each vertex of Γ at distance b from γ_0. Typically in the amalgam method, one considers the normal closure V_{γ_1} of V under G_2; for $\gamma_1 h \in \Gamma$, let $V_{\gamma_1 h} := V_{\gamma_1}^h$. The usual amalgam method analyzes the relationship between V_α and V_β for various $\alpha, \beta \in \Gamma$, particularly the relationship between V and V_γ for $\gamma \in \Gamma$ at distance b from γ_0 such that V does not fix some vertex adjacent to γ. Section F.7 contains the basic lemmas necessary for this analysis.

In our approach, the normal closure U in G_2 of a suitable subgroup V_+ of V is often of more interest. We choose V_+ to contain a nontrivial normal subgroup V_1 of G_2, and so that $G_1 \cap G_2$ is irreducible on V_+/V_1. In most cases where we apply the amalgam method, V_1 is of order 2, so we assume that $|V_1| = 2$ in the remainder of this exposition.

We usually partition our analysis into two cases: U nonabelian and U abelian. In the first case, U is "almost extraspecial": that is, $\Phi(U) = V_1$ is of order 2 and $U = U_0 Z(U)$, where U_0 is an extraspecial 2-group. This allows us to use some of the techniques for studying groups with a large extraspecial 2-subgroup; see for example section G.2 and the discussion below of the related $\mathbf{F}_2$-representations.

The case where U is abelian is more subtle and difficult. We develop some machinery in sections F.8 and F.9 to handle this case. Ultimately we achieve some control over G_2 by showing that $q(G_1/C_{G_1}(U/V_1), U/V_1) \leq 2$.

0.11. Properties of $\mathcal{K}$-groups

Most of the results discussed above depend upon various properties of the simple quasithin and strongly quasithin groups L in $\mathcal{K}$ that appear in Theorems B and C. In this section we discuss some of these properties.

0.11.1. Schur multipliers, covering groups, and 1-cohomology. In section I.1, we list the Schur multipliers of the quasithin $\mathcal{K}$-groups, and some 1-cohomology groups of small $\mathbf{F}_2$-modules for those groups; these results come from the literature. We also prove various facts about the covering groups of the groups for which there is no convenient reference. In B.4.9 and in section I.1, we prove that certain $\mathbf{F}_2$-extensions split when appropriate noncyclic elementary abelian 2-groups act quadratically.

0.11.2. FF-modules and 2F-modules. In [**GM02**] and [**GM04**], Guralnick and Malle determine all pairs (L, V) such that L is a quasisimple $\mathcal{K}$-group with $O_2(G) = 1$ and V is a faithful 2F-module for a finite group G such that $F^*(G) = L$. We consider the cases where G is a SQTK-group, and determine which modules are FF-modules, which satisfy $q(G, V) \leq 2$, and which satisfy $\hat{q}(G, V) < 2$. In many cases we determine the offending subgroups: that is, the nontrivial elementary abelian 2-subgroups A such that (essentially) $r_{A,V} = q(G, V)$ or $\hat{q}(G, V)$. The pairs that arise are listed in Theorems B.4.2 and B.4.5. Also FF-offenders are described in Theorm B.4.2, and some information about q and $\hat{q}$ is given in Theorem B.4.5. Proofs of many of the facts appearing in these theorems can be found in chapter K.

Offenders for the alternating groups on their natural modules are discussed in detail in section B.3. Detailed information about some of the less accessible modules for various groups appears in chapter H, in particular for the code and cocode modules of the Mathieu groups.

The literature contains a fairly well developed theory describing the general structure of FF-modules. This literature is summarized in section B.1. No such literature exists for 2F-modules, so section D.2 introduces the necessary conceptual base and proves various basic facts about such modules.

In section B.5, we determine the representations $\varphi : G \to GL(V)$ of SQTK-groups G such that $O_2(G\varphi) = 1$ and V is an FF-module for $G\varphi$. This treatment uses the list of possiblities with $F^*(G)$ quasisimple and irreducible on V as well as the theory from section B.1.

In section D.3, we determine the representations $\varphi : G \to GL(V)$ of SQTK-groups G such that $O_2(G\varphi) = 1$ and V is an 2F-module for $G\varphi$ satisfying suitable minimality conditions. Again we use the results on the case $F^*(G)$ quasisimple and irreducible on V, and the theory from section D.2.

0.11.3. $\mathbf{F_2}$-representations on extraspecial groups. As is well known, many finite simple groups G contain a *large extraspecial 2-subgroup*; that is, G possesses an involution z such that $F^*(G)$ is an extraspecial 2-subgroup. Recall a somewhat weaker condition arose in our approach to the amalgam method as described in section 0.10, where in our simple QTKE-group G we produce an "almost extraspecial" subgroup U in the centralizer of some involution z: that is, $U \trianglelefteq C_G(z)$ and $U = U_0Z(U)$ with U_0 extraspecial and $\langle z \rangle = \Phi(U)$.

The literature contains a classification of simple groups with a large extraspecial subgroup due to Timmesfeld, S. Smith, and others. We do not appeal to this literature, but we do borrow some of the elementary arguments from the literature. We use our $\mathcal{K}$-group hypothesis to avoid the most difficult parts of the analysis of large extraspecial subgroups. Most particularly, we consider the representation of $C_G(z)$ on the quotient $U/Z(U)$, and translate elementary notions from the theory of large extraspecial subgroups into restrictions on this representation. Then in sections G.6–G.11, we classify the $\mathbf{F}_2$-modules for SQTK-groups satisfying those restrictions. The groups and representations that survive are all realized via some large extraspecial subgroup of some group, although not always in a simple QTKE-group.

Presumably it is possible to extend this approach to arbitrary finite simple groups G.

0.11.4. Permutation modules. We determine the structure and special properties of various permutation modules over $\mathbf{F}_2$. Most of this information is well known, and all but some of the specialized information is at least in the folklore. For example the structure of the permutation module of degree n for S_n (cf. section B.3) and the multiply transitive permutation modules of degree m for the Mathieu groups M_m are certainly well known and decribed in detail in the literature. On the other hand, chapter H contains some specialized information about the modules for the Mathieu groups that may be new.

We also obtain detailed information about the 7-dimensional and 21-dimensional modules for $L_3(2)$ in sections H.5 and H.6, and the 15-dimensional modules for A_6 in section G.5.

0.12. Recognition theorems

Recall a "recognition theorem" is a result which says that if G and $\bar{G}$ are groups such that the subgroup structure of G "resembles" that of $\bar{G}$, then $G \cong \bar{G}$. Volume I proves a number of recognition theorems which are of interest independent of the study of quasithin groups.

0.12.1. Amalgams. The notion of an "amalgam" supplies one set of hypotheses which makes possible group recognition.

An *amalgam*

$$\alpha := (X_a, (\alpha_{a,b} : X_a \to X_b) : a, b \in D,\ a \leq b),$$

is a functor from some poset D (regarded as a small category) to the category of groups. For example if D is a collection of subgroups of G partially ordered by inclusion, then the inclusion maps among these subgroups define a *subgroup amalgam* on D. There is an obvious notion of morphisms of amalgams. A *completion* of α is a morphism $\xi = (\xi_d : X_d \to G_d)$ of α with some subgroup amalgam $(G_d : d \in D)$ such that ξ_d is surjective for each $d \in D$ and $G = \langle G_d : d \in D \rangle$. The completion is *faithful* if the maps are injective.

Moreover there is a universal completion $grp(\alpha)$. See section F.2 for more details.

Associated to each completion G is a simplical complex $\Gamma(G)$, the coset complex on $\coprod_{d \in D} G/G_d$. Under weak constraints on D, $G = grp(\alpha)$ iff $\Gamma(G)$ is simply connected. Then if also $F^*(G)$ is simple, there is a unique faithful completion of α, so we can recognize G via the family α of subgroups.

By now this approach is a well understood tool in the group theoretic literature. However many subtleties have not been written down explicitly; we write out proofs of some of these facts (presumably for the first time) in section F.2.

0.12.2. Groups of Lie type of Lie rank 2. Let G be an extension of a finite group of Lie type of Lie rank 2, and $\alpha = (G_1, G_{1,2}, G_2)$ the amalgam of parabolics over a fixed Borel subgroup of G. The universal completion of α is infinite, and its coset complex is an infinite regular tree, so the approach of the previous subsection is not sufficient here. Instead we must introduce extra constraints on a completion $\hat{G}$ of α or its bipartite graph $\hat{\Gamma}$, to prove that $G \cong \hat{G}$.

The graph Γ of G is a Moufang (see F.4.17) generalized $2m$-gon for some integer m. Define the completion $\hat{G}$ to be *small* if for some pair x, x' of vertices at distance $2m$ in $\hat{\Gamma}$, there is more than one geodesic from x to x'. We prove in Theorem F.4.8 that if $\hat{G}$ is small, then $\hat{G} \cong G$. The proof uses the Tits-Weiss classification [**TW02**] of Moufang generalized polygons, and is not difficult when G is Moufang on Γ. However the theorem is also true when G is the Tits group ${}^2F_4(2)'$ or $G_2(2)' \cong U_3(3)$, where G is *not* Moufang on Γ, and hence Tits-Weiss can not be applied directly. In these two cases the proof is more subtle.

Theorem F.4.8 and its corollaries F.4.24 and F.4.26 are of interest as recognition theorems independent of the study of quasithin groups. For example they are used to prove 2-local recognition theorems in [**Asc02a**] and [**Asc02b**]. We also use Theorem F.4.8 to prove Theorem F.4.31, a 2-local recognition theorem for extensions of groups of Lie type, Lie rank 2, and characteristic 2. In addition Theorem F.4.8 is used in the proof of our recognition theorem for the sporadic group Ru, discussed in the next subsection.

0.12.3. The Rudvalis group Ru. In chapter J we prove a 2-local recognition theorem for the Rudvalis group Ru. This recognition theorem is also of independent interest. Moreover we prove that the usual list of conjugacy classes of subgroups of Ru of prime order and their normalizers is correct. See section J.1 for a discussion of the history of this problem and an exact statement of the theorem.

0.13. Background References

The "original proof" of the Classification requires the theorem that each QTK-group of characteristic 2-type is isomorphic to a group in $\mathcal{K}$. Our Main Theorem includes the first proof of this latter result, so it is part of the original proof of the Classification. However it is also part of an ongoing program to produce a simplified, self-contained proof of the Classification. Work in this program should meet a different standard from that for work in the "original proof". In particular, it should be made clear which references the work depends on, and these "background references" should be either standard texts, or papers in the literature which have the same small controlled set of background references.

We have borrowed the term "background references" from GLS, although our collection of background references is different from that of the GLS series. On the one hand, we appeal to a few papers not included in the GLS background, but on the other, we do not use all of the GLS background. Also we explicitly list references for the recognition theorems which we do not prove, whereas in the case of sporadic groups, GLS takes recognition as an axiom—that is, they regard their treatment of a case involving a sporadic group to be complete when they produce a suitable family of subgroups, leaving it to others to provide a body of recognition theorems for the sporadics meeting the standards alluded to above.

Moreover we have divided our bibliography into two parts: The first part contains our list of background references; this is further subdivided into those used in the GLS series, and those used by us but not by GLS. The second part consists of "expository references" (again following the terminology of GLS) and is much lengthier—but our work does not depend on these expository references; instead they are mentioned either for comparison, or in our discussion of the history of various parts of the work.

Volume I contains most of the technical machinery needed for the proof of the Main Theorem. Often the theorems we require do not exist in the literature, although sometimes the results do appear in the literature, but do not meet the standards outlined above for background references.

Our basic elementary reference is the text of Aschbacher [**Asc86a**]; other texts on finite groups are also used on occasion, like those of Gorenstein [**Gor80**], Huppert-Blackburn [**Hup67**] [**HB85**], and Suzuki [**Suz86**], as well as Aschbacher's more advanced texts [**Asc94**] and [**Asc97**]. For the convenience of the reader, we have stated the results from these texts that we use most often in section A.1.2 and chapter I. We also make frequent appeals to results in the GLS series; see [**GLS94**] and the succeeding numbers (through 5 so far) of that series in the bibliography. Many of these results are elementary, but there are appeals to some deep results proved there, like the Bender-Suzuki classification of groups with a strongly embedded subgroup (see Theorems SE and ZD on pp. 20 and 21 of [**GLS99**]). In lemma 16.2.10 in Volume II, we appeal to lemma 3.4 from [**Asc75**]; but that result is a fairly easy corollary to Theorem ZD in [**GLS99**], so the appeal could be removed without too much effort.

We use the Odd Order Theorem of Feit-Thompson [**FT63**] that a finite group of odd order is solvable. We follow the convention of GLS in our Background References by quoting the modern revision of Bender-Glauberman [**BG94**] as our reference for the proof of this result.

We need either the Fong-Seitz papers [**FS73**] on split BN-pairs of rank 2, or the Tits-Weiss book [**TW02**] on Moufang generalized polygons, to identify groups of Lie type of characteristic 2 and Lie rank 2. We regard the latter as our primary approach, and correspondingly list [**TW02**] in our Background References. We also use the "Green Book" [**DGS85**] of Goldschmidt, Delgado, and Stellmacher for the classification of the amalgams of parabolics in those groups, supplemented by papers of Goldschmidt [**Gol80**] and Fan [**Fan86**]. This makes possible an appeal to Theorem F.4.31 for the final identification.

Various other appeals to recognition theorems in the literature are listed in section I.4, so we do not repeat that list here.

We use the Glauberman-Niles Theorem [**GN83**]; there is also a different proof of this result due to Campbell in [**Cam79**]. We use the difficult paper of Meierfrankenfeld and Stellmacher [**MS93**] on pushing up suitable subgroups of rank-2 groups. Actually we need only a special case of their result, so that some simplification of our work might be possible here. We use more elementary pushing up results from [**Asc81b**] and [**Asc81a**], particularly in sections C.1 and C.2.

We need various facts about the subgroup structure and $\mathbf{F}_2$-representations of almost-quasisimple groups whose nonabelian simple sections appear in Theorem C. The remainder of the section lists the major sources of such information:

We appeal to work of Guralnick and Malle in [**GM02**] and [**GM04**] for a list of the pairs (G, V) where G is a nearly simple SQTK-group and V a faithful irreducible $\mathbf{F}_2G$-module such that $\hat{q}(G, V) \leq 2$.

We use [**Asc80**] (only invoked in the case of SQTK-groups) and [**Asc86b**] to determine which groups listed in Theorem C are abstract minimal parabolics (see Definition B.6.1) We occasionally use [**GLS98**] (e.g. 6.3.1) and [**Asc86b**] for other facts about subgroup structure of the groups in Theorem C.

We use James' paper [**Jam73**] for facts on the irreducible $\mathbf{F}_2$-representations of the Mathieu groups. We appeal to [**Asc87**] and [**Asc88**] for some detailed facts about the Weyl module for $G_2(2^n)$.

We appeal to [**Asc81a**] for knowledge of FF-offenders on the natural modules for the alternating groups, but this is easy to work out independently. Similarly we appeal to [**Asc82a**] for some specialized facts about $\mathbf{F}_2$-representations in section E.4. The material in section E.1 comes from [**Asc81c**] and [**Asc82a**].

We need some facts about the 1-cohomology of certain $\mathbf{F}_2$-modules for the groups in Theorem C; see I.1.6 and the references there, primarily the paper [**JP76**] by Jones and Parshall on representations of groups of Lie type.

We also need the list of Schur multipliers of the simple groups in Theorem C; see I.1.3, and the reference there to [**GLS98**].

Section I.8 contains some material from the literature, including statements, proofs, and discussion of relationships among the references.

Finally we list some references used occasionally for particular facts, most of which are "well known":

We appeal several times to the classical paper of Zsigmondy [**Zsi92**].

At one point we quote the theory of fundamental subgroups in a group of Lie type and odd characteristic from [**Asc80**].

We use [**AS76a**] for facts about centralizers of involutions in groups of Lie type in characteristic 2. Also in the proof of our corollary to the Main Theorem that supplies a bridge to the GLS program, we quote [**AS76b**] for the structure of certain involution centralizers in sporadic groups.

We use [**Asc82b**] for some facts about the Tits group.

At one point we quote [**AS85**] for a result on 1-cohomology of modules, but this could be proved by ad hoc arguments in that special case.

CHAPTER A

Elementary group theory and the known quasithin groups

In this initial chapter of Volume I, we first collect some standard, elementary results from the literature, and derive various easy consequences of these results.

We also prove some basic facts about the structure of quasithin $\mathcal{K}$-groups; for example in Theorem C (A.2.3) of section A.2, we give the list of simple SQTK-groups. Since these groups are the nonabelian simple sections in 2-local subgroups of simple QTKE-groups, this list will be used throughout our work. In particular, a main theme of the subsequent chapters of Volume I is the development of more detailed properties on the structure and representations of the groups in Theorem C, and of more general SQTK-groups; details which will be needed in the proof of the Main Theorem.

A.1. Some standard elementary results

In this section, for convenience and in order to maintain a reasonably self-contained treatment, we provide the statements (and references to proofs) of certain fairly standard elementary results from the literature. We concentrate on those results we use most frequently. This section provides reference numbers for such results, though in the main text of the paper we may often reference the results just by the standard name indicated here.

We also include some other results which appear to be well-known, but do not seem to be proved in the literature. We supply proofs of these results.

Furthermore we mention certain elementary results which we do *not* formally reference by a number or a citation in the main text—since they are probably familiar (e.g. by name like Sylow's Theorem) to most readers.

In this section G denotes a finite group.

A.1.1. Basic group theory.

Here are several results that we do not formally reference by a number in the main text:

(Dedekind) Modular Law—1.14 in [**Asc86a**]

Frattini Argument—6.2 in [**Asc86a**]

Hall's Theorem on solvable groups (extending the theory of Sylow subgroups for a prime to Hall subgroups for a set of primes)—18.5 in [**Asc86a**]

We also make frequent use, without reference, of the standard properties of the generalized Fitting subgroup $F^*(G)$ of a group G, particularly its self-centralizing property:

$$C_G(F^*(G)) = Z(F^*(G)) \leq F^*(G).$$

(See chapter 11 of [**Asc86a**]). Of course if G is solvable, then $F^*(G) = F(G)$ is the usual Fitting subgroup. In particular we often concentrate on the situation where $F^*(G) = O_p(G)$ for some prime p; observe that

$$F^*(G) = O_p(G) \text{ iff } O^p(F^*(G)) = 1 \text{ iff } C_G(O_p(G)) \leq O_p(G).$$

LEMMA A.1.1 (Three-Subgroup Lemma). *Let X, Y, Z be subgroups of a group G with $[X, Y, Z] = [Y, Z, X] = 1$. Then $[Z, X, Y] = 1$.*

PROOF. See 8.7 in [**Asc86a**]. □

Recall that a group generated by two involutions is dihedral.

LEMMA A.1.2 (Baer-Suzuki Theorem). *Let X be a p-subgroup of G. Then either $X \leq O_p(G)$, or there exists $g \in G$ with $\langle X, X^g \rangle$ not a p-group. In particular if X is of order 2, then X inverts an element of odd order in G.*

PROOF. See 39.6 in [**Asc86a**]. □

LEMMA A.1.3 (Dickson's Theorem (on subgroups of $L_2(q)$)). *The subgroups of $L_2(p^f) = PSL_2(p^f)$ are of the form:*

(1) Elementary abelian p-groups of rank at most f.
(2) Cyclic groups of order z dividing $(p^f \pm 1)/k$ where $k := (p^f - 1, 2)$.
(3) Dihedral groups of order $2z$ where z is as in (2).
(4) The alternating group A_4 for $p = 2$ and f even.
(5) The symmetric groups S_4 for $p^f - 1 \equiv 0 \mod 16$.
(6) The alternating group A_5 for $p^{2f} - 1 \equiv 0 \mod 5$.
(7) Semidirect products of an elementary abelian p-group of rank m with a cyclic subgroup of order t, where t divides $p^m - 1$ and $(p^f - 1)/k$.
(8) $L_2(p^m)$ for m dividing f, and $PGL_2(p^m)$ for $2m$ dividing f.

PROOF. See II.8.27 in [**Hup67**, I]. □

As usual $\pi(X)$ denotes the set of primes dividing the order of the group X.

LEMMA A.1.4. *Suppose that K is solvable, and for all $p \in \pi(F(K)) =: \pi$ that $O^p(K) \leq C_K(O_p(K))$. Then K is a π-group and $K = F(K)$ is nilpotent.*

PROOF. The hypothesis that $O^p(K) \leq C_K(O_p(K))$ for all $p \in \pi = \pi(F(K))$ implies $O^\pi(K) \leq C_K(F(K)) = Z(F(K))$ as K is solvable. Hence K is a π-group. Applying the hypothesis for each $q \neq p$ gives $O^{p'}(K) \leq C_K(O^p(F(K)) =: Y$. Then the hypothesis shows that

$$O^p(Y) \leq O^p(C_Y(O_p(K))) = O^p(C_Y(F(K))) = O^p(Y \cap Z(F(K))) \leq Z(Y).$$

Therefore Y is p-closed, and hence $O_p(Y) = O^{p'}(K)$ is Sylow in K for each $p \in \pi$, so K is nilpotent. □

LEMMA A.1.5 (Thompson's Dihedral Lemma). *If X is a group of odd order admitting the faithful action of an elementary abelian 2-group A of rank n, then $A \leq Y \leq AX$ with $Y = Y_1 \times \cdots \times Y_n$ and $Y_i \cong D_{2p_i}$ for suitable odd primes p_i.*

PROOF. This lemma of Thompson appears in [**Tho68**, 5.34]; we prove a slightly stronger version later in G.8.8. □

LEMMA A.1.6. *Suppose $T \leq H \leq G$ with $T \in Syl_2(G)$. Then $O_2(G) \leq O_2(H)$. Further if $F^*(G) = O_2(G)$, then $F^*(H) = O_2(H)$.*

Proof. As $T \in Syl_2(G)$, $O_2(G) \leq T \leq H$, so $O_2(G) \trianglelefteq H$ and hence $O_2(G) \leq O_2(H)$. If $F^*(G) = O_2(G)$, then $C_G(O_2(G)) \leq O_2(G)$, so that $C_H(O_2(H)) \leq O_2(H)$. □

Recall X is a *TI-set in* G if $X \cap X^g = 1$ for $g \in G - N_G(X)$.

Lemma A.1.7. *Let $x \in X \leq G$. Then*

(1) $x^G \cap X = x^{N_G(X)}$ iff $C_G(x)$ is transitive on $\{Y \in X^G : x \in Y\}$.

(2) If $x^G \cap X = x^{N_G(X)}$ and $C_G(x) \leq N_G(X)$, then x is contained in a unique member of X^G.

(3) Assume that X is a TI-set in $M \leq G$, $x^G \cap X = x^{N_G(X)}$, and $C_G(x) \leq M$. Then x is in a unique member of X^G.

Proof. Each of the two statements in (1) is equivalent to the transitivity of G on pairs $\{(Y, y) \in X^G \times x^G : y \in Y\}$, so (1) holds. Then (1) implies (2), and (2) implies (3), since then $C_G(x) = C_M(x) \leq N_M(X)$ as X is a TI-set in M. □

Lemma A.1.8. *If $F^*(G) = O_p(G)$ and $X \leq Z(G)$, then $F^*(G/X) = O_p(G/X)$.*

Proof. Set $\bar{G} := G/X$. The preimage Y of $O^p(F^*(\bar{G}))$ centralizes the factors of the series $O_p(G)X \geq X \geq 1$, and hence by Coprime Action (cf. the next subsection) lies in $C_G(O_p(G)) = Z(O_p(G))$, so that $\bar{Y} = 1$. □

Lemma A.1.9. *If a p-group P normalizes a group Y with $O_p(Y) = 1$, and P centralizes $F^*(Y)$, then P centralizes Y.*

Proof. By hypothesis P centralizes $F^*(Y)$, so $[Y, P] \leq C_Y(F^*(Y)) \leq F^*(Y)$, and hence Y acts on $O_p(F^*(Y)P) = P$ since $O_p(Y) = 1$. Therefore $[Y, P] \leq Y \cap P \leq O_p(Y) = 1$. □

Lemma A.1.10. *Let $L \trianglelefteq G = LM$ with $F^*(L) = O_2(L)$ and $F^*(M) = O_2(M)$. Assume M contains a Sylow 2-subgroup T of G, and $N_G(Q) \leq M$ for some $Q \leq O_2(M)$. Then $F^*(G) = O_2(G)$.*

Proof. Assume otherwise; then there is $X \leq F^*(G)$ such that either X is a component of G, or X is a normal p-subgroup of G for some odd prime p. By hypothesis, $F^*(M) = O_2(M)$, so $X \not\leq M$, and similarly $X \not\leq L$. If X is a component of G, then as $X \not\leq L$, X centralizes L by 31.4 in [**Asc86a**]. If X is a p-group, then $X \leq O_p(LX)$, so $[L, X] \leq X \cap L \leq O_p(LX) \cap L \leq O_p(L) = 1$, so $LX = L \times X$. Thus in either case, X centralizes L.

Let $Y := M \cap LX$. As $G = LM$, $LX = LX \cap LM = LY$ using the Dedekind Modular Law. Then as L centralizes X, for $U \leq X$, $[U, X] = [U, LX] = [U, LY] = [U, Y]$.

Suppose first that X is a component of G. As $T \in Syl_2(G)$ and X is subnormal in G, $T \cap X \in Syl_2(X)$. Hence as X is quasisimple, $X = [X, T \cap X]$, so $X = [Y, T \cap X]$ by the previous paragraph. But then $X \leq M$ as Y and T lie in M, contrary to paragraph one.

Therefore X is a normal p-subgroup of G with p odd, and $LX = L \times X$. Let $\pi : Y \to X$ be the projection with respect to this decomposition; as $LX = LY$, π is a surjection. Now $[O_2(M), Y] \leq O_2(M) \cap Y$, so as π is M-equivariant, $[O_2(M), y\pi]$ is a 2-group for each $y \in Y$. Then as $X \leq O_p(G)$ for p odd and π is a surjection, $O_2(M)$ centralizes X. Therefore $X \leq C_G(Q) \leq M$, again contrary to paragraph one, which completes the proof. □

Recall from Definition B.2.11 and B.2.12 that $R_2(G)$ is the product of the members of $\mathcal{R}_2(G)$; that is, of all the 2-reduced subgroups of G. In particular, $R_2(G) \leq \Omega_1(Z(O_2(G)))$.

LEMMA A.1.11. *Assume $T \in Syl_2(G)$ with $F^*(G) = O_2(G)$, and $T \leq H \leq G$. Then $R_2(H) \leq R_2(G)$.*

PROOF. By A.1.6, $O_2(G) \leq O_2(H)$, so $O_2(G)$ centralizes $R_2(H) =: R$. Then as $F^*(G) = O_2(G)$, $R \leq C_G(O_2(G)) = Z(O_2(G))$, so $S := \langle R^G \rangle \leq \Omega_1(Z(O_2(G)))$. Set $\bar{G} := G/C_G(S)$, and let Q denote the preimage in G of $O_2(\bar{G})$. Then $Q \trianglelefteq G$, so $P := T \cap Q \in Syl_2(Q)$ and hence $\bar{Q} = \bar{P}$. Therefore $Q = C_G(S)P$, so as $T \leq H$, $Q = C_G(S)(H \cap Q)$, and hence $(H \cap Q)/C_H(S) \cong Q/C_G(S) = \bar{Q}$ is a 2-group. As $R \leq S$, $C_H(S) \leq C_H(R)$, so $(H \cap Q)/C_H(R)$ is a 2-group. Thus as $H \cap Q \trianglelefteq H$ and $O_2(H/C_H(R)) = 1$, $H \cap Q$ centralizes R, so $Q = C_G(S)(H \cap Q)$ centralizes R. Then since $S = \langle R^G \rangle$ and $Q \trianglelefteq G$, $Q = C_G(S)$, so $S \in \mathcal{R}_2(G)$, and therefore $R \leq S \leq R_2(G)$. □

Recall that for a prime p, $m_p(G)$ denotes the p-rank of G: the largest dimension of an elementary abelian p-subgroup of G, regarded as an $\mathbf{F}_p$-space.

LEMMA A.1.12. *Let V be a finite-dimensional vector space over $\mathbf{F}_2$, X a cyclic subgroup of $G := GL(V)$ regular [1] on $V^\#$, and Y an overgroup of X of odd order in G. Then $X \trianglelefteq Y$.*

PROOF. As Y is of odd order, Y is solvable by the Odd Order Theorem, so $C_Y(F(Y)) = Z(F(Y))$. Let $n := \dim(V)$; if $n = 1$ then G is cyclic, so the lemma is trivial. Thus we may assume $n > 1$, so by Zsigmondy's Theorem [**Zsi92**], either

(i) $n = 6$, or

(ii) there is a Zsigmondy prime divisor r of $2^n - 1$; that is there is a subgroup R of order r in X which is irreducible on V.

Assume case (ii) holds. Then R is irreducible on V, and $End_{\mathbf{F}_2 R}(V) \cong \mathbf{F}_{2^n}$, so $|C_G(R)| = 2^n - 1 = |X|$, and hence $C_G(R) = X$. Thus $m_r(G) = 1$. Suppose that $R \leq F(Y)$. Then as $m_r(G) = 1$, $R \trianglelefteq Y$, so $X = C_G(R) \trianglelefteq Y$, and hence the lemma holds in this case. Thus we may assume that $R \not\leq F(Y)$, and hence as R is of prime order and $m_r(G) = 1$, $F(Y)$ is an r'-group and R is faithful on $F(Y)$. Then as Y is of odd order and $C_V(R) = 0$, we contradict 36.2 in [**Asc86a**].

Thus we may take $n = 6$. Here let $W \leq X$ be of order 7. Again using 36.2 in [**Asc86a**], W centralizes $O^7(F(Y))$, so as a Sylow 7-subgroup of G is abelian, $W \leq C_Y(F(Y)) = Z(F(Y))$. As $X \leq C_G(W)$, $G_W := C_G(W)$ is irreducible on V, so by Schur's Lemma (cf. subsection A.1.4), $End_{\mathbf{F}_2 G_W}(V)$ is a finite field, and the subfield generated by W is $\mathbf{F}_8$. Thus G_W lies in the subgroup $GL_2(8)$ of units of $End_{\mathbf{F}_2 W}(V)$. By Dickson's Theorem A.1.3, X is maximal among subgroups of $GL_2(8)$ of odd order, so $X = C_Y(W)$. Therefore $W = O_7(Y) \trianglelefteq Y$, so $X = C_Y(W) \trianglelefteq Y$, completing the proof. □

Recall A is *weakly closed* in B with respect to G if $A^g \leq B$ for some $g \in G$ implies $A = A^g$.

LEMMA A.1.13. *Suppose the p-subgroup X of G is not weakly closed in $N_G(X)$. Then there exists $g \in G$ such that $X^g \neq X$, with $X^g \leq N_G(X)$ and $X \leq N_G(X^g)$.*

[1] We follow the usage in the permutation-group literature, so that by "regular" we mean not just "free" but also "transitive".

PROOF. If each member of $X^G \cap T$ is normal in $T \in Syl_p(G)$, then the lemma holds, so we may assume $S := N_T(X) < T$. Thus there is $g \in N_T(S) - S$, and X and X^g are distinct and normal in S, so again the result holds. □

Recall that for V a vector space of dimension n over a field F of characteristic 2, with $V_1 \leq V_{n-1}$ subspaces of dimension 1 and $n-1$, the group of *transvections* with *center* V_1 and *axis* V_{n-1} consists of those involutions $a \in GL(V)$ with $[V, a] = V_1$ and $C_V(a) = V_{n-1}$.

LEMMA A.1.14. *Let G be a finite group, $E_4 \cong V \leq G$, v_1 and v_2 distinct involutions in V, $Q_i := O_2(C_G(v_i))$, and $Y := \langle Q_1, Q_2 \rangle$. Assume $[V, Q_i] = \langle v_i \rangle$ for $i = 1$ and 2. Then*

(1) Y induces $GL(V)$ on V with kernel $O_2(Y) = C_{Q_1}(V)C_{Q_2}(V)$.
(2) $Y = \langle Q_1^{N_G(V)} \rangle \trianglelefteq N_G(V)$.
(3) $N_G(V)/O_2(Y) = Y/O_2(Y) \times C_G(V)/O_2(Y)$.
(4) If G is quasithin, then $m_3(C_G(V)) \leq 1$.

PROOF. Let $M := N_G(V)$ and $M^* := M/C_M(V)$. By hypothesis, Q_i^* is the group of transvections in $GL(V)$ with center $\langle v_i \rangle$, so $Y^* = \langle Q_1^*, Q_2^* \rangle = GL(V)$. Thus Y is transitive on $V^\#$, so $Q_1^M = Q_1^Y \subseteq Y$ and hence (2) holds.

Let $P := C_{Q_1}(V)C_{Q_2}(V)$. Now $C_G(V) \leq C_G(v_i) \leq N_G(Q_i)$, so $C_{Q_i}(V) \trianglelefteq C_G(V)$, and hence P is a normal 2-subgroup of $C_G(V)$. Further $[C_G(V), Q_i] \leq C_{Q_i}(V) \leq P$, so $Y = \langle Q_1, Q_2 \rangle$ acts on P. Let $Y^+ := Y/P$. As $|Q_i : C_{Q_i}(V)| = |Q_i^*| = 2$, $|Q_i^+| = 2$, so $Y^+ = \langle Q_1^+, Q_2^+ \rangle \cong D_{2n}$ for some n. As $[C_Y(V), Q_i] \leq P$, $C_Y(V)^+ \leq Z(Y^+)$, so as $Y^+ \cong D_{2n}$, $|C_Y(V)^+| \leq 2$. Then as $Y^* \cong D_6$, $n = 3$ or 6, and as Q_1^+ is conjugate to Q_2^+ in Y^+, $n \neq 6$, so $P = C_Y(V) = O_2(Y)$, completing the proof of (1).

As $M = C_G(V)Y$, as Y and $C_M(V)$ are normal in M, and as $C_Y(V) = O_2(Y)$, (3) holds. By (3), $m_3(M) = m_3(C_M(V)) + m_3(Y)$, while $m_3(Y) = 1$ by (1); thus (4) holds. □

Here is a group-theoretic Krull-Schmidt Theorem:

LEMMA A.1.15 (Krull-Schmidt Theorem). *If X is a direct product of indecomposable groups X_i, then $Aut(X)$ permutes the groups $X_iZ(X)$.*

PROOF. See 1.6.18.ii and 2.4.8 in [**Suz86**, 1] □

LEMMA A.1.16. *Assume that $Y \trianglelefteq G$, and set $\bar{G} := G/Y$. Let $Y \leq D \leq G$ and $C \leq G$. Then*

(1) $C \cap D$ is the preimage in C of $\bar{C} \cap \bar{D} = \overline{C \cap D}$.
(2) If $C \in Syl_2(G)$ and $\bar{C} \cap \bar{D} \in Syl_2(\bar{D})$, then $C \cap D \in Syl_2(D)$.

PROOF. Part (1) is easy. Assume the hypothesis of (2). By (1), $\overline{C \cap D} \in Syl_2(\bar{D})$, and as $Y \trianglelefteq G$, $C \cap Y \in Syl_2(Y)$. Thus $|C \cap D| = |\overline{C \cap D}||C \cap Y| = |D|_2$, establishing (2). □

A.1.2. Coprime action, critical subgroups, and p-groups of small rank. This subsection primarily contains a discussion of coprime action, and especially of critical subgroups; and applications of such results to SQTK-groups.

A.1.2.1. *Coprime action and supercritical subgroups.* Some standard results not formally referenced by number in the main text:

Coprime Action—we use this name (capitalized) to refer to various standard results, such as 18.7, 24.1, 24.3, 24.4–24.6, 24.8 in [**Asc86a**], and 4.3ii, 11.5 , 11.13 in [**GLS96**].

LEMMA A.1.17 (Generation by Centralizers of Hyperplanes). *Assume A is an abelian r-group acting on an r'-group G and*

$$\mathcal{B} := \{B \leq A : A/B \text{ is cyclic and } C_G(B) \neq 1\}.$$

Then $G = \langle C_G(B) : B \in \mathcal{B}\rangle$, and if G is abelian and $G = [G, A]$, then $G = \bigoplus_{B\in\mathcal{B}} C_G(B)$.

PROOF. Standard; cf., Exercise 4.1 or 6.5 or 8.1 in [**Asc86a**], or 11.13 in [**GLS96**]. □

LEMMA A.1.18 (Thompson $A\times B$ Lemma). *Let AB be a finite group represented as a group of automorphisms of a p-group G, with $[A, B] = 1$, B a p-group, and $A = O^p(A)$.*

(1) If $[C_G(B), A] = 1$ then $[G, A] = 1$.

(2) If A is faithful on G and $O_p(A) = 1$, then A is faithful on $C_G(B)$.

PROOF. See 24.2 in [**Asc86a**] for (1). Next assume the hypotheses of (2) hold. Then $O^p(C_A(C_G(B))) \leq C_A(G) = 1$, so that $C_A(C_G(B)) \leq O_p(A) = 1$. □

LEMMA A.1.19. *If $K \trianglelefteq G$ and U is a normal elementary abelian 2-subgroup of G with $C_K(U) = O_2(K)$, then $C_K(R_2(G)) = O_2(K)$.*

PROOF. Pick U minimal subject to the hypotheses of the lemma, and let $G^* := G/C_G(U)$. As $C_K(U) = O_2(K)$, $O_2(K^*) = 1$. Thus as $K \trianglelefteq G$, $F^*(K^*)$ centralizes $O_2(G^*)$. Then by the Thompson $A \times B$-Lemma A.1.18.2, $F^*(K^*)$ is faithful on $C_U(O_2(G^*))$, so $U = C_U(O_2(G^*))$ by minimality of U, and hence $O_2(G^*) = 1$. Thus $U \in \mathcal{R}_2(G)$, so $C_K(R_2(G)) \leq C_K(U) = O_2(K)$, and of course $O_2(K)$ centralizes $R_2(G)$. □

Given an odd prime p and a p-group P, recall (e.g. Section 24 of [**Asc86a**]) that a *critical subgroup* of P is a characteristic subgroup Y of P such that $\Phi(Y) \leq Z(Y) \geq [Y, P]$ and $C_P(Y) \leq Y$.

DEFINITION A.1.20. Define a *supercritical subgroup* of P to be a characteristic subgroup X of P such that $\Phi(X) \leq Z(X) \geq [X, P]$, X is of exponent p, and X contains all elements of order p in $C_P(X)$.

LEMMA A.1.21 (Supercritical Subgroups Lemma). *Let p be an odd prime and P a p-group. Then*

(1) P possesses a supercritical subgroup X.

(2) X is of class at most 2.

(3) Each p'-automorphism of P acts faithfully on $X/\Phi(X)$.

(4) If $m_p(P) > 1$, then $m_p(X) > 1$.

PROOF. By 24.9 in [**Asc86a**], P possesses a critical subgroup Y such that $\Omega_1(Y)$ contains each element of order p in $C_P(\Omega_1(Y))$. Set $X := \Omega_1(Y)$. We claim X is a supercritical subgroup of P. First $\Phi(X) \leq \Phi(Y) \cap X \leq Z(Y) \cap X \leq Z(X)$, and $[P, X] \leq [P, Y] \cap X \leq Z(Y) \cap X \leq Z(X)$. In particular (2) holds. Then as $X =$

$\Omega_1(X)$ and X is of class at most 2, X is of exponent p by 23.11 in [**Asc86a**]. Hence X is supercritical in P, so (1) holds. By 24.9 in [**Asc86a**], each p'-automorphism α of P is faithful on X. Then by Coprime Action, α is faithful on $X/\Phi(X)$, so (3) holds. As X contains each element of order p in $C_P(X)$, (4) holds. □

A.1.2.2. *Lemmas on small p-rank.* In this subsubsection we establish some technical lemmas. As our quasithin hypothesis restricts the rank of certain p-subgroups for odd primes p, we concentrate on p-groups of rank at most 2. The lemmas in this subsection are used throughout the work, but the most immediate applications will be made in the next sections A.2 and A.3 on quasithin $\mathcal{K}$-groups.

First we recall that in a p-group of rank 2, elements of order p can generate only a few possible p-groups. Throughout the work, we use the convention:

NOTATION A.1.22. For odd p, p^{1+2} denotes the extra-special group of order p^3 and exponent p.

For this and other extraspecial groups we often use the standard fact 23.8 in [**Asc86a**]:

LEMMA A.1.23. *Assume P is an extraspecial p-group for some prime p. Then $Inn(P) = C_{Aut(P)}(P/\Phi(P))$.*

LEMMA A.1.24. *Let p be an odd prime, and $P = \Omega_1(P)$ a p-group of p-rank 2 and class at most 2. Then $P \cong E_{p^2}$ or p^{1+2}.*

PROOF. By 23.11 in [**Asc86a**], P is of exponent p. Thus a normal subgroup E of P of order p^2 is isomorphic to E_{p^2} and $E = C_P(E)$. As $|P : C_P(E)| \le p$, the lemma follows. □

LEMMA A.1.25. *Let p be an odd prime and P a p-group with $m_p(P) \le 2$. Then*

(1) $X \cong \mathbf{Z}_p$, E_{p^2}, or p^{1+2} for each supercritical subgroup X of P, and $m_p(X) = m_p(P)$.

(2) $Aut(P)/O_p(Aut(P))$ is a subgroup of $GL_2(p)$.

(3) If $p = 3$, then $Aut(P)$ is a $\{2,3\}$-group.

PROOF. Let X be a supercritical subgroup of P and set $\bar{X} := X/\Phi(X)$. By definition, X is of exponent p, and by A.1.21.2, X is of class at most 2. By hypothesis, $m_p(X) \le 2$, so $m_p(X) = m_p(P)$ using A.1.21.4. Therefore (1) follows from A.1.24.

Let $A := Aut(P)$ and $B := O_p(A)$. By A.1.21.3, $C_A(\bar{X}) \le B$. On the other hand by (1), $\bar{X}$ is of rank $m \le 2$, so $A/C_A(\bar{X}) \le GL_m(p) \le GL_2(p)$. Thus (2) holds. Finally (2) implies (3). □

LEMMA A.1.26. *Assume $X = O^2(X) \le H$ and $V = [V, X]$ is a 2-subgroup of H. Then*

(1) If $m_p(O_p(H)) \le 2$ for all odd primes p and $\Phi(V) = 1$, then V centralizes $O(H)$.

(2) If H is a solvable SQTK-group and $X = O^{\{2,3\}}(X)$, then $V \le O_2(H)$.

PROOF. Assume the hypotheses of (1) or (2), and $[O(H), V] \ne 1$. Then $[O_p(H), V] \ne 1$ for some odd prime p by A.1.9. Let $P := O_p(H)$, $\bar{H} := H/C_H(P)$, and $H^* := \bar{H}/O_p(\bar{H})$. As $m_p(O_p(H)) \le 2$, A.1.25.2 says

$$1 \ne V^* = [V^*, X^*] \le H^* \le GL_2(p).$$

This is impossible, as using Dickson's Theorem A.1.3, $O^2(N_{GL_2(p)}(A))$ centralizes A for each elementary abelian 2-subgroup A of $GL_2(p)$, and $O^{\{2,3\}}(N_{GL_2(p)}(S))$ centralizes S for each 2-subgroup S of $GL_2(p)$. Thus V centralizes $O(H)$. In particular, (1) holds. Similarly under the hypotheses of (2), $m_p(H/O_2(H)) \leq 2$ for each odd prime p, so passing to $H/O_2(H)$, we may assume $O_2(H) = 1$. Then as H is solvable and V centralizes $O(H)$, $V = 1$, establishing (2). □

We sometimes use the following result without explicit reference:

LEMMA A.1.27. *Let X be an SQT-group, p an odd prime, and $E_{p^2} \cong P \leq X$.*

(1) If P acts on $H \leq X$ and $P \cap H = 1$, then H is a p'-group.

(2) If P acts faithfully on an elementary abelian 2-subgroup U of X, then $C_X(U)$ is a p'-group.

PROOF. If (1) fails then P centralizes some subgroup Q of H of order p, so $m_p(PQ) = 3$, contradicting the hypothesis that X is an SQT-group. Thus (1) holds, while (2) is the special case of (1) where $H = C_X(U)$. □

Recall for an odd prime p that $m_{2,p}(G)$ denotes the *2-local p-rank* of G: that is the maximum of $m_p(X)$ over 2-locals X of G. In a quasithin group, $m_{2,p}(G) \leq 2$ for all odd primes P, but the p-rank of subgroups which are not 2-locals can exceed 2. Still we obtain restrictions on such subgroups; see for example the next lemma and A.1.31.

LEMMA A.1.28. *Let p be an odd prime, and G a finite group with $m_{2,p}(G) \leq 2$. Suppose $H \leq G$ with $m_p(O_p(H)) > 2$. Then $m_2(H) \leq 3$, and in case of equality $Z^*(H) \neq 1$.*

PROOF. This is 3.4 in [**Asc81d**]; we reproduce the proof here for completeness:

By A.1.21.1, we may choose a supercritical subgroup P of $Q := O_p(H)$. Pick an elementary abelian 2-subgroup A of H with $m_2(A) = m_2(H) =: n$. As $m_p(Q) > 2$ but $m_{2,p}(G) \leq 2$, $C_H(Q)$ is of odd order. Thus $C_H(P)$ is of odd order by A.1.21.3, so $C_H(P) \leq O(H)$. In particular A is faithful on P. Therefore by A.1.5, there is a subgroup of AP which is a direct product of n copies of a dihedral group D_{2p}. Then as $m_{2,p}(G) \leq 2$, $n \leq 3$.

Suppose $n = 3$; we will show that some $a \in A$ inverts $P^* := P/\Phi(P)$. Then $aC_H(P) \in Z(H/C_H(P))$, so as $C_H(P) \leq O(H)$, $Z^*(H) \neq 1$, completing the proof of the lemma.

Let $R := [P, A]$. Then by Generation by Centralizers of Hyperplanes A.1.17,

$$R^* = \bigoplus_{D \in \Delta} C_{R^*}(D),$$

where $\Delta := \{D_i : 1 \leq i \leq r\}$ is the set of hyperplanes of A with $C_{R^*}(D) \neq 1$. Let $\langle a_{i,j} \rangle := D_i \cap D_j$ for $i \neq j$. Then

$$2 \leq m_p(C_{R^*}(D_i)) + m_p(C_{R^*}(D_j)) \leq m_p(C_{R^*}(a_{i,j})),$$

while $m_p(C_{R^*}(a_{i,j})) \leq 2$ using A.1.24 since $m_{2,p}(G) \leq 2$. Thus all inequalities are equalities, so $|C_{R^*}(D_i)| = p$ and hence $P = R$, $m_p(P^*) = |\Delta| = r$, and each $a_{i,j}$ is in exactly two members of Δ. If $r = 3$, then $a := a_{1,2}a_{1,3}a_{2,3}$ inverts P^*, and we are done as mentioned earlier; so we may take $r > 3$. Then for $1 \leq i < j \leq 3$, $a_{i,j} \notin D_k$ for $k > 3$, so $D_k = \langle a_{1,2}a_{2,3}, a_{1,2}a_{1,3} \rangle$ as D_k is a hyperplane of A. Hence $r = 4$, and again a inverts P^*. □

LEMMA A.1.29. *Let p be an odd prime and G a finite group. Assume $P = \Omega_1(P)$ is a normal p-subgroup of G of class at most 2 with $m_p(P) = 2$. Assume $H = H^\infty \leq G$ with $[P, H] \neq 1$. Then*

(1) $H/C_H(P/\Phi(P)) \cong SL_2(p)$, or possibly $SL_2(5)$ if $p \equiv \pm 1 \mod 5$.

(2) If $H/O(H)$ is quasisimple, then $O(H) = C_H(P/\Phi(P))$; in particular, $m_2(H) = 1$.

PROOF. By A.1.24, $P \cong E_{p^2}$ or p^{1+2}, so $\bar{H} := H/C_H(P/\Phi(P))$ is a subgroup of $GL_2(p)$. Then as H is perfect by hypothesis, $\bar{H} \leq SL_2(p)$. From Dickson's Theorem A.1.3, the only possible perfect proper subgroup of $SL_2(p)$ is $SL_2(5)$, which occurs when $p \equiv \pm 1 \mod 5$. Thus $\bar{H} \cong SL_2(r)$, for $r = p$ or 5, establishing (1).

Therefore we may assume $H/O(H)$ is quasisimple, and it remains to show that $C_H(P/\Phi(P)) = O(H)$. As $\bar{H} \cong SL_2(r)$, $O(\bar{H}) = 1$, so that $O(H) \leq C_H(P/\Phi(P))$. So $\bar{H} \cong SL_2(r)$ is a homomorphic image of $H/O(H)$, which is quasisimple by hypothesis. However the the Schur multiplier of $SL_2(r)$ for r prime is trivial by I.1.3, completing the proof. □

The next lemma will be used to simplify the structure of members of $\mathcal{C}(H)$ and $\mathcal{L}(G,T)$ in A.3.6 and 1.2.1.4.

LEMMA A.1.30. *Let $p > 3$ be prime and $P \cong \mathbf{Z}_{p^2} \times \mathbf{Z}_{p^2}$. Then each element of order p in $Aut(P)$ centralizes $\Omega_1(P)$, so in particular $Aut(P)$ has no $SL_2(p)$-subgroup.*

PROOF. This is Lemma 4.9 in [**MS**]; we reproduce their proof here for completeness.

Write P additively, and set $Q := \Omega_1(P)$; then $pP = Q$. Let $t \in Aut(P)$ be of order p, and set $\theta := t - 1 \in End(P)$. Then t is quadratic on P/Q, so $\theta^2 = 0$ on P/Q. Similarly $\theta^2 = 0$ on Q, so $\theta^4 = 0$. Then as $p \geq 5$:

(a) $\theta^p = 0$.

As $pP = Q$ and $\theta^2 = 0$ on Q:

(b) $p\theta^2 = 0$.

Set $f(x) := (x^p - 1)/(x - 1) \in \mathbf{Z}[x]$; then $f(1) = p$. Now $x^p - 1 \equiv (x-1)^p \mod p$, so $x^p - 1 = (x-1)^p + pG(x)$ for some $G(x) \in \mathbf{Z}[x]$ such that $G(x) = (x-1)g(x)$ for some $g(x) \in \mathbf{Z}[x]$. Thus $f(x) = (x-1)^{p-1} + pg(x)$, so $p = f(1) = pg(1)$, and hence $g(1) = 1$. Next $g(x) = (x-1)h(x) + g(1)$ for some $h(x) \in \mathbf{Z}[x]$, so $g(x) = (x-1)h(x) + 1$. Therefore

(c) $x^p - 1 = (x-1)^p + p(x-1)^2 h(x) + p(x-1)$.

Evaluating (c) at $x = t$:

(d) $0 = t^p - 1 = \theta^p + p\theta^2 h(t) + p\theta$,

so $0 = p\theta$ by (a) and (b). Hence t centralizes $pP = Q$. As p'-automorphisms of P are faithful on Q by Coprime Action, while $SL_2(p) = O^{p'}(SL_2(p))$ and $O_p(SL_2(p)) = 1$, the lemma follows. □

The next technical result is crucial for restricting the structure of solvable sections of strongly quasithin groups, at many points in our analysis.

LEMMA A.1.31. *Let G be a group and p an odd prime with $m_p(G) \leq 2$. Let $H \trianglelefteq G$, set $\bar{G} := G/H$, and assume $F^*(\bar{G}) = O_p(\bar{G})$. Then*

(1) $C_{O_p(\bar{G})}(\bar{T})$ *is cyclic for each nontrivial* 2*-subgroup* $\bar{T}$ *of* $\bar{G}$.
(2) $m_2(\bar{G}) \leq 2$.
(3) If $O_p(\bar{G}) \cong E_{p^3}$ *and* $m_2(\bar{G}) > 1$*, then* $\bar{G}$ *is not irreducible on* $O_p(\bar{G})$.

PROOF. First observe that if $m_2(\bar{G}) > 2$, then there exists an involution $\bar{t} \in \bar{G}$ with $C_{O_p(\bar{G})}(\bar{t})$ noncyclic: Namely since $F^*(\bar{G}) = O_p(\bar{G})$, an E_8-subgroup $\bar{A}$ of $\bar{G}$ is faithful on $O_p(\bar{G})$, so A.1.5 says $\bar{A}O_p(\bar{G})$ contains a direct product of 3 dihedral groups. Thus (1) implies (2), so it remains to prove (1) and (3).

Let r be a prime divisor of $|H|$ and $R \in Syl_r(H)$. By a Frattini Argument, $G = HN_G(R)$ with $N_G(R)/N_H(R) \cong \bar{G}$. If $N_G(R) < G$, we may apply induction on the order of G to $N_G(R)$ to obtain the conclusions of (1) and (3) for $N_G(R)/N_H(R)$—which suffices, since (1) and (3) are statements about $\bar{G} \cong N_G(R)/N_H(R)$. Thus we may assume that H is nilpotent. Similarly if $r \neq p$ then $(G/R)/(H/R) \cong \bar{G}$ and $m_p(G/R) = m_p(G) \leq 2$, so again by induction on $|G|$, we may assume H is a p-group. Therefore as $F^*(\bar{G}) = O_p(\bar{G})$, also $F^*(G) = O_p(G) =: P$, say.

We now prove (1). Without loss $T = \langle t \rangle$, where t is an involution. As $C_{\bar{P}}(t)$ is noncyclic, so is $C_P(t)$. As $F^*(G) = P$, t is faithful on P, so by the Thompson $A \times B$-Lemma A.1.18, t is also faithful on $C_P(C_P(t))$. But then t inverts Y of order p in $C_P(C_P(t))$, so $YC_P(t) = Y \times C_P(t)$ is of p-rank at least 3, contradicting the hypothesis that $m_p(G) \leq 2$. This establishes (1) and hence (2).

Thus we may assume G is a counterexample to (3). Here we may take $T \cong E_4$ and set $G^* := G/P$, so that $G^* \cong \bar{G}/\bar{P}$. By (1), $C_{\bar{P}}(\bar{t})$ is cyclic for each $\bar{t} \in \bar{T}^\#$, so as $\bar{P} \cong E_{p^3}$, we conclude from Generation by Centralizers of Hyperplanes A.1.17 that $C_{\bar{P}}(\bar{t}) \cong \mathbf{Z}_p$ for each $\bar{t}$. In particular as $\bar{G}$ is irreducible on $\bar{P}$, $t^* \notin Z(G^*)$.

Let Q be a supercritical subgroup of P, $\tilde{Q} := Q/\Phi(Q)$, and $G^+ := G/C_G(\tilde{Q})$. As $P = F^*(G)$, $C_G(\tilde{Q}) \leq P$ by A.1.21.3; thus $G^* = G^+/P^+$ and as $T^* \cap Z(G^*) = 1$, $T^+ \cap Z(G^+) = 1$. Also T is faithful on $\tilde{Q}$, so $\tilde{Q}$ is noncyclic, and hence $\tilde{Q} \cong E_{p^2}$ by A.1.24. Then as T is faithful on $\tilde{Q}$, some $t \in T^\#$ inverts $\tilde{Q}$. Therefore $t^+ \in Z(G^+)$, contrary to $T^+ \cap Z(G^+) = 1$. This completes the proof of (3). □

LEMMA A.1.32. *Let* r*,* p *be odd primes,* G *a finite group, and* $R = \Omega_1(R)$ *a normal* r*-subgroup of* G *of class at most* 2 *with* $m_r(R) \leq 2$*. Assume* $P = \Omega_1(P)$ *is a* p*-subgroup of* G *of class at most* 2*, with* $m_p(P) = 2$ *and* $N_G(P)$ *irreducible on* $P/\Phi(P)$*. Then*

(1) Either $[R, P] = 1$*, or* $p = r$*,* $R \cong p^{1+2}$*, and* $P \leq RC_G(R)$.

(2) If $r = p$ *and* $m_p(G) = 2$*, then either* $R = P$*, or* $R \cong \mathbf{Z}_p$*,* $P \cong p^{1+2}$*, and* $R = Z(P)$.

PROOF. By A.1.24, $R \cong \mathbf{Z}_r$, E_{r^2}, or r^{1+2}, and $P \cong E_{p^2}$ or p^{1+2}. In particular $\bar{G} := G/C_G(R/\Phi(R))$ is a subgroup of $GL(R/\Phi(R)) \cong GL_k(r)$, $k \leq 2$. Let $\bar{H} := \bar{G} \cap SL(R/\Phi(R))$ and observe $\bar{G}/\bar{H}$ is cyclic and $m_p(\bar{H}) \leq 1$.

Suppose $\bar{P} \neq 1$. By hypothesis $N_G(P)$ is irreducible on $P/\Phi(P)$, so it follows that $C_P(R/\Phi(R)) \leq \Phi(P)$, and so $\bar{P} \cong E_{p^2}$ or p^{1+2}. Hence as $\bar{G}/\bar{H}$ is cyclic and $m_p(\bar{H}) \leq 1$, we conclude $|\bar{P} : \bar{P} \cap \bar{H}| = p = |\bar{P} \cap \bar{H}|$, contradicting $N_G(P)$ irreducible on $\bar{P}/\Phi(\bar{P})$.

So $\bar{P} = 1$. Thus if $r \neq p$, then $[R, P] = 1$ by Coprime Action, and (1) holds. So we may assume $r = p$. If $\Phi(R) = 1$, then $\bar{P} = 1$ just says $P \leq C_G(R)$ and again (1) holds. Otherwise $R \cong p^{1+2}$, and as $Inn(R) = C_{Aut(R)}(R/\Phi(R))$ by A.1.23, at least $P \leq RC_G(R)$, completing the proof of (1).

Thus it remains to establish (2), so we assume $m_p(G) = 2$. If $[R, P] = 1$ then $R \leq \Omega_1(C_{RP}(P)) \leq P$ since $m_p(G) = 2$; then as $N_G(P)$ is irreducible on $P/\Phi(P)$ while R is normal in G and central in P, either $R = P$ is abelian, or $P \cong p^{1+2}$ and $R = Z(P)$, so (2) holds. Thus we may assume $[R, P] \neq 1$, so $R \cong p^{1+2}$ by (1). This time $\Omega_1(C_{RP}(R)) \leq R$ as $m_p(G) = 2$. If $P \leq R$, then as $N_G(P)$ is irreducible on $P/\Phi(P)$, $R = P$ and (2) holds. Thus we may assume there is $x \in P - R$. By (1), $RP = RC_{RP}(R)$, so $x = yz$ with $y \in R$ and $z \in C_{RP}(R) - R$. Then as R and P are of exponent p, $1 = x^p = (yz)^p = y^p z^p = z^p$, so $z \in \Omega_1(C_{RP}(R)) \leq R$, contrary to the choice of z. This contradiction completes the proof of (2). □

LEMMA A.1.33. *Let p be an odd prime and $P \trianglelefteq L$ with $P \cong p^{1+2}$. Then each of the following imply $m_p(L) > 2$:*

(1) There exists $t \in L$ inverting $P/Z(P)$ such that $C_L(t)$ contains an element of order p nontrivial on $P/Z(P)$.

(2) $p > 3$ and $L/P \cong SL_2(p)$.

(3) $p = 3$, $L/P \cong SL_2(3)$, and an involution in L is nontrivial on P and centralizes a subgroup of L of order 3 distinct from $Z(L)$.

PROOF. Let $Z := Z(P)$, $V := P/Z(P)$, and identify V with $Inn(P)$. Then $A := O^{p'}(Aut(P))$ is the split extension of V by $SL_2(p)$ acting naturally on V.

Assume the hypotheses of (1); then there is X of order p in $C_L(t)$ nontrivial on V. Let P_1 denote the preimage of $C_V(X)$ in P. Then

$$E_{p^2} \cong P_1 = [P_1, t] \times Z(P) = C_P(X),$$

and as X centralizes t, X acts on $[P_1, t]$, so $XP_1 \cong E_{p^3}$ and hence $m_p(L) \geq 3$, establishing (1).

Next assume $L/P \cong SL_2(p)$, and let t be an involution in L. By a Frattini Argument, $L = PL_t$, where $L_t := C_L(t)$ and $L_t/Z \cong SL_2(p)$. Suppose $p > 3$. Then $K := L_t^\infty \cong SL_2(p)$ contains an element of order p, so if K centralizes P, then $m_p(L) > 2$. On the other hand if K does not centralize P, then K is faithful on V by paragraph one, so $m_p(L) > 2$ by (1).

Finally assume the hypotheses of (3) and let t be an involution in L. As t is nontrivial on P and $L/P \cong SL_2(3)$, t inverts V by paragraph one, and again $m_p(L) > 2$ by (1). □

We record the next result here, although its proof involves an appeal to Theorems A (A.2.1) and C (A.2.3) from the following section A.2; however the result will not be applied until section A.3, after those Theorems are established.

LEMMA A.1.34. *Assume G is quasithin and H is a $\mathcal{K}$-subgroup of G such that H contains distinct isomorphic components L_1 and L_2. Then*

(1) There exists an odd prime r dividing the order of a 2-local subgroup of $L_1/Z(L_1)$, and for each such r either

(i) $L_1L_2 = O^{r'}(H)$, or

(ii) $r = 3$, G is not a quotient of an SQTK-group, $L_1 \cong SU_3(8)$, and $C_G(L_1L_2)$ is of odd order.

(2) If L is a component of H isomorphic to L_1, then $L = L_1$ or L_2.

(3) Either

(a) $L_1/Z(L_1) \cong L_2(2^n)$, $Sz(2^n)$, J_1, or $L_2(p^e)$ for some prime $p > 3$ and positive integer $e \leq 2$, or

(b) G is not a quotient of an SQTK-group, $m_3(L_1L_2) = 3$, $L_1 \cong SU_3(8)$, and $Z(L_1) = Z(L_2)$.

PROOF. Set $\overline{L_1L_2} := L_1L_2/Z(L_1L_2)$. As $\bar{L}_1$ centralizes involutions in $\bar{L}_2$, $\bar{L}_1$ is described in Theorem C (A.2.3) in the following section, whose proof is independent of this lemma.

Let $\mathcal{R}$ be the set of odd primes r dividing the order of a 2-local subgroup of $\bar{L}_1$; i.e., $m_{2,r}(\bar{L}_1) > 0$. Then $\mathcal{R} \neq \emptyset$ by the Frobenius Normal p-Complement Theorem (39.4 in [**Asc86a**]) for $p = 2$. Let $r \in \mathcal{R}$. Since $L_2 \cong L_1$ and H is a quasithin $\mathcal{K}$-group, Theorem A (A.2.1) in the following section says that either $\bar{L}_1\bar{L}_2$ is quasithin and hence $m_r(\bar{L}_2) = 1$, or (3b) holds. Further in the latter case $m_3(L_1L_2) = 3$, so as G is quasithin, $C_G(L_1L_2)$ is of odd order; hence (1.ii) and (2) of the lemma hold.

Thus in the remainder of the proof we may assume:

$$\text{For each } \ r \in \mathcal{R}, \ m_{2,r}(\bar{L}_1) = 1. \qquad (*)$$

By (*) and I.1.2, $Z(L_i)$ is an r'-group, so $C_H(L_1L_2)$ is an r'-group since $m_{2,r}(H) \leq 2$. By inspection of the list of groups in Theorem C, either $\bar{L}_1$ contains A_4 or $SL_2(3)$, so that $m_{2,3}(\bar{L}_1) > 0$; or $\bar{L}_1$ is $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$, for suitable odd n. But if $\bar{L}_1 \cong U_3(2^n)$, then there is a prime divisor q of 2^n+1 with $m_{2,q}(\bar{L}_1) = 1 < m_q(\bar{L}_1)$, contrary to (*). Assume next that $m_{2,3}(\bar{L}_1) > 0$. Then $m_3(\bar{L}_1) = 1$ by (*), so appealing to the list of Theorem C, we conclude that $\bar{L}_1 \cong L_2(2^n)$, $Sz(2^n)$, $L_2(q^e)$, $L_3^{\epsilon}(q)$, for some prime $q > 3$ and $e \leq 2$, or J_1. However if $\bar{L}_1 \cong L_3^{\epsilon}(q)$, then the q-rank of the centralizer of an involution in $\bar{L}_1$ is 1 so that $m_{2,q}(\bar{L}_1) > 0$, while $m_q(\bar{L}_2) = 2$, contrary to (*). Thus we are left with the groups appearing in (3.a), so (3) is established.

Next if $L_1L_2 < O^{r'}(H)$, then as $C_H(L_1L_2)$ is an r'-group, some r-element x induces an outer automorphism on L_1L_2. But then from the structure of $Aut(L_1)$ for L_1 described in (3.a), $\bar{L}_1 \cong L_2(2^n)$ or $Sz(2^n)$, and we may choose x to induce a field automorphism of order r on at least one L_i, say L_1. Then x acts on a Borel subgroup B of L_1 and centralizes an element of order r in B. But x also centralizes an element of order r in L_2, so $m_{2,r}(BL_2\langle x\rangle) \geq 3$, contrary to G quasithin. Thus (1) is established.

Assume the hypothesis of (2). Then $L \leq O^{r'}(H) = L_1L_2$ by (1.i), so that (2) holds. □

A.1.3. Transfer and Fusion.

LEMMA A.1.35 (Burnside's Fusion Lemma). *Let p be a prime, $T \in Syl_p(G)$, $W \leq T$ with W weakly closed in T with respect to G, and $D := C_G(W)$. Then $N_G(W)$ controls G-fusion of subsets of G normalized by W, and in particular of elements of D.*

PROOF. See 37.6 in [**Asc86a**], or 16.2 and 16.9 in [**GLS96**]. □

LEMMA A.1.36 (Thompson Transfer Lemma). *Assume that $P \in Syl_2(X)$, Q is of index 2 in P, and u is an involution in X with $u^X \cap Q = \emptyset$. Then $u \notin O^2(X)$.*

PROOF. See 15.16 in [**GLS96**]. □

We ordinarily apply the following extension of the previous result with $p = 2$ and $H \in Syl_2(G)$:

LEMMA A.1.37 (Generalized Thompson Transfer). *Let G be a finite group, p a prime, $H \leq G$ with $(p, |G : H|) = 1$, $K \trianglelefteq H$ with H/K abelian, and g a p-element in $H - K$.*

(1) If $g^{ma} \in g^m K$ for all integers m, and all $a \in G$ such that $g^{ma} \in H$, then $g \notin [G, G]$.
(2) If $p = 2$ and g is an involution such that $g^G \cap H \subseteq gK$, then $g \notin [G, G]$.

PROOF. See 37.4 in [**Asc86a**]. □

LEMMA A.1.38 (Cyclic Sylow 2-Subgroups). *If p is the smallest prime divisor of the order of G, and G has cyclic Sylow p-subgroups, then G has a normal p-complement.*

PROOF. See 39.2 in [**Asc86a**]. □

A.1.4. Representation Theory.
Some results not formally referenced by number in the main text:
Schur's Lemma—12.4 in [**Asc86a**]
Maschke's Theorem—12.9 in [**Asc86a**]
Clifford's Theorem—12.13 in [**Asc86a**]

LEMMA A.1.39 (Gaschütz's Theorem). *Let p be a prime, V an abelian normal p-subgroup of a finite group G, and $P \in Syl_p(G)$. Then G splits over V if and only if P splits over V.*

PROOF. See 10.4 and 12.8 in [**Asc86a**]. □

Let G be a finite group, V an $\mathbf{F}_2G$-module, $X = O^2(X) \leq G$, and $\tilde{V} := V/C_V(X)$. We record some basic definitions:

DEFINITION A.1.40. $Irr(X, V)$ consists of the irreducible X-submodules of V. Let $Irr_+(X, V)$ consist of the X-submodules I of V such that $I = [I, X]$ and X is irreducible on $I/C_I(X)$. For $Y \leq N_G(X)$, let $Irr_+(X, V, Y)$ consist of those $I \in Irr_+(X, V)$ such that $\tilde{I}$ is an X-homogeneous component of $\langle \tilde{I}^Y \rangle$.

When Y is a 2-group, A.1.42.3 below says there exists $I \in Irr_+(X, V, Y)$, and the modules $\langle \tilde{I}^Y \rangle$ are the irreducible XY-submodules of $\tilde{V}$ containing an X-submodule isomorphic to $\tilde{I}$.

LEMMA A.1.41. *Let $I \in Irr_+(X, V)$ with $O_2(Aut_X(I)) = 1$, and set $\tilde{I} := I/C_I(X)$. Then $C_{GL(I)}(Aut_X(I))$ is isomorphic to a subgroup of $C_{GL(\tilde{I})}(Aut_X(\tilde{I}))$ $= End_X(\tilde{I})^\#$, and $End_X(\tilde{I})$ is a finite field of characteristic 2, so $C_{GL(I)}(Aut_X(I))$ is cyclic of odd order.*

PROOF. Without loss, $V = I$ and $G = GL(V)$; thus $O_2(X) = 1$ by hypothesis. Let $S \in Syl_2(C_G(X))$. By the Thompson $A \times B$-Lemma, $O^2(C_X(C_V(S))) = 1$, so $C_X(C_V(S)) \leq O_2(X) = 1$, and hence X is faithful on $C_V(S)$. So as X is irreducible on $\tilde{V}$, $V = C_V(S)$. Thus $S = 1$, so $C_G(X)$ is of odd order. Let $Y := C_{C_G(X)}(\tilde{V})$. As Y is of odd order, $V = C_V(Y) \oplus [V, Y]$ by Coprime Action. Then as $[V, Y] \leq C_V(X)$, $V = [V, X] \leq C_V(Y)$, so $Y = 1$. Thus $C_G(X)$ is faithful on $\tilde{V}$, so the lemma follows from Schur's Lemma and Wedderburn's Theorem that a finite division ring is a field. □

LEMMA A.1.42. *Let $T \in Syl_2(N_G(X))$, $I \in Irr_+(X,V)$, $I_T := \langle I^T \rangle$, and $\tilde{V} := V/C_V(X)$. Then*

(1) The stabilizer in T of the X-equivalence class of $\tilde{I}$ acts on I iff $I \in Irr_+(X,V,T)$.

(2) $Irr_+(X,I_T,T) \neq \emptyset$.

(3) If $I \in Irr_+(X,I_T,T)$ then distinct T-conjugates of $\tilde{I}$ are not isomorphic, and $\tilde{I}_T$ is the direct sum of these conjugates.

PROOF. Let S be the stabilizer in T of the X-equivalence class of $\tilde{I}$. Assume S acts on I. Then $\tilde{I}^t$ is not X-isomorphic to $\tilde{I}^r$ whenever $t,r \in T$ with $St \neq Sr$. But as $\tilde{I}_T = \langle \tilde{I}^T \rangle$ and $\tilde{I}$ is simple, $\tilde{I}_T$ is semisimple and the direct sum of some subset Δ of these conjugates. As each simple submodule is isomorphic to some member of Δ, Δ is the set of *all* conjugates, and the conjugates are the homogeneous components, so $I \in Irr_+(X,V,T)$, proving one implication in (1). The other implication is trivial, so (1) is established, as is (3).

Let $\tilde{J}$ be the homogeneous component of $\tilde{I}_T$ containing $\tilde{I}$. Let $\mathbf{F}_q := End_X(\tilde{I})$ and $e := \dim(\tilde{J})/\dim(\tilde{I})$. Then $\tilde{J}$ is the direct sum of e copies of $\tilde{I}$, and $N := (q^e-1)/(q-1) = |Irr_+(X,\tilde{I}_T)|$. In particular as N is odd, $N_T(\tilde{J})$ acts on some $I' \in Irr_+(X,J)$, so (2) follows from (1). □

LEMMA A.1.43. *Assume G is irreducible on V and $S \subseteq V$ is G-invariant with $|S| > 1$. Then any hyperplane W of V contains a member of S.*

PROOF. If $S \subseteq V - W$, then as $m(V/W) = 1$, W contains the G-invariant set $\{s+s' : s,s' \in S\}$, which is nonzero since $|S| > 1$, contradicting G irreducible on V. □

LEMMA A.1.44. *Assume G is dihedral of twice odd order, generated by distinct involutions t and u. Let $x := tu$ and V a faithful $\mathbf{F}_2G$-module. Then $V = W \oplus C_V(x)$, where $[V,x] =: W = [W,t] \oplus [W,u]$ and $C_W(t) = [W,t]$.*

PROOF. First $V = W \oplus C_V(x)$ by Coprime Action. Next as $G = \langle t,u \rangle$ and $W = [W,G]$, $W = [W,t] + [W,u]$. As $[W,t] \leq C_W(t)$ and $[W,u] \leq C_W(u)$, $C_W(t) \cap [W,u] \leq C_W(x) = 0$, so $W = C_W(t) \oplus [W,u]$ and $[W,t] = C_W(t)$. □

A.2. The list of quasithin $\mathcal{K}$-groups: Theorems A, B, and C

In this section G is a finite group. We discuss $\mathcal{K}$-groups G satisfying hypotheses related to the quasithin hypothesis, but without any reference to the even-characteristic hypothesis (E).

Recall from the Introduction to Volume I that G is *quasithin* if

(QT) $e(G) \leq 2$,

and that we define G to be *strongly quasithin* if

(SQT) $m_p(G) \leq 2$ for all odd primes p.

Of course G is quasithin iff all its 2-locals are strongly quasithin.

In this section we prove three related theorems (simultaneously). First, we will need to know that the conditions (QT) and (SQT) are preserved in semisimple sections:

THEOREM A.2.1 (Theorem A). *Let G be a quasithin $\mathcal{K}$-group, and assume $S = E(S)$ is a semisimple section of G. Then for each odd prime p:*

(1) Either

(a) $m_{2,p}(S) \leq m_{2,p}(G) \leq 2$, or

(b) $p = 3$, $S = H/Z(H)$ for some subgroup H of G such that $m_3(H) = 3$, $H = H_1H_2$ where each H_i is a component of H, $H_i \cong SU_3(2^{n_i})$ with $n_i \geq 3$ odd, $Z(H_1) = Z(H_2)$, and $(n_1, n_2) = 1$ or 3.

(2) If $m_p(G) \leq 2$, then $m_p(S) \leq m_p(G)$. In particular if G is strongly quasithin, then so is S.

Second, under the hypotheses of our Main Theorem, arbitrary proper sections will be known QT-groups, but not necessarily QTKE-groups. We determine the list of simple QTK-groups in the following theorem.

THEOREM A.2.2 (Theorem B (QTK-list)). *Let G be a simple quasithin $\mathcal{K}$-group. Then one of the following holds:*

(1) $G \cong A_n$ for $n \leq 9$.

(2) $G \cong L_2(q)$, q odd; $L_3^\epsilon(p^e)$, $PSp_4(p)$, $G_2(p)$, p an odd prime, $e \leq 2$; $L_4(r)$, $r = 2^a + 1$ a Fermat prime; or $U_4(s)$, $s = 2^b - 1$ a Mersenne prime.

(3) G is a group of Lie type, characteristic 2, and Lie rank at most 2, but G is $U_5(q)$ only when $q = 4$.

(4) $G \cong L_4(2)$, $L_5(2)$, or $Sp_6(2)$.

(5) G is a Mathieu group, a Janko group, HS, He, Ru, or Mc.

Third, in a quasithin $\mathcal{K}$-group, 2-local subgroups H will satisfy

(SQTK) H is a $\mathcal{K}$-group satisfying (SQT),

although simple sections of H will not necessarily satisfy (E); we will need to refer to the corresponding list below frequently:

THEOREM A.2.3 (Theorem C (SQTK-list)). *Let G be a simple strongly quasithin $\mathcal{K}$-group. Then one of the following holds:*

(1) $G \cong A_n$ for $n \leq 8$.

(2) $G \cong L_2(p^e)$ or $L_3^\epsilon(p)$, p an odd prime, $e \leq 2$.

(3) G is group of Lie type, characteristic 2, and Lie rank at most 2, but not $U_4(q)$ or $U_5(q)$.

(4) $G \cong L_4(2)$ or $L_5(2)$.

(5) G is a Mathieu group, J_1, J_2, J_4, HS, He, or Ru.

REMARK A.2.4. The Lie types appearing in (C.3) are: The groups of rank 1 are the Bender groups, namely of type $A_1 = L_2$, ${}^2A_2 = U_3$, ${}^2B_2 = Sz$; the groups of rank 2 are of type $A_2 = L_3$, $B_2 = Sp_4$, G_2, ${}^3D_4$, or ${}^2F_4$. In (B.3), one adds rank 2 groups of type ${}^2A_3 = U_4$ and ${}^2A_4(4) = U_5(4)$.

NOTATION A.2.5. As indicated in Notation 16.1.3, for outer automorphisms of groups of Lie type in characteristic 2, we follow the convention of Definition 2.5.13 in [**GLS98**]—which differs from the widely-used original convention of Steinberg, in which twisted groups have no graph automorphisms. Instead in the convention of [**GLS98**], a graph automorphism of odd order of a twisted group does occur for order 3 in type ${}^3D_4$; and we show in the corollary below that this automorphism cannot arise in an SQTK-group.

COROLLARY A.2.6. *Assume G is a strongly quasithin $\mathcal{K}$-group and $L \trianglelefteq G$ such that $L/O_2(L)$ is a quasisimple group of Lie type and characteristic 2. Then if $x \in G$ is of odd order, x induces an automorphism on $L/Z(L)$ which is the product of inner-diagonal and field automorphisms.*

PROOF. Set $G^* := G/O_{2,Z}(L)$. Then by (2) of Theorem A, L^* appears in Theorem C. If x is an element of odd order which does not satisfy the conclusion of the lemma, then by 2.5.12 in [**GLS98**], $L^* \cong {}^3D_4(2^n)$ and xL contains a 3-element inducing a graph automorphism on L^*, which we may take to be x. By I.1.3, $Z(L/O_2(L)) = 1$, so as $m_3(L) = 2$ and G is strongly quasithin, $C_G(L^*)$ is a $3'$-group. But then x is of order 3 and $m_3(C_L(x)\langle x \rangle) = 3$, contradicting G strongly quasithin. □

We will eventually prove Theorems A, B, and C together by induction on the order of G, but first we establish some reductions needed to establish Theorem A; see especially Remark A.2.14 below.

LEMMA A.2.7. *Let $K \trianglelefteq G$, $T := G/K$, and p an odd prime. Then*
(1) If $m_p(G) \leq 1$ then $m_p(T) \leq m_p(G)$.
(2) If $m_{2,p}(G) \leq 1$ then $m_{2,p}(T) \leq m_{2,p}(G)$.
(3) If $m_p(T) \leq 2$ then $m_p(T) \leq m_p(G)$.
(4) If $m_{2,p}(T) \leq 2$ then $m_{2,p}(T) \leq m_{2,p}(G)$.
(5) If $m_p(T) \leq 2$ then $m_p(T) \leq m_p(G)$ and $m_{2,p}(T) \leq m_{2,p}(G)$.

PROOF. Let $J \trianglelefteq H \leq G$, $H^* := H/J$, and $P \in Syl_p(H)$. Then $P^* \in Syl_p(H^*)$. If $m_p(H) = 0$ then $P = 1$, so $P^* = 1$ and hence $m_p(H^*) = 0$. If $m_p(H) = 1$ then P is cyclic, so P^* is cyclic and hence $m_p(H^*) \leq 1$. Thus

$$\text{if } m_p(H) \leq 1 \text{ then } m_p(H^*) \leq m_p(H), \qquad (*)$$

and (*) implies:

$$\text{if } m_p(H^*) \leq 2 \text{ then } m_p(H^*) \leq m_p(H). \qquad (**)$$

Applying (*) and (**) in the case $G = H$ and $J = K$, we get (1) and (3). If M/K is a 2-local of T, let U be a Sylow 2-group of the preimage of $O_2(M/K)$ in M; then by a Frattini Argument, $M = KN_M(U)$ and $M/K \cong N_M(U)/N_K(U)$, so applying (*) and (**) to $H = N_M(U)$ and $J = N_K(U)$ for the various choices of M/K, we get (2) and (4). Finally as $m_{2,p}(T) \leq m_p(T)$, (3) and (4) imply (5). □

In our next three lemmas we work under the following hypothesis:

In lemmas A.2.8 through A.2.11: (G, S, p) is a counterexample to Theorem A which is minimal in the following sense: Theorem A holds in each proper quasithin section of G.

LEMMA A.2.8. *G is minimal subject to covering the section S. In particular S is a quotient of G.*

PROOF. Suppose $S = H/K$ with $H < G$. As G is a counterexample to Theorem A, either $m_{2,p}(S) > m_{2,p}(G)$ and S does not satisfy conclusion (b) of part (1) of Theorem A, or $m_p(G) \leq 2$ and $m_p(S) > m_p(G)$. Then H is a quasithin $\mathcal{K}$-group with $m_{2,p}(H) \leq m_{2,p}(G)$ and $m_p(H) \leq m_p(G)$, so S also exhibits the failure of Theorem A for H. Thus minimality of G supplies a contradiction. □

NOTATION A.2.9. By A.2.8, there is $K \trianglelefteq G$ with $G^* := G/K \cong S$.

LEMMA A.2.10. $O_2(G) = 1$.

PROOF. Assume $O_2(G) \neq 1$ and let $\bar{G} := G/O_2(G)$. Note that

$$m_p(\bar{G}) = m_p(G) = m_{2,p}(G) \leq 2, \qquad (a)$$

as G is quasithin. Now $T := \bar{G}/\bar{K} \cong G/KO_2(G) \cong S/S_0$, where $S_0 \leq O_2(S)$. Thus as S is semisimple, so is T, and

$$m_p(S) = m_p(T). \qquad (b)$$

By (a), $\bar{G}$ is a proper quasithin section of G, so by minimality of G:

$$m_p(T) \leq m_p(\bar{G}) \leq 2. \qquad (c)$$

Now by (b) and (c), $m_p(S) \leq 2$, so A.2.7.5 contradicts the choice of S, G as a counterexample. □

LEMMA A.2.11. *$G/O_p(G) = S/Z(S)$ for p the prime exhibiting the failure of Theorem A.*

PROOF. Let H be the preimage of $Z(S)$ in G, and $Q \in Syl_q(H)$ for some prime q. By a Frattini Argument, $G = HN_G(Q)$. Thus $S = G^* = H^*N_G(Q)^*$, while as S is semisimple, $H^* = Z(S) \leq \Phi(S)$, so $S = N_G(Q)^*$. Therefore $N_G(Q)$ covers S, so by A.2.8, $Q \trianglelefteq G$. Suppose $1 \neq Q$ and $q \neq p$. By A.2.10, q is odd. Set $\hat{G} := G/Q$. Then $m_p(\hat{G}) = m_p(G)$ and $m_{2,p}(\hat{G}) = m_{2,p}(G)$, and $\dot{S} := S/O_q(S)$ is a semisimple section of $\hat{G}$ with $m_p(\dot{S}) = m_p(S)$ and $m_{2,p}(\dot{S}) = m_{2,p}(S)$. Hence by minimality of G and A.2.8, $\dot{S} = \hat{G}/Z(\hat{G})$ with $\hat{G}$ in the role of "H" satisfying conclusion (b) of part (1) of Theorem A. Let $X := G^\infty$. Observe then as $n_i \geq 3$, that $m_2(X) = m_2(S) \geq 6$, so $m_q(Q) \leq 2$ by A.1.28. Thus X centralizes Q by A.1.25, so $X = E(X)$. But now $G = XQ$, contrary to A.2.8.

Thus H is a p-group, so $H = O_p(G)$ since $S = E(S)$. □

At this point, we begin our parallel proof of the three Theorems by induction on the group order so we assume:

In the remainder of the section, G is a minimal counterexample to the union of Theorems A, B, and C; that is, Theorems A, B, and C hold in each proper section of G satisfying the hypotheses of the respective theorem.

Observe that if G is a counterexample to Theorem A, then G is a minimal counterexample to Theorem A in our earlier sense. Thus choosing a section S and a prime p exhibiting that failure, we may apply the earlier results to (G, S, p).

PROPOSITION A.2.12. *If the counterexample G is to Theorem A, then G is quasisimple.*

PROOF. Adopt Notation A.2.9. Then $S = G^*$ is the product of n components G_i^*, $1 \leq i \leq n$; set $G_i := H_i^\infty$ where H_i is the preimage of G_i^* in G, so that $G_i = G_i^\infty \trianglelefteq G$. By A.2.8, $G = G_1 \cdots G_n$. We need to show that $n = 1$, and each G_i is quasisimple, so we assume otherwise and derive a contradiction.

Set $m_i := m_2(G_i)$ and let $A_i \leq G_i$ be an elementary abelian 2-group of rank m_i. By A.2.11, $G_i/O_p(G_i)$ is simple, so $m_i = m_2(G_i^*) > 1$, and we can choose notation so that

$$A := \langle A_i : 1 \leq i \leq n \rangle = A_1 \times \cdots \times A_n.$$

By A.1.21, there is a supercritical subgroup P of $O_p(G)$. As $G/O_p(G) = S/Z(S)$ by A.2.11, $C_G(O_p(G)) \leq O_p(G)E(G)$, and then $C_G(P/\Phi(P)) \leq O_p(G)E(G)$ by A.1.21.

We show next that $G = E(G)$—that is, G is semisimple. Assume otherwise; then we can assume $G_1 \not\leq C_G(P)$ and hence A_1 is faithful on P. We apply A.1.29.2 with G_1 in the role of "H": If $m_p(P) \leq 2$ the lemma says that $m_1 = 1$, contrary to the previous paragraph, so we conclude that $m_p(P) > 2$. Then as $m_{2,p}(G) \leq 2$, A is faithful on $P/\Phi(P)$, so $m_2(G) \leq 3$ by A.1.28; indeed since $G/O_p(G) = S/Z(S)$ is a product of simple groups, we see that $Z^*(G) = 1$, so $m_2(G) \leq 2$ by A.1.28. As $m_1 > 1$, we conclude that $n = 1$, $G = G_1$, and $m_1 = 2$. As $m_p(G) \geq m_p(P) > 2$, part (2) of Theorem A is vacuous, so part (1) fails. Hence $m_{2,p}(S) > 2$ by A.2.7.4. Since $m_2(G^*) = m_1 = 2$, we can refer to lists of groups in $\mathcal{K}$ of 2-rank 2. This information can be deduced from tables of p-ranks such as in 5.2.10 and 5.6.1 in [**GLS98**], but it may be quicker just to quote the list (not the proof, as we are assuming that G is a $\mathcal{K}$-group) from the general 2-rank 2 classification 48.1 in [**Asc86a**] to conclude that $S/Z(S)$ is one of: $L_2(q)$ or $L_3^\epsilon(q)$, q odd; M_{11}, $U_3(4)$, or A_7. [2] Hence as $m_{2,p}(S) > 2$, $S \cong L_3^\epsilon(q)$ with $q = p^e$ and $e > 2$. But then there is a quasisimple section $SL_2(p^e) \cong L^* < G^*$, with $m_{2,p}(L^\infty) > 2$, exhibiting failure of Theorem A in the preimage L of L^* which is proper in G, and hence contradicting the minimality of G.

This contradiction shows that $G = E(G)$, so it remains to assume that $n > 1$, and derive a contradiction.

Let T_j be a nontrivial 2-subgroup of G_j. For $i \neq j$, $G_i \leq N_G(T_j)$, so as G is quasithin, $m_r(G_i) \leq 2$ for each odd prime r. Thus G_i is strongly quasithin, so by minimality of G, $m_r(G_i^*) \leq 2$ for each odd prime r, and G_i^* is on the list of Theorem C.

We claim:

(I) If $Z(G_i) \neq 1$ for some i, then $p = 3$, $|Z(G_i)| = 3$, $G_i \cong \hat{A}_6$, $\hat{A}_7$, $\hat{M}_{22}$, or $SL_3^\epsilon(q)$ for q an odd prime or a power of 2, with $q \equiv \epsilon \mod 3$, $m_3(G_i) = m_3(G_i^*) = 2$, and either

(i) $1 \leq m_{2,3}(G_i^*) \leq m_{2,3}(G_i) = 2$, or

(ii) $G_i \cong SU_3(2^{n_i})$ with $n_i \geq 3$ odd, and $m_{2,3}(G_i^*) \leq m_{2,3}(G_i) = 1$.

(Recall that $\hat{H}$ denotes the perfect central extension of $\mathbf{Z}_3$ by H, as defined in Notation A.3.5). Namely if $Z(G_i) \neq 1$, then inspecting the list I.1.3 of Schur multipliers for the groups in Theorem C, and keeping in mind that $Z(G_i)$ is a p-group with p odd, we conclude that $p = 3$, $|Z(G_i)| = 3$, and G_i is one of the groups in the initial assertion of (I). Further $m_3(G_i^*) = 2 = m_3(G_i)$ using I.2.2.2. In particular $m_{2,3}(G_i^*) \leq 2$; but on the other hand, for each group G_i in (I) (other than $SU_3(2^m)$ for suitable odd m, which would appear in (I.ii)), G_i^* contains a subgroup isomorphic to A_4 or $SL_2(3)$, so that $1 \leq m_{2,3}(G_i^*)$. Hence (I.i) holds whenever $m_{2,3}(G_i) > 1$: since then $m_{2,3}(G_i) = 2$ as G is quasithin, ans $m_{2,3}(G_i^*) = 1$. So we may assume that $m_{2,3}(G_i) \leq 1$; and hence as $Z(G_i)$ is of order 3 that $m_{2,3}(G_i) = 1$. Now if $G_i \cong \hat{A}_6$, $\hat{A}_7$, or $\hat{M}_{22}$, then a Sylow 3-subgroup P of G_i is isomorphic to 3^{1+2} of exponent 3; so as $m_{2,3}(G_i^*) \geq 1$, $m_{2,3}(G_i) > 1$, contrary to our assumption. So G_i is $SL_3^\epsilon(q)$ for q an odd prime or a power of 2, with $q \equiv \epsilon \mod 3$. Now G_i contains an $SL_2(q)$-subgroup L, which in particular has a center of order 2. Unless $q = 2^{n_i}$

[2] A_7 was inadvertently omitted from the list in 48.1 of [**Asc86a**].

with $n_i \geq 3$ odd, L has 3-rank 1 so that $m_{2,3}(L) = 1$; and then $m_{2,3}\big(LZ(G_i)\big) = 2$, so that $m_{2,3}(G_i) > 1$, contrary to our assumption. Therefore $G_i = SL_3^{\epsilon}(2^{n_i})$ with $n_i \geq 3$ odd. As $q \equiv \epsilon \mod 3$, $G_i = SU_3(2^{n_i})$, and $m_{2,3}(G_i^*) \leq m_{2,3}(G_i)$ by A.2.7.5; so as $m_{2,3}(G_i) = 1$ by our assumption, (I.ii) holds, completing the proof of (I).

We next claim:

(II) If $Z(G_i) \neq 1$, then $Z(G_j) = 1$ for each $j \neq i$.

Assume otherwise. Without loss, $Z(G_r) \neq 1$ for $r = 1, 2$. By (I), $p = 3$, $Z_r := Z(G_r)$ is of order 3, and $m_3(G_r) = m_3(G_r^*) = 2$. Hence if $m_{2,3}(G_r) > 1$ or $Z_1 \neq Z_2$, then $m_{2,3}(G_r P) > 2$ for $P \in Syl_3(G_{3-r})$, contradicting G quasithin. Therefore $m_{2,3}(G_r) = 1$ for each r and $Z_1 = Z_2$, and (ii) rather than (i) of (I) holds, so that $G_r \cong SU_3(2^{n_r})$ for $r = 1, 2$. Now $m_3(G_1G_2) = 3$, so as G is quasithin, $C_G(G_1G_2)$ is of odd order, and hence $n = 2$ and $G = G_1G_2$.

If $d := (n_1, n_2)$ is not 1 or 3, then there is a prime divisor $s > 3$ of $2^d + 1$ such that $m_{2,s}(G_1) = 1$ and $m_s(G_2) = 2$. Thus $m_{2,s}(G) > 2$, contradicting G quasithin. Thus $d = 1$ or 3, so (1.b) of Theorem A holds, contrary to the choice of G as a counterexample to Theorem A. This completes the proof of (II).

Finally observe:

(III) For all j, $m_p(G_j) = m_p(G_j^*)$ and $m_{2,p}(G_j) \geq m_{2,p}(G_j^*)$.

For (III) is trivial when $Z(G_j) = 1$, and (III) follows from (I) when $Z(G_j) \neq 1$.

By (II), the semisimple group G is not just the central product, but in fact the direct product of the subgroups G_j, and hence S is either G or $G/Z(G_i)$ using A.2.11. As the product is direct, (III) says that $m_p(G) = m_p(S)$ and $m_{2,p}(G) \geq m_{2,p}(S)$, contrary to our choice of G, S as a counterexample to Theorem A. $\square$

REMARK A.2.13. In view of Proposition A.2.12, we have the following dichotomy:

Case I. G is simple and G is a counterexample to Theorem B or C.
Case II. G is quasisimple but not simple, and (G, S, p) is a counterexample to Theorem A, where $S := G/Z$ for some $1 \neq Z \leq Z(G)$ and $Z(G)$ is a p-group.

For Theorem A is trivial when $G = S$, so A.2.11 says that G is not simple if G is a counterexample to Theorem A. Further in that event, G is quasisimple by A.2.12. Also by A.2.11, there is a unique prime p for which Theorem A fails, $Z(G)$ is a p-group, and $S = G/Z$ for some subgroup Z of $Z(G)$. On the other hand if G is not simple, then Theorems B and C hold vacuously, so G is simple when G is a counterexample to Theorem B or C.

In the remainder of the section, set $G^* := G/Z(G)$.

REMARK A.2.14. In Case II, $m_p(S) > 2$ by A.2.7.5. Thus if $|Z(G)| = p$, then $m_p(G^*) > 2$.

LEMMA A.2.15. *Assume that $M < G$, $R = E(R)$ is a semisimple section of M, and r is an odd prime. Then*

(1) $m_r(R) \leq m_r(M) \leq m_r(G)$ if $m_r(G) \leq 2$.

(2) $m_r(R) \leq m_r(M) \leq m_{2,r}(G) \leq 2$ if M is a 2-local subgroup of G.

(3) If R is simple, then R is on the list of Theorem B; if in addition M is a 2-local subgroup of G, then R is on the list of Theorem C.

PROOF. We invoke the minimality of G as a counterexample to the union of the three Theorems: As G is quasithin, so is M, so that Theorem A holds in M by minimality of G. Further if $m_r(G) \leq 2$, then $m_r(M) \leq 2$, so (1) follows as (2) of Theorem A holds in M. The inequality $m_r(M) \leq m_{2,r}(G)$ in (2) holds since M is a 2-local; of course $m_{2,r}(G) \leq 2$ as G is quasithin. Thus again $m_r(R) \leq m_r(M)$ as (2) of Theorem A holds in M, completing the proof of (2).

Suppose R is simple. Then R is quasithin as Theorem A holds in M. Further if M is a 2-local, then R is strongly quasithin by (2). Thus (3) follows from the minimality of G. □

In the remainder of the section, we complete the proof of our three Theorems, by considering the possible choices for G^* from the various families of simple $\mathcal{K}$-groups: alternating groups, groups of Lie type, and sporadic groups. Our standard reference for the structure of these groups is [**GLS98**]. In particular for the structure of parabolic subgroups of Lie-type groups, see section 2.6 of [**GLS98**] or section 43 of [**Asc86a**]. For the sporadic groups, we will appeal to the list of maximal overgroups of a Sylow 2-subgroup in [**Asc86b**]; for further reference we mention also the Atlas [**C$^+$85**], and the list of references for the 2-radical subgroups ($\mathcal{B}_2(G)$) recently given by Yoshiara [**Yos**, Table 3]. In Case II of Remark A.2.13 the p-part of the Schur multiplier of G^* is nontrivial, so we frequently quote the list of Schur multipliers of simple $\mathcal{K}$-groups in Definition 6.1.1 and Tables 6.1.2 and 6.1.3 of [**GLS98**]. In the remainder of this section, we will abbreviate this reference by using the (capitalized) phrase "the LIST" (of multipliers).

LEMMA A.2.16. *G^* is not an alternating group.*

PROOF. Assume $G^* \cong A_n$. First assume G is not simple. Then we are in Case II of Remark A.2.13, so $Z(G)$ is a nontrivial p-group for the unique odd prime p exhibiting the failure of Theorem A. From the LIST of multipliers for the alternating groups, $n = 6$ or 7 and $|Z(G)| = p = 3$. But then $m_3(G^*) = 2$, contrary to Remark A.2.14.

Therefore $G^* = G$ is simple, so G is a counterexample to Theorem B or C. If $n > 10$, then G has a proper A_{10}-subgroup, contradicting A.2.15.3. If $n = 10$, G has an $A_4 \times A_6$-subgroup of 3-rank 3, contradicting $m_{2,3}(G) \leq 2$. If $n = 9$, then G appears in Theorem B and $m_3(G) = 3$, so G is not strongly quasithin; thus G is a counterexample to neither Theorem B nor Theorem C. Thus $n \leq 8$, so again G appears in Theorems B and C, contrary to the choice of G as a counterexample. □

LEMMA A.2.17. *G^* is not of Lie type and odd characteristic.*

PROOF. Assume otherwise, so that G^* is defined [3] over $\mathbf{F}_q$ where $q = r^e$ for an odd prime r and $e \geq 1$. We exclude the case $G^* \cong {}^2G_2(3)' \cong L_2(8)$, as it is treated in the next lemma.

We first consider the cases where G^* is $L_2(q)$ or ${}^2G_2(q)$. By A.2.16, G^* is not $A_6 \cong L_2(9)$. Then from the LIST, the multiplier of G^* is a 2-group, so $G^* = G$ is simple, and a counterexample to Theorem B or C. Now if $G \cong {}^2G_2(q)$, then G has a $\mathbf{Z}_2 \times L_2(q)$-subgroup, so as $q > 3$ is an odd power of 3 and G is quasithin, we have a contradiction. Next if $G \cong L_2(q)$, then G appears in Theorem B, while if G is also strongly quasithin, then $e \leq 2$, so that G also appears in Theorem C. This completes the treatment of $G \cong L_2(q)$ or ${}^2G_2(q)$.

[3] For twisted groups, we follow the usual convention of writing $X(q)$, with the field of definition $\mathbf{F}_q$ given by the fixed field of the underlying field automorphism; cf. definition 2.2.4 in [**GLS98**].

We claim that $O_r(G) = 1$. Assume otherwise. Then Case II holds with $r = p$ and the p-part of the multiplier of G^* is nontrivial, so we conclude from the LIST of multipliers of G^* that $p = 3$ and G^* is $G_2(3)$, $U_4(3)$, or $\Omega_7(3)$. However in each case $m_{2,3}(G) > 2$ by I.2.1, a contradiction.

As G^* is not $L_2(q)$ or a Ree group, G^* has a proper *fundamental subgroup* $K_0^* \cong SL_2(q)$ (see [**Asc80**, p.401]). Let K_0 be the preimage of K_0^* in G. By the claim, $O_r(G) = 1$, so $K := O^{r'}(K_0) \cong SL_2(q)$ and $K_0^* = K^*$. Let z be the involution generating $Z(K)$. As $K \leq C_G(z)$ and $m_{2,3}(G) \leq 2$,

$$e \leq 2.$$

Let $J := O^{r'}(C_G(K))$; as $K = O^{r'}(K)$ and $Z(G)$ is an r'-group, $J^* = O^{r'}(C_{G^*}(K^*))$ by Coprime Action. Thus J^* is (essentially) described in Theorem 1 of [**Asc80**, p.402]. As $Z(K) = \langle z \rangle$ is of order 2, for each odd prime $s \in \pi(K)$, $2 \geq m_{2,s}(G) \geq m_s(KJ) = 1 + m_s(J)$, so that $m_s(J) \leq 1$. Thus a Sylow s-subgroup of J is cyclic, so that

$$m_s(J^*) \leq 1 \quad \text{for } s \in \{3, r\} \; . \tag{$*$}$$

We first consider those cases where G^* is classical.

Assume that $G^* \cong L_n^\epsilon(q)$. Then $n > 2$ and $J^* \cong SL_{n-2}^\epsilon(q)$ (by the reference mentioned above). Hence by (*), either $n = 3$, or $n = 4$ and $q = r$.

Suppose first that $n = 3$. Then $m_s(G^*) \leq 2$ for each odd prime s, and from the LIST, $|Z(G)| = 1$ or 3, so Case I holds by Remark A.2.14. As $e \leq 2$, G appears in Theorem B. Further if G is strongly quasithin, then $m_r(G) \leq 2$, so that $e = 1$ and hence G also appears in Theorem C. Thus G is not a counterexample to Theorems B or C, contrary to the choice of G in Case I.

Therefore we may suppose that $n = 4$ and $q = r$ is prime. As $O_r(G) = 1$, the LIST says G is simple, so Case I holds. Now a maximal torus H of G of maximal rank centralizes z, and $m_s(H) = 3$ for all primes s dividing $r - \epsilon$. Thus as G is quasithin, $r - \epsilon$ is a power of 2, so $G \cong L_4^\epsilon(r)$ is not a counterexample to Theorem B. Further $m_r(G) > 2$, so G is not strongly quasithin, and hence G is also not a counterexample to Theorem C. This contradiction completes our treatment of the case $G^* \cong L_n^\epsilon(q)$.

Next suppose that $G^* \cong PSp_{2n}(q)$; we may take $n > 1$ as $PSp_2(q) \cong L_2(q)$. From the LIST the multiplier of G^* is a 2-group, so Case I holds. Then $J \cong Sp_{2n-2}(q)$, so we conclude from (*) that $n = 2$, and $e = 1$ so that $q = r$. But then $G \cong PSp_4(r)$ appears in Theorem B, and $m_r(G) = 3$, so that G is also not a counterexample to Theorem C.

Finally suppose that $G^* \cong P\Omega_n^\epsilon(q)$. We may take $n \geq 7$, since for $n \leq 6$, G^* is isomorphic to one of the groups already eliminated. In this case $J^* \cong SL_2(q) * \Omega_{n-4}^\epsilon(q)$, so as $n \geq 7$, $m_3(J^*) \geq 2$, contrary to (*).

At this stage we have shown that G^* is not classical. In case $G^* \cong F_4(q)$, $J^* \cong Sp_6(q)$, so that $m_3(J^*) > 1$, contrary to (*). Further

$$F_4(q) < {}^2E_6(q) \quad \text{and} \quad F_4(q) < E_6(q) < E_7(q) < E_8(q),$$

so A.2.15.3 shows that G^* is none of these groups. This leaves the cases where $G^* \cong G_2(q)$ or ${}^3D_4(q)$. As $O_r(G) = 1$, we conclude from the LIST that the multiplier of G^* is trivial, so Case I holds. Now $J \cong SL_2(q)$ or $SL_2(q^3)$ when $G \cong G_2(q)$ or ${}^3D_4(q)$, respectively, so we conclude from (*) that $G = G_2(r)$. Then G appears in Theorem B, and as $m_r(G_2(r)) > 2$, G is also not a counterexample to Theorem C. □

Lemma A.2.18. *G^* is not of Lie type and characteristic 2.*

Proof. Assume otherwise. By A.2.15.3, nonabelian simple sections of proper parabolic subgroups of G^* appear in Theorem C. This restriction rules out all groups of Lie rank greater than 3, except possibly $L_n(2)$ for $n \leq 6$; it also rules out $U_6(q)$, $U_7(q)$, and $\Omega_8^-(q)$ of Lie rank 3.

Suppose first that G^* is of Lie rank 3; hence G^* is $L_4(q)$ or $Sp_6(q)$ by the previous paragraph. Then from the LIST, the multiplier of G^* is a 2-group, so Case I of Remark A.2.13 holds. Further as G is quasithin, a Cartan subgroup H of G satisfies $m_r(H) \leq 2$ for all odd primes r; so as $m_s(H) = 3$ for all odd primes s dividing $q-1$, we conclude that $q = 2$, and hence $G = L_4(2)$ or $Sp_6(2)$. We already eliminated the case $G \cong A_8 \cong L_4(2)$, while $Sp_6(2)$ appears in Theorem B, and $m_3(Sp_6(2)) = 3$ so that G is not a counterexample to Theorem C. Therefore G^* is not of Lie rank 3.

Next assume that $G^* \cong L_n(2)$, so that $3 \leq n \leq 6$. Again from the LIST, the multiplier of G^* is a 2-group, so Case I holds. Further $n = 5$ or 6, since otherwise G is isomorphic to a group $L_3(2) \cong L_2(7)$ or $L_4(2) \cong A_8$ already eliminated. Next G has a parabolic with Levi complement $L_2(2) \times L_{n-2}(2)$, so as the 3-rank of this Levi subgroup is at most $m_{2,3}(G) \leq 2$, we conclude that $n \neq 6$. Finally $L_5(2)$ appears in Theorems B and C.

This leaves the cases where G^* is of Lie rank at most 2. Assume for the moment that G is not simple. Then from the LIST, the multiplier of G^* is a 2-group unless $Sp_4(2)' \cong A_6$ (a case already eliminated in A.2.16), or:

(i) $G^* \cong L_3^\epsilon(q)$, $q \equiv \epsilon \mod 3$, $Z(G) \cong \mathbf{Z}_3$, and $G \cong SL_3^\epsilon(q)$, or
(ii) $G^* \cong U_5(q)$, $q \equiv -1 \mod 5$, $Z(G) \cong \mathbf{Z}_5$, and $G \cong SU_5(q)$.

We now treat the cases $G^* \cong L_3^\epsilon(q)$ or $U_5(q)$ in both Case I and Case II.

First assume G^* is $U_5(q)$. Then there is a parabolic of G^* containing a subgroup

$$L^* \cong \mathbf{Z}_{(q+1)/d} \times SU_3(q), \text{ where } d := (q+1, 5).$$

If $Z(G) = 1$ then $L \cong L^*$, while if $Z(G) \neq 1$ then $L \cong \mathbf{Z}_{q+1} \times SU_3(q)$. Thus in each case $m_r(L) > 2$ for every prime divisor r of $(q+1)/d$, so $q+1 = d = 5$. Moreover if $Z(G) \neq 1$ then $m_5(L) > 2$, contrary to $m_{2,5}(L) \leq 2$; therefore Case I holds. But $U_5(4)$ appears in Theorem B, and as $m_5(U_5(4)) > 2$, $U_5(4)$ is also not a counterexample to Theorem C.

Assume $G \cong (S)L_3^\epsilon(q)$. Then $m_r(G^*) \leq 2$ for all odd primes r and $|Z(G)| = 1$ or 3, so Case I holds by Remark A.2.14. As $L_3(q)$ appears in Theorems B and C, this contradicts the choice of G in Case I.

We have shown that G is simple of Lie rank at most 2, but G is not $U_5(q)$ or $L_3^\epsilon(q)$. If G is $U_4(q)$, then G appears in Theorem B, and $m_r(G) > 2$ for r a prime divisor of $q+1$, so that G is not a counterexample to Theorem C. Finally the remaining groups of Lie rank at most 2 appear in both Theorems B and C, contrary to G a counterexample. □

Lemma A.2.19. *G^* is not a sporadic group.*

Proof. Assume otherwise. We use the list of centralizers of involutions of sporadics (see [**GLS98**, Sec 5.3]) and A.2.15.3 to eliminate the eleven sporadic groups with an involution centralizer having a nonabelian simple section not appearing in Theorem C. We are left with the thirteen sporadics in Theorem B, as well as $O'N$ and F_5.

If G^* is F_5, then G^* contains an A_{12}-subgroup, contrary to A.2.15.3. So suppose G^* is $O'N$. Then G^* has an $E_9 \times A_6$ subgroup containing an $E_9 \times A_4$ subgroup L^*, so $m_{2,3}(G^*) > 2$. As G is assumed to be quasithin, we conclude that G is not simple; thus Case II of Remark A.2.13 holds. But by I.2.1, $m_{2,3}(G) > 2$, so G^* is not $O'N$. Therefore we have shown that G^* is on the list of Theorem B.

Suppose Case II holds. From the LIST (or I.1.3, as G^* appears in Theorem B), G^* is M_{22}, J_3, or Mc, and $|Z(G)| = 3$. If $G^* \cong M_{22}$ then $|Z(G)| = 3$ and $m_3(G^*) = 2$, contrary to A.2.14. In the remaining two cases, $m_{2,3}(G) > 2$ by I.2.1, again a contradiction.

Therefore Case I holds. But as $G \cong G^*$ is on the list of Theorem B, G must be a counterexample to Theorem C. If $G \cong J_3$ or Mc, then $m_3(G) = 3$ or 4, respectively, so that G is not a counterexample to Theorem C. The remaining eleven possibilities for G appear in both Theorems B and C, contrary to the choice of G as a counterexample. This completes the proof. □

Notice that lemmas A.2.16 through A.2.19 cover all possible choices for a simple group G^* in $\mathcal{K}$, so the proof of our three theorems is complete.

A.3. A structure theory for Strongly Quasithin $\mathcal{K}$-groups

In this section, we define the notion of a $\mathcal{C}$-component, which generalizes the usual notion of component. Some of the properties depending on subnormality go back to Wielandt. Also $\mathcal{C}$-components of 2-locals in thin groups were exploited by Aschbacher in [**Asc78b**]. As mentioned in the Introduction to Volume I, we analyze 2-locals in QTKE-groups in terms of their $\mathcal{C}$-components, and also use these $\mathcal{C}$-components to produce uniqueness subgroups in chapter 1. In the present section, we will see that an SQTK-group G with $O_2(G) = 1$ has a very restricted structure, which can be described in terms of its $\mathcal{C}$-components. Then we "pull back" this structure to a general 2-local in a QTKE-group in chapter 1.

We will begin by developing some purely formal properties of $\mathcal{C}$-components, independent of the hypothesis that G is quasithin. Thus until our second lemma A.3.6, G can be any finite group.

The following definition of $\mathcal{C}$-component includes the usual notions both of component and p-component for a prime p:

Definition A.3.1. Let $\mathcal{C}(G)$ be the set of subgroups L of G minimal subject to

$$1 \neq L = L^\infty \trianglelefteq \trianglelefteq G.$$

The members of $\mathcal{C}(G)$ are called the *$\mathcal{C}$-components* of G.

Observe that Proposition A.3.3 provides a partial analogue of the usual theory of ordinary components, such as in [**Asc86a**, Sec 31]; in particular, compare A.3.3 to [**Asc86a**, 31.3,31.4,31.6]. Indeed in A.3.3 we obtain a characterization of $O_{\infty,E}(G)$—which in view of terminology in the literature, we might think of as the ∞-layer of G.

Recall that $O_\infty(G)$ denotes the largest normal solvable subgroup of G. Also we use a brief but technically abusive notation for intersections, for $\mathcal{C}(G)$ and indeed for other sets of subgroups we will define:

Notation A.3.2. If H is a subgroup of G, let $\mathcal{C}(G) \cap H$ denote the set of members of $\mathcal{C}(G)$ lying in H.

PROPOSITION A.3.3. *(1) For each $L \in \mathcal{C}(G)$, $O_\infty(L)$ is the unique maximal proper subnormal subgroup of L, and $L/O_\infty(L)$ is a nonabelian simple group. In particular, L is the unique member of $\mathcal{C}(L)$.*

(2) If $H \leq G$ then $\mathcal{C}(G) \cap H \subseteq \mathcal{C}(H)$. If $H \trianglelefteq \trianglelefteq G$, then $\mathcal{C}(G) \cap H = \mathcal{C}(H)$.

(3) If $L \in \mathcal{C}(G)$ and N is an L-invariant solvable subgroup of G, then $[L, N] \leq L$ and $(LN)^\infty = L$.

(4) Let N be a solvable normal subgroup of G and $\bar{G} := G/N$. Then $\overline{\mathcal{C}(G)} = \mathcal{C}(\bar{G})$, and the map $L \mapsto \bar{L}$ is a 1:1 correspondence between $\mathcal{C}(G)$ and $\mathcal{C}(\bar{G})$.

(5) $\mathcal{C}(G/O_\infty(G))$ is the set of components of $G/O_\infty(G)$, so $\langle \mathcal{C}(G) \rangle = O_{\infty,E}(G)^\infty$.

(6) $L \trianglelefteq \langle \mathcal{C}(G) \rangle$ for each $L \in \mathcal{C}(G)$.

(7) If $X \leq G$ and $L \in \mathcal{C}(G)$, then either $[L, X] \leq O_\infty(G)$ so that $[L, X]$ is solvable, or $[L, X] = \langle L^X \rangle$.

PROOF. Throughout the proof, L denotes a $\mathcal{C}$-component of G, if $\mathcal{C}(G)$ is nonempty. If N is a proper subnormal subgroup of L, then as the subnormality relation is transitive, we conclude that $N^\infty = 1$ from the minimality of L. In particular $O_\infty(L)$ is the unique maximal proper subnormal subgroup of L, so $L/O_\infty(L)$ is simple and (1) holds.

Suppose $L \leq H \leq G$; then also $L \trianglelefteq \trianglelefteq H$. If $L \geq N$ with $1 \neq N = N^\infty \trianglelefteq \trianglelefteq H$, then also $N \trianglelefteq \trianglelefteq L$, so N contains a member of $\mathcal{C}(L)$, and hence $N = L$ by (1). Thus $L \in \mathcal{C}(H)$, so $\mathcal{C}(G) \cap H \subseteq \mathcal{C}(H)$. Let $K \in \mathcal{C}(H)$ and assume that $H \trianglelefteq \trianglelefteq G$. Then K is a minimal subnormal perfect subgroup of G by transitivity of the subnormality relation, so $K \in \mathcal{C}(G)$ and hence $\mathcal{C}(H) \subseteq \mathcal{C}(G) \cap H$, completing the proof of (2).

Assume (3) is false, and let G be a minimal counterexample. Then $L < G$. By (2), $L \in \mathcal{C}(LN)$, so $G = LN$ by minimal order of G, and hence $N \trianglelefteq G$. Set $H := \langle L^G \rangle$. As $L < G$ and $L \trianglelefteq \trianglelefteq G$, $H < G$. Further by the Dedekind Modular Law, $H = L(H \cap N)$ with $H \cap N$ a solvable normal subgroup of H, so by minimality of G, $L = H^\infty \trianglelefteq G$. Now $G/L = LN/L \cong N/(N \cap L)$ is solvable, so $(LN)^\infty = G^\infty = L$, proving (3).

Assume the hypotheses of (4). As N is solvable but L is not, $L \not\leq N$, so $\bar{L} \neq 1$. As $L \trianglelefteq \trianglelefteq G$, $\bar{L} \trianglelefteq \trianglelefteq \bar{G}$ and as $L = L^\infty$, $\bar{L} = \bar{L}^\infty$. If $1 \neq \bar{K} = \bar{K}^\infty \leq \bar{L}$ with $\bar{K} \trianglelefteq \trianglelefteq \bar{G}$, then the preimage K of $\bar{K}$ in G lies in LN and $1 \neq K^\infty \trianglelefteq \trianglelefteq G$. By (3), $K^\infty \leq L$, so by minimality of L, $L = K^\infty$. Thus $\bar{K} = \bar{K}^\infty = \bar{L}$, so $\bar{L} \in \mathcal{C}(\bar{G})$.

Let φ be the map in (4). If $\bar{L} = \bar{L}_1$ for some $L_1 \in \mathcal{C}(G)$, then by (3),

$$L = (LN)^\infty = (L_1 N)^\infty = L_1,$$

so φ is injective. To see φ is onto, pick $\bar{K} \in \mathcal{C}(\bar{G})$; as above $K^\infty \trianglelefteq \trianglelefteq G$ and $K^\infty \neq 1$, so there exists some $L \leq K^\infty$ minimal subject to $1 \neq L = L^\infty \trianglelefteq \trianglelefteq G$, and then $L \in \mathcal{C}(G)$. We just saw that $\bar{L} \in \mathcal{C}(\bar{G})$, so (2) applied in $\bar{G}$ shows that $\bar{L} \in \mathcal{C}(\bar{K})$, and hence $\bar{L} = \bar{K}$ by (1). Thus φ is surjective, completing the proof of (4).

Now specialize to the case $N := O_\infty(G)$. As the first part of (5) is a statement about $\bar{G}$, in proving that part we may take $O_\infty(G) = 1$. Thus for $L \in \mathcal{C}(G)$, $O_\infty(L) \leq O_\infty(G) = 1$, so L is a simple subnormal subgroup of G by (1) and hence L is a component of G. Conversely if K is a component of G, then K is a minimal nontrivial subnormal subgroup of G and K is perfect, so $K \in \mathcal{C}(G)$, completing the proof of the first part of (5). By (4) and the first part of (5), the components of $E(\bar{G})$ have preimages $LO_\infty(G)$ for $L \in \mathcal{C}(G)$, so the second part of (5) follows from (3).

In particular the components of $\bar{G}$ commute and so are normal in the group $E(\bar{G})$ they generate. We conclude from (4) and (5) that $LN \trianglelefteq \langle \mathcal{C}(G) \rangle N$. Then (6) follows from (3).

It remains to prove (7), so assume $[\bar{L}, \bar{X}] \neq 1$. Then $[\bar{X}, \bar{L}, \bar{L}] \neq 1$ by 8.9 in [**Asc86a**]. So as $\bar{L}$ is a component of $\bar{G}$, and $[\bar{L}, \bar{X}] \trianglelefteq \langle \bar{L}, \bar{X} \rangle$, $\bar{L}$ is a component of $[\bar{L}, \bar{X}]$ by 31.4 in [**Asc86a**]. Then $\langle \bar{L}^{\bar{X}} \rangle \leq [\bar{L}, \bar{X}]$; while as $\langle \bar{L}^{\bar{X}} \rangle \trianglelefteq \langle \bar{L}, \bar{X} \rangle$, $[\bar{L}, \bar{X}] \leq \langle \bar{L}^{\bar{X}} \rangle$. That is, $[\bar{L}, \bar{X}] = \langle \bar{L}^{\bar{X}} \rangle$. Hence $[L, X]N = \langle L^X \rangle N$. As $\langle L^X \rangle \trianglelefteq \langle L, X \rangle$, $[L, X] \leq \langle L^X \rangle$. Therefore as $\langle L^X \rangle \leq [L, X]N$, $\langle L^X \rangle = [L, X](\langle L, X \rangle \cap N)$ by the Dedekind Modular Law. So as N is solvable and $[L, X] \trianglelefteq \langle L, X \rangle$, $\langle L^X \rangle = \langle L^X \rangle^\infty \leq [L, X]$. This completes the proof of (7). □

With these general facts about $\mathcal{C}$-components in place, we can obtain stronger results in strongly quasithin $\mathcal{K}$-groups G. Passing to $G/O_2(G)$, it makes sense to first study groups with no nontrivial normal 2-subgroup. So for the remainder of this section, we assume

HYPOTHESIS A.3.4. *G is a strongly quasithin finite $\mathcal{K}$-group, with $O_2(G) = 1$.*

Then in chapter 1, we will apply these results to $H/O_2(H)$, to obtain information about 2-local subgroups H of QTKE-groups.

We first use Theorems A (A.2.1), B (A.2.2), and C (A.2.3) of the previous section to see how the SQT hypothesis restricts the possible structure of the $\mathcal{C}$-components of G.

NOTATION A.3.5. For $L \in \{A_6, A_7, M_{22}\}$, write $\hat{L}$ for the quasisimple group with center of order 3, such that $\hat{L}/Z(\hat{L}) \cong L$.

PROPOSITION A.3.6. *Assume $L \in \mathcal{C}(G)$. Then one of the following holds:*

(1) L is a simple component of G appearing in Theorem C.

(2) L is a quasisimple component of G, $Z(L) \cong \mathbf{Z}_3$, and L is $SL_3^\epsilon(q)$ for $q = 2^e \equiv \epsilon \mod 3$ or q an odd prime, $\hat{A}_6$, $\hat{A}_7$, or $\hat{M}_{22}$.

(3) $F^*(L) = F(L) \cong E_{p^2}$ for some prime $p > 3$, and $L/F^*(L) \cong SL_2(p)$ acts naturally on $F^*(L)$.

(4) $F^*(L) = F(L)$ is nilpotent with $L/F^*(L) \cong SL_2(5)$, and for each $p \in \pi(F^*(L))$,

(a) Either $p \equiv \pm 1 \mod 5$, or $p = 5$; and

(b) Either $O_p(L) \cong p^{1+2}$, or $O_p(L)$ is homocyclic of rank 2.

In each of the cases (1)–(4), $F(L) = O(L)$, $L/F(L)$ is quasisimple, and either

(i) $F(L) = F^*(L)$ and $F(L)$ is of index 2 in $O_\infty(L)$, or

(ii) $F(L) = Z(L) = O_\infty(L)$ has order 1 or 3.

Thus $O_\infty(L)' = [F(L), L]$, and $L/O_\infty(L)'$ is quasisimple.

PROOF. If L centralizes $F(L)$, then $F^*(L) > F(L)$, so $L = E(L)$ is quasisimple by A.3.3.1. Now (2) of Theorem A guarantees that $L/Z(L)$ is also strongly quasithin, so $L/Z(L)$ appears on the list of Theorem C. Then by inspection of the list I.1.3 of multipliers of the groups appearing in Theorem C, we conclude that either (1) or (2) holds.

So we may assume that L does not centralize $O_p(L)$ for some prime p. As $O_2(G) = 1$ by Hypothesis A.3.4, p is odd. As $L = L^\infty$, L acts nontrivially on a supercritial subgroup P of $O_p(G)$ by A.1.21. Set $K := C_L(P/\Phi(P))$. As $L = L^\infty$

is nontrivial on P, P is not cyclic; so by A.1.24, $P \cong E_{p^2}$ or p^{1+2}. By A.1.29.1, either $L/K \cong SL_2(p)$, or $p \equiv \pm 1 \mod 5$ and $L/K \cong SL_2(5)$. In particular if $L/K \cong SL_2(p)$ for $p \neq 5$, then L centralizes $O^p(F(L))$. Further in either case, $P/\Phi(P) = [P/\Phi(P), L]$, so $P = [P, L]$. As K is a proper normal subgroup of L, K is solvable by A.3.3.1. As $O_p(L/K) = 1$, $O_p(K) = O_p(L)$.

Let $\pi := \{p_1, \cdots, p_n\}$ be the set of primes $q \in \pi(F(G))$ with L nontrivial on $O_q(L)$, let P_i be a supercritical subgroup of $O_{p_i}(L)$, and set $K_i := C_L(P_i/\Phi(P_i))$. We claim $K_i = K_j$ for all i, j: First if $L/O_\infty(L)$ is $L_2(p)$ for $p > 5$, then we showed that $\pi = \{p\}$, so we may assume that $L/O_\infty(L)$ is $L_2(5)$. Now $|O_\infty(L) : K_i| = 2$ as $L/K_i \cong SL_2(5)$. Thus if $K_i \neq K_j$, then $L/(K_i \cap K_j)$ is a perfect central extension of $\mathbf{Z}_2 \cong K_i/(K_i \cap K_j)$ by $L/K_i \cong SL_2(5)$, impossible as the Schur multiplier of $SL_2(5)$ is trivial. So as claimed, $K := K_i = K_j$ is independent of i. As $|O_\infty(L) : K| = 2$ and $O_2(G) = 1$, $F(L) \leq K$, so $F(L) = F(K)$.

Consider any $p \in \pi$. Recall $P \cong E_{p^2}$ or p^{1+2}, with L irreducible on $P/\Phi(P)$. Further K centralizes $P/\Phi(P)$, so that $K = C_K(P)P$ by A.1.23. Therefore $O^p(K)$ centralizes P, so that $O^p(K) \leq C_K(O_p(K))$ by A.1.21. This also holds for any prime $p \in \pi(F(K)) - \pi$, since L centralizes $O_p(L)$ for such a prime p by definition.

Applying A.1.4 to K, we conclude that $K = F(K) = F(L)$ is a nilpotent $\pi(F(K))$-group. We next establish:

$$\text{If } N \trianglelefteq L \text{ and } K = Z(L)N, \text{ then } N = K. \tag{$*$}$$

Set $L^* := L/N$; we want to show that $K^* = 1$. As $K = Z(L)N$, $K^* = Z(L)^*$; and as $O_2(G) = 1$, $Z(L)$ is of odd order. Now $K_+ := O_\infty(L)$ is the preimage in L of $Z(L/K)$. As $|K_+ : K| = 2$ and $K^* = Z(L)^*$, K_+^* is abelian. But then the Sylow 2-group of K_+^* of order 2 is normal, and hence central, in L^*, so $K_+^* = F(L^*)$ is central in L^*. It follows that L^* is a covering group of $L/K_+ \cong L_2(r)$; so as $L/K \cong SL_2(r)$ for r prime has trivial Schur multiplier, we have established (*).

Now K is nilpotent, and $[O_{\pi'}(F(L)), L] = 1$ by definition of π, so $K = Z(L)O_\pi(F(K))$, and then (*) yields

$$F(L) = K = O_\pi(F(K)). \tag{$**$}$$

Consider the case where $L/K \cong SL_2(p)$ for $p > 5$. Here by earlier remarks $\pi = \{p\}$, so by (**), $K = O_p(L)$. Also as noted earlier,

$$K = C_K(P)P. \tag{$***$}$$

Suppose first that $P \cong p^{1+2}$. As P is supercritical, P contains each element of order p in $C_K(P)$, so by (***), $\Omega_1(K) = P$. Therefore $\Omega_1(C_K(P)) = Z(P)$ is of order p, so $C_K(P)$ is cyclic and hence $C_K(P) = Z(L)$ as L is perfect. Thus $K = Z(L)P$, so $K = P$ by (*). Then $L/P \cong SL_2(p)$, so that $m_p(L) = 3$ by A.1.33, contradicting our hypothesis that G is strongly quasithin.

Therefore $P \cong E_{p^2}$. Let t denote an involution of L. Then t inverts P, so $C_K(t)P = C_K(t) \times P$, and hence $C_K(t) = 1$ as G is strongly quasithin. Therefore t inverts K, so K is abelian—and of rank 2 as G is strongly quasithin. Thus $P = \Omega_1(K)$, so as L is irreducible on P, K is homocyclic of rank 2. If $K > P$, applying A.1.30 to $\Omega_2(K)$, we obtain a contradiction. That is, (3) holds.

This leaves the case where $L/K \cong SL_2(5)$. Here $K = O_\pi(F(K))$ by (**), and as noted earlier, $K = C_K(P_i)P_i$ for each prime p_i in π, so

$$K = C_K(P_0)P_0, \tag{!}$$

where $P_0 := P_1 \times \cdots \times P_n$.

If $P \cong p^{1+2}$, an earlier argument using (!) in place of (***) shows that $O_p(L) = P$. However this time we do not obtain a contradiction, as $SL_2(5)$ does not contain a subgroup X of order p when $p > 5$. When $P \cong E_{p^2}$, another earlier argument shows that $O_p(L)$ is homocyclic abelian of rank 2. Again for $p > 5$ we cannot apply A.1.30 to conclude that $O_p(L)$ is elementary abelian. However, we have established all the assertions of (4), so the proof of A.3.6 is complete. □

Of course distinct components of G commute (e.g., 31.5 in [**Asc86a**]). Using the restrictions on the structure of G in the previous two lemmas, we next obtain the analogue of that fact: Distinct $\mathcal{C}$-components of our group G commute. To see that both parts of Hypothesis A.3.4 are actually needed for this result, consider a group H which is the extension of an elementary abelian p-group V by the direct product of two copies of $SL_3(p)$ acting on V as the tensor product of natural modules, for $p = 2, 3$; the two $\mathcal{C}$-components commute only modulo $O_p(H)$.

PROPOSITION A.3.7. *If L_1 and L_2 are distinct members of $\mathcal{C}(G)$ then $[L_1, L_2] = 1$. So $\langle \mathcal{C}(G) \rangle$ is the central product of the members of $\mathcal{C}(G)$.*

PROOF. By A.3.3.6, L_1 and L_2 are normal in $\langle \mathcal{C}(G) \rangle$, so L_1L_2 is a subnormal subgroup of G. Then as $O_2(G) = 1$, also $O_2(L_1L_2) = 1$. By A.3.3.2, $L_1, L_2 \in \mathcal{C}(L_1L_2)$, so without loss $G = L_1L_2$. Then by A.3.3.6, $L_i \trianglelefteq G$. As distinct components of G commute, we may assume that some L_i, say L_1, is not a component of G. Thus L_1 is described in case (3) or (4) of A.3.6, so $L_1/F_1 \cong SL_2(p)$ for some prime p, where $F_1 := F^*(L_1)$ is nilpotent of odd order.

Let $\bar{G} := G/O(G)$, $q \in \pi(O(L_1))$, $Q := O_q(L_1)$, and $\tilde{Q} := Q/\Phi(Q)$. Then $\tilde{Q} \cong E_{q^2}$, and $SL_2(p) \cong Aut_{L_1}(\tilde{Q}) = N_{GL(\tilde{Q})}(Aut_{L_1}(\tilde{Q}))^\infty$, so $G = C_G(\tilde{Q})L_1$. By A.3.6, $\bar{L}_i$ is a component of $\bar{G}$. Thus as $\bar{G} = \bar{L}_1\overline{C_G(\tilde{Q})}$, and $\overline{C_G(\tilde{Q})} \trianglelefteq \bar{G}$, $\bar{L}_2 \leq \overline{C_G(\tilde{Q})}$. By Coprime Action, L_2 centralizes Q. As this holds for each $q \in \pi(F_1)$, L_2 centralizes $F_1 = F^*(L_1)$, so L_2 centralizes L_1, completing the proof. □

The condition in A.3.7 that distinct $\mathcal{C}$-components of an SQTK-group G commute allows us to conclude that there are at most two conjugates of a $\mathcal{C}$-component of G:

PROPOSITION A.3.8. *Let H be a quasithin $\mathcal{K}$-group and K a component of H, with $O_2(K) = 1$ and K not normal in H. Then*

(1) $|K^H| = 2$ and $\langle K^H \rangle = KK^t$ for $t \in H - N_H(K)$, and

(2) K is $L_2(2^n)$, $Sz(2^n)$, J_1, $L_2(p^e)$ for a prime $p > 3$ and $e \leq 2$, or $SU_3(8)$.

(3) If $L \in \mathcal{C}(G)$ then either $L \trianglelefteq G$, or L is $L_2(2^n)$, $Sz(2^n)$, J_1, or $L_2(p)$ for a prime $p > 3$, and $\langle L^G \rangle$ is the direct product of two copies of L. Thus for $T \in Syl_2(G)$, either

(i) $L \trianglelefteq G$, so $L^T = \{L\} = \mathcal{C}(\langle L, T \rangle)$ and $\langle L, T \rangle = LT$, or

(ii) $L^T = \{L, L^t\} = \mathcal{C}(\langle L, T \rangle)$ for $t \in T - N_T(L)$, and $\langle L, T \rangle = (L \times L^t)T$.

PROOF. Notice we assume only that H is quasithin, rather than that H is strongly quasithin. On the other hand, we are assuming that K is an ordinary component of H, not a more general $\mathcal{C}$-component.

Let $X := \langle K^H \rangle$. As distinct components of H commute, X is the central product of the conjugates of K. By hypothesis, $|X^H| > 1$, so $|X^H| = 2$ by A.1.34.2. Thus (1) holds, and as $O_2(H) = 1$, (2) follows from A.1.34.3 and I.1.3, which shows that the multiplier of $K/Z(K)$ is a 2-group in case (a) of A.1.34.3.

Assume the hypotheses of (3), and let $L \in \mathcal{C}(G)$; thus G is strongly quasithin by Hypothesis A.3.4. By A.3.7, $\langle L^G \rangle$ is the central product of the G-conjugates $L_1, \ldots, L_N$ of L, and we may assume that $N > 1$. Thus $N = 2$ as G is strongly quasithin. Hence $L_1 \cap L_2 \leq Z(L_1)$. Thus if L is not quasisimple, A.3.6 implies $m_p(O_p(L_1)O_p(L_2)) > 2$ for each prime divisor p of $|O(L)|$, contrary to G strongly quasithin. Therefore L is quasisimple, and hence an ordinary component of G. Further $O_2(L) = 1$ as $O_2(G) = 1$ by Hypothesis A.3.4. Now (1) and (2) complete the proof of (3)—noting that $e \neq 2$ in (2), and L is not $SU_3(8)$ as G is strongly quasithin. □

In a general finite group, the induced permutation action of G on $\mathcal{C}(G)$ can be quite complicated; but for a strongly quasithin group, A.3.8.3 shows that the orbits of G on its $\mathcal{C}$-components are of length 1 or 2. This leads to a very strong restriction: $G/\langle \mathcal{C}(G) \rangle$ is solvable.

PROPOSITION A.3.9. *(1)* $\langle \mathcal{C}(G) \rangle = G^\infty$.
(2) $\langle \mathcal{C}(G) \rangle / O(\langle \mathcal{C}(G) \rangle) = E(\langle \mathcal{C}(G) \rangle / O(\langle \mathcal{C}(G) \rangle))$.

PROOF. Let $L \in \mathcal{C}(G)$, and set $H := \langle \mathcal{C}(G) \rangle$ and $\bar{H} := H/O(H)$. As $L \trianglelefteq \trianglelefteq G$, $O(L) \leq O(H)$, so by A.3.6, $\bar{L}$ is quasisimple. Thus $\bar{L}$ is a component of $\bar{H}$, so (2) is established.

By A.3.8.3, $|L^G| \leq 2$, G^∞ fixes L^G pointwise. Therefore

$$H = H^\infty \leq G^\infty \leq G_0 := \bigcap_{L \in \mathcal{C}(G)} N_G(L).$$

We must prove $H = G^\infty$, so without loss we may assume $G = G^\infty$. Thus also $G = G_0$. Now by A.3.6, $L/O(L)$ is quasisimple, so as $G = G_0$ we have $G = N_G(L) = LC_G(L/O(L))$, using the Schreier property (i.e., $Out(L)$ is solvable, which holds for groups in $\mathcal{K}$) for the last equality. Then by induction on $|\mathcal{C}(G)|$, $G = HC_G(\bar{H})$. If $C_G(\bar{H})$ is not solvable, pick $L_1 \leq C_G(\bar{H})$ minimal subject to $1 \neq L_1 = L_1^\infty$ subnormal in G. Then $L_1 \in \mathcal{C}(G) \subseteq H$, so $\bar{L}_1 \leq Z(\bar{H})$ and hence $L_1 \leq O_\infty(H)$. But this contradicts $L_1 = L_1^\infty$. We conclude $C_G(\bar{H})$ is solvable, so

$$G^\infty = H^\infty C_G(\bar{H})^\infty = H$$

as asserted. □

We next put in place some machinery to begin the study of the family $\mathcal{L}(H,T)$ (described in the Introduction to Volume II) of subgroups of a QTKE-group H determined by a Sylow 2-subgroup T of H. (Cf. chapter 1). As a first step, we define an analogue of that family (using that same notational convention) in our SQTK-group G with $O_2(G) = 1$.

DEFINITION A.3.10. So for the remainder of the section let $T \in Syl_2(G)$ and define $\mathcal{L}(G,T)$ to be the set of subgroups B of G such that $B \in \mathcal{C}(\langle B, T \rangle)$.

We use (typically without reference) the following elementary relation between $\mathcal{C}$ and $\mathcal{L}$:

LEMMA A.3.11. *If* $T \leq H \leq G$, *then* $\mathcal{C}(H) \subseteq \mathcal{L}(G,T)$.

PROOF. Given $L \in \mathcal{C}(H)$, we have $\langle L, T \rangle \leq H$, so $L \in \mathcal{C}(\langle L, T \rangle)$ by A.3.3.2. □

The next result says that each member of $\mathcal{L}(G,T)$ embeds in a unique member of $\mathcal{C}(G)$, and determines the possible proper embeddings by examining the overgroups of Sylow 2-subgroups of the groups in Theorem C. This result is used frequently throughout the proof of the Main Theorem.

PROPOSITION A.3.12. *Let $B \in \mathcal{L}(G,T)$. Then $B \leq L$ for a unique $L \in \mathcal{C}(G)$, and one of the following holds:*

(1) $B = L$; that is $B \in \mathcal{C}(G)$.

(2) $B/O_2(B) \cong L_2(2^n)$, L is a rank-2 group of Lie type of characteristic 2 in Theorem C, and $B = P^\infty$ for some maximal parabolic P of L.

(3) $B/O_2(B) \cong Sz(q)$, $L \cong {}^2F_4(q)$, and $B = P^\infty$ for some parabolic P of L.

(4) $B/O_2(B) \cong L_3(2)$, $L \cong L_4(2)$ or $L_5(2)$, and $B = P^\infty$ for some parabolic P of L.

(5) $B/O_2(B) \cong L_4(2)$, $L \cong L_5(2)$, and $B = P^\infty$ for some parabolic P of L.

(6) $B \cong A_5$ or $L_3(2)$, and $L \cong A_7$ or $\hat{A}_7$.

(7) $B \cong A_6$ and $L \cong A_7$, or $B \cong \hat{A}_6$ and $L \cong \hat{A}_7$.

(8) $B \cong L_2(p)$ and $L \cong L_2(p^2)$ for an odd prime $p > 3$; or $B \cong A_5$ and $L \cong L_2(p)$ when $p^2 \equiv 1 \mod 5$ and $p \equiv \pm 3 \mod 8$.

(9) $B \cong SL_2(p)$ and $L \cong L_3^\epsilon(p)$ or $SL_3^\epsilon(p)$ for an odd prime $p > 3$; $B \cong A_6$ and $L \cong U_3(5)$; or $B \cong \hat{A}_6$ and $L \cong SU_3(5)$.

(10) $B \cong SL_2(p)/E_{p^2}$ and $L \cong L_3(p)$ or $SL_3(p)$ for an odd prime $p > 3$.

(11) $B \cong A_6$ and $L \cong M_{11}$.

(12) $B \cong A_5/E_{16}$ or A_6/E_{16} and $L \cong M_{22}$; or $B \cong A_5/E_{16}$ or $\hat{A}_6/E_{16}$ and $L \cong \hat{M}_{22}$.

(13) $B \cong M_{22}$, $L_3(4)$, A_7/E_{16}, $L_3(2)/E_{16}$, $L_3(2)/E_8$, A_6/E_{16}, or A_5/E_{16}; and $L \cong M_{23}$.

(14) $B \cong L_4(2)/E_{16}$, $L_3(2)/E_{64}$, $L_3(2)/2^{1+6}$, or $\hat{A}_6/E_{64}$; and $L \cong M_{24}$.

(15) $B \cong A_5$ and $L \cong J_1$.

(16) $B \cong A_5/2^{1+4}$ and $L \cong J_2$.

(17) $L \cong J_4$ and $B \cong \hat{M}_{22}/2^{1+12}$ or $M_{24}/E_{2^{11}}$, or $B/O_2(B) \cong A_5$, $\hat{A}_6$, $L_4(2)$, or $L_3(2)$.

(18) $B \cong A_5/(\mathbf{Z}_4 * 2^{1+4})$ or $L_3(2)/\mathbf{Z}_4^3$, and $L \cong HS$.

(19) $B \cong L_3(2)/2^{1+6}$ or $\hat{A}_6/E_{64}$, and $L \cong He$.

(20) $B/O_2(B) \cong A_5$ or $L_3(2)$, with $|O_2(B)| = 2^{11}$, and $L \cong Ru$.

(21) $[O(L), L] \neq 1$, $L = BO(L)$, and $L/O(L) \cong SL_2(p)$ for an odd prime p.

(22) $[O(L), L] \neq 1$, $L/O(L) \cong SL_2(p)$ for an odd prime p with $p^2 \equiv 1 \mod 5$ and $p \equiv \pm 3 \mod 8$, and $BO(L)/O(L) \cong SL_2(5)$.

PROOF. Let $M := \langle B, T \rangle$ and $A := \langle B^M \rangle$. By definition of $\mathcal{L}(G,T)$, $B \in \mathcal{C}(M)$, so we conclude from A.3.8.3 that $M = AT$, and A is either B, or BB^t for $t \in T - N_T(B)$. As $B \trianglelefteq\trianglelefteq M$ and $T \in Syl_2(M)$, $T_B := T \cap B \in Syl_2(B)$. Thus $[B, T_B] \not\leq O_\infty(B)$ as $B/O_\infty(B)$ is simple by A.3.3.1. Therefore $B = [B, T_B]$ by A.3.3.7.

Let $H := \langle \mathcal{C}(G) \rangle$ and $\bar{H} := H/O_\infty(H)$. By A.3.3.4, $\bar{H}$ is the direct product of the groups $\bar{L}$, $L \in \mathcal{C}(G)$. Also $B = B^\infty \leq G^\infty = H$ by A.3.9.1. Thus there is some $L \in \mathcal{C}(G)$ such that $\bar{B}$, and hence also $\bar{T}_B$, projects nontrivially on $\bar{L}$. Let $\bar{T}_0$ be the projection of $\bar{T}_B$ on $\bar{L}$. For $K \in \mathcal{C}(G)$, $K \trianglelefteq\trianglelefteq G$, so $T_K := T \cap K \in Syl_2(K)$ and $T \cap H = \prod_{K \in \mathcal{C}(G)} T_K$. Thus for $t \in T_B$, $t = \prod_K t_K$ with $t_K \in T_K$, so $\bar{t}_L$ is the projection of $\bar{t}$ on $\bar{L}$, and hence $\bar{T}_0 \leq \bar{T}_L$; therefore we can choose a preimage

T_0 of $\bar{T}_0$ in T_L. Next $B \leq H \leq N_G(L)$ by A.3.3.6, so $[T_L, B] \leq L$. As $B = [B, T_B]$ and $\bar{T}_0 \leq \bar{T}_L$, $[\bar{B}, \bar{T}_L]$ contains the projection $\bar{B}_L$ of $\bar{B}$ on $\bar{L}$, and hence $[\bar{B}, \bar{T}_L]$ is nonsolvable. Therefore as $B \in \mathcal{C}(M)$ and $T_L \leq T \leq M$, $B \leq [T_L, B]$ by A.3.3.7, so $B \leq [T_L, B] \leq L$. Further by A.3.7, L is unique, establishing the first assertion of the proposition.

During the remainder of the proof, which determines the possible proper embeddings $B < L$, we may as well replace G by L, so that now $T_L = T$; that is $B \in \mathcal{C}(\langle B, T_L \rangle)$ using A.3.3.2, so $B \in \mathcal{L}(L, T_L)$.

Assume first that L is simple. Then L is one of the simple groups in Theorem C. Furthermore B is described in A.3.6, and in particular $B/O_\infty(B)$ also appears in Theorem C.

Now overgroups of Sylow 2-groups in known simple groups are described in results in the literature; indeed in most cases, they can be obtained from suitable tables in our Background References:

For example, for L a group of Lie type and characteristic 2, each such overgroup contains $O^{2'}(P)$ for some parabolic subgroup P, so that $B \in \mathcal{C}(P)$. Further parabolic subgroups are indexed by, and can be described in terms of, subdiagrams of the Dynkin diagram, as in standard references such as [**Car72**].

Suppose L is an alternating group A_n. Then $5 \leq n \leq 8$; and $A_5 \cong L_2(4)$, $A_6 \cong Sp_4(2)'$, and $A_8 \cong L_4(2)$ are groups of Lie type of characteristic 2, so we may assume $n = 7$. Next B is a subgroup of A_7 iff B has a faithful permutation representation of degree 7, in which case $\bar{B}$ has dihedral Sylow 2-groups of order 4 or 8 as Sylow 2-groups of A_7 of dihedral of order 8. Finally it is easy to determine all such representations for candidates for B with such Sylow 2-groups.

When L is sporadic we appeal to [**Asc86b**], which specifically describes the overgroups of T_L.

For the groups $L_2(p^e)$ with p an odd prime, Dickson's Theorem A.1.3 describes all subgroups. The condition $p \equiv \pm 3 \mod 8$ in (8) (and then also in (9), (10), and (22)) comes from the fact that B is T_L-invariant.

Thus of the groups in Theorem C, only $L_3^\epsilon(p)$ with p an odd prime seems not to have a convenient reference. We determine the possible embeddings of our candidates for B in $L_3^\epsilon(p)$ in lemma A.3.21; we postpone that lemma to the end of the section.

If L is quasisimple but not simple, then L is one of the groups described in A.3.6.2, and the embedding of $\bar{B}$ appears on the list in our lemma from the discussion above. We use the description of covering groups of $\bar{L}$ in I.1.3 and I.2.2.1 (observe that $\hat{M}_{22}$ contains $\hat{A}_6$) to lift $\bar{B}$ to B, and hence complete the treatment of the case L quasisimple. We remark that the references for overgroups which we are quoting typically give just the maximal overgroups; but chains of embeddings can be recovered by an inductive argument, and such chains are few and short.

Now assume instead that L is not quasisimple. Then case (3) or (4) of A.3.6 holds, so that $L/O(L) \cong SL_2(p)$ or $SL_2(5)$. As $B = B^\infty$, $B \not\leq O_\infty(L)$. If $BO(L) = L$, then conclusion (21) holds, so we may assume that $BO(L) < L$. Then $\bar{B}$ is a proper $\bar{T}_L$-invariant subgroup of $\bar{L} \cong L_2(p)$, so as p is prime, (8) says that $\bar{B} \cong A_5$, $p \equiv \pm 1 \mod 5$, and $p \equiv \pm 3 \mod 8$. Hence case (22) holds. $\square$

The last two conclusions of A.3.12 summarize several types of proper inclusions $B < L$, which we describe in more detail in the following lemma. In particular we classify the cases where both B and L satisfy (3) or (4) of A.3.6.

LEMMA A.3.13. *Assume that $B \in \mathcal{L}(G,T)$, and $B < L \in \mathcal{C}(G)$ such that L is not quasisimple. Then $L/O(L) \cong SL_2(p)$ for some prime $p > 3$; $BO(L)/O(L) \cong SL_2(r)$; either $r = p$, or $r = 5$ with $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$; and one of the following holds:*

(1) $B \cong SL_2(r)$ is quasisimple.

(2) B is not quasisimple, and either:

(i) $L = BO(L)$, $O(B) < O(L)$, and $r = p = 5$, or

(ii) $L > BO(L)$, $L \cong SL_2(p)/E_{p^2}$ with $p > r = 5$, and $O(B) = O(L)$.

PROOF. As L is not quasisimple, the embedding of B in L is described in case (21) or (22) of A.3.12. Recall that $B/O_2(B)$ appears in A.3.6; In particular the initial conclusions of the lemma hold and L appears in case (3) or (4) of A.3.6. We may assume B is not quasisimple, so B also appears in case (3) or (4) of A.3.6.

Suppose $L = BO(L)$. Then $B/O(B) \cong L/O(L) \cong SL_2(p)$, so that $r = p$, and as $B < L$, $O(B) < O(L)$ so $O(L)$ is not E_{p^2}. Therefore L satisfies case (4) rather than (3) of A.3.6, so that $p = 5$ and (2i) holds.

Therefore we may assume case (22) of A.3.12 holds. Then $r = 5 < p$, so L satisfies case (3) rather than (4) of A.3.6. Thus $E_{p^2} \cong O(L)$, so $O(L) = O(B)$, and hence (2ii) holds. □

Embeddings of B in L with $B/O_\infty(B) \cong A_5$ arise frequently in our work, and many such embeddings appear in A.3.12. Thus in the next lemma, we provide the sublist of those embeddings for easy reference.

The definition and properties of *blocks* are given in section C.1 on pushing up; in particular for the notion of $L_2(4)$-block, see definition C.1.12.

LEMMA A.3.14. *Let $B \in \mathcal{L}(G,T)$ with $B/O_\infty(B) \cong A_5$, and suppose $B < L$ for some $L \in \mathcal{C}(G)$. Then one of the following holds:*

(1) $F^*(B) = O_2(B) = O_\infty(B)$, $L \cong (S)L_3(4)$, $Sp_4(4)$, $G_2(4)$, or ${}^3D_4(4)$, and $B = P^\infty$ for some maximal parabolic P of L.

(2) $B \cong A_5$, $L \cong A_7$ or $\hat{A}_7$, and $N_L(B)$ is the global stabilizer of 5 points in the representation of L on 7 points.

(3) $B \cong A_5$; and either $L \cong L_2(25)$, or $L \cong L_2(p)$ for $p^2 \equiv 1 \mod 5$ and $p \equiv \pm 3 \mod 8$.

(4) $B \cong SL_2(5)$, $L \cong (S)L_3^\epsilon(5)$, and $B = C_L(O_2(B))^\infty$.

(5) $B \cong SL_2(5)/E_{25}$, $L \cong L_3(5)$, and $B = P^\infty$ for some maximal parabolic P of L.

(6) $B \cong A_5/E_{16}$ is an $L_2(4)$-block, and $L \cong M_{22}$ or $\hat{M}_{22}$.

(7) $B \cong A_5/E_{16}$ is an $L_2(4)$-block and $L \cong M_{23}$.

(8) $B \cong A_5$, $L \cong J_1$, and $B = C_L(z)^\infty$ for some involution $z \in L$.

(9) $B \cong A_5/Q_8D_8$, $L \cong J_2$, and $B = C_G(z)$ for z a 2-central involution in L.

(10) $B/O_2(B) \cong A_5$ with $F^*(B) = O_2(B)$ and $L \cong J_4$.

(11) $B \cong A_5/(\mathbf{Z}_4 * Q_8D_8)$, $L \cong HS$, and $B = C_L(z)^\infty$ for some 2-central involution $z \in L$.

(12) $B/O_2(B) \cong A_5$, $|O_2(B)| = 2^{11}$, $L \cong Ru$, and $B = C_L(z)^\infty$ for some 2-central involution $z \in L$.

(13) $B/O(B) \cong SL_2(5)$ *and* $L = BO(L)$.

(14) $B/O(B) \cong SL_2(5)$ *and* $L/O(L) \cong SL_2(p)$, $p^2 \equiv 1 \mod 5$ *and* $p \equiv \pm 3 \mod 8$.

PROOF. This follows by extracting from A.3.12 the cases with $B/O_\infty(B) \cong A_5$. □

The next result allows us to control the embedding of certain T-invariant solvable subgroups of G. Control of such embeddings is useful for studying the family $\Xi(H,T)$ of subgroups of a QTKE-group H determined by a Sylow 2-group T, which is discussed in the Introduction to Volume II and has properties described in 1.3.4. More generally, we use the result in various situations where a group of odd order permutes with a Sylow 2-subgroup of an almost-simple SQT-group. We remark that 3-groups are explicitly excluded in the lemma, because of their frequent occurrence inside parabolic subgroups (but outside the Cartan subgroup) of groups of Lie type defined over $\mathbf{F}_2$, and in sporadic groups.

LEMMA A.3.15. *Let* $F^*(G) = L$ *be simple, and* B *a subgroup of* G *of odd order such that* $BT = TB$ *and* $O^3(B) \neq 1$. *In addition assume either*

(a) B *is abelian, or*

(b) B *is an* r*-group for some prime* r *and* $B \leq L$.

Then one of the following holds (where p denotes a suitable odd prime):

(1) $L \cong L_2(p^e)$, *for* p *an odd prime and* $e \leq 2$, $B \leq L$, B *is cyclic,* $|B|$ *divides* $p^e - \epsilon$, *where* $\epsilon = \pm 1 \equiv p^e \mod 4$, *and* B *is inverted in* $T \cap L$.

(2) $L \cong L_3(p)$, $B \leq L$, $B \cong E_{p^2}$, *and* T *acts irreducibly on* B.

(3) $L \cong L_3^\delta(p)$, $B \leq L$, $B = (B \cap K)C_B(K)$, *where* $SL_2(p) \cong K \leq L$, $B \cap K$ *and* $C_B(K)$ *are cyclic,* $C_B(K)$ *is a Cartan subgroup of* L *with order dividing* $p - \delta$ *and* $|B \cap K|$ *dividing* $p - \epsilon$, *where* $\epsilon = \pm 1 \equiv p \mod 4$, *and* $B \cap K$ *is inverted in* $T \cap K$.

(4) $L \cong J_1$ *and* $B \leq N_L(T)$ *is of order* 7.

(5) L *is of Lie type over* $\mathbf{F}_{2^n}$, $n > 1$, *and* $B \leq FH \leq Aut(L)$, *where* F *is cyclic of order* f *dividing* n *and induces field automorphisms on* L, H *is an* F*-invariant Cartan subgroup of* $N_I(T \cap L)$, *and* I *is the subgroup of* $FBL = HBL$ *inducing inner-diagonal automorphisms on* L.

(6) $L = {}^3D_4(2)$ *and either* B *is a Cartan subgroup* D *of* L *of order* 7, *or* $B = D \times A$ *where* A *is of order* 3.

(7) L *is the Tits group* ${}^2F_4(2)'$ *and* B *is of order* 5.

PROOF. As G is strongly quasithin, the possibilities for L are listed in Theorem C.

First assume L is of Lie type over $\mathbf{F}_{2^n}$ in Theorem C. Since G is strongly quasithin by Hypothesis A.3.4, Corollary A.2.6 says that BL is contained in the subgroup I_0 of $Aut(L)$ generated by inner-diagonal and field automorphisms, in the convention of Notation A.2.5 for these terms. Let FL/L and HL/L be the projections of BL/L on the field and diagonal automorphisms of $Out(L)$, respectively: we may choose F to induce a group of field automorphisms on L of order f dividing n, and H to be a Cartan subgroup of $N_I(T_L)$, where $T_L := T \cap L$, and I is the subgroup of FHL inducing inner-diagonal automorphisms on L. By construction, $FBL = HBL$. Let $D := B \cap I$. Then $DT_L = T_L D$, so as the only subgroups of DL containing T_L are of the form $O^{2'}(P)H_0$ for some parabolic P of L and $H_0 \leq H$, either we may choose H so that $D \leq H$, or P is a solvable parabolic of G distinct from a Borel subgroup so that G is defined over $\mathbf{F}_2$. In the former case either (5) holds, or the first possibility of (6) holds with $B = D$ of order 7. By hypothesis $O^3(B) \neq 1$, so in the latter case, either (7) holds, or the Cartan subgroup of L is not a 3-group, in which case L is ${}^3D_4(2)$ and B is of order 7 or 21, so one of the possibilities in (6) holds.

So we may assume L is not of Lie type and characteristic 2. By 6.1 in [**Asc80**], L is not an alternating group in Theorem C. If L is a sporadic group in Theorem C, we appeal to the list of maximal overgroups of T in [**Asc86b**], and use induction on the order of G (that is, appeal to smaller cases treated earlier in this result), to see which such subgroups can have a subgroup like B permuting with T. The only case with $O^3(B) \neq 1$ is J_1 as in (4).

If $L \cong L_3^\delta(p)$, p an odd prime, then in case (a) we appeal to 6.2.4 in [**Asc80**] to conclude that (2) or (3) holds; the condition that $|B \cap K|$ divides $p - \epsilon$ selects the split or nonsplit torus of $SL_2(p)$ permuting with T. In case (b), we may assume B is not abelian; further $r > 3$ as $O^3(B) \neq 1$. But then $BT \neq TB$, contrary to hypothesis. Finally if $L \cong L_2(p^e)$, p an odd prime, $e = 1$ or 2, then (1) holds from the list of subgroups of L in Dickson's Theorem A.1.3.

This completes the proof of A.3.15. □

NOTATION A.3.16. Consider $L := L_3(2^{6n})$, $A := Aut(L)$, identify L with $Inn(L)$, and set $A^* := A/L = Out(L)$. From 2.5.12 in [**GLS98**], a Sylow 3-group of A^* is of the form $\langle f_0^* \rangle \times \langle d^* \rangle$, where d^* is of order 3 and induces a diagonal automorphism on L, and f_0 is a field automorphism of L of order $n_3 = |f_0^*|$. Let f be one of the two elements of order 3 in $\langle f_0 \rangle$, and write $L_3^\circ(2^{6n})$ or $L_3^{+,\circ}(2^{6n})$ for the preimage A° of $\langle f^* d^* \rangle$ in A. As d^* is inverted by a field automorphism in A centralizing f^*, the isomorphism type of the extension is independent of the choice of f. If there were a of order 3 in fdL, then by 4.9.1.d in [**GLS98**], a and f would be conjugate in A, impossible as $f^* \in Z(A^*)$. Thus the extension is nonsplit. Finally let $\sigma \in A$ be a graph-field automorphism and write $U_3^\circ(2^{3n})$ or $L_3^{-,\circ}(2^{3n})$ for the extension $C_{A^\circ}(\sigma)$ of $C_L(\sigma) \cong U_3(2^{3n})$. As $L_3^\circ(2^{6n})$ is nonsplit, so is $U_3^\circ(2^{3n})$.

We often (but not always) use the notation $\theta(H)$ for the following group:

DEFINITION A.3.17. Let $\theta(H)$ denote the subgroup generated by all elements of order 3 in H.

We use the next two results frequently; they show that in most cases, a $\mathcal{C}$-component of p-rank 2 for an odd prime p of a 2-local H contains all p-elements of H.

LEMMA A.3.18. *Let H be an SQTK-group and $L \in \mathcal{C}(H)$ with $L/O_2(L)$ quasisimple. Assume p is an odd prime with $m_p(L) > 1$, and set $H^* := H/O_{p'}(H)$. Then one of the following holds:*

(1) $L = O^{p'}(H)$.

(2) $p = 3$, $L^* \cong L_3^\epsilon(q)$ with $q \equiv \epsilon \mod 3$, and either $O^{3'}(H^*) \cong PGL_3^\epsilon(q)$ or $q = 2^{3m}$ and $O^{3'}(H^*) \cong L_3^{\epsilon,\circ}(q)$.

(3) $p = 3$, $L^* \cong SL_3^\epsilon(q)$, $q \equiv \epsilon \mod 3$, $\hat{A}_6$, $\hat{A}_7$, or $\hat{M}_{22}$. Further either

(a) L is the subgroup $\theta(H)$ generated by all elements of H of order 3, or

(b) $L^* \cong SL_3^\epsilon(q)$, $O^{3'}(H^*)$ is the split extension of L^* by x^* of order 3 inducing a diagonal outer automorphism on L^*, $C_{L^*}(x^*)$ is cyclic of order $q^2+\epsilon q+1$,

and each subgroup of H^* of order 3 not contained in L^* is conjugate to $\langle x^* \rangle$ under L^*.

PROOF. As the lemma is a statement about H^*, to simplify notation during the proof we assume $O_{p'}(H) = 1$. Thus L is quasisimple. As $m_p(L) > 1$, $L \trianglelefteq H$ by A.3.8.3. We first observe:

(*) L contains each subgroup of $C_H(L)$ of order p.

For as $m_p(L) > 1$, there is $E_{p^2} \cong E \leq L$, so if X is of order p in $C_H(L)$, then X centralizes E, so $X \leq E \leq L$ as $m_p(H) = 2$ by hypothesis.

Assume first that $Z(L) = 1$. Then by (*), $C_H(L)$ is a p'-group, so $C_H(L) \leq O_{p'}(H) = 1$; thus $L = F^*(H)$ is simple and hence $H \leq Aut(L)$. Thus if (1) fails, some nontrivial p-element $x \in H$ induces an outer automorphism on L. In particular p is an odd prime divisor of $|Out(L)|$, so from the structure of $Aut(L)$ for the groups L in Theorem C with $m_p(L) > 1$, either L is of Lie type and even characteristic, or $L \cong L_3^\epsilon(q)$ for some odd prime q and so (2) holds. Of course we may assume the former, so by Corollary A.2.6, $x = fd$, where f induces a field automorphism on L and d induces an inner-diagonal automorphism. When $f \neq 1$, we must show $O^{3'}(H) \cong L_3^{\epsilon,\circ}(2^{3m})$, so replacing x by a suitable power we may take f of order p; and if d is inner, replacing x by f, we may take $x = f$. Next by 2.5.12 in [**GLS98**], $Outdiag(L) = 1$ (the subgroup of diagonal outer automorphisms) unless $L \cong L_3^\epsilon(q)$ with $q \equiv \epsilon \mod 3$, in which case $|Outdiag(L)| = 3$. Recalling the discussion in Notation A.3.16, we conclude that either L is of some Lie type $X(q^p)$ for q a power of 2 with x inducing a field automorphism of order p on L; or $O^{3'}(H) \cong PGL_3^\epsilon(q)$ or $L_3^{\epsilon,\circ}(q)$ with $q \equiv \epsilon \mod 3$. Again we may assume that (2) fails, so the former case holds. Then by 4.9.1.a in [**GLS98**], $C_L(x) \cong X(q)$ with $m_p(X(q)) = m_p(X(q^p)) > 1$, contradicting $m_p(H) = 2$.

Therefore we may assume that $Z := Z(L) \neq 1$. Thus as $O_{p'}(H) = 1$, Z is a nontrivial p-group, so A.3.6 says that $p = 3$ and L is one of the groups listed in (3). In this case we will show that (3) holds, so we may assume that (3a) fails with $x \in H - L$ of order 3, and it remains to show that (3b) holds.

Suppose first that x induces an inner automorphism on L. Then $x = yc$ for some 3-elements $y \in L$ and $c \in C_H(L)$. As $x \notin L$, $c \notin Z$, so $Z = \Omega_1(\langle c \rangle)$ by (*). Thus $|c| > 3$ and $y \in L - Z$. Now $1 = x^3 = y^3c^3$, so $c^3 = y^{-3} \in L \cap C_H(L) = Z$ and therefore $Z = \langle c^3 \rangle$ as $|Z| = 3$. Hence $|y| = |c| = 9$, so that a Sylow 3-group P of L is not of exponent 3. When $|P| = 27$, $P \cong 3^{1+2}$ is of exponent 3, so $L \cong SL_3^\epsilon(q)$ with $|P| > 27$. Now $|L| = (q^3 - \epsilon)q^3(q^2 - 1)$ with $q = 3r + \epsilon$, so $|L|_3 = (q^2 + \epsilon q + 1)_3(q - \epsilon)_3^2$. Also

$$q^2 + \epsilon q + 1 = 9r^2 + 9r\epsilon + 3 \equiv 3 \mod 9$$

so $|L|_3 = 3(q - \epsilon)_3^2$. As $|P| > 27$, $q \equiv \epsilon \mod 9$. Thus if V is the natural module for L over the field $F := \mathbf{F}_q$ or $\mathbf{F}_{q^2}$ for $\epsilon = 1$ or -1, respectively, then F contains elements of order 9, so y is diagonalizable on V. Indeed if $\epsilon = -1$, then as $q \equiv \epsilon \mod 9$, the eigenspaces for y on V are nondegenerate. So in either case, y is contained in a maximal torus T of L of order $(q - \epsilon)^2$. In particular $T \leq C_L(y) = C_L(x)$ and $m_3(T) = 2$, contradicting $m_3(H) = 2$.

Therefore x induces an outer automorphism on L. From the structure of $Aut(L)$, this again reduces us to the case where $L \cong SL_3^\epsilon(q)$. Further x induces a field or diagonal automorphism on L since the extension $L_3^{\epsilon,\circ}(2^{3m})$ is nonsplit from the discussion in Notation A.3.16. We eliminate the case where x induces a field automorphism just as we did when $Z(L) = 1$. Thus x induces a diagonal automorphism, and $\overline{L\langle x\rangle} := L\langle x\rangle/Z \cong PGL_3^\epsilon(q)$.

Define F and the natural module V for L over F as above, and set $G := GL^\epsilon(V)$, $Z_G := Z(G)$, and $\hat{G} := G/Z_G$. Thus $L = SL^\epsilon(V)$, $\hat{G} = \hat{L}\langle\hat{x}\rangle \cong \bar{L}\langle\bar{x}\rangle$, and we can work in $\hat{G}$ rather than in $\bar{L}\langle\bar{x}\rangle$, which will be more convenient.

Let $\gamma \in G$ be a Singer cycle, of order $q^3 - \epsilon$ and $\alpha \in \langle\gamma\rangle$ of order $3(q-\epsilon)_3$; then $\alpha \in G - Z_G L$ and $\alpha^3 \in Z_G$, so $\hat{\alpha}$ is of order 3 and induces a diagonal outer automorphism on L. Let β and δ denote elements of order 3 in G such that β has a single nontrivial eigenvalue ζ on V and δ has eigenvalues $1, \zeta, \zeta^{-1}$. Observe that $\delta \in L$; while $\hat{\beta} \in \hat{L}$ iff $\beta \in Z_G L$ iff $q \equiv \epsilon \mod 9$. Thus β induces a diagonal outer automorphism on L iff $q \not\equiv \epsilon \mod 9$.

We claim there are one or two orbits of $\hat{L}$ with representatives $\hat{\alpha}$ or $\hat{\alpha}, \hat{\beta}$, for $q \equiv \epsilon \mod 9$ or $q \not\equiv \epsilon \mod 9$, respectively. By the previous paragraph, $\hat{\alpha}^{\hat{L}}$ is an orbit of $\hat{L}$ on the set $\mathcal{X}$ of elements of order 3 in the coset $\hat{L}\hat{x}$, and $\hat{\beta}^{\hat{L}} \subseteq \mathcal{X}$ iff $q \not\equiv \epsilon \mod 9$.

Let T_G be a maximal torus of G of order $(q-\epsilon)^3$, and $X := \Omega_1(O_3(T_G))$. The Weyl group of T_G has two orbits on subgroups of order 3 in $\hat{X}$ with representatives $\langle\hat{\beta}\rangle$ and $\langle\hat{\delta}\rangle$, so by the previous paragraph, the set $\mathcal{X}_2$ of members of $\mathcal{X}$ fused into $\hat{T}_G$ is nonempty iff $q \not\equiv \epsilon \mod 9$, in which case $\mathcal{X}_2 = \hat{\beta}^{\hat{L}}$.

Observe that $1 \neq \hat{g} \in \hat{G}$ is of order 3 iff $g^3 \in Z_G$, and we may take g to be a 3-element. Let $\mathcal{Y}_i$ for $i = 1, 2$ consist of those $\hat{g}$ such that $|g| = 3(q-\epsilon)_3$, $|g| \leq (q-\epsilon)_3$, respectively. Observe that $\mathcal{Y}_1 \cup \mathcal{Y}_2$ is an L-invariant partition of the set of elements in $\hat{G}$ of order 3, $\hat{\alpha} \in \mathcal{Y}_1$, and $\hat{\beta} \in \mathcal{Y}_2$. Thus by the previous paragraph, to complete the proof of the claim it suffices to show that $\mathcal{Y}_2$ consists of those elements of order 3 fused into $\hat{T}_G$, and $\mathcal{Y}_1$ is the set of elements conjugate to either $\hat{\alpha}$ or $\hat{\alpha}^{-1}$.

Let $\hat{g} \in \mathcal{Y}_2$. Then $g^3 = z^3$ for some 3-element $z \in Z_G$, so that gz^{-1} has order 3 in $GL_3^\epsilon(q) = G$, and hence lies in X. On the other hand let $\hat{g} \in \mathcal{Y}_1$. Then $\langle g\rangle$ is irreducible on V, with $E := End_{F\langle g\rangle}(V)$ a cubic extension of F, so $C_L(g) = E \cap L$ is cyclic of order $q^2 + \epsilon q + 1$. Then as L is transitive on E-structures on V, $\hat{g}$ is conjugate to $\hat{\alpha}$, completing the proof of the claim.

We observe next that we may take $\hat{x} = \hat{\alpha}$: For by the claim, the only other possibility is that $q \not\equiv \epsilon \mod 9$ and $\hat{x} = \hat{\beta}$. Then $C_{\hat{L}}(\hat{x})^\infty = C_{\hat{L}}(\hat{\beta})^\infty \cong SL_2(q)$, so $C_L(x)^\infty \cong SL_2(q)$ contains an element of order 3 not in Z. But now $m_3(C_L(x)) > 1$, contradicting $m_3(H) = 2$.

Next we show that $C_H(L) = Z$. Let Q be an x-invariant Sylow 3-subgroup of $C_H(L)$. It suffices to show that $Q = Z$, for then $C_H(L) = Z \times O_{3'}(C_H(L)) = Z$ since $O_{3'}(H) = 1$. So assume that $Q > Z$. By (*), Q is cyclic. Thus if $q \equiv \epsilon \mod 9$, then there is y of order 9 in L with $y^3 \in Z$, so $\langle y, Q\rangle$ contains an element of order 3 not in L inducing an inner automorphism on L, contrary to an earlier reduction. Therefore $q \not\equiv \epsilon \mod 9$, so there is $u \in xL$ with $\hat{u} = \hat{\beta}$, and hence by an argument in the previous paragraph, Z is the unique subgroup of order 3 in $\langle u, Q\rangle$. But this is impossible, as $\langle u, Q\rangle$ is not cyclic since $u^3 \in L$.

To complete the proof that (3b) holds, it remains to show that $L\langle x\rangle = O^{3'}(H)$. By the previous paragraph $L = O^{3'}(LC_H(L))$, so if $L\langle x\rangle < O^{3'}(H)$ then some 3-element $h \in H$ induces a field automorphism of order 3 on L. Let $J := N_{L\langle x\rangle}(\langle h\rangle Z)$. Then $\bar{J} \cong PGL_3^\epsilon(q^{1/3})$. As $O^3(J)$ centralizes h and is of 3-rank 2, h has order 9 and $Z = \langle h^3\rangle$. But applying the arguments above to J in the role of $L\langle x\rangle$, there is j of order 9 in J with $Z = \langle j^3 \rangle$; so as $\langle h, j\rangle$ is noncyclic, there is k of order 3 in $\langle h, j\rangle - L$ inducing a field automorphism on L, again a case already eliminated.

Thus (3b) holds, and the proof of the lemma is complete. □

However before we end this analysis, observe that:

LEMMA A.3.19. *Assume H is an SQTK-group, $L \in \mathcal{C}(H)$ with $L/O_2(L) \cong SL_3(2^n)$, n even, and V is a normal elementary abelian 2-subgroup of H such that V is the sum of $s \leq 2$ isomorphic natural modules for $L/O_2(L)$. Then $L = O^{3'}(H)$.*

PROOF. Assume the lemma fails, let $H^* := H/O_{3'}(H)$, $\bar{H} := H/O_2(H)$, $\bar{Z} := Z(\bar{L})$, as in Definition A.3.17 write $\theta(Y)$ for the subgroup of Y generated by all elements of Y of order 3. As $\bar{Z} \neq 1$, conclusion (3) of A.3.18 holds. Let $\mathcal{J}$ be the set of subgroups K of H containing L such that $|K : L| = 3$. As $L < O^{3'}(H)$, $\mathcal{J}$ is nonempty. Pick $K \in \mathcal{J}$. As $m_3(\bar{H}) = 2$, $\bar{Z} = \theta(C_{\bar{K}}(\bar{L})$, so as $V \trianglelefteq H$ and $\bar{L}$ is faithful on V, $\bar{K}$ is faithful on V. Indeed L has 1 or $2^n + 1$ irreducibles on V for $s := 1$ or 2, respectively, and as n is even, $(3, 2^n + 1) = 1$, so $\bar{K}$ acts faithfully on some irreducible I for L in V.

Continue the notation established during the proof of A.3.18. Using that discussion, we will establish the claims that:

(i) $Z(L^*) = C_{H^*}(L^*)$.
(ii) $\bar{K}/\bar{Z} \cong PGL_3(2^n)$.
(iii) $\bar{K}$ splits over $\bar{L}$, $2^n \not\equiv 1 \mod 9$, and $\beta \in \bar{K}$.

First assume that $\bar{K}$ induces inner automorphisms on $\bar{L}$. Then $Z < Z(\bar{K})$, so as $Z(\bar{K})$ is faithful on I, $2^n \equiv 1 \mod 9$. Then an argument in the proof of A.3.18 shows that $m_3(\bar{K}) > 2$, a contradiction. This establishes claim (i), and shows that $F^*(\bar{K}) = \bar{L}$.

If $x \in K - L$ induces a field automorphism of $\bar{L}$, then x has order 3, whereas there is no such element in the two cases in conclusion (3) of A.3.18. Thus $\bar{K}/\bar{Z} \cong PGL_3(2^n)$ or $L_3^\circ(2^n)$.

Assume the latter case holds. Then $n = 6m$ for suitable m. Let $q := 4^m$, $\xi \in F$ of order $(q^3 - 1)_3$, and Y the stabilizer in $G := GL(I)$ of the three points Fv_i, $1 \leq i \leq 3$ determined by a basis $\{v_1, v_2, v_3\}$ of I. Let $h, k, z \in Y$ have eigenvalues $(\xi, \xi, 1)$, $(\xi, \xi^{-1}, 1)$, (ξ, ξ, ξ) with eigenvectors v_1, v_2, v_3, respectively. Let f be a field automorphism of order 3 fixing the eigenvectors; then for $j \in Y$, $j^f = j^q = jj_0$, where $j_0^3 = 1$ and f centralizes j_0. Further

$$\bar{L}Z_G \cap Y = \langle h^3, k, z\rangle,$$

so we may take $\bar{x} \in fh\langle h^3, k, z\rangle$. Thus $\bar{x} = fi$, $i := h^{1+3a}k^bz^c$, and $i^f = ii_0$, so

$$\bar{x}^3 = f^3 i^{f^2} i^f i = i^3 i_0^3 = i^3.$$

Therefore as $\bar{x}^3 \in \bar{L}$, $i^3 \in \bar{L} \cap Y = \langle k, h^3 z^{-2}\rangle$, which is visibly not the case. This establishes claim (ii).

From claim (ii) and the proof of A.3.18 there is a 3-element $\bar{k} \in \bar{K}$ whose image in $\bar{K}/\bar{Z}$ lies in $\mathcal{Y}_1$; that is, $|\bar{k}| = 3(2^n - 1)_3$ and $\bar{k}^3 \in Z(GL(I))$. As $|Z(GL(V))| = 2^n - 1$, $\langle \bar{k}^3\rangle = O_3(Z(GL(I)))$, so as $\bar{Z} = O_3(Z(GL(I))) \cap \bar{K}$, $2^n \not\equiv 1 \mod 9$. Thus from the proof of A.3.18, $\beta \in \bar{K} - \bar{L}$ has order 3, and so $\bar{K}$ splits over $\bar{L}$. Thus (iii) is established.

By (ii), $\bar{K}/\bar{Z} \cong PGL_3(2^n)$, and by (iii), $\bar{K} > \bar{L}$ is split, so conclusion (3b) of A.3.18 holds. This is a contradiction as $\beta \in \bar{K}$ by (iii), which is explicitly excluded in (3b). This contradiction completes the proof. □

LEMMA A.3.20. *Let H be an SQTK-group and $L \in \mathcal{C}(H)$ with $L \trianglelefteq H$ and $m_3(L) \geq 1$. Then*

(1) $m_3(C_H(L/O_\infty(L))) \leq 1$.

(2) If $m_3(L) = 2$, then $C_H(L/O_\infty(L))$ has no S_3-section.

PROOF. Set $C := C_H(L/O_\infty(L))$ and $\bar{H} := H/O_2(H)$. As $L \in \mathcal{C}(H)$, $\bar{L}$ is described in A.3.6, and in all cases where $m_3(L) \geq 1$, there is a subgroup X of order 3 in $L - O_\infty(L)$. In particular $\bar{X} \not\leq \bar{L} \cap \bar{C} = Z(\bar{L})$, so since H is an SQTK-group, we conclude $m_3(C) \leq 1$, establishing (1). Now assume that $m_3(L) = 2$, and (2) fails. Then a 2-element $\bar{t} \in \bar{C}$ inverts a 3-element $\bar{y}$ of $\bar{C}$, and hence inverts $\bar{Y} := \Omega_1(\langle \bar{y} \rangle)$. Thus $\bar{Y} \not\leq \bar{L}$, so as $m_3(L) = 2$, $m_3(YL) > 2$, contradicting H an SQTK-group. This contradiction completes the proof. □

Recall that to complete the proof of A.3.12, we needed certain information about the subgroup structure of $L_3^\epsilon(p)$ for p an odd prime, which we promised to provide at the end of this section. The final result of the section supplies that information:

LEMMA A.3.21. *Assume L is quasisimple with $L/Z(L) \cong L_3^\epsilon(p)$ for p an odd prime, $T \in Syl_2(L)$, and $B \in \mathcal{L}(L,T)$ with $B < L$. Then B is described in case (9) or (10) of A.3.12.*

PROOF. The result can be retrieved from Mitchell [**Mit11**], but to keep our treatment self-contained, we supply our own proof, omitting some of the elmentary details.

Set $L^* := L/Z(L)$. By I.1.3, either L is simple, or $Z(L)$ is of order 3 and $L \cong SL_3^\epsilon(p)$. Let $F := \mathbf{F}_p$ or $\mathbf{F}_{p^2}$, for $\epsilon = +1$ or -1, respectively. We assume $L = SL^\epsilon(V)$ for some 3-dimensional F-space; the result for the projective group follows immediately from this case. As $B \in \mathcal{L}(L, T_L)$, T acts on B, so $T_B := T \cap B \in Syl_2(B)$, and $B/O_2(B)$ is described in A.3.6, so in particular $B/O_\infty(B)$ appears in Theorem C.

We appeal to the following properties of $SL^\epsilon(V)$:

(a) L has one conjugacy class z^L of involutions, and $L_z := O^{p'}(C_L(z)) \cong SL_2(p)$, with $C_L(z)^\infty \leq L_z$.

(b) T is semidihedral or wreathed, so $m_2(T) = 2$.

(c) If $q \neq p$ is an odd prime and Q is a q-subgroup of L, then either Q is abelian with $Aut_L(Q) \leq S_3$; or $q = 3$, $p \equiv \epsilon \mod 3$, and $Aut_L(Q)$ is solvable.

Let $R := O_2(BT)$ and let z be an involution in $Z(T)$. By (b), $Z(T)$ is cyclic, so as T acts on B, $z \in B$, and if $R \neq 1$, then $z \in E := \Omega_1(Z(R))$ and $m_2(E) \leq 2$. Suppose $R \neq 1$. As $m_2(E) \leq 2$, B centralizes E. Thus $B \leq C_L(z)^\infty$, so $B \leq L_z$ by (a), and $B \in \mathcal{L}(L_zT, T)$. Then applying A.3.12 to the embedding of B in L_z, either conclusion (9) of A.3.12 holds as required, or $p > 5$ and $B \cong SL_2(5)$. The latter is impossible as $TL_z/O_2(TL_z) \cong PGL_2(p)$, so T acts on no A_5-subgroup of $L_z/\langle z \rangle$.

Thus we may assume $R = 1$, so T is faithful on B and B is described in A.3.6. Assume B is not quasisimple; then by A.3.6, $B/O(B) \cong SL_2(r)$ for some odd prime r. Then as $z \in B$, $B = O(B)B_z$, where $B_z := C_B(z)^\infty$. By the previous paragraph, $B_z = L_z$. By A.3.6, $O(B)$ is nilpotent and B_z is faithful on $O_q(B)$ for each $q \in \pi(O(B))$. Therefore $O(B)$ is a p-group by (c). Then $O(B) \cong E_{p^2}$ by A.3.6, so conclusion (10) of A.3.12 holds in this case, as required.

Therefore we may assume B is quasisimple. As T_B is a subgroup of a semidihedral or wreathed 2-group, and as T_B is Sylow in the group $B/Z(B)$ appearing in Theorem C, we conclude T_B is dihedral, semidihedral, or wreathed, and $B/Z(B)$ is A_7, M_{11}, $L_2(q^e)$, or $L_3^\mu(q)$ for some odd prime q and $e \leq 2$. As T is faithful on B and semidihedral or wreathed, $B/Z(B)$ is not A_7 or $L_2(q)$. As M_{11} has Frobenius subgroups of order 55 and 20, $B/Z(B)$ is not M_{11} by (c).

Suppose $B/Z(B)$ is $L_3^\mu(q)$. Applying (c) to the prime q, either $q = p$, or $q = 3$ with $p \equiv \epsilon \mod 3$. In the first case $B = L$, contrary to the hypotheses of the lemma. In the second there is $3^{1+2} \cong Q \leq B$, and as Q is faithful on V of dimension 3, a generator of $Z(Q)$ induces scalar action on V. This is impossible as $Z(Q) \not\leq Z(B)$.

Therefore $B/Z(B) \cong L_2(q^2)$. Applying (c) to the prime q, either $q = p$, or $q = 3$ and $p \equiv \epsilon \mod 3$. In the first case, $(q^4 - 1)/2$ divides $|B|$ but not $|L|$, so the second case holds. As T is semidihedral or wreathed and acts faithfully on B, we conclude that T is semidihedral of order 16 and $BT \cong M_{10}$. Therefore BT has a Frobenius subgroup of order 20, so $p = 5$ by (c). Then as $5 \equiv \epsilon \mod 3$, $L \cong SU_3(5)$ and $Z(L)$ is of order 3. As $m_3(L) = 2$, $BZ(L)$ does not split over $Z(L)$, so $Z(L) = Z(B)$ and $B \cong \hat{A}_6$. Therefore conclusion (9) of A.3.12 holds, as required.

This completes the proof of A.3.21, and hence also of A.3.12. □

A.4. Signalizers for groups with $\mathbf{X = O^2(X)}$

Recall that the families $\mathcal{L}(H,T)$ and $\Xi(H,T)$ of subgroups of a QTKE-group H determined by a Sylow 2-subgroup T of H, defined in chapter 1, play a central role in our analysis of QTKE-groups. If X is a member of either of these families, then $X = O^2(X)$ is subnormal in $\langle X, T\rangle$, $F^*(X) = O_2(X)$, and all noncentral 2-chief factors of X are contained in $O_2(X)$. In this section, we investigate subgroups of a finite group sharing some of these properties, and then apply our results in other sections to QTKE-groups.

The first lemma consists of some fairly elementary properties of 2-signalizers for a subgroup X—that is, 2-groups invariant under X. The characterization A.4.2.4 of maximal signalizers is one that we use frequently throughout the work.

We mention also that the proof of most of the results goes through with 2 replaced by any prime p, but we have no need for the results for odd primes.

In this section G is just a finite group; that is, we don't require G to be quasithin.

Definition A.4.1. Recall that $\mathsf{M}_M(X,2)$ denotes the set of 2-subgroups of M invariant under X; and $\mathsf{M}^*_M(X,2)$ denotes the maximal members of that set.

In the literature the members of $\mathsf{M}_M(X,2)$ are often called 2-*signalizers* for X in M. (This relaxes the original definition of Thompson, which would require that the invariant subgroups intersect X trivially).

Lemma A.4.2. *Assume $X = O^2(X) \trianglelefteq \trianglelefteq M \leq G$ and let R be a 2-subgroup of M. Then*

(1) $R \in \text{И}_M(X,2)$ *iff* $[X,R] \leq R$.

(2) If $RX = XR$, *then* $X = O^2(XR)$, *so* $X \trianglelefteq XR$.

(3) If $R \in \text{И}_M(X,2)$, *then* $[X,R] \leq O_2(X)$.

(4) $\text{И}^*_M(X,2) = Syl_2(C_M(X/O_2(X)))$.

(5) If $T \in Syl_2(N_M(X))$, *then* $O_2(XT) \in \text{И}^*_M(X,2)$.

(6) Assume $R \in \text{И}^*_M(X,2)$. *If* $|N_G(R) : N_M(R)|$ *is odd, then* $R = O_2(N_G(R))$, *and if* $N_G(R) \leq M$, *then* $R \in \text{И}^*_G(X,2)$.

(7) Assume $X \trianglelefteq M$, $T \in Syl_2(M)$, $R := O_2(XT)$, *and* $N_G(R) \leq M$. *Then* $R \in Syl_2(C_G(X/O_2(X))) \subseteq \text{И}^*_G(X,2)$, $R \in \text{И}^*_M(X,2) = Syl_2(C_M(X/O_2(X))$, $R = O_2(N_G(R))$, *and* $R \in Syl_2(\langle R^M \rangle)$.

Proof. First $R \in \text{И}_M(X,2)$ iff $X \leq N_M(R)$ iff $[X,R] \leq R$, so (1) holds.

Assume $RX = XR$. As $X \trianglelefteq \trianglelefteq M$ by hypothesis, there is a series

$$X = X_0 \trianglelefteq \cdots \trianglelefteq X_n = RX.$$

We show $X = O^2(X_i)$ by induction on i; the assertion holds at $i = 0$ by the hypothesis $X = O^2(X)$. Suppose the assertion holds at $i-1$; then $X = O^2(X_{i-1})$ char X_{i-1} so $X \trianglelefteq X_i$. But $X \leq X_i \leq XR$, so by the Dedekind Modular Law, $X_i = X(X_i \cap R)$. As $X_i \cap R$ is a 2-group, $O^2(X_i) \leq X$; then $X = O^2(X_i)$ using the hypothesis $X = O^2(X)$. So (2) holds.

Let $R \in \text{И}_M(X,2)$. In general $[X,R] \triangleleft \langle X,R \rangle$, and $\langle X,R \rangle = XR$ by hypothesis. By (2), $[X,R] \leq X$, so $[X,R] \trianglelefteq X$. Further by (1), $[X,R]$ is a 2-group, so $[X,R] \leq O_2(X)$. That is, (3) holds.

Let $R \in \text{И}^*_M(X,2)$. By (3), $R \leq M_0 := C_M(X/O_2(X))$, so $R \leq S \in Syl_2(M_0)$. As $S \leq M_0$, $[X,S] \leq O_2(X)$, while $O_2(X) \leq M_0$, so $O_2(X) \leq S$ by A.1.6. Hence $[S,X] \leq S$, so $S \in \text{И}_M(X,2)$ by (1). Therefore $R = S$ by maximality of R, so $\text{И}^*_M(X,2) \subseteq Syl_2(M_0)$. On the other hand if $S_1 \in Syl_2(M_0)$, then we've seen $S_1 \in \text{И}_M(X,2)$, so $S_1 \leq R_1 \in \text{И}^*_M(X,2) \subseteq Syl_2(M_0)$, and hence $S_1 = R_1$ as $R_1 \leq M_0$. This proves the opposite inclusion, establishing (4).

Assume the hypotheses of (5). As $T \in Syl_2(N_M(X))$, $T_C := T \cap M_0 \in Syl_2(M_0)$, so $T_C \in \text{И}^*_M(X,2)$ by (4). In particular $T_C \trianglelefteq XT_C$, so as $T_C \trianglelefteq T$, we have $T_C \leq O_2(XT)$. Conversely $O_2(XT) \in \text{И}_M(X,2)$, so $O_2(XT) = T_C$ by maximality of T_C, establishing (5).

Next let $R \in \text{И}^*_M(X,2)$. Then $X \leq N_G(R) \leq N_G(O_2(N_G(R)))$, and so $RO_2(N_G(R)) \in \text{И}_G(X,2)$. Now if $|N_G(R) : N_M(R)|$ is odd then $O_2(N_G(R)) \leq M$, so $RO_2(N_G(R)) \in \text{И}_M(X,2)$ and thus $O_2(N_G(R)) \leq R$ by maximality of R—giving the first assertion of (6). Similarly if $N_G(R) \leq M$ and $R \leq S \in \text{И}^*_G(X,2)$ then $N_S(R) \in \text{И}_M(X,2)$, so $R = N_S(R)$ by maximality of R, and hence $R = S$ by standard properties of nilpotent groups. This establishes the second assertion of (6).

Finally assume the hypotheses of (7). By (4) and (5), $R \in \text{И}^*_M(X,2) = Syl_2(M_0)$, and then by (6), $R = O_2(N_G(R))$ and $R \in \text{И}^*_G(X,2)$. Then $R \in Syl_2(C_G(X/O_2(X)))$ by (4) applied to $N_G(X)$ in the role of "M". As $X \trianglelefteq M$, also $M_0 \trianglelefteq M$, so $\langle R^M \rangle = O^{2'}(M_0)$ and in particular $R \in Syl_2(\langle R^M \rangle)$, the final assertion of (7).

This completes the proof of A.4.2. □

Lemma A.4.3. *Assume* $X = O^2(X) \trianglelefteq \trianglelefteq M \leq G$, *and let* $Y \leq M$ *with* $Y = (X \cap Y)O_2(Y)$ *and* $X = (Y \cap X)O_2(X)$. *Then*

(1) $\text{И}_M(Y,2) \subseteq C_M(X/O_2(X))$ *and* $Y \leq N_M(X)$.

(2) $И^*_M(Y,2) \subseteq И^*_M(X,2)$.

Proof. Let $B \in И_M(Y,2)$ and set $Y_+ := Y \cap X$. Now $O_2(X) \leq O_2(M)$ as $X \trianglelefteq \trianglelefteq M$, and $C := \langle O_2(X), B\rangle$ is a 2-group. Both $O_2(X)$ and B are invariant under $Y_+ = Y \cap X$, and C is $O_2(X)$-invariant as $O_2(X) \leq C$. Thus as $X = Y_+O_2(X)$, $C \in И_M(X,2)$. Now $B \leq C$, and by A.4.2.3, $C \leq M_0 := C_M(X/O_2(X))$, giving the first assertion of (1). Specializing to the case $B := O_2(Y)$, we conclude $O_2(Y)$ normalizes X, so $Y = Y_+O_2(Y) \leq N_M(X)$, completing the proof of (1).

If $B \in И^*_M(Y,2)$ then $B \leq S \in Syl_2(M_0)$ by (1). Also by (1) both B and Y normalize $O_2(X)$, so that $O_2(X)B \in И_M(Y,2)$, and hence $O_2(X) \leq B$ by maximality of B. Next

$$[Y_+, S] \leq [X,S] \leq O_2(X) \leq B \leq S,$$

so $Y_+ \leq N_M(S)$. By maximality of B, $O_2(Y) \leq B$, so that $O_2(Y) \leq S$. Then the hypothesis $Y = Y_+O_2(Y)$ gives $S \in И_M(Y,2)$, which forces $S = B$ by maximality of B. Thus (2) holds, as $S \in И^*_M(X,2)$ by A.4.2.4. □

The next result will be applied in conjunction with A.4.3 in various pushing up arguments; cf. section 4.1 for example.

Recall $C(A,B)$ from Definition C.1.5, and that $\mathcal{M}$ denotes the set of maximal 2-locals of G.

Lemma A.4.4. *Let $H, K \leq G$, set $H^* := H/O_2(H)$, define $\Theta(H^*)$ as in G.8.9, and suppose $O_2(H) \leq X \leq H \cap K$ such that either*

(a) $X^* \in \Theta(H^*)$, or

(b) X^* is subnormal in $F^*(H^*)$, and contains each element of prime order in $C_{F^*(H^*)}(X^*)$.

Then

(1) $O_2(H) = O_2(H \cap K)$.

(2) If $H \in \mathcal{M}$, then $C(K, O_2(H \cap K)) \leq H \cap K$.

(3) If $H, K \in \mathcal{M}$ and $O_{2,F^*}(K) \leq H$, then $H = K$.

Proof. Set $L := H \cap K$ and $Q := O_2(H \cap K)$. By hypothesis $X \leq L$ and $O_2(H) \leq K$, so $O_2(H) \leq Q$. In (a), $X^* \trianglelefteq \trianglelefteq F^*(H^*)$ by definition of $\Theta(H^*)$ in G.8.9, and in (b) this holds by hypothesis. Thus $X^* = O^2(X^*) \trianglelefteq \trianglelefteq H^*$, so $O_2(X^*) = 1$, and hence X^* centralizes each member of $И_{H^*}(X^*,2)$ by A.4.2.3, so $[Q^*, X^*] = 1$. Then in (a), G.8.9.2 says $Q^* = 1$, while in (b), $Q^* = 1$ by G.8.9.1. Thus in either case (1) holds.

Assume the hypothesis of (3). Then using symmetry between H and K, we apply (1) to get $O_2(K) = Q = O_2(H)$. Then as $H, K \in \mathcal{M}$, (3) holds. Finally if $H \in \mathcal{M}$, then $H = N_G(C)$ for each nontrivial normal 2-subgroup C of H, so as $Q = O_2(H)$ by (1), $C(G,Q) = H$, establishing (2). □

As in the Introduction to Volume II:

Definition A.4.5. We define $\mathcal{X}$ to be the set of non-trivial subgroups $Y = O^2(Y)$ of G such that $F^*(Y) = O_2(Y)$.

The bulk of the analysis in the proof of the Main Theorem consists of analyzing nontrivial internal $\mathbf{F}_2$-modules for such subgroups, so we develop some notation to discuss some of these modules.

DEFINITION A.4.6. For $Y \in \mathcal{X}$ and $R \in \mathcal{U}_{N_G(Y)}(Y, 2)$, define

$$V(Y, R) := [\Omega_1(Z(R)), Y].$$

Some basic properties are immediate from the definitions: by hypothesis both Y and R are normal in YR, so

$$V(Y, R) \leq Y \cap \Omega_1(Z(R)).$$

As $Y = O^2(Y)$, Coprime Action says

$$V(Y, R) = [V(Y, R), Y]. \qquad (*)$$

As $O_2(Y) \in \mathcal{U}_{N_G(Y)}(Y, 2)$, we can define

DEFINITION A.4.7.

$$V(Y) := V(Y, O_2(Y)).$$

In our applications, $Y/O_2(Y)$ usually acts faithfully on $V(Y)$ if $V(Y) \neq 1$, so we write $\mathcal{X}_f$ for the set of all $Y \in \mathcal{X}$ such that $V(Y) \neq 1$; that is, the subscript f stands for "faithful".

The next lemma does not actually make any direct use of the hypothesis that $F^*(X) = O_2(X)$—it is just that we chose to define $V(Y, R)$ only for $Y \in \mathcal{X}$, rather than under the less restrictive hypothesis $Y = O^2(Y)$.

The lemma supplies criteria for X to act nontrivially on abelian 2-groups; its corollary A.4.9 will often be used to show that suitable members of $\mathcal{X}$ are in $\mathcal{X}_f$.

LEMMA A.4.8. *Let $X \in \mathcal{X}$, with $X \trianglelefteq\trianglelefteq M \leq G$. Assume $R, S \in \mathcal{U}_M(X, 2)$ with $V(X, S) \leq R \leq S \geq O_2(X)$, and set $X^*S^* := XS/C_{XS}(V(X))$. Then*

(1) $O_2(X)$ centralizes $V(X)$, and R^* centralizes X^*.

(2) If all noncentral 2-chief factors of X are in $O_2(X)$, then $O_2(X^*) \leq Z(X^*)$; and if all 2-chief factors are in $O_2(X)$, then $O_2(X^*) = 1$.

(3) $V(X, S) = [C_{V(X,R)}(S), X]$.

(4) $V(X, S) = [C_{V(X)}(S), X]$.

(5) $V(X, S) \neq 1$ iff $V(X) \neq 1$.

PROOF. As $X \trianglelefteq\trianglelefteq M$, A.4.2.3 says $[X, R] \leq O_2(X) \geq [X, S]$ and hence $R, S \leq N_M(X)$, so that $V(X, R)$ and $V(X, S)$ are indeed defined. By definition, $V(X) \leq Z(O_2(X))$, so $O_2(X) \leq C_X(V(X))$. As $[X, R] \leq O_2(X) \leq C_{XR}(V(X))$, $[X^*, R^*] = 1$, completing the proof of (1).

Under the hypothesis of (2), the noncentral 2-chief factors of X^* are in $O_2(X)^*$, while by (1), $O_2(X)^* = 1$. Thus $O_2(X^*) \leq Z(X^*)$ by Coprime Action as $X^* = O^2(X^*)$. Similarly the second statement of (2) holds.

By hypothesis, $V(X, S) \leq R \leq S$, so $V(X, S) \leq Z(S) \cap R \leq Z(R)$. Then using (*) above

$$V(X, S) = [V(X, S), X] \leq [\Omega_1(Z(R)), X] = V(X, R).$$

Then as $V(X, S) \leq Z(S)$, (*) says

$$V(X, S) = [V(X, S), X] \leq [C_{V(X,R)}(S), X].$$

By hypothesis $R \leq S$, so $C_{V(X,R)}(S) \leq Z(S)$, and hence $[C_{V(X,R)}(S), X] \leq V(X, S)$, so that (3) holds.

Next $V(X, S) = [V(X, S), X] \leq [X, S] \leq O_2(X) \leq S$, so we can apply (3) to $O_2(X)$ in the role of R, to conclude (4) holds. By (1), X^* centralizes S^*, so by the Thompson $A \times B$-Lemma, $1 \neq [V(X), X]$ iff $1 \neq [C_{V(X)}(S), X]$. Then (4) and (*) imply (5). $\square$

LEMMA A.4.9. *Assume that $X \in \mathcal{X}$ is subnormal in some finite group M. Then $[\Omega_1(Z(O_2(M))), X] \neq 1$ iff $[\Omega_1(Z(O_2(X))), X] \neq 1$.*

PROOF. Apply A.4.8.5 with $S = R = O_2(M)$. □

We use the next result in 1.2.9 and 1.3.9 to control chains of inclusions in $\mathcal{L}(H,T) \cup \Xi(H,T)$ in a QTKE-group H.

LEMMA A.4.10. *Let $X \leq Y \leq G$ with $X, Y \in \mathcal{X}$. Assume T normalizes Y and $T \in Syl_2(YT)$, with $O_2(YT) \leq N_T(X) \in Syl_2(N_{YT}(X))$. Then*

(1) $C_T(X/O_2(X)) := S \in \text{И}^*_{N_{YT}(X)}(X,2)$.
(2) $V(X,S) \leq V(Y, O_2(YT))$.
(3) If $X \in \mathcal{X}_f$, then $Y \in \mathcal{X}_f$.

PROOF. As $N_T(X) \in Syl_2(N_{YT}(X))$, $S := N_T(X) \cap C_G(X/O_2(X))$ is Sylow in $C_{YT}(X/O_2(X)) = C_{N_{YT}(X)}(X/O_2(X))$. Then we apply A.4.2.4 with $N_{YT}(X)$ in the role of M to obtain $S \in \text{И}^*_{N_{YT}(X)}(X,2)$. Thus (1) holds.

By hypothesis $SX \leq N_{YT}(X)$ normalizes $O_2(YT)$, and of course X normalizes S by definition of $C_G(X/O_2(X))$, so $O_2(YT)S \in \text{И}_{N_{YT}(X)}(X,2)$; thus $O_2(YT) \leq S$ by (1). As $Y \in \mathcal{X}$, $F^*(YT) = O_2(YT)$, so $Z(S) \leq C_{YT}(O_2(YT)) \leq Z(O_2(YT))$. Then using (*),

$$V(X,S) = [V(X,S), X] \leq [\Omega_1(Z(O_2(YT))), Y] = V(Y, O_2(YT)),$$

establishing (2). Further as $V(Y, O_2(YT)) \leq V(Y)$ by A.4.8.4, also $V(X,S) \leq V(Y)$. Finally if $X \in \mathcal{X}_f$, then $V(X) \neq 1$. By (1), S contains $O_2(X)$, and it contains $V(X,S)$ by definition. Now we apply A.4.8.5 (again with $N_{YT}(X)$ in the role of M) to conclude $V(X,S) \neq 1$. Then as $V(X,S) \leq V(Y)$, $V(Y) \neq 1$, so $Y \in \mathcal{X}_f$. This establishes (3). □

Our final lemma will be used in 1.2.10 to get a criterion for a faithful, 2-reduced action.

We recall Definition B.2.11: A normal elementary abelian 2-group V of M is 2-reduced if $O_2(M/C_M(V)) = 1$, $\mathcal{R}_2(M)$ denotes the set of 2-reduced subgroups of M, and $R_2(M) = \langle \mathcal{R}_2(M) \rangle$. In fact $R_2(M)$ is the unique maximal member of $\mathcal{R}_2(M)$ by B.2.12.

LEMMA A.4.11. *Let $F^*(M) = O_2(M)$. Suppose $X = O^2(X) \trianglelefteq \trianglelefteq M$, and all non-central 2-chief factors of X lie in $O_2(X)$. Then the following are equivalent:*

(1) $X \in \mathcal{X}_f$.
(2) There exists $V \in \mathcal{R}_2(M)$ with $[V, X] \neq 1$.
(3) $[R_2(M), X] \neq 1$.

PROOF. We will see that the hypothesis on the 2-chief factors of X is needed only for the proof that (1) implies (2).

By B.2.12, (2) and (3) are equivalent.

By hypothesis $X = O^2(X)$, $X \trianglelefteq \trianglelefteq M$, and $F^*(M) = O_2(M)$. Thus (cf. 1.1.3.1) $F^*(X) = O_2(X)$—so in particular, $X \in \mathcal{X}$.

We first show that (2) implies (1): Assume (2), and set $\bar{M} := M/C_M(V)$. As $X \trianglelefteq \trianglelefteq M$, $O_2(X) \leq O_2(M)$, while $O_2(\bar{M}) = 1$ as V is 2-reduced. Then $O_2(X) \leq C_M(V)$, so if we set $S := O_2(X)V$ then $V \leq \Omega_1(Z(S))$ and $S \in \text{И}_M(X,2)$, so $[V,X] \leq V(X,S)$. By hypothesis, $1 \neq [V,X]$, so as $[V,X] \leq V(X,S)$, $V(X,S) \neq 1$.

We have the hypotheses of A.4.8 with $R = S$. Therefore A.4.8 says $V(X) \neq 1$, so $X \in \mathcal{X}_f$. This completes the proof that (2) implies (1).

Conversely assume (1); that is, $V(X) \neq 1$. Set $R := O_2(M)$ and $U := \Omega_1(Z(R))$. Applying A.4.9, we conclude that $[U, X] \neq 1$. Thus U is a member of the set $\mathcal{V}$ of subgroups $1 \neq V \trianglelefteq M$ with $V \leq U$ and $[V, X] \neq 1$. Choose V minimal in $\mathcal{V}$ and set $\bar{M} = M/C_M(V)$. Now

$$O_2(X) \leq R \leq C_M(U) \leq C_M(V).$$

We apply A.4.2 to $1 \neq \bar{X} = O^2(\bar{X})$ which is subnormal in $\bar{M}$, and conclude from A.4.2.3 that $O_2(\bar{M})$ centralizes $\bar{X}/O_2(\bar{X})$. Then we invoke Coprime Action and our hypothesis that all noncentral 2-chief factors of X lie in $O_2(X)$, to guarantee that $O_2(\bar{X}) \leq Z(\bar{X})$. Then $\bar{X}$ centralizes $O_2(\bar{M})/O_2(\bar{X})$ and $O_2(\bar{X})$, so again by Coprime Action, $\bar{X}$ centralizes $O_2(\bar{M})$. Then by the Thompson $A \times B$-Lemma, $\bar{X}$ is faithful on $C_V(O_2(\bar{M})) \neq 1$, so by minimality of V, $V = C_V(O_2(\bar{M}))$. Therefore $O_2(\bar{M}) = 1$. This shows that (1) implies (2), completing the proof of the lemma. □

A.5. An ordering on $\mathcal{M}(T)$

In chapter 15 of the proof of the Main Theorem, the uniqueness theorems from chapter 1 are unavailable, since the set $\mathcal{L}_f(G, T)$ is empty. In results such as A.5.7 and A.5.10.3 of this section, we provide some alternative uniqueness theorems, based instead on a certain partial order on the set $\mathcal{M}(T)$ of maximal 2-local subgroups of G containing $T \in Syl_2(G)$.

We recall the definition of $\mathcal{H}(T)$ from the Introduction to Volume II.

In this section we assume G is a simple group of even characteristic, $T \in Syl_2(G)$, and $Z := \Omega_1(Z(T))$.

Notation A.5.1. The results of this section are applied only in the final chapter 15, and in section 14.1 which provides preliminary analysis for that chapter. We now define a notation $V(H)$, which will be used only for this section and for those applications; thus for all other parts of our proof, $V(H)$ has the meaning given earlier in definition A.4.7. However in this section, for $H \in \mathcal{H}(T)$, we set

$$V(H) := \langle Z^H \rangle;$$

and set $\bar{H} := H/C_H(V(H))$.

As G is of even characteristic and $T \in Syl_2(G)$, B.2.14 says that $V(H)$ is a normal elementary abelian 2-subgroup of H, and $O_2(\bar{H}) = 1$.

Definition A.5.2. Define a relation $\lesssim$ on $\mathcal{H}(T)$ by $H_1 \lesssim H_2$ iff

$$H_1 = (H_1 \cap H_2)C_{H_1}(V(H_1)). \quad (*)$$

Observe in particular that $\lesssim$ is reflexive, and indeed extends ordinary inclusion: that is, if $H_1 \leq H_2$ then $H_1 \lesssim H_2$. Write $\mathcal{H}_v(T)$ for the subset of those $H \in \mathcal{H}(T)$ such that $H = N_G(V(H))$. For example, $\mathcal{M}(T) \subseteq \mathcal{H}_v(T)$.

Lemma A.5.3. *Assume $H_1, H_2 \in \mathcal{H}(T)$ with $H_1 \lesssim H_2$. Then*

(1) $V(H_1) \leq V(H_2)$.

(2) $C_{H_2}(V(H_2)) \leq C_{H_2}(V(H_1))$.

(3) If $H_1 \in \mathcal{H}_v(T)$, then $C_{H_2}(V(H_2)) \leq C_{H_1}(V(H_1))$.

PROOF. As $H_1 \lesssim H_2$, by the definition (*)

$$V(H_1) = \langle Z^{H_1} \rangle = \langle Z^{H_1 \cap H_2} \rangle \leq \langle Z^{H_2} \rangle = V(H_2),$$

so that $C_{H_2}(V(H_2)) \leq C_{H_2}(V(H_1))$. Also if $H_1 \in \mathcal{H}_v(T)$, then $H_1 = N_G(V(H_1))$, so $C_{H_2}(V(H_1)) \leq C_{H_1}(V(H_1))$. □

LEMMA A.5.4. *Assume $H_1, H_2 \in \mathcal{H}_v(T)$. Then the following are equivalent:*
(1) $H_1 = H_2$.
(2) $V(H_1) = V(H_2)$.
(3) $H_1 \lesssim H_2$ and $H_2 \lesssim H_1$.

PROOF. Trivially, (1) implies (2) and (3). As $H_i \in \mathcal{H}_v(T)$, $H_i = N_G(V(H_i))$, so (2) implies (1). Finally by A.5.3.1, (3) implies (2). □

LEMMA A.5.5. *$\lesssim$ is a partial order on $\mathcal{H}_v(T)$, and hence also on $\mathcal{M}(T)$.*

PROOF. Trivially $\lesssim$ is reflexive. By A.5.4, $\lesssim$ is antisymmetric. Suppose $H_1 \lesssim H_2 \lesssim H_3$. Then $H_2 = (H_2 \cap H_3)C_{H_2}(V(H_2))$, and by A.5.3.3, $C_{H_2}(V(H_2)) \leq C_{H_1}(V(H_1))$, so $H_2 \leq (H_2 \cap H_3)C_{H_1 \cap H_2}(V(H_1))$. Therefore using the Dedekind Modular Law,

$$H_1 \cap H_2 \leq H_1 \cap ((H_2 \cap H_3)C_{H_1 \cap H_2}(V(H_1))) = (H_1 \cap H_2 \cap H_3)C_{H_1 \cap H_2}(V(H_1))$$
$$\leq (H_1 \cap H_3)C_{H_1}(V(H_1)).$$

Then using (*),

$$H_1 = (H_1 \cap H_2)C_{H_1}(V(H_1)) = (H_1 \cap H_3)C_{H_1}(V(H_1)),$$

and therefore $\lesssim$ is transitive. □

LEMMA A.5.6. *Assume $H \in \mathcal{H}(T)$ and $M \in \mathcal{M}(T)$ with $H \lesssim M$. Suppose there is V satisfying*

$$1 \neq V = \langle (V \cap Z)^{H \cap M} \rangle \trianglelefteq M.$$

Then $H \leq M$.

PROOF. As $M \in \mathcal{M}$ and $V \trianglelefteq M$, $M = N_G(V)$. Also

$$V = \langle (V \cap Z)^{H \cap M} \rangle \leq \langle Z^H \rangle = V(H),$$

while as $H \lesssim M$, $H = (H \cap M)C_H(V(H))$, so

$$V = \langle (V \cap Z)^{H \cap M} \rangle = \langle (V \cap Z)^H \rangle \trianglelefteq H,$$

and hence $H \leq N_G(V) = M$. □

LEMMA A.5.7. *Let $M \in \mathcal{M}(T)$ be maximal in $\mathcal{M}(T)$ with respect to $\lesssim$. Then*
(1) $M = !\mathcal{M}(X)$ for each $X \leq \mathcal{H}(T) \cap M$ with $M = XC_M(V(M))$.
(2) Assume either
(a) $V = V(M)$, or
(b) $V = \langle (Z \cap V)^M \rangle$ and M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.
Let $R := C_T(V)$. Then we have $R = F^*(N_M(R))$, $V \in \mathcal{R}_2(N_M(R))$, $Aut_M(V) = Aut_{N_M(R)}(V)$, and $M = !\mathcal{M}(N_M(R))$.

(3) $H \in \mathcal{H}(T)$ is maximal in $\mathcal{H}_v(T)$ with respect to $\lesssim$ iff $H \in \mathcal{M}(T)$ and H is maximal in $\mathcal{M}(T)$ with respect to $\lesssim$.

Proof. Assume that $H \in \mathcal{H}_v(T)$, and H is maximal with respect to $\lesssim$ in either $\mathcal{M}(T)$ or $\mathcal{H}_v(T)$. Assume further that $X \in \mathcal{H}(T) \cap H$ with $H = XC_H(V(H))$. If $M_2 \in \mathcal{M}(X)$, then as $T \leq X$, $M_2 \in \mathcal{M}(T)$, and as $H = XC_H(V(H))$, $H \lesssim M_2$; therefore $H = M_2 \in \mathcal{M}$ by maximality of H. Thus (1) holds, as does the forward implication in (3).

Assume one of the hypotheses of (2). As G is of even characteristic $F^*(N_M(R)) = O_2(N_M(R))$ by 1.1.3.2. By a Frattini Argument, we have $M = C_M(V)N_M(R)$, so $Aut_{N_M(R)}(V) = Aut_M(V)$. In case (a), $V = \langle Z^M \rangle$, so in either case $V = \langle (Z \cap V)^M \rangle$. Therefore as $M = C_M(V)N_M(R)$,

$$V = \langle (Z \cap V)^{N_M(R)} \rangle, \qquad (!)$$

so that $V \in \mathcal{R}_2(N_M(R))$ by B.2.14. Thus $O_2(N_M(R)) \leq C_T(V) = R$, so $R = O_2(N_M(R)) = F^*(N_M(R))$.

In (a), $M = !\mathcal{M}(N_M(R))$ by (1); so assume (b) holds, and let $M_1 \in \mathcal{M}(N_M(R))$. Then $M_1 \lesssim M$ by (b), so by (!) we may apply A.5.6 to conclude that $M_1 \leq M$; hence $M_1 = M$ since $M_1 \in \mathcal{M}$. This completes the proof of (2).

Let $H \in \mathcal{H}(T)$. If H is maximal in $\mathcal{H}_v(T)$ with respect to $\lesssim$, then $H \in \mathcal{M}(T)$ by the first paragraph of the proof. Conversely if H is maximal in $\mathcal{M}(T)$, then $H \lesssim M$ for some M maximal in $\mathcal{H}_v(T)$; but $M \in \mathcal{M}(T)$ by paragraph one, so $H = M$ by maximality of H in $\mathcal{M}(T)$. Thus (3) holds. □

Definition A.5.8. Let $\mathcal{V}(T) := \{V(H) : H \in \mathcal{H}(T)\}$, and partially order $\mathcal{V}(T)$ by inclusion.

Lemma A.5.9. *Let $H, K \in \mathcal{H}(T)$. Then*

(1) If $H \leq K$, then $H \lesssim K$.
(2) If $H \lesssim K$ and $K \lesssim H$, then $V(H) = V(K)$.

Proof. Here (1) just recalls that $\lesssim$ extends ordinary inclusion, and (2) follows from A.5.3.1. □

Lemma A.5.10. *Let $H, K \in \mathcal{H}(T)$ with $V(H)$ maximal in $\mathcal{V}(T)$. Then*
(1) If $V(H) \leq V(K)$, then $K \leq N_G(V(H))$.
(2) If $H \lesssim K$, then $K \leq N_G(V(H))$.
(3) $N_G(V(H)) = !\mathcal{M}(H)$.
(4) $N_G(V(H))$ is maximal in $\mathcal{M}(T)$ with respect to $\lesssim$.

Proof. If $V(H) \leq V(K)$, then by maximality of $V(H)$, we have $V(H) = V(K)$. Then as $V(K) \trianglelefteq K$, (1) holds. Then A.5.3.1 and (1) imply (2), while A.5.9.1 and (2) imply (3).

Finally set $N := N_G(V(H))$. Then $N \in \mathcal{M}(T)$ by (3). Also $H \lesssim N$ by A.5.9.1, so $V(H) = V(N)$ by A.5.3.1 and maximality of $V(H)$, and hence N also satisfies the hypothesis for "H". Now if $N \lesssim M \in \mathcal{M}(T)$, then by (2) applied to N, $M \leq N_G(V(N)) = N_G(V(H)) = N$; hence $M = N$ by maximality of M, and (4) holds. □

A.6. A group-order estimate

In this section, G is a finite group with exactly two classes of involutions, with representatives z and t. For x an involution in G, define

$$\mathcal{A}(x) := \{(u,v) \in z^G \times t^G : x \in \langle uv \rangle\},$$

and set $a(x) := |\mathcal{A}(x)|$. The Thompson Order Formula for groups with exactly two classes of involutions says:

Lemma A.6.1 (Thompson Order Formula). $|G| = |C_G(z)|a(t) + |C_G(t)|a(z)$.

Proof. See for example 45.6 in [**Asc86a**]. □

Next for x an involution in G, define

$$\mathcal{A}_1(x) := \{(u,v) \in \mathcal{A}(x) : |uv| \equiv 0 \mod 4\}$$

and

$$\mathcal{A}_2(x) := \{(u,v) \in \mathcal{A}(x) : |uv| \equiv 2 \mod 4\},$$

and let $a_i(x) := |\mathcal{A}_i(x)|$. As $z^G \neq t^G$, uv has even order for each $(u,v) \in \mathcal{A}(x)$, so $\mathcal{A}(x) = \mathcal{A}_1(x) \cup \mathcal{A}_2(x)$ is a disjoint union, and $a(x) = a_1(x) + a_2(x)$. Also let y^G denote the class of involutions distinct from x^G, and define:

$$\mathcal{B}_1(x) := \{u \in x^G \cap C_G(x) : ux \in u^G\}, \quad \mathcal{B}_2(x) := \{u \in x^G \cap C_G(x) : ux \notin u^G\},$$

$$\mathcal{C}_1(x) := \{u \in y^G \cap C_G(x) : ux \in u^G\}, \quad \mathcal{C}_2(x) := \{u \in y^G \cap C_G(x) : ux \notin u^G\}.$$

Let $b_i(x) := |\mathcal{B}_i(x)|$ and $c_i(x) := |\mathcal{C}_i(x)|$.

Lemma A.6.2. *$a_i(x) \leq b_i(x)c_i(x)$ for each involution x in G and $i = 1, 2$.*

Proof. Let $(u,v) \in \mathcal{A}(x)$. Then x is the involution in $\langle uv \rangle$, so $X := \langle u, v \rangle \leq C_G(x)$. Further $ux \in u^X$ if and only if $|uv| \equiv 0 \mod 4$, so as $v^G \neq u^G$, it follows that $(u,v) \in \mathcal{A}_1(x)$ iff $ux \in u^X$ iff $vx \in v^X$. Therefore

$$\mathcal{A}_1(x) \subseteq \{(u,v) \in (z^G \cap C_G(x)) \times (t^G \cap C_G(x)) : ux \in u^G \text{ and } vx \in v^G\},$$

so $a_1(x) \leq b_1(x)c_1(x)$. Similarly $a_2(x) \leq b_2(x)c_2(x)$. □

Lemma A.6.3. *We have*

$$c_i(t) = \frac{|C_G(t)|c_{3-i}(z)}{|C_G(z)|} \quad \textit{for} \quad i = 1, 2.$$

Proof. Let $\Omega := \{(u,v) \in z^G \times t^G : uv \in t^G\}$. Then

$$|G : C_G(z)|c_1(z) = |z^G|c_1(z) = |\Omega| = |t^G|c_2(t) = |G : C_G(t)|c_2(t),$$

establishing the lemma for $i = 2$. A similar argument on the set Ω', defined similarly but with $uv \in z^G$, gives the lemma when $i = 1$. □

Lemma A.6.4. $|G : C_G(z)| \leq c_1(t)(b_1(t) + b_2(z)) + c_2(t)(b_2(t) + b_1(z))$.

Proof. We apply the Thompson Order Formula A.6.1, A.6.2, A.6.3, and our observation that $a(x) = a_1(x) + a_2(x)$, to obtain:

$$|G : C_G(z)| = a(t) + \frac{|C_G(t)|a(z)}{|C_G(z)|} = a_1(t) + a_2(t) + \frac{|C_G(t)|}{|C_G(z)|}(a_1(z) + a_2(z))$$

$$\leq b_1(t)c_1(t) + b_2(t)c_2(t) + \frac{|C_G(t)|}{|C_G(z)|}(b_1(z)c_1(z) + b_2(z)c_2(z))$$

$$= b_1(t)c_1(t) + b_2(t)c_2(t) + b_1(z)c_2(t) + b_2(z)c_1(t)$$

$$= c_1(t)(b_1(t) + b_2(z)) + c_2(t)(b_2(t) + b_1(z)).$$

□

LEMMA A.6.5. *Let G be a finite group with exactly two classes z^G and t^G of involutions. Assume*

$$\textit{If } u, v \in z^G \textit{ with } uv = vu \neq 1, \textit{ then } uv \in z^G. \tag{$*$}$$

Then

$$|G : C_G(z)| \leq |z^G \cap C_G(t)| \cdot (|z^G \cap C_G(t)| + |z^G \cap C_G(z)|).$$

PROOF. This is a special case of A.6.4: For by (*), $c_1(t) = 0$, $c_2(t) = |z^G \cap C_G(t)|$, and $b_1(z) = |z^G \cap C_G(z)| - 1$. Further

$$\mathcal{B}_2(t) = \{u \in t^G \cap C_G(t) : ut \notin u^G\} = \{u \in t^G \cap C_G(t) : ut \in z^G\} \cup \{t\},$$

and the map $u \mapsto ut$ is a bijection of

$$\{u \in t^G \cap C_G(t) : ut \in z^G\} \quad \text{with} \quad z^G \cap C_G(t)$$

by (*), so $b_2(t) = |z^G \cap C_G(t)| + 1$. □

CHAPTER B

Basic results related to Failure of Factorization

The techniques used in this work might be called *unipotent techniques*, in that they are useful in the study of local subgroups of characteristic 2—which in a group G of Lie type and characteristic 2, reflect the unipotent structure of G. The first and most important tool in the arsenal of the unipotent group theorist is Thompson Factorization:

When a 2-local subgroup H of G with Sylow 2-subgroup T admits a Thompson factorization (see B.2.15.2), we have some control over H in terms of the 2-local subgroups $N_G(J(T))$ and $C_G(\Omega_1(Z(T)))$ of G. On the other hand if Thompson factorization fails (see B.2.15.1), then we obtain useful information about the action of H on each of the 2-reduced normal subgroups in $\mathcal{R}_2(H)$ (see Definition B.2.11 and B.2.12).

In this chapter, we collect and develop the most basic ideas related to failure of factorization, and discuss briefly the connection between failure of factorization and some other notions, such as the theory of "small" $\mathbf{F}_2$-representations and pushing-up, which are treated in other chapters.

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B.1. Representations and FF-modules

In this short introductory section B.1, we first develop the more purely representation theoretic aspects of failure of factorization. We postpone to the subsequent section B.2 the case where the module is *internal* to the group, including the corresponding connections with the Thompson subgroup.

So in this section, let G be a finite group, and V a faithful $\mathbf{F}_2G$-module.

DEFINITION B.1.1. Write $\mathcal{A}^2(G)$ [1] for the set of nontrivial elementary abelian 2-subgroups of G. For $A \in \mathcal{A}^2(G)$, define the "ratio" $r_{A,V}$ of A on V by

$$r_{A,V} := \frac{m(V/C_V(A))}{m(A)},$$

and define the "global quadratic action ratio" $q(G,V)$ of G on V by

$$q(G,V) := \min\{r_{A,V} : A \in \mathcal{A}^2(G) \text{ and } [V,A,A] = 0\},$$

with the parameter set equal to ∞ if $\mathcal{A}^2(G)$ is empty (i.e., if G is of odd order). We say A is *quadratic* on V if $[V,A,A] = 0$. We say V is a *failure of factorization*

[1] We use this superscript-variant of the standard notation $\mathcal{A}_2(G)$ of Quillen, since we want to use subscript-notation $\mathcal{A}_k(G)$ for the subgroups of rank $m_2(G) - k$ which define the higher Thompson subgroups in Definition B.2.2.

module (FF-module) if $r_{A,V} \leq 1$ for some $A \in \mathcal{A}^2(G)$. We say V is a *strong* FF-module if $r_{A,V} < 1$ for some $A \in \mathcal{A}^2(G)$.

We will see in results such as B.1.4 below (as in the literature, e.g. section 32 of [**Asc86a**]) that we may study FF-modules by refining the condition $r_{A,V} \leq 1$, as follows:

Definition B.1.2. Let $\mathcal{P}(G, V)$ denote the set of $A \in \mathcal{A}^2(G)$ such that

$$m(A) + m(C_V(A)) \geq m(B) + m(C_V(B)) \qquad (*)$$

for all $B \leq A$. Notice that the case $B = 1$ provides the the inequality more typically associated with failure of factorization in the literature, namely

$$m(A) \geq m(V/C_V(A)), \qquad (**)$$

which is just a reformulation of the condition $r_{A,V} \leq 1$.

We sometimes consider the slightly weaker condition $m(A) \geq m(V/C_V(A)) - 1$; typically in the literature, V is then called an $(F-1)$-*module*, and A an $(F-1)$-*offender*.

Define a partial order $\lesssim$ on $\mathcal{A}^2(G)$ by $B \lesssim A$ if $B \leq A$ and (*) is an equality. Let $\mathcal{P}^*(G, V)$ be the set of minimal members of $\mathcal{P}(G, V)$ under $\lesssim$.

In the following section B.2, we will encounter elementary abelian subgroups of 2-locals H called "FF-offenders". The images of these FF-offenders acting on "internal" modules V for H will lie in $\mathcal{P}(Aut_H(V), V)$. To distinguish the elementary abelian subgroups from their images in $GL(V)$, we introduce a different but related term for offenders in our present strictly module-theoretic context.

Definition B.1.3. Thus we call the members of $\mathcal{P}(G, V)$ FF*-*offenders* on V. A *strong* FF*-offender is an FF*-offender A such that $r_{A,V} < 1$.

Set $J(G, V) := \langle \mathcal{P}(G, V) \rangle$.

Lemma B.1.4. *(1) If $A \in \mathcal{P}(G, V)$, then $r_{A,V} \leq 1$.*

(2) If $B \lesssim A \in \mathcal{P}(G, V)$, then $B \in \mathcal{P}(G, V)$.

(3) (Thompson Replacement Lemma) If $A \in \mathcal{P}^*(G, V)$, then A is quadratic on V. In particular if A is minimal in $\mathcal{P}(G, V)$ under inclusion, then A is quadratic on V.

(4) If $A \in \mathcal{A}^2(G)$ with $r_{A,V} \leq 1$ (resp. $r_{A,V} < 1$), then A contains a member B of $\mathcal{P}(G, V)$, with $r_{B,V} \leq 1$ (resp. $r_{B,V} < 1$).

(5) The following are equivalent:

(i) V is an FF-module.
(ii) $\mathcal{P}(G, V) \neq \emptyset$.
(iii) $q(G, V) \leq 1$.

(6) If V is not a strong FF-module then

$$\mathcal{P}(G, V) = \{A \in \mathcal{A}^2(G) : r_{A,V} = 1\}.$$

Proof. Part (1) re-states our earlier observation that (*) implies (**). Part (2) is a straightforward consequence of the definition. To prove (3), one can show (cf. 6.7 in [**Asc82a**]) for $v \in V$ that (*) is an equality for $B := C_A([v, A])$. Then $B \in \mathcal{P}(G, V)$ by (2), so $B = A$ by minimality of A, giving (3).

To prove (4), we may assume A is minimal subject to $A \in \mathcal{A}^2(G)$ and $r_{A,V} \leq 1$, $r_{A,V} < 1$, respectively. As $r_{A,V} \leq 1$, (*) holds when $B = 1$, and if (*) fails for some $1 \neq B < A$, then

$$m(B) > m(V/C_V(B)) + m(A) - m(V/C_V(A)) \geq m(V/C_V(B)),$$

so $r_{B,V} < 1$, contrary to the minimality of A. Thus $A \in \mathcal{P}(G,V)$, establishing (4).

Assume V is not a strong FF-module. Then

$$\mathcal{P}(G,V) \subseteq \mathcal{S} := \{A \in \mathcal{A}^2(G) : r_{A,V} = 1\},$$

and

$$m(B) + m(C_V(B)) \leq m(V) \text{ for all } B \in \mathcal{A}^2(G). \tag{!}$$

On the other hand, if $A \in \mathcal{S}$, then $m(A) + m(C_A(V)) = m(V)$, so (*) is satisfied in view of (!), and hence $\mathcal{S} \subseteq \mathcal{P}(G,V)$, establishing (6).

It remains to prove (5). That (iii) implies (i) is immediate from the definitions, and that (i) implies (ii) follows from (4). Finally (1) and (3) show that (ii) implies (iii). □

Recall if $X = O^2(X) \leq G$, that $Irr_+(X,V)$ is defined in Definition A.1.40 to be the set of X-submodules I of V such that $I = [I,X]$ and $I/C_I(X)$ is an irreducible X-module.

The following lemma shows that if V is an FF-module for G, then suitable sections of V are also FF-modules for suitable sections of G.

Lemma B.1.5. *Let $A \in \mathcal{P}(G,V)$ and W an A-invariant subspace of V with $[W,A] \neq 0$. Then*

(1) $Aut_A(W) \in \mathcal{P}(Aut_G(W),W)$.

(2) If $A \in \mathcal{P}^*(G,V)$ then $Aut_A(W) \in \mathcal{P}^*(Aut_G(W),W)$. If in addition $Aut_A(W)$ is not a strong FF-offender on W, then $V = W + C_V(A)$ and A is faithful on W.*

(3) If $A \in \mathcal{P}^*(G,V)$, and $X = O^2(X) = [X,A] \leq G$ such that each 2-chief factor of X is central in X, then A fixes $Irr_+(X,V)$ pointwise.

(4) $\mathcal{P}(G,V)$ normalizes each component of G.

(5) If $A \in \mathcal{P}^*(G,V)$ and $q(Aut_G(W),W) = 1$, then $Aut_A(W) \in \mathcal{P}^*(Aut_G(W),W)$, $V = W + C_V(A)$, and A is faithful on W.

(6) If $m(W/C_W(A)) = m(A/C_A(W))$, then either A is faithful on W or $C_A(W) \in \mathcal{P}(G,V)$.

(7) If A is faithful on W and $q(Aut_G(W),W) = 1$, then $V = W + C_V(A)$ so $[A,V] \leq W$.

(8) Assume $L := F^*(G)$ is quasisimple, $O_2(G) = 1$, and S is a G-invariant section of V with $[S,L] \neq 1$. Then A is faithful on S and $A \cong Aut_A(S)$ contains a member of $\mathcal{P}(Aut_G(S),S)$. Assume further that $q(Aut_G(S),S) = 1$ and let $S = W/U$. Then $Aut_A(S) \in \mathcal{P}(Aut_G(S),S)$, AL centralizes U and V/W, and $V = W + C_V(A)$.

Proof. Parts (1), (2),and (6) are essentially contained in 3.2 in [**Asc81e**], or 26.20 and 26.22 in [**GLS96**]; for completeness we prove the former result as B.7.1 at the end of this chapter. Thus these three conclusions follow from parts (1), (2), and (3) of B.7.1. Also see D.2.6 for a refinement of (1). Part (3) is 3.3.2 in [**Asc81e**]; again for completeness we have provided a proof as B.7.2.2. Further (4) is 26.24 in [**GLS96**] (due to Timmesfeld). Under the hypotheses of

(5), $m(Aut_A(W)) = m(W/C_W(A))$ as $q(Aut_G(W), W) = 1$, so (5) follows from (2). Under the hypotheses of (7), $m(A) = m(W/C_W(A))$, so we have

$$m(A) \geq m(V/C_V(A)) \geq m(W/C_W(A)) = m(A),$$

so that $m(V/C_V(A)) = m(W/C_W(A))$, and hence (7) holds.

Assume the hypotheses of (8). As L is quasisimple and $[L, S] \neq 1$, $C_G(S) \leq Z(L)$; so as $O_2(G) = 1$, A is faithful on S and $L = [L, A]$. Therefore

$$m(Aut_A(S)) = m(A) \geq m(V/C_V(A)) \geq m(S/C_S(A)),$$

so $Aut_A(S)$ contains a member of $\mathcal{P}(Aut_G(S), S)$ by B.1.4.4. Assume further that $q(Aut_G(S), S) = 1$ and let $S = W/U$. Then $A \in \mathcal{P}(Aut_G(S), S)$ by B.1.4.6, and

$$m(A) = m(W/C_W(A)) \geq m(S/C_S(A)) + m(U/C_U(A)) = m(A) + m(U/C_U(A)),$$

so A centralizes U and $m(W/C_W(A)) = m(A)$. Then as in (7), $V = W + C_V(A)$, so $[A, V] \leq W$. As A centralizes U and V/W, so does $L = [L, A]$, completing the proof of (8). □

We next introduce a notion whose importance will not be evident until the next section, in results such as B.2.5:

Definition B.1.6. A nonempty subset $\mathcal{P}$ of $\mathcal{P}(G, V)$ is *stable* if $\mathcal{P}$ is closed in $\mathcal{A}^2(G)$ under both G-conjugation and $\lesssim$; that is,

(a) $A^g \in \mathcal{P}$ for each $A \in \mathcal{P}$ and $g \in G$, and

(b) whenever $A \in \mathcal{P}$ and $B \lesssim A$, then also $B \in \mathcal{P}$.

Given a stable set $\mathcal{P}$ and $H \leq G$, define $J_{\mathcal{P}}(H) := \langle \mathcal{P} \cap H \rangle$.

Example B.1.7. $\mathcal{P}^*(G, V)$ is a stable subset of $\mathcal{P}(G, V)$.

The next lemma is the representation-theoretic version of Glauberman's result on Solvable Thompson Factorization (see B.2.16 later).

Lemma B.1.8. *Assume $O_2(G) = 1$, V is an FF-module, and G is solvable. Then*

(1) $J(G, V) = G_1 \times \cdots \times G_s$ and $[V, J(G, V)] = V_1 \oplus \cdots \oplus V_s$, where $V_i := [V, G_i]$.

(2) $G_i \cong L_2(2)$, and V_i is the natural module for G_i.

(3) $q(G, V) = 1$. In particular, G contains no strong FF-offenders.*

(4) A is an FF-offender iff $A = A_1 \times \cdots \times A_s$, where either $A_i = 1$ or $A_i \in Syl_2(G_i)$, and $A_j \neq 1$ for at least one j.*

(5) If $\mathcal{P}$ is a stable subset of $\mathcal{P}(G, V)$, then $J_{\mathcal{P}}(G) = \prod_{i \in I} G_i$ for some $I \subseteq \{1, \ldots, s\}$, and $\mathcal{P} = \mathcal{P}(G, V) \cap J_{\mathcal{P}}(G)$.

Proof. See Glauberman in [**Gla73**] for the original version of (1)–(3); there is a proof of (1)–(3) in our context in 32.3 of [**Asc86a**].

From the action of $J(G, V)$ on V, $Syl_2(G_i) = \mathcal{P}(G_i, V_i)$, so all the products in (4) are FF*-offenders. Conversely if B is an FF*-offender, there is $A \in \mathcal{P}^*(G, V)$ contained in B and A is nontrivial on some V_i; so applying B.1.5.2 with V_i in the role of "W", $Aut_A(V_i) \in \mathcal{P}(Aut_G(V_i), V_i)$, and A centralizes V_k for $k \neq i$ as $Aut_A(V_i)$ is not a strong FF*-offender by (3). Thus $A \in Syl_2(G_i)$ and $B = AC_B(V_i)$. By B.1.5.6, either $C_B(V_i) = 1$ or $C_B(V_i)$ is an FF*-offender. Then A has the desired form by induction on $m(A)$. Thus (4) holds, and (5) follows from (4) and the fact that $G_i = \langle S_i^{G_i} \rangle$ for $S_i \in Syl_2(G_i)$. □

It is convenient to state the next result here, though its proof invokes a result D.2.12 in a later section; however the proof of that lemma is elementary, and in particular does not depend on the present section. This is not the first instance of interdependence between sections, and it will not be the last. We will usually not comment on such occurrences, since to do so would be too unwieldy.

LEMMA B.1.9. *Set* $\mathcal{P} := \mathcal{P}(G,V)$, $\mathcal{P}^* := \mathcal{P}^*(G,V)$, $J_{\mathcal{P}} := J(G,V)$, *and* $J_{\mathcal{P}^*} := J_{\mathcal{P}^*}(G)$. *Assume that* $O_2(G) = 1$ *and* $F := [F(G), J_{\mathcal{P}^*}] \neq 1$. *Then*

(1) $J_{\mathcal{P}} = G_1 \times \cdots \times G_s \times J_{\mathcal{P}}(C_G(F))$, *where* $G_i \cong L_2(2)$.

(2) $F = F_1 \times \cdots \times F_s$ *with* $F_i := O(G_i) \cong \mathbf{Z}_3$, *and* $F(G) = F \times C_{F(G)}(J_{\mathcal{P}^*})$.

(3) $[V,F] = V_1 \oplus \cdots \oplus V_s$, *where* $V_i := [V,F_i] = [V,G_i]$ *is of rank* 2.

(4) We have

$$\mathcal{P}^*_f := \{A \in \mathcal{P}^* : [F(G),A] \neq 1\} = \bigcup_{i=1}^{s} Syl_2(G_i).$$

(5) Suppose $A \in \mathcal{P}^*$ *and* $K = [K,A] \leq G$. *Then* $K \leq C_{J_{\mathcal{P}^*}}(F(G))$.

PROOF. Recall that both $\mathcal{P}$ and $\mathcal{P}^*$ are stable in $\mathcal{P}$, so that $J_{\mathcal{P}}$ and $J^*_{\mathcal{P}}$ are normal in G; in particular $[F(G), J_{\mathcal{P}^*}] = F$ is also normal in G.

Since $F \neq 1$ by hypothesis, $\mathcal{P}^*_f \neq \emptyset$; for $A \in \mathcal{P}^*_f$ let $F_A := [F(G),A]$. Fix $A \in \mathcal{P}^*_f$ and set $B := C_A(F_A)$. Now F_A is faithful on $C_V(B)$ by the Thompson $A \times B$ Lemma, so $0 \neq [C_V(B), F_A] =: W = [W, F_A]$. Therefore since $1 \neq F_A = [F_A, A]$, also $[W,A] \neq 0$. Then by B.1.5.1, $Aut_A(W) \in \mathcal{P}(Aut_G(W),W)$. Now $H := F_A A$ is solvable, and $H = \langle A^H \rangle$, so $H = J(H,W)$. Thus the action of H on W is described in B.1.8, and in particular $q(Aut_H(W),W) = 1$. Then as $A \in \mathcal{P}^*(G,V)$ by hypothesis, it follows from B.1.5.2 that A is faithful on W, so that $Aut_A(W) = A \in \mathcal{P}^*(Aut_H(W),W)$ and $V = W + C_V(A)$. As $W \leq C_V(B)$, we conclude $B = 1$; so $W = [V,F_A]$, and H is faithful on W. using B.1.8. As $A \in \mathcal{P}^*(H,W)$, B.1.8 says F_A is of order 3 and W is of rank 2. As $V = W + C_V(A)$ with $W = [V,F_A]$, $W = [V,H]$.

We have shown that for $A \in \mathcal{P}^*_f$, F_A is a member of the collection $\mathcal{X}$ of *all* subgroups X of $F(G)$ of order 3 satisfying $m([V,X]) = 2$. Thus $F \leq \langle \mathcal{X} \rangle =: Y$. By parts (1) and (2) of D.2.12, $Y = X_1 \times \cdots \times X_r$ is the direct product of the members of $\mathcal{X}$, and

$$[V,Y] = \prod_{i=1}^{r} [V,X_i],$$

so the factorizations of F and $[V,F]$ in (2) and (3) follow.

Let $\mathcal{P}^*_i := \{A \in \mathcal{P}^*_f : F_i = F_A\}$ and $G_i := \langle \mathcal{P}^*_i \rangle$. For $A \in \mathcal{P}^*_i$, $F(G) = F_i C_{F(G)}(A)$ by Coprime Action; then it follows from the factorization of F and the definition of $\mathcal{P}^*_f$ that $F(G) = FC_{F(G)}(J_{\mathcal{P}^*})$. Since the two factors are normal in $F(G)$ and intersect trivially, the product is direct, completing the proof of (2).

Let $A := \langle a \rangle$. As $F_i A = GL(V_i)$, for $C := \langle c \rangle \in \mathcal{P}^*_i$, there exists $g \in F_i$ with $a^g c \in C_G(V_i) \cap C_G(V/V_i) \leq O_2(G) = 1$. Thus $C \leq F_i A$, so $G_i = F_i A$, establishing (4), and visibly $\langle \mathcal{P}^*_f \rangle = G_1 \times \cdots \times G_s$.

Now consider any $B \in \mathcal{P}$; by B.1.5.1, either $[V,F,B] = 0$ or

$$Aut_B([V,F]) \in \mathcal{P}(Aut_G([V,F]),[V,F]).$$

In the former case B centralizes each V_i, and in the latter B.1.8 shows that B acts on V_i. Thus $J_{\mathcal{P}}$ acts on G_i, so as $G_i = Aut(G_i)$, (1) follows.

Finally assume the hypotheses of (5). If $A \leq C_G(F(G))$, then $K = [K, A] \leq C_{J_{\mathcal{P}^*}}(F(G))$ and we are done. Thus we may assume $A \in \mathcal{P}_f^*$, and hence by (4), $A \leq G_i$ for some i, and $K = [K, A] \leq [J_{\mathcal{P}^*}, A] = F_i$. But by (2), $F_i \leq Z(F(G))$, so again $K \leq C_{J_{\mathcal{P}^*}}(F(G))$, completing the proof of (5). □

LEMMA B.1.10. *Assume that* $O_2(G) = 1$, *and set* $\mathcal{P} := \mathcal{P}(G, V)$ *and* $J_{\mathcal{P}} := J_{\mathcal{P}}(G) = J(G, V)$. *Assume* $K \leq G$ *such that* $K = [K, A]$ *for some* $A \in \mathcal{P}$ *with* K *either quasisimple or of order* 3. *Then* $K \leq C_{J_{\mathcal{P}}}(F(G))$.

PROOF. A simplification in the following proof was suggested by John Thompson.

As $K = [K, A]$, $K \leq \langle A^K \rangle \leq J_{\mathcal{P}}$. Thus it remains to prove that K centralizes $F(G)$; we will use induction on $|G|$, so assume G, K, A is a counterexample with G of minimal order. Then $[F(G), K] \neq 1$, so

$$\text{if } K \text{ is quasisimple, then } C_K(F(G)) \leq Z(K). \quad (*)$$

Set $H := F(G)KA$, $Q := O_2(H)$, and $U := C_V(Q)$. Then $Q \leq C_H(U)$, and we claim in fact that $Q = C_H(U)$: For $F(G)$ is of odd order as $O_2(G) = 1$, so a Sylow 2-subgroup of KA is Sylow in H, and hence contains Q. Then $[K, Q] \leq K \cap Q \leq O_2(Z(K))$, so that Q centralizes K by Coprime Action, while $[Q, F(G)] \leq Q \cap F(G) = 1$. Thus Q centralizes $F(G)K = O^2(H)$, and then by the Thompson $A \times B$-Lemma, elements of H of odd order act faithfully on $C_V(Q) = U$. Thus $C_H(U)$ is a 2-group, completing the proof of the claim.

Set $H^* := H/C_H(U)$. We verify the inductive hypothesis for H^*, K^*, A^*: First $O_2(H^*) = 1$, since $C_H(U) = Q = O_2(H)$ by the claim. Next $F(G) \cong F(G)^* \leq F(H^*)$, so $[K^*, F(H^*)] \neq 1$. Further $K^* \cong K$ if $|K| = 3$, while if K is quasisimple, then K^* is also quasisimple, and $K^* = [K^*, A^*]$ since $K = [K, A]$. Finally $A^* \in \mathcal{P}(H^*, U)$ by B.1.5.1, completing the verification. Thus H^*, K^*, A^* is a counterexample to the lemma, so by minimality of $|G|$, $|H^*| = |G|$, and therefore $Q = 1$ and $G = KAF(G)$.

If $K \cong \mathbf{Z}_3$ then G is solvable, so $F^*(G) = F(G)$. If K is quasisimple then as $G = KAF(G)$ and $C_K(F(G)) \leq Z(K)$ by (*), again $F^*(G) = F(G)$.

Thus $F^*(G) = F(G)$ is of odd order, so each member of $\mathcal{P}^* := \mathcal{P}^*(G, V)$ is faithful on $F(G)$, and hence the hypotheses of B.1.9 are satisfied; adopt the notation of that lemma, with $J_{\mathcal{P}}^* := J_{\mathcal{P}}^*(G)$. In particular $F := [F(G), J_{\mathcal{P}}^*] \neq 1$, but as $K \leq O^2(J_{\mathcal{P}})$, K centralizes F by B.1.9.1. Set $E := C_{F(G)}(J_{\mathcal{P}^*})$; by B.1.9.2, $F(G) = F \times E$, so that $E \geq [E, K] \neq 1$. Arguing by induction as before, now with E in the role of "$F(G)$", we conclude $G = KAE$. Thus $F \leq O^2(G) = KE$.

As KA contains a Sylow 2-subgroup of $KAE = G$, there is $B \in \mathcal{P}^*$ with $B \leq KA$. If $K = [K, B]$, then $K \leq J_{\mathcal{P}^*}$, so K centralizes $F(G)$ by B.1.9.5, contrary to our assumption that $[F(G), K] \neq 1$. Thus B centralizes K. But by definition of E, B centralizes E, so B centralizes $KE \geq FE = F(G)$, whereas we saw earlier that each member of $\mathcal{P}^*$ is faithful on $F(G)$. This contradiction completes the proof. □

LEMMA B.1.11. *Assume that* L *is a quasisimple subgroup of* G *such that* $L = [L, A]$ *for some* $A \in \mathcal{P}(G, V)$. *Then there is a subgroup* H *of* G *containing* L, *and* $I \in Irr_+(L, V)$ *with* $H \leq N_G(I)$, *such that* $\tilde{I} := I/C_I(L)$ *is an FF-module for* $Aut_H(\tilde{I})$ *with* $C_H(\tilde{I}) \leq C_H(L)$ *and* L *irreducible on* $\tilde{I}$.

PROOF. Without loss $G = LA$. Set $Q := O_2(G)$ and $U := C_V(Q)$. As in the proof of B.1.10, $C_G(U) = Q$ and $C_L(U) = O_2(L)$. Thus as $L = [L, A]$, A is nontrivial on U, so $Aut_A(U) \in \mathcal{P}(Aut_G(U), U)$ by B.1.5.1. Therefore replacing G, V by $Aut_G(U)$,U, we may assume that $O_2(G) = 1$. Thus as $G = LA$, $L = F^*(G)$, so each member of $\mathcal{P}^*(G, V)$ is faithful on L and hence we may take $A \in \mathcal{P}^*(G, V)$. Choose $I \in Irr_+(L, V)$. By B.1.5.3, A acts on I, so G acts on I. As L is quasisimple and $I \in Irr_+(L, V)$, $C_L(\tilde{I}) \leq Z(L)$ and L is irreducible on $\tilde{I}$. As $F^*(G) = L$, $C_G(\tilde{I}) = C_L(\tilde{I})$. By B.1.5.8, $\tilde{I}$ is an FF-module. □

LEMMA B.1.12. *Assume that $T \in Syl_2(G)$, $\mathbf{Z}_3 \cong K \leq G$ with $K = [K, J(T, V)]$, and $K \trianglelefteq\trianglelefteq H := \langle T, K\rangle$. Set $K_0 := \langle K^T\rangle$, $M := J(T, V)K_0$, and $\bar{M} := M/O_2(M)$. Then*

(1) $K_0 = K_1 \times \cdots \times K_n$ where $\{K_1, \ldots, K_n\} = K^T$.

(2) $\bar{M} = \bar{M}_1 \times \cdots \times \bar{M}_n$ with $\bar{K}_i = O(\bar{M}_i)$, $\bar{M}_i \cong L_2(2)$, and T transitively permutes $\{\bar{M}_1, \ldots, \bar{M}_n\}$.

PROOF. Without loss, $G = H$. Then K is subnormal in G by hypothesis, so that $K_0 = O_3(G) = O^2(G)$. Let $Q := O_2(G)$ and $U := C_V(Q)$. As in the proof of B.1.10, $C_G(U) = Q$. Thus setting $G^+ := G/Q$, $O_2(G^+) = 1$ and $F^*(G^+) = F(G^+) = K_0^+ \cong K_0$. Further by B.1.5.1,

$$\mathcal{P}^+ := \{A^+ : A \in \mathcal{P}(G, V) \text{ and } A^+ \neq 1\}$$

is a subset of $\mathcal{P}(G^+, U)$, and stable since $\mathcal{P}(G, V)$ is stable. Thus as G is solvable, by B.1.8, $\langle P^+\rangle = G_1^+ \times \cdots \times G_n^+$ with $G_i^+ \cong L_2(2)$. Therefore as $K = [K, J(T, V)]$, we may take $K^+ = O(G_1^+)$. By B.1.8, G^+ permutes the G_i^+, so $K^{+T} = \{O(G_i^+) : 1 \leq i \leq n\}$. Then $M^+ = \langle \mathcal{P}^+\rangle \cong \bar{M}$ as $K_0^+ \cong K_0$, so the lemma holds. □

LEMMA B.1.13. *Assume that G is a $\mathcal{K}$-group with $O_2(G) = 1$, set $\mathcal{P} := \mathcal{P}(G, V)$, and let $T \in Syl_2(G)$. Assume $K \leq G$ such that K is either quasisimple or of order 3, $K \trianglelefteq \trianglelefteq \langle K, T\rangle$, and $K = [K, J_{\mathcal{P}}(T)]$. Then one of the following holds:*

(1) $K \cong \mathbf{Z}_3$ and $K \leq Z(F(G))$.

(2) K is a component of G, and K is of Lie type, an alternating group, or $\hat{A}_6$.

(3) There is a component L of G with $K \leq L$ and $L \cong A_m$ for some m. Further either $K \cong A_n$ with $n \leq m$, or $K \cong L_3(2)$ and $m = 7$.

PROOF. By hypothesis there is $A \in \mathcal{P}$ acting nontrivially on K, so since K is quasisimple or of order 3, $K = [K, A]$. Thus K centralizes $F(G)$ by B.1.10. Suppose K also centralizes $E(G)$. Then $K \leq C_G(F^*(G)) = Z(F(G))$, so (1) holds. Thus we may assume that $[E(G), K] \neq 1$.

By B.1.5, $J_{\mathcal{P}}(G)$ acts on each component of G, so as $K = [K, J_{\mathcal{P}}(T)]$, K does also. Thus there exists a component L of G with $L = [L, K]$. As $J_{\mathcal{P}}(T)$ acts on L and $K = [K, J_{\mathcal{P}}(T)]$, also $L = [L, J_{\mathcal{P}}(T)]$. Then by B.1.11, there is an irreducible module U for L such that U is an FF-module for $N_{GL(U)}(Aut_L(U))$. To give a self-contained proof of the special case of the present result which is used in this work: under our SQTK-hypothesis, we may apply Theorem B.4.2 to determine $L/Z(L)$. To prove the result in general, as G is a $\mathcal{K}$-group, this determination follows from Table 2 of Guralnick-Malle [**GM04**]. Thus it follows that either L is of Lie type and characteristic 2 or $L/Z(L)$ is an alternating group. Further if $L/Z(L)$ is alternating, then as $O_2(G) = 1$, I.1.3 says that L is an alternating group or $\hat{A}_6$ or $\hat{A}_7$. In the last case as L is faithful on V, $Z(L)$ is faithful on some member of $Irr_+(L, V)$, so from the proof of B.1.11, we can choose L faithful on U; then B.4.2 supplies a contradiction. Thus if $L/Z(L)$ is an alternating group, then L is an alternating group or $\hat{A}_6$.

Let $H := \langle K, T\rangle$ and observe that $T_L := T \cap L \in Syl_2(L)$ and $H_L := H \cap L$ is subnormal in H. If K is quasisimple then by 31.4 in [**Asc86a**], either K is a component of H_L, or K centralizes H_L. We next establish the similar claim that when K is of order 3, either $K \leq H_L$ or K centralizes H_L: We apply B.1.12, and choose notation as in that lemma, so that $K_0 := \langle K^T\rangle = K_1 \times \cdots K_n$ with $K = K_1$. Set $M := K_0 J(T)$. As $J_{\mathcal{P}}(G)$ acts on L, so does M, so M acts on H_L and of course H_L acts on the normal subgroup M of $K_0 T = H$. Thus $[M, H_L] \leq M \cap H_L$. Suppose first that H_L does not centralize K. Then $H_L \not\leq O_2(H)K_0$, so as $H = K_0 T$, there is $t \in T_L$ with $[K, t] \neq 1$. If $K^t \neq K$, then $[t, \bar{M}]$ contains $\bar{s} := \bar{s}_1 \cdots \bar{s}_r$ for suitable $r \leq n$, where $\bar{s}_i$ is the involution in $\overline{T \cap M_i}$. But the preimage s of $\bar{s}$ lies in L, so $K = [K, s] \leq L$, and our claim holds in this case. On the other hand if $K = K^t$, then $K = [K, t] \leq L$, completing the proof of the claim in the remaining case.

Now if K centralizes H_L, then K centralizes T_L; this is impossible, as in a group of Lie type in characteristic 2 or an alternating group, the centralizer of a Sylow 2-group is a 2-group. Thus $K \leq H_L \leq L$.

By an earlier reduction, $L/Z(L)$ is either of Lie type and characteristic 2 or A_m for some m. Thus if $L = K$ then (2) holds, so we may assume K is proper in L. Thus H_L is proper in L, and as $K \trianglelefteq\trianglelefteq H_L$, $F^*(H_L) \neq O_2(H_L)$.

Suppose first that L is of Lie type and characteristic 2. Then $F^*(X) = O_2(X)$ for each proper subgroup X of L containing T_L, contradicting $F^*(H_L) \neq O_2(H_L)$. Thus $L/Z(L) \cong A_m$. From the structure of the overgroups of T_L in L (we may apply A.3.12 under our SQTK-hypothesis), this forces $K \cong A_n$ for $n \leq m$, or $K \cong L_3(2)$ and $m = 7$. Thus (3) holds, completing the proof of the lemma. □

B.2. Basic Failure of Factorization

What we now call failure of factorization arguments originated in work of Thompson, particularly in [**Tho68**, 5.53]; the basic observations below are mostly due to him. For general background, we will quote from the more modern treatments in section 32 of [**Asc86a**], and [**GLS96**, Sec 26].

Many of the results in this section hold for arbitrary p, but our applications will be for $p = 2$. Thus throughout this section, we assume:

HYPOTHESIS B.2.1. *G is a finite group, V a normal elementary abelian 2-subgroup of G, and $G^* := G/C_G(V)$.*

We refer to V as an *internal module* for G. We will study conditions guaranteeing that such a module is an FF-module for G^* in the sense of Definition B.1.1 of the previous section.

We will see that these connections involve the Thompson subgroup and related subgroups:

DEFINITION B.2.2. Recall that for $H \leq G$ and j a nonnegative integer, $\mathcal{A}_j(H)$ is the set of all elementary abelian 2-subgroups of H of rank $m_2(H) - j$, and $J_j(H) := \langle \mathcal{A}_j(H)\rangle$. Write $\mathcal{A}(H)$ for $\mathcal{A}_0(H)$, and $J(H)$ for $J_0(H)$. The group $J(H)$ is called the *Thompson subgroup* of H. The *Baumann subgroup* of a 2-group

H is

$$\mathrm{Baum}(H) := C_H(\Omega_1(Z(J(H)))).$$

Visibly $J_i(H)$ and $\mathrm{Baum}(H)$ are characteristic subgroups of H. We begin with some well-known elementary properties of the Thompson subgroup, its variants, and the Baumann subgroup.

LEMMA B.2.3. *Let S be a 2-group and $0 \le i \le j < m_2(S)$. Then*

(1) $J_i(S) \le J_j(S)$.

(2) $\Omega_1(Z(J_j(S))) \le \Omega_1(C_S(J_i(S))) = \Omega_1(Z(J_i(S)))$.

(3) If $J_i(S) \le R \le S$, then $J_i(S) = J_i(R)$.

(4) If $Baum(S) \le R \le S$, then $Baum(S) = Baum(R)$.

(5) If $J(S) \le C_S(U)$ for some elementary abelian subgroup U of S, then $Baum(S) = Baum(C_S(U))$.

(6) For $A \in \mathcal{A}(S)$, $A = \Omega_1(C_S(A))$.

(7) We have

$$\Omega_1(C_S(J(S))) = \Omega_1(Z(J(S))) = \bigcap_{A \in \mathcal{A}(S)} A.$$

PROOF. For each $A \in \mathcal{A}_i(S)$, $A = \langle \mathcal{A}_j(S) \cap A \rangle \le J_j(S)$, so (1) holds. By (1), $C_S(J_j(S)) \le C_S(J_i(S))$. Let D be of order p in $\Omega_1(C_S(J_i(S)))$. Then $AD \in \mathcal{A}_k(S)$ for some $k \le i$, so by (1), $AD \le J_i(S)$ and then $D \le Z(J_i(S))$. It follows that $\Omega_1(C_S(J_i(S)) \le \Omega_1(Z(J_i(S)))$, completing the proof of (2).

Under the hypotheses of (3), $J(S) \le J_i(S) \le R$ by (1), so $m_2(R) = m_2(S)$. Then it follows that $\mathcal{A}_i(S) = \mathcal{A}_i(R)$, so (3) holds. Similarly under the hypotheses of (4), $J(S) \le \mathrm{Baum}(S) \le R$, so $J(S) = J(R)$ by (3), and hence $E := \Omega_1(Z(J(S))) = \Omega_1(Z(J(R))) =: F$. Then as $\mathrm{Baum}(S) \le R$ by hypothesis, we conclude

$$\mathrm{Baum}(S) = C_S(E) = C_S(F) = C_R(F) = \mathrm{Baum}(R),$$

proving (4).

Assume the hypotheses of (5). Then $U \le \Omega_1(C_S(J(S))$, so $U \le E$ by (2). Thus $\mathrm{Baum}(S) \le C_S(U)$, and therefore (5) follows from (4).

Let $A \in \mathcal{A}(S)$. Then maximality of $m(A)$ shows for D of order p in $C_S(A)$ that $DA = A$, so that $D \le A$. Hence (6) holds and

$$\Omega_1(C_S(J(S)) \le \bigcap_{A \in \mathcal{A}(S)} A \le \Omega_1(Z(J(S))) \le \Omega_1(C_S(J(S))),$$

establishing (7). □

LEMMA B.2.4. *Let S be a 2-subgroup of G containing V, j a nonnegative integer, and $A \in \mathcal{A}_j(S)$ with $A^* \ne 1$. Then*

(1) $m(A^*) \ge m(V/C_V(A)) - j$. In particular if $j = 0$, then $r_{A,V} \le 1$, so that V is an FF-module for G.

(2) If equality holds in (1), then $C_A(V)V \in \mathcal{A}(S)$.

(3) If $A \in \mathcal{A}(S)$ and $m(V/C_V(A)) = m(A^*)$, then $C_A(V)V \in \mathcal{A}(S)$, $m_2(C_S(V)) = m_2(S)$, $\mathcal{A}(C_S(V)) \subseteq \mathcal{A}(S)$, and $V \le \Omega_1(Z(J(C_S(V)))) \ge \Omega_1(Z(J(S)))$.

PROOF. Let $B := C_A(V)V$. Then

$$m(B) = m(C_A(V)) + m(V/(V \cap A)) \geq m(C_A(V)) + m(V/C_V(A)). \tag{!}$$

Further as $A \in \mathcal{A}_j(S)$,

$$m(B) \leq m(A) + j, \text{ with equality only if } B \in \mathcal{A}(S). \tag{!!}$$

Therefore by (!) and (!!),

$$m(A^*) = m(A) - m(C_A(V)) \geq m(A) - (m(B) - m(V/C_V(A)))$$

$$= m(V/C_V(A)) - (m(B) - m(A)) \geq m(V/C_V(A)) - j,$$

with equality only if $B \in \mathcal{A}(S)$. Hence (1) and (2) hold.

Assume the hypotheses of (3). Then by (2), $B = C_A(V)V \in \mathcal{A}(S) \cap C_S(V)$, so the first three statements in (3) hold. Next V centralizes, and hence is contained in, each member of $\mathcal{A}(C_S(V))$, so that $V \leq \Omega_1(Z(J(C_S(V)))) =: E$. Similarly $F := \Omega_1(Z(J(S))) \leq B \leq C_S(V)$, so $F \leq E$. □

Recall from the previous section B.1 the discussion of FF-modules, and in particular, the definitions B.1.3 and B.1.6 of $\mathcal{P}(G^*, V)$ and its stable subsets. The following observation, which appears as 32.2 in [**Asc86a**], begins to show the connection between those notions for an internal module V, and the Thompson subgroup:

LEMMA B.2.5. *Define*

$$\mathcal{P}_G := \{A^* : A \in \mathcal{A}(G) \quad \textit{and} \quad A^* \neq 1\}.$$

Then $\mathcal{P}_G$ is a stable subset of $\mathcal{P}(G^*, V)$.

DEFINITION B.2.6. Members A of $\mathcal{A}(G)$ with $A^* \neq 1$ are called *FF-offenders* on V. More generally for $V \leq H \leq G$, we may say A is an FF-offender on V *relative to H* if $A \in \mathcal{A}(H)$ and $A^* \neq 1$.

Notice that if A is an FF-offender on V, then by B.2.5, A^* is an FF*-offender on V in the sense of Definition B.1.3 of the previous section. An FF-offender A is a *strong* FF-offender if A^* is a strong FF*-offender as in that previous definition: that is if $r_{A^*,V} < 1$.

PROPOSITION B.2.7. *Let $T \in Syl_2(G)$. Then either*

(1) $J(G) \leq C_G(V)$ and $G = N_G(J(T))C_G(V)$, or

(2) $J(G) \not\leq C_G(V)$, V is an FF-module for G^*,

$$\mathcal{P}_G := \{A^* : A \in \mathcal{A}(G) \textit{ and } A^* \neq 1\} \textit{ is a stable subset of } \mathcal{P}(G^*, V),$$

and $J(G)^* = J_{\mathcal{P}_G}(G^*) \leq J(G^*, V)$.

PROOF. As $V \trianglelefteq G$, $C_G(V) \trianglelefteq G$, so $S := C_T(V) \in Syl_2(C_G(V))$. Then by a Frattini Argument, $G = C_G(V)N_G(S) = C_G(V)N_G(J(S))$. Further if $J(T) \leq S$, then $J(S) = J(T)$ by B.2.3.1, so $G = C_G(V)N_G(J(T))$. In particular (1) holds in this case. On the other hand if $J(T) \not\leq S$, then the set $\mathcal{P}_G$ of Lemma B.2.5 is nonempty and stable in $\mathcal{P}(G^*, V)$, so V is an FF-module for G^* by B.2.5 and B.1.4.5. Thus the proposition is established. □

REMARK B.2.8. Notice that for A as in B.2.7.2, by the definition B.1.6 of stability, if $B^* \lesssim A^*$ then there exists $D \in \mathcal{A}(G)$ with $D^* = B^*$. Indeed from the proof of B.2.5 in 32.2 in [**Asc86a**], $B_0C_V(B_0) \in \mathcal{A}(G)$, where B_0 is the preimage in A of B^*. Notice in addition that if A is not a strong FF-offender, then $m(A^*) + m(C_V(A^*)) = m(V)$. But if $1 \neq B^* \leq A^*$ with $r_{B^*,V} \leq 1$, then $m(B^*) + m(C_V(B^*)) \geq m(V)$, so $B^* \lesssim A^*$.

Therefore:

LEMMA B.2.9. *If A is an FF-offender but not a strong FF-offender, then*

(1) For any FF-offender B^* contained in A^*, there exists an FF-offender D with $D^* = B^*$.*

(2) For each $1 \neq B^* \leq A^*$ with $r_{B^*,V} \leq 1$, there exists an FF-offender D with $D^* = B^*$.

Next we see that lemma B.2.5 on FF-offenders in G relative to T, can be extended to FF-offenders relative to R for suitable subgroups R of T:

LEMMA B.2.10. *Assume $O_2(G) \leq R \leq T \in Syl_2(G)$, and set $S := Baum(R)$. Then*

(1) If $O_2(G) = C_R(V)$, then either

(a) $J(R) = J(O_2(G))$ and $S = Baum(O_2(G))$ are normal in G, or

(b) $\mathcal{P}_{R,G} := \{1 \neq A^* : A^g \in \mathcal{A}(R)$ for some $g \in G\}$ is a stable subset of $\mathcal{P}(G^*, V)$, so $J_{\mathcal{P}_{R,G}} \leq J(G^*, V)$.

(2) Assume $O_2(G^*) = 1$, and $G = LT$ with $[V, L] \neq 1$, where either $L \in \mathcal{C}(G)$ with $L/O_2(L)$ quasisimple, or $L/O_2(L) \cong \mathbf{Z}_3$. Then $C_G(V)$ is 2-closed, so that $O_2(G) = C_R(V)$. Further $L^* = F^*(G^*)$, and either $S = Baum(O_2(G))$ or $L^* = F^*(J_{\mathcal{P}_{R,G}}(G^*))$.

PROOF. Let $Q := O_2(G)$, and suppose first that $Q = C_R(V)$. If $[V, J(R)] = 1$, then $J(R) = J(Q)$ and $S = \mathrm{Baum}(Q)$ by parts (3) and (5) of B.2.3, so (1a) holds. Therefore we may assume that $[V, J(R)] \neq 1$, so applying B.2.7 to R in the role of "G":

$$\mathcal{P}_R := \{A^* : A \in \mathcal{A}(R) \text{ and } A^* \neq 1\} \text{ is a stable subset of } \mathcal{P}(R^*, V).$$

It follows by taking the union of the G-conjugates of $\mathcal{P}_R$ that $\mathcal{P}_{R,G}$ is G-stable, so (1b) holds in this case.

Next assume the hypotheses of (2). As $O_2(G^*) = 1$, $Q \leq C_T(V)$. Further $L/O_2(L)$ is quasisimple or $\mathbf{Z}_3$, $G = LT$, and $[V, L] \neq 1$, so we conclude $Q = C_T(V) \in Syl_2(C_G(V))$, and hence $C_G(V)$ is 2-closed. Then as $G = LT$ and $O_2(G^*) = 1$, $L^* = F^*(G^*)$. As $C_T(V) = Q \leq R$, $Q = C_R(V)$, so we have the hypotheses of (1). If $[V, J(R)] = 1$, then (1a) holds—so $S = \mathrm{Baum}(O_2(G))$, and (2) holds. Otherwise $[V, J(R)] \neq 1$, and (1b) holds; so as $F^*(G)$ is quasisimple or $\mathbf{Z}_3$, $L^* = F^*(J_{\mathcal{P}_{R,G}}(G^*))$, and again (2) holds. □

The condition that V is an FF-module and other related conditions place strong restrictions on G^* and it action on V—but only when $O_2(G^*) = 1$; we use the following standard terminology:

DEFINITION B.2.11. We say that V is *2-reduced* if $O_2(G/C_G(V)) = 1$, and we write $\mathcal{R}_2(G)$ for the set of all 2-reduced normal elementary abelian 2-subgroups of G.

Notice that if $V \in \mathcal{R}_2(G)$, then as $O_2(G)^* \leq O_2(G^*) = 1$, we have $O_2(G) \leq C_G(V)$, so $V \leq \Omega_1(Z(O_2(G)))$. Also it is a well-known elementary fact (26.22 and 26.23 in [**GLS96**]) that:

LEMMA B.2.12. *The product of any two members of $\mathcal{R}_2(G)$ is itself in $\mathcal{R}_2(G)$—so the product $R_2(G)$ of all the members of $\mathcal{R}_2(G)$ is the unique maximal member of $\mathcal{R}_2(G)$.*

It is also well known that if $O_2(G) \neq 1$, then $R_2(G) \neq 1$:

LEMMA B.2.13. *Assume $T \in Syl_2(G)$ and $O_2(G) \neq 1$. Then for any $1 \neq Z \leq O_2(G) \cap \Omega_1(Z(T))$, $W := \langle Z^G \rangle$ and $[W, G]$ are in $\mathcal{R}_2(G)$, and $W = [W, G]C_W(G)$.*

PROOF. Set $H := C_G(W)$, and let D be the preimage of $O_2(G/H)$. As D/H is a 2-group, $D = H(D \cap T) \leq C_G(Z)$; then as $D \trianglelefteq G$, D centralizes $\langle Z^G \rangle = W$, so that $W \in \mathcal{R}_2(G)$. Further $W = [W, G]Z = [W, G]C_W(G)$ using Gaschütz's Theorem A.1.39. Hence $C_G(W) = C_G([W, G])$, so that $[W, G] \in \mathcal{R}_2(G)$ also. □

We will be concerned most often with the situation where $F^*(G) = O_2(G)$. In that case, as $C_G(F^*(G)) \leq F^*(G)$, it follows that for $T \in Syl_2(G)$, $Z(T) \leq O_2(G)$; therefore by B.2.13:

LEMMA B.2.14. *Assume $F^*(G) = O_2(G)$, and let $T \in Syl_2(G)$, $1 \neq Z \leq \Omega_1(Z(T))$, and $U := \langle Z^G \rangle$. Then*

(1) U and $[U, G]$ are in $\mathcal{R}_2(G)$, and $U = [U, G]C_U(G)$.
(2) $U \leq \Omega_1(Z(O_2(G)))$.
(3) $O_2(G/C_G(U)) = 1 = O_2(G/C_G([U, G]))$.

When $F^*(G) = O_2(G)$, B.2.13 provides a 2-reduced internal module to which we can apply B.2.7. The first alternative in that lemma leads to Thompson Factorization:

LEMMA B.2.15 (Thompson Factorization Lemma). *Assume $F^*(G) = O_2(G)$, and let $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, and $U := \langle Z^G \rangle$. Then either*

(1) $J(T) \not\leq C_G(U)$, U is an FF-module for $G/C_G(U)$, and $O_2(G/C_G(U)) = 1$, or

(2) $J(T) \leq C_G(U)$, and $G = C_G(Z)N_G(J(T))$.

PROOF. By B.2.14, $U \in \mathcal{R}_2(G)$, so U is elementary abelian and normal in G with $O_2(G/C_G(U)) = 1$. Then by B.2.7, either $J(T) \not\leq C_G(U)$ and so (1) holds, or $J(T) \leq C_G(U)$ and $G = C_G(U)N_G(J(T))$. In the latter case as $Z \leq U$, (2) holds. □

The factorization $G = C_G(Z)N_G(J(T))$ in the second case of the Thompson Factorization Lemma B.2.15.2 is called the "Thompson factorization of G". In the first case of the lemma, we say that "Thompson factorization fails".

Classical Thompson factorization describes conditions for a *solvable* group G with Sylow 2-subgroup T and $F^*(G) = O_2(G)$ to admit a Thompson factorization. Thompson's result can be stated in the form of the following lemma. This statement and its original proof are due to Glauberman in [**Gla73**]. As in standard elementary references such as 32.5 in [**Asc86a**], the result follows easily from Thompson Factorization B.2.15 and our earlier representation-theoretic version B.1.8. The final two remarks follow from B.2.4.3.

THEOREM B.2.16 (Thompson Factorization for Solvable Groups). *Assume G is solvable with $F^*(G) = O_2(G)$, and let $T \in Syl_2(G)$ and $Z := \Omega_1(Z(T))$. Then $V := \langle Z^G \rangle \in \mathcal{R}_2(G)$, and either*

(1) $J(T) \leq C_G(V)$ and $G = N_G(J(T))C_G(\Omega_1(Z(T)))$, or

(2) $J(T) \not\leq C_G(V)$, $J(T)^*$ is Sylow in $L^* := J(G)^*$, $L^* = L_1^* \times \cdots \times L_n^*$, $V = C_V(L^*) \oplus V_1 \oplus \cdots \oplus V_n$, where $V_i := [L_i^*, V]$ is the natural module for $L_i^* \cong L_2(2)$, and G^* permutes the sets $\{L_1^*, \ldots, L_n^*\}$ and $\{V_1, \ldots, V_n\}$.

In any case, $m_2(C_T(V)) = m_2(T)$ and $J(C_T(V)) \leq J(T)$.

While we are often able to use Thompson factorization on solvable groups, more frequently we wish to factorize a non-solvable group G. Hence we need to know the structure of G^* and its action on $V \in \mathcal{R}_2(G)$, for more general groups G when Thompson factorization fails. For this work, the important case is when G is an SQTK-group, where we will be able to give a fairly precise description of the possibilities for G^* and its action on V in Theorems B.5.1 and B.5.6. Certain minimal situations are particularly important in the context of pushing up; we collect various results on pushing up in chapter C of Volume I.

The next lemma was suggested by John Thompson, for use in the subsequent result.

LEMMA B.2.17. *Suppose $X_1, \cdots, X_n$ are subgroups of G, $I := \langle X_1, \cdots, X_n \rangle$, $L \leq I$, and V is an $\mathbf{F}_2G$-module such that*

$$m(V/C_V(L)) \geq \sum_{i=1}^{n} m(V/C_V(X_i)). \qquad (*)$$

Then

(i) $C_V(L) = C_V(I)$, and

(ii) $m(V/C_V(L)) = \sum_{i=1}^{n} m(V/C_V(X_i))$.

PROOF. Since $L \leq I$, we have $C_V(I) \leq C_V(L)$, so

$$m(V/C_V(I)) \geq m(V/C_V(L)). \qquad (**)$$

Since $I = \langle X_1, \cdots, X_n \rangle$,

$$C_V(I) = \bigcap_{i=1}^{n} C_V(X_i).$$

There is an injection

$$V/\bigcap_{i=1}^{n} C_V(X_i) \to \bigoplus_{i=1}^{n} V/C_V(X_i),$$

$$v + \bigcap_{i=1}^{n} C_V(X_i) \mapsto (v + C_V(X_1), v + C_V(X_2), \cdots, v + C_V(X_n)).$$

Hence

$$\sum_{i=1}^{n} m(V/C_V(X_i)) \geq m(V/C_V(I)). \qquad (+)$$

Applying (**), (*), and (+) respectively yields

$$m(V/C_V(I)) \geq m(V/C_V(L)) \geq \sum_{i=1}^{n} m(V/C_V(X_i)) \geq m(V/C_V(I)).$$

All inequalities are thus equalities, so (i) and (ii) hold. □

The next lemma is technical but important, and is used repeatedly throughout the proof of the Main Theorem. We will wait until chapter C on pushing up to try to motivate the lemma; its real utility will not become clear until we finally apply it frequently in the proof of the Main Theorem. The argument goes back to Baumann, cf. the special case in 2.11.1.4 of [**Bau76**].

LEMMA B.2.18 (Baumann's Argument). *Let $L := O^2(G)$, R a 2-subgroup of G containing V, and $S := Baum(R)$. Assume*

(a) $O_2(L) \leq C_R(V)$ and $L \leq N_G(C_R(V))$.

(b) $C_L(V) \leq O_{2,\Phi}(L)$.

(c) R^* contains no strong FF^*-offenders on V.

(d) There exist L-conjugates $X_1, \ldots, X_n$ of subgroups of R such that $X_i = J(X_i)$ with $m_2(X_i) = m_2(R)$, $L^* \leq \langle X_1^*, \ldots, X_n^* \rangle$, and

$$m(V/C_V(L^*)) \geq \sum_{i=1}^{n} m(V/C_V(X_i^*)). \qquad (*)$$

(e) $J(R) \leq I := \langle X_1, \cdots, X_n \rangle$.

For X a 2-subgroup of G, define $\alpha(X) := \Omega_1(Z(J(X)))$. Then

(1) $O_2(L) \leq S$.

(2) $\alpha(R) = C_{\alpha(R)}(L) C_V(J(R))$.

(3) If $C_R(V) \in Syl_2(C_{LR}(V))$, then $C_S(V) \in Syl_2(C_{LS}(V))$. Thus if in addition $S^* \in Syl_2(L^*S^*)$, then $S \in Syl_2(LS)$.

PROOF. We extend an argument of Baumann, from his original proof of Baumann's Lemma B.6.10; we will use this extension later when we give our proof of B.6.10.

As the lemma is a statement about LR, and LR satisfies the hypotheses of the lemma in the role of "G", replacing G by LR we may assume $G = LR$.

In the first two paragraphs of the proof below, we assume only hypotheses (a)–(d) of the lemma; later we add hypothesis (e). In particular under hypotheses (a)–(d), we prove $Q := C_R(V) \trianglelefteq G$ and $J(R)^G = J(R)^I$; we can then use these facts in the proof of B.2.19 after we establish (a)–(d) there.

First $Q \trianglelefteq R$ and L acts on Q by (a), so $Q \trianglelefteq LR = G$. In particular IQ is a subgroup of G. Also $J(R)^G = J(R)^{RL} = J(R)^L$.

By (d), $L^* \leq I^*$, so as $L = O^2(G)$, $L = O^2(I)C_L(V)$. Set $L^+ := L/O_2(L)$. Then $L^+ = O^2(I)^+ C_L(V)^+$, while by (b), $C_L(V)^+ \leq \Phi(L^+)$, so $L^+ = O^2(I)^+$. Thus $L \leq IO_2(L)$. Then as $O_2(L) \leq Q$ by (a), $L \leq IQ$, so

$$J(R)^G = J(R)^L \subseteq J(R)^{QI} = J(R)^I \subseteq J(R)^G,$$

and hence $J(R)^G = J(R)^I$.

We now assume hypothesis (e). Thus $J(R) \leq I$, so $\langle J(R)^G \rangle = \langle J(R)^I \rangle \leq I$. On the other hand by (d), there is $l_i \in L$ with $X_i^{l_i} \leq R$; where $X_i = J(X_i)$ and $m_2(X_i) = m_2(R)$. As $J(R)^G = J(R)^I$, there is $g_i \in I$ with $X_i^{g_i} = J(X_i)^{g_i} \leq J(R)$, and so $I = \langle J(R)^I \rangle$. Therefore $I = \langle J(R)^G \rangle \trianglelefteq G$. Hence $O^2(I) = O^2(IQ)$, so as $L = O^2(L) \leq IQ$, $L \leq O^2(IQ) \leq I$.

Set $W := \alpha(R)V$. By (c), R^* contains no strong FF^*-offenders on V, so by B.2.4.3, for any $A \in \mathcal{A}(R)$ with $A^* \neq 1$ we have $C_A(V)V = (A \cap Q)V \in \mathcal{A}(R) \cap Q$, so that $\mathcal{A}(Q) \subseteq \mathcal{A}(R)$ and $J(Q) \leq J(R)$. Then using the characterization of $\alpha(S)$ as an intersection in B.2.3.7,

$$\alpha(R) \leq \alpha(Q) \leq (A \cap Q)V.$$

Then

$$[\alpha(Q), A] \leq [(A \cap Q)V, A] = [V, A] \leq V,$$

and hence $[\alpha(Q), J(R)] \leq V$. Therefore as $I = \langle J(R)^G \rangle$ and $Q \trianglelefteq G$, $[\alpha(Q), I] \leq V$. Then as $\alpha(R) \leq \alpha(Q)$, $[W, I] \leq V$, and hence $G = IR$ acts on W. Observe that $W = C_W(J(R))V$ as $J(R)$ centralizes $\alpha(R)$. Thus $m(V/C_V(J(R)^*)) = m(W/C_W(J(R)))$. Similarly as $X_i^{g_i} \leq J(R)$, and g_i acts on W, $m_i := m(V/C_V(X_i^*)) = m(W/C_W(X_i))$. Define

$$U := \bigcap_{i=1}^{n} C_W(X_i).$$

Then $U = C_W(I)$, and by B.2.17 applied to V and W, $C_W(L) = C_W(I) = U$ and

$$m(V/C_V(L)) = \sum_{i=1}^{n} m_i = m(W/U).$$

Thus $W = UV$. By (e), $U = C_W(I) \leq \alpha(R)$, so as $\alpha(R) \leq W = UV$, we conclude that $\alpha(R) = U(\alpha(R) \cap V) = UC_V(J(R)) = C_W(I)C_V(J(R))$, so that (2) holds. By (a), $O_2(L) \leq C_R(V)$, so as $L \leq I$, $O_2(L)$ centralizes $C_W(I)C_V(J(R)) = \alpha(R)$. Hence as $O_2(L) \leq R$, (1) holds.

Next assume $C_R(V) \in Syl_2(C_{LR}(V))$. Then as $LS \trianglelefteq LR$, $P := C_R(V) \cap LS \in Syl_2(C_{LS}(V))$. Then for $x \in P$, x centralizes $C_V(J(R))$ as x centralizes V, while x centralizes $C_{\alpha(R)}(L)$ as $x \in LS \leq C_G(C_{\alpha(R)}(L))$. Therefore $x \in C_R(\alpha(R)) = S$ using (2), so $P \leq S$ and hence $P = C_S(V)$ is Sylow in $C_{LS}(V)$, establishing (3). □

In most of our applications of Baumann's Argument B.2.18, we can choose $X_1 := J(R)$, so that hypothesis (e) of B.2.18 is trivially satisfied. However sometimes this choice is not possible, notably in some cases of C.1.37; in those cases we will appeal to the following variant of B.2.18:

LEMMA B.2.19. *Let $L := O^2(G)$, R a 2-subgroup of G containing V, and $S := Baum(R)$. Assume*

(A) $O_2(L) \leq Q := C_R(V) \in Syl_2(C_{LR}(V))$, and $L \leq N_G(Q)$.

(B) $C_L(V) \leq O_{2,\Phi}(L)$.

(C) R^* contains no strong FF^*-offenders on V.

(D) There exist L^*-conjugates $Y_1^*, \ldots, Y_n^*$ of subgroups of R^*, such that for R_i a Sylow 2-subgroup of the preimage Y_i in LR of Y_i^*, we have $Y_i^* = J(R_i)^*$, with $m_2(R_i) = m_2(R)$, $L^* \leq \langle Y_1^*, \ldots, Y_n^* \rangle$, and

$$m(V/C_V(L^*)) \geq \sum_{i=1}^{n} m(V/C_V(Y_i^*)). \qquad (*)$$

(E) For each FF^*-offender A^* in R^*, $A^* = A_1^* \cdots A_r^*$, with A_i^* an FF^*-offender, and $A_i^* \leq Y_{j_i}^{*g_{j_i}^*} \leq R^*$ for some index j_i and $g_{j_i}^* \in L^*$.

For X a 2-subgroup of G, define $\alpha(X) := \Omega_1(Z(J(X)))$. Then

(1) $O_2(L) \leq S$.

(2) $\alpha(R) = C_{\alpha(R)}(L)C_V(J(R))$.

(3) $C_S(V) \in Syl_2(C_{SL}(V))$.

(4) *If $S^* \in Syl_2(L^*S^*)$, then $S \in Syl_2(LS)$.*

PROOF. Set $X_i := J(R_i)$ for $1 \leq i \leq n$, so that $X_i^* = Y_i^*$, and set $I := \langle X_1, \ldots, X_r \rangle$. We will verify the hypotheses (a)–(e) of B.2.18 for this family of subgroups of G. Then the lemma follows from B.2.18. Just as in the proof of B.2.18, we may assume $G = LR$.

First hypotheses (A), (B), and (C) imply hypotheses (a), (b), and (c) of B.2.18, respectively.

Let Y be the preimage in G of R^*; by (A), $R \in Syl_2(Y)$, so $Y = O^2(Y)R = (Y \cap L)R$. By (D), $L^* \leq I^*$, and there exist $g_i \in L$ with $X_i^{*g_i^*} \leq R^*$. Then $X_i^{g_i} \leq Y$, so as $Y = (Y \cap L)R$, adjusting g_i by an element of $Y \cap L$ if necessary, we may assume $X_i^{g_i} \leq R$. Thus $X_i^{g_i} = J(R \cap Y_i^{g_i})$. Since $m_2(X_i) = m_2(R_i) = m_2(R)$ by (D), $X_i^{g_i} \leq J(R)$ and hypothesis (d) of B.2.18 is satisfied. Thus to complete the proof, it remains to verify hypothesis (e) of B.2.18; that is, we must show that for each $A \in \mathcal{A}(R)$, $A \leq I$.

As we mentioned during the proof of B.2.18, since we have established (a)–(d), $Q \trianglelefteq G$ and $J(R)^G = J(R)^I$. Thus as $X_i^{g_i} \leq J(R)$, we may take $g_i \in I$, so as $X_i \leq I$, also $X_i^{g_i} \leq I$. Without loss $1 \neq X_i^*$ for each i, so again as $X_i = J(R_i)$ and $m_2(R_i) = m_2(R)$, there is $B \in \mathcal{A}(R)$ with $B^* \neq 1$. By (C) and B.2.4.3, $C_B(V)V \in \mathcal{A}(R) \cap Q$, so $m_2(Q) = m_2(R) = m_2(R_i)$. As $Q \trianglelefteq G$, Q is contained in the Sylow 2-group R_i of Y_i, so as $m_2(R_i) = m_2(Q)$, $J(Q) \leq J(R_i) = X_i \leq I$.

Thus $A \leq I$ for each $A \in \mathcal{A}(R)$ with $A \leq Q$, so we may assume $A^* \neq 1$. By B.2.10, A^* is an FF^*-offender on V, so we may choose FF^*-offenders A_i^*, $1 \leq i \leq r$ as in (E). Let B_i be the preimage in A of A_i^*; by Remark B.2.8, $D_i := B_iC_V(B_i) \in \mathcal{A}(R)$. Thus if each $D_i \leq I$, then $A \leq B_1 \cdots B_r \leq I$, as desired. Therefore we may assume that $A = D_i$ for some i. Then by (E), $A^* \leq X_j^{*g^*}$ where $j := j_i$ and $g := g_j$. Then $A \leq J(R \cap Y_j^g) = X_j^{g_j} \leq I$, contrary to $A \not\leq I$. This contradiction completes the proof. □

LEMMA B.2.20. *Assume $O_2(G) \leq R \leq T$ with $J(R) \not\leq C_R(V) = O_2(G)$. Suppose $\mathcal{P}(R^*, V)$ contains a unique FF^*-offender A^*, and $A^* = C_{R^*}(C_V(A^*))$. Then $A^* = J(R)^* = Baum(R)^*$.*

PROOF. By hypothesis there is $B \in \mathcal{A}(R)$ with $B^* \neq 1$, and $B^* \in \mathcal{P}(R^*, V)$ by B.2.7 applied to R in the role of "G". Hence $B^* = A^* = J(R)^*$ by hypothesis. Then $C_V(A^*) \leq \Omega_1(Z(J(R))))$, so $\mathrm{Baum(R)}^* \leq C_{R^*}(C_V(A^*)) = A^*$, and hence $\mathrm{Baum}(R)^* = A^*$. □

Occasionally we need to know that the preimage of an FF*-offender (that is, a member of $\mathcal{P}(G^*, V)$), contains a *unique* FF-offender in $\mathcal{A}(G)$; the following condition suffices:

LEMMA B.2.21. *Assume $C_G(V) = V$ and $A \in \mathcal{A}(G)$ such that $C_V(A) = C_V(a)$ for some $a \in A$. Then A is the unique member B of $\mathcal{A}(G)$ such that $A^* = B^*$.*

PROOF. Assume $B \in \mathcal{A}(G)$ with $A^* = B^*$. Thus $C_V(A) = C_V(B)$ and as $A, B \in \mathcal{A}(G)$, $C_V(A) = C_V(B) \leq A \cap B$. Therefore as $C_V(A) = C_V(a)$ by hypothesis, $C_{AV}(a) = AC_V(a) = AC_V(A) = A$. Now as $A^* = B^*$, there is $b \in B$ with

$a^* = b^*$. Hence as $V = C_G(V)$, $a = bv$ for some $v \in V$, and as $a^2 = b^2 = v^2 = 1$, $\langle b, v\rangle = \langle a, b\rangle$ is abelian. Thus $v \in C_V(a) = C_V(A) \leq B$, so that $a = bv \in B$. But now $B \leq C_{AV}(a) = A$, so $A = B$ as $|A| = |B|$. □

B.3. The permutation module for A_n and its FF*-offenders

In this section, we study failure of factorization for the alternating group on its natural module, primarily quoting [**Asc81a**, 2.4] as adapted for our situation.

So we set $\Omega := \{1, \ldots, n\}$ and $G := Sym(\Omega) \cong S_n$, with $L := E(G) \cong A_n$. Since we are interested in strongly quasithin groups, Theorem C (A.2.3) tells us that we may restrict attention to the cases $n = 5$, 6, 7, or 8.

Let U be the n-dimensional permutation module for G over $\mathbf{F}_2$, and U_0 the *core* of U: If we regard U as the power set 2^Ω of subsets of Ω, with addition given by the symmetric difference of sets, then U_0 is the subspace of U of subsets of even order. For $S \subseteq \Omega$, let e_S denote the subset S regarded as an element of U. In particular e_Ω is a fixed point of G, and we set $\tilde{U} := U/\langle e_\Omega\rangle$ and write $\tilde{X}$ for the image of $X \subseteq U$ in the quotient space $\tilde{U}$.

As just observed, the map

$$\begin{aligned} 2^\Omega &\to U \\ S &\mapsto e_S \end{aligned}$$

is a bijection of the power set of Ω with U. Define the *weight* of a vector $e_S \in U$ to be the order $|S|$ of the set S. By a slight abuse we extend this notion to the quotient $\tilde{U}$, by "defining" the weight of $\tilde{e}_S$ to be $|S|$; thus $\tilde{e}_S$ also has weight $n-|S|$, as $\tilde{e}_S$ and $\tilde{e}_{\Omega-S}$ have been identified.

We define the *natural module* for L as the module $\tilde{U}_0$. The following statements are elementary and well-known; see for example Exercise 6.3 in [**Asc86a**].

LEMMA B.3.1. *(1) L is irreducible on its natural module $\tilde{U}_0$.*

(2) If n is odd, then $U = U_0 \oplus \langle e_\Omega\rangle$, so the natural module $\tilde{U}_0$ is of rank $n-1$ and isomorphic to U_0.

(3) If n is even then

$$\langle e_\Omega\rangle < U_0 < U$$

is the unique L-chief series for G, so the natural module is of rank $n-2$.

It is well known that the module U is an FF-module; for example a transposition induces a transvection on U. We next quote a result which gives a complete description of the set $\mathcal{P}(G, V)$ of offenders (as in B.2.7) for the various sections of U. The behavior of $\mathcal{P}(G, V)$ differs according to the following two cases:

Case (a). Either

(a1) n is odd, or

(a2) n is even and $V = U$ or $\tilde{U}$.

Case (b). n is even and $V = U_0$ or $\tilde{U}_0$.

PROPOSITION B.3.2. *Let V be one of $U, U_0, \tilde{U}, \tilde{U}_0$ and $A \in \mathcal{P}(G, V)$. Let $t_i := (2i-1, 2i)$, $1 \leq i \leq n/2$, be transpositions in G, and define $A_m := \langle t_i : 1 \leq i \leq m\rangle$ for $1 \leq m \leq n/2$,* [2] *$D := A_{[n/2]}$, and*

$$A_0 := \langle (1, 2)(3, 4), (1, 3)(2, 4), t_i : 2 < i \leq n/2\rangle.$$

[2]This notation of A_m for 2-subgroups should not be confused with the alternating subgroup $L \cong A_n$ of $G \cong S_n$.

We may choose $T \in Syl_2(G)$ *to contain these subgroups; then*

(1) For $1 \leq m \leq n/2$, $m(A_m) = m$, *and* $m(V/C_V(A_m)) = m$, *except that* $m(V/C_V(A_{n/2})) = \frac{n}{2} - 1$ *in case (b).*

(2) In case (b), $m(A_0) = n/2 = m(V/C_V(A_0))$.

(3) If $n = 8$, $V = \tilde{U}_0$, *and* A *is an* E_8*-subgroup of* L *regular on* Ω, *then* $m(V/C_V(A)) = 3$.

(4) In case (a),

$$\mathcal{P}(G, V) = \bigcup_{1 \leq m \leq n/2} A_m^G,$$

with $J(T, V) = D$, *and* $q(G, V) = 1$.

(5) Assume case (b) holds but $n \neq 8$ *when* $V = \tilde{U}_0$. *Then*

$$\mathcal{P}(G, V) = (D \cap L)^G \cup [\bigcup_{0 \leq m \leq n/2} A_m^G] \cup \mathcal{H},$$

where $\mathcal{H}$ *is the set of conjugates of hyperplanes of* D *other than* $D \cap L$. *Further* $J(T, V) = D\langle\, (t_1 t_2)^G \cap T \rangle$ *and* $q(G, V) = (n-2)/n$.

(6) Assume $n = 8$ *and* $V = \tilde{U}_0$. *Then*

$$\mathcal{P}(G, V) = (D \cap L)^G \cup [\bigcup_{0 \leq m \leq 4} A_m^G] \cup \mathcal{H} \cup \mathcal{R},$$

where $\mathcal{R}$ *is the set of regular* E_8*-subgroups of* L. *Further* $J(G, T) = T$ *and* $q(G, V) = 3/4$, *with* D *the unique strong* FF^**-offender in* T.

Proof. See 2.4 in [**Asc81a**] for the determination of $\mathcal{P}(G, V)$. The subgroups in $\mathcal{H}$ are mistakenly omitted from the conclusion of 2.4 in [**Asc81a**], but the proof is easily repaired. The asserted value of $q(G, V)$ then follows from the list of offenders. □

Next we recall that the $\mathbf{F}_2L$-modules V considered in B.3.2 include the indecomposables whose only non-central chief factor is a natural module:

Lemma B.3.3. *Assume* H *is either* L *or* G, *let* W *be the natural module for* L, *and* V *an* $\mathbf{F}_2H$*-module which is indecomposable as an* L*-module, and whose only non-central* L*-chief factor is* W. *Then*

(1) If n *is odd, then* $H^1(L, W) = 0$, *so* $V = W = U_0 \cong \tilde{U} \cong \tilde{U}_0$. *If* n *is even, then* $H^1(L, W) \cong \mathbf{Z}_2$.

Now assume further that n *is even.*

(2) If $V/C_V(H) \cong W$, *then* $V \cong U_0$ *or* $V = W \cong \tilde{U}_0$.

(3) If $[V, L] = W$, *then* $V \cong \tilde{U}$ *or* $V = W \cong \tilde{U}_0$.

(4) $V \cong U$, U_0, $\tilde{U}$, *or* $\tilde{U}_0$.

Proof. These results are well known; see for example Exercise 6.3 in [**Asc86a**] for (1) and (3). Then as W is self-dual, (2) follows from (3); cf. the discussion in Remark I.1.7. If $V = [V, L]$ then (4) follows from (2), so we may assume otherwise. Hence setting $\hat{V} := V/C_V(L)$, $\hat{V} \cong \tilde{U}_0$ by (3), so there is $v \in V - [V, L]$ such that $C_L(\hat{v}) \cong A_{n-1}$. Then $C_L(\hat{v}) = O^2(C_L(\hat{v}))$, so $C_L(\hat{v}) = C_L(v)$. Thus $\langle v^L \rangle$ is a quotient of the induced module U as an L-module, so as V is an indecomposable L-module, $V = \langle v^L \rangle$. It follows that V is U or $\tilde{U}$, completing the proof of (4). □

LEMMA B.3.4. *Assume $n = 6$, $L \leq H \leq G$, and $V := U_0$ or $\tilde{U}_0$. Let $\mathcal{P}$ be a nonempty stable subset of $\mathcal{P}(H,V)$, $T \in Syl_2(H)$, H_i the stabilizer of an i-dimensional subspace of $V/C_V(H)$ stabilized by T for $i = 1,2$, $R_i := O_2(H_i)$, and $L_i := H_i \cap L$. Then*

(1) If $H = L \cong A_6$, then $J_{\mathcal{P}}(T) = R_2$ has rank 2 and contains no strong FF-offenders, and $J_{\mathcal{P}}(R_1) = 1$.

(2) If $H = G \cong S_6$, then

(i) R_2 of rank 3 is the unique strong FF^*-offender in T.

(ii) R_1 contains no strong FF^*-offender.

(iii) If all offenders in $\mathcal{P}$ are strong, then $J_{\mathcal{P}}(R_1) = 1$ and $J_{\mathcal{P}}(T) = R_2$.

(iv) If $J_{\mathcal{P}}(R_1) = 1$, then $J_{\mathcal{P}}(T) \trianglelefteq L_2T$ and $L_1 = [L_1, J_{\mathcal{P}}(T)]$.

(v) If $J_{\mathcal{P}}(R_1) \neq 1$, then $L_1 = [L_1, J_{\mathcal{P}}(T)]$, and either

(a) $J_{\mathcal{P}}(R_1) = \langle(5,6)\rangle$ and $J_{\mathcal{P}}(T) \trianglelefteq L_2T$, or

(b) $J_{\mathcal{P}}(R_1) = R_1$ and $J_{\mathcal{P}}(T) = T$.

(vi) Either

(a) all members of $\mathcal{P}$ are of order 2, or

(b) there exists $A \in \mathcal{P}$, and a hyperplane B of A (not necessarily in $\mathcal{P}$) with $B \leq T$ but $B \not\leq R_2$.

PROOF. Assume first that $H = L$. Then R_2 is of rank 2 and by B.3.2.4, R_2 is the unique FF*-offender in T and is not strong, so (1) holds. Thus we may assume that $H = G$. Then by parts (1) and (2) of B.3.2, A_3 is the unique strong FF*-offender in T, so parts (i) and (ii) of (2) hold, and imply (iii).

By B.3.2.5, each offender is conjugate to one of

(I) A_r for $1 \leq r \leq 3$.

(II) $O_2(L_2)$ of rank 2.

(III) R_1.

(IV) $A_+ := \langle(1,2)(3,4),(5,6)\rangle$.

Observe $A_3 = R_2$ and let $A_1' := \langle(5,6)\rangle$ denote the conjugate of A_1 in R_1. Then $J_{\mathcal{P}}(R_1) = 1$ iff none of A_1', R_1 and A_+ is in $\mathcal{P}$. Further $A_1 \stackrel{<}{\sim} A_+$, R_1, and A_2, as the latter subgroups are not strong offenders. Hence as $\mathcal{P}$ is stable, if $A_1' \notin \mathcal{P}$, then A_+, R_1, and A_2, are not in $\mathcal{P}$. Therefore $J_{\mathcal{P}}(R_1) = 1$ iff $A_1' \notin \mathcal{P}$ iff $\mathcal{P} \cap T \subseteq \{R_2, O_2(L_2)\}$. Thus (iv) holds. Further if $J_{\mathcal{P}}(R_1) \neq 1$, then $A_1' \in \mathcal{P}$, so that $R_2 \leq J_{\mathcal{P}}(T)$, and either $A_1' = J_{\mathcal{P}}(R_1)$ so that (v.a) holds, or one of R_1 or A_+ lies in $\mathcal{P}$ so that (v.b) holds.

Finally suppose that there exists $A \in \mathcal{P}$ not of order 2. If $A = A_2$ or $O_2(L_2)$, then some conjugate of $\langle(1,2)(3,4)\rangle$ is contained in T but not in R_2. If $A = R_1$ then $A \leq T$ but $A \not\leq R_2$, so some hyperplane B of A does not lie in R_2. Finally some conjugate of A_+ is contained in T but not R_2, and A_+ is a hyperplane of A_3, so (vi) is established. □

B.4. $\mathbf{F}_2$-representations with small values of q or $\hat{q}$

In this section, we consider strongly quasithin groups G, and begin to describe the $\mathbf{F}_2G$-modules V such that $\hat{q}(G,V) \leq 2$, where $\hat{q}(G,V)$ is the "cubic" analogue of the parameter $q(G,V)$:

DEFINITION B.4.1. Define $\hat{q}(G,V)$ as the minimum of $r_{A,V}$ over elementary 2-subgroups A of G satisfying $[V,A,A,A] = 0$.

The parameter is of importance in 3.1.8, which is based on the *qrc*-lemma D.1.5.

We begin with the list of all SQTK-groups G such that $L := F^*(G)$ is quasisimple and G possesses an FF-module V. As mentioned in B.1.4.5, these pairs satisfy $q(G,V) \leq 1$. If $q(G,V) \leq 1$ for some module V, then there exists an overgroup G_0 of L in G and a faithful G_0-module W such that L is irreducible on W and $q(G,W) \leq 1$. Thus in the next lemma we describe all pairs (G,V) for which $F^*(G) = L$ is quasisimple, V is a faithful FF-module for G, and L is irreducible on V. Then later in Theorems B.5.1 and B.5.6 we consider general FF-modules for general SQTK-groups. In view of B.2.7, we also are interested in describing the set $\mathcal{P}(G,V)$ of FF*-offenders in V. Finally, we also record the exact value of the ratio $q(G,V)$ for each pair (G,V), since we will need this information later.

THEOREM B.4.2. *Let G be an SQTK-group with $L := F^*(G)$ quasisimple, and V a faithful $\mathbf{F}_2G$-module with L irreducible on V. Assume $A \in \mathcal{P}(G,V)$ and set $H := J(G,V)$. Then one of the following holds:*

(1) $H = L \cong L_2(2^n)$, V is the natural module, $A \in Syl_2(L)$, and $q(G,V) = 1$.

(2) $H = L \cong SL_3(2^n)$, V is a natural module, and $q(G,V) = 1/2$.

(3) $H \cong Sp_4(2^n)$, V is a natural module, and $q(G,V) = 2/3$. Further $H = L$ if $n > 1$.

(4) $H \cong G_2(2^n)$, V is the natural module, $m(A) = 3n$, and $q(G,V) = 1$. Further $H = L$ if $n > 1$.

(5) $H \cong S_5$ or S_7, V is the natural module, A is generated by commuting tranpositions, and $q(G,V) = 1$.

(6) $H = L \cong A_6$, V is a natural module, $m(A) = 2$, and $q(G,V) = 1$. If notation is chosen as in section B.3, then A is conjugate to $\langle(1,2)(3,4),(3,4)(5,6)\rangle$.

(7) $G = H = L \cong A_7$, $m(V) = 4$, $q(G,V) = 1$, and A is conjugate to $\langle(1,2)(3,4),(1,3)(2,4)\rangle$.

(8) $H = L \cong \hat{A}_6$, $m(V) = 6$, $q(G,V) = 1$, and A is the centralizer of an $\mathbf{F}_4$-line in V.

(9) $H = L \cong L_n(2)$, $n = 4$ or 5, V is a natural module, and $q(G,V) = 1/(n-1)$.

(10) $L \cong L_4(2)$, V is the 6-dimensional orthogonal module, and either

(i) $G = H = L$, $m(A) = 3$, and $q(G,V) = 1$, or

(ii) $G = H \cong S_8$ and $q(G,V) = 3/4$.

(11) $G = H = L \cong L_5(2)$, $m(V) = 10$, A is the unipotent radical of an end-node maximal parabolic, and $q(G,V) = 1$.

PROOF. The proof appears in chapter K. Lists of FF-modules satisfying the hypotheses of B.4.2 appear in unpublished papers of Cooperstein, Mason, and McClurg, e.g. [**CM**] and [**McC82**]. But more recently, Guralnick and Malle in [**GM02**] and [**GM04**, Table 2] have produced a more general treatment which we appeal to as the basis for our proof. □

From the cases in B.4.2 where $L < H$ we obtain:

COROLLARY B.4.3. *Under the hypothesis of B.4.2, if $\mathcal{P}(G,V) \not\subseteq L$, then $H \cong S_n$ for $5 \leq n \leq 8$ or $H \cong G_2(2)$, and V is the natural module, with $q(G,V) \geq 2/3$.*

Let G be a finite group with $F^*(G) =: L$ quasisimple and V a faithful $\mathbf{F}_2G$-module such that L is irreducible on V. If $q(G,V) \leq 1$, then certainly $\hat{q}(G,V) \leq 1$. However we also need information about pairs which satisfy the weaker condition $\hat{q}(G,V) \leq 2$. More precisely, we need to know the pairs such that either $q(G,V) \leq 2$, or $q(G,V) > 2$ but $\hat{q}(G,V) < 2$. The latter class of representations arises in D.1.5.1 via an appeal to E.2.15; this is applied especially in the proof of 3.2.9.

REMARK B.4.4. Our convention in the statement of B.4.5 below is to regard the alternating groups A_5, A_6, A_8 as the groups $L_2(4)$, $Sp_4(2)'$, $L_4(2)$ of Lie type and characteristic 2. Similarly we regard the groups $L_2(5)$, $L_2(7)$, $L_2(9)$, $U_3(3)$ as the groups $L_2(4)$, $L_3(2)$, $Sp_4(2)'$, $G_2(2)'$ of Lie type and characteristic 2. Notation for subgroups of the Mathieu groups is as in chapter H of Volume I.

THEOREM B.4.5. *Let G be an SQTK-group with $F^*(G) =: L$ quasisimple, and V a faithful $\mathbf{F}_2G$-module with L irreducible on V. Set $\hat{q} := \hat{q}(G,V)$ and $q := q(G,V)$. Assume that $\hat{q} \le 2$. Then one of the following holds:*

(i) $\hat{q} \le 1$, so that G and V are described in Theorem B.4.2.

(ii) $\hat{q} > 1$ and L, V, and bounds on q, $\hat{q}$ are listed in Table B.4.5:

ThmC	L	$\dim V$	q	$\hat{q}$
C.1	A_7	6	$3/2$	$3/2$
C.3	$L_2(2^{2n})$	$4n$ *orthog.*	≤ 2	$\le 3/2$
	$U_3(2^n)$	$6n$	2	2
	$Sz(2^n)$	$4n$	2	2
	$L_3(2^{2n})$	$9n$	> 2	$5/4$
	$G_2(2)'$	6	$3/2$	$3/2$
C.5	M_{12}	10	> 2	> 1
	$\hat{M}_{22}$	12	> 1	> 1
	M_{22}	10 *cocode*	> 2	> 1
		10 *code*	≥ 2	> 1
	M_{23}	11 *cocode*	> 2	> 1
		11 *code*	> 2	> 1
	M_{24}	11 *cocode*	> 2	> 1
		11 *code*	> 2	> 1

(iii) $\hat{q} = 2$ but $q > 2$, and either

(a) $L \cong Sp_4(F)$ where $F := \mathbf{F}_{2^{2a}}$, $K := End_{\mathbf{F}_2L}(V) = F_{2^a}$, and $V^K \otimes_K F$ is the tensor product of a natural 4*-dimensional FL-module with a conjugate by the generator of $Gal(F/K)$, where V^K is V regarded as a KL-module, or*

(b) $L \cong J_2$ with V of dimension 12.

PROOF. The proof of this theorem also appears in chapter K. Some results can be found (without proof) in [**MS**, 6.15], referring to [**MS90**]; others appear in Stroth [**Str**, 1.35]. However our proof is again based on the work of Guralnick and Malle in [**GM02**] and [**GM04**]. □

Most of the remaining results in this section collect detailed information about pairs (G,V) for which $q(G,V) \le 1$. In each case V is "small" but usually V is reducible.

In the first of these lemmas, we consider the group $G_2(2^n)$ on its natural Cayley-algebra module (i.e. the 7-dimensional Weyl module) and on the 6-dimensional irreducible quotientof the Weyl module. We call the latter module the *natural* *module* for $G_2(2^n)$ or the $G_2(2^n)$-*module*. In particular we show in B.4.6.13 that there is a unique FF*-offender up to conjugacy on this module.

The set $\mathcal{A}_k(G,V)$ and the parameter $a(G,V)$ appear in Definitions E.3.8 and E.3.9. For the usual theory of root groups in Lie type groups, see e.g. pages 13, 45–46, 103–104 in [**GLS98**].

LEMMA B.4.6. *Let* $L := G_2(q)$*, with* $q := 2^n$*, let* G *be a finite group with* $F^*(G) = E(L)$*, and let* $V = [V, E(L)]$ *be an* $\mathbf{F}_2G$*-module such that* $\tilde{V} := V/C_V(L)$ *is the natural 6-dimensional* $\mathbf{F}L$*-module, where* $F := \mathbf{F}_{2^n}$*. Then*

(1) $End_{\mathbf{F}_2L}(\tilde{V}) = F$, $\dim_{\mathbf{F}_2}(V) \le 7n$*, and in case of equality* V *is the Weyl module for* L *over* F*.*

(2) L *is transitive on* $\tilde{V}^{\#}$*.*

(3) Let $\tilde{V}_1$ *be a* 1*-dimensional* F*-subspace of* $\tilde{V}$*. Then* $N_L(\tilde{V}_1) =: P_1$ *is a maximal parabolic of* L*,* $R_1 := O_2(P_1)$ *is of order* q^5 *with* $Z(R_1) =: Z_1$ *the natural module for* $P_1/R_1 \cong GL_2(q)$*,* $\tilde{V}_3 := C_{\tilde{V}}(Z_1)$ *is of* F*-dimension 3,* $A_1 := C_G(\tilde{V}_3) \cong E_{q^3}$*, and* R_1/A_1 *is the natural module for* P_1/R_1*. Also*

$$\tilde{E} := \langle C_{\tilde{V}}(Z) : Z \text{ is a long root subgroup of } L \text{ and } Z \le Z_1\rangle$$

is an F*-hyperplane of* $\tilde{V}$*, and* $[E, A_1] = V_3$*.*

(4) Let $T \in Syl_2(G)$ *with* $T_L := T \cap L \le P_1$ *and* $Z_2 := Z(T_L)$*. Then* $Z_2 \cong E_q$ *is a long-root group of* L*,* $P_2 := N_L(Z_2)$ *is the maximal parabolic over* T_L *distinct from* P_1*,* $R_2 := O_2(P_2)$ *is special of order* q^5 *with center* Z_2*,* $C_{\tilde{V}}(R_2) =: \tilde{V}_2$ *is the natural module for* $P_2/R_2 \cong GL_2(q)$*,* $\tilde{V}_2 = [\tilde{V}, z]$*, and* $C_{\tilde{V}}(Z_2) = C_{\tilde{V}}(z)$ *is of* F*-dimension* 4 *for each* $z \in Z_2^{\#}$*.*

(5) L *has two classes of involutions, the long root involutions* z^L *and the short root involutions* r^L*. Further* $C_L(z) = C_L(Z_2) = O^{2'}(P_2)$*.*

(6) $C_L(r) = C_L(R)$*, where* R *is the root group of* r *and we may choose* $A_1 = Z_1 \times R$*. Also* $C_{\tilde{V}}(r) = C_{\tilde{V}}(R) = \tilde{V}_3 = [\tilde{V}, r]$*, and* $N_L(R) = A_1L_1$*, where* L_1 *is a Levi complement in* P_1*.*

(7) $C_{\tilde{V}}(i) = \widetilde{C_V(i)}$ *for each involution* $i \in L$*.*

(8) If $q > 2$ *then* $C_V(L) \le [V, R]$*.*

(9) We have $a(G,V) = 2n$*,* $\mathcal{A}_2(G,V) \subseteq L$*,* $\mathcal{A}_{2n}(G,V) = A_1^L$*, and each member of the set* $\mathcal{A}_{n+1}(G,V)$ *is fused into* A_1 *under* L*.*

(10) V_3/V_1 *is partitioned by* $q+1$ P_1*-conjugates of* V_2/V_1*.*

(11) If $B \le G$ *with* $B^{\#} \subseteq z^L$*, then* B *is fused into* Z_1 *under* L*. Further* Z_1 *is strongly closed in* R_1 *with respect to* G*.*

(12) For $n > 1$*,* $\mathcal{A}_{2n-1}(G,V) \cap R_1 \subseteq A_1$*.*

(13) $\mathcal{P}(G,V) = A_1^L$*,* $J(T,V) = R_2$*, and* $J(R_1,V) = A_1$*.*

(14) Assume that $q = 4$*, and* X *is of order* 3 *with* $XT = TX$*. Then* $X \le P_1 \cap P_2^\infty$ *if and only if* $[X, P_1^\infty] \le R_1$*.*

PROOF. Part (1) follows from I.1.6 and I.2.3; we mention that the latter quotes [**Asc87**] which in turn uses (5.2) of [**Asc88**]. Parts (2)–(6) and (10) are probably well known; in any case details are easily retrievable from [**Asc87**].

By (6), $C_{\tilde{V}}(r) = [\tilde{V}, r] = \widetilde{[V,r]}$, so as $[V,r] \le C_V(r)$ since r is an involution, while $\widetilde{C_V(r)} \le C_{\tilde{V}}(r)$, it follows that $C_{\tilde{V}}(r) = \widetilde{C_V(r)}$. From 2.3 in [**Asc87**], there is a z-invariant complement to $C_W(L)$ in the Weyl module W, so $C_{\tilde{V}}(z) = \widetilde{C_V(z)}$. Hence (7) follows from (5).

To see that (8) holds, see the displayed equations on page 212 in Lemma 4.4 of [**Asc87**]: note that our R is the group called "D_1" there, and that if $q > 2$, then commutators of D_1 with x_1' cover $Fx_0 = C_V(L)$.

Suppose that $B \leq G$ with $B^\# \subseteq z^G$. Without loss $z \in B$. By a standard argument based on 43.9 in [**Asc86a**], [3] P_2 is transitive on root groups $Z_2^g \neq Z_2$ centralizing Z_2, so if $b \in B - Z_2$ we may take $Z_1 = Z_2 \times Z_2(b)$, where $Z_2(b)$ is the root group of b. Now

$$B \leq C_L(\langle z, b\rangle) = C_L(Z_2) \cap C_L(Z_2(b)) = C_L(Z_1) = R_1.$$

We will show that Z_1 is strongly closed in R_1 with respect to G, which will force $B \leq Z_1$; that is, the second assertion of (11) suffices to establish (11). Now (cf. 5.3 in [**Asc87**]) P_1 has three orbits on L/P_2, and hence also on Z_2^L, with representatives: Z_2, $Z_2^{g_2} \leq P_1$ but $Z_2^{g_2} \cap R_1 = 1$, and $Z_2^{g_3} \cap P_1 = 1$. Thus $Z_1 = \langle Z_2^{P_1}\rangle$ is the strong closure of Z_1 in R_1. So as noted above, (11) holds.

We now turn to the assertions related to $a(G, V)$ in (9) and (12). Suppose that $A \in \mathcal{A}_2(G, V)$; recall this means that $C_V(A) = C_V(B)$ for each hyperplane B of A. If $A \not\leq L$, then some $a \in A$ induces a field automorphism on L and V, so $[C_V(A \cap L), a] \neq 0$, contrary to the previous remark. That is $\mathcal{A}_2(G, V) \subseteq L$.

Next assume that $A \in \mathcal{A}_{n+1}(G, V)$. Suppose $A^\# \subseteq z^G$. Then by (11) we may take $A \leq Z_1$. As $m(A) \geq n + 1$, A is not contained in a root group of Z_1, so $C_V(A) = C_V(Z_1) = V_3$ using (3). On the other hand, $m(A/A \cap Z_2) \leq m(Z_1/Z_2) = n < n + 1$, so $V_3 = C_V(A) = C_V(A \cap Z_2) \geq C_V(Z_2)$, whereas $C_V(Z_2) > V_3$ by (4) and (7). Therefore we may suppose that $r \in A$. So $A \leq C_L(r) \leq A_1L_1$ by (6). Then $m(A/C_A(V_3)) \leq m_2(L_1) = n$, so $C_V(A) = C_V(C_A(V_3)) = V_3$, as $r \in C_A(V_3)$ and $C_V(r) = V_3$ by (6). Thus $A \leq C_G(V_3) = A_1$. Hence we have shown that each member of $\mathcal{A}_{n+1}(G, V)$ is fused into A_1 under L, establishing the last statement in (9).

Next we verify that $A_1 \in \mathcal{A}_{2n}(G, V)$: For given $B \leq A_1$ with $m(A_1/B) < 2n$, we have $B \cap R \neq 1$, so $C_V(B) \leq C_V(B \cap R) = V_3$ using (6), and hence $C_V(B) = V_3 = C_V(A_1)$. Thus $a(G, V) \geq 2n$. On the other hand, $C_V(Z_1) > V_3 = C_V(A_1)$, so $A_1 \notin \mathcal{A}_{2n+1}(G, V)$. Since we saw $\mathcal{A}_{n+1}(G, V)$ is fused into A_1 and $2n \geq n + 1$, to complete the proof of (9) it remains to show that if $B \in \mathcal{A}_{2n}(A_1, V)$, then $B = A_1$.

Suppose first that $BZ_2 < A_1$. Then

$$m(B/B \cap Z_2) = m(BZ_2/Z_2) < m(A_1/Z_2) = 2n,$$

so

$$V_3 = C_V(B) = C_V(B \cap Z_2) \geq C_V(Z_2),$$

contrary to (4) and (7). Thus $BZ_2 = A_1$; then from the Dedekind Modular Law we get $Z_1 = Z_2B_1$, where $B_1 := B \cap Z_1$. As $Z_2 \leq Z_1$, $BZ_1 = A_1$, so $m(B_1) = m(B) - n$. Similarly the arguments above can be applied to any root group S of Z_1, so $Z_1 = SB_1$, and therefore $m(B_1 \cap S) = m(B) - 2n =: k$, say. Therefore as the $2^n + 1$ root subgroups in Z_1 partition the nontrivial elements of Z_1,

$$2^{n+k} - 1 = |B_1^\#| = (2^n + 1)(2^k - 1) = 2^{n+k} - 2^n + 2^k - 1,$$

so $n = k$ and hence $B = A_1$. So (9) is established.

[3]But notice there is a misprint in the statement of 43.9 of [**Asc86a**]: $(G_JwG_K) \cap W$ should be $(G_JwG_K) \cap UWU$.

Assume that $n > 1$, and suppose $A \in \mathcal{A}_{2n-1}(G,V) \cap R_1$. By (11), $Z_A := \langle z^G \cap A \rangle \leq Z_1$. As $n > 1$, $n < 2n-1 \geq n+1$, so by (9), $A \leq A_1^g$ for some $g \in G$. Hence $Z_A \leq Z_1 \cap Z_1^g$. Also

$$m(A/Z_A) \leq m(A_1^g/Z_1^g) = n < 2n-1,$$

so as $A \in \mathcal{A}_{2n-1}(G,V)$, $C_V(A) = C_V(Z_A)$. Now if $Z_A \leq Z_2^g$, then by (4) and (7), $\dim_F(V/C_V(A)) = 2$. However as $m(A) \geq 2n-1 > n$, A contains an element a of $A_1^g - Z_1^g$, so $\dim_F(V/C_V(A)) \geq \dim_F(V/C_V(a)) = 3$ by (6) and (7). Thus Z_A intersects at least two distinct root groups of Z_1 and Z_1^g, so by (3), $V_3 = C_V(Z_A) = V_3^g$—and hence $g \in P_1 = N_G(V_3)$. Thus $A \leq A_1^g = A_1$, completing the proof of (12).

Now assume $D \in \mathcal{P}(G,V)$. If $d \in D^\#$, then either $d \in r^G$ so that $m([V,d]) \geq 3n$ by (6), or $d \in z^L$ and $m([V,d]) = 2n$ by (4). Therefore $m(D) \geq 2n$, and $m(D) \geq 3n$ unless $D^\# \subseteq z^L$. But in the latter case $D \leq Z_1^g$ for some $g \in G$ by (11), so as $m(Z_1) = 2n$, we conclude $D = Z_1^g$ and $m(V/C_V(D)) = 3n > m(D)$, contradicting the definition of $\mathcal{P}(G,V)$. Hence $m(D) \geq 3n$, and we may take $r \in D$, so $C_V(D) \leq C_V(r) = V_3$ and hence $m(V/C_V(D)) \geq 3n$. Also $D \leq C_G(r)$ and $m_2(C_G(r)) = 3n$ from (6), so $m(D) = 3n$ and $V_3 = C_V(D)$. Thus $D \leq C_G(V_3) = A_1$ of rank $3n$, so $D = A_1$, completing the determination of $\mathcal{P}(G,V)$. By (11), Z_1 is strongly closed in R_1, so as $A_1 = C_G(C_{\tilde{V}}(Z_1))$, A_1 is the only FF*-offender in R_1. Further P_1 is transitive on $Z_2^L \cap Z_1$, so P_2 is transitive on $Z_1^L \cap P_2$, and hence $J(T,V) = \langle A_1^{P_2} \rangle = R_2$. This completes the proof of (13).

Assume the hypotheses of (14), let H be a Cartan subgroup of $N_L(T_L)$, Δ the set of long roots in the root system determined by H, and K the subgroup generated by the root groups in Δ. Then Δ is an A_3-root system, so $K \cong SL_3(4)$. Then $Y := Z(K) = C_H(L_1)$, where L_1 is a Levi complement in P_1: since L_1 is generated by long root subgroups, and Y is invariant under the Weyl group. Further Y centralizes no short root groups, but is invariant under the reflection in the Levi complement L_2 of P_2 generated by short root groups, so $Y \leq L_2$, and hence $Y \leq P_2^\infty$. Thus Y satisfies the conditions on "X" in (14), and is the only subgroup of order 3 in P_1 permuting with T which also lies in P_2^∞. This establishes (14). □

We digress briefly from our study of specific FF-modules to interject lemma B.4.7, which shows that the quadratic subgroups on a $\mathbf{F}_2$-module and its dual are the same. This facilitates calculation of $q(G,V)$ in terms of $q(G,V^*)$. In particular, in case (3) of the *qrc*-lemma Theorem D.1.5, V^* is an FF-module; in lemma B.5.13, we will use B.4.7 to show that this forces $q(G,V) \leq 2$. We also use the lemma in various other places.

LEMMA B.4.7. *Let F be a field of characteristic 2, V an n-dimensional F-space, and A an elementary abelian 2-subgroup of $GL(V)$. For $U \leq V$ define $\alpha(U)$ to be the annihilator of U in the dual space V^* of V. Then*

(1) $\alpha : PG(V) \to PG(V^*)$ is a $GL(V)$-equivariant anti-isomorphism of projective geometries with $\dim(\alpha(U)) = n - \dim(U)$.

(2) $\alpha([V,A]) = C_{V^*}(A)$ and $\alpha(C_V(A)) = [V^*,A]$.

(3) A is quadratic on V iff A is quadratic on V^*.

(4) Assume $F = \mathbf{F}_2$, A is quadratic on V, and $\dim([V,A]) \leq 2m(A)$; then $q(A,V^*) \leq 2$.

PROOF. Part (1) is easy linear algebra and well known. Set $U := C_V(A)$. As A is unipotent on V and V^*, (1) says that for $v \in V^\#$,

$v \in U$ iff A centralizes the 1-space Fv iff A normalizes the hyperplane $\alpha(Fv)$ of V^* iff $[V^*, A] \leq \alpha(Fv)$.

Thus

$$[V^*, A] = \bigcap_{u \in U^\#} \alpha(Fu) = \alpha(\langle Fu : u \in U^\# \rangle) = \alpha(U).$$

The dual argument shows that $\alpha([V, A]) = C_{V^*}(A)$, so (2) holds.

If A is quadratic on V, then $[V, A] \leq U$, so by (1) and (2),

$$[V^*, A] = \alpha(U) \leq \alpha([V, A]) = C_{V^*}(A)$$

and hence A is quadratic on V^*. As $(V^*)^* = V$, by symmetry A is quadratic on V if A is quadratic on V^*, so (3) is established. Finally (1)–(3) imply (4). □

We return to our study of selected FF-modules: It is well known (see I.1.6.4) that the natural module for $SL_3(2^n)$ has nontrivial 1-cohomology only when $n = 1$. The next lemma determines the FF*-offenders on the corresponding indecomposable modules.

LEMMA B.4.8. *Let $G \cong L_3(2)$, V a faithful indecomposable $\mathbf{F}_2G$-module, and W a natural 3-dimensional module for G. Then*

(1) $|H^1(G, W)| = 2$.

(2) If $\dim(V) = 4$ and $W \cong V/C_V(G)$ then $\mathcal{P}(G, V) = A^G$, where A is the group of transvections on W with fixed center, and $q(G, V) = 1$. Further for $T \in Syl_2(G)$, $C_V(T) = C_V(G)$, and for each involution a of G, $[V, a] = C_V(a)$ is of rank 2.

(3) If $\dim(V) = 4$ and $W = [V, G]$ then $\mathcal{P}(G, V) = B^G$ where B is the group of transvections on W with fixed axis, and $q(G, V) = 1$. Further for each involution a of G, W contains $[V, a] = C_V(a)$ of rank 2.

(4) If V has a unique noncentral chief factor which is natural, then $\dim(V) = 3$ or 4.

PROOF. Part (1)–(3) are standard: see I.1.6.4 (or the underlying Background Reference [**JP76**]) for (1). Under the hypotheses of (2) or (3), if G were to contain a transvection on V, then as G is generated by three involutions we would have $m([V, G]) = 3 = m(V/C_V(G))$ in the respective cases, contrary to the assumption that V is indecomposable. Thus for each involution $a \in G$, $[V, a] = C_V(a)$ is of rank 2, so in (3), $C_V(a) \leq W$. Then it is easy to see that (2) and (3) hold.

Assume the hypothesis of (4). As the unique noncentral chief factor of V is natural, $\dim([V, G]) \leq 4$ by (1). If $\dim([V, G]) = 3$ then $\dim(V) \leq 4$ by (1), so we may assume that $\dim([V, G]) = 4$. Indeed we may assume that $\dim V = 5$, and it remains to derive a contradiction. By Gaschütz's Theorem A.1.39, no $v \in V - [V, G]$ is centralized by $T \in Syl_2(G)$, so $\dim C_V(G) = 1$. Let X be of order 7 in G and Y of order 3 in $N_G(X)$. Then $\dim C_V(X) = 2$, $C_V(G) < C_V(X) \not\leq [V, G]$, and Y centralizes $C_V(X)$ by Coprime Action. So as XY is maximal in G, $C_G(v) = XY$ for each $v \in C_V(X) - C_V(G)$. Thus V is a quotient of the 8-dimensional permutation module P on the cosets of XY in G, by some submodule S of dimension 3. Observe that T acts regularly on P, so that $\dim C_P(T) = 1$, and hence $C_P(T) = C_P(G)$. Then P has no natural submodule N since $C_N(T) \neq 0$, so S has only trivial composition factors, and hence S is trivial since G is perfect, contradicting $C_P(T)$ of dimension 1. □

Next we consider an $\mathbf{F}_2G$-module V for $G \cong SL_n(2^f)$ possessing a natural submodule U and such that V/U is isomorphic to U or its dual. We establish a sufficient condition for V to split over U, and show that this condition is satisfied when V is an FF-module. The proof was suggested by Mason's proof of 1.3.9 in [**Mas**], although both his result and proof differ from ours.

LEMMA B.4.9. *Let $G \cong L_n(2)$, $n \geq 4$, or $SL_3(2^f)$, V a faithful $\mathbf{F}_2G$-module, and U a natural $\mathbf{F}_2G$-submodule of V such that $\tilde{V} := V/U$ is $\mathbf{F}_2G$-isomorphic to U or its dual U^*. Then*

(1) Assume G contains a 4-subgroup E of transvections with a fixed axis on $\tilde{V}$ such that E is quadratic on V. If $G \cong SL_3(2^f)$, assume further that E is not contained in a root group of G; and if $f = 2$, assume in addition that all such 4-subgroups not in a root group are quadratic on V. Then V splits over U as an $\mathbf{F}_2G$-module.

(2) Assume $T \in Syl_2(G)$ and there is $A \in \mathcal{P}(T, V)$. Then

(i) V splits over U. That is, non-split extensions of a natural module by a second natural module are not FF-modules.

(ii) If $n = 3$, then $\tilde{V} \cong U$ and $q(G, V) = 1$. So the sum of the natural module and its dual is not an FF-module.

(iii) If $n = 4$ and $\tilde{V} \cong U^*$, then $A = J(T)$ is of rank 4, and $q(G, V) = 1$.

(iv) If $n = 5$ and $\tilde{V} \cong U^*$, then $q(G, V) = 5/6$ and either $A \cong E_{16}$ is contained in a Levi complement of an end-node maximal parabolic—or $A \cong E_{32}$ or E_{64} and A is contained in the unipotent radical of an interior-node maximal parabolic.

PROOF. Assume the hypotheses of (1). For $e \in E^\#$, set $\tilde{I} := C_{\tilde{V}}(e)$, the common axis for the transvections in E; thus $Q := C_G(\tilde{I})$ is the full group of transvections with axis $\tilde{I}$ and so contains E. Then Q is partitioned by root groups, and we denote by Q_e the root group containing e. Let K_0 denote a Levi complement of the parabolic $N_G(Q)$, and $K := O^{2'}(K_0)$. Observe that KQ contains $T \in Syl_2(G)$, so by Gaschütz's Theorem A.1.39 to show the splitting required for (1), it suffices to exhibit a KQ-complement to U in V.

Further let

$$\mathcal{E}(e) := \{E^g : g \in G \text{ and } e \in E^g\},$$

except when G is $SL_3(4)$, in which case $\mathcal{E}(e)$ is defined to consist of all 4-subgroups in Q containing e but not contained in Q_e. We observe that $Q = \langle \mathcal{E}(e) \rangle$: This is because $E \leq Q$, but $E \not\leq Q_e$ (using the first additional hypothesis when $f > 1$), and $Q = \langle e, x^{C_K(e)} \rangle$ for $x \in Q - Q_e$—unless $G \cong SL_3(4)$, where the observation holds as $\mathcal{E}(e) = \{\langle e, x \rangle : x \in Q - Q_e\}$.

For $E_e \in \mathcal{E}(e)$, E_e is quadratic on V by our hypotheses, so as $Q = \langle \mathcal{E}(e) \rangle$,

$$[V, e, Q] = \langle [V, e, E_e] : E_e \in \mathcal{E}(e) \rangle = 0.$$

Thus Q centralizes $[V, e]$, so by transitivity of K on $Q^\#$, Q centralizes $[V, Q]$—so Q is quadratic on V.

Set $F := \mathbf{F}_{2^f}$, where $f := 1$ if $G \cong L_n(2)$. Observe that K contains a conjugate $P := H^g$ of a hyperplane H of Q—unless $f > 1$, where K contains a conjugate

$P := Q_e^g$ of Q_e. Also K fixes a complement $F\tilde{v}$ to $\tilde{I}$ in $\tilde{V}$. Now $P \le Q^g$ with

$$\tilde{v} \in [\tilde{V}, Q^g] = \widetilde{[V, Q^g]} \le \widetilde{C_V(Q^g)} \le \widetilde{C_V(P)}$$

since Q is quadratic on V. Therefore we may pick $v \in C_V(P)$. As $[\tilde{v}, K] = 0$, K acts on the preimage U_0 of $F\tilde{v}$ in V, and then on $\hat{U}_0 := U_0/C_U(K)$.

We will show that K centralizes v. Suppose first that $[\hat{v}, K] = 0$. Then $[v, O^2(K)] = 0$ by Coprime Action. Now $K = O^2(K)$, except when $G \cong L_3(2)$, where $K = O^2(K)P$ and P centralizes v by our choice of v—so in any event, we conclude that $[v, K] = 0$ in this case, as desired. So we turn to the case $[\hat{v}, K] \neq 0$; then $\hat{U}_0$ is indecomposable for K of rank fn with $[\hat{U}, K]$ a natural module of rank $f(n-1)$. However (see I.1.6, and also recall B.4.8.1) the natural module for K has trivial 1-cohomology unless $K \cong L_3(2)$ or $SL_2(2^f)$ for $f > 1$—that is, unless $G \cong L_4(2)$ or $SL_3(2^f)$ for $f > 1$. However, in those explicit non-split extensions (cf. B.4.8.3 when $K = L_3(2)$, and the dual of I.2.3 when $K = SL_2(2^f)$), the subgroup P does not fix a vector outside $[\hat{U}_0, K] = \hat{U}$, whereas P fixes v.

We have shown that K fixes some $v \in V - ([V, Q] + U)$. Set $W := [v, Q]$ and for $x \in Q$ define

$$\begin{aligned} \alpha : Q &\to W \\ x &\mapsto [v, x] \end{aligned}$$

Now for $y \in Q$,

$$[v, xy] = [v, x]^y + [v, y] = [v, x] + [v, y]$$

as Q is quadratic on V, so α is a homomorphism. Now $\ker(\alpha) = C_Q(v) = 1$, since elements of Q have their fixed points in $[V, Q]$; so α is an isomorphism. Thus $m(W) = f(n-1)$, with $W \cap U = 0$.

So when L is $L_n(2)$ where $f = 1$, the vector v spans a 1-space over $F = \mathbf{F}_2$, and hence $W + \langle v \rangle$ is a KQ-complement to U in V; as mentioned earlier, this establishes (1) in this case.

So we have reduced to the case $G = SL_3(2^f)$ with $f > 1$; set $H := C_{K_0}(K)$, so that H is cyclic of order $2^f - 1$ and $K_0 = KH$. As K acts on W and $\tilde{W} = [\tilde{W}, K]$, also $W = [W, K]$.

Suppose $\tilde{V} \cong U^*$. Then $C_U(Q)$ is a 1-space over F centralized by K, so as $C_V(Q) = C_U(Q) + W$, $W = [C_V(Q), K]$ and hence W is H-invariant; therefore

$$W = W^h = [v, Q]^h = [v^h, Q]$$

for each $h \in H$. Now choosing an H-complement W_H to $C_U(K)$ in $C_V(K)$, $W + W_H$ is spanned by W and the vectors v^h for $h \in H$. Thus $W + W_H$ gives a K_0Q-complement to U in V, establishing (1) in this case.

Thus we may assume finally that $\tilde{V} \cong U$. Then $C_U(Q)$ and W are natural K-submodules of $C_V(Q)$, so as $C_{\tilde{V}}(Q) = \tilde{I} = \tilde{W}$, $C_V(Q) = C_U(Q) \oplus W$ is a homogeneous K-module. Therefore the set $\mathcal{W}$ of irreducible K-submodules of $C_V(Q)$ is of order $2^f + 1$, and the map

$$\begin{aligned} \varphi : C_V(K)^{\#} &\to \mathcal{W} \\ v &\mapsto [v, Q] \end{aligned}$$

is H-equivariant. As $[u + v, x] = [u, x] + [v, x]$ for $x \in Q$ and $u, v \in V$, $\varphi^{-1}(W)$ is a subgroup of $C_U(K)$ for each $W \in \mathcal{W}$. Further $\varphi^{-1}(W) \cap U = 0$ for $W \neq C_U(Q)$, so $|\varphi^{-1}(W)| \le 2^f - 1$, and in case of equality $\varphi^{-1}(W) \cup \{0\}$ is a complement to $C_U(K)$ in $C_V(K)$. Therefore as

$$|C_U(K)^\#| = 2^{2f} - 1 = |\mathcal{W}| \cdot (2^f - 1)$$

we conclude from the pigeonhole principle that $V_W := \varphi^{-1}(W)$ is a complement to $C_U(K)$ in $C_V(K)$ for each $W \neq C_U(Q)$. So $W + V_W$ is a KQ-complement to U in V, completing the proof of (1).

We turn to the proof of (2). Let $\mathcal{E} = \mathcal{E}(G)$ be the set of 4-subgroups E of G centralizing a hyperplane of $\tilde{V}$, with E not contained in a root group if $n = 3$. Assume $A \in \mathcal{P}(T, V)$, and let $\tilde{V}_i$ be the T-invariant i-dimensional F-subspace of $\tilde{V}$.

By the Thompson Replacement Lemma B.1.4.3, there is $B \le A$ with $B \in \mathcal{P}(G, V)$ and B quadratic on V; and in particular if A is minimal in $\mathcal{P}(G, V)$ under inclusion, then $B = A$ itself is quadratic on V. Further if B contains a member of $\mathcal{E}$, then V splits over U by (1)—except possibly in case G is $SL_3(4)$, where we must require B to contain all 4-subgroups with a fixed axis on $\tilde{V}$, in order to guarantee this splitting.

Set $m_1 := m(U/C_U(A))$ and $m_2 := m(\tilde{V}/C_{\tilde{V}}(A))$. Then as $r_{A,V} \le 1$ by B.1.4,

$$k := m(A) \ge m(V/C_V(A)) \ge m_1 + m_2 \ge 2, \tag{!}$$

so for $q_i := m_i/k$,

$$q_1 + q_2 \le 1. \tag{$*$}$$

In particular either $q_i < 1/2$ for some $i = 1$ or 2, or $q_1 = q_2 = 1/2$.

Assume that $n = 3$. Then it follows from B.4.2.2 that $q_1 = q_2 = 1/2$, and from the proof of B.4.2, A must be the group of transvections with fixed axis, since that class of subgroups is the unique class which achieves this minimum ratio. This must be true on both modules, so $\tilde{V} \cong U$, and $A = B$ is the group of transvections with axis $\tilde{V}_2$. As B contains all 4-subgroups with axis $\tilde{V}_2$, B contains all the necessary members of $\mathcal{E}$, and hence V splits over U by earlier remarks. Thus (2.i) holds in this case, with (2.ii) established along the way.

Assume next that $n = 4$. We first consider the case where $\tilde{V} \cong U^*$. Then as $q_1 + q_2 \le 1$, A does not centralize a hyperplane in either module. Hence

$$m_1 \ge 2 \le m_2, \quad \text{so} \quad k \ge 4 \tag{$**$}$$

by (*). As

$$C_T(\tilde{V}_2) \cap C_T(\tilde{V}/\tilde{V}_2) = J(T)$$

is of rank 4, this forces $J(T) = A = B$. Again B contains a member of $\mathcal{E}$, so as before the extension splits as required for (2.i), and we have also established the additional requirement of (2.iii) in this case. This leaves the case where $\tilde{V} \cong U$. If A centralizes $\tilde{V}_3$ then as $k \ge 2$, B contains a member of $\mathcal{E}$, so the extension splits. If A does not centralize $\tilde{V}_3$, then as $U \cong \tilde{V}$, A does not centralize V_3 either, so $m_i \ge 2$ for $i = 1$ and 2. Thus $k \ge 4$ and then as earlier $A = B = J(T)$ and the extension splits. Thus (2.i) and hence also (2) is established for $n = 4$.

Finally take $n = 5$. Again we first consider the case $\tilde{V} \cong U^*$. Arguing as in the previous paragraph, A does not centralize a hyperplane of either $\tilde{V}$ or U, so again (**) holds.

Assume first that $k = 4$. Then A is minimal in $\mathcal{P}(G, V)$ under inclusion, and hence as mentioned earlier $B = A$ is quadratic on V. Further as $k = 4$, $m_1 = 2 = m_2$ by (**), so we may assume that $C_{\tilde{V}}(A) = \tilde{V}_3$, and A centralizes a 3-subspace of U.

Then as $\tilde{V} \cong U^*$, $[\tilde{V}, A]$ is a 2-subspace of $\tilde{V}$ by B.4.7.2. As A is quadratic on $\tilde{V}$, $[\tilde{V}, A] \leq \tilde{V}_3$. Thus A lies in the center of the unipotent radical R of the parabolic stabilizing the flag $[\tilde{V}, A] < \tilde{V}_3$, so as $|Z(R)| = 16 = |A|$, $A = Z(R)$. But R lies in a Levi complement $L \cong L_4(2)$ of the parabolic P stabilizing a 4-subspace $\tilde{W}$ of $\tilde{V}$ such that $\tilde{V}_3 \cap \tilde{W} = [\tilde{V}, A]$, and in fact R is the unipotent radical of the parabolic of L stabilizing the 2-subspace $[\tilde{V}, A]$ of $\tilde{W}$. Therefore from our treatment of the case $n = 4$, A contains a member of $\mathcal{E}(L)$, so as $\mathcal{E}(L) \subseteq \mathcal{E}(G)$, V splits over U by (1). Thus we have established (2.i), and that part of (2.iv) referring to the case $m(A) = 4$. The equality $q(G, V) = 5/6$ will be obtained in the discussion of the case $m(A) = k > 4$.

Thus we may assume that $k > 4$. If $k = 6$, then we may assume that A is the unipotent radical of the parabolic $N_G(\tilde{V}_i)$ for $i = 2$ or 3, so A contains a member of $\mathcal{E}$; and as A is quadratic, the extension splits by (1), giving (2.i). Also A is a quadratic offender with ratio 5/6, so $q(G, V) \leq 5/6$. Thus we may take $k = 5$, so that $m_1 + m_2 \leq 5$ by (!). Therefore as $m_1, m_2 \geq 2$ by (**), we may assume that $C_{\tilde{V}}(A)$ and $C_U(A)$ are given by either

(a) $\tilde{V}_3$ and a line or plane in U, or
(b) $\tilde{V}_2$ and a plane in U.

In case (a) if $[\tilde{V}, A] \not\leq \tilde{V}_3$, then $[\tilde{V}, a] \not\leq \tilde{V}_3$ for some $a \in A^{\#}$, so $\tilde{V}_3 + [\tilde{V}, a]$ is a hyperplane centralized by a, and a is a transvection with center $[\tilde{V}, a]$; then A centralizes $[\tilde{V}, a]$, contradicting $\tilde{V}_3 = C_{\tilde{V}}(A)$. Thus $[\tilde{V}, A] \leq \tilde{V}_3$, so A is contained in the radical of $N_G(\tilde{V}_3)$, and hence as $k = 5$, A contains a member of $\mathcal{E}$, and $C_U(A)$ is a line rather than a plane. To prove splitting, we may take A minimal in $\mathcal{P}(G, V)$, and hence quadratic on V, so the extension splits by (1), and the offender A satisfies $r_{A,V} = 1$. In case (b), we obtain the same conclusions by arguing on U in place of $\tilde{V}$. This completes the proof of (2.i), and also of (2.iv) since we have now shown that $q(G, V) \geq 5/6$.

Finally we turn to the case where $\tilde{V} \cong U$. We only need to establish (2.i) in this case, so we may choose A minimal in $\mathcal{P}(G, V)$ under inclusion, and hence quadratic on V; further we may assume that A contains no member of $\mathcal{E}$. As $k \geq m_1 + m_2 \geq 2$, A is noncyclic. Thus as A contains no member of $\mathcal{E}$, A does not centralize a hyperplane of $\tilde{V}$ or U, so again (**) holds. If A does not centralize a 3-subspace of $\tilde{V}$, then $m_i \geq 3$ and so $k \geq 6$; hence we may assume that A is the radical of $N_G(\tilde{V}_i)$ for $i = 2$ or 3, contradicting the assumption that A contains no member of $\mathcal{E}$. So we may assume that A centralizes $\tilde{V}_3$. Then as A is quadratic, $[\tilde{V}, A] \leq \tilde{V}_3$, so A is contained in the radical R of $N_G(\tilde{V}_3)$, and $k = 4$ since A contains no member of $\mathcal{E}$. Now $R = R_1 \times R_2$ where $R_i := C_R(\tilde{W}_i)$ for $\tilde{W}_1$ and $\tilde{W}_2$ distinct hyperplanes of $\tilde{V}$ over $\tilde{V}_3$. As A contains no member of $\mathcal{E}$, $A_i := A \cap R_i \cong \mathbf{Z}_2$. Notice it suffices to show that R is quadratic on V, since then R contains a member of $\mathcal{E}$ acting quadratically, so the module splits by (1). As R is generated by $N_G(R)$-conjugates of A_1 and $[V, A_1, A] = 0$, it even suffices to show that $R = \langle A^X \rangle$, where $X := N_G(R) \cap N_G(A_1)$. Set $D := C_R(O^2(X))$—a 4-group of transvections with fixed center, containing A_1. Observe that $[R, O_2(X)] = R_1 D$, and X is irreducible on $R/R_1 D$, so that every proper X-submodule of R is contained in $R_1 D$; thus $R = \langle A^X \rangle$ as desired, unless $A \leq R_1 D$. But in that event, as $m(A) = 4 = m(R_1 D)$ we have $A = R_1 D$, whereas $A_1 = R_1 \cap A$ is of order 2 rather than 8. This contradiction completes the proof of B.4.9. □

The following lemma will be used for example in D.2.9.

LEMMA B.4.10. *Let V be a faithful $\mathbf{F}_2G$-module, and assume $u \in V^\#$ such that the group T of transvections on V with center $\langle u \rangle$ is contained in G. Let $U := \langle u^G \rangle$ and $L := \langle T^G \rangle$. Then $Aut_L(U) = GL(U)$.*

PROOF. Observe that $Aut_T(U)$ is the group of transvections on U with center $\langle u \rangle$; so replacing G, V by $Aut_G(U)$, U, we may assume that $V = \langle u^G \rangle$. As $V = \langle u^G \rangle$, there is a basis $X := \{x_1, \ldots, x_n\}$ of V contained in u^G. Let $x_i := u^{g_i}$, $T_i := T^{g_i}$, and $L_m := \langle T_1, \ldots, T_m \rangle$ for $1 \leq m \leq n$. It suffices to show that $L_n = GL(V)$, which we prove by induction on n.

If $n = 1$, then $L_n = 1 = GL(V)$, giving the base step. So assume that $n > 1$. Then by induction on n, $Aut_{L_{n-1}}(W) = GL(W)$, where $W := \langle x_1, \ldots, x_{n-1} \rangle$. Next $C_{T_1}(W) =: \langle t \rangle$ is of order 2, and as L_{n-1} is irreducible on W, $Q := \langle t^{L_{n-1}} \rangle$ is the group of transvections with axis W. Thus $Q = C_{GL(V)}(W) \leq L_{n-1}$, so $L_{n-1} = N_{GL(V)}(W)$ is a maximal parabolic of $GL(V)$. Finally $T_n \not\leq L_{n-1}$, so by maximality of L_{n-1}, $L_n = \langle L_{n-1}, T_n \rangle = GL(V)$, completing the proof. □

The following result is well known, and can be obtained from James [**Jam78**] or the Modular Atlas [**JLPW95**]; however as usual to avoid such outside appeals we sketch a proof:

LEMMA B.4.11. *Let $G \cong A_7$ and V a nontrivial irreducible $\mathbf{F}_2G$-module with $\dim(V) \leq 8$. Then $dim(V) = 4$ or 6.*

PROOF. Let $n := \dim(V)$. As G contains an element of order 5, $n \geq 4$, so we may assume that $n = 5$, 7 or 8.

Let $\mathcal{A}$, $\mathcal{L}$ be the set of subgroups H of G isomorphic to A_6 or $L_3(2)$, respectively, and pick $H \in \mathcal{A} \cup \mathcal{L}$. Inspecting the character of the permutation module M for G on G/H in characteristic 0, we find that the nontrivial composition factors for G on M are of dimension 6 when $H \in \mathcal{L}$, and of dimension 6 or 4 when $H \in \mathcal{A}$. Thus V is not a quotient of the reduction of M modulo 2, so that $C_V(H) = 0$.

Recall next that up to quasiequivalence (conjugacy in $Out(H)$), the nontrivial irreducibles for H over $\mathbf{F}_2$ (cf. G.5.1 and H.6.1) are a projective P of dimension 8 (resp. 16), and the natural module N of dimension 3 (resp. 4), for $H \in \mathcal{L}$ (or $\mathcal{A}$, respectively). Further (cf. I.1.6) $\dim(H^1(H, N)) = 1$. Then since H has no fixed points on V (or on its dual V^*), we conclude from the existence of $H \in \mathcal{A}$ that $n \neq 5, 7$, and hence that $n = 8$, Similarly, for $A \in \mathcal{A}$, $V|_A$ has two 4-dimensional composition factors; and for $L \in \mathcal{L}$, $V|_L$ is the 8-dimensional projective P. So for $T \in Syl_2(G)$, $V|_T$ is the regular T-module, and hence $\dim C_V(T) = 1$. Now for Y of order 3 in L, $\dim(C_V(Y)) = 2$ by H.6.3.3; so by G.5.1.2, the two composition factors of $V|_A$ are non-isomorphic, and then $\dim(C_V(X)) = 1$ for X in either conjugacy class of subgroups of order 3 in A. But we may choose X from the class not in L to be T-invariant, and then $V = [V, X] \oplus C_V(X)$ by Coprime Action, so that T has nonzero fixed points on each of the two summands, contrary to $\dim(C_V(T)) = 1$. □

LEMMA B.4.12. *Let G be a group, V an $\mathbf{F}_2G$-module, and $v \in V$, such that*

(i) $V = \langle v^G \rangle$, and

(ii) there exists $U \leq V$ with $\dim(U) > 1$ such that $U^\# \subseteq v^G$.

Then

(1) If G is 2-transitive on v^G then $V^{\#} = v^G$.

(2) Assume $G \cong L_4(2)$ or A_7, $|v^G|$ is odd, and $C_G(v)/O_2(C_G(v)) \cong L_3(2)$. Then $\dim(V) = 4$ and $V^{\#} = v^G$.

PROOF. Assume the hypotheses of (1), and let $W := v^G \cup \{0\}$; to prove (1) if suffices by (i) to show that W is a subspace of V. Thus we must show that if $u, w \in v^G$ are distinct, then $u + w \in v^G$. Pick U as in (ii); as G is 2-transitive on v^G, we may assume $u, w \in U$, so $u + w \in U^{\#} \subseteq v^G$, completing the proof of (1).

Next assume the hypotheses of (2) and let $H := C_G(v)$. By hypothesis, $|G : H| = |v^G|$ is odd, so as $H/O_2(H) \cong L_3(2)$, we conclude H is an end-node maximal parabolic of G if $G \cong L_4(2)$, and $H \cong L_3(2)$ when $G \cong A_7$. In either case, G is 2-transitive on G/H of order 15, so $V^{\#} = v^G$ by (1). Then as $|v^G| = 15$, $\dim(V) = 4$. $\square$

LEMMA B.4.13. *Assume that $G \cong L_4(2)$ and*

(1) $V = \langle v^G \rangle$ for some $v \in V^{\#}$ such that $C_G(v)$ is the parabolic P stabilizing a point of the 6-dimensional orthogonal module W for $G \cong \Omega_6^+(2)$, and

(2) $\langle v^H \rangle$ is of rank 3, where H is a parabolic of G isomorphic to $L_3(2)/E_8$ sharing a Sylow 2-subgroup T of G with P.

Then V is isomorphic to W as an $\mathbf{F}_2G$-module.

PROOF. Let Q be the quadratic form on W, and $(\ ,\ )$ the associated bilinear form. Let Ω be the set of 35 nonzero Q-singular vectors of W, and U the permutation module on Ω, viewed as usual as the power set of Ω. Let f be the symmetric bilinear form on U with $f(x, y) = (x, y)$ for all $x, y \in \Omega$, and q the quadratic form on U with bilinear form f such that $q(x) = 0$ for each $x \in \Omega$. By definition, $P = G_x$ for some $x \in \Omega$; and by (2), $|H : (H \cap P)| = 7$, with H the stabilizer of some totally singular 3-subspace W_H of W containing x, where $W_H \cong \langle v^H \rangle$ as $\mathbf{F}_2H$-module.

By (1) there is a surjective G-homomorphism $\varphi : U \to V$ with $\varphi(x) = v$; let $K := \ker(\varphi)$. Similarly there is a G-surjection $\psi : U \to W$ with kernel K_W, and $f(u, u') = (\psi(u), \psi(u'))$ and $q(u) = Q(\psi(u))$.

Let l be a projective line in W_H, viewed as a 3-subset which is an element of U. By (2), K contains the subspace J of U generated by all G-conjugates of l, so as G is transitive on totally singular lines of W, J contains all such lines. Let $\hat{U} := U/J$; then $J \leq K \cap K_W$ since W also satisfies the hypotheses for V, so φ and ψ induce maps from $\hat{U}$ to V and W, which we also denote by φ and ψ. Each $y \in \Omega$ is orthogonal to one or all of the points on l, so $l \in Rad(f)$, and hence $J \leq Rad(f)$. Similarly $J \leq Rad(q)$. Thus f and q induce forms $\hat{f}$ and $\hat{q}$ on $\hat{U}$ preserved by G such that $\hat{f}(\hat{u}, \hat{u}') = (\psi(\hat{u}), \psi(\hat{u}'))$ and $\hat{q}(\hat{u}) = Q(\psi(\hat{u}))$.

Next there is a conjugate W'_H of W_H under P with $x^\perp = W_H + W'_H$ and $W_H \cap W'_H = \langle x \rangle$. Let

$$\hat{U}_H := \{\hat{w} : w \in W_H \cap \Omega\} \cup \{0\} \text{ and } \hat{U}'_H := \{\hat{w} : w \in W'_H \cap \Omega\} \cup \{0\}.$$

By H.5.5, $\psi : \hat{U}_H \to W_H$ is an $\mathbf{F}_2H$-isomorphism and $\psi : \hat{U}'_H \to W'_H$ is an $\mathbf{F}_2N_G(W'_H)$-isomorphism. Let $\hat{U}(x) := \hat{U}_H + \hat{U}'_H$. As W_H is a maximal totally singular subspace of W, no $w \in W'_H - W_H$ is orthogonal to W_H, so applying ψ, $\hat{w} \notin \hat{U}_H$. Thus ψ induces an isometry of $(\hat{U}(x), \hat{q})$ with $(x^\perp, Q)$, so $\{\hat{w} : w \in x^\perp \cap \Omega\}$ is the set of singular vectors in $\hat{U}(x)^{\#}$.

Let $y \in \Omega - x^{\perp}$ and set $\hat{U}_0 := \hat{U}(x) + \langle \hat{y} \rangle$. As $x^{\perp} \cap y^{\perp}$ is a complement to $\langle y \rangle$ in $y^{\perp}$, $\hat{U}_0 = \langle \hat{x}, \hat{y} \rangle \perp (\hat{U}(x) \cap \hat{U}(y))$, and indeed the same holds for each $y' \in \Omega - x^{\perp}$ with $\hat{y}' \in \hat{U}_0$. In particular it holds for each $y' \in (\Omega - x^{\perp}) \cap y^{\perp}$. Then as for each $y'' \in \Omega - x^{\perp}$, y'' is orthogonal to some y' orthogonal to y, it follows that $\hat{z} \in \hat{U}_0$ for each $z \in \Omega$. Hence $\hat{U} = \hat{U}_0$ by (1). But then ψ is an isometry of $(\hat{U}, \hat{q})$ with (W, Q), so $K_W = J$ and G is irreducible on U/J. Thus as $J \leq K < U$, $J = K$, so the lemma holds. □

LEMMA B.4.14. *Let $G \cong PGL_3(4)$, V a faithful irreducible $\mathbf{F}_2G$-module, P a maximal parabolic of G, $L := P^{\infty}$, and X of order 3 in P centralizing $L/O_2(L)$. Assume*

Each noncentral chief factor for L on V is a natural $L_2(4)$-module for $L/O_2(L)$. (*)

Then V is the adjoint module, and as an $\mathbf{F}_4C_L(X)$-module, $C_V(X)$ is a tensor product of two natural $L_2(4)$-modules, and so is a uniserial module with trivial submodule and quotient module.

PROOF. Let $F := \mathbf{F}_4$, $V^F := V \otimes_{\mathbf{F_2}} F$, $L_X := C_L(X)$, and $K := SL_3(4)$. By the Steinberg Tensor Product Theorem (cf. 2.8.5 in [**GLS98**]), the irreducible FK-modules are the tensor products $M_1 \otimes M_2^{\sigma}$, where M_1 and M_2 are basic irreducible modules and $Gal(F/\mathbf{F}_2) =: \langle \sigma \rangle$. Moreover the nontrivial basic modules for K are the natural module N, its dual N^*, and the adjoint module A of dimension 3,3,8 over F. Further $XL_X \cong GL_2(4)$, and as an XL_X-module, $N = N_1 \oplus N_2$ where N_1 is 1-dimensional, N_2 is the natural L_X-module, and a generator x of X has eigenvalue ω of order 3 on N_1 and ω^{-1} on N_2. Next $N \times N^* = F \oplus A$, so as an XL_X-module, $A = A_0 \oplus A_1 \oplus A_2$, where A_i is the ω^i-eigenspace for x on A, A_1 and A_2 are natural modules for L_X, and $A_3 \cong N_2 \otimes N_2$ as an L_X-module. In particular $A_3 = C_A(X)$ is the uniserial module for L_X described in the lemma.

If both M_1 and M_2 are nontrivial then $N_2 \otimes N_2^{\sigma}$ is a section of V^F, and $N_2 \otimes N_2^{\sigma} = S^F$ for some L_X-section of V such that S is the A_5-module for L_X. This is contrary to (*), so V^F is a Galois conjugate of one of the basic irreducibles. Therefore as $Z(K)$ is trivial on V, V^F is A or A^{σ} and hence as an $\mathbf{F}_2G$-module, V is isomorphic to A. □

B.5. FF-modules for SQTK-groups

In section B.4 we concentrated on the case where $F^*(G)$ was quasisimple and irreducible on V, but in this section we remove those restrictions. We will give a complete description of FF-representations φ of SQTK-groups $\hat{G}$ with $O_2(\hat{G}\varphi) = 1$ in Theorems B.5.1 and B.5.6.

Recall the definition B.1.2 of $\mathcal{P}(G, V)$, and the set $\mathcal{P}^*(G, V)$ of minimal members of $\mathcal{P}(G, V)$ under the partial ordering defined by the relation $\lesssim$.

Given a group X and an $\mathbf{F}_2X$-module U, we will be concerned with the set $Irr_+(X, U)$ (cf. A.1.40) of all X-submodules I of U such that $I = [I, X]$ and $I/C_I(X)$ is X-irreducible. These appear in the literature for example in [**Asc81e**] and [**Asc82a**].

Our first main result of the section essentially maintains the requirement in B.4.2 that $F^*(G)$ be quasisimple, but removes the restriction that V should be an irreducible $F^*(G)$-module:

THEOREM B.5.1. *Let G be a finite group with $F^*(G) = LZ(G)$, where L is a quasisimple strongly quasithin $\mathcal{K}$-group, and V is a faithful $\mathbf{F}_2G$-module. Assume $O_2(G) = 1$ and $G = J(G,V)$. Let $U := [V,L]$ and $T \in Syl_2(G)$. Then*

(1) $L = F^*(G)$, and one of the following holds:

(i) $U \in Irr_+(L,V)$, and L and its action on $U/C_U(L)$ are described in B.4.2. Moreover either $G = L$; or $G = LT \cong S_n$, $5 \leq n \leq 8$, or $G_2(2)$, with $U/C_U(L)$ the natural module.

(ii) $G = L \cong SL_3(2^n)$, and U is the sum of two isomorphic natural modules. The FF*-offenders on U are the conjugates of the full group of transvections with a fixed axis on each summand, and so $q(G,V) = 1$.

(iii) $G = L \cong L_n(2)$ with $n = 4$ or 5, and U is the sum of the natural module and its dual.

(iv) $G = L \cong L_n(2)$ with $n = 4$ or 5, and U is the sum of at most $n-1$ isomorphic natural modules.

(2) $V \neq U \oplus C_V(L)$ iff $C_U(L) > 0$ or $V > U + C_V(L)$, in which case $U \in Irr_+(L,V)$ and $U/C_U(L)$ is a natural module for $L \cong L_2(2^n)$, $Sp_4(2^n)$, $G_2(2^n)$, A_6, A_8, $G_2(2)'$, or $L_3(2)$.

(3) Either $\tilde{U} := U/C_U(L)$ is homogeneous under L, or U is the sum of the natural module and its dual for $L \cong L_n(2)$, $n = 4$ or 5.

(4) If $V \neq U \oplus C_V(G)$, then $U \in Irr_+(L,V)$, and either

(a) $V \neq U \oplus C_V(L)$, so that L and its action on V are described in (2), or

(b) $V = U \oplus C_V(L)$, U is the natural module for $G \cong S_6$ or S_8, and $|C_V(L) : C_V(G)| = 2$.

(5) Either $V = U + C_V(G)$; or $U \in Irr_+(L,V)$ and $U/C_U(L)$ is a natural module for $Sp_4(2^n)$, A_6, A_8, or $L_3(2)$. In the latter case if L is $L_3(2)$, then $C_U(L) = 0$.

(6) Assume B.5.1.1.iii holds, let $T \in Syl_2(G)$, and set $Z := C_V(T)$. Then $C_L(Z) \cong L_{n-2}(2)/2^{1+2(n-2)}$ and $\mathcal{P}(G,V) \cap O_2(C_G(Z)) = \emptyset$.

PROOF. Let $T \in Syl_2(G)$ and $A \in \mathcal{P}(T,V)$. As $F^*(G) = LZ(G)$ and $O_2(G) = 1$, T is faithful on L, so $L = F^*(LT)$. Further $G = J(G,V) = \langle J(T,V)^G \rangle$, so:

(α) If $J(T,V) \leq L$, then $G = L$.

(β) If $Out(L)$ is abelian, then $G = LT$.

We will reduce to (α) or (β) in various cases of the proof of (1).

Assume first that $U \in Irr_+(L,V)$; we will show that (i) holds. Set $S := U/C_U(L)$. Then S is an irreducible L-module, and L is faithful on S by Coprime Action, so LT is faithful on S as $F^*(LT) = L$. By B.1.5.8, A contains some $B \in \mathcal{P}(T,S)$, and we may take $A = B$ if $q(T,S) = 1$. Hence setting $J := J(LT,S)$, the pair J,S is described in B.4.2, and in particular either

(α') $J = L$, or

(β') $J = S_n$ or $G_2(2)$, with S the natural L-module.

In (β'), $Out(L)$ is abelian, so $G = LT$ by (β). Further $Aut(G) \cong J$ unless L is A_6, and G is not $Aut(A_6)$ since the natural module $S = U/C_U(L)$ does not admit $Aut(A_6)$. Thus (i) holds in this case.

So we may assume that (β') does not hold, and hence $L = J$. Now we already observed that we may take $B = A$ if $q(T, S) = 1$; and we may also take $B = A$ when $C_U(L) = 0$ by B.1.5.1. Thus under either of these assumptions, $A = B \leq J = L$ for all $A \in \mathcal{P}(T, V)$, so that $L = G$ by (α), and again (i) holds.

Thus we may finally assume that $C_U(L) \neq 0$ and $q(T, S) < 1$. Then from the sublist of B.4.2 with $q(T, S) < 1$ and (β') removed, and inspecting I.1.6 for cases with $C_U(L) > 0$, we conclude that L is $L_3(2)$ or $Sp_4(2^n)$ for $n > 1$, and S is a natural module with $\mathcal{P}(T, S) \subseteq L$. Further the nonsplit extension U is described in B.4.8.2 or I.2.3. We check in these cases (see especially the explicit computation of FF*-offenders in B.4.8.2) that $\mathcal{P}(T, U) \subseteq L$, so again $L = G$ by (α), and (i) holds. This completes the treatment of the case $U \in Irr_+(L, V)$.

Thus in the remainder of the proof of (1), we may assume that $U \notin Irr_+(L, V)$, and we will show that one of (ii)–(iv) holds. Pick $A \in \mathcal{P}^*(G, V)$ and let $U_1, \ldots, U_r$ denote the noncentral chief factors for LA on V. As $A \in \mathcal{P}^*(G, V)$, A acts on each member of $Irr_+(L, V)$ by B.1.5.3. Therefore as $U \notin Irr_+(L, V)$:

$$r > 1.$$

Further as A is faithful on L, and L is quasisimple and nontrivial on U_i, LA is faithful on U_i. Set $m_i := m(U_i/C_{U_i}(A))$. Now $r_{A,V} \leq 1$ as $A \in \mathcal{P}(G, V)$, so (since $\overline{C_X(A)} \leq C_{\bar{X}}(A)$ for any quotient $\bar{X}$ of any A-submodule X of V)

$$m(A) \geq m(V/C_V(A)) \geq \sum_{i=1}^{r} m_i.$$

Further $r_{A,U_i} \geq q(Aut_{LA}(U_i), U_i)$, so setting

$$q := \min\{q_i : 1 \leq i \leq r\},$$

we have

$$1 \geq m(V/C_V(A))/m(A) \geq (\sum_{i=1}^{r} m_i)/m(A) = \sum_{i=1}^{r} r_{A,U_i} \geq \sum_{i=1}^{r} q_i \geq rq. \qquad (*)$$

Then as $r > 1$, we conclude from (*) that

$$q \leq 1/r \leq 1/2, \ q_i < 1 \ \text{ for each } i, \text{ and } \ q_i + q_j \leq 1 \ \text{ for each } i \neq j. \qquad (+)$$

Further by B.1.5.8, A contains a member B_i of $\mathcal{P}(Aut_{LA}(U_i), U_i)$. So as $r > 1$ and $U_i = [U_i, L]$, by induction on the dimension of V, the pair $Aut_{LB_i}(U_i)$, U_i satisfies one of conclusions (i)–(iv) of (1); and in particular the LA-chief factor U_i is a semisimple L-module. Hence to describe the value of q_i, we may use sums of the corresponding values for the L-irreducible modules in B.4.2. Pick k with $q_k = q$. Then since $q_k = q \leq 1/2$ by (+), we conclude that

$$Aut_L(U_k) \cong SL_3(2^m) \ \text{ or } L_n(2) \text{ for } n = 4 \text{ or } 5. \qquad (**)$$

Next B.4.2 also provides the following lower bounds:

$$\text{Each } \ q_i \geq 1/2, 1/3, 1/4 \ \text{ in the respective cases of } (**). \qquad (++)$$

Since $q_i + q_j \leq 1$ by (+), it follows from (++) that for each i, $q_i \leq 1/2, 2/3, 3/4$ in the respective cases of (**). Hence for any i, U_i cannot contain an L-irreducible in case (10) or (11) of B.4.2. This leaves cases (2) and (9), so that the L-constituents of U_i are (possibly non-isomorphic) natural L-modules. Indeed from B.4.9.2, U_i is a single natural L-module when $Aut_L(U_i) = SL_3(2^m)$, or a sum of at most $n - 2$ isomorphic natural L-modules when $Aut_L(U_i) = L_n(2)$. Furthermore by A.3.6.2,

$Aut_L(U_i)$ has no proper covering over a center of odd order, so that LA is faithful on U_i.

In proving that one of (ii)–(iv) holds, we must also show that $G = L$; we now prove this in two special cases:

(I) The L-socle $Soc(U)$ of U is a direct sum of isomorphic natural L-modules I.

(II) $Soc(U)$ is the direct sum of natural modules and their duals for $L \cong L_n(2)$.

Later we will show that one of these cases must always arise. To obtain $G = L$ in these cases, we will use the following observation: If LT acts on some nontrivial irreducible L-submodule I of V, then I is some U_i, and so by our discussion above, I is a natural L-module in case (2) or (9) of B.4.2; hence $\mathcal{P}(LT, I) \subseteq L$, so as $\mathcal{P}(LT, V) \subseteq \mathcal{P}(LT, I)$ using B.1.5.1, we have $G = L$ by (α).

Now in case (I), T acts on such an irreducible by A.1.42, so $G = L$ by the observation.

Notice since we saw that each U_i is L-homogeneous, this reference also shows (independently of case (I)) that A acts on some irreducible L-submodule of U_i; hence as U_i is an LA-chief factor,

$$U_i \text{ is an irreducible natural } L\text{-module.}$$

Next in case (II), if $B \in \mathcal{P}(G, V)$ and there is $b \in B - N_B(I)$, then $m(U/C_U(b)) \geq m(I) = n$ and $m_2(C_G(b)) = 4$ as $b \notin L$. Thus as $r_{B,V} \leq 1$, $n = m(B) = 4$ and B centralizes $C_U(b)$, impossible as $B \cap L$ is faithful on $C_U(b)$. Therefore each member of $\mathcal{P}(G, V)$ normalizes I, and hence LT normalizes I since $G = J(G, V)$, so again $G = L$ by the observation.

We begin by considering the first case in (**), where $L \cong SL_3(2^m)$. Then our earlier discussion shows that

(a) $r = 2$ and $q_1 = 1/2 = q_2$,

(b) U_i is a natural module for L, with LA faithful on U_i, and

(c) $2m_i \leq m(A)$, so that A is the full group of transvections on each U_i with a fixed axis, and $m_i = n = m(A)/2$.

By (b) and (c), U_1 and U_2 are isomorphic natural modules. If $n > 1$, then by I.1.6, $H^1(G, U_i) = 0$, so U has no trivial chief factors. Assume now that U has no trivial chief factors. Then we may take U_1 to be a submodule of U, and $U_2 = U/U_1$. So by B.4.9.2.i, the L-module U splits. Then the L-socle $Soc(U)$ is the direct sum of two natural modules isomorphic to U_1, so $G = L$ from the discussion of case (I), and hence conclusion (ii) of (1) holds. Thus we have reduced to the case where $n = 1$ so that G is $L_3(2)$, and also there is a nonsplit 1-dimensional submodule of some U_i. Thus U involves a 4-dimensional section S as in B.4.8.2, and therefore $q(G, S) = 1$, so that $q(G, V) \geq q(Aut_G(S), S) + q(Aut_G(U_{3-i}), U_{3-i}) > 1$, contrary to $q(G, V) \leq 1$.

Finally we consider the remaining cases in (**), where $L \cong L_n(2)$ with $n = 4$ or 5, and each U_i is a natural module for L. Pick U_0 to be a maximal LA-submodule of U. As $U = [U, L]$, we may choose our notation so that $U_r = U/U_0$, and $U_1, \ldots, U_{r-1}$ are the noncentral LA-chief factors of U_0. Further $H^1(L, U_i) = 0$ for any $i = 1, \ldots, r$ by I.1.6; so $U_0 = [U_0, L]$ and $C_{U_0}(L) = 0$. As $q_i \geq 1/(n-1)$ by (++), $r \leq n - 1$. By B.1.5.2, $Aut_A(U_0) \in \mathcal{P}^*(Aut_G(U_0), U_0)$, so as $U_0 = [U_0, L]$, by induction on $\dim(V)$, the pair $Aut_{LA}(U_0)$, U_0 satisfies (i), (iii), or (iv) of (1). Further if U_0

satisfies (i), then U_0 and $U/U_0 \cong U_r$ are natural L-modules, as we saw this is true for each U_i. Hence one of the following holds:

(A) U_0 and U/U_0 are natural L-modules.

(B) $r \geq 3$, and U_0 is the direct sum of $r-1 \leq n-2$ isomorphic natural L-modules.

(C) $r = 3$, and U_0 is the direct sum of a natural L-module and its dual.

In particular case (I) or (II) holds, so $G = L$ by our earlier discussion of those cases.

Now if (A) holds, then U splits over U_0 by B.4.9.2.i, so conclusion (iii) or (iv) of (1) holds.

If (C) holds, then from the values of 1 or 5/6 when $n = 4$ or 5 in (iii) and (iv) of B.4.9.2,

$$m(U_0/C_{U_0}(A)) > 1 - \frac{1}{n-1},$$

so as $q_3 \geq q \geq 1/(n-1)$ by (++), we contradict $q(G,V) \leq 1$.

Therefore we may assume that (B) holds. As $G = L$ we may choose notation so that U_0 is the direct sum of irreducibles realizing the factors $U_1, \ldots, U_{r-1}$ of U_0, with $U_i \cong U_1$ for all $i < r$.

Next for $i < r$, set $U^i := U/U_i$. Then by B.1.5.8, A contains a member of $\mathcal{P}(Aut_G(U^i), U^i)$, and hence U^i is an FF-module. Therefore as $r - 1 > 1$, by induction on $\dim(V)$, U^i is described in (iii) or (iv), so U^i is the direct sum of irreducible L-submodules. Further the preimage W_0 of a maximal L-submodule of U^i is a maximal L-submodule of U, so applying our arguments to W_0 in the role of U_0, W_0 is homogeneous. As $r > 2$ it follows that U is a homogeneous module for $L = G$, so conclusion (iv) of (1) holds. This completes the treatment of (B), and hence completes the proof of (1).

The equivalence in (2) is immediate, so assume that either $C_U(L) > 0$ or $V > U + C_V(L)$; then $H^1(L, U_i) \neq 0$ for some i. Applying the cohomological results in I.1.6 to the list in B.4.2 reduces that list to the groups given in (2), and shows furthermore that for some $I \in Irr_+(L,V)$, $I/C_I(L)$ is the natural module. Suppose that $U \notin Irr_+(L,V)$. Among the groups in (2), the only example appearing in cases (ii)–(iv) of (1) is $L \cong L_3(2)$ in case (ii), where $C_U(L) = 0$. Thus $V > U$. But notice that the argument used in the branch of the proof of (1) leading to case (ii) can be applied via B.4.8.3 rather than B.4.8.2, to give $V = U$—contradicting $V > U$. Thus $U = I \in Irr_+(L,V)$, and (2) holds.

Part (3) follows by inspection of the examples that occur in (1): Namely in case (i) of (1), $\tilde{U}$ is irreducible and hence certainly homogeneous under L, while in the remaining cases, U is homogeneous, apart from the the exceptional case (iii)—which is allowed in (3).

Assume the hypothesis of (4), and also $V = U \oplus C_V(L)$. Thus $C_V(L) > C_V(G)$ and hence $G > L$. By hypothesis $G = J(G,V)$, so $A \not\leq L$ for some $A \in \mathcal{P}(G,V)$. Thus case (i) of (1) holds, so $U \in Irr_+(L,V)$ and $G \cong S_n$ or $G_2(2)$ with $U/C_U(L)$ the natural module. Thus $G/L \cong A/(A \cap L) \cong \mathbf{Z}_2$. But unless G is S_6 or S_8, we have $q(G,U) = 1 = q(G,\tilde{U})$ by B.3.2.4 and B.4.6.13—in which case $V = U + C_V(A)$ by B.1.5.8, so in particular $C_V(L) \leq C_V(A)$ for each $A \in \mathcal{P}(G,V)$, contradicting $C_V(L) > C_V(G)$. Thus $G = S_6$ or S_8, so $m(U/C_U(A)) \geq m(A) - 1$ by B.3.2. Hence $|C_V(L) : C_V(G)| = 2$, and thus conclusion (b) of (4) holds.

Suppose that $V > U + C_V(G)$. Then by (4), $U \in Irr_+(L,V)$, and L and its action on V are described in (2) or (4b). But if $L \cong L_2(2^n)$ or $G_2(2^n)'$, the duals of the nonsplit extensions in I.2.3 are not FF-modules, since for U the irreducible submodule of V, $q(G,U)=1$, so $V=U+C_V(A)$ for each $A\in\mathcal{P}(G,V)$ by B.1.5.2, whereas $C_U(A)=C_V(A)$. Further if $L\cong L_3(2)$ then $G=L$ by B.4.2.2, and by B.4.8.4, $C_U(G)=0$. So (5) holds.

Assume the hypotheses of (6). By (4), $V=C_V(G)\oplus U$; and in (1.iii), $U=U_1\oplus U_2$ with U_1 natural and U_2 dual to U_1. So $Z=C_V(G)\oplus Z_U$ where $Z_U:=C_U(T)$, and either

(γ) T acts on U_1, U_2, and $Z_U=Z_1\oplus Z_2$ with $Z_i:=C_{U_i}(T)$ of rank 1; or

(δ) T contains an involution inducing an outer automorphism on L switching U_1 and U_2, so that Z_U is of rank 1.

In either case $C_L(Z)\cong L_{n-2}(2)/2^{1+2(n-2)}$ is the parabolic obtained by removing the end nodes. Now none of the FF*-offenders determined in (iii) and (iv) of B.4.9.2 are contained in $O_2(C_L(Z))$, establishing the final statement of (6). This completes the proof of Theorem B.5.1 □

The following lemma is useful in analyzing representations of 2-locals in QTKE-groups on internal modules of the 2-local:

LEMMA B.5.2. *Let G be a finite group which is a quotient of an SQTK-group. Then there exists an SQTK-group $\hat{G}$ and $\hat{K}\trianglelefteq\hat{G}$ such that $G=\hat{G}/\hat{K}$, and*

(1) The preimage of $F(G)$ in $\hat{G}$ is nilpotent.

(2) $\hat{K}$ is nilpotent of odd order.

(3) For each component L of G, $L=\hat{L}\hat{K}/\hat{K}$ for some component $\hat{L}$ of $\hat{G}$.

(4) $F^*(G)=F^*(\hat{G})/\hat{K}$.

PROOF. By hypothesis, $G=\hat{G}/\hat{K}$ for some SQTK-group $\hat{G}$ and $\hat{K}\trianglelefteq\hat{G}$. Let $\hat{F}$ be the preimage in $\hat{G}$ of $F(G)$, p a prime, and $P\in Syl_p(\hat{F})$. Then $O_p(G)=P\hat{K}/\hat{K}$ and $\hat{G}=\hat{K}N_{\hat{G}}(P)$ by a Frattini Argument. Hence

$$G=\hat{G}/\hat{K}=\hat{K}N_{\hat{G}}(P)\cong N_{\hat{G}}(P)/N_{\hat{K}}(P);$$

so replacing $\hat{G}$ by $N_{\hat{G}}(P)$, we may assume $P\trianglelefteq\hat{G}$. That is, we may choose $\hat{G}$ and $\hat{K}$ so that (1) holds.

Next as $\hat{G}$ is an SQTK-group, so is $\hat{G}/O_2(\hat{K})$ and

$$\hat{G}/\hat{K}\cong\frac{\hat{G}/O_2(\hat{K})}{\hat{K}/O_2(\hat{K})}.$$

Thus replacing $\hat{G}$ by $\hat{G}/O_2(\hat{K})$, we may assume $O_2(\hat{K})=1$, and with this choice (2) follows from (1).

Let L be a component of G, $\hat{L}_1$ its preimage in $\hat{G}$, $\hat{L}:=\hat{L}_1^\infty$, and $\hat{R}:=\hat{K}\cap\hat{L}$. As L is quasisimple, $\hat{L}_1=\hat{K}\hat{L}$, so $\hat{L}/\hat{R}\cong L$ is quasisimple. Observe $O_2(\hat{L})\le Z(\hat{L})$: for $O_2(\hat{L})$ centralizes $\hat{R}$ as $\hat{R}$ is of odd order, and $O_2(L)\le Z(L)$. Let $\hat{L}^*:=\hat{L}/O_2(\hat{L})$. Then by A.3.6, either $\hat{L}^*$ is quasisimple, or $\hat{L}^*$ is described in case (3) or (4) of A.3.6. In the first case, (3) holds; so we may assume $\hat{L}^*$ is described in case (3) or (4) of A.3.6. Thus $\hat{L}^*/O(\hat{L}^*)\cong SL_2(p)$ for some prime p, so as $\hat{R}$ is of odd order and $SL_2(p)$ has trivial multiplier (e.g. I.1.3), $\hat{R}^*=O(\hat{L})^*$. Now (1) supplies a contradiction, as $Z(L)\le F(G)$ but $O_{2',2}(\hat{L}^*)$ is not nilpotent. Thus the proof of (3) is complete.

By (1), $\hat{F} \leq F(\hat{G})$. On the other hand, $F(\hat{G})/\hat{K}$ is a nilpotent normal subgroup of G, so $F(\hat{G}) = \hat{F}$. Similarly if $\hat{E}$ is the preimage of $E(G)$, then $\hat{E} \leq E(G)\hat{K}$ by (3), and of course $E(G) \leq \hat{E}$. Thus (4) is established. □

Theorem B.5.1 determines the FF-modules for an SQTK-group G with $F^*(G)$ quasisimple. In the next few lemmas (including our second main result Theorem B.5.6 and its proof), we remove the restriction on $F^*(G)$, and instead assume the following hypothesis:

HYPOTHESIS B.5.3. *G is a finite group and V is a faithful $\mathbf{F}_2G$-module such that $O_2(G) = 1$, $G = J(G, V)$, and G is a quotient of an SQTK-group.*

Notice that Hypothesis B.5.3 holds when $G = H/C_H(V)$ for some SQTK-group H and $V \in \mathcal{R}_2(H)$, and of course this is exactly the situation of interest when we study failure of factorization in 2-local subgroups of a QTKE-group.

NOTATION B.5.4. Under Hypothesis B.5.3, choose $\hat{G}$ and $\hat{K}$ with $G = \hat{G}/\hat{K}$ as in B.5.2. Let $\alpha : \hat{G} \to G$ be the natural surjection.

In the next lemma B.5.5, we record some elementary consequences of Hypothesis B.5.3, such as the possible components L of G.

LEMMA B.5.5. *Assume Hypothesis B.5.3. Then*

(1) If L is a component of G, then

(i) $L \trianglelefteq G$.

(ii) There exists $A \in \mathcal{P}(G, V)$ with $L = [L, A]$, and for each such A, L is faithful on $W := C_V(O_2(LA))$, $C_A(W) = C_A(L)$, and $Aut_A(W) \in \mathcal{P}(Aut_G(W), W)$.

(iii) There is $A \in \mathcal{P}(G, V)$ and an A-invariant faithful L-chief section S of V such that $L = [L, A]$, $F^*(Aut_{LA}(S)) = L$, and $Aut_A(S)$ contains a member of $\mathcal{P}(LAut_A(S), S)$.

(iv) $L \cong L_2(2^n)$, $SL_3(2^n)$, $Sp_4(2^n)'$, $G_2(2^n)'$, A_7, $\hat{A}_6$, $L_4(2)$, or $L_5(2)$, and the section S of (iii) is described in B.4.2.

(2) $O_p(G) \leq Z(G)$ for each prime $p \neq 3$. So $F^*(G) = E(G)O_3(G)Z(G)$.

PROOF. If $X \trianglelefteq G$, then as $G = J(G, V)$ by Hypothesis B.5.3, either $[X, A] \neq 1$ for some $A \in \mathcal{P}(G, V)$ or $X \leq Z(G)$.

In particular if $O_p(G) \not\leq Z(G)$ for some prime p, then as p is odd by Hypothesis B.5.3, taking $X := O_p(G)$, we have $Y := [X, A] \neq 1$ for some $A \in \mathcal{P}(G, V)$. Let $G_0 := YA$, $B := C_A(Y) = O_2(YA)$, $W := C_V(B)$ and $G_0^* := G_0/C_{G_0}(W)$. Then by the Thompson $A \times B$ Lemma, Y is faithful on W, so as $B = C_A(Y)$, also $B = C_A(W)$. Therefore $O_2(G_0^*) = C_{A^*}(Y^*) = B^* = 1$, and by B.1.5.1, $A^* \in \mathcal{P}(G_0^*, W)$. Thus by B.1.8, $p = 3$, establishing (2).

It remains to prove (1), so we may assume that L is a component of G. By B.1.5.4, $\mathcal{P}(G, V) \subseteq N_G(L)$, so using Hypothesis B.5.3, $G = \langle \mathcal{P}(G, V) \rangle \leq N_G(L)$, proving (i). Further L is not central in G, so as noted at the outset, $[L, A] \neq 1$ for some $A \in \mathcal{P}(G, V)$, and hence $L = [L, A]$, proving the first part of (ii).

Now $O_2(L) \leq O_2(G) = 1$. Set $B := C_A(L) \leq O_2(LA)$ and $W := C_V(O_2(LA))$. Since $O_2(L) = 1$, the Thompson $A \times B$ Lemma says L is faithful on W; so $B = C_A(W)$ and $A/B = Aut_A(W) \in \mathcal{P}(Aut_{LA}(W), W)$, completing the proof of (ii). So replacing (G, V) by $(Aut_{LA}(W), W)$, we may assume that $F^*(G) = L$. Now as $F^*(G) = L$ and $O_2(G) = 1$, $L = [L, A]$ for *all* $A \in \mathcal{P}(G, V)$. Therefore we may take $A \in \mathcal{P}^*(G,V)$, and hence by B.1.5.3, A acts on each $I \in Irr_+(L,V)$. So replacing V by I, we may take $V \in Irr_+(L,V)$. Finally setting $\tilde{V} := V/C_V(L)$, by B.1.5.8, $A \cong Aut_A(\tilde{V})$ contains a member of $\mathcal{P}(G,\tilde{V})$. Thus we may even assume that V is an irreducible L-module; that is, $I/C_I(L)$ is the section S required by (iii). In particular, G and its action on V are described in B.4.2. Finally the groups described in B.4.2 are just those in (iv). □

We are now ready to state the second main result of the section, which describes FF-representations φ for the general SQTK-group $\dot{G}$ with $O_2(\dot{G}\varphi) = 1$.

THEOREM B.5.6. *Assume Hypothesis B.5.3. Then one of the following holds:*

(1) $F^*(G)$ is quasisimple, so G and its action on V are described in Theorem B.5.1. In particular, $F^*(G) \cong L_2(2^n)$, $SL_3(2^n)$, $Sp_4(2^n)$, $G_2(2^n)$, A_6, A_7, $\hat{A}_6$, $G_2(2)'$, $L_4(2)$, or $L_5(2)$.

(2) $G \cong S_3$ and $V = [V,G] \oplus C_V(G)$ with $m([V,G]) = 2$.

(3) $G = G_1 \times G_2$ and $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$ as a G-module, where $\tilde{V} := V/C_V(G)$, and each G_i is one of $L_2(2^{n_i})$, $n_i \geq 1$, S_5, or $L_3(2^{m_i})$, $m_i \geq 1$ odd. Moreover one of the following holds:

(a) $V_i := [V,G_i]$ and $\tilde{V}_i$ is the natural module for G_i.

(b) $V_i := [V,G_i]$ and $\tilde{V}_i$ is the sum of two isomorphic natural modules for $G_i \cong L_3(2^{m_i})$.

(c) $\tilde{V}_i$ is a 4-dimensional indecomposable for $G_i \cong L_3(2)$ described in B.4.8.3, and $[V,G_i] \leq V_i$.

(4) $G = (G_1 \times G_2)\langle t\rangle$ and $V = C_V(G) \oplus V_1 \oplus V_2$, where $V_i := [V,G_i]$, $G_1 \cong A_6$, $m(V_1) = 4$ or 5, $G_2 \cong \mathbf{Z}_3$, $m(V_2) = 2$, and t induces an outer automorphism on G_1 and G_2.

(5) $G = G_1 \times G_2$ and $[V,G] = V_1 \oplus V_2$, where $V_i := [V,G_i]$, $G_1 \cong L_3(2)$, V_1 is of rank 3, $[V,G] + C_V(G)$ is a hyperplane of V, and either

(a) V_2 is the sum of at most two isomorphic natural modules for $G_2 \cong L_3(2)$, or

(b) $V_2/C_{V_2}(G_2)$ is the natural module for $G_2 \cong L_2(2^n)$.

The proof proceeds via a short series of reductions. So until the proof of the theorem is complete, we assume G, V satisfy the hypotheses of Theorem B.5.6, but not its conclusion. We first see that the existence in G of a component of 3-rank 2 leads to conclusions (1) or (4):

LEMMA B.5.7. *If L is a component of G, then $m_3(L) = 1$.*

PROOF. Assume $m_3(L) > 1$. By B.5.2, $L = \hat{L}\alpha$ for some component $\hat{L}$ of $\hat{G}$. Applying A.3.18 to $\hat{G}$, we conclude $\hat{L} = E(\hat{G})$, and either

(i) $C_{\hat{G}}(\hat{L})$ is a $3'$-group, or

(ii) $\hat{L} \cong SL_3^\epsilon(q)$, $\hat{A}_6$, or $\hat{A}_7$, and a Sylow 3-subgroup $\hat{X}$ of $O_3(\hat{G})$ is cyclic with $\Omega_1(\hat{X}) = Z(\hat{L})$. In this case, set $X := \hat{X}\alpha$.

As $\hat{L} = E(\hat{G})$, $L = E(G)$ by B.5.2.4, so it follows from B.5.5.2 that $F^*(G) = LO_3(G)Z(G)$. Thus if $O_3(G) \leq Z(G)$, then $F^*(G) = LZ(G)$, so case (1) of Theorem B.5.6 holds by Theorem B.5.1, contrary to our choice of G, V as a counterexample. Therefore $O_3(G) \not\leq Z(G)$. In particular, case (ii) holds. Then by B.5.5.1.iv, either

(a) $L \cong \hat{L} \cong SL_3(2^n)$, n even, or $\hat{A}_6$, or

(b) $\hat{L} \cong \hat{A}_6$ or $\hat{A}_7$ and $L \cong A_6$ or A_7.

Next as $O_2(G) = 1$ and $F^*(G) = LO_3(G)Z(G)$, a Sylow 2-subgroup S of $C := C_G(L)$ is faithful on $O_3(G)$. Thus if $S \neq 1$, then as $\hat{X}$ is cyclic, there is an involution $s \in S$ inverting X, so there is an involution $\hat{s} \in \hat{C}$ inverting $\hat{X}$. This is impossible as $\hat{s}$ centralizes $\hat{L}$ and $\Omega_1(\hat{X}) = Z(\hat{L})$. Thus C is of odd order.

As C is of odd order, every $A \in \mathcal{P}(G, V)$ acts faithfully on L, and $F^*(LA) = L$. Thus since $A \leq L$ in cases (ii)–(iv) of B.5.1.1 and since (a) or (b) holds, either $A \leq L$; or else $U := [V, L] \in Irr_+(L, V)$, and using B.4.3, $\hat{U} := U/C_U(L)$ is the natural module for $LA \cong S_6$ or S_7, so in particular (b) holds. But as $G = J(G, V)$ and $X \not\leq Z(G)$, some $A \in \mathcal{P}(G, V)$ acts nontrivially on X, so in particular $A \not\leq L$. Thus $LA \cong S_6$ or S_7 and $\hat{U}$ is the natural module.

If L is not irreducible on U, then by B.5.1.2, $L \cong A_6$ and U is the 5-dimensional core of a 6-dimensional permutation module. Otherwise, U is the natural irreducible module. In particular in each case, $End_{\mathbf{F}_2 L}(\hat{U}) = \mathbf{F}_2$, so $[U, X] = 0$ and hence $[V, LX] = U \oplus [V, X]$. Further if $AL \cong S_7$, then $m(U) = 6$ and $q(LA, U) = 1 = q(LA, \hat{U})$ by B.3.2.4, so $V = U + C_V(A)$ by B.1.5.8, and hence A centralizes $[V, X]$, contradicting $X = [X, A]$. Thus $L \cong A_6$. By B.3.4.2.i, $r_{A,U} = 1$ unless $m(A) = 3$ and $m(U/C_U(A)) = 2$. Therefore as $C_{[V,X]}(A) < [V, X]$, the latter case holds with $m([V, X]) = 2$, so conclusion (4) of Theorem B.5.6 holds, contrary to the choice of G, V as a counterexample. □

Lemma B.5.8. *$\mathcal{P}^*(G, V)$ centralizes $F(G)$.*

Proof. Assume $A \in \mathcal{P}^*(G, V)$ is nontrivial on $F(G)$. Then by B.1.9, $G = G_1 \times G_2$ and $V = V_1 \oplus V_2$, where $A \leq G_1 \cong L_2(2)$, $V_1 := [V, G_1]$ is of rank 2, $G_2 = J(G_2, V)$ centralizes V_1, and $V_2 := C_V(G_1)$. If $G_2 = 1$ then case (2) of B.5.6 holds, contrary to our choice of G as a counterexample. Therefore $G_2 \neq 1$, and as G_2 centralizes V_1, G_2 is faithful on V_2. Thus the pair G_2, V_2 satisfies the hypotheses of Theorem B.5.6, so by induction on the order of G, G_2 and its action on V_2 are described in that Theorem. As $G_1 \cong L_2(2)$, $m_3(G_2) = 1$ by A.1.31.1. Thus G_2, V_2 satisfy conclusion (1) or (2) of B.5.6, and in case (1), $F^*(G_2) \cong L_2(2^n)$ or $L_3(2^m)$ with m odd. If (2) holds, then G satisfies conclusion (3) of B.5.6. If (1) holds, then the action of G_2 on V_2 is described in Theorem B.5.1. If $V_2 = [V, G_2] + C_V(G_2)$, then G satisfies (a) or (b) of conclusion (3) of Theorem B.5.6. Otherwise $V_2 > [V, G_2] + C_V(G_2)$. Then by B.5.1.5 and B.4.8, $\tilde{V}_2$ is the 4-dimensional indecomposable for $G_2 \cong L_3(2)$ described in B.4.8.3, so that G satisfies conclusion (3c) of B.5.6. Thus in each case G, V is not a counterexample. □

For the rest of the proof of B.5.6, we take

$$A \in \mathcal{P}^*(G, V);$$

that is, we take A to be minimal under the relation $\lesssim$ in Definition B.1.2. By B.5.8, $\mathcal{P}^*(G, V)$ centralizes $F(G)$, so using B.5.5.1.iv,

$$A \text{ is faithful on } E(G) = O^{3'}(E(G)),$$

and hence $[L, A] \neq 1$ for some component L of G. By B.5.5.1.i, $L \trianglelefteq G$, so $L = [L, A]$. Let $W := [V, L]$ and $\tilde{W} := W/C_W(L)$.

By B.5.7,

$$m_3(L_1) = 1 \text{ for each component } L_1 \text{ of } G.$$

Thus inspecting the list of B.5.5.1.iv, we conclude that L is either $L_2(2^n)$, or $SL_3(2^m)$ for m odd.

We can now more or less repeat the proof of 14.8 in [**Asc82a**]. The next lemma is a start toward pinning down the modules that are involved.

LEMMA B.5.9. *One of the following holds:*

(1) $L \cong L_2(2^n)$, $A \in Syl_2(L)$, $\tilde{W}$ *is the natural module for* L, *and* $V = W + C_V(A)$.

(2) $LA \cong S_5$, $|A| = 2$, W *is the* A_5*-module, and* $V = W + C_V(A)$.

(3) $L \cong L_3(2^m)$ *for* m *odd,* W *is the direct sum of two isomorphic natural* $\mathbf{F}_2L$*-modules,* $A \leq L$, *and* $V = W + C_V(A)$.

(4) $L \cong L_3(2^m)$ *for* m *odd,* W *is a natural* L*-module, and* A *induces inner automorphisms on* L.

(5) $L \cong L_3(2)$, $E_4 \cong A \leq L$, W *is the 4-dimensional indecomposable* L*-module described in B.4.8.2, and* $V = W + C_V(A)$.

PROOF. Set $B := C_A(L)$, $U := [C_V(O_2(LA)), L]$, and $(LA)^+ := LA/C_{LA}(U)$; by B.5.5.1, L is faithful on U and $B = C_A(U)$. By B.1.5, $A^+ \in \mathcal{P}^*(L^+A^+, U)$; with $B = 1$ and $V = U + C_V(A)$ in case $q(L^+A^+, U) = 1$. In this last case $U = W$ since $L = [L, A]$. Also as we just observed, $L \cong L_2(2^n)$ or $L_3(2^m)$ for m odd; we examine the possible representations of L^+A^+ on U appearing in Theorem B.5.1 and B.4.2.

Suppose first that $L \cong L_2(2^n)$. Then by B.5.1.1, $U \in Irr_+(L, V)$. Therefore by B.4.2, either:

(I) $L \cong L_2(2^n)$, $\tilde{U}$ is the natural module for L, $A^+ \in Syl_2(L^+)$, and $m(A^+) = m(U/C_U(A)) = n$, or

(II) $L^+A^+ \cong S_5$, U is the A_5-module, A^+ is generated by a transposition since it is a minimal FF^*-offender, and $m(U/C_U(A)) = 1$.

In cases (I) and (II), $q(L^+A^+, U) = 1$, so $U = W$ by an earlier remark.

Suppose next that $L \cong L_3(2^m)$ for m odd. Then either case (i) of B.5.1.1 holds with $U \in Irr_+(L, V)$, or case (ii) holds with U a sum of two isomorphic natural modules. In case (ii), $m(A^+) = 2m$ and $q(L^+A^+, U) = 1$, so in that case:

(III) $L \cong L_3(2^m)$, $W = U$ is the sum of two isomorphic natural modules, A induces inner automorphisms on L, and $m(A^+) = 2m = m(W/C_W(A))$.

In case (i) by B.4.2, $\tilde{U}$ is a natural module for L. By I.1.6, either $C_U(L) = 0$, or $m = 1$ and U is described in B.4.8.2. In particular $q(L^+A^+, U) = 1$ in the latter case. This leads to our final two cases:

(IV) U is natural and $A^+ \leq L^+$.

(V) $L \cong L_3(2)$, $U = W$, $m(W) = 4$, and $m(W/C_W(A)) = m(Aut_A(W)) = 2$.

In all cases other than (IV), $U = W$ and $m(A^+) = m(W/C_W(A))$, so as we saw in the first paragraph of the proof, $V = W + C_V(A)$ and $B = 1$. In particular, (2) holds in case (II), so we may assume we are in one of the remaining cases. Therefore A induces inner automorphisms on L, so $A \leq A_L \times A_C$, where A_L and A_C are the projections of A on L and $C_G(L)$, respectively.

Assume case (IV) does not hold. Then $V = W + C_V(A)$, so AL centralizes V/W, and hence A_C centralizes V/W. Also in (I) and (V), $W \in Irr_+(L, V)$, so

A_C centralizes W by the Thompson $A \times B$-Lemma. Thus

$$A_C \leq C_G(W) \cap C_G(V/W) \leq O_2(G) = 1$$

and hence $A = A_L$ so that $A \leq L$, giving conclusion (1) or (5).

Thus we may assume that case (III) holds, and in view of the argument we just made, we may also assume $A_C \neq 1$ is faithful on W. But A_L fixes $Irr_+(L,V)$ pointwise, as does A by B.1.5.3, so A_C fixes $Irr_+(L,V)$ pointwise, contradicting A_C faithful on W. This contradiction shows $A \leq L$ in (III), completing the proof that (3) holds in this case.

It remains to consider case (IV), where we may assume $U < W$, so $A_C \neq 1$. As $C_G(A) \cap C_G(L) = C_G(A_C) \cap C_G(L)$, it follows that there is a component K distinct from L with $K = [K, A]$. Then as $A \not\leq K$ and $|A| > 2$, it follows from symmetry between L and K that $K \cong L_3(2^k)$ and A induces inner automorphisms on K. By (2) of Theorem A (A.2.1), $LK = O^{3'}(E(G))$, so $A \leq LK$ and hence $B \leq A_C \leq K$. As $U < W$, $[W, A_C] \neq 0$, so $[W, K] \neq 0$. By symmetry between L and K we may take $m \geq k$, so

$$m(A) \leq m_2(LK) = 2(m+k) \leq 4m, \tag{$*$}$$

with equality only if $m = k$ and $B = A_C$ is of rank $2m$.

Let c be the number of noncentral factors in a chief series for V as L-module. Observe for each such chief factor I that $m(I) \geq 3m$ and $m(I/C_I(A)) \geq m$. Thus

$$m(A) \geq m(V/C_V(A)) \geq cm. \tag{!}$$

Next if $c = 2$, then $C_{GL(W)}(L)$ contains no $L_3(2^k)$-subgroup, so $[W, K] = 0$, contrary to an earlier remark. Thus $c > 2$, so as $m(A^+) \leq 2m$, $B \neq 0$ by (!). Therefore $K = [K, B]$ so as $[W, K] \neq 0$, $[W, B] \neq 0$. Thus as L centralizes B, L is nontrivial on $V/C_V(B)$, so $m(V/C_V(B)) \geq 3m$. Hence

$$m(A) \geq m(V/C_V(A)) \geq m(V/C_V(B)) + m(U/C_U(A)) \geq 4m, \tag{!!}$$

with equality only if L has $c-1$ noncentral chief factors on $C_V(B)$. It follows from (*) and (!!) that $B = A_C$ and L has $c-1$ noncentral chief factors on $C_V(B)$. As $B = A_C$, $U = C_V(B)$, so L has one noncentral chief factor on $C_V(B)$. Thus $c = 2$, contrary to an earlier reduction. □

Set $G_1 := L$, unless case (2) of B.5.9 holds, where we set $G_1 := LA$. Let $V_1 := W$, $V_2 := C_V(G_1)$, and $G_2 := C_G(G_1)$.

Define G_1 to be *exceptional* if W is a natural module for $G_1 \cong L_3(2)$, and $V_1 + V_2$ is a hyperplane of V.

LEMMA B.5.10. *(1) $G_i \trianglelefteq G$ and G acts on V_i.*
(2) $G = G_1 \times G_2$ and $G_2 = C_G(V_1)$.
(3) Either $V = V_1 + V_2$, or G_1 is exceptional.

PROOF. Recall $L \trianglelefteq G$. Thus G acts on $[V, L] = W = V_1$. Further if case (2) of B.5.9 does not hold, then $G_1 = L$, $G_2 = C_G(L)$, and $V_2 = C_V(L)$, so (1) holds in those cases.

Suppose case (2) of B.5.9 holds. Then W is a projective L-module, so $V = W \oplus C_V(L)$. Then as $V = W + C_V(A)$, A centralizes $C_V(L)$, so $V_2 = C_V(G_1) = C_V(L)$ is G-invariant and $V = V_1 \oplus V_2$. Then as $G_1 = N_{GL(V_1)}(L)$, $G_1 = C_G(V_2)$, so G_1 and $G_2 = C_G(G_1)$ are normal in G. Thus (1) and (3) hold in this case, and in particular the proof of (1) is complete.

Observe $[C_G(W), G_1] \leq C_{G_1}(W) = 1$, so $C_G(W) \leq G_2$. Set $H_1 := G_1$ and $H_2 := C_G(W)$. Let $D \in \mathcal{P}(G, V)$ with $[W, D] \neq 0$. By B.1.5.1, $Aut_D(W) \in \mathcal{P}(Aut_G(W), W)$, so from the description of G_1 and its action on W in B.5.9, and from B.4.2, $Aut_D(W) \leq Aut_{G_1}(W)$. Thus $D \leq D_1 \times D_2$, where D_i is the projection of D on H_i with respect to the decomposition $H_1 \times H_2$. So as $G = J(G, V)$ and $G_1 = J(G_1, V)$, $G = H_1 \times H_2$. Then as $H_2 \leq G_2$ and $G_1 \cap G_2 = 1$, (2) holds.

Suppose $V > V_1 + V_2$; then $H^1(L, W) \neq 0$, so either case (1) of B.5.9 holds, or $L \cong L_3(2)$ and one of the last three cases of B.5.9 holds. But in case (1) of B.5.9, that result also says that $A \in Syl_2(L)$ and $V = W + C_V(A)$, so $V = V_1 + V_2$ holds by Gaschütz's Theorem A.1.39, contrary to assumption. Similarly if $L \cong L_3(2)$ and $V = W + C_V(A)$ (which holds in cases (3) and (5) of B.5.9), then we obtain the same contradiction from B.4.8.3. This leaves case (4) of B.5.9, where as $H^1(X, W) \cong \mathbf{Z}_2$ by B.4.8.1, we have $m(V/(V_1 + V_2)) \leq 1$. As $V > V_1 + V_2$, this inequality is an equality. Thus G_1 is exceptional, so (3) is established. □

LEMMA B.5.11. *(1)* $1 \neq G_2$ *is faithful on* V_2*, and* $G_2 = J(G_2, V_2)$.

(2) $G_2 \cong L_2(2^n)$, $L_3(2^m)$, *m odd, or* S_5, *and* $V_2 = V_2' + C_{V_2}(G_2)$ *where* V_2' *satisfies one of conclusions (a), (b), or (c) of case (3) of B.5.6.*

PROOF. If $G_2 = 1$, then $G = G_1$ by B.5.10.2; then conclusion (1) of B.5.6 holds, contrary to our choice of G, V as a counterexample. Thus $G_2 \neq 1$.

Set $U := V_2$, $\bar{G} := G/C_G(U)$, and let $\pi : G \to \bar{G}$ be the natural surjection. By construction, $G_1 \leq \ker(\pi)$, and G_2 centralizes V_1 by B.5.10.2. Thus if $V = W + U$ then G_2 is faithful on U. If $V > W + U$, then by B.5.10.3, $m(V/U + W) = 1$, so as G_2 centralizes W, $C_{G_2}(U) \leq O_2(G) = 1$. Thus again G_2 is faithful on U, and the restriction $\pi : G_2 \to \bar{G}$ an isomorphism as $G = G_1 \times G_2$ by B.5.10.2. In particular $O_2(\bar{G}) = \pi(O_2(G_2)) = 1$, as $O_2(G) = 1$ and $G_2 \trianglelefteq G$. Further for $D \in \mathcal{P}(G, V)$ with D nontrivial on U, $\bar{D} \in \mathcal{P}(\bar{G}, U)$ by B.1.5.1, so as $G = J(G, V)$, also $\bar{G} = J(\bar{G}, U)$ and hence $G_2 = J(G_2, U)$. Thus (1) is established. Since by construction $G_1 \neq 1$, we may apply induction on the order of G to G_2, and conclude G_2 and its action on U are described in B.5.6.

Since G_1 contains a subgroup isomorphic to S_3, $m_3(G_2) = 1$ by A.1.31.1. Thus G_2 must satisfy conclusion (1) or (2) of B.5.6: Either $F^*(\bar{G}) \cong F^*(G_2)$ is quasisimple or $G_2 \cong \bar{G} \cong L_2(2)$. In the latter case, the requirements of conclusion (2) are immediate.

So assume the former case holds. Then B.5.6.1 also says that G_2 and its action on U are described in B.5.1. In particular as $m_3(G_2) = 1$, G_2 is one of the groups in conclusion (2) of B.5.11. In these cases, (1) and (2) of B.5.1 show that $U = V_2' + C_{V_2}(G_2)$ with V_2' described in conclusion (2). □

LEMMA B.5.12. G_1 *is exceptional.*

PROOF. Assume G_1 is not exceptional. By B.5.10, $G = G_1 \times G_2$ and $V = V_1 + V_2$. By B.5.11, $G_2 = J(G_2, V_2)$ is faithful on V_2, and G_2 and its its action on V_2 are described in B.5.11.2; in particular the action of G_2 on V_2 is described in conclusion (3) of B.5.6. The same holds for G_1 by B.5.9. Moreover

$$V_1 \cap V_2 \leq C_{V_1}(G_1) \cap C_{V_2}(G_2) \leq C_V(G_1) \cap C_V(G_2) = C_V(G)$$

as $G = G_1 G_2$ and G_i centralizes V_{3-i}. Thus conclusion (3) of Theorem B.5.6 is satisfied, contrary to the choice of G, V as a counterexample. □

We are now in a position to complete the proof of Theorem B.5.6. By B.5.12, V/V_2 is the 4-dimensional indecomposable for $G_1 \cong L_3(2)$ described in B.4.8.3. Pick $u \in V - (V_1 + V_2)$ and let $U_1 := \langle W, u\rangle$. Thus $V = U_1 \oplus V_2$, and as $W = [G_1, V]$, U_1 is a G_1-submodule isomorphic to V/V_2.

Let $L_2 := F^*(G_2)$. Then from B.5.11.2, $G_2 = L_2\langle t\rangle$, where either $t = 1$, or $G_2 \cong L_2(2)$ or S_5 with t an involution.

Let $U_2 := [V, L_2]$, and suppose first that $V = U_2 \oplus C_V(L_2)$. As G_1 is self-normalizing in $GL(V/V_2)$, $[V, G_2] \le V_2$. Further we may choose $u \in C_V(L_2)$, so $[U_1, t] \le C_{V_2}(L_2) = C_{V_2}(G_2)$, and hence $\tilde{U}_1$ is a G-submodule of $\tilde{V}$. By construction V_2 is a G-submodule, and $\tilde{V} = \tilde{U}_1 \oplus \tilde{V}_2$, so conclusion (3) of B.5.6 is satisfied by B.5.11.2. This contradicts the choice of G, V as a counterexample.

Thus $V \neq U_2 \oplus C_V(L_2)$. Therefore by the cohomology of the modules appearing in B.5.11, either $G_2 \cong L_2(2^n)$, or $G_2 \cong L_3(2)$ with $\tilde{U}_2$ the natural module. Then by B.5.1.5 and B.4.8, $V_2 = W_2 + C_{V_2}(G_2)$, where either $W_2 = U_2$ satisfies conclusion (a) or (b) of case (3) of B.5.6, or W_2 is a 4-dimensional indecomposable described in B.4.8.3. In the former case $V_1 + V_2 = W \oplus U_2 + C_V(G)$ is a hyperplane of V, so conclusion (5) of B.5.6 is satisfied, contrary to the choice of G, V as a counterexample. In the latter case, by B.4.8.4, $V = W_2 \oplus C_V(G_2)$, so we can pick $u \in C_V(G_2)$, and hence U_1 is a G-submodule of V with $V = U_1 \oplus W_2 \oplus C_V(G)$, so that conclusion (3) of B.5.6 is satisfied, again contrary to our choice of a counterexample.

This final contradiction establishes Theorem B.5.6.

We mention next that in conclusion (3) of the *qrc*-lemma D.1.5 in the following chapter, there is an $\mathbf{F}_2G$-module V such that the dual V^* of V is an FF-module. In this case, the next lemma says that $q(G, V) \le 2$.

LEMMA B.5.13. *Let G be a finite group and V a faithful $\mathbf{F}_2G$-module. Assume Hypothesis B.5.3 is satisfied by the pair G, V^*, where V^* is the dual of V as an $\mathbf{F}_2G$-module. Then $q(G, V) \le 2$.*

PROOF. By hypothesis B.5.3, $G = J(G, V^*)$. Therefore G, V^* is described in Theorem B.5.6.

We must show that $q(G, V) \le 2$. By B.4.7.4, it suffices to exhibit a nontrivial elementary abelian 2-subgroup A of G which is quadratic on V^* with

$$m([V^*, A]) \le 2m(A). \qquad (*)$$

If G, V^* satisfies conclusion (5) of Theorem B.5.6, then for G_1, V_1 as defined there, choose $E_4 \cong A \le G_1$ with $m(C_{V_1^*}(A)) = 2$. Then for $a \in A^{\#}$,

$$[V^*, a] \le [V^*, G_1] \cap C_{V^*}(a) = V_1^* \cap C_{V^*}(a) = C_{V_1^*}(A),$$

so $m([V^*, A]) \le 2 = m(A)$, verifying (*). Similarly in case (4) of B.5.6, pick $A \in \mathcal{P}(G, V^*)$ and observe $m(A) = 3$ and $[V^*, A] = C_{V^*}(A)$ is of rank 3 or 4, which is less than $2m(A)$, verifying (*). In the decomposable case (3) of B.5.6, it suffices to establish the bound for a suitable FF*-offender inside one of the factors G_1 or G_2, so replacing G by G_i, we reduce to the case of a single factor in case (1) or (2) of B.5.6, with $F^*(G)$ quasisimple or $\mathbf{Z}_3$. If $F^*(G) \cong \mathbf{Z}_3$, then $m([V^*, A]) = 1 = m(A)$, so (*) holds.

Thus we are reduced to the case where $L = F^*(G)$ is quasisimple, where we can appeal to Theorem B.5.1. Let $U^* := [V^*, L]$. If $V^* = V_1^* \oplus V_2^*$ with $[V_2^*, G] = 0$, then $V = V_1 \oplus V_2$ with $[V_2, G] = 0$ and V_1^* is an FF-module, so the lemma holds by induction on $\dim(V)$. Thus

$$\text{There is no nonzero direct summand } V_2^* \text{ of } V^* \text{ centralized by } G. \qquad (**)$$

First consider cases (ii)–(iv) of B.5.1.1; in these cases $V^* = U^* \oplus C_{V^*}(G)$ by B.5.1.4, so $V^* = U^*$ by (**). Now in case (iii), V^* is self-dual, while in cases (ii) and (iv), V is quasiequivalent (conjugate under $Out(G)$) to V^*, so in all of these cases, $q(G, V) = 1$, and we are done.

Thus we have reduced to case (i) of B.5.1.1, where by (**), V^* is indecomposable under G with one noncentral L-chief factor $W := U^*/C_{U^*}(L)$. If V^* is an irreducible L-module, then by inspection of the list in B.4.2, V is quasiequivalent to V^*, so $q(G, V) \le 1$. Thus we may assume that L, V^* is one of the exceptional cases of B.5.1.4.

Suppose case (b) of B.5.1.4 holds. If $G \cong S_8$ then V^* is self-dual, while if $G \cong S_6$ then V is quasiequivalent to V^*, so in either case, $q(G, V) = 1$. Thus case (a) of B.5.1.4 holds, with V^* not of form $U^* \oplus C_{V^*}(L)$, and L and its action on V^* are described in B.5.1.2.

Suppose first that W is the natural module for $G \cong L_2(2^n)$ or $G_2(2^n)$. Then $V^* = U^* + C_{V^*}(G)$ by B.5.1.5, so $V^* = U^*$ as V^* is indecomposable. Let $D \in \mathcal{P}^*(G, V^*)$; thus D is quadratic on V^* by B.1.4.3. By B.5.5.1.iii, there is $A \le D$ with $A \in \mathcal{P}(G, W)$. Thus $m := m(A) = m(C_W(A)) = n$ or $3n$, respectively, by B.4.2, and $m(C_{U^*}(L)) \le n$ by I.1.6. As A is quadratic on V^*, $[V^*, A] \le C_{U^*}(A)$, so $m([V^*, A]) \le m + n \le 2m$, establishing (*).

Thus W is a natural module for $L \cong Sp_4(2^n)$, A_6, A_8, or $L_3(2)$. (Recall that if $L = U_3(3) \cong G_2(2)'$, then $G = G_2(2)$, a case we just treated.) If $L \cong L_3(2)$, then by B.4.8.4, $m(V^*) = 4$ and $q(G, V^*) = 1$. As V is also a 4-dimensional indecomposable, $q(G, V) = 1$ by B.4.8, handling this case.

Suppose $G \cong Sp_4(2^n)$ with $n > 1$. Then $G = L$ by B.4.2.3, and as V^* is indecomposable we may describe V^* more explicitly as follows: Let J denote a 6-dimensional orthogonal space over $\mathbf{F}_{2^n}$; we may choose G to be the stabilizer in $\Omega(J)$ of a non-singular point N of J, U^* to be a quotient of $I := N^\perp$, and $V^*/C_{V^*}(G)$ to be a submodule of J/N. Let W_2 be a totally singular 2-dimensional $\mathbf{F}_{2^n}$-subspace of I/N, let V_2^* be the preimage in U^* of W_2, and set $B := C_L(W_2)$. Then $m(B) = 3n$, while $[V^*, B] \le V_2^*$ of rank at most $3n$, so (*) is satisfied.

This leaves the cases $L \cong A_n$, $n = 6$ or 8. We first consider the case where V^* is a decomposable L-module. Then $[C_{V^*}(L), A] \ne 1$ by (**). Let $\tilde{V}^* := V^*/C_{V^*}(L)$. Then

$$\begin{aligned} m(A) \ge m(V^*/C_{V^*}(A)) &\ge m(\tilde{V}^*/C_{\tilde{V}^*}(A)) + m(C_{V^*}(L)/C_{V^*}(LA)) \\ &\ge m(\tilde{V}^*/C_{\tilde{V}^*}(A)) + 1, \end{aligned}$$

so

$$m(\tilde{V}^*/C_{\tilde{V}^*}(A)) \le m(A) - 1. \qquad (!)$$

In B.3.4.2i and B.3.2.6, a strong FF-offender A exists only when $m(A) = n/2$, $\tilde{V}^* = [\tilde{V}^*, L]$, and the inequality in (!) is an equality. As $[\tilde{V}^*, L] = \tilde{V}^*$, $V^* = U^* + C_{V^*}(L)$. As (!) is an equality, $m([C_{V^*}(L), A]) = 1$. But then

$$m([V^*, A]) \le m([U^*, A]) + m([C_{V^*}(L), A]) \le n/2 + 1 = m(A) + 1 \le 2m(A),$$

verifying (*).

Thus we have reduced to the case where V^* is an indecomposable L-module. Therefore by B.3.3.4, V^* is a submodule or quotient of the full n-dimensional permutation module; that module is self-dual, so V is also a submodule or quotient. Finally each such module is an FF-module for S_n, so as $L \le G \le S_n$, with $|S_n : A_n| = 2$, there is $B \le L \cap A$ with $m(V/C_V(B)) \le m(B)+1$. Thus $q(G, V) \le 2$, completing the proof of the lemma. □

B.6. Minimal parabolics

The object of this section is to define a collection of subgroups of an abstract finite group, which resemble the minimal parabolic subgroups (ie. parabolics of Lie rank 1) in a group of Lie type over a field of characteristic 2.

Thus, following ideas of McBride, we define:

Definition B.6.1. A *minimal parabolic* of a finite group H (with respect to the prime 2) is a subgroup P of H such that some Sylow 2-subgroup T of H is contained in a unique maximal subgroup of P, but T is not normal in P.

In particular if H is a minimal parabolic in the Lie-theoretic sense of a group of Lie type over a field of characteristic 2, then H is also a minimal parabolic as an abstract group according to this definition.

As a consequence of our first three results below (which are entirely elementary), we will see that if $H = O^{2'}(H)$ is a finite group, then H is generated by the minimal parabolics over a fixed Sylow 2-subgroup. In chapter E on generation and weak closure later in the Volume I, we will go on to describe in more detail the structure of minimal parabolics in SQTK-groups. The remainder of this section will study FF-modules for minimal parabolics—information which will be required in the following chapter on pushing up.

In this section H is a finite group.

Definition B.6.2. Write $\mathcal{N}$ for the set of maximal subgroups of H, and for $X \subseteq H$ let

$$\mathcal{N}(X) = \mathcal{N}_H(X) := \{N \in \mathcal{N} : X \subseteq N\}$$
$$\mathcal{U}(X) = \mathcal{U}_H(X) := \{U \le H : X < U \text{ and } |\mathcal{N}_U(X)| = 1\}$$

$$\hat{\mathcal{U}}(X) = \hat{\mathcal{U}}_H(X) := \{U \in \mathcal{U}(X) : X \not\trianglelefteq U\}$$

Thus $\mathcal{U}(X)$ consists of the subgroups of H in which X is contained in a unique maximal subgroup.

Lemma B.6.3 (McBride's Lemma). *If $X < H$ then $H = \langle \mathcal{U}_H(X) \rangle$.*

Proof. Let H be a minimal counterexample. If $|\mathcal{N}(X)| = 1$, then $H \in \mathcal{U}(X)$, and the result holds trivially. So assume M, N are distinct members of $\mathcal{N}(X)$; thus $M > X < N$. Now for any $X < K < H$, $\mathcal{U}_K(X) \subseteq \mathcal{U}_H(X)$, and by minimality of H, $K = \langle \mathcal{U}_K(X) \rangle$. Therefore applying this observation to M and N in the role of "K", we have

$$H = \langle M, N \rangle = \langle \mathcal{U}_M(X), \mathcal{U}_N(X) \rangle = \langle \mathcal{U}_H(X) \rangle,$$

as desired. □

LEMMA B.6.4. *Let* $X \leq H$ *and* $\hat{H} := \langle \hat{\mathcal{U}}_H(X) \rangle$. *Then*

(1) $\hat{H} \trianglelefteq H = \hat{H} N_H(X)$.

(2) If $X < \langle X^H \rangle = H$, *then* $H = \hat{H}$.

PROOF. If $X = H$, then $\hat{\mathcal{U}}_H(X) = \emptyset$ so that $\hat{H} = 1$ and (1) is trivial. So assume $X < H$. By B.6.3, $H = \langle \mathcal{U}_H(X) \rangle$. Now $\mathcal{U}_H(X) - \hat{\mathcal{U}}_H(X) \subseteq N_H(X)$, and $N_H(X)$ acts on $\hat{\mathcal{U}}_H(X)$. Therefore $N_H(X)$ acts on $\hat{H}$, and

$$H = \langle \mathcal{U}_H(X) \rangle \leq \langle \hat{\mathcal{U}}_H(X), N_H(X) \rangle = \hat{H} N_H(X) \leq H$$

with $\hat{H} \trianglelefteq H$. That is, (1) holds.

Further if $X < \langle X^H \rangle = H$, then as $H = \langle \hat{\mathcal{U}}_H(X), N_H(X) \rangle$ by (1), we conclude $\hat{\mathcal{U}}_H(X) \neq \emptyset$, so $X \leq \hat{H}$. Now by (1), $\hat{H} \trianglelefteq H$, so $H = \langle X^H \rangle \leq \hat{H}$. □

In the remainder of this section let $T \in Syl_2(H)$.

In the terminology of Definition B.6.1, $\hat{\mathcal{U}}_H(T)$ is the set of minimal parabolics of H above T. Recall that T is *not* normal in the members of $\hat{\mathcal{U}}_H(T)$ by definition. Notice that $\langle T^H \rangle = O^{2'}(H)$, so by B.6.4.2:

LEMMA B.6.5. *If* $H = O^{2'}(H)$ *then* H *is generated by the minimal parabolics above* T.

REMARK B.6.6. In the proof of our Main Theorem, after Theorem 2.1.1 is established there exists a pair H, M of 2-local subgroups containing T with $H \not\leq M$. Furthermore usually $N_G(T) \leq M$, so that $N_H(T) < H$; see for example Theorem 3.3.1. Since $H = O^{2'}(H) N_H(T)$ by a Frattini Argument, B.6.5 tells us that H is generated by $N_H(T)$ and minimal parabolics above T. Hence we can reduce many problems to the case where our group is a minimal parabolic.

So in the remaining results in this section, our hypotheses will usually include the assumption that $H \in \hat{\mathcal{U}}_H(T)$.

DEFINITION B.6.7. If $H \in \hat{\mathcal{U}}_H(T)$, we write $!\mathcal{N}_H(T)$ (or just $!\mathcal{N}(T)$) for the unique maximal subgroup of H containing T.

Recall for $A \leq B$ that $\ker_A(B) := \cap_{b \in B} A^b$ (in the literature this is often denoted $\mathrm{core_B(A)}$).

We obtain some elementary restrictions on the structure of H:

LEMMA B.6.8. *Assume* H *is a minimal parabolic and* $M := !\mathcal{N}(T)$. *Set* $J := \ker_M(H)$, $K := O^2(H)$, $\bar{H} := H/O_2(H)$, *and* $H^* := H/J$. *Then*

(1) $O_2(H) = O_2(J)$, *but* $K \not\leq J$.

(2) If H *is solvable,* [4] *then* $H = O_{2,p,2}(H)$ *for some odd prime* p, $\bar{J} = \Phi(O_p(\bar{H}))$, *and* T *irreducible on* $O_p(H^*)$.

(3) If H *is not solvable then* $J = O_{2,F}(H)$, $H = KT$, *and* K^* *is the direct product of the* T*-conjugates of a nonabelian simple group* L^* *such that*

$$Aut_{H^*}(L^*) \in \hat{\mathcal{U}}_{Aut_{H^*}(L^*)}(Aut_{T^*}(L^*)) \quad \text{and} \quad N_{M^*}(L^*) = !\mathcal{N}_{N_{H^*}(L^*)}(N_{T^*}(L^*)).$$

Further either

(i) $J = O_{2,F}(H) = O_{2,F^*}(H)$, *or*

[4]This result holds for general p. It may be regarded as a descendant of [**FT63**, 7.6], showing generation by $\{p, q\}$-subgroups of p-length at most 2.

(ii) $J = O_\infty(H)$ *and* $H = O_{2,E,2}(H)$.

(4) If $Y \trianglelefteq H$, *then either* $Y \leq J$ *or* $K \leq Y$.

(5) J *is* 2*-closed; that is,* $O_2(H) \in Syl_2(J)$.

(6) Let W *be an* $\mathbf{F}_2H$*-module.*

(a) If $K \not\leq C_H(W)$ *then* $C_H(W) \leq J$.

(b) If $C_H(W) \leq M$ *then* $C_H(W) \leq J$, *so* $C_{O_2(H)}(W) \in Syl_2(C_H(W))$.

(c) If $O_2(H) \leq C_H(W) \leq M$, *then* $O_2(H/C_H(W)) = 1$.

(d) Assume $V \in \mathcal{R}_2(H)$ *with* $[V, H] \neq 1$, *and* $N \trianglelefteq T$. *Then* $[O^2(H), N] < O^2(H)$ *iff* $N \leq O_2(H)$ *iff* $N \leq C_T(V)$. *In particular* $[V, J(T)] = 1$ *iff* $J(T) = J(O_2(H))$ *iff* $Baum(T) = Baum(O_2(H))$. *Thus if* $[V, J(T)] \neq 1$, *then* $J(H) = O^2(H)J(T)$, *and* V *is an FF-module for* $H/C_H(V)$.

(7) $\bar{J} = \Phi(\bar{H})$.

Proof. As $J \trianglelefteq H$, certainly $O_2(J) \leq O_2(H)$. On the other hand, $O_2(H) \leq T \leq M$ as T is Sylow in H, so $O_2(H) \leq J$; then $O_2(H) = O_2(J)$, giving the first part of (1). As our other conclusions are essentially statements about $H/O_2(H)$, passing to $H/O_2(H)$ we may assume $O_2(H) = 1$.

We claim that J is nilpotent of odd order. For if J is of odd order but not nilpotent, then some T-invariant Sylow r-subgroup R of J is not normal in H, while if J has even order then $1 \neq R := T \cap J \in Syl_2(J)$—and as $O_2(H) = 1$, R is not normal in H. In either case $T \leq N_H(R) < H$, so $N_H(R) \leq M$; then by a Frattini Argument, $H = JN_H(R) \leq M$, contradicting $M < H$. This contradiction establishes the claim. Now as J is of odd order, J is 2-closed, establishing (5), and also showing that $J \leq O^2(H) = K$. If $K \leq J$, we again obtain a contradiction as $H = TO^2(H) = TJ \leq M < H$, completing the proof of (1).

Now let K_0 be any subgroup normal in H, but not contained in J, and hence not contained in M. Then $T \leq TK_0 \not\leq M$, so as $M = !\mathcal{N}(T)$, we conclude $H = TK_0$. Therefore $K = O^2(H) = O^2(K_0)$, so

$$K \text{ is the unique subgroup minimal with respect to } K \trianglelefteq H \text{ and } K \not\leq J. \qquad (*)$$

In particular this establishes (4). Further (*) also shows that $K^* = K/J$ is a minimal normal subgroup of H^*. As $K = O^2(H)$ and $J < K$, K/J is not a 2-group.

Consider the case where H is solvable; we need to prove (2). Suppose there are distinct odd prime divisors p, q of $|H|$; for $r := p$ or q, by Hall's Theorem we can choose $H_r \geq T$ a Hall r'-subgroup of H, and $M_r \in \mathcal{N}(H_r)$. Then $M_p \neq M_q$, as $|H : M_r|$ is a power of r, contrary to the hypothesis that $H \in \mathcal{U}(H)$.

Thus H is a $\{2, p\}$-group for some odd prime p, and hence so is K. Now the solvable minimal normal subgroup K/J is not a 2-group, so it must be a p-group—and J is a p-group as J is of odd order. Therefore as $K = O^2(H)$ we conclude $K = O_p(H)$.

As $\Phi(K) < K$, (*) shows $\Phi(K) \leq J$. Let $K_1/\Phi(K)$ be an irreducible T-submodule of $K/\Phi(K)$. If $K_1 < K$, then $K/\Phi(K) = K_1/\Phi(K) \times K_2/\Phi(K)$ where K_2 is T-invariant. Then each K_i is normal in H, so again by (*), $K_i \leq J$, and then $K = K_1K_2 \leq J$, contrary to (1). Therefore $K_1 = K$ so that T is irreducible on $K/\Phi(K)$. Then as $\Phi(K) \leq J < K$, $J = \Phi(K)$, establishing (2).

We turn to the case where H is not solvable; thus K is not solvable, and we need to prove (3). The minimal normal subgroup K^* of H^* is the direct product of the conjugates under T of some nonabelian simple group L^*, so that $F(K^*) = 1$.

If $J < F(H)$ then $F(H) \geq K$ by (*), contradicting $F(K^*) = 1$, so we conclude $J = F(H)$. If $F(H) = F^*(H)$, we have conclusion (i) of (3). Otherwise from the transitivity of T on the components of H^*, we get $K = E(H)J$; and then by (*), $K = E(H)$, so conclusion (ii) of (3) holds. The remaining assertions of (3) are essentially about H^*, so passing to H/J, we may assume $J = 1$. Now if $X \in \mathcal{N}_{LN_T(L)}(N_T(L))$, then $\langle X, T\rangle \in \mathcal{N}(T)$, so $M = \langle X, T\rangle$. Therefore $X = N_M(L)$ and $Aut_X(L) = !\mathcal{N}_{Aut_H(L)}(Aut_T(L))$, completing the proof of (3).

Next we prove (6). First (6a) and (6b) are just (4) and (5) applied to $C_H(W)$ in the role of "Y". Assume the hypotheses of (c), let I be the preimage of $O_2(H/C_H(W))$ in H and $S := T \cap I$. Then $S \in Syl_2(I)$ with $I = C_H(W)S$, so by a Frattini Argument $H = C_H(W)N_H(S)$. Hence as $C_H(W) \leq M < H$ by the hypotheses of (c), $N_H(S) \not\leq M$, so as $T \leq N_H(S)$ and $M = !\mathcal{N}(T)$, $N_H(S) = H$. Hence $S \leq O_2(H)$, while $O_2(H) \leq C_H(W)$ by the hypotheses of (c). That is $O_2(H/C_H(W)) = 1$, as asserted.

Now assume the hypotheses of (d). If $N \leq O_2(H)$ then $[N, O^2(H)] \leq O_2(H)$, and so $[O^2(H), N] < O^2(H)$. Conversely if $[O^2(H), N] < O^2(H)$, then because $[O^2(H), N]N \trianglelefteq O^2(H)T = H$, $[O^2(H), N]N \leq J$ by (4), so $N \leq O_2(H)$ by (5).

Next since $V \in \mathcal{R}_2(H)$, $O_2(H/C_H(V)) = 1$, so in particular $O_2(H) \leq C_H(V)$. By hypothesis, $[V, H] \neq 1$, so $O^2(H) \not\leq C_H(V)$. Therefore $C_H(V) \leq J$ by (a), so by (b), $C_H(V)$ is 2-closed, with Sylow group $O_2(H)$. Thus $N \leq C_T(V)$ iff $N \leq O_2(H)$, and the proof of the first statement of (d) is complete. Then the remaining statements follow using B.2.3. This completes the proof of (6).

Assume (7) fails. Then there is a maximal subgroup X of G with $J \not\leq X$; thus $G = XJ$ so as J is of odd order, X contains a Sylow 2-subgroup of G, which we may take to be T. But then $X = M$ as $M = !\mathcal{N}(T)$, so $G = XJ \leq M$, for our final contradiction. □

The following result determines the FF-modules for minimal parabolics which are $\mathcal{K}$-groups. (A later result E.2.3 gives further details in the case of an SQTK-group). Notice the similarity with the representation-theoretic version of Solvable Thompson Factorization B.1.8.

Lemma B.6.9. *Assume H is a minimal parabolic, H is a $\mathcal{K}$-group, $O_2(H) = 1$, and V is a faithful FF-module for H. Let $\tilde{V} := V/C_V(J(H,V))$. Then*

(1) $q(H,V) = 1$. In particular, H contains no strong FF-offenders.

(2) $J(H,V) = H_1 \times \cdots \times H_s$, and $[\tilde{V}, J(H,V)] = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_s$, where $V_i := [V, H_i]$ and $[V_i, H_j] = 0$ for $i \neq j$.

(3) T permutes the set $\{H_1, \ldots, H_s\}$ transitively.

(4) $H = J(H,V)T$.

(5) Set $M := !\mathcal{N}_H(T)$. Then either

(i) $H_i \cong L_2(2^e)$, $M \cap H_i = N_{H_i}(T \cap H_i)$, and $\tilde{V}_i$ is the natural module for H_i, or

(ii) $H_i \cong S_{2^k+1}$ is a symmetric group, $M \cap H_i \cong S_{2^k}$, and V_i is the core of the (2^k+1)-dimensional permutation module of rank 2^k.

Proof. Adopt the notation of B.6.8 (in particular $J = \ker_M(H)$) and let $Y := J(H,V)$. If $Y \leq J$ then as $Y = O^{2'}(Y)$, $Y \leq O_2(H)$ by B.6.8.5, contradicting $O_2(H) = 1$. Thus $Y \not\leq J$, so $O^2(H) = K \leq Y$ by B.6.8.4. Thus (4) holds and $K = O^2(Y)$. Indeed let $\mathcal{P} := \mathcal{P}^*(H,V)$; then the same argument shows $K = O^2(J_{\mathcal{P}}(H))$.

Next H is described in B.6.8.2 or B.6.8.3. If H is solvable, then (1), (2), and (5i) with $e := 1$ hold by B.1.8. In particular H permutes $\Delta := \{H_1, \dots, H_s\}$, so as T is irreducible on K by B.6.8.2, (3) holds. Thus the lemma holds in this case, so we may assume H is not solvable, and hence H is described in B.6.8.3.

Suppose that $[F(H), K] \neq 1$. Then as $K = O^2(J_{\mathcal{P}}(H))$, B.1.9 says there is a normal subgroup $G_1 \cong L_2(2)$ of Y with $[V, G_1]$ of rank 2. Hence $K := K^\infty$ centralizes $[V, G_1]$, so $O^2(G_1) \leq O^2(H) = K$ centralizes $[V, G_1]$, a contradiction.

Therefore K centralizes $F(H)$, so since $O_2(H) = 1$, $F(H) = Z(K)$. Therefore by B.6.8.3, $K = F^*(H) = K_1 \cdots K_s$ is the central product of the components of H, permuted transitively by T. By B.1.5.4, $K_i \trianglelefteq Y$. Let $L := K_1$ and $A \in \mathcal{P} = \mathcal{P}^*(H, V)$. As T is transitive on $\{K_1, \dots, K_s\}$, we may assume $L = [L, A]$. Let $W \in Irr_+(L, V)$ and set $\tilde{W} := W/C_W(L)$. By B.1.5.3, A acts on W. Then by B.1.5.1, $Aut_A(W) \in \mathcal{P}(Aut_H(W), W)$. So by B.1.5.8, $Aut_A(\tilde{W})$ contains a member of $\mathcal{P}(Aut_H(\tilde{W}), \tilde{W})$, so that $\tilde{W}$ is an FF-module for $Aut_{LA}(\tilde{W})$, and by construction L is irreducible on $\tilde{W}$.

As $\tilde{W}$ is an FF-module for $Aut_{LA}(\tilde{W})$, $L/Z(L)$ is of Lie type and characteristic 2 or an alternating group: this is a standard consequence of our $\mathcal{K}$-group hypothesis; in particular, it follows from B.4.2 in the case where L is an SQTK-group, which is the only case we need in this work. By B.6.8.3, $Aut_H(L)$ is also a minimal parabolic. If L is of Lie type it follows that either

(a) L is of Lie rank 1 and $M \cap L = N_L(T \cap L)$, or

(b) $L \cong (S)L_3(2^e)$ or $Sp_4(2^e)$, and $N_T(L)$ is nontrivial on the Dynkin diagram of L.

In case (a), as $\tilde{W}$ is an FF-module, using the $\mathcal{K}$-group hypothesis (in the form of B.4.2 when H is an SQTK-group), we conclude $L \cong L_2(2^e)$, $\tilde{W}$ is the natural module (or else the A_5-module for $Aut_{LA}(W) \cong S_5$—which we treat below in the symmetric case), A induces inner automorphisms on L, and $q(Aut_H(W), W) = 1$. By B.1.5.5, $V = W + C_V(A)$, so $W = [V, LA] = V_1$. As $[V, L] \in Irr_+(L, V)$, K_i centralizes V_j for $i \neq j$, and then as $V = W + C_V(A)$, A centralizes K_iV_i for $i > 1$. Then as A induces inner automorphisms on L, $A \leq L$; indeed $A \in Syl_2(L)$. Moreover an argument in the proof of B.1.8 shows that each member of $\mathcal{P}(H, V)$ is a product of members of $\mathcal{P}^*(H, V) = \mathcal{P}$, so $K = Y$. This completes the proof of the lemma in case (a).

In case (b), $Aut_{LA}(W)$ has no FF-modules by B.4.2, so this case does not arise.

This leaves the case where $L \cong A_n$ for some n. As $Aut_H(L)$ is a minimal parabolic we conclude that $n = 2^k + 1$ and $M \cap L \cong A_{2^k}$; this follows from the structure of S_n, and in particular from E.2.2 when L is an SQTK-group, the only case we need in this work. Then as $\tilde{W}$ is an FF-module, it follows (again using the $\mathcal{K}$-group hypothesis, and in particular B.4.2 when L is an SQTK-group) that W is the core of the permutation module of degree $2^k + 1$, $A \in \mathcal{P} = \mathcal{P}^*(H, V)$ induces a transposition on L, and $q(Aut_H(W), W) = 1$. Then applying B.1.5.5 as above, $W = V_1$ and AL centralizes K_iV_i for $i > 1$. Set

$$H_i := \bigcap_{j \neq i} C_H(V_j).$$

Then as $AL = N_{GL(V_1)}(L)$, $AL = H_1$, and as H is transitive on $\{K_1, \dots, K_s\}$, $H_i \cong H_1$ for each i. Finally arguing as in the proof of B.1.8, each member of

$\mathcal{P}(H, V)$ is a product of members of $\mathcal{P}$, so $Y = H_1 \times \cdots \times H_s$, and hence the lemma holds in this final case as well. □

The final result of this section is due to Baumann; its proof depends on our first use of Baumann's Argument B.2.18. (The original result is the special case given in 2.11.1.4 of [**Bau76**]). Part (2) is particularly important, in that it guarantees that the Baumann subgroup of T is Sylow in certain crucial subgroups.

PROPOSITION B.6.10 (Baumann's Lemma). *Assume H is a minimal parabolic with $F^*(H) = O_2(H)$, let $V \in \mathcal{R}_2(H)$, set $H^* := H/C_H(V)$, and assume that L is subnormal in H with $L^* \cong L_2(2^n)'$, $[V, L]/C_{[V,L]}(L)$ is the natural module for L^*, and $[V, J(T)] \neq 1$. Then*

(1) $Baum(T) \leq N_H(L^*)$.
(2) $Baum(T)$ is Sylow in $\langle Baum(T)^L \rangle$.
(3) Let $E := \Omega_1(Z(J(T)))$. Then $E = C_{[V,L]}(J(T))C_E(L)$.

PROOF. Let $S := \mathrm{Baum}(T)$, $E := \Omega_1(Z(J(T)))$, and $Q := O_2(H)$. By hypothesis $[J(T), V] \neq 1$, so by B.6.8.6d, $J(H) = KJ(T)$. Further defining $\mathcal{P} := \mathcal{P}_H$ as in B.2.7, by that result $\mathcal{P}$ is a stable subset of $\mathcal{P}(H^*, V)$ and $K^* \leq J(H)^* \leq J(H^*, V)$. Next B.6.9 gives a description of $J(H^*, V)$, and in particular by transitivity of T in B.6.9, no proper normal subgroup of $J(H^*, V)$ is generated by a subset of FF*-offenders, so $J(H^*, V) = J(H)^*$. By B.6.9, $J(H)^* = H_1^* \times \cdots \times H_s^*$ and $[\tilde{V}, J(H)] = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_s$. As L is subnormal in H and $[\tilde{V}, L]$ is the natural module for $L^* \cong L_2(2^n)'$, it follows that $L^* = O^2(H_i)^*$ and $[V, L] = V_i$ for some i. Now $J(T) \leq J(H) \leq N_H(V_i)$, and as $\tilde{V}_i$ is the natural module for L^*, $Z_i := C_{V_i}(J(T)) \not\leq C_{V_i}(L)$. Thus $Z_i \leq E$, and as T permutes the V_j with $[V, J(H)]$ the direct sum of the $\tilde{V}_j$, it follows that S acts on V_i and hence also on L^*, establishing (1).

Set $M := !\mathcal{N}(T)$, $J := \ker_M(H)$. Observe using B.6.8.6a that $C_H(V) = J$. Let $T \cap LS \leq H_0 \leq LS$ be minimal subject to $LS = H_0(J \cap L)$. Then as L^*S^* is a minimal parabolic, H_0 is a minimal parabolic, and if (2) and (3) are satisfied in H_0, they are satisfied in H as J/Q is of odd order. Thus replacing H by H_0, we may assume that $L = O^2(H)$ and $H = LS$. As $\tilde{V}$ is the natural module for $J(H)^* \cong L_2(2^e)$, B.2.20 says that $S^* = J(T)^* \in Syl_2(J(H)^*)$, so $H^* = L^*S^* = J(H)^*$.

To complete the proof, we appeal to Baumann's argument B.2.18 applied to T in the role of "R". First as $V \in \mathcal{R}_2(H)$ with $[V, H] \neq 1$, $C_T(V) = O_2(H)$ by part (6d) of B.6.8, so hypothesis (a) of B.2.18 is satisfied. Hypothesis (c) follows from B.6.9.1. As $\tilde{V}$ is the natural module for $H^* \cong L_2(2^e)$ and $J(T)^*$ is Sylow in H^*, $2m(V/C_V(J(T))) = m(\tilde{V})$ and $H^* = \langle J(T)^*, J(T)^{*g} \rangle$ for $g \in L - M$, so hypothesis (d) is satisfied with $n = 2$, $X_1 = J(T)$, and $X_2 = J(T)^g$, while hypothesis (e) is satisfied as $J(T) = X_1$. Let $\bar{H} := H/Q$. If H is solvable, then $\bar{J} = \Phi(\bar{L})$ by B.6.8.2. If H is not solvable, then $H^* = L^*$, so $H = LJ$ and $\Phi(\bar{L}) = \bar{J}$ by B.6.8.7. Therefore as $C_H(V) \leq J$ by B.6.8.6b, hypothesis (b) is satisfied.

Thus the hypotheses of Baumann's Argument B.2.18 are satisfied; now part (2) of that result gives conclusion (3). Finally, $C_T(V) \in Syl_2(C_H(V))$ as $T \in Syl_2(H)$, and $C_{T^*}(C_V(J(T))) = J(T)^*$, so $S^* = J(T)^*$ is Sylow in $J(H)^*$. Then part (3) of B.2.18 shows that $S \in Syl_2(KS)$, establishing (2). □

B.7. Chapter appendix: Some details from the literature

In this short section, for completeness we state and prove 3.2 and 3.3 from [**Asc81e**], quoted in the proof of B.1.5. In keeping with the style of exposition in this work, the proofs are somewhat more expansive than the originals.

Assume G is a finite group, V is a faithful $\mathbf{F}_2G$-module, $A \in \mathcal{P}(G,V)$, and let $U = U(A) := [V,A]$.

The following lemma is 3.2 in [**Asc81e**]; a new part (3) has been added to make the proof of B.1.5.6 explicit.

LEMMA B.7.1. *Let W be an A-invariant subspace of V with $[W,A] \neq 0$. Then:*

(1) $Aut_A(W) \in \mathcal{P}(Aut_G(W),W)$.

(2) If $A \in \mathcal{P}^*(G,V)$, then $Aut_A(W) \in \mathcal{P}^*(Aut_G(W),W)$. If in addition $m(A/C_A(W)) = m(W/C_W(A))$, then $V = W + C_V(A)$ and A is faithful on W; in particular, this holds if A is not a strong FF^*-offender on W.

(3) Assume $m(A/C_A(W)) = m(W/C_W(A))$. Then either A is faithful on W with $V = W + C_V(A)$; or $C_A(W) \in \mathcal{P}(G,V)$ with $C_A(W) \lesssim A$.

PROOF. Assume that $B \leq A$ with $C_A(W) \leq B$; then $C_B(W) = C_A(W)$. Since $A \in \mathcal{P}(G,V)$, condition (*) in section B.1 says

$$|A||C_V(A)| \geq |B||C_V(B)|. \qquad (!)$$

As $B \leq A$, $C_V(A) \cap C_W(B) = C_W(A)$, so that $(C_V(A) + C_W(B))/C_W(B) \cong C_V(A)/C_W(A)$, and hence

$$|C_V(B) : C_W(B)| = |C_V(B) : C_V(A) + C_W(B)||C_V(A) + C_W(B) : C_W(B)|$$
$$= |C_V(B) : C_V(A) + C_W(B)||C_V(A) : C_W(A)|.$$

It follows that

$$|C_V(B)|/|C_V(A) : C_W(A)| = |C_W(B)||C_V(B) : C_V(A) + C_W(B)|.$$

Dividing (!) by $|C_A(W)| = |C_B(W)|$ and $|C_V(A) : C_W(A)|$, and then using the previous equality,

$$|A/C_A(W)||C_W(A)| \geq |B/C_B(W)||C_W(B)||C_V(B) : C_V(A) + C_W(B)| \qquad (!!)$$
$$\geq |B/C_B(W)||C_W(B)|,$$

and (!!) is an equality only if $|A||C_V(A)| = |B||C_V(B)|$ and $C_V(B) = C_V(A) + C_W(B)$. In particular, $Aut_A(W) \in \mathcal{P}(Aut_G(W),W)$, establishing (1).

Observe that if $|A/C_A(W)||C_W(A)| = |B/C_B(W)||C_W(B)|$, then (!!) is an equality, so by the previous paragraph

$$C_V(B) = C_V(A) + C_W(B), \quad \text{and} \qquad (+)$$

$$|A||C_V(A)| = |B||C_V(B)|. \qquad (++)$$

Then either $B = 1$, so that $V = C_V(A)+W$ by (+), or (++) says that $B \in \mathcal{P}(G,V)$ and $B \lesssim A$.

Assume that $m(A/C_A(W)) = m(W/C_W(A))$, and set $B := C_A(W)$. Then $|A/C_A(W)||C_W(A)| = |W| = |B/C_B(W)||C_W(B)|$, so by the previous paragraph either $B = 1$ (so that A is faithful on W) and $V = C_V(A) + W$, or $B \in \mathcal{P}(G,V)$ and $B \lesssim A$. Thus (3) holds.

It remains to prove (2), so we assume that $A \in \mathcal{P}^*(G, V)$. Suppose that $D \leq Aut_G(W)$ with $D \lesssim Aut_A(W)$. Then $D = Aut_B(W)$ for some $B \leq A$ with $C_A(W) \leq B$. As $Aut_B(W) \lesssim Aut_A(W)$, $|A/C_A(W)||C_W(A)| = |B/C_B(W)||C_W(B)|$. So by an earlier observation either $B = 1$, or $B \in \mathcal{P}(G, V)$ with $B \lesssim A$, so that $B = A$ since we are now assuming $A \in \mathcal{P}^*(G, V)$. Thus either $D = 1$ or $D = Aut_A(W)$. We conclude that $Aut_A(W) \in \mathcal{P}^*(Aut_G(W), W)$, establishing the first statement in (2).

Finally assume in addition that $m(A/C_A(W)) = m(W/C_W(A))$, and again set $B := C_A(W)$. Then as $A \in \mathcal{P}^*(G, V)$, $B \notin \mathcal{P}(G, V)$, so that (3) completes the proof of (2). □

The following lemma is 3.3 in [**Asc81e**]:

LEMMA B.7.2. *Assume* $1 \neq H = O^2(H) = [H, A] \leq G$ *such that each* 2*-chief factor of* H *is central. Then:*

(1) If $a \in A$ *with* $0 = [V, a, A]$*, then* a *fixes* $Irr_+(H, V)$ *pointwise.*

(2) If $A \in \mathcal{P}^*(G, V)$*, then* A *fixes* $Irr_+(H, V)$ *pointwise.*

PROOF. Notice that (1) implies (2): For if $A \in \mathcal{P}^*(G, V)$, then by Thompson Replacement B.1.4.3, $[V, A, A] = 0$.

So assume that $a \in A$ with $[V, a, A] = 0$, and $I \in Irr_+(H, V)$ with $I \neq I^a$; it remains to derive a contradiction. As $I \neq I^a$, $I \neq 0$. Set $W := I + I^a$; then $W = [W, H] \neq 0$ by definition of $I \in Irr_+(H, V)$.

Suppose H normalizes $[I, a]$. Then as $H = [H, A]$ and A centralizes $[V, a]$, H centralizes $[I, a]$, and hence H also centralizes W since $[I, a] = \{ii^a : i \in I\}$ is the a-diagonal of $I + I^a = W$. This is a contradiction as $0 \neq W = [W, H]$, so H does not normalize $[I, a]$. Thus $W = I + I^a = [I, a, H]$ is A-invariant, since $[I, a]$ and H are A-invariant. Appealing to B.7.1.1 to conclude that $Aut_A(W) \in \mathcal{P}(Aut_{HA}(W), W)$, we may replace the pair G, V by $Aut_{HA}(W)$, W; hence we may assume $G = HA$ and $V = W$.

Set $Z := C_V(H)$, $\tilde{V} := V/Z$, and $n := m(\tilde{I})$. Then $\tilde{V} = \tilde{I} \oplus \tilde{I}^a$ has rank $2n$, and $n = m([\tilde{V}, a]) \leq m(V/C_V(A)) \leq m(A)$ since $A \in \mathcal{P}(G, V)$. Suppose $b \in A^\#$ normalizes I. Then b also normalizes I^a, and as $[V, a, b] = 0$, b centralizes the diagonal $[\tilde{I}, a]$, so b centralizes $\tilde{V}$. Thus $[H, b] \leq C_H(\tilde{V}) \leq O_2(H)$ using Coprime Action, while $O_2(H) \leq Z(H)$ since all 2-chief factors of H are central by hypothesis; so we conclude from Coprime Action that $H = O^2(H)$ centralizes b. Hence $[VH, b] \leq [V, b] \leq Z \leq Z(VH)$. Recall $V = [V, H]$ since $V = W$, so that $VH = O^2(VH)$; hence VH centralizes b by Coprime Action, so $b \in C_G(V) = 1$. Thus $N_A(I) = 1$, so that A is regular on I^A. Now $\tilde{I}_1 \cap \tilde{I}_2 = 0$ for distinct members I_i of $Irr_+(H, V)$, and $|\tilde{I}_i^\#| = 2^n - 1$, so $|Irr_+(H, V)| \leq (2^{2n} - 1)/(2^n - 1) = 2^n + 1$. Then as $m(A) \geq n$, $|A| \geq 2^n$, so we conclude $|A| = 2^n = |I^A|$. Furthermore A centralizes $[\tilde{V}, a]$, so that $\tilde{I}^A \cup [\tilde{V}, a]$ is a partition of $\tilde{V}$ by subspaces. Since I is H-invariant, I^A is HA-invariant, and hence so is $[\tilde{V}, a] = [\tilde{I}, a]$, a case eliminated in the previous paragraph. This completes the proof. □

CHAPTER C

Pushing-up in SQTK-groups

In this chapter, we recall the fundamentals of pushing up, and develop a number of pushing up results—primarily of local subgroups which are SQTK-groups. Further results on pushing up in QTKE-groups used in the proof of the Main Theorem appear in chapter 4.

C.1. Blocks and the most basic results on pushing-up

In this section, we record and review some of the most fundamental notions and theorems from the theory of pushing up. For the most part we only state results for SQTK-groups, since we will only apply the theory to such groups, and since the theorems are stronger and easier to state for SQTK-groups.

In particular, we prove a version of the Aschbacher Local $C(G,T)$-Theorem, and quote a version of the result of Meierfrankenfeld-Stellmacher on pushing up weak BN-pairs of rank 2, but only for SQTK-groups. These results identify the "obstructions" to pushing up in two of the most basic cases. Sometimes the $C(G,T)$-Theorem and the Meierfrankenfeld-Stellmacher Theorem are sufficient for our purposes; but often we also require deeper results on pushing up, like those in section C.2 (using ideas from [**Asc81b**]), and later sections of this chapter. Still the mathematics in the present section supplies the foundation for the proofs of the more sophisticated theorems, and since the SQTK hypothesis is quite restrictive, that foundation is sufficient.

C.1.1. The pushing up hypotheses (PU) and (CPU). We begin with some general background and notation. We will be considering the normalizers of various nontrivial subgroups of a 2-subgroup R of a finite group G.

Definition C.1.1. As in the literature, $\mathcal{S}_2(G)$ denotes the set of nontrivial 2-subgroups of G.

To obtain interesting conclusions, we must restrict the embedding of R in G. In particular, we often require that R is in the set $\mathcal{B}_2(G)$, of 2-subgroups of G called *2-radical* or *2-stubborn* in the literature; that is, R satisfies:

$$1 \neq R = O_2(N_G(R)).$$

We begin with a few elementary observations about 2-radical subgroups:

Lemma C.1.2. *Let G be a finite group and $R \in \mathcal{S}_2(G)$. Then*

(1) If $R \in Syl_2(G)$ then $R \in \mathcal{B}_2(G)$.

(2) There is $B \in \mathcal{B}_2(G)$ with $R \leq B$ and $N_G(R) \leq N_G(B)$.

(3) If $H \leq K \leq G$ with $R \in \mathcal{B}_2(H)$ and $N_K(R) \leq H$, then $R \in \mathcal{B}_2(K)$.

(4) If $M \leq G$, $L = O^2(L) \trianglelefteq M$, and $R \in Syl_2(C_M(L/O_2(L)))$, then $R \in Syl_2(\langle R^M \rangle)$ and $R \in \mathcal{B}_2(M)$.

PROOF. If $R \in Syl_2(G)$, then $|G : R|$ is odd so certainly (1) holds. Let $B \in \mathcal{S}_2(G)$ be maximal subject to $R \leq B$ and $N_G(R) \leq N_G(B)$. Then since $N_G(B) \leq N_G(O_2(N_G(B)))$, $B \in \mathcal{B}_2(G)$ by the maximal choice of B, establishing (2). Under the hypotheses of (3), $R = O_2(N_H(R))$, so as $N_K(R) \leq H \leq K$, $N_K(R) = N_H(R)$, and hence $R \in \mathcal{B}_2(K)$, so (3) holds. Assume the hypotheses of (4), and let $Q := O_2(N_M(R))$ and $H := C_M(L/O_2(L))$. Then $H \trianglelefteq M$, and we are assuming $R \in Syl_2(H)$, so $R = Q \cap H$. By A.4.2.4, L normalizes R, and hence also normalizes $Q = O_2(N_M(R))$; thus as $L \trianglelefteq M$, $[L, Q] \leq L \cap Q \leq O_2(L)$. Therefore $Q \leq H$, so $Q = Q \cap H = R$. This completes the proof of the second conclusion of (4); the first conclusion follows as $H \trianglelefteq M$. □

Now suppose C is some nontrivial characteristic subgroup of $R \in \mathcal{S}_2(G)$. Then $N_G(R) \leq N_G(C)$, so if $N_G(R) < N_G(C)$, we have "pushed up" $N_G(R)$ to the larger local subgroup $N_G(C)$. A special case of particular interest occurs when $N_G(C) = G$, so that C is normal in G. Unfortunately, there are pairs (G, R) for which no such characteristic subgroup exists, so we are led to study the following situation:

DEFINITION C.1.3. We say that the pair (G, R) satisfies (PU) if

$$1 \neq R \in \mathcal{B}_2(G), \text{ and no nontrivial characteristic subgroup of } R \text{ is normal in } G.$$

Hypothesis (PU) is the weakest of the standard pushing up hypotheses. For example, when $O_2(G) = 1$, (PU) is obviously satisfied for all $R \in \mathcal{B}_2(G)$. However, we will be most interested in the case where $F^*(G) = O_2(G) \neq 1$, where (PU) begins to have some bite. The pairs G, R satisfying (PU) with $F^*(G) = O_2(G)$ are said to be *obstructions* to pushing up.

REMARK C.1.4. The existence of pairs satisfying (PU) restricts internal modules for G: As $R \in \mathcal{B}_2(G)$, $O_2(G) \leq R$, so $J(R) \not\leq O_2(G)$, since otherwise $J(R) = J(O_2(G)) \trianglelefteq G$ by B.2.3.3. Then if $V \in \mathcal{R}_2(G)$ with $O^{2'}(C_G(V)) = O_2(G)$, $J(R) \not\leq C_G(V)$, so by B.2.4.1, the 2-reduced internal module V is an FF-module for $G/C_G(V)$.

Under (PU), the normalizers of all nontrivial characteristic subgroups of R are proper subgroups of G. It is also of interest to study a stronger pushing up condition, in which a *single* proper subgroup contains all these normalizers. Recall that for $S \in \mathcal{S}_2(G)$:

DEFINITION C.1.5. $C(G, S) := \langle N_G(C) : 1 \neq C \text{ char } S \rangle$.

Even when $C(G, R)$ is proper in G, in order to prove strong results we must again restrict the embedding of R in $C(G, R)$. Thus we are led to the following stronger pushing up situation:

DEFINITION C.1.6. The pair (G, R) satisfies (CPU) if

$$C(G, R) \leq M < G \text{ for some } 1 \neq R \in \mathcal{B}_2(G) \text{ with } R \in Syl_2(\langle R^M \rangle).$$

In this context, if C is a nontrivial characteristic subgroup of R such that $N_G(C) \not\leq M$, then we have "pushed up" $N_G(R)$ to $N_G(C)$ outside of M; so in (CPU) we are studying obstructions G, R to the conclusion $C(G, R) = G$. This stronger condition occurs for example when R is normal in a uniqueness subgroup, such as in 1.4.1 in the proof of our Main Theorem, where $T \in Syl_2(G)$, $L_0 = O^2(L_0) \trianglelefteq M < G$, M is the unique maximal 2-local of G containing L_0T, and $R := C_T(L_0/O_2(L_0))$. For then by C.1.2.4, $R \in \mathcal{B}_2(M)$, and by 1.4.1.1, $C(G,R) \leq M$. We will develop the basic theory of triples (R, M, G) satisfying (CPU) under the SQTK-hypothesis in the subsequent sections C.2, C.3, and C.4 of this chapter.

C.1.2. Blocks and their elementary properties. We will return to (PU) and (CPU) later in the section; we turn now to a discussion of short subgroups and blocks, since such subgroups arise naturally as obstructions to pushing up.

When hypothesis (PU) holds with $F^*(G) = O_2(G)$, the typical input is knowledge of the quotient $G/O_2(G)$, and the output consists of restrictions on the structure of the 2-group R, and hence also on the structure of $O_2(G)$. Often (see C.1.29 etc.) the only obstruction to obtaining a nontrivial characteristic subgroup of R normal in G is the case where there is a unique non-central composition factor of G in $O_2(G)$.

Thus we are led to the following definition:

Definition C.1.7. A group L is *short* if
$L = O^2(L)$,
$F^*(L) = O_2(L)$,
$U(L) := [O_2(L), L] \leq \Omega_1(Z(O_2(L)))$,
$L/O_2(L)$ is quasisimple or of order 3, and
L is irreducible on $\tilde{U}(L) := U(L)/C_{U(L)}(L)$.

Admitting the possibility that $L/O_2(L)$ is of order 3 allows us to treat the solvable linear group $L/Z(L) \cong SL_2(2)'$ in parallel with the cases where $L/O_2(L) \cong L_2(2^n)$ with $n > 1$ is quasisimple.

A *block* of a finite group G is a short subnormal subgroup of G. These are sometimes called *Aschbacher blocks* in the literature.

We follow the convention in the literature that the expression "L is a block" (that is, with no overgroup G of L specified) implicitly assumes that $G := L$—that is, this expression just means that L is short.

The next few lemmas establish some basic properties of blocks; all are presumably known at least to experts.

Lemma C.1.8. *If X is a block, then $X/U(X)$ is quasisimple or of order* 3.

Proof. By definition:
(a) $U(X) = [O_2(X), X]$,
(b) $X/O_2(X)$ is quasisimple or of order 3, and
(c) $X = O^2(X)$.

By (a), $O_2(X/U(X)) \leq Z(X/O_2(X))$, so the lemma follows from (b) and (c), using 31.1 in [**Asc86a**] in the case where $X/O_2(X)$ is quasisimple. □

See Definition C.1.12 for the notion of A_3-block.

Proposition C.1.9. *If G has an A_3-block, assume $F^*(O_{2,3}(G)) = O_2(O_{2,3}(G))$. Then distinct blocks in G commute.*

Proof. See 3.4 in [**Asc81a**]. □

In the following lemma, if π is empty, then $O^{\pi}(Y) = Y$.

LEMMA C.1.10. *Assume X is a normal block of H, and let $F := End_X(\tilde{U}(X))$ and π the set of prime divisors of $|F^\#|$. Assume $Y \le H$ satisfies $Y = O^2(Y) = O^\pi(Y)[Y,Y]$, and Y centralizes $X/O_2(X)$. Then Y centralizes X.*

PROOF. Set $\bar{H} := H/C_H(\tilde{U}(X))$, and recall $O_2(X) \le C_X(U(X)) \le C_H(\tilde{U}(X))$ by definition as X is short. As $\bar{X}$ is irreducible on $\tilde{U}(X)$, F is a field. By hypothesis $\bar{Y} \le C_{Aut_H(\tilde{U}(X))}(\bar{X})$), and the latter group is a subgroup of the abelian group $F^\#$ of odd order. Therefore as $Y = O^\pi(Y)[Y,Y]$ by hypothesis, Y centralizes $\tilde{U}(X)$. Also $[X,Y] \le O_2(X) \le C_X(U(X))$, so by the Three-Subgroup Lemma, Y centralizes $[U(X),X] = U(X)$.

Next $[Y,X] \le O_2(X)$ by hypothesis, and $X/U(X)$ is either quasisimple or of order 3 by C.1.8. In the latter case, $U(X) = O_2(X)$, so that Y centralizes $X/U(X)$; and in the former, Y centralizes $X/U(X)$ by 31.6.1 in [**Asc86a**]. Thus as Y centralizes $U(X)$, the lemma holds by Coprime Action as $Y = O^2(Y)$. □

LEMMA C.1.11. *Let $H = KS$ with K a product of blocks and $S \in Syl_2(H)$. Set $U := [O_2(K), K]$. Then $C_H(U) = O_2(H)$.*

PROOF. As $U = [O_2(K), K]$ and $F^*(K) = O_2(K)$, $C_K(U) \le O_2(K)$. Thus as $H = KS$, $C_H(U) \le O_2(H)$, so it remains to show $O_2(H) \le C_H(U)$. By induction on the number of blocks of H, we may assume K is a block. Furthermore $[O_2(H), K] \le O_2(K) \le C_K(U)$, so $O_2(H) \le C_H(U)$ by A.1.41, completing the proof. □

We next consider the blocks that arise in results like the $C(G,T)$-theorem (see C.1.29 below).

DEFINITION C.1.12. We say a short group L is of *type* $L_2(2^n)$ (for $n > 1$) if $L/O_2(L) \cong L_2(2^n)$ and $\tilde{U}(L)$ is the natural module for $L/O_2(L)$. Similarly L is of *type* A_n if $L/O_2(L) \cong A_n$ and $\tilde{U}(L)$ is the natural module (that is, the unique noncentral chief factor in the n-dimensional permutation module described in section B.3). Write χ for the set of short groups of type $L_2(2^n)$ or A_m, m odd, and write χ_0 for the set of short groups of type $L_2(2^n)$, A_3, or A_5.

A χ-*block* of G is a block in χ, and an $L_2(2^n)$-*block* is a block of type $L_2(2^n)$, etc.

To motivate the definitions of χ and χ_0, first recall from [**Asc81a**] that $\mathcal{Y}$ is the subset of χ consisting of the blocks of type $L_2(2^n)$ and A_m with m of form $2^a + 1$. In an SQTK-group, Theorem C (A.2.3) rules out sections which are alternating groups of degree 9 or more, leaving only A_3 and A_5. The collection $\mathcal{Y}$ appeared in our earlier lemma B.6.9; while χ_0 will appear later on in the present section in the $C(G,T)$-Theorem C.1.29, and χ will appear sooner in C.1.16 and C.1.28.

LEMMA C.1.13. *Let $T \in Syl_2(LT)$ with L a normal block of LT, and set $Q := O_2(LT)$ and $U := U(L)$. Then*

(a) $\Phi(Q) \le C_T(L)$.

(b) $m(Q/UC_T(L)) \le m(H^1(L/O_2(L), \tilde{U}(L)))$, and $O_2(L)/U$ is a quotient of the the Schur multiplier of $L/O_2(L)$ if $L/O_2(L)$ is quasisimple, and is trivial if $L/Z(L) \cong \mathbf{Z}_3$.

(c) If L is an A_3-block or A_5-block then $Q = U \times C_T(L)$ and U is the natural module. If L is an A_3-block then $L \cong A_4$. If L is an A_5-block, then $L/U \cong A_5$ or $SL_2(5)$, with $Z(L)$ of order 2 in the latter case. In any case, Q centralizes $O_2(L)$.

(d) If L is an A_6-block, then Q centralizes $O_2(L)$.

PROOF. As L is a block, $[Q, L] = U$, while $U \leq Z(Q)$ by C.1.11. Thus for $l \in L$ and $q \in Q$, we have $l^q = lv$ for some $v \in U$, and then $l^{q^2} = lv^2 = l$, since U is elementary abelian, so (a) holds.

Now set $\bar{L}\bar{T} := LT/C_T(L)$. Then in particular $\bar{U} \cong \tilde{U}(L)$. Further $[\bar{L}, \bar{Q}] = \overline{[L, Q]} = \bar{U}$ and by (a), $\bar{Q}$ is elementary abelian. Setting $\bar{D} := C_{\bar{Q}}(\bar{L})$, $m(\bar{Q}/\bar{U}\bar{D}) \leq m(H^1(L/O_2(L), \tilde{U}(L)))$. Let D denote the preimage of $\bar{D}$. Then L centralizes $\bar{D} = D/C_T(L)$ and $C_T(L)$, and so $L = O^2(L)$ centralizes D by Coprime Action; that is, $C_T(L) = D$, establishing the first part of (b).

Next set $L^* := L/U$. By C.1.8, L^* is of order 3 or L^* is quasisimple. Hence $O_2(L)^*$ is a quotient of the Schur multiplier of $L/O_2(L)$ when $L/O_2(L)$ is quasisimple, and trivial if $L/O_2(L)$ is of order 3, completing the proof of (b).

Now assume that L is an A_n-block for $n = 3$, 5, or 6. When $n = 3$, L^* is of order 3 and $O_2(L) = U$ is the natural module for L^* by (b), so that $L \cong A_4$. Thus we may take $n = 5$ or 6. By (b), $L^* \cong A_n$, $SL_2(5)$, or $SL_2(9)$, as the latter two groups are the universal 2-covering groups of A_5, A_6, respectively using I.1.3. If $n = 5$, the irreducible natural module for $L/O_2(L)$ is projective by I.1.6, so $C_U(L) = 1$ and $Q = U \times D$ by (b). This completes the proof of (c).

It remains to prove (d), so we may take $n = 6$. Recall $U \leq Z(Q)$, so we may assume that $U < P := O_2(L)$ and hence $L^* \cong SL_2(9)$ and $|P : U| = 2$ by the previous paragraph. Let K_0/P be the stabilizer of a point in the representation of L/P on 6 points, and set $K := O^2(K_0)$. Now $U/C_U(L)$ is the 4-dimensional permutation module for A_6, so $U_K := [U, K]$ is the A_5-module for K_0/P, and hence K is an A_5-block, so that $O_2(K) = Z(K) \times U_K$ by (c). As $L^* \cong SL_2(9)$, $K^* \cong SL_2(5)$ with $Z(K^*) = Z(L^*)$, so $K/U_K \cong SL_2(5)$ with $Z(K)$ of order 2, and $P = UZ(K)$. Further $[Q, K] \leq [Q, L] = U$, so Q acts on $O^2(KU) = K$, and hence centralizes $Z(K)$ as $|Z(K)| = 2$. Therefore Q centralizes $UZ(K) = P$, establishing (d). $\square$

We often specialize to the case where $F^*(G) = O_2(G)$ and work with some internal module V; in that case we adopt the following notation:

HYPOTHESIS C.1.14. *G is a finite group with $F^*(G) = O_2(G)$. Choose $V \in \mathcal{R}_2(G)$ and $T \in Syl_2(G)$, and set $G^* := G/C_G(V)$ and $Z := \Omega_1(Z(T))$.*

REMARK C.1.15. Recall that as $V \in \mathcal{R}_2(G)$, V is normal in G, V is an elementary abelian 2-group, and V is 2-reduced: that is, $O_2(G^*) = 1$. When $J(T) \not\leq C_G(V)$, V is an FF-module for G^* by B.2.4.1. Then since $O_2(G^*) = 1$, we can apply various results on the Thompson and Baumann subgroups in section B.2, and in particular results determining FF-modules, such as Theorem B.5.6.

The next technical lemma shows that $J(T)$ normalizes the components of G^*, and that the Baumann subgroup $\mathrm{Baum}(T)$ normalizes suitable blocks.

PROPOSITION C.1.16. *Assume Hypothesis C.1.14. Then*

(1) $J(T)^*$ acts on each component of G^*.

(2) $Baum(T)$ acts on each χ-block L of G such that $[L^*, J(T)^*] \neq 1$.

(3) If G is an $SQTK$-group, $L \in \mathcal{C}(L)$ with L^* quasisimple, and $[L^*, J(T)^*] \neq 1$, then $Baum(T)$ acts on L.

PROOF. Part (1) appears as Theorem 26.24 in [**GLS96**], but the proof is easy given B.1.5.4: Namely by B.2.5, $J(T)^* \le J(G^*, V)$, and by B.1.5.4, each component of $J(G^*, V)$ is normal in $J(G^*, V)$. On the other hand, if K^* is a component of G^* not contained in $J(G^*, V)$, then $[K^*, J(G^*, V)] = 1$ by 31.4 in [**Asc86a**]. Thus (1) is established.

Assume the hypotheses of (2) or (3); thus in either case, L is subnormal in G and $[L^*, J(T)^*] \neq 1$. We may assume $G = \langle L, T\rangle$. Thus if L is an $L_2(2^n)$-block or an A_3-block, then G is a minimal parabolic in the sense of Definition B.6.1, and (2) follows from Baumann's Lemma B.6.10.1. Therefore we assume that either L is an A_n-block for $n > 3$, or the hypotheses of (3) hold. In either case, $L \in \mathcal{C}(G)$ and L^* is a component of G^*, so by (1), $J(T)$ acts on L^* and hence on $[V, L] =: U$; the action is nontrivial since $[L^*, J(T)^*] \neq 1$. Let $K := \langle L^T\rangle$, $U_K := \langle U^T\rangle$, and $\tilde{U}_K := U_K/C_{U_K}(K)$. Then $K = K_1 \cdots K_m$ is the central product of the conjugates of L under T, and we may assume that $m > 1$. Let $U_i := [K_i, V]$.

In (2), L is irreducible on $\tilde{U}$, so $\tilde{U}_K$ is the direct sum of the $\tilde{U}_i$. In (3) as $m > 1$, (1) and (3) of A.3.8 say that $m = 2$ and L^* is $L_2(2^n)$, $Sz(2^n)$, $L_2(p)$, or J_1. Then by Theorem B.5.6, $\tilde{U}_K = \tilde{U}_1 \oplus \tilde{U}_2$, and $\tilde{U}$ is the natural module for $L^* \cong L_2(2^n)$ or A_5, or the sum of at most two isomorphic natural modules for $L_3(2)$.

Thus in any event, $\tilde{U}_K = \tilde{U}_1 \oplus \cdots \tilde{U}_m$. But in each case, we check that $C_U(L) < C_U(J(T)^*)$, so as $C_U(J(T)) \le \Omega_1(Z(J(T)))$ by B.2.3.7, and as $\tilde{U}_K$ is the direct sum of the $\tilde{U}_i$, Baum(T) acts on U and hence also on L. □

C.1.3. Pushing-up in certain minimal situations. In the remainder of the section, we collect a number of the most basic results from the literature determining the obstructions to pushing up. For many of these results, we provide short, modern proofs under the hypothesis that G is an SQTK-group.

Typically the results show that the obstruction to pushing up must be a block, or occasionally some other highly restricted group.

In this subsection, we focus on results describing obstructions to pushing up when $G/O_2(G)$ is a rank-1 group, or one of a number of other "small" groups. We assume, and record here as C.1.18 below, the fundamental result of Glauberman-Niles and Campbell on pushing up a minimal parabolic G in the sense of Definition B.6.1 which satisfies Hypothesis C.1.14, with $G^* \cong L_2(2^n)$ and $V/C_V(G)$ the natural module.

DEFINITION C.1.17. Given a 2-group S, denote by $\mathcal{G}(S)$ the set of finite groups H such that: $S \in Syl_2(H)$, $F^*(H) = O_2(H)$, H is a minimal parabolic in the sense of Definition B.6.1, and setting $V := \langle \Omega_1(Z(S))^H\rangle$ and $H^* := H/C_H(V)$, we have $F^*(H^*) \cong L_2(2^n)'$, with $[V, F^*(H^*)]/C_{[V,F^*(H^*)]}(F^*(H^*))$ the natural module for $F^*(H^*)$.

THEOREM C.1.18 (Glauberman-Niles/Campbell). *Let S be a 2-group. Then there exist nontrivial characteristic subgroups $C_i(S)$, $i = 1, 2$, of S, such that $C_1(S) \le \Omega_1(Z(S))$, $C_2(S)$ char Baum(S), and for all $H \in \mathcal{G}(Baum(S))$, one of the following holds:*

(1) $C_1(S) \le Z(H)$.
(2) $C_2(S) \trianglelefteq H$.
(3) $O^2(H)$ is an $L_2(2^n)$-block or A_3-block of H.

PROOF. This appears in [**GN83**] in our Background References. (We mention that it was also obtained by Campbell in his unpublished thesis [**Cam79**]). □

REMARK C.1.19. Notice that it suffices to prove Theorem C.1.18 for $B :=$ Baum(S) in the role of "S": Namely given $C_i(B)$ satisfying the conclusions of C.1.18 for B, define $C_2(S) := C_2(B)$ and $C_1(S) := Z(S) \cap C_1(B)$, and observe that these groups satisfy the conclusions of C.1.18 for S.

In the remainder of this subsection, we will apply C.1.18 in treating a number of more general minimal situations, culminating in the $C(G,T)$-theorem C.1.29. Since the two characteristic subgroups defined in C.1.18 will play a prominent role, we establish the following notation:

NOTATION C.1.20. Given a 2-group T, let $C_1(T)$ and $C_2(T)$ be characteristic subgroups of T supplied by the Glauberman-Niles/Campbell Theorem C.1.18. Thus $C_1(T) \leq \Omega_1(Z(T))$ and $C_2(T)$ char Baum(T). Indeed because of Remark C.1.19, we can and will choose these subgroups so that $C_2(T) := C_2(\mathrm{Baum}(T))$ and $C_1(T) := Z(T) \cap C_1(\mathrm{Baum}(T)) \leq C_1(\mathrm{Baum}(T))$.

The following argument, due originally to Glauberman, is our main technical tool for extending Theorem C.1.18 to groups other than those in $\mathcal{G}(T)$. Notice that we leave the 2-group T unspecified; in particular T is not necessarily a subgroup of G. In some applications, G is a subgroup of some larger group containing T.

LEMMA C.1.21 (Glauberman's Argument). *Assume*

(i) G is a finite group with $F^*(G) = O_2(G)$ and $L := O^2(G) \in \mathcal{C}(G)$ with $L/O_2(L)$ quasisimple.

(ii) T is a finite 2-group and $S :=$ Baum$(T) \leq R \leq T$, with $O_2(G) \leq R \leq G$. Let $\mathcal{Y}(G,R)$ denote the set of all subgroups $Y = O^2(Y)$ of G invariant under $R_0 := SO_2(G)$, such that $R \cap Y \in Syl_2(Y)$ and $R_2(R_0Y)/C_{R_2(R_0Y)}(Y)$ is the natural module for $Y/O_2(Y) \cong L_2(2^n)$ or $\mathbf{Z}_3$. Then

(1) For each $Y \in \mathcal{Y}(G,R)$, one of the following holds:

(a) Y normalizes $C_2(T)$.
(b) Y centralizes $C_1(T)$ and $C_1(S)$.
(c) Y is an $L_2(2^n)$-block or A_3-block.

Moreover either Y acts on S, or $S \in Syl_2(YS)$ and $Y = [Y, J(R)]$.

(2) If there is $\mathcal{Y} \subseteq \mathcal{Y}(G,R)$ such that $G = \langle \mathcal{Y}, C_G(\Omega_1(Z(R)))\rangle = \langle N_G(S), Y\rangle$ for each $Y \in \mathcal{Y}$, then one of the following holds:

(a) $C_1(T) \leq Z(G)$.
(b) $C_2(T) \trianglelefteq G$.
(c) L is a block, and $U(L)$ is an FF-module for $LS/C_{LS}(U(L))$.

(3) If $G = \langle N_G(S), Y\rangle$ for some $Y \in \mathcal{Y}(G,R)$, then either

(I) some nontrivial characteristic subgroup of S is normal in G, or
(II) L is a block, and $U(L)$ is an FF-module for $LS/C_{LS}(U(L))$.

(4) Let $V \in \mathcal{R}_2(G)$ and set $G^* := G/C_G(V)$ and $Z_V := C_V(J(R))$. Assume $J(R)^* = C_R(Z_V)^*$, and $X = O^2(X)$ is a $J(R)C_G(V)$-invariant subgroup of G satisfying $C_X(V) = O_2(X)$, $X^*J(R)^*/O_2(X^*J(R)^*) \cong L_2(2^n)$, $J(R)^* \in Syl_2(X^*J(R)^*)$, and $[Z_V, X] \neq 1$. Then $S^* = J(R)^*$ and $X \in \mathcal{Y}(G,R)$.

PROOF. Let $Q := O_2(G)$ and $Z := \Omega_1(Z(T))$. Now $Z \leq C_T(\Omega_1(Z(J(T)))) = \mathrm{Baum}(T) = S$, and by hypothesis $S \leq R$, so $Z \leq \Omega_1(Z(R)) =: Z_R$. Second, $J(R) = J(S)$ and $S = \mathrm{Baum}(R)$ by parts (3) and (4) of B.2.3.

Assume $Y \in \mathcal{Y}(G, R)$; then by definition, QS acts on Y and $R \cap Y \in Syl_2(Y)$. Thus as $Q \leq R$, $P := QS(R \cap Y) \leq R$ and P is Sylow in $YQS = YP$. Again

$$J(R) = J(P) \text{ and } S = \mathrm{Baum}(P)$$

by parts (3) and (4) of B.2.3.

Let $U := R_2(YP)$, and $Y^*P^* := YP/C_{YP}(U)$. We will verify that the hypotheses of B.2.10.2 are satisfied, with YP, Y, P, P, U in the roles of "G, L, T, R, V"; since $S = \mathrm{Baum}(P)$, S still plays the role of "S". First, $O_2(YP) \leq P$ since $P \in Syl_2(YP)$. As $U \in \mathcal{R}_2(YP)$, $O_2(Y^*P^*) = 1$. Further $Y = O^2(Y)$ with $Y/O_2(Y) \cong L_2(2^n)'$, so either $n > 1$ and $Y \in \mathcal{C}(YP)$, or $n = 1$ and $Y/O_2(Y) \cong \mathbf{Z}_3$. Finally $[U, Y] \neq 1$ as $U/C_U(Y)$ is the natural module for $Y^* \cong L_2(2^n)'$ by hypothesis. This completes the verification of the hypotheses of B.2.10.2, so we can appeal to Lemma B.2.10.

We turn to the proof of (1). If $S = \mathrm{Baum}(O_2(YP))$, then Y normalizes S, and hence also normalizes the characteristic subgroup $C_2(T)$ of S in view of C.1.20, so that conclusion (1a) holds.

Thus we may assume that case (b) of B.2.10.1 holds: $Y^* = F^*(J_{\mathcal{P}_{P,YP}}(Y^*P^*))$. We saw that $J(P) = J(R)$, so that $J(R)$ does not centralize U, and hence $Y = [Y, J(R)]$. Furthermore as YP is a minimal parabolic in the sense of Definition B.6.1, and $U/C_U(Y)$ is the natural module for Y^*, the hypotheses of Baumann's Lemma B.6.10 are satisfied with YP, P, Y^*, U in the roles of "H, T, L^*, V". Then we conclude from B.6.10.2 that S is Sylow in $\langle S^Y \rangle = YS$. In particular this establishes the final statement in (1). Further as $U/C_U(Y)$ is the natural module for Y^*, $U \cap \Omega_1(Z(S)) \not\leq C_U(Y)$ by I.2.3, so $\langle \Omega_1(Z(S))^{YS} \rangle = UC_{\Omega_1(Z(S))}(Y)$ by B.2.13. Therefore $YS \in \mathcal{G}(S)$.

Thus we may apply C.1.18 with YS, S, S in the roles of "H, S, $\mathrm{Baum}(S)$", and appealing to C.1.20, in the three cases of C.1.18 we obtain the three conclusions of our lemma: As $S = \mathrm{Baum}(T)$ and $C_2(T) = C_2(S)$, case (2) of C.1.18 gives conclusion (1a). In case (1), Y centralizes $C_1(S)$, and hence also centralizes $C_1(T) = Z(T) \cap C_1(S)$, giving conclusion (1b). Finally in case (3) conclusion (1c) holds for $O^2(G) = L$. This completes the proof of (1).

Next we make a reduction which will simplify the verification of conclusions (2) and (3): Suppose Y satisfies neither case (a) nor (b) of (1), so that Y satisfies case (c) of (1). Set $W := \Omega_1(Z(Q))$, and recall that U centralizes $O_2(YP)$ and $Q \leq O_2(YP)$; hence as $F^*(G) = Q$, we conclude that $U \leq W$. Thus $U(Y) \leq W$, so $[Q, Y] \leq W$. Hence $L = \langle Y^L \rangle$ centralizes Q/W, so $[Q, L] \leq W$. Indeed as Y has only one noncentral chief factor on W, so does L, so L is also a block with $U(L) = [Q, L] \leq W$. Thus the hypotheses of B.2.10.2 are satisfied, with LP, L, P, P, $U(L)$ in the roles of "G, L, T, R, V"; so by that result either $S = \mathrm{Baum}(Q)$, or $L = [L, J(R)]$. In the former case, conclusion (a) of (1) holds, contrary to assumption; in the latter case, $U(L)$ is an FF-module for $LS/C_{LS}(U(L))$, so that conclusions (2c) and (3II) hold. Therefore in our proof of (2) and (3), we may assume that each $Y \in \mathcal{Y}(G, R)$ satisfies case (a) or (b) of (1).

Now assume the hypotheses of (2). Then there is $\mathcal{Y} \subseteq \mathcal{Y}(G, R)$ with $G = \langle Y, N_G(S) \rangle$ for each $Y \in \mathcal{Y}$. If case (a) of (1) holds for some $Y \in \mathcal{Y}$, then Y acts on $C_2(T)$. So as $C_2(T)$ is characteristic in S, $G = \langle Y, N_G(S) \rangle$ acts on $C_2(T)$, and hence conclusion (b) of (2) holds. Hence by the reduction in the previous paragraph, we may assume that each $Y \in \mathcal{Y}$ satisfies case (b) of (1). But then by the hypothesis of (2), $G = \langle \mathcal{Y}, C_G(Z_R)\rangle$ centralizes $C_1(T)$, as $C_1(T) \leq Z \leq Z_R$. Thus conclusion (a) of (2) holds, and the proof of (2) is complete.

Assume the hypotheses of (3). Our earlier reduction showed that Y satisfies case (a) or (b) of (1), so that either $C_2(S)$ or $C_1(S)$ is normalized by Y, and hence is normalized by $G = \langle Y, N_G(S)\rangle$. Thus conclusion (I) of (3) holds, so the proof of (3) is also complete.

Finally assume the hypotheses of (4). By B.2.3.2, $Z_V \leq \Omega_1(Z(J(R)))$, so that $S^* \leq C_R(Z_V)^* = J(R)^*$ by hypothesis. Hence $S^* = J(R)^*$, establishing one of the conclusions of (4), and also $C_G(V)S = C_G(V)J(R)$.

As $X = O^2(X) \leq O^2(G) = L$ and $[V, X] \neq 1$, also $[V, L] \neq 1$. As before, the hypotheses of B.2.10.2 are satisfied with LP, L, P, P in the roles of "G, L, T, R"; so Q is Sylow in $C_G(V)$ by that result. Thus as $Q \leq R$ and $J(R)^* \in Syl_2(X^*J(R)^*)$ by hypothesis, $R \cap X \in Syl_2(X)$. Also as $C_G(V)J(R) = C_G(V)S$, $QJ(R) = QS = R_0$. By hypothesis X is $C_G(V)J(R)$-invariant and $J(R)^* \in Syl_2(X^*J(R)^*)$, so X is R_0-invariant, and R_0 is Sylow in XR_0. Further $R_0 \leq R$ centralizes Z_V, and by hypothesis $[Z_V, X] \neq 1$, so $[R_2(XR_0), X] \neq 1$ by B.2.14. Then as $XR_0/O_2(XR_0) \cong X^*J(R)^*/O_2(X^*J(R)^*) \cong L_2(2^n)$, the FF-module $R_2(XR_0)/C_{R_2(XR_0)}(X)$ is the natural module by Theorem B.5.1.1. Thus $X \in \mathcal{Y}(G, R)$, completing the proof of (4). □

We now begin our series of results extending C.1.18. The following lemma considers the only case where G is a minimal parabolic with $O^2(G/O_2(G)) \cong L_2(2^n)$ and $R_2(G)$ is an FF-module for G, but $G \notin \mathcal{G}(S)$: the case where $[R_2(G), O^2(G)]$ is the A_5-module for $O^2(G/O_2(G)) \cong L_2(4)$. For technical reasons, we also treat one of the two cases where $O^2(G/O_2(G)) \cong A_7$. In the that case, we need the first of the following definitions, and we will need the second shortly thereafter.

Definition C.1.22. A block L is an *exceptional A_7-block* if $L/O_2(L) \cong A_7$ and $m(U(L)/C_{U(L)}(L)) = 4$; by I.1.6.10, this implies that $C_{U(L)}(L) = 0$ and hence $m(U(L)) = 4$. Thus $U(L)$ is the natural module for $L_4(2) \cong A_8$ restricted to its subgroup A_7. Define L to be a *$\hat{A}_6$-block* if $L/O_2(L) \cong \hat{A}_6$, $m(U) = 6$, and $C_L(U) = O_2(L)$; thus U is the natural module for $SL_3(4)$ restricted to its subgroup $\hat{A}_6$.

Just as in C.1.21, the 2-group T in the following result need not be a subgroup of G. In at least one later application, T will be Sylow in an overgroup of G.

Lemma C.1.23. *Assume T is a finite 2-group, $S := Baum(T) \leq R \leq T$, G is a finite group with $R \in Syl_2(G)$, and $G = LR$ for some $L \in \mathcal{C}(G)$, with $F^*(G) = O_2(G)$, and $L/O_2(L) \cong A_5$ or A_7. Then*

(1) If $L/O_2(L) \cong A_5$, then one of the following holds:

(a) $C_1(T) \leq Z(G)$.

(b) $C_2(T) \trianglelefteq G$.

(c) L is an A_5-block or $L_2(4)$-block and $L = [L, J(T)]$.

(2) If L is an exceptional A_7-block then $G = \langle C_G(C_1(T)), N_G(C_2(T))\rangle$.

Proof. Let $Z := \Omega_1(Z(T))$, $V := \langle Z^G\rangle$, $U := [V, L]$, and $G^* := G/C_G(V)$.

First $Z \leq C_T(\Omega_1(Z(J(T)))) = \mathrm{Baum}(T) = S$, and by hypothesis $S \leq R \leq T$, so $Z \leq \Omega_1(Z(R)) =: Z_R$, and hence $V \in \mathcal{R}_2(G)$ by B.2.14. Also observe that

$J(R) = J(S)$ and $S = \mathrm{Baum}(R)$ by parts (3) and (4) of B.2.3. Hence the hypotheses of B.2.10 hold, with R serving in the role of both "R" and "T".

Suppose first that $[V, L] = 1$. Then the hypothesis of (2) does not hold. Furthermore by C.1.20 $C_1 := C_1(T) \leq Z \leq V$, so that C_1 is central in $LR = G$, and conclusion (a) of (1) holds.

So we may assume that $[V, L] \neq 1$. Hence the hypotheses for part (2) of B.2.10 hold, so we can apply that result and adopt its notation. If $S = \mathrm{Baum}(O_2(G))$ then $C_2 := C_2(T)$ char $S \trianglelefteq G$, so conclusion (2) holds, as does conclusion (b) of (1).

Thus we may assume that the second case of B.2.10.2 holds. Since $R \in Syl_2(G)$, $\mathcal{P}_{R,G} = \mathcal{P}_G$ is a stable set of FF*-offenders on V, and V is an FF-module with $F^*(J(G)^*) = L^*$ quasisimple, so we may apply Theorem B.5.1 to conclude $U \in Irr_+(L, V)$. Then we apply B.4.2 to conclude $\tilde{U}$ is a natural module for $J(G)^* \cong L_2(4)$ or S_5 in (1), while in (2) $\tilde{U}$ is a 4-dimensional module for $G^* \cong A_7$ by hypothesis.

We first prove (1). We will appeal to part (2) of Glauberman's Argument C.1.21; more precisely, we will exhibit $Y \in \mathcal{Y}(G, R)$ such that $G = \langle Y, R\rangle$, so (1) follows from C.1.21.2.

First if $\tilde{U}$ is the $L_2(4)$-module, we take $Y := L$; as we are assuming $R \in Syl_2(G)$, visibly $L \in \mathcal{Y}(G, R)$, and by hypothesis $G = LR$, so (1) holds in this case.

Next suppose $\tilde{U}$ is the natural module for S_5. In this case $C_U(L) = 1$ as the natural module is projective (e.g., I.1.6.1), so $U \cong \tilde{U}$, and we adopt the notation of section B.3. We will use part (4) of Glauberman's Argument C.1.21 to produce Y. By B.3.2.4, $J(R)^* = \langle (1, 2), (3, 4)\rangle$ and $Z_U := C_U(J(R)) = \langle e_{1,2}, e_{3,4}\rangle$. Hence $J(R)^* = C_{G^*}(C_U(J(R)))$. Next we define $Y := O^2(X)$, where X is the preimage in L of $\langle (3, 4), (3, 5)\rangle$. Visibly the hypotheses of C.1.21.4 are satisfied with Y in the role of "X", so $Y \in \mathcal{Y}(G, R)$ by that lemma. Then as $G^* = \langle Y^*, R^*\rangle$, $G = \langle Y, R\rangle$, completing the proof of (1).

Finally suppose L is an exceptional A_7-block. Then $U = [V, L]$ is a 4-dimensional module for $G^* \cong A_7$. Represent G^* on $\Omega := \{1, \ldots, 7\}$ as in section B.3. In this case by B.4.2.7 there is a unique FF*-offender $A^* := \langle (1, 2)(3, 4), (1, 3)(2, 4)\rangle$ in R^*, and $C_U(A)$ is the unique R^*-invariant 2-subspace of U. Then $C_{G^*}(C_U(A)) \cong A^* \times \mathbf{Z}_3$, and hence $S^* = J(R)^* = A^*$. Thus $N_G(S)^*$ is the global stabilizer of $\{1, 2, 3, 4\}$, and since C_2 char S, $N_G(S)^* \leq N_G(C_2)^*$. Further by Theorem B.5.1.2, $UZ = UC_Z(L)$, so that $C_L(U \cap Z) \leq C_G(C_1)$ as $C_1 \leq Z$. Then as $C_L(U \cap Z)^* \cong L_3(2)$, $G = \langle C_G(C_1), N_G(C_2)\rangle$, establishing (2). □

LEMMA C.1.24. *Assume Hypothesis C.1.14 with $G = LT$ for some $L \in \mathcal{C}(G)$, $F^*(L) = O_2(L)$, $[V, L] \neq 1$, and $L/O_{2,Z}(L) \cong A_m$. Let $O_2(G) \leq R \leq T$, and set $S := Baum(R)$. Further assume that one of the following holds:*

(a) $m = 5$ or 7.

(b) $m = 6$ and $R^* \leq L^*$.

(c) $m = 6$ and $|R^*| = 2$.

Then either

(1) some nontrivial characteristic subgroup of S is normal in G, or

(2) L is a χ-block, an exceptional A_7-block, an A_6-block, or an $\hat{A}_6$-block.

PROOF. Set $U := [V, L]$, and $G^* := G/C_G(V)$. Observe that the hypotheses of B.2.10.2 are satisfied, so $C_R(V) = O_2(G) = C_T(V)$ by that lemma. Furthermore if $S = \mathrm{Baum}(O_2(G))$ then conclusion (1) holds, so by B.2.10.2 we may assume $\mathcal{P} := \mathcal{P}_{R,G}$ is a stable set of FF^*-offenders on V, and setting $H^* := J_{\mathcal{P}}(G^*)$, $F^*(H^*) = L^*$ is quasisimple. Hence $U \in Irr_+(L, V)$ by Theorem B.5.1, and B.4.2 says that $\tilde{U} := U/C_U(L)$ is either a natural module for $H^* \cong L_2(4)$, S_5, S_7, A_6, S_6, or $\hat{A}_6$, or the 4-dimensional module for $H^* \cong A_7$.

In each case we will appeal to part (3) of C.1.21 with R in the roles of "R" and "T". Specifically we will exhibit $Y \in \mathcal{Y}(G, R)$ such that $G^* = \langle Y^*, N_{G^*}(S^*)\rangle$, and hence $G = \langle Y, N_G(S)\rangle$ since $O_2(G) \leq R$, so $S = \mathrm{Baum}(O_2(G)S)$ by B.2.3.4. This verifies the hypothesis of C.1.21.3, so by that lemma either conclusion (1) holds, or L is a block. In the latter case from the description of $\tilde{U}$ in the previous paragraph, the block is of one of the types described in conclusion (2), completing the proof of the lemma.

Thus it remains to produce Y. To do so, we will appeal to part (4) of C.1.21. Thus we will need to locate $J(R)^*$ and $Z_U := C_U(J(R))$.

We first consider the case where $\tilde{U}$ is the natural module for $H^* \cong L_2(4)$. Here $T^* \cap L^* \in Syl_2(L^*)$ is the unique FF^*-offender on U in T^* by B.4.2.1, and $T^* \cap L^* = C_{G^*}(C_U(T^* \cap L^*))$, so that $J(R)^* = T^* \cap L^* = S^*$ and $Z_U = C_U(T^* \cap L^*)$. In this case we take $Y := L$. Visibly L satisfies the hypotheses for "X" in C.1.21.4, so $L \in \mathcal{Y}(G, R)$. As $S^* = T^* \cap L^* \trianglelefteq T^*$, $G^* = L^*T^* = L^*N_{G^*}(S^*)$, so the lemma holds in this case.

We next consider the case where $\tilde{U}$ is the natural module for $H^* \cong S_m$, $m := 5$, 6, or 7, and adopt the notation of section B.3. If $m = 5$ or 7, then since $\mathcal{P}$ is stable, we conclude from B.3.2.4 that $J(R)^* \cong E_{2^k}$ is generated by $k \leq \lfloor m/2 \rfloor$ commuting transpositions. On the other hand if $m = 6$, then as we are assuming $H^* \not\leq L^*$, hypothesis (c) holds, so R^* is of order 2, and hence $R^* = J(R)^*$ is generated by a single transposition; in this case we set $k := 1$. In each case we check that $J(R)^* = C_{G^*}(C_U(J(R)))$. Let $X^* := \langle r^*, t^*\rangle \cong S_3$, where $r^* := (i, j)$ is a transposition in R^*, and t^* a transposition moving a point of Ω fixed by $J(R)^*$. Let $Y_0/O_2(G)$ be a $J(R)$-invariant complement to $O_{2,Z}(L)O_2(G)/O_2(G)$ in $X/O_2(G)$ and $Y := O^2(Y_0)$. Then $Y^* = O^2(X^*)$ moves 3 points of Ω, and does not centralize $e_{i,j} \in Z_U$. Thus the hypotheses of C.1.21.4 are satisfied with Y in the role of "X", so $S^* = J(R)^*$ and $Y \in \mathcal{Y}(G, R)$ by that result. Further $G^* = \langle N_{G^*}(S^*), Y^*\rangle$: for example, $N_{G^*}(S^*)$ is maximal in G^* when $k = 1$ or 3, and when $k = 2$ this group is contained in a unique maximal subgroup, which fixes a point moved by Y^*. Thus the lemma also holds in this case.

Finally we consider the cases where either $\tilde{U}$ is a 4-dimensional module for $G^* \cong A_7$, or hypothesis (b) holds. In case (b) as $R^* \leq L^*$, $H^* \cong A_6$ or $\hat{A}_6$, and $\tilde{U}$ is the natural module of rank 4 or 6. Represent $G^*/Z(L^*)$ on $\{1, \ldots, m\}$, and recall from cases (6)–(8) of B.4.2 that there is a unique FF*-offender A^* in $T^* \cap L^*$: namely $A^* := \langle (1,2)(3,4), (1,3)(2,4)\rangle$ (modulo the right choice of notation when $m = 6$). Hence $J(R)^* = A^*$. Moreover in each case $J(R)^* = O^{2'}(C_{G^*}(Z_U))$ and $m(U/Z_U) = m(A^*) = 2$. This time let $X^*/Z(L^*)$ be the pointwise stabilizer of $\{6\}$ or $\{6, 7\}$, when $m = 6$ or 7, respectively, X the preimage of $X^*/Z(L^*)$ in G, and $Y := X^\infty$. Then $Y^* \cong L_2(4)$ with $C_Y(V) = O_2(Y)$, $J(R)^* \in Syl_2(Y^*)$, and Y does not centralize the subspace Z_U of U of codimension 2. Thus once again the hypotheses of C.1.21.4 are satisfied, so $Y \in \mathcal{Y}(G, R)$ and $S^* = J(R)^*$. As $N_{G^*}(S^*)$ is maximal in G^*, $G^* = \langle N_{G^*}(S^*), Y^*\rangle$, completing the proof of the lemma. □

REMARK C.1.25. The following lemma shows that under the SQTK-hypothesis, χ_0-blocks are the obstruction to pushing up minimal parabolics in the sense of Definition B.6.1.

PROPOSITION C.1.26. *Assume that H is an SQTK-group with $T \in Syl_2(H)$, $F^*(H) = O_2(H)$, and H is a minimal parabolic over T. Then one of the following holds:*

(1) $C_1(T) \leq Z(H)$.

(2) $C_2(T) \trianglelefteq H$.

(3) $O^2(H)$ is the central product of χ_0-blocks K_i of H permuted transitively by T, with $K_i = [K_i, J(T)]$.

PROOF. Let $Z := \Omega_1(Z(T))$, $V := \langle Z^H \rangle$, $H^* := H/C_H(V)$, $E := \Omega_1(Z(J(T)))$, and $S := \text{Baum}(T)$. Adopt the notation of B.6.8. By Thompson Factorization B.2.15, either $J(T)^* \neq 1$ or $H = C_H(Z)N_H(J(T))$. Assume the second case holds. Then as $T \leq C_H(Z) \cap N_H(J(T))$ and H is a minimal parabolic, either $H = C_H(Z)$ or $H = N_H(J(T))$. In the first case (1) holds since $C_1 := C_1(T) \leq Z$, so we may assume $Z \not\leq Z(H)$, and hence $J(T) \trianglelefteq H$. Then by B.6.8.6d, $S \trianglelefteq H$, so (2) holds as $C_2 := C_2(T)$ char S.

Thus we may assume that $J(T)^* \neq 1$, so V is an FF-module for H^* and hence H^* and its action on V are described in B.6.9.

Suppose first that case (i) of B.6.9.5 holds. Then from the description in that result, there is $L^* \trianglelefteq \trianglelefteq H^*$ with $L^* \cong L_2(2^n)$ or $\mathbf{Z}_3$, $[V, L]/C_{[V,L]}(L)$ the natural module for L^*, where L is the preimage of L^*, and $L^* = [L^*, J(T)^*]$. Thus H satisfies the hypotheses of Baumann's Lemma B.6.10, so by that lemma, S normalizes L^*, and S is Sylow in $H_0 := \langle S^L \rangle$. Thus $H_0^* = L^*S^*$. Then $H_0 \in \mathcal{G}(S)$, so we may apply The Glauberman-Niles/Campbell Theorem C.1.18 with H_0, T, S in the roles of "H, S, Baum(S)", to conclude that $C_1 \leq Z(H_0)$, or $C_2 \trianglelefteq H_0$, or $O^2(H_0) =: L_0$ is an $L_2(2^n)$-block or A_3-block. In the first case $H = \langle H_0, T \rangle$ centralizes C_1, and in the second $C_2 \trianglelefteq \langle H_0, T \rangle = H$. In the third case L_0 is a χ_0-block, with $L_0 = [L_0, J(T)]$ since $L^* = [L^*, J(T)^*]$. Thus one of the conclusions of the lemma holds in each case.

Thus we may assume that (5ii) of B.6.9 holds, so $K^* = K_1^* \times \cdots \times K_s^*$ is the direct product of alternating groups $K_i^* \cong A_n$ (with $n := 2^k + 1 \geq 5$) permuted transitively by T, $[V, K] = V_1 \oplus \cdots \oplus V_s$ with $V_i := [V, K_i]$ the natural module for K_i^*, and $K_i^* = [K_i^*, J(T)^*]$. Then $1 \neq C_{V_i}(J(T)) \leq E$, so S also acts on each K_i. As H is an SQTK-group, $n = 5$, and there is $L_1 \in \mathcal{L}(K_1, N_T(K_1))$ with $L_1/O_2(L_1)$ quasisimple and $L_1^* = K_1^*$. Then applying C.1.23 with $H_0 := L_1N_T(K_1)$, L_1, $N_T(K_1)$, T in the roles of "G, L, R, T", we conclude that $C_1 \leq Z(H_0)$, or $C_2 \trianglelefteq H_0$, or $O^2(H_0) = L_1 = [L_1, J(T)]$ is a block. Finally we finish the proof as in the previous paragraph. □

LEMMA C.1.27. *Assume $F^*(G) = O_2(G)$, $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, G is a $\mathcal{K}$-group, and H is a minimal parabolic of G over T such that $H = T\langle K^T \rangle$ for some block K of H with $K = [K, J(T)]$. Then there is a block L of G containing K, and $U(L) \leq \langle Z^G \rangle =: V$. Further either $K = L$, or $L/C_L(V)$ and $K/C_K(V)$ are alternating groups of odd degree.*

PROOF. Set $G^* := G/C_G(V)$; as $V \in \mathcal{R}_2(G)$ by B.2.14, $O_2(G^*) = 1$. As $F^*(G) = O_2(G)$ and $T \leq H$, $F^*(H) = O_2(H)$ by A.1.6.

Let $K_0 := \langle K^T \rangle$. As we are assuming K is a block, C.1.9 says K_0 is the central product of the members of K^T. As $K_0 = \langle K^T \rangle$ and $K = [K, J(T)]$, $K_0 \leq J(H)$. Let $U_0 := \langle U(K)^T \rangle$; as H is a minimal parabolic and $K_0 \leq J(H)$, K and its action on $U(K)$ are described in B.6.9. In particular $K/O_2(K) \cong L_2(2^n)$ or A_{2^k+1} and $C_{U(K)}(N_T(K)) \not\leq C_G(K)$, so $Z \cap U_0 \not\leq C_{U_0}(K)$, and hence $U(K) \leq \langle Z^G \rangle = V$. Therefore as $K/U(K)$ is quasisimple or of order 3 by C.1.8, $K^* \cong K/C_K(V)$ is quasisimple or of order 3. Furthermore as K is a block, $[V, K] \leq [O_2(KT), K] = U(K)$, so that $[V, K] = U(K)$. In particular either K^* is a component of H^* or $K^* \cong \mathbf{Z}_3$. Thus the hypotheses of B.1.13 are satisfied with G^*, V, K^* in the roles of "G, V, K". Thus as K^* is $L_2(2^n)$ or A_{2^k+1}, B.1.13 says that either $K^* \leq Z(F(G^*))$ (in which case we set $L^* := K^*$), or there is a component L^* of G^* containing K^* such that either $K^* = L^*$, or K^* and L^* are alternating groups. In the latter case K^* is alternating of odd degree and subnormal in $\langle K^*, T^* \rangle$, so L^* is also alternating of odd degree.

Let L_0 be the preimage of L^* in G, and set $L := \langle K^{L_0} \rangle$. As $U(K) \leq V$, K centralizes $C_T(V)/V$, so that L centralizes $O_2(G)/V$. Then $[C_G(V), L] \leq C_G(V) \cap C_G(O_2(G)/V) \leq O_2(G)$ as $F^*(G) = O_2(G)$. Thus $C_G(V)$ acts on $O^2(LO_2(G)) = L$, and hence $L \trianglelefteq LC_G(V) \trianglelefteq \trianglelefteq G$, so $L \trianglelefteq \trianglelefteq G$. As the unique noncentral 2-chief factor for K is in V, the same holds for L, so $U(L) = [O_2(L), L] \leq V \leq Z(O_2(G))$, and hence $U(L) \leq Z(O_2(L))$. Then L is a block of G. □

THEOREM C.1.28. *Assume $F^*(G) = O_2(G)$ and G is an SQTK-group. Let $T \in Syl_2(G)$ and $M := \langle C_G(C_1(T)), N_G(C_2(T)) \rangle$. Then either*

(1) $G = M$, or

(2) $G = ML_1 \cdots L_s$, where L_i is a χ-block of G with $L_i = [L_i, J(T)]$, and $s \leq 2$.

PROOF. This is a consequence of Theorem 3 in [**Asc81b**] and the Glauberman-Niles/Campbell Theorem C.1.18, but is it not difficult to give a proof here using our $\mathcal{K}$-group hypothesis. Note that $s \leq 2$ in conclusion (2) because $m_3(G) \leq 2$ since G is an SQTK-group.

Let $C_i := C_i(T)$. First $N_G(T) \leq N_G(C_i) \leq M$, so by a Frattini Argument $G = O^{2'}(G)M$, and we may assume that $G = O^{2'}(G) > T$. Thus by B.6.5, $G = \langle \mathcal{H} \rangle$, where $\mathcal{H}$ is the set of minimal parabolics over T. If $H \leq M$ for all $H \in \mathcal{H}$, then (1) holds, so we may assume there is $H \in \mathcal{H}$ with $H \not\leq M$. Therefore by C.1.26, $H = TK_1 \cdots K_r$, where $K_i = [K_i, J(T)]$ is a χ_0-block, $\{K_1, \ldots, K_r\}$ is permuted transitively by T, and $K_i \not\leq M$.

Let $V := \langle Z^G \rangle$ and $G^* := G/C_G(V)$. Let $U_1 := U(K_1)$; as G is an SQTK-group, G is a $\mathcal{K}$-group, so as $K_i = [K_i, J(T)]$, C.1.27 says that there is a block L of G with $K := K_1 \leq L$ and $U_1 \leq U(L) \leq V$, such that either $L = K$, or $K/O_2(K)$ and $L/O_2(L)$ are alternating groups of odd degree. In the former case as K is a χ_0-block, L is a χ-block; we claim L is a χ-block in the latter case as well: As $K = [K, J(T)]$, $\tilde{U}(L)$ is an FF-module for $L^* \cong A_m$. Therefore as G is an SQTK-group, we may apply Theorem B.4.2 to conclude that $m = 5$ or 7, and that the block L is a χ-block or an exceptional A_7-block. However in the latter case as G is an SQTK-group, L is normalized by T by A.3.8.3. Thus we may apply C.1.23.2 with T in the roles of "T" and "R" to conclude $L \leq M$, contradicting $K \not\leq M$. This establishes the claim that L is a χ-block of G.

Let $X := \langle L^G \rangle$. Then $S := \mathrm{Baum}(T)$ acts on each member of L^G by C.1.16.2, and $Out(L/O_2(L))$ is 2-closed, so $SXC_G(X) \trianglelefteq G$. Now by B.2.3.4, $S = \mathrm{Baum}(T_0)$ for $T_0 \in Syl_2(SXC_G(X))$. Further $N_G(S) \leq N_G(C_2) \leq M$ as C_2 char S. Hence by a Frattini Argument, $G = XC_G(X)M$. Now by induction on the order of G, $C_G(X)M$ satisfies one of the conclusions of the theorem, so G satisfies conclusion (2). □

This subsection culminates in the Aschbacher Local $C(G,T)$-theorem, stated here in the case of SQTK-groups. It applies to the optimal case for pushing up, where our 2-subgroup R is the *full* Sylow group T of G. The $C(G,T)$-theorem shows that in this optimal case, the only obstruction to the conclusion $G = C(G,T)$ is the existence of χ_0-blocks of G.

Theorem C.1.29 ($C(G,T)$-Theorem). *Assume $F^*(G) = O_2(G)$, $T \in Syl_2(G)$, and G is an SQTK-group. In addition assume $C(G,T) \leq M \leq G$. Then $G = ML_1 \cdots L_r$, where $0 \leq r \leq 2$, and L_i is a χ_0-block with $(L_i \cap M)/O_2(L_i)$ a Borel subgroup of $L_i/O_2(L_i)$ and $L_i = [L_i, J(T)]$ for each $0 < i \leq r$.*

Proof. The Local $C(G,T)$-theorem for the general group G with $F^*(G) = O_2(G)$ is Theorem 1 in [**Asc81a**], but we can derive the theorem for SQTK-groups as a consequence of Theorem C.1.28. We may assume that $M < G$, and hence $M_0 := \langle N_G(C_1), N_G(C_2) \rangle \leq C(G,T) \leq M < G$. Therefore by Theorem C.1.28, $G = M_0L_1 \cdots L_r = ML_1 \ldots L_r$, where the L_i are χ-blocks of G not contained in M, $0 < r \leq 2$, and $L_i = [L_i, J(T)]$. Observe if L_i is a χ_0-block, then as G is an SQTK-group, L_i is of type $L_2(2^n)$, A_3 or A_5. Thus in each case, L_i is a minimal parabolic in the sense of Definition B.6.1, so as $L_i \not\leq M$, it follows that $L_i \cap M$ is a Borel subgroup of L_i.

Thus we may assume that some L_i, say L_1, is a χ-block but not a χ_0-block, and it remains to derive a contradiction. As G is an SQTK-group, Theorem C (A.2.3) forces L_1 to be an A_7-block and $L_1 \trianglelefteq G$ by A.3.8.3. By induction on the order of G, we may assume that $G = L_1T$. Represent G on $\Omega := \{1, \ldots, 7\}$ as in section B.3 so that T has orbits $\{1,2,3,4\}$, $\{5,6\}$, $\{7\}$. As G_7 is an A_6-block, $G_7 \leq M$ by induction on the order of G, so $M = G_7$ by maximality of G_7 in G. Let Y_0 be the preimage in G of $\langle (5,6,7) \rangle$, and $Y := O^2(Y_0)$; then $YT \not\leq M$ and $Y \cong A_4$, with $YT = (Y \times D)\langle t \rangle$ where $D := C_T(Y)$, t induces a transposition on Ω, and $D/C_D(L_1)$ is the extension of $W := C_{U(L_1)}(Y) \cong E_{16}$ by a Sylow 2-group D_8 of $G_{5,6,7}$ with W a permutation module for $G_{5,6,7}/O_2(G) \cong S_4$. In particular $D/C_D(L_1)$ has class 4; while $T/D \cong D_8$ has class 2. Thus if T_i is the i-th term in the descending central series for T, then $1 \neq T_3 \leq D$, and hence $Y \leq C_G(T_3) \leq C(G,T) \leq M$, contrary to our observation that $M = G_7$. □

C.1.4. Pushing up rank-2 groups. The $C(G,T)$-Theorem C.1.29 pins down the obstructions to the (CPU) version of pushing up when $R \in Syl_2(G)$. However we also need to push up in situations where R is *not* Sylow in G, as well as under the weaker (PU) hypothesis. In the next most accessible case, T is in just *two* maximal subgroups G_1 and G_2, and (PU) holds for $R := O_2(G_i)$ or T. An important subcase was treated by Meierfrankenfeld and Stellmacher in [**MS93**], which considers the situation where $F^*(G/O_2(G))$ is (essentially) a rank-2 group of Lie type.

There are some further classes of blocks that arise in this wider context:

DEFINITION C.1.30. A block L is an *$Sp_4(4)$-block* or an *$SL_3(2^n)$-block* if $L/O_2(L) \cong Sp_4(4)$ or $SL_3(2^n)$, respectively, and $\tilde{U}(L)$ is a natural module. Define L to be a *$G_2(2)$-block* if $L/O_2(L) \cong G_2(2)'$ and $\tilde{U}(L)$ is the natural module. For use in other contexts, we define a block L to be an *$L_n(2)$-block* if $L/O_2(L) \cong L_n(2)$ and $\tilde{U}(L)$ is a natural module; and an *$\Omega^\epsilon_m(2^n)$-block* if $L/O_2(L) \cong \Omega^\epsilon_m(2^n)$ and $\tilde{U}(L)$ is a natural module.

In order to quote the main result of [**MS93**], we first adapt their hypotheses to our situation:

DEFINITION C.1.31 (MS-pairs). Define an *MS-pair* to be a pair (G, R) where

(MS1) G is an SQTK-group with $F^*(G) = O_2(G)$ and $G = LR$ with $L \in \mathcal{C}(G)$, and $L/O_{2,Z}(L)$ is the commutator subgroup of a group of Lie type in characteristic 2 and Lie rank 2.

(MS2) Either

(a) $R = O_2(H)$ for some proper subgroup H of G of odd index, or

(b) $G/O_2(G) \cong G_2(2)$ and $O_2(G) \leq R$ with $R/O_2(G)$ a normal E_8-subgroup of some maximal parabolic subgroup of $G/O_2(G)$.

(MS3) No nontrivial characteristic subgroup of R is normal in G.

Observe that in (MS1) we allow $L/O_{2,Z}(L)$ to be one of the commutator groups $A_6 \cong Sp_4(2)'$, $U_3(3) \cong G_2(2)'$, and the Tits group ${}^2F_4(2)'$.

Define G to be an *MS-group* if (G, R) is an MS-pair for some $R \leq G$.

We can now state the theorem of Meierfrankenfeld and Stellmacher classifying MS-groups:

THEOREM C.1.32 (Meierfrankenfeld-Stellmacher rank-2 pushing up). *If (G, R) is an MS-pair then one of the following holds:*

(1) L is a $G_2(2)$-block.
(2) L is an A_6-block.
(3) L is an $Sp_4(4)$-block.
(4) L is a $\hat{A}_6$-block.
(5) $L/O_2(L) \cong SL_3(2^n)$.

REMARK C.1.33. Much more precise information about the groups arising in case (5) of Theorem C.1.32 is given in several lemmas below, particularly C.1.34. Notice that when n is even, only the full universal group $SL_3(2^n)$ arises, not just the simple quotient $L_3(2^n)$; this distinction is important in the proof and in later applications.

PROOF. This is a special case of the Main Theorem of [**MS93**]. (That result in turn assumes Theorem A of [**DGS85**]).

We begin with a prelimary observation: Set $E := O_{2,Z}(L)O_2(G)$ and $G^* := G/E$. We claim that G^* is trivial on the Dynkin diagram of L^*. If not, then L^* is $L_3(2^n)$ or $Sp_4(2^n)'$, so by (MS2), $R^* = O_2(H^*)$ for some proper subgroup H^* of G^* of odd index. Thus as $G = LR$ by (MS1), $R \in Syl_2(G)$ (cf. C.2.2.4) and $N_{L^*}(R^* \cap L^*)$ is the unique maximal subgroup of G^* containing R^*, so that $G \in \hat{\mathcal{U}}(R)$ in the sense of definition B.6.2. Further by (MS3), neither $\Omega_1(Z(R))$ nor $J(R)$ is normal in G; so as $G \in \hat{\mathcal{U}}(R)$, G has an FF-module by Thompson

Factorization B.2.15. Now Theorem B.5.1.1 supplies a contradiction since R is nontrivial on the Dynkin diagram of L^*, completing the proof of the claim.

We next check that the hypotheses of the Main Theorem of [**MS93**] hold, with R, G in the roles of "T, M". In our setup, E is the normal subgroup "E" of that theorem, and $F^*(G) = O_2(G)$ by (MS1), which makes their hypothesis (A) immediate. By (MS1), $G = LR$, giving their hypothesis (B). By (MS2), $O_2(G) \leq R$, and then (MS3) affords their hypothesis (P). Finally we verify that G^* possesses a weak BN-pair of rank 2 (cf. Definition F.1.7): Let P_i^*, $i = 1, 2$ be the maximal parabolics of L^* and $L_i^* := O^{2'}(P_i^*)$. As G^* is trivial on the Dynkin diagram of L^*, R^* acts on L_i^*, and we set $B^* := N_{L^*}(R^* \cap L^*)R^*$ and $\alpha := (L_1^* B^*, B^*, L_2^* B^*)$. It is straightforward to check that Hypothesis F.1.1 is satisfied by L_1^*, L_2^*, and R^*, so α is a weak BN-pair by F.1.9. This completes our verification of the hypotheses of [**MS93**].

Next, our hypotheses also exclude some of their conclusions: As $F^*(G) = O_2(G)$, conclusion (1) of their result does not hold. Our assumption that G is an SQTK-group excludes the groups $L/O_2(L) \cong U_4(2^n) \cong \Omega_6^-(2^n)$, avoiding their conclusion (9). As $p = 2$, conclusion (12) does not hold. As $L \in \mathcal{C}(G)$ and $G = LR$, $L \trianglelefteq G$, which eliminates the two cases appearing in their conclusion (7) in which $F^*(G/O_2(G))$ is the product of two components. The remaining case in their conclusion (7) where $V := [O_2(E), L] \cong E_{2^6}$ and $G/C_G(V) \cong 2^8 L_3(2)$ does not occur: For since the Schur multiplier of $L_3(2)$ is a 2-group by I.1.3, $O_{2,Z}(L) = O_2(L)$, so that $E = O_{2,Z}(L)O_2(G) = O_2(G)$, and hence $G/O_2(G) \cong L_3(2)$. Thus $V = [O_2(E), L]$ contains a Steinberg-module section 2^8 from $2^8 L_3(2)$: observe by page 932 of [**MS93**] that (7) arises via their (3.6) and (1.8), where the latter shows that this module is identified using (1.4)(iii) on page 842. This section of rank 8 contradicts $V \cong E_{2^6}$. Their conclusions (4), (5), (8), (11), and (13) involve $SL_3(2^n)$, and are summarized in conclusion (5) of our Theorem; as we remarked, those cases are described in more detail in lemma C.1.34 below as our conclusions (1), (2), (5), (3), and (4), respectively. Finally their conclusions (2), (3), (6), and (10) become our conclusions (2), (3), (4), and (1), respectively. □

The examples involving $SL_3(2^n)$ which appear in C.1.32.5 need not have blocks. The next result describes in further detail the possibilities that *can* occur in this case.

THEOREM C.1.34 (Meierfrankenfeld-Stellmacher). *Assume (G, R) is an MS-pair such that $L/O_2(L) \cong SL_3(2^n)$, and let $Q := [O_2(G), L]$, $R \leq S \in Syl_2(G)$, and $Z_S := \Omega_1(Z(S))$. Then one of the following holds:*

(1) L is an $SL_3(2^n)$-block and Q is the natural module for $L/O_2(L)$.

(2) Q is the direct sum of two isomorphic natural modules for $L/O_2(L)$ and $R \neq O_2(C_G(Z_S))$.

(3) $Z(Q)$ is a natural module for $L/O_2(L)$, $Q/Z(Q)$ is the direct sum of two copies of the dual of $Z(Q)$, and $R \neq O_2(C_G(Z_S))$.

(4) $E_{2^{2n}} \cong C_Q(L) \leq Z(G)$, $Z(Q) = C_Q(L) \oplus U$ where $U = [Z(Q), L]$ is a natural module for $L/O_2(L)$, $\Phi(Q)/Z(Q)$ is dual to U, $C_Q(\Phi(Q))/\Phi(Q)$ is the sum of two copies of the dual of U, $Q/C_Q(\Phi(Q))$ is the sum of two copies of U, and $R \neq O_2(C_G(Z_S))$.

(5) $L/O_2(L) \cong L_3(2)$, $C_Q(L) \cong \mathbf{Z}_2$, $Q/C_Q(L)$ is a natural module for $L/O_2(L)$, $Z_S \leq Z(G)$, and $R = O_2(G_1)$, where G_1 is the subgroup of G of index 7 containing S and fixing a point of $Q/C_Q(L)$.

PROOF. This is again a special case of the Main Theorem of [**MS93**]. Earlier we gave the correspondence between their conclusions and ours. The statement that $R \neq O_2(C_G(Z_S))$ in our cases (3) and (4) comes from the final remark about their cases (11)–(13) in their Main Theorem. That same statement in our case (2) can be obtained as follows: Setting $\bar{G} := G/C_G(Q)$, we see that $SL_3(2^n) \cong \bar{L} = F^*(\bar{G})$, so that $O_2(\bar{G}) = 1$ and $Q \in \mathcal{R}_2(G)$. Recall that $O_2(G) \leq R$ using (MS2); further $\mathrm{Baum}(R) \neq \mathrm{Baum}(O_2(G))$ in view of (MS3). Then by B.2.10, $\overline{J(R)} \neq 1$, so that $\bar{R}$ contains an FF*-offender on Q. As Q is a sum of two isomorphic natural modules for $L/O_2(L)$, case (ii) of B.5.1.1 holds; but that result shows that the unique conjugacy class of FF*-offenders is given by the transvections with a common axis on each summand of Q—rather than those with a common center as required if $R = O_2(C_G(Z_S))$. The statement in (4) that $C_Q(L) \leq Z(G)$ can be recovered from their Lemma 4.5, the lemma in which this conclusion arises: Namely by 4.5.iii there, $C_Q(L) = H_\alpha$ in the notation of that lemma. Then in their Hypothesis 4.0 we find $H_\alpha = [V_\alpha^\circ, Q_\alpha]$, while by their 4.4, $[V_\alpha, Q_\alpha] \leq Z(G)$. Therefore $C_Q(L) = H_\alpha = [V_\alpha^\circ, Q_\alpha] \leq [V_\alpha, Q_\alpha] \leq Z(G)$. □

In applications involving cases (2) and (3) of theorem C.1.34, it turns out we will require still more precise information under the additional hypothesis that $C_G(L) = 1$. A lengthier argument is required in case (3):

LEMMA C.1.35. *Let G be an MS-group with $L/O_2(L) \cong SL_3(2^n)$, $S \in Syl_2(G)$, $Q := O_2(L)$, $U := Z(Q)$, and $G^* := G/U$. Assume further that $C_G(L) = 1$, and L has 3 noncentral 2-chief factors, so that we are in the case (3) of C.1.34. Then*

(1) Q is special, $Q^* = W_1^* \oplus W_2^*$, U is a natural L/Q-module, W_1^* is the dual of U as an L/Q-module, and there exists an L/Q-isomorphism $\alpha : W_1^* \to W_2^*$.

(2) The preimage W_i of W_i^* in Q is isomorphic to $E_{2^{6n}}$, and for $w \in W_1 - U$, $|C_Q(w) : W_1| = 2^n$ with $C_{W_2}(w)^* = Z_w^*\alpha$, where Z_w^* is the 1-dimensional $\mathbf{F}_{2^n}$-subspace of W_1^* containing w^*.

(3) $O_2(G) = Q$.

(4) $m_2(G) = 6n$.

(5) $Z_2(S) \leq U$.

PROOF. As L has 3 noncentral 2-chief factors, case (3) of C.1.34 holds, establishing conclusion (1) which is just a restatement of that case. We are considering modules over the field $F := \mathbf{F}_{2^n}$ of definition of $L/O_2(L)$, and dimensions discussed below are over F.

As L is transitive on $W_1^{*\#}$, either $W_1 \cong E_{2^{6n}}$ or $U = \Omega_1(W_1)$. In the latter case each $w \in W_1 - U$ has order 4, with $\Phi(\langle w, U\rangle) = \langle w^2\rangle$ since $U = Z(Q)$; then $C_L(w^*) \leq C_L(w^2)$, contradicting W_1^* dual to U. Thus $W_1 \cong E_{2^{6n}}$.

Next as W_1^* is a natural module for L/Q, $End_{\mathbf{F}_2L}(W_1^*) = F$, so for each $a \in F^\#$, the map

$$\alpha_a := \alpha \circ a : W_1^* \to W_2^*$$

is an $F[L/O_2(L)]$-isomorphism, and then since Q/U is a sum of two isomorphic modules,

$$W_a^* = \{w^*(w^*\alpha_a) : w^* \in W_1^*\}$$

is another irreducible for L on Q^*. (cf. 27.14 in [**Asc86a**])

Then by symmetry between W_1 and the preimage W_a, $W_a \cong E_{2^{6n}}$. Now let Z_w^* be the 1-dimensional F-subspace of W_1^* containing $w^* \in W_1^{*\#}$. We have just shown that wu is an involution for $u^* := w^*\alpha_a$, so $Z_w Z_{w,2}$ is elementary abelian, where $Z_{w,2}^* := Z_w^*\alpha$. Hence $Q_w \le C_Q(w)$, where $Q_w^* := W_1^* Z_{w,2}^*$. On the other hand as $U = Z(Q)$ by definition, $C_Q(w) < Q$; so as the parabolic $C_L(w^*)$ is irreducible on the 2-dimensional quotient $Q/Q_w \cong W_2/Z_{w,2}$, $Q_w = C_Q(w)$. That is, (2) holds. In particular, notice that $C_Q(W_1) = W_1$.

Next as $C_G(L) = 1$ by hypothesis, $G \le Aut(L)$. Let $H_1 := N_G(W_1)$ and $\bar{H}_1 := H_1/C_G(W_1)$; then $H_1 = LN_R(W_1)$ as $G = LR$ by (MS1). Now $\bar{H}_1$ is contained in the parabolic P stabilizing U in the linear group $GL(W_1)$, and $P \cong (GL_3(F) \times GL_3(F))/F^9$, with $O_2(P)$ centralizing U and W_1/U. We claim $C_{O_2(P)}(\bar{L}) = 1$: For if $1 \ne x \in C_{O_2(P)}(\bar{L})$, then as L is irreducible on U and W_1/U, $U = C_{W_1}(x)$. But now $C_{\bar{L}}(w^*)$ acts on $[U + w, x] = [w, x]$, impossible as W_1^* is dual to U, so $C_{\bar{L}}(w^*)$ fixes no vector of U. So indeed $C_{O_2(P)}(\bar{L}) = 1$. We claim in fact that $N_{O_2(P)}(\bar{L}) = O_2(\bar{L})$. If $n > 1$, then by I.1.6.4 $H^1(\bar{L}, O_2(\bar{L})) = 0$, so as $C_{O_2(P)}(\bar{L}) = 1$, the claim holds in this case. So suppose $n = 1$. Then in the language of the weight theory for irreducible representations of Lie type groups (cf. section 2.8 of [**GLS98**], briefly summarized in section H.6), where the basic natural modules are denoted by $M(\lambda_i)$, we have $U \cong M(\lambda_1)$ and $W_1/U \cong M(\lambda_2)$ as L/Q-modules. Therefore L/Q is diagonally embedded in $P/O_2(P)$, so $O_2(P)$ is a tensor product of the form $M(\lambda_1) \otimes M(\lambda_1)$ or $M(\lambda_1) \otimes M(\lambda_2)$ as an $\bar{L}/O_2(\bar{L})$-module. As $O_2(\bar{L})$ is a natural L-submodule of $O_2(P)$, it must be the former, as the latter has only adjoint and trivial sections. But then all L-sections of $O_2(P)$ are of dimension 3 and hence nontrivial, so as $[N_{O_2(P)}(L), L] \le O_2(\bar{L})$, again $N_{O_2(P)}(\bar{L}) = O_2(\bar{L})$, completing the proof of the claim. Next as L is irreducible on U and W_1^*, $O_2(N_P(\bar{L})) \le O_2(P)$, so $O_2(\bar{H}_1) = O_2(\bar{L}) = \bar{Q}$. Therefore $O_2(H_1) = C_R(W_1)Q$.

Now our Sylow 2-group S acts on some L-irreducible of $Q^* := Q/U$ which we may take to be W_1^*, so as $G = LR$, $G = H_1$. From the previous paragraph $O_2(G) = QV$, where $V := C_{O_2(G)}(W_1)$. Now $[V, L] \le V \cap L = W_1$ since we saw earlier that $C_Q(W_1) = W_1$. Further $W_1 = [W_1, X]$ for X a Hall $2'$-subgroup of the preimage of a suitable torus of $L/O_2(L)$, so $V = W_1 \times C_V(X)$. Then $\Phi(C_V(X)) = \Phi(V) \trianglelefteq G$, so $\Phi(V) = 1$ since $L = [L, X]$ and $C_G(L) = 1$ by hypothesis. Thus V is elementary, and we may write $V = W_1 \oplus C_V(X)$.

Assume temporarily that we have established (3); that is, assume that $Q = O_2(G)$. Then $Z(S)$ is contained in a 1-dimensional F-subspace Z of U, and using (2) we see for w^* fixed by S^* that

$$Z_2(S)^* \le Z(S^*) = Z_w^*(Z_w^*\alpha),$$

so if (5) fails then replacing W_1 by a suitable L-invariant subgroup, we may take $w \in Z_2(S)$. Now by (2), $Q_w = C_Q(w)$ has corank $2n$ in Q, so $[Q, w]$ is not contained in Z of rank n, contradicting $w \in Z_2(S)$. Thus (3) implies (5), so it only remains to establish (3) and (4).

If $J(S) \le O_2(G)$, then since $O_2(G) \le R \le S$, $J(S) = J(R) = J(O_2(G))$ normal in G by B.2.3.3, contrary to (MS3). Therefore there is $A \in \mathcal{A}(S) - \mathcal{A}(O_2(G))$. Let $\hat{G} := G/O_2(G)$. Now $C_V(A) = V \cap A$ since $A \in \mathcal{A}(S)$, so

$$m(V/(A \cap V)) = m(V/C_V(A)) \ge m(W_1/C_{W_1}(A))$$

$$\geq m(U/C_U(A)) + m(W_1^*/C_{W_1^*}(A)) =: m. \qquad (*)$$

We claim that

$$m \geq 2n, \text{ and } m \geq 3n \text{ if } m(\hat{A}) > n. \qquad (+)$$

For as U is a natural module, $m(U/C_U(\hat{a})) \geq n$ for each involution $\hat{a} \in \hat{G}$ and similarly $m(W_1^*/C_{W_1^*}(\hat{a})) \geq n$. On the other hand if $m(\hat{A}) > n$, then as W_1^* is dual to U, $m(X/C_X(A)) \geq 2n$ for $X := U$ or W_1^*. This establishes (+).

As $m(A) \geq m(V)$ and $m(\hat{A}) \leq 2n$, we conclude from (*) and (+) that

$$N := m((A \cap O_2(G))/(A \cap V)) \geq m(V/(A \cap V)) - m(\hat{A}) \geq m - m(\hat{A}) \geq n. \quad (**)$$

In case $N = n$, all inequalities in (*) and (**) are equalities, so in particular $m(A) = m(V)$ and $V = W_1 C_V(A)$.

But in any case $N \geq n > 0$, so $B := A \cap O_2(G) \not\leq V$. Let $A_2 := BV \cap W_2$ and $A_1 := A \cap W_1$. Then $A_2U/U = A_2^* \cong A_2V/V$, so that $A_2V = BV \leq C_{QV}(A_1)$, and hence $A_2^* \leq Z_w^*$ for each $1 \neq w^* \in A_1^*$ by (2). In particular as $A_2^* \neq 1$, $m(A_1^*) \leq n$, and if $A_1^* \neq 1$, then $m(A_2^*) \leq n$; so $m(A_2V/V) \leq n$. But also as $BV = A_2V$, $N = m(BV/V) = m(A_2V/V)$ and hence either $N \leq n$ or $A_1^* = 1$. When $N \leq n$, we saw that $N = n$, $m(A) = m(V)$, and $V = W_1 C_V(A)$.

Notice also that if $V = W_1$, then $O_2(G) = VQ = W_1Q = Q$, so (3) holds. So if in addition $m(A) = m(V)$ for all $A \in \mathcal{A}(S) - \mathcal{A}(O_2(G))$, then $m_2(S) = m(A) = m(V) = m(W_1) = 6n$, establishing (4), and completing the proof of the lemma. Thus it suffices to show that $V = W_1$ and $m(A) = m(V)$.

We first show that $m(A) = m(V)$ and that $V = W_1 C_V(A)$. If not then by an earlier remark, $A_1^* = 1$. Thus $A_1 = A \cap W_1 \leq U$. As $\hat{A}$ is non-trivial, $A \cap W_1 = A \cap U = C_U(A)$ and B^* have rank at most $2n$. Therefore

$$m(V/V \cap A) \geq m(W_1/W_1 \cap A) \geq m(W_1^*) + m(U/C_U(A)) \geq 4n$$
$$\text{and } N = m(B^*) = m(A_2^*) \leq 2n. \qquad (!)$$

Combining the inequalities in (!) with those in (**), we get:

$$2n \geq N \geq m(V/(V \cap A)) - m(\hat{A}) \geq 4n - 2n = 2n. \qquad (!!)$$

Now all inequalities in (!) and (!!) are equalities. In particular from (!!), $m(\hat{A}) = 2n$, while from (!), $m(C_U(A)) = m(A_2^*) = 2n$, which is impossible as the subgroup $\hat{A}$ of rank $2n$ in $\hat{G}$ cannot centralize an $\mathbf{F}_2$-space of dimension $2n$ in both U and the dual module W_2^*.

We have shown $m(A) = m(V)$, so it remains to show $V = W_1$. Further we have shown $V = W_1 C_V(A)$, so $V = W_1(V \cap A)$, since $C_V(A) \leq A$ as $A \in \mathcal{A}(S)$. Recall $O_2(G) = VQ$; then $Z(O_2(G)) \leq C_{O_2(G)}(Q) \leq C_{O_2(G)}(W_1) = V$, so as V is elementary, $Z(O_2(G)) = C_V(Q)$. First suppose $V \not\leq Z(O_2(G))W_1 = C_V(Q)W_1$. As $[L, V] \leq W_1$, L acts on $V_1 := \langle v_1, W_1 \rangle$ for each $v_1 \in V - C_V(Q)W_1$, so as L is irreducible on $O_2(G)/V$ and $C_Q(v_1) < Q$, $C_{O_2(G)}(v_1) = V$. As $V = W_1(V \cap A)$, we may replace v_1 by $v_1' := w_1 v_1 \in V \cap A$ for suitable $w_1 \in W_1$, and we still have $C_{O_2(G)}(v_1') = V$, since $v_1' = w_1 v_1 \notin C_V(Q)W_1$. But we saw earlier that $A \cap O_2(G) \not\leq V$, and $A \cap O_2(G)$ centralizes $v_1' \in V \cap A$, contradicting $C_{O_2(G)}(v_1') = V$.

Therefore $V \leq Z(O_2(G))W_1 = C_V(Q)W_1$, so $V = EW_1$, for $E := C_V(O_2(G))$. Assume that $V > W_1$, so that $E > U$. Now $E \cap L = U$, and as $C_G(L) = 1$ by hypothesis, E is indecomposable under L, so $m(E/U) \leq \dim(H^1(L, U'))$ for U' the dual of U. Recall from I.1.6.4 that this dimension is 0 for $n > 1$ and 1 for $n = 1$. Therefore $n = 1$, $m(E) = 4$, $m(V) = m(W_1) + 1 = 7$, and E is the unique indecomposable module with submodule U described in B.4.8.3. By that result, we get $C_E(a) = C_U(a)$ of rank 2 for any $a \in A - O_2(G)$. As $O_2(G)$ centralizes E, $A \cap O_2(G) = C_A(E)$; and as $A \in \mathcal{A}(S)$, $m(A) \geq m(EC_A(E))$. This forces $m(\hat{A}) = 2$, with A centralizing a hyperplane U_0 of U. But then $\hat{A}$ centralizes just a 1-space in each of the dual modules W_i^* in Q^*, so we conclude that

$$m(A) \leq m(\hat{A}) + m(C_E(A)) + m(C_{Q^*}(A)) \leq 6 < 7 = m(V),$$

contradicting $A \in \mathcal{A}(S)$. Thus the proof of C.1.35 is at last complete. □

LEMMA C.1.36. *Let G be an MS-group with $L/O_2(L) \cong SL_3(2^n)$, $S \in Syl_2(G)$ and $Q := O_2(L)$. Assume further that $C_G(L) = 1$ and L has two noncentral 2-chief factors, so that we are in the case (2) of C.1.34. Then*

(1) $Q = C_G(Q)$.
(2) $\mathcal{A}(S) = \{Q, A\}$ with $A \leq L$ and $m(A \cap Q) = 4n$.

PROOF. The proof is much like that of C.1.35, only substantially easier. As L has 2 noncentral 2-chief factors, case (2) of C.1.34 holds. Thus $Q = U_1 \oplus U_2$ with U_1 and U_2 isomorphic natural modules for $L/O_2(L)$, and $S \leq N_G(U_1)$ (cf. the argument in A.1.42). Set $V := C_G(Q)$. Arguing as in the proof of C.1.35, $[V, L] \leq V \cap L = Q$ and $Q = [Q, X]$ for a suitable subgroup X of L of odd order, so $V = Q \times C_V(X)$; hence $\Phi(C_V(X)) = \Phi(V) \trianglelefteq G$, and as before $\Phi(V) = 1$ since $C_G(L) = 1$. As $C_G(L) = 1$, V is an indecomposable $\mathbf{F}_2L$-module, so applying I.1.6.4 as in the proof of C.1.35, we get $V = Q$ if $n > 1$, establishing (1) in this case. On the other hand if $n = 1$ then by B.4.8.3, $|V : Q| \leq 4$, with $C_V(a) \leq Q$ for $a \in (S \cap L) - C_{S \cap L}(Q)$.

Set $\hat{G} := G/O_2(G)$; then $\hat{G} = \hat{L} \cong SL_3(2^n)$ since $G = LR$ and $\hat{R}$ is the radical of a parabolic since (G, R) is an (M, S)-pair. Set $\bar{G} := G/V$, $S_L := S \cap LV$, and $S_C := O_2(G)$. Then $\bar{S} = \bar{S}_L \times \bar{S}_C$ and as $C_{GL(Q)}(\bar{L}) \cong GL_2(2^n)$, $\bar{S}_C$ is elementary abelian of rank at most n, with $C_Q(\bar{s}) = U_1$ for each $1 \neq \bar{s} \in \bar{S}_C$, and $\bar{S}_C$ is semiregular on $Irr_+(L, Q) - \{U_1\}$.

Using B.2.3.3 as in the proof of C.1.35, there is some $A \in \mathcal{A}(S) - \mathcal{A}(O_2(G))$. Then $C_V(A) = A \cap V$ since $A \in \mathcal{A}(S)$, and $m(\hat{A}) \leq m_2(\hat{L}) = 2n$. If $A_C := A \cap S_C > A \cap V$, then $C_Q(A_C) = U_1$ by the previous paragraph, so $C_Q(A) = C_{U_1}(A)$ is of rank at most $2n$, and then

$$m(A) \leq m(\hat{A}) + m(\bar{A}_C) + m(A \cap V) \leq 2n + n + (m(V) - 4n) < m(V),$$

contradicting $A \in \mathcal{A}(S)$. Therefore $A \cap O_2(G) = A_C = A \cap V \leq C_A(Q)$, so by B.2.4.1,

$$m(\hat{A}) \geq m(Q/C_Q(A)) \geq 2n = m_2(\hat{L}).$$

Hence $m(\hat{A}) = 2n = m(Q/C_Q(A))$, so $QC_A(Q) = Q(A \cap V) = V \in \mathcal{A}(S)$ by B.2.4.2. Further $m(V/(A \cap V)) = 2n$, so as each $I \in Irr_+(L, Q)$ satisfies $m(I) \geq 3n$, $A \cap I \neq 0$. Therefore $A \leq C_G(I \cap A) \leq N_G(I)$, so A lies in the kernel S_L of the action of S on $Irr_+(L, Q)$.

If $n = 1$ we saw earlier that $A \cap V \leq C_V(a) \leq Q$ for $a \in A - C_A(Q) = A - V$, so $V = (A \cap V)Q = Q$, completing the proof of (1).

Now as $V = Q$, $S_L = S \cap L$ and $C_Q(A) = A \cap Q$. Further as $m(\hat{A}) = 2n = m(Q/C_Q(A))$, we conclude that $A \cap Q = W_1 \oplus W_2$, where $W_i := A \cap U_i$ is the S_L-invariant subgroup of Q of rank $2n$, and $A \cap Q = C_Q(a)$ for each $a \in A - Q$.

Thus A and V are the maximal elementary abelian subgroups of S_L, establishing (2). $\square$

We conclude the subsection with a useful but more technical result which, in many of the cases in (MS1) where $L/O_{2,Z}(L)$ is defined over the smallest field $\mathbf{F}_2$, shows roughly that we may replace R in (MS3) with the Baumann subgroup of a suitable unipotent radical. The main idea is to use Baumann's Argument B.2.18.3 to extend the reach of the $C(G,T)$-Theorem C.1.29, much as we earlier used Glauberman's Argument C.1.21 to extend C.1.18 in the lemmas that followed it.

LEMMA C.1.37. *Assume $G = LT$ with $T \in Syl_2(G)$, $L \in \mathcal{C}(G)$, $F^*(G) = O_2(G)$, and $L/O_2(L) \cong L_3(2)$, A_6, $\hat{A}_6$, or $U_3(3)$, with T trivial on the Dynkin diagram of $L/O_2(L)$. Let $T \leq P \leq M$ where M is a maximal subgroup of G; and either $P = M$—or $L/O_2(L) \cong \hat{A}_6$ and P is of index 3 in M with $O^2(P) \not\leq O_{2,Z}(L)$. Let $R := O_2(P)$, $S := Baum(R)$, and assume $V \in \mathcal{R}_2(G)$ with $[V,L] \neq 1$. Set $G^* := G/C_G(V)$, $U := [V,L]$, and $\tilde{V} := V/C_V(L)$. If $L/O_2(L) \cong L_3(2)$, assume either that U is the sum of two isomorphic natural modules for $L/O_2(L)$, or that U is the natural module and P is the stabilizer of a point in U. Then one of the following holds:*

(1) Some nontrivial characteristic subgroup of S is normal in G.

(2) L is a block of type $L_3(2)$, A_6, $\hat{A}_6$, or $G_2(2)$, and if L is of type $\hat{A}_6$ then P^* is the stabilizer of an E_{16}-subgroup of U.

(3) $L/O_2(L) \cong L_3(2)$, $U = [O_2(L), L]$ is the sum of two isomorphic natural modules, and P is the stabilizer of a line in each of those submodules.

(4) $\tilde{U}$ is the natural module for $G^* \cong G_2(2)$, and P is the stabilizer of a line in $\tilde{U}$.

(5) $\tilde{U}$ is a natural module for $G^* \cong A_6$, P^* is the stabilizer of a line in $\tilde{U}$, $O^2(O_{2,Z}(L)) \not\leq P$ if $L/O_2(L) \cong \hat{A}_6$, and the image in G^* of some member of $\mathcal{A}(T)$ has order greater than 2.

PROOF. Let $Q := O_2(G)$. Notice that $R \trianglelefteq T$ from the hypothesis.

If $[V, J(R)] = 1$ then $J(R) = J(Q)$ by B.2.3.3. Thus in this case $J(R)$ is a characteristic subgroup of Baum$(R) = S$ normal in G, so (1) holds.

Therefore we may assume that $[V, J(R)] \neq 1$. Since $O_2(G) \leq O_2(P) = R$ by hypothesis, the hypotheses of B.2.10.2, and hence also of B.2.10.1, are satisfied and so we can apply that lemma and adopt its notation. In particular, $Q = C_R(V)$. Let $\mathcal{P} := \mathcal{P}_{R,G}$ and set $L_0^* := J_{\mathcal{P}}(G^*)$. By B.2.10, $L^* = F^*(L_0^*)$. Thus we can apply Theorem B.5.1 to conclude either $U \in Irr_+(L,V)$, or $L/O_2(L)$ is $L_3(2)$ and U is the sum of two isomorphic natural modules. Then we apply B.4.2 to conclude $\tilde{U}$ is a natural module for $L_0^* \cong L_3(2)$, A_6, S_6, $\hat{A}_6$, or $G_2(2)$, or U is the sum of two isomorphic natural modules for $L_0^* \cong L_3(2)$.

We next determine the embedding of $J(R)^*$ and S^* in G^*, and $Z_U := C_U(J(R))$, in the various cases:

If $L_0^* \cong G_2(2)$ then $L_0^* = Aut(L_0^{*\infty})$, and so (4) holds—unless P is the stabilizer of a point of $\tilde{U}$, so we may assume P has that form. In this case by B.4.6.13, there is a unique FF^*-offender A^* in $R^* = O_2(P^*)$, $m(A^*) = 3 = m(U/Z_U)$, and $A^* = C_{L_0^*}(Z_U)$. Therefore $S^* = J(R)^* = A^*$, and as $r_{A^*,V} = 1$ there are no strong FF-offenders in R.

If $L_0^* \cong \hat{A}_6$, then by B.4.2.8, there is a unique FF*-offender A^* in T^*: namely $O_2(P_1^*)$, where P_1^* is the maximal parabolic stabilizing a 4-dimensional subspace of U. But then the hypotheses of part (b) of C.1.24 are satisfied, so (1) or (2) holds by that lemma. Thus we may assume L_0^* is not $\hat{A}_6$. Similarly we may assume L_0^* is not A_6, since there too the hypotheses of C.1.24.b hold, so that conclusion (1) or (2) holds.

Assume that $L_0^* \cong L_3(2)$. Suppose first that U is the sum of two isomorphic natural modules. Here we check easily that there is a unique FF*-offender $A^* := O_2(P_1^*)$, where P_1^* is the stabilizer of a line in each member of $Irr_+(L, U)$, so that $m(U/C_U(A)) = 2$. Thus if $P \neq P_1$, $J(R) = J(Q)$ and (1) holds, so we may assume $P = P_1$. Thus $R^* = O_2(P_1^*) = A^* = J(R)^*$ and $Z_U = C_U(A)$ with $A^* = C_{L_0^*}(Z_U)$, so $S^* = A^*$. Again A is not a strong FF-offender as $r_{A^*,V} = 1$.

Next suppose that U is not the sum of two natural modules, so that by hypothesis, U is a natural module and P is the stabilizer of a point of U. This time $\mathcal{P}(G^*, V)$ contains conjugates of the subgroups of L^* of order 2, and the conjugates of $R^* = O_2(P^*)$, so $J(R)^* = O_2(P^*)$. In particular $Z_U = C_U(R)$ is a point of U, $S^* = J(R)^* = R^*$, and there are no strong FF-offenders.

Finally suppose $L_0^* \cong S_6$, and adopt the notation of section B.3. As $L_0^* \cong S_6$, $R^* \not\leq L^*$, so $O^2(O_{2,Z}(L)) \not\leq P$ if $L/O_2(L) \cong \hat{A}_6$. Assume first that P^* is the stabilizer of a point in $\tilde{U}$. Then by B.3.4.2v, one of the following holds:

(i) $J(Q) = J(R)$.

(ii) $S^* = \langle (5, 6) \rangle$.

(iii) $S^* = R^*$.

In case (i), (1) holds as before. In case (ii), the hypotheses of C.1.24.c hold with $R_0 := SO_2(G)$ in the role of "R", so as $S = \mathrm{Baum}(R_0)$ by B.2.3.4, again (1) or (2) holds by that lemma. Thus we may assume when P^* is the stabilizer of a point that (iii) holds: that is, $S^* = R^*$, so that again $S^* = J(R)^* = R^*$ and $\tilde{Z}_U$ is a point in $\tilde{U}$. In that event by B.3.4.2ii, R contains no strong FF-offender. Further since $J(R)^* = R^*$, and $\langle (5, 6) \rangle$ is the only FF*-offender of order 2 contained in R^*, there is an FF-offender $A \in \mathcal{A}(R)$ with $m(A^*) \geq 2$. Indeed since the set $\mathcal{P}_{R,LR}$ of B.2.10 is stable, and hence closed below under the relation $\lesssim$, we see if FF-offenders with image R^* occur, then also we may choose our FF-offender A with $m(A^*) = 2$. Thus for such an A we have $m(U/C_U(A)) = 2$.

Finally, assume that P^* is the stabilizer of a line in $\tilde{U}$. We may assume that (5) does not hold, so all FF-offenders have images in R^* of rank 1, and hence are generated by a transposition. Therefore as R^* is the subgroup of T^* generated by transpositions, $J(R)^* = R^*$. As before, $R^* = O^{2'}(C_{L_0^*}(Z_U))$, so that $S^* = J(R)^* = R^*$, and of course R contains no strong FF-offenders since we are assuming all FF-offenders have image of rank 1. Finally observe that $m(U/Z_U) = 2$.

We have determined the possible embeddings of S^* in G^*, in those cases where the lemma may fail. We will use Baumann's Argument B.2.18.3 or B.2.19.3 to show that $S_C := C_S(V) \in Syl_2(C_{LS}(V))$. This will suffice to prove the lemma: For in the cases where $S^* = R^*$, S is then Sylow in $O_2(P_0)$, where $P_0 := P \cap LS$. On the other hand if $S^* < R^* = O_2(P^*)$, then from the discussion above, $G^* \cong G_2(2)$, P is the stabilizer of a point in $\tilde{U}$, and S^* is the unique normal E_8-subgroup of P^*. In either case, hypothesis (MS2) holds with S in the role of "R". Of course hypothesis (MS1) also holds from our hypothesis, and if (1) fails, then hypothesis (MS3) holds. Thus we may assume (LS, S) is an MS-pair. Then when $L/O_2(L) \not\cong L_3(2)$, (2) holds by Theorem C.1.32. When $L/O_2(L) \cong L_3(2)$ that result shows that LS is described in Theorem C.1.34, and as our hypothesis that P is the point stabilizer when U is natural rules out cases (3) and (4) of C.1.34, conclusion (2) or (3) holds.

So it remains to verify the hypotheses of B.2.18.3 or B.2.19. First $Q = C_R(V)$, so hypothesis (a) of B.2.18 holds, and $C_R(V) \in Syl_2(C_{LR}(V))$, as required in B.2.18.3 and hypothesis (A) of B.2.19. As $C_L(V) = O_2(L)$ or possibly $L/O_2(L) \cong \hat{A}_6$ and $C_L(V) = O_{2,Z}(L)$, hypothesis (b) holds, and hypothesis (B) is satisfied as $L/O_2(L)$ is quasisimple. From the discussion above, R has no strong FF-offenders, so hypothesis (c) holds, as does (C) of B.2.19. Therefore it remains to exhibit subgroups X_i, $1 \leq i \leq n$, satisfying hypothesis (d) and (e) of B.2.18, or (D) and (E) of B.2.19.

First consider the cases where either $L_0^* \cong G_2(2)$; or $L_0^* \cong S_6$, and P is the stabilizer of a line. Here we set $X_1 := J(R)$ and take $n := 2$ with $X_2 := X_1^g$ for $g \in L$ with P^{*g} an opposite parabolic to P^*. Thus $L_0^* = \langle X_1^*, X_2^* \rangle$. Then we observe in both of these cases that $m(\tilde{U}) = 2m(U/C_U(J(R)))$, completing the verification of (d) and (e) in these cases. Next if L_0^* is $L_3(2)$ and U is the sum of two natural modules, we take $n := 3$ and choose the X_i to be conjugates of $J(R)$ whose images generate L_0^*, completing the verification of (d) and (e) in this case. In the remaining two cases, we verify (D) and (E) of B.2.19: Suppose first that L_0^* is $L_3(2)$, with U the natural module and P the stabilizer of a point. Here we take $n := 3$, and X_i to be a conjugate of an FF-offender in R with X_i^* of order 2 and $L_0^* = \langle X_i^* : 1 \leq i \leq 3 \rangle$, verifying (D) in this case. Any FF^*-offender X^* in R is L^*-conjugate either to X_1^* or to $R^* = X_1^* X_1^{*l}$ for suitable $l \in L$, verifying (E). Finally suppose $\tilde{U}$ is the natural module for $L_0^* \cong S_6$ and P is the stabilizer of a point. Here we take $n := 2$, X_1 an offender in R with X_1^* of order 4, and X_2 a conjugate of X_1 with $L_0^* = \langle X_1^*, X_2^* \rangle$, verifying (D). This time by B.3.2.5, any FF^*-offender in R^* either is generated by a transvection (and hence lies in an L^*-conjugate of X_1^*) or is L^*-conjugate to X_1^* or R^*. As R^* is the product of X_1^* and the subgroup generated by a transvection, (E) is verified in this case. Thus the proof of the lemma is complete. □

C.2. More general pushing up in SQTK-groups

As mentioned in section C.1, we will often encounter SQTK-groups H with a 2-subgroup R and a proper subgroup M_H satisfying the pushing up hypothesis (CPU) in Definition C.1.6:

$$C(H, R) \leq M_H < H \ \text{ for some } \ 1 \neq R \in \mathcal{B}_2(H) \ \text{ with } \ R \in Syl_2(\langle R^{M_H} \rangle).$$

In this section we derive consequences of this condition along the lines of sections 8–10 of [**Asc81b**], but now under the SQTK-hypotheses. In particular, we will see in C.2.13 that obstructions to pushing up H can usually be detected in $O_{2,E}(H)$.

We postpone consideration of (CPU) until later, when Hypothesis C.2.3 is introduced. Instead we first record, as in [**Asc81b**, Sec 8], a few elementary properties (beyond those already listed in C.1.2) of the set $\mathcal{B}_2(H)$ of 2-radical subgroups of a finite group H. Recall from Definition C.1.1 that $\mathcal{B}_2(H)$ consists of the nontrival 2-subgroups R of H such that $1 \neq R = O_2(N_H(R))$.

Lemma C.2.1. *Assume H is a finite group and $R \in \mathcal{B}_2(H)$. Then*

(1) $\{R\} =$И$^*_H(N_H(R), 2)$.

(2) $O_2(H) \leq R$.

PROOF. Part (1) is easy; cf. 8.1 in [**Asc81b**]. Then (2) follows from (1). □

The following technical lemma from [**Asc81b**] shows how $\mathcal{B}_2(H)$ interacts with the $\mathcal{C}$-components of H; see section A.3 and chapter 1 for a discussion of the properties of $\mathcal{C}$-components.

LEMMA C.2.2. *Assume H is an SQTK-group, $R \in \mathcal{B}_2(H)$, and $L \in \mathcal{C}(H)$. Then*

(1) $R \cap LC_H(L/O_2(L)) = (R \cap L)C_R(L/O_2(L))$.

(2) $R \cap L = O_2(N_L(R)) = O_2(N_L(R \cap L))$. *That is,* $R \cap L \in \mathcal{B}_2(L)$.

(3) Let $X := LN_R(L)$. *Then* $N_R(L) = O_2(N_X(N_R(L)))$. *That is,* $N_R(L) \in \mathcal{B}_2(X)$.

(4) If $L/O_2(L)$ is of Lie type and characteristic 2, then $N_L(R \cap L)/O_2(L)$ is a parabolic subgroup of $L/O_2(L)$.

PROOF. By C.2.1.2, $O_2(H) \leq R$; then we pass to $H/O_2(H)$, observing that $H/O_2(H)$ is an SQTK-group, $R/O_2(H) \in \mathcal{B}_2(H/O_2(H))$, and $LO_2(H)/O_2(H) \in \mathcal{C}(H/O_2(H))$ by A.3.3.4. Thus we may assume that $O_2(H) = 1$, giving Hypothesis A.3.4, so that we can apply the subsequent results from section A.3.

By A.3.7, distinct members of $\Delta := L^H$ commute. Hence the hypotheses of 8.2 in [**Asc81b**] are satisfied, and lemma C.2.2 follows from that lemma. □

We now impose the (CPU) hypothesis from Definition C.1.6, in the form studied in [**Asc81b**, Sec 10]:

HYPOTHESIS C.2.3. *H is an SQTK-group; further $1 \neq R \in \mathcal{B}_2(H)$ and M_H are subgroups of H such that*

$$C(H, R) \leq M_H \quad \textit{and} \quad R \in Syl_2(\langle R^{M_H} \rangle).$$

Hypothesis C.2.3 is satisfied in a 2-local H in the QTKE-groups G appearing in the proof of the Main Theorem, when $R := O_2(L_0T)$ for a suitable uniqueness subgroup L_0T and $M_H := N_H(L_0)$; see in particular 1.4.1. Later in sections 4.1 and 4.2, we will see how to use this initial example of (CPU) to get the condition in other situations.

Eventually we will see in C.2.13 that under the stronger Hypothesis C.2.8, the obstructions to pushing up H are certain $\mathcal{C}$-components L of H with $L/O_2(L)$ quasisimple. The case where L is an ordinary component will be treated in section C.3. We begin with some preliminary results restricting triples R, M_H, H satisfying Hypothesis C.2.3, concentrating on the case where $F^*(L) = O_2(L)$ and $L/O_2(L)$ is quasisimple. See Definition C.1.12 for the definition of χ_0-blocks.

LEMMA C.2.4. *Assume Hypothesis C.2.3, and assume $L \in \mathcal{C}(H)$ with $F^*(L) = O_2(L)$ and $L/O_2(L)$ quasisimple. Set $L^* := L/O_2(L)$. Then*

(1) If $R \cap L \in Syl_2(L)$ then either $L \leq M_H$; or L is a χ_0-block, $N_R(L)$ contains an FF-offender on $R_2(LN_R(L))$, and $(L \cap M_H)^*$ is a Borel subgroup of L^*.

(2) Either $R \cap L \in Syl_2(L)$ or $R \leq N_H(L)$.

PROOF. Part (1) follows from C.1.29; see also 10.2 in [**Asc81b**]. Part (2) is 10.3 in [**Asc81b**]. □

The next result is just a restatement of the Local $C(G,T)$-theorem C.1.29 in our present context:

LEMMA C.2.5. *Assume Hypothesis C.2.3 with* $R \in Syl_2(H)$ *and* $F^*(H) = O_2(H)$. *Then* $H = M_H L_1 \cdots L_r$ $(r \geq 0)$, *where* $L_1, \ldots, L_r$ *are subnormal* χ_0*-blocks of* H.

Similarly by C.2.1.2 and C.1.29:

LEMMA C.2.6. *Assume Hypothesis C.2.3 with* $F^*(O_{2,F}(H)) = O_2(H)$. *Then*

(1) $R \in Syl_2(O_{2,F}(H)R)$.

(2) If $O_{2,F}(H) \not\leq M_H$, *then* $O_{2,F}(H) = (M_H \cap O_{2,F}(H))\ L_1 \cdots L_r$ *with* L_i *an* A_3*-block not contained in* M_H.

In view of C.2.4, we can focus on the case where R normalizes some $\mathcal{C}$-component K of H with $K \not\leq M_H$. The next result provides some initial restrictions on the structure of such $\mathcal{C}$-components, utilizing results from [**Asc81b**] along with C.1.32. We will obtain much stronger restrictions later in Theorem C.4.1. See Definition C.1.31 for MS-pairs.

LEMMA C.2.7. *Assume Hypothesis C.2.3 and* $K \in \mathcal{C}(H)$ *such that* $K/O_2(K)$ *is quasisimple,* $F^*(K) = O_2(K)$, $K \not\leq M_H$, *and* $R \leq N_H(K)$. *Also define* $V := \Omega_1(Z(O_2(KR)))$, $M_K := M_H \cap K$, *and* $(KR)^* := KR/C_{KR}(V)$. *Then*

(1) $O_2(KR) \leq C_{KR}(V) \leq O_2(KR)O_{2,Z}(K)$ *and* $V \in \mathcal{R}_2(KR)$.

(2) $J(R) \not\leq J(O_2(KR))$, V *is an FF-module for* K^*R^*, *and* R^* *contains FF*-offenders on* V.

(3) One of the following holds:

(a) K *is a* χ*-block, and one of the following holds:*

(i) K *is an* $L_2(2^n)$*-block,* $J(R)^* \in Syl_2(K^*)$, *and* M_K^* *is a Borel subgroup of* K^*.

(ii) K *is an* A_5*-block,* $R \in Syl_2(KR)$, $K^*R^* \cong S_5$, *and* M_K^* *is a Borel subgroup of* K^*.

(iii) K *is an* A_7*-block,* $R^* \cong D_8$, $K^*R^* \cong S_7$, *and* M_K^* *is the stabilizer of a partition of type* $4, 3$.

(iv) K *is an* A_7*-block,* $R^* \cong E_8$, $K^*R^* \cong S_7$, *and* M_K^* *is the stabilizer of a partition of type* $2^3, 1$.

(b) K *is an* A_n*-block for* $5 \leq n \leq 8$, R^* *is generated by an involution inducing a transposition on* K^*, *and* $M_K^* = C_{K^*}(R^*)$.

(c) K *is an* $\hat{A}_6$*-block and* $|K : M_K| = 15$, *with* M_K *the stabilizer of a* 2*-dimensional* $\mathbf{F}_4$*-subspace of the* 3*-dimensional* $\mathbf{F}_4$*-space* $U(K)$.

(d) K *is an exceptional* A_7*-block and* $|K : M_K| = 35$.

(e) K *is an* A_6*-block or* $Sp_4(4)$*-block and* M_K^* *is a maximal parabolic of* K^*.

(f) K *is a* $G_2(2)$*-block,* $K^*R^* \cong G_2(2)$, *and* $M_K^*R^*$ *is a maximal parabolic of* K^*R^*.

(g) $K^* \cong SL_3(2^n)$, M_K^* *is a maximal parabolic of* K^*, *and* (KR, R) *is an MS-pair described in Theorem C.1.34.*

(h) $K^* \cong L_4(2)$ *or* $L_5(2)$ *and* M_K^* *is a parabolic subgroup of* K^* *of semisimple rank at least* 1.

PROOF. By hypothesis R acts on K, so by C.2.2.3, $R \in \mathcal{B}_2(RK)$. Then the triple R, $KR \cap M_H$, KR satisfy Hypothesis C.2.3, so we may assume $H = KR$.

Now parts (1) and (2) are easy consequences of B.2.10, or see 10.4 in [**Asc81b**]. Then Theorem B.5.1 implies that

$$K^* \text{ is } L_r(2),\ L_2(2^n),\ Sp_4(2^n)',\ SL_3(2^n),\ G_2(2^n)',\ \hat{A}_6, \text{ or } A_7. \tag{!}$$

It remains to show that one of conclusions (a)–(h) of (3) holds.

Let $J := J(R)$ and $K_J := N_K(J)$. Since $C(H,R) \leq M_H$ by Hypothesis C.2.3, and $O_2(H) \leq R$ by C.2.1.2, (1) says that

$$N_{K^*}(J^*) = K_J^* \leq M_K^* < K^*. \tag{$*$}$$

Suppose first that K is a χ-block. If K is an $L_2(2^n)$-block then by B.4.2.1, the Sylow 2-subgroups of K^* are the only FF*-offenders, so that $J^* \in Syl_2(K^*)$. Then as Borel subgroups are maximal in K^*, we conclude from (*) that $M_K^* = K_{J^*}$ is Borel in K^*, so that case (i) of (a) holds. Therefore we may assume that K is an A_n-block for $n = 5$ or 7, and represent K^*R^* on $\Omega := \{1, \ldots, 7\}$ as in section B.3. By B.4.2.5, J^* is generated by k commuting transpositions. If $k = 1$ then K_J^* is maximal in K^*, so $M_K^* = K_J^*$ and (b) holds. Assume $k = 2$. Then the unique maximal subgroup of K^* containing K_J^* is the global stabilizer of the set Δ of 4 points of Ω moved by J^*, which we will denote by M_J^*, and $|M_J : K_J| = 3$, so M_K is M_J or K_J by (*). However in the latter case $O_2(K_J^*) \leq R^*$ by C.2.1.1, so R is Sylow in $K_{\Omega-\Delta}R$ (where $K_{\Omega-\Delta}$ denotes the pointwise stabilizer of $\Omega - \Delta$), and $X := O^2(K_{\Omega-\Delta})$ is not an A_3-block, so that $X \leq M_K$ by C.2.6.2, contrary to our assumption that $M_K = K_J$. Therefore $M_K = M_J$, so as $R \in Syl_2(\langle R^{M_K} \rangle)$ by Hypothesis C.2.3, and $k = 2$, it follows that $R^* \cong D_8$ and hence case (ii) or (iii) of (a) holds. Finally assume $k = 3$. Then $n = 7$ and K_J^* is the stabilizer of a partition of Ω of type $2^3, 1$. The only proper subgroup of K^* properly containing K_J^* is the stabilizer M_J^* of a point of Ω, so $M_K^* = K_J^*$ or M_J^* by (*). But if $M_K = M_J$, then as $R \in Syl_2(\langle R^{M_K} \rangle)$, $R \in Syl_2(H)$, and C.1.29 supplies a contradiction since the preimage M_J of M_J^* is not a χ_0-block. Hence $M_K = K_J$, so that case (iv) of (a) holds.

Thus we may assume that K is not a χ-block, so by C.1.29,

$$R \notin Syl_2(H), \tag{$**$}$$

and case (1) of 10.5 in [**Asc81b**] does not hold. In case (2) of that result $K^*/Z(K^*)$ is of Lie rank at least 3 and M_K^* is a parabolic subgroup of K^*. As K^* appears in (!), K^* is $L_4(2)$ or $L_5(2)$, and by (**), M_K^* is a parabolic of rank at least 1, so conclusion (h) holds. In case (3) of 10.5 in [**Asc81b**], K^* is an alternating group of degree at least 10, and no such group appears in (!). Thus we may assume that case (4) of 10.5 in [**Asc81b**] holds; thus $M_H = N_H(R)$, so that $R = O_2(M_H)$ by C.2.1.1. Furthermore the hypotheses of sections 8 and 9 of [**Asc81b**] are satisfied, so K^* described in 9.3 of [**Asc81b**].

Assume K^* is A_7. Then as K is not a χ-block, K^* is an exceptional A_7-block by 9.3 in [**Asc81b**], so by B.4.2.7, J^* is a regular 4-subgroup of K^*. Thus K_J^* is maximal in K^*, so $M_K^* = K_J^*$ and hence (d) holds. Thus we may assume that K^* is not A_7, so by 9.3 in [**Asc81b**], K^* is one of the groups of Lie type of characteristic 2 appearing in (!), and hence 8.9 of [**Asc81b**] applies. In the first case of that result, 8.8 of [**Asc81b**] says that (b) holds, so we may assume that the second case of 8.9 of [**Asc81b**]holds: that is M_K^* is a parabolic subgroup of K^*. We may assume K^* is of

Lie rank 2 since we have already handled the other cases, and then M_K^* a parabolic of rank 1 by (**). Further (H, R) is an MS-pair; for example (MS3) follows from Hypothesis C.2.3 and the hypothesis that $K \not\leq M_H$. Now C.1.32 completes the proof of C.2.7. □

Next we strengthen Hypothesis C.2.3 so as to achieve the hypotheses attained in the proof of Theorem 4.2.5; in particular, that hypothesis holds for our standard uniqueness situation in 1.2.9.2. Thus in the remainder of this section we assume:

HYPOTHESIS C.2.8. *H is an SQTK-group, and R is a 2-group with $R \leq M_H \leq H$ such that*

(1) $F^*(M_H) = O_2(M_H)$ and $O_2(H) \neq 1$.

(2) There exists $L_H \in \mathcal{C}(M_H)$ such that

$$R \in Syl_2(C_{M_H}(M_0/O_2(M_0)) \quad and \quad C(H,R) \leq M_H$$

where $M_0 := \langle L_H^{M_H} \rangle$.

(3) There exists a nontrivial elementary abelian 2-subgroup V of M_0 such that $V = [V, M_0] \leq Z(O_2(M_0 R))$ and $N_H(V) \leq M_H$.

We first obtain some easy initial consequences of Hypothesis C.2.8. In the remainder of the section, we will take

$$R \leq T_H \in Syl_2(M_H).$$

LEMMA C.2.9. *(1) $M_0/O_2(M_0)$ is a direct product of at most two $\mathcal{C}$-components, with $L_H^{M_H} = L_H^{T_H}$. In particular, $M_0 = M_0^\infty$.*

(2) $M_0 \leq N_H(R)$ and $R \in \mathcal{B}_2(H)$.

(3) Hypothesis C.2.8 is inherited by any subgroup X with $M_0 T_H \leq X \leq H$.

PROOF. As in earlier arguments, we may apply the results of section A.3 in $M_H/O_2(M_H)$; in particular A.3.8.1 implies (1). Applying A.4.2.4 to M_0, M_H in the roles of "X, M", and appealing to C.2.8.2, M_0 normalizes R. Also $R \in \mathcal{B}_2(H)$ by C.1.2.4, establishing (2). Part (3) is straightforward. □

LEMMA C.2.10. *(1) $O(H) = 1$.*

(2) No component of H is contained in M_H.

(3) $R \in Syl_2(C_H(M_0/O_2(M_0))$.

(4) Hypothesis C.2.3 is satisfied.

(5) $\{R\} = \text{И}^*_{N_H(R)}(M_0, 2)$.

PROOF. Embed $R \leq S \in Syl_2(C_H(M_0/O_2(M_0)))$. By C.2.8.2, $N_H(R) \leq M_H$—while R is Sylow in $C_{M_H}(M_0/O_2(M_0))$, so $N_S(R) = R$ and hence (3) holds. As M_0 normalizes R by C.2.9.2, R is contained in each $S \in \text{И}^*_{N_H(R)}(M_0, 2)$, and $S \leq N_H(R) \leq M_H$ so $S = R$ by C.2.8.2 and A.4.2.4, proving (5). Next $R \in \mathcal{B}_2(H)$ by C.2.9.2. As $M_0 \trianglelefteq M_H$ by C.2.8.2, also $C_{M_H}(M_0/O_2(M_0)) \trianglelefteq M_H$, and of course R is Sylow in the former subgroup so $R \in Syl_2(\langle R^{M_H} \rangle)$. Therefore Hypothesis C.2.3 is satisfied, establishing (4).

As $F^*(M_H) = O_2(M_H)$ by Hypothesis C.2.8, (2) holds and $O(H) \cap M_H \leq O(M_H) = 1$. By A.1.26, $V = [V, M_0] \leq C_H(O(H))$, so by (3) of Hypothesis C.2.8,

$$O(H) = O(H) \cap C_H(V) = O(H) \cap M_H = 1,$$

completing the proof of (1), and hence of the lemma. □

LEMMA C.2.11. *$O_{2,F}(H) \leq M_H$.*

PROOF. By C.2.10.1, $O(H) = 1$, so $F^*(O_{2,F}(H)) = O_2(H)$. Thus if $O_{2,F}(H) \not\leq M_H$, then by C.2.6.2 there is some A_3-block X of $O_{2,F}(H)$ with $X \not\leq M_H$. As H is an SQTK-group, $m_3(H) \leq 2$, so $|X^H| \leq 2$. Therefore as M_0 is perfect and $Aut(X)$ is solvable, $M_0 \leq C_H(X)$. Then $V = [V, M_0] \leq C_H(X)$, so $X \leq C_H(V) \leq M_H$ by Hypothesis C.2.8.3, contrary to assumption. □

LEMMA C.2.12. *$N_H(M_0) = M_H$.*

PROOF. Set $X := C_H(M_0/O_2(M_0))$; by C.2.10.3, $R \in Syl_2(X)$. By a Frattini Argument, $N_H(M_0) \leq XN_H(R)$, so as $N_H(R) \leq M_H$, it remains to show that $X \leq M_H$. Assume otherwise. By C.2.9.3, we may assume $H = M_0XT_H$.

We first show that $F^*(X) = O_2(X)$. By C.2.10.1, $O(X) = 1$. Suppose L is a component of X. Then by definition of X, $[M_0, L] \leq O_2(M_0) \leq C_X(L)$, so L centralizes M_0 by the Three-Subgroup Lemma. Thus $L \leq C_H(M_0) \leq C_H(V) \leq M_H$ using Hypothesis C.2.8.3, contrary to C.2.10.2. Therefore $F^*(X) = O_2(X)$.

Since R is Sylow in X, the Local $C(G,T)$-Theorem in the form C.2.5 says there is a χ_0-block K of X not contained in M_H. Therefore from Hypothesis C.2.8.3, K does not centralize V, and so $[M_0, K] \neq 1$. However $[M_0, K] \leq O_2(M_0) \cap K \leq O_2(K)$, so a Sylow 2-group T_0 of M_0 is contained in $O_2(KT_0)$. Thus T_0 centralizes $U := U(K)$ by C.1.11, so as M_0 acts on K, $M_0 = \langle T_0^{M_0} \rangle$ centralizes U. Then M_0 centralizes K by Coprime Action, completing the proof. □

We now arrive at the main result C.2.13 of this sequence of lemmas: if M_H is proper in H, then (leaving aside the very special situation in C.2.13.2, which causes little difficulty) there is a $\mathcal{C}$-component K of H such that $K/O_2(K)$ is quasisimple and $K \not\leq M_H$. In particular either K is a component of H (the case treated in section C.3) or $F^*(K) = O_2(K)$ (the case treated in section C.4).

PROPOSITION C.2.13. *Assume $O_{2,F^*}(H) \leq M_H$. Then either*

(1) $H = M_H$, or

(2) $M_0 = L_H \in \mathcal{C}(H)$ and $M_H = N_H(M_0)$ is of index 2 in H.

PROOF. By C.2.9.1, $M_0 = \langle L_H^{T_H} \rangle$ for some $L_H \in \mathcal{C}(M_H)$ and $R \leq T_H \in Syl_2(M_H)$. As $O_{2,F^*}(H) \leq M_H$ by hypothesis, $L_H \in \mathcal{C}(H)$ using A.3.3.2. Now if $M_H < H$, M_0 is not normal in H by C.2.12. By A.3.8.1 there are exactly two H-conjugates of L_H; thus $M_0 = L_H$. Now (2) holds using C.2.12. □

LEMMA C.2.14. *If $H_1 \leq H$ is solvable with $M_0T_H \leq N_H(H_1)$, then $H_1 \leq M_H$.*

PROOF. As $M_0T_H \leq N_H(H_1)$, by C.2.9.3 we may assume $H = H_1M_0T_H$, so $H_1 \trianglelefteq H$. As H_1 is solvable, $O_{2,F^*}(H_1) = O_{2,F}(H_1) \leq M_H$ by C.2.11. Now as $H = H_1M_0T_H$, $M_0 = H^\infty \trianglelefteq H$, and then C.2.12 shows $M_H = H \geq H_1$, completing the proof. □

C.3. Pushing up in nonconstrained 2-locals

Recall that because of C.2.13 and C.2.11, the obstruction to pushing up H (i.e., to showing $M_H = H$) under hypothesis C.2.8 is some $\mathcal{C}$-component K of H with $K/O_2(K)$ quasisimple. In this section, we show that when K is quasisimple, the triple M_0, $M_H \cap K$, K is on a short list of configurations:

THEOREM C.3.1. *Assume Hypothesis C.2.8 and that K is a component of H. Let $M_K := K \cap M_H$. Then $L_H = M_0 \trianglelefteq M_H$, $M_0 \leq K$, M_H acts on K, $Z(K)$ is a 2-group, and one of the following holds:*

(1) $K/Z(K)$ is of Lie type and Lie rank 2 over $\mathbf{F}_{2^n}$, M_K is a maximal parabolic of K, $M_0 = M_K^\infty$, $M_0/O_2(M_0) \cong L_2(2^n)$, and either V is the natural module for $M_0/O_2(M_0)$, or $K \cong Sp_4(2^n)$ and $V/C_V(M_0)$ is the natural module.

(2) $K \cong L_n(2)$, $n = 4$ or 5, M_K is a parabolic of K, $M_0 = M_K^\infty$, and either

(a) $M_0/O_2(M_0) \cong L_{n-1}(2)$ and $V = O_2(M_0)$ is the natural module for M_0/V, or

(b) $n = 5$, $M_0 \cong L_3(2)/E_{64}$, and V is the natural module for $M_0/O_2(M_0)$.

(3) $K/Z(K) \cong M_{22}$ and $M_0/Z(K)$ is $L_2(4)/E_{16}$, A_6/E_{16}, or $L_3(2)/E_8$.

(4) $K \cong M_{23}$ and M_0 is $L_2(4)/E_{16}$ or A_7/E_{16}.

(5) $K \cong M_{24}$ and M_0 is $\hat{A}_6/E_{64}$, $L_4(2)/E_{16}$, or $L_3(2)/E_{64}$.

(6) $K \cong J_4$ and M_0 is $M_{24}/E_{2^{11}}$, $L_3(2)/2^{3+12}$, or $L_5(2)/E_{2^{10}}$.

(7) $K/Z(K) \cong HS$ and $M_0/Z(K)$ is $L_3(2)/\mathbf{Z}_4^3$ or A_6/E_{16}.

(8) $K \cong He$ and M_0 is $\hat{A}_6/E_{64}$.

(9) $K/Z(K) \cong Ru$ and $M_0/Z(K)$ is $L_3(2)/2^{3+8}$ or $G_2(2)'/E_{64}$.

(10) K is the double cover group of A_8, $Z(K) = C_V(M_0)$ and V is the 4-dimensional indecomposable for $M_0/V \cong L_3(2)$ described in B.4.8.2.

Recall by C.2.10.1 that $O(H) = 1$, so that $Z(K)$ is a 2-group.

The proof of Theorem C.3.1 involves a short series of reductions. In the remainder of the section assume the theorem fails. Let $R \leq T_H \in Syl_2(M_H)$ and set $K_0 := \langle K^{T_H} \rangle$. Applying A.3.8.1 in $H/O_2(H)$, we see that $|K^H| \leq 2$. As M_0 is perfect by C.2.9.1, $M_0 \leq N_H(K)$. By C.2.9.3 we may assume $H = M_0 K_0 T_H$. In particular $K_0 \trianglelefteq H$.

LEMMA C.3.2. *$M_0 \leq K_0$ and, replacing K by some conjugate if necessary, $L_H \leq K$.*

PROOF. By C.2.10.2, $K \not\leq M_H$, so as $C_H(V) \leq M_H$ by C.2.8.3, $[V, K] \neq 1$. Then as $V \leq M_0$ by C.2.8.3, $[K, M_0] \neq 1$.

Recall M_0 is perfect by C.2.9.1; so by the Schreier property for $\mathcal{K}$-groups, $M_0 \leq K_0 C_H(K_0)$. Let M_1 be the projection of L_H on K; we saw $[K, M_0] \neq 1$, so replacing K by a conjugate if necessary, $M_1 \neq 1$. Let $M_2 := \langle M_1^{T_H} \rangle$ and observe $O_2(M_2) \in \mathcal{U}_H(M_H, 2)$, so by C.2.1.1, $O_2(M_2) \leq R$. Further $[M_0, R] \leq O_2(M_0)$ by C.2.8.2, so $[M_2, R] \leq O_2(M_2) \leq R$, and therefore $M_2 \leq N_H(R) \leq M_H \leq N_H(M_0)$. As L_H is perfect, so is M_1, so M_1 acts on L_H using A.3.8.1. Then as M_1 is the projection of L_H, $L_H = [L_H, M_1] \leq M_1 \leq K$, so $M_0 = \langle L_H^{T_H} \rangle \leq K_0$. □

LEMMA C.3.3. *$K = K_0$.*

PROOF. Assume otherwise; then as we noted earlier, $K_0 = KK^t$ for some $t \in T_H - N_H(K)$. By A.3.8.3 $K/Z(K)$ is $L_2(2^n)$, $Sz(2^n)$, J_1, or $L_2(p)$, p an odd prime. Let $X := N_H(K)$ and $\bar{X} := X/C_X(K)$. Since

$$1 \neq V = [V, M_0] \leq M_0 \leq K_0 \leq N_H(K) = X$$

by C.3.2, $1 \neq \bar{V} \trianglelefteq \bar{M}_0$ with $\bar{V} = [\bar{V}, \bar{M}_0]$. But by inspection of the maximal subgroups of the groups just listed, $\bar{K}_0$ has no such perfect subgroup $\bar{M}_0$ with $1 \neq \bar{V} = [\bar{V}, \bar{M}_0]$. □

LEMMA C.3.4. *$V \leq R \cap K$ and $V \not\leq C_H(K)$.*

PROOF. By C.3.2 and C.3.3, $V \leq M_0 \leq K$, so as $V = [V, M_0]$, V acts nontrivially on K. Further $V \leq O_2(M_0) \leq R$ using C.2.10.5. □

LEMMA C.3.5. *If $K/Z(K)$ is of Lie type and characteristic 2, then conclusion (1) or (2) of Theorem C.3.1 holds.*

PROOF. Assume that $\bar{K} := K/Z(K)$ is of Lie type and characteristic 2 and set $R_K := R \cap K$ and $Y := N_K(R_K)$. By C.3.4, $V \leq R_K$, so $\bar{R}_K \neq 1$. Then by (4) and (2) of C.2.2, $\bar{Y}$ is a parabolic subgroup of $\bar{K}$ and $O_2(\bar{Y}) = \bar{R}_K$. By C.2.9.2, $M_0 \leq N_K(R_K) = Y$. As $M_0 = M_0^\infty$ acts nontrivially on V, $\bar{Y}$ is not a Borel subgroup of $\bar{K}$, so $\bar{K}$ is of Lie rank at least 2. From Theorem C (A.2.3),

$$\bar{K} \text{ is of Lie rank 2, } L_4(2), \text{ or } L_5(2),$$

and then as M_0 is a nonsolvable subgroup of Y, either $\bar{Y}$ is a maximal parabolic, or $Y/O_2(Y) \cong L_3(2)$ and $\bar{K} \cong L_5(2)$. In either case $M_0 \leq X := Y^\infty \in \mathcal{C}(Y)$, and either $X/O_2(X)$ is of Lie rank 1 or $Y/O_2(Y)$ is $L_3(2)$ or $L_4(2)$.

We claim that $M_0 = X$. Suppose not; then by C.2.9.3 and C.2.10.4, the hypotheses of C.2.7 are satisfied with X in the role of "K". Therefore as $X/O_2(X)$ is of Lie rank 1, $L_3(2)$, or $L_4(2)$, and M_0 is a nonsolvable subgroup of $M_H \cap X$ we conclude from the list in C.2.7.3 that $X/O_2(X) \cong L_4(2)$, so $\bar{K} \cong L_5(2)$ and $Y = X$ is a maximal parabolic. Then as R acts on Y, R induces inner automorphisms on K, so $R = R_K C_R(K)$ by C.2.2.1. Thus

$$Y \leq N_H(R) \leq N_H(M_0),$$

so $M_0 = X$ by A.3.3.1 applied to $Y \in \mathcal{C}(Y)$, completing the proof of the claim.

Therefore $M_0 \trianglelefteq Y$, so $Y \leq M_H$ by C.2.12. Thus $M_K := M_H \cap K$ is a parabolic subgroup of K containing Y. In particular if $\bar{Y}$ is a maximal parabolic, then $Y = M_K$, and thus $M_0 = M_K^\infty$. Suppose on the other hand that $\bar{Y}$ is not maximal. Recall in this case that $\bar{K} \cong L_5(2)$ and $M_0/O_2(M_0) \cong L_3(2)$. If Y is the parabolic of K centralizing an involution, then $Z(R_K) \leq Z(Y)$, contradicting the hypothesis in C.2.8 that $V = [V, M_0] \leq Z(R)$. In the remaining cases Y is contained in a maximal parabolic P of K with $P/O_2(P) \cong L_3(2) \times S_3$, so $P \leq M_K$ by C.2.14, and hence $P = M_K$. In this case $Z(R_K)$ is the natural module for $X/O_2(X)$, so as $V \leq Z(R_K)$ by Hypothesis C.2.8.3, $V = Z(R_K)$.

Suppose $\bar{K} \cong L_n(2)$ with $n = 4$ or 5. As $Z(K)$ is a 2-group, I.1.3 says that either $Z(K) = 1$ or K is the double cover group of A_8. In the former case the module V is as claimed from the structure of the maximal parabolics, and our observation in the previous paragraph in the case $n = 5$ and $M_K/O_2(M_K) \cong L_3(2) \times S_3$. In the latter case (10) holds. Thus we have completed the treatment of the case $\bar{K} \cong L_4(2)$ or $L_5(2)$.

Finally if $\bar{K}$ is of Lie rank 2, then as M_0 acts nontrivially on $\Omega_1(Z(O_2(R_K)))$, we conclude from the structure of the parabolics of K that $M_0/O_2(M_0) \cong L_2(2^n)$, and indeed V is the natural module—unless we are in the exceptional case that conclusion (1) allows when $K \cong Sp_4(2^n)$ (where $\bar{K} = K$ by I.1.3 since $n > 1$). So conclusion (1) of Theorem C.3.1 holds in all cases for $\bar{K}$ of rank 2. This completes the proof of C.3.5. □

We now complete the proof of Theorem C.3.1. By C.3.2 and C.3.3, $M_0 \leq K_0 = K$. Now $M_0 \trianglelefteq N_K(V) \leq M_K$ using C.2.8.2 and C.2.8.3. By C.3.4, $V \leq R$, so that $V = [V, M_0] \leq R \cap Z(O_2(M_0))$ using C.2.8.3. We get $L_H \in \mathcal{C}(N_K(V))$ using C.2.8.3 and A.3.3.2. Set $H^* := H/C_H(K)$. By C.3.4, $V^* \neq 1$, so as $V = [V, M_0]$ and $M_0^* = M_0^{*\infty}$, $3 \leq m(V^*) \leq m_2(K^*)$. This rules out A_7 and the groups of odd characteristic in Theorem C (A.2.3); the other alternating groups in Theorem C are of Lie type and characteristic 2, so they have already been handled in C.3.5. Therefore K^* is sporadic. We inspect the list of 2-locals for each of those sporadics (cf. the references for sporadics in the proof of Theorem C), seeking locals M_H^* with $V^* = [V^*, M_0^*] \leq Z(O_2(R^*M_0^*))$, $M_0^* = M_0^{*\infty} \trianglelefteq M_H^*$, and $N_{H^*}(V^*) \leq M_H^* \geq N_{H^*}(R^*)$. We conclude that M_0^* is contained in an R-invariant 2-local X^* of K^* such that the pair K^*, X^* is described in one of the conclusions (3)–(9) of Theorem C.3.1. In particular if $X^* = M_0^*$ then the Theorem holds, using I.1.3 to determine the cases where $Z(K) \neq 1$. So assume that $M_0^* < X^*$. Then by C.2.9.3 and C.2.10.4, the hypotheses of C.2.7 are satisfied with X in the role of "K". Comparing the list of groups in C.2.7.3 to the list of conclusions in Theorem C.3.1 with K sporadic, and recalling that M_0 is a nonsolvable subgroup of $M_H \cap X$, we conclude that $X/O_2(X) \cong L_4(2)$ or $L_5(2)$. and $M_0^*R^*$ is a parabolic of X^*. If $X/O_2(X) \cong L_4(2)$, then $K \cong M_{24}$ and $M_0 \cong L_3(2)/E_{64}$, so the pair K, M_0 appears on the list of Theorem C.3.1. If $X/O_2(X) \cong L_5(2)$, then $K \cong J_4$ and $X \cong L_5(2)/2^{10}$. From C.2.7.2, $R/O_2(X)$ contains an FF*-offender on $O_2(X)$, so from B.4.2.11, $M_0 \cong L_4(2)/2^{6+8}$. But now $M_K = N_K(M_0) \cong L_4(2)/E_{2^4}/E_{2^{11}}$, so $O_2(M_K) = R$ by C.2.1. This is a contradiction as $O_2(M_K)$ does not act on X in J_4.

C.4. Pushing up in constrained 2-locals

Again recall from C.2.13 and C.2.11 that we can detect when M_H is proper in H under Hypothesis C.2.8 from the existence of a suitable $\mathcal{C}$-component K in $O_{2,E}(H)$. In section C.3 we dealt with the case where K is quasisimple. In this section we deal with the case $F^*(K) = O_2(K)$. We are able to improve considerably on the initial restrictions for this case given in C.2.7:

THEOREM C.4.1. *Assume Hypothesis C.2.8 and that there is* $K \in \mathcal{C}(H)$ *with* $F^*(K) = O_2(K)$, $K/O_2(K)$ *quasisimple, and* $K \not\leq M_H$. *Let* $K_0 := \langle K^H \rangle$ *and* $M_K := M_H \cap K$. *Then* $L_H = M_0 \leq K_0 = K$ *and one of the following holds:*

(1) M_0 *is an* A_n*-block,* $n = 5$ *or* 6, $V = U(M_0)$, K *is an* A_{n+2}*-block,* R *induces a transposition on* $K/U(K)$, *and* $M_K = N_K(R)$.

(2) K *is an* $Sp_4(4)$*-block,* $M_K/U(K)$ *is the maximal parabolic of* $K/U(K)$ *stabilizing the* $\mathbf{F}_4$*-line* $V/C_{U(K)}(K)$ *of* $U(K)/C_{U(K)}(K)$, *and* $M_0/O_2(M_0) \cong L_2(4)$.

(3) $K/O_2(K) \cong SL_3(2^n)$ *for* $n > 1$, *and* (KR, R) *is an MS-pair described in one of conclusions (1)–(4) of Theorem C.1.34. Further* $M_K/O_2(K)$ *is the maximal parabolic of* $K/O_2(K)$ *stabilizing a* $\mathbf{F}_{2^n}$*-line* V_1 *in a* K*-irreducible on* $[\Omega_1(Z(O_2(RK))), K]$, *and* $M_0/O_2(M_0) \cong L_2(2^n)$*; and either* $V = V_1$, *or case (2) of Theorem C.1.34 holds with* V *the sum of two lines.*

(4) $K/O_2(K) \cong L_4(2)$ *or* $L_5(2)$, $M_K/O_2(K)$ *is a parabolic of* $K/O_2(K)$, *and*

$$M_0/O_2(M_0) \cong L_3(2) \quad \text{or} \quad L_4(2).$$

The proof involves a series of reductions. Until it is finished, assume that the theorem fails. Let $R \leq T_H \in Syl_2(M_H)$, and set $R_K := R \cap K$. Passing to $H/O_2(H)$, $|K^H| \leq 2$ by A.3.8.1. Therefore $M_0 = O^2(M_0) \leq N_H(K)$. By C.2.9.3 we may assume:

Lemma C.4.2. *$H = K_0 M_0 T_H$.*

Set $H^* := H/C_H(K_0/O_2(K_0))$.

Lemma C.4.3. *$M_0^* \neq 1$.*

Proof. By C.2.1.2, $O_2(K_0) \leq O_2(H) \leq R \leq N_H(M_0)$, so using C.2.9.1, M_0 is the product of members of $\mathcal{C}(O_2(K_0)M_0)$. Thus if M_0 centralizes $K_0/O_2(K_0)$, then K_0 normalizes M_0 using A.3.3.3, so $K_0 \leq M_H$ by C.2.12, contradicting $K \not\leq M_H$. □

Lemma C.4.4. *R normalizes K.*

Proof. If not then by C.2.4.2, $R_K \in Syl_2(K)$, so as $K \not\leq M_H$, K is a χ_0-block of H by C.2.4.1. By C.4.2, $H = T_H M_0 K_0$, so as M_0 and K_0 normalize K, $K_0 = KK^t$ for some $t \in T_H - N_{T_H}(K)$. Let $B := N_{K_0}(R \cap K_0)$; then B^* is a Borel subgroup of K_0^*. Next B is normalized by $T_H M_0$ in view of C.2.9.2, so $B \leq M_H$ by C.2.14. As B^*R^* is maximal in $K_0^*R^*$ and $K_0 \not\leq M_H$, $B = M_H \cap K_0$. But by C.4.3, $1 \neq M_0^* \leq B^*$ which is solvable, contradicting $M_0 = M_0^\infty$. □

By C.4.4, the hypotheses of C.2.7 are satisfied, and hence K satisfies one of the conclusions of that lemma. Recall that $M_0 \leq N_H(K)$ and set $X := RM_0K$, $M_X := M_H \cap X$, and $\bar{X} := X/C_X(K/O_2(K))$. As $K_0 = \langle K^{T_H} \rangle$, T_H normalizes M_0, and $M_0^* \neq 1$, we conclude that

$$\bar{M}_0 \neq 1.$$

Of course $\bar{M}_0 \trianglelefteq \bar{M}_X$, and $\bar{M}_X$ is a proper subgroup of $\bar{X}$ because $K \not\leq M_H$. By the Schreier property, $\bar{M}_0 \leq \bar{K}$, so $N_{\bar{K}}(\bar{R})$ is not solvable.

We claim that $K = K_0$; for if not, then by A.3.8.3 and C.2.7, $K/O_2(K)$ is $L_2(2^n)$ or $L_3(2)$ and $N_{\bar{K}}(\bar{R})$ is solvable, contrary to the previous paragraph.

Let M_1^* be the projection of M_0^* on K^*. As M_0 acts on R, M_1^* acts on R^*, so as $O_2(K) \leq R$, $M_1 \leq N_H(R) \leq M_H$. Then as $\bar{M}_0 = \bar{M}_1$, $M_1 = [M_0, M_1] \trianglelefteq M_0$ by A.3.3.7, and hence $M_1 = M_0$. Similarly $N_{\bar{K}}(\bar{R}) = \overline{N_K(R)} \leq \bar{M}_K$, where $M_K := M_H \cap K$. Thus we have shown:

Lemma C.4.5. *(1) $M_0 \trianglelefteq M_K$.*
(2) $\bar{M}_0 \leq N_{\bar{K}}(\bar{R}) \leq \bar{M}_K$.
(3) $K = K_0$.

We begin to consider the various possibilities listed in C.2.7:

Lemma C.4.6. *If K is a χ-block, then K is an A_7-block and conclusion (1) of Theorem C.4.1 holds.*

Proof. Assume that K is a χ-block. Then by Theorem C (A.2.3), K is of type $L_2(2^n)$ or A_m with $m = 5$ or 7. By C.4.5, $M_0 \leq M_K$, so M_K is not solvable. Thus by inspection of the list in C.2.7.3, K is an A_7-block, $\bar{R}$ is generated by a transposition, and $\bar{M}_0 \cong A_5$. As K is an A_7-block, M_0 is an A_5-block and $U(M_0) = U(K)$. Therefore $V = U(M_0)$, as $U(M_0)$ is the unique elementary abelian subgroup of M_0 satisfying $V = [V, M_0]$. That is, conclusion (1) of Theorem C.4.1 holds. □

Lemma C.4.7. *If $K/O_2(K) \cong SL_3(2^n)$, then conclusion (3) of Theorem C.4.1 holds.*

PROOF. Assume $K/O_2(K) \cong SL_3(2^n)$; then case (g) of C.2.73 holds. Thus $\bar{M}_K$ is a maximal parabolic of $\bar{K}$, and K is described in Theorem C.1.34. As $M_0 = M_0^\infty \trianglelefteq M_K$ by C.4.5, we conclude from the structure of the maximal parabolic $\bar{M}_K$ that $n > 1$, $M_0 = M_K^\infty$, and $M_0/O_2(M_0) \cong L_2(2^n)$. From C.2.8.3, $V = [V, M_0] \leq Z(O_2(M_0R))$, so as $F^*(K) = O_2(K) \leq R$, $V \leq Z(O_2(K))$. Therefore from the description of $Z(O_2(K))$ in C.1.34, $M_K/O_2(M_K)$ is the parabolic stabilizing a line in a K-chief factor W of $Z(O_2(K))$. Furthermore W is unique in all cases except case (2) of C.1.34, where $Z(O_2(K))$ is the sum of two isomorphic natural modules—and then V may be a sum of two lines. This completes the proof. □

We are now in a position to complete the proof of Theorem C.4.1. By C.4.5, $M_0^\infty = M_0 \trianglelefteq M_K$ and $M_0^* \leq N_{K^*}(R^*) \leq M_K^*$. We inspect the possibilities for K on the list of C.2.7.3 not yet eliminated, for the existence of such a perfect normal subgroups of M_K and $N_{K^*}(R^*)$. In case (b) as $N_{K^*}(R^*)$ is not solvable, $K^* \cong A_8$ and conclusion (1) of Theorem C.4.1 holds, just as in C.4.6. Similarly cases (c), (d), and (f), and the subcase of (e) with $K^* \cong A_6$ are eliminated as $N_{K^*}(R^*)$ is not solvable. Therefore either case (h) or the subcase of (e) with $K^* \cong Sp_4(4)$ holds. If $K/O_2(K) \cong Sp_4(4)$, we deduce that conclusion (2) of Theorem C.4.1 holds, just as in C.4.7. When $K/O_2(K) \cong L_n(2)$, M_0^* is a parabolic of K^* by C.2.7.3, and the possible choices for $M_0/O_2(M_0)$ are $L_3(2)$ and $L_4(2)$, which is all that conclusion (4) of Theorem C.4.1 asserts.

We close this section with a corollary to Theorems C.3.1 and C.4.1:

THEOREM C.4.8. *Assume Hypothesis C.2.8 with $M_H < H$. Then $M_0 = L_H$ and $L_H \leq K \in \mathcal{C}(H)$ with $K/O_2(K)$ quasisimple, and one of the following holds:*

(1) $L_H/O_2(L_H) \cong L_2(2^n)$, and $V/C_V(L_H)$ is the natural module for L_HR/R or the A_5-module with $n = 2$.

(2) $L_H/O_2(L_H) \cong L_3(2)$ or $L_4(2)$, and V is either the direct sum of one or more isomorphic natural modules for L_HR/R, or the 6-dimensional orthogonal module for $L_H/O_2(L_H) \cong L_4(2)$.

(3) $K/Z(K) \cong M_{22}$ or HS and L_H is an A_6-block.

(4) $K \cong M_{23}$ and L_H is an exceptional A_7-block.

(5) $K \cong M_{24}$ or He and L_H is an $\hat{A}_6$-block.

(6) $K \cong J_4$ and L_H is a block with $L_H/O_2(L_H) \cong M_{24}$ or $L_5(2)$ described in case (6) of Theorem C.3.1.

(7) $K/Z(K) \cong Ru$ and L_H is a $G_2(2)$-block.

(8) K is an A_8-block and L_H is an A_6-block.

(9) K is the double cover group of A_8 and L_H is an $L_3(2)$-block with $m(V) = 4$.

(10) $O_{2,F^*}(H) \leq M_H$, $L_H/O_2(L_H) \cong L_2(2^n)$, $Sz(2^n)$, J_1, or $L_2(p)$ for some odd prime p, and $M_H = N_H(M_0)$ is of index 2 in H.

In cases (1)–(9), $K \not\leq M_H$, K is invariant under M_H, and K is described in C.3.1 or C.4.1.

PROOF. Assume first that $O_{2,F^*}(H) \leq M_H$; since $M_H < H$ by hypothesis, C.2.13.2 says that $|H : M_H| = 2$, and then conclusion (10) of Theorem C.4.8 holds by A.3.8.3 applied in $H/O_2(H)$.

Thus we may assume that $O_{2,F^*}(H) \not\leq M_H$. Hence by C.2.11, there is $K \in \mathcal{C}(H)$ with $K/O_2(K)$ quasisimple and $K \not\leq M_H$. Now by Theorems C.3.1 and C.4.1, $M_0 = L_H$, K is M_H-invariant, and one of the conclusions of Theorem C.4.8

holds—except that in case (4) of C.4.1, we must show that V is as claimed in conclusion (2).

Assume that case (4) of Theorem C.4.1 holds, and let $Q := O_2(KR)$, $U := \Omega_1(Z(Q))$, and $(KR)^* := KR/C_{KR}(U)$. Thus $K/O_2(K) \cong L_n(2)$, with $n = 4$ or 5, and M_0 is a $\mathcal{C}$-component of the parabolic M_K of K. By C.2.7.1, $Q = C_{KR}(U)$ so $O_2(K^*R^*) = 1$, and by C.2.7.2 there is $A \in \mathcal{A}(R)$ with A^* an FF*-offender on U. Now $Q \leq R$ by C.2.1.2, and $V \leq Z(O_2(M_0R))$ by Hypothesis C.2.8.3, so $V \leq C_{KR}(Q) \leq Q$. Therefore $V \leq W := [C_U(O_2(M_0)), M_0]$. As U is an FF-module for K^*R^* and $K^* \cong L_n(2)$, U is described in case (i), (iii), or (iv) of B.5.1.1. In each case, for each $\mathcal{C}$-component M_0 in a parabolic of K, W is either a sum of isomorphic natural modules for $M_0/O_2(M_0) \cong L_m(2)$, or the 6-dimensional orthogonal module for $M_0/O_2(M_0) \cong L_4(2)$, so the proof is complete. □

C.5. Finding a common normal subgroup

As mentioned in the Introduction to Volume I, in the proof of the Main Theorem, outcome (5) of the Meierfrankenfeld-Stellmacher *qrc*-lemma D.1.5.5 leads to a certain pushing up situation 3.1.6.1—which we can rule out via Theorem C.5.8 of this section. Formally this is accomplished by restating Theorem C.5.8 in a slightly modified form as Theorem 3.1.1, and applying Theorem 3.1.1 to eliminate that configuration. Theorem 3.1.1 is used repeatedly throughout the proof of the Main Theorem. Some of its most immediate corollaries are recorded and derived in section 3.1.

We will show roughly that if a 2-subgroup R is normal in some subgroup M_0, and Sylow in a minimal parabolic H in the sense of Definition B.6.1, then some nontrivial subgroup of R must be normal in both M_0 and H. We prove the contrapositive, showing that if no such common normal subgroup exists, then condition (PU) of Definition C.1.3 is satisfied in H, and then more detailed analysis leads to a contradiction. In many situations arising in the proof of our Main Theorem, such normal subgroups are ruled out as $O_2(\langle M_0, H\rangle) = 1$—often because M_0 is a uniqueness subgroup with $M = !\mathcal{M}(M_0)$, while $H \in \mathcal{H}_*(T, M)$. (Cf. 1.4.1 and 3.3.2.4).

The main result of this section is Theorem C.5.8; it relies on some detailed lemmas established under more general hypotheses, which will be useful elsewhere.

So we will assume in this section:

HYPOTHESIS C.5.1. *G is an SQTK-group and H and M_0 are subgroups of G such that:*

(1) $F^*(H) = O_2(H)$, and $T_H \in Syl_2(H)$ is in a unique maximal subgroup of H. Set $K := O^2(H)$.

(2) $R \leq T_H \leq M_0 \leq G$ with $R \in Syl_2(RK)$ and $R \trianglelefteq M_0$.

We add a further condition useful in pushing up, and assume it holds in lemmas C.5.3 through C.5.6—which give us a detailed list of consequences of these two hypotheses.

HYPOTHESIS C.5.2. *There is no nontrivial subgroup R_0 of R such that $R_0 \trianglelefteq \langle M_0, H\rangle$.*

LEMMA C.5.3. *$K = K_1 \cdots K_s$ is the product of a set $\Delta := \{K_1, \ldots, K_s\}$ of $s \leq 2$ blocks of type $L_2(2^n)$, A_3, or A_5, permuted transitively by T_H. Further $K_i = [K_i, J(R)]$ for each $i = 1, \cdots, s$.*

PROOF. If $1 \neq C$ char R and $C \trianglelefteq KR$, then from C.5.1.2, $C \trianglelefteq \langle M_0, H\rangle$, contrary to C.5.2. Hence no such characteristic subgroup exists. As $F^*(H) = O_2(H)$, $F^*(KR) = O_2(KR)$. Then since R is Sylow in KR, the lemma follows from C.1.26. □

Recall the Baumann subgroup $\mathrm{Baum}(R)$ of R from Definition B.2.2, and set $S := \mathrm{Baum}(R)$. The next lemma collects some properties of the Baumann subgroup which we will need.

Set

$$U_i := U(K_i),\ Q_i := O_2(\langle K_i, S\rangle),\ D_i := C_S(K_i), \text{ and } E_i := [U_i, S].$$

LEMMA C.5.4. *(1) S acts on each K_i.*

(2) If K_i is an $L_2(2^n)$-block or A_3-block then $S \cap K_i \in Syl_2(K_i)$, and $S = (S \cap K_i)Q_i$ when K_i is an $L_2(2^n)$-block, while $K_iS/Q_i \cong L_2(2)$ if K_i is an A_3-block.

(3) If K_i is an A_5-block then $Q_i = U_i \times D_i$, and $S/Q_i \cong E_4$ is generated by two transpositions in $K_iS/Q_i \cong S_5$.

(4) $U_i \leq Q_i = C_S(U_i)$, and setting $U := U_1 \cdots U_s$, $C_R(U) = O_2(KR)$.

PROOF. By C.5.3, $K_i = [K_i, J(R)]$ for each i, so (1) follows from C.1.16.2.

Suppose K_i is an $L_2(2^n)$-block or an A_3-block. Then by Baumann's Lemma B.6.10, S is Sylow in $\langle S^{K_i}\rangle$, while as $K_i = [K_i, J(R)]$, $K_i \leq \langle S^{K_i}\rangle$, so S is Sylow in K_iS. In particular $Q_i = O_2(K_iS) \leq S$. By B.6.9, $K_iS/Q_i \cong L_2(2^n)$, where we set $n := 1$ if K_i is an A_3-block. For $n > 1$, $L_2(2^n)$ is perfect, so that $S \leq K_iQ_i$, and hence since $Q_i \leq S$ we conclude $S = (S \cap K_i)Q_i$. This completes the proof of (2).

Assume that K_i is an A_5-block. By B.3.2.4, $J(R)Q_i/Q_i$ is generated by two commuting transpositions in $K_iJ(R)Q_i/Q_i \cong S_5$. Thus the hypotheses of Baumann's argument B.2.18 are satisfied by K_i, U_i, R in the roles of "L, V, R" and with $n = 2$, $X_1 = J(R)$, and $X_2 = J(R)^g$ for suitable $g \in K_i$. Therefore by B.2.18.1, $Q_i \leq S$, and then $Q_i = U_i \times D_i$ by C.1.13.c, completing the proof of (3).

By (2) and (3), $Q_i \leq S$ so $O_2(KR) \leq R$. Then (4) follows from C.1.11. □

Now recall that by Hypothesis C.5.1.2, M_0 normalizes the Sylow group R of RK, and hence normalizes the characteristic subgroup S; as $U_1 \leq Q_1 \leq S$ by C.5.4.4, we have $U_1^{M_0} \subseteq S \leq H$. However:

LEMMA C.5.5. $\langle U_1^{M_0}\rangle \not\leq O_2(H)$.

PROOF. Suppose $U_0 := \langle U_1^{M_0}\rangle \leq O_2(H)$. Then as $[O_2(H), K_i] = U_i \leq U_0$, K normalizes U_0, as does M_0, so we may take $R_0 := U_0$ to obtain a contradiction to C.5.2. □

By C.5.5, we may choose $x \in M_0$ with $U_1^x \not\leq O_2(H)$; observe since $S \leq H$ that $x \in M_0 - S$. By C.5.4.4, $O_2(H) \cap R$ is the centralizer of $U = U_1 \cdots U_s$ in S. Therefore $[U_1^x, U_i] \neq 1$ for some i, so as $T_H \leq M_0$ is transitive on Δ, we may assume $[U_1, U_1^x] \neq 1$.

LEMMA C.5.6. *(1) $Q_1 = U_1D_1$ with $D_1 \leq C_{Q_1}(U_1^x)$.*

(2) $S = U_1^xQ_1 = Q_1Q_1^x$ and $E_1 = [U_1, U_1^x] = C_{U_1}(U_1^x) = U_1 \cap U_1^x$.

(3) $E_1^x = E_1$.

(4) $\Phi(D_1)^x = \Phi(D_1)$.

(5) x is of even order.

(6) If $T_H \in Syl_2(M_0)$ then $s = 2$ and we may choose $x \in N_{M_0}(T_1)$ with $x^2 \in T_1$, where $T_1 := N_{T_H}(U_1)$.

(7) Assume $|M_0 : T_H| = 2$ and set $D := C_S(K)$ and $U := U_1 \cdots U_s$. Then $U = O_2(K)$, $\Phi(D) = 1 = D \cap D^x$, $D = C_R(K)$, $O_2(KR) = DU \in \mathcal{A}(R)$, $S = DUU^x = J(R)$, and one of the following holds:

(i) $s = 1$, K is an $L_2(2^n)$-block or A_3-block, and $\mathcal{A}(R) = \{Q_1, Q_1^x\}$.

(ii) $s = 1$, K is an A_5-block, and $\mathcal{A}(R) = \{Q_1, Q_1^x, A, A^r\}$, where $|A : A \cap Q_1| = 2$ and $r \in R \cap K - S$.

(iii) $s = 2$, $S = UU^xD$, and $\{DU, DU^x\}$ are the T_H-invariant members of $\mathcal{A}(R)$.

Proof. We begin by establishing (1), (2), and (3).

Suppose first that K_1 is an $L_2(2^n)$-block or A_3-block, and set $n := 1$ in the latter case. We argue as in Stellmacher's proof of lemma 3.4 in [**Ste86**]: By C.5.4.4, $Q_1 = C_S(U_1)$, and from C.5.4.2, S/Q_1 is Sylow in $L_1S/Q_1 \cong L_2(2^n)$. Then S centralizes a 1-dimensional $\mathbf{F}_{2^n}$-subspace of the 2-dimensional natural module $\tilde{U}_1$. Further by the structure of the extension U_1 in I.2.3.1, $\widetilde{C_{U_1}(Y)} = C_{\tilde{U}_1}(Y)$ for any $1 \neq Y \leq S/Q_1$, and $C_{U_1}(S) = [U_1, S]$, so $C_{\tilde{U}_1}(S) = C_{\tilde{U}_1}(U_1^x)$ as $U_1^x \not\leq Q_1$, and

$$|S : C_S(U_1)| = 2^n = |U_1 : C_{U_1}(U_1^x)|$$

$$\text{and if } S = U_1^x Q_1, \text{ then } [U_1, U_1^x] = U_1 \cap U_1^x = C_{U_1}(U_1^x). \tag{$*$}$$

Notice next that $x \in M_0 \leq N_G(R) \leq N_G(S)$, and we have symmetry between U_1 and U_1^x. By (*) and this symmetry, $|U_1^x : C_{U_1^x}(U_1)| = 2^n$, so again by (*), $|U_1^x : C_{U_1^x}(U_1)| = |S : C_S(U_1)|$ and hence

$$S = U_1^x C_S(U_1) = U_1^x Q_1 \tag{!}$$

using C.5.4.4. Further since $U_1 \leq Q_1$, $S = Q_1 Q_1^x$. As $S = U_1^x Q_1$, $E_1 = [U_1, S] = [U_1, U_1^x]$, and as U_1 and U_1^x are normal in S, $[U_1, U_1^x] \leq U_1 \cap U_1^x \leq C_{U_1}(U_1^x)$, so that $[U_1, U_1^x] = C_{U_1}(U_1^x)$ by (*), completing the proof of (2). By symmetry, $[S, U_1^x] = [U_1, U_1^x] = E_1$, so $E_1^x = [S, U_1]^x = [S, U_1^x] = E_1$, and hence (3) holds.

It remains to establish (1) in this case. We first obtain some intermediate facts useful in the proof of (1) and later. Applying (*) to both U_1 and U_1^x, we have $|S : C_S(U_1)| = |S : C_S(U_1^x)| = 2^n$, so $|S : C_S(U_1 U_1^x)| \leq 2^{2n}$. But notice $C_S(U_1) \geq U_1 C_S(U_1 U_1^x)$, with $U_1 \cap C_S(U_1 U_1^x) = C_{U_1}(U_1^x)$, so

$$|U_1 C_S(U_1 U_1^x) : C_S(U_1 U_1^x)| = |U_1 : C_{U_1}(U_1^x)| = 2^n;$$

combined with the first equality of (*), this forces $|S : C_S(U_1 U_1^x)| \geq 2^{2n}$. So in fact all inequalities above must be equalities, and in particular

$$U_1 C_S(U_1 U_1^x) = C_S(U_1) = Q_1 \text{ and } |Q_1 : C_{Q_1}(U_1^x)| = 2^n. \tag{$**$}$$

Then as $S = U_1^x Q_1$, (**) says

$$S = U_1 U_1^x C_S(U_1 U_1^x).$$

We turn to (1). Note $K_1 \leq LQ_1$, where $L := \langle U_1^x, U_1^{xk} \rangle$ for some $k \in K_1$. Now $Q_1 \leq S \leq N_H(U_1^x)$, and hence $Q_1 = Q_1^k$ also normalizes U_1^{xk}; thus $L \trianglelefteq LQ_1 = K_1 Q_1$, and so $K_1 = O^2(K_1 Q_1) \leq L$. Hence $C_H(U_1^x) \cap C_H(U_1^{xk}) \leq C_H(K_1)$. Now by (**) and (*),

$$|Q_1 : C_{Q_1}(U_1^x)| = |U_1 : C_{U_1}(U_1^x)| = 2^n,$$

so as $C_H(U_1^x) \cap C_H(U_1^{xk}) \leq C_H(K_1)$ it follows that

$$|Q_1 : C_{Q_1}(K_1)| \leq |Q_1 : C_{Q_1}(U_1^x) \cap C_{Q_1}(U_1^{xk})| \leq 2^{2n} = |U_1 : C_{U_1}(K_1)|,$$

and hence $Q_1 = U_1 C_{Q_1}(K_1)$ with $C_{Q_1}(K_1) \leq C_{Q_1}(U_1^x)$. Now by definition $D_1 = C_S(K_1) \leq Q_1$, so $C_{Q_1}(K_1) = D_1$, establishing (1). This completes the proof (1)–(3) in the case where K_1 is an $L_2(2^n)$-block or A_3-block.

So assume instead that K_1 is an A_5-block. Then from C.5.4.3, $Q_1 = U_1 \times D_1$. We saw in the proof of C.5.4.3 that $J(R)Q_1/Q_1 = S/Q_1$ is a 4-group generated by commuting transpositions.

Assume first that $S = U_1^x Q_1 = U_1 Q_1^x$. Then as $S = U_1^x Q_1$, $[U_1, S] = C_{U_1}(S)$ is of rank 2, so (*) and (!) hold with $n := 2$, and of course $S = U_1 Q_1^x$ supplies the analogous statement for the action of S on U_1^x. The proofs of (2) and (3) follow as before. We can then use (*) and (!) to repeat the deduction of (**); and use then (*) and (**) together with the fact that $K_1 S/Q_1 \cong S_5$ is generated by two conjugates of S/Q_1 to repeat the proof of (1).

Thus it remains to show that $S = U_1^x Q_1 = U_1 Q_1^x$. Suppose R normalizes K_1; then $U_1 \trianglelefteq R$, so $U_1^x \trianglelefteq R$ as $M_0 \leq N_G(R)$. Therefore as $Aut_{KR}(U_1) \cong S_5$, $Aut_{U_1^x}(U_1)$ is not generated by a transvection, so appealing to B.3.2.4, we conclude $Aut_{U_1^x}(U_1) = Aut_S(U_1)$, and hence $S = U_1^x Q_1$. Further $R = R^x$ normalizes K_1^x, so by symmetry, $S = U_1 Q_1^x$, and we are done in this case.

So it remains to eliminate the case where $|U_1^x : Q_1 \cap U_1^x| = 2$ with $Aut_{U_1^x}(K_1/U_1)$ generated by a transposition, and R does not normalize K_1, so that $s = 2$. Let $E := \Omega_1(Z(S))$. Then $E = C_E(K) \times E_1 \times E_2$ with E_i elementary of order 4, as $D_i = Q_i \times U_i$. Let R_+/S be the subgroup of R/S generated by all transvections on E. Each such transvection centralizes an element of $E_1^\#$, so $R_+ \leq N_R(K_1)$. Further $m([R \cap K_i, E_i]) = 1$ and $R \cap K_i$ centralizes E_{3-i}, so $R \cap K_i \leq R_+$. Also x normalizes E and hence R_+, so as $U_1 \trianglelefteq R_+$, also $U_1^x \trianglelefteq R_+$. This is a contradiction as $|U_1^x : U_1^x \cap Q_1| = 2$; namely R_+ contains $R \cap K_1$ which does not act on the subgroup $Aut_{U_1^x}(K_1/U_1)$ generated by a transposition. This contradiction completes the proof of (1)–(3) in all cases.

Next by (1) and (**), $U_1 D_1 = Q_1 = U_1 C_S(U_1 U_1^x)$, so

$$\Phi(C_S(U_1 U_1^x)) = \Phi(U_1 C_S(U_1 U_1^x)) = \Phi(Q_1) = \Phi(U_1 D_1) = \Phi(D_1),$$

and by symmetry, $\Phi(D_1^x) = \Phi(C_S(U_1 U_1^x))$. Therefore $\Phi(D_1)^x = \Phi(D_1^x) = \Phi(D_1)$, so that (4) holds.

We next prove (5) and (6). Set $\bar{S} := S/\Phi(D_1)$. Again suppose first that K_1 is an $L_2(2^n)$-block or A_3-block. Then $Aut_S(U_1)$ is the unique FF*-offender on U_1 and $|Aut_S(U_1)| = |U_1 : C_{U_1}(s)|$ for each $s \in S - Q_1$, so as $S = U_1^x Q_1$ with $Q_1 = D_1 U_1$ and $D_1 \leq C_{Q_1}(U_1^x)$ by (1), B.2.21 says that $\mathcal{A}(\bar{S}) = \{\bar{Q}_1, \bar{U}_1^x \bar{D}_1\}$ is of order 2. Thus $M_1 := N_{M_0}(\Phi(D_1))$ acts on $\Gamma := \{Q_1, U_1^x D_1\}$. By (4), $x \in M_1$, and as $U_1 \leq Q_1$ but $U_1^x \not\leq Q_1$, M_1 is transitive on Γ. Thus $|M_1 : N_{M_1}(Q_1)| = 2$, so x has even order and (5) holds.

In proving (6), we assume that $T_H \in Syl_2(M_0)$. If $s = 1$, then $T_H \leq M_1$, contradicting $|M_1 : N_{M_1}(Q_1)| = 2$. Thus $s = 2$, so $T_1 := N_{T_H}(U_1)$ is of index 2 in the Sylow subgroup T_H of M_0. But as $T_1 \leq N_{M_1}(Q_1)$ which is of index 2 in M_1, $T_1 = T_H \cap M_1$ is of index 2 in some $T_0 \in Syl_2(M_1)$, and $T_1 \in Syl_2(N_{M_1}(Q_1))$. We claim for $x_0 \in T_0 - T_1$ that $[U_1^{x_0}, U_1] \neq 1$: For $M_1 \leq M_0$ acts on S, and hence M_1 acts on $\bar{S}$. Also D_1 centralizes $U_1 U_1^x$ and $S = U_1 U_1^x D_1$ by (1) and (2), so as $\bar{D}_1$ is abelian, $\bar{D}_1 \leq Z(\bar{S})$. Then $\bar{D}_1^{x_0} \leq Z(\bar{S}) \leq \bar{Q}_1$ as $F^*(\bar{K}_1 \bar{S}) = O_2(\bar{K}_1 \bar{S})$. Since

$Q_1 = U_1D_1$ and $x_0 \notin N_{M_1}(Q_1)$, $U_1^{x_0} \not\leq Q_1 = C_S(U_1)$, as claimed. So replacing x by x_0, we may assume that $x \in N_{M_0}(T_1)$. Hence (6) holds in this case.

Therefore in proving (5) and (6), we may assume that K_1 is an A_5-block. This time the analysis of FF*-offenders using B.3.2.4 yields

$$\mathcal{A}(\bar{S}) = \{\bar{Q}_1, \bar{U}_1^x\bar{D}_1, \bar{A}_1, \bar{A}_1^r\},$$

where $\bar{A}_1/(\bar{A}_1 \cap \bar{Q}_1)$ is of order 2 and induces a transposition on K_1/U_1, and $r \in R \cap K_1 - S$. Thus $N_R(U_1)$ induces the transposition r on the 4-set $\mathcal{A}(\bar{S})$. If $s = 1$, then $R = N_R(U_1) = N_R(\Phi(D_1))$. If $s = 2$, then $S_2 := S \cap K_2 \leq D_1$, with $U_2 \cap \Phi(S_2) \neq 1$, while $D_1 \cap U_1 = 1$ by C.5.4.3, so $N_R(\Phi(D_1))$ cannot interchange K_1 and K_2, and hence $N_R(\Phi(D_1)) = N_R(U_1)$. Thus in either case, $N_R(U_1) = N_R(\Phi(D_1))$. But $R \trianglelefteq M_0$ so $N_R(\Phi(D_1)) \trianglelefteq M_1$. Then as $N_R(\Phi(D_1)) = N_R(U_1)$ induces only the transposition r on $\mathcal{A}(\bar{S})$, and $N_R(\Phi(D_1)) \trianglelefteq M_1$, it follows that M_1 acts on $\{Q_1, U_1^xD_1\}$. Then (5) and (6) follow just as above.

It remains to establish (7), so we may assume $|M_0 : T_H| = 2$ and set $D := C_S(K)$. Then $M_0 = T_H\langle x\rangle$. Since $T_H \leq H \leq N_G(O_2(H))$ and $U_1^x \not\leq O_2(H)$, we can in fact choose x to be any element of $M_0 - T_H$. Assume first that $s = 1$; then $D = D_1$ and $\Phi(D) = 1$ by (4) and Hypothesis C.5.2. Hence $\bar{S} = S$, so now

$$\mathcal{A}(S) = \{Q_1, U_1^xD_1\} \text{ or } \{Q_1, U_1^xD_1, A_1, A_1^r\}.$$

In particular, Q_1 is elementary abelian as $Q_1 \in \mathcal{A}(S)$. Notice also that $x \in N_G(R) \leq N_G(S)$, so $Q_1^x \in \mathcal{A}(S)$ and Q_1^x is R-invariant as $s = 1$, so $U_1^xD_1 = Q_1^x$. Therefore $\mathcal{A}(S) = \{Q_1, Q_1^x\}$ when K is not an A_5-block, and $\mathcal{A}(S) = \{Q_1, Q_1^x, A, A^r\}$ when K is an A_5-block. In particular $S = U_1^xQ_1 = \langle\mathcal{A}(S)\rangle = J(R)$. Finally $x^2 \in T_H \leq N_G(D)$, so both $M_0 = \langle x, T_H\rangle$ and H normalize $D \cap D^x$, and hence $D \cap D^x = 1$ by Hypothesis C.5.2. That is, (7) holds in this case—except possibly for the assertions that $U = O_2(K)$, $D = C_R(K)$, and $O_2(KR) = DU$, which we will return to at the end of the proof.

So we turn to the case $s = 2$. Let $t \in T_H - N_H(K_1)$. Then $|T_H : T_1| = 2$, with $T_1 := N_{T_H}(U_1)$ as earlier, so $|M_0 : N_{M_0}(U_1)| = 4$. Let I be the kernel of the action of M_0 on $U_1^{M_0}$. By the hypothesis for (7), M_0 induces a transitive 2-subgroup of S_4, so M_0/I is E_4, $\mathbf{Z}_4$, or D_8.

Recall we showed that (*) holds in all cases, so $|S : C_S(U_1^x)| = |U_1 : C_{U_1}(U_1^x)|$, and therefore U_1^x must centralize $U_2 = U_1^t$—since U_1^x acts on U_2, and by I.2.3.1, $[U_1, a] \cap C_{U_1}(K_1) = 1$ for $a \in U_1^x - Q_1$.

First assume that $M_0/I \cong D_8$. Then T_1/I fixes U_1 and U_2, but acts nontrivially on $U_1^{M_0}$, so there is also an element in M_0 fixing U_1^x and interchanging U_1 and U_2, contradicting $[U_1^x, U_1] \neq 1 = [U_1^x, U_2]$. Assume next $M_0/I \cong \mathbf{Z}_4$. Then $t, x^2 \in T_H - I$ so $tI = x^2I$ and hence in its action on U^{M_0}, $x = (U_1, U_1^x, U_2, U_2^x)$; so as $[U_1, U_2] = 1$, also $[U_1^x, U_2^x] = 1$. Therefore as $[U_1^x, U_2] = 1$, M_0 must permute the noncommuting pairs $\{U_1, U_1^x\}$ and $\{U_2, U_2^x\}$, whereas x does not.

Therefore $M_0/I \cong E_4$, and and M_0/I is regular on $U_1^{M_0}$ as $|U_1^{M_0}| = 4$. So as in the previous paragraph,

$$[U_1U_1^x, U_2U_2^x] = 1.$$

Recall the factorizations $S = U_iU_i^xC_S(U_iU_i^x) = U_iU_i^xD_i$ obtained after (**), and observe $C_S(U_iU_i^x) = E_iD_i$ by (1) and (2), for $i = 1, 2$; these factorizations give

$$S = U_1U_1^xU_2U_2^xC_S(U_1U_1^xU_2U_2^x) = U_1U_1^xU_2U_2^xD = UU^xD,$$

with $C_S(UU^x) = E_1E_2D$. Thus by symmetry,

$$E_1E_2D^x = C_S(UU^x) = E_1E_2D,$$

and hence

$$\Phi(D) = \Phi(C_S(UU^x)) = \Phi(D^x) = \Phi(D)^x.$$

So as $\Phi(D) \trianglelefteq H$ we conclude $\Phi(D) = 1$ from Hypothesis C.5.2. Similarly we get $D \cap D^x = 1$ just as in the case $s = 1$, using $x^2 \in N_G(D)$. Now it is easy to see that (7) holds, again modulo the same three assertions mentioned earlier, to which we now turn.

Thus to complete the proof of (7), we must show in each of the cases (i), (ii), and (iii) that $U = O_2(K)$, $O_2(KR) = DU$ and $C_R(K) \leq D$. We recall from C.5.4.4 that $C_R(U) = O_2(KR)$.

To show $U = O_2(K)$, we must show $U_1 = O_2(K_1)$. By C.1.8, K_1/U_1 is quasisimple or of order 3, and in the latter case $U_1 = O_2(K_1)$. If K is an $L_2(2^n)$-block, then $U_1 = O_2(K_1)$ by Gaschütz's Theorem A.1.39 since Q_1 is abelian and U_1^x contains a complement to Q_1 in $S \in Syl_2(K_1Q_1)$. Finally if K_1 is an A_5-block, then by C.1.13.c, K_1/U_1 is A_5 or $SL_2(5)$, and we may assume the latter. But as $S = U_1^xQ_1$, there is an involution $t \in U_1^x$ inducing a nontrivial inner automorphism on $K_1/O_2(K_1)$, and as $\Phi(\langle j, D_1\rangle) = 1$, $jD_1 \cap K_1$ contains an involution, contradicting $K_1/U_1 \cong SL_2(5)$.

Next S/Q_1 is self-centralizing in $Aut(K_1)$, so $C_R(U^x) \leq U^xO_2(KR)$. If K_1 is an A_3-block or an A_5-block, then $O_2(KR) = U \times C_R(K)$ by C.1.13.c. On the other hand if K_1 is an $L_2(2^n)$-block, then from the action of R on the module U and the description of $U_1 \cap U_1^x$ in (2), we have

$$C_R(U \cap U^x) = U^xC_R(U) \ \text{ and } \ |C_R(U \cap U^x) : C_R(U)| = |U : U \cap U^x|.$$

Then as x normalizes $U \cap U^x$ and R, also

$$C_R(U \cap U^x) = UC_R(U^x) \ \text{ and } \ |C_R(U \cap U^x) : C_R(U^x)| = |U^x : U \cap U^x|.$$

Therefore $U \leq C_R(U) \leq UC_R(U^x)$, so $C_R(U) = UC_R(UU^x)$. Then because $U^xO_2(KR)/O_2(KR)$ is Sylow in $KO_2(KR)/O_2(KR)$, we conclude $UC_R(K) = C_R(U) = O_2(KR)$ by Gaschütz's Theorem A.1.39.

So in either case, $C_R(U) = O_2(KR) = UC_R(K)$, with $C_R(U^x) \leq U^xO_2(KR)$. Therefore it suffices to show $C_R(K) \leq S$: for if so, $C_R(K) = C_S(K) = D$ by its definition, so

$$O_2(KR) = C_R(K)U = DU,$$

as desired.

In fact, it will suffice to show $C_R(U^x) = C_R(K)U^x$: For this implies that

$$\Phi(C_R(U)) = \Phi(UC_R(K)) = \Phi(C_R(K)) = \Phi(C_R(U^x)) = \Phi(C_R(U))^x,$$

and hence $\Phi(C_R(K)) = 1$ by Hypothesis C.5.2. Then $C_R(K)$ is abelian, so it centralizes $C_S(K) = D$, and hence as $U \leq K$ and (by assumption) $C_R(K) \leq C_R(U^x)$, also $C_R(K)$ centralizes $UU^xD = S$ using (2). But $S = C_R(\Omega_1(Z(J(R))))$ by the definition of the Baumann subgroup, so that S contains $\Omega_1(Z(J(R)))$, and hence $C_R(K) \leq S$, as desired.

To prove the sufficient condition $C_R(U^x) = C_R(K)U^x$, notice that $P := O_2(KR) = C_R(U)$ normalizes U^x as $U \trianglelefteq R$ and x acts on R. So as $Aut_U(U^x)$ is self-centralizing in $Aut_R(U^x)$ in each of our three cases, it follows that $P = UC_P(U^x)$.

Therefore $[P, u^x] = [U, u^x]$ for each $u \in U^\#$, and $[U, u^x] \cap C_U(K) = 1$, so

$$[C_R(K), u^x] \leq C_R(K) \cap [P, u^x] = C_R(K) \cap [U, u^x] = 1,$$

and hence $C_R(K) \leq C_R(U^x)$. But we showed earlier that $C_R(U^x) \leq U^x P$, so as $C_U(U^x) = U \cap U^x$ and recalling $P = C_R(U) = C_R(K)U$,

$$C_R(U^x) = U^x C_P(U^x) = U^x C_{UC_R(K)}(U^x) = U^x C_R(K) C_U(U^x) = U^x C_R(K),$$

completing the proof of (7) and hence of C.5.6. □

There is a special case of C.5.6 which we will encounter from time to time:

LEMMA C.5.7. *Assume Hypotheses C.5.1 and C.5.2. Assume further that* $s = 1$ *and* $N_{M_0}(K)$ *is a maximal subgroup of* M_0*. Then* $|M_0 : N_{M_0}(K)| = 2$*,* $\Phi(D_1) = 1$*,* $S = Q_1 Q_1^x$ *for* $x \in M_0 - N_{M_0}(K)$*, and either*

(1) K *is an* $L_2(2^n)$*-block or* A_3*-block and* $\mathcal{A}(S) = \{Q_1, Q_1^x\}$*, or*

(2) K *is an* A_5*-block and* $\mathcal{A}(R) = \{Q_1, Q_1^x, A, A^r\}$ *for* $r \in R \cap K - S$*.*

PROOF. As $N_{M_0}(K)$ normalizes Q_1 but $U_1^x \not\leq Q_1$, $x \notin N_{M_0}(K)$. Thus by maximality of $N_{M_0}(K)$, $M_0 = \langle N_{M_0}(K), x\rangle$, so no nontrivial subgroup of R is invariant under K, $N_{M_0}(K)$, and x in view of C.5.2. However as $s = 1$ by hypothesis, $\Phi(D_1)$ is normalized by K and $N_{M_0}(K)$, and also by x in view of C.5.6.4. Therefore $\Phi(D_1) = 1$. Now as we saw during the proof of C.5.6.5, either (1) or (2) holds, with $S = Q_1 Q_1^x$ by C.5.6.2. In either case $\{Q_1, Q_1^x\}$ is the set of R-invariant members of $\mathcal{A}(S)$; then $M := N_{M_0}(Q_1)$ is of index 2 in M_0, as $x \notin M$. As $N_{M_0}(K) \leq N_{M_0}(Q_1) = M < M_0$, from maximality of $N_{M_0}(K)$ we conclude $N_{M_0}(K) = M$. □

We come to the main result of this section. It is stated under the hypothesis of our Main Theorem, which we ordinarily avoid in Volume I; however it seems most natural to place it here, since the proof of the theorem continues to use the notation and point of view developed in this section. The proof will also make use of the machinery developed in chapter 1 from the proof of the Main Theorem.

THEOREM C.5.8. *Assume Hypotheses C.5.1, and in addition assume* $T_H = T \in Syl_2(G)$ *and* G *is a simple QTKE-group. Then there is* $1 \neq R_0 \leq R$ *with* $R_0 \trianglelefteq \langle M_0, H\rangle$*.*

Until the proof of Theorem C.5.8 is complete, assume we are working in a counterexample to that Theorem. We begin a short series of reductions.

As we are in a counterexample to Theorem C.5.8, Hypothesis C.5.2 is satisfied, and we can appeal to C.5.3 through C.5.6. In particular $K = K_1 \cdots K_s$ as in C.5.3 and we adopt the rest of the notation established just after C.5.3 and in C.5.6, especially C.5.6.7.

By the hypothesis of C.5.8, we have $T_H = T$ Sylow in G, and hence as $T_H \leq M_0$ by C.5.1.2, T is also Sylow in M_0. In particular we may apply C.5.6.6 to see that

$$s = 2,$$

so that $T_1 := N_T(U_1)$ is of index 2 in T. By C.5.6.6 we may also choose $x \in N_{M_0}(T_1)$ with $x^2 \in T_1$. Thus as T_1 is of index 2 in $T \in Syl_2(G)$, T and $T_0 := T_1\langle x\rangle$ are Sylow in $Y := N_G(T_1)$. Notice as $|T : T_1| = 2$, $|Y : O^2(Y)T_1| = 2$.

LEMMA C.5.9. *Hypotheses C.5.1 and C.5.2 are satisfied by the quadruple* $\gamma := (Y, T_1, T_1, K_1T_1)$ *in the role of "*(M_0, R, T_H, H)*".*

PROOF. As H satisfies hypothesis C.5.1.1, $H \in \mathcal{H}^e$, in the language of chapter 1, so $K_1T_1 \in \mathcal{H}^e$ as $O_2(H) \leq T_1$ and hence $O_2(H) \leq O_2(K_1T_1)$. In particular K_1T_1 is an SQTK-group as G is a QTKE-group. By construction T_1 is Sylow in K_1T_1. Finally if T_1 were in distinct maximal subgroups of K_1T_1, then adjoining $t \in T \backslash T_1$ would give distinct maximal subgroups over T in $K_1K_1^tT = H$, contradicting C.5.1.1 for H. Thus we have verified C.5.1.1 for our quadruple γ.

Next $T_1 \trianglelefteq Y$ by construction, and $K_1 = O^2(K_1T_1)$ plays the role of "K" in γ, so our earlier observation that T_1 is Sylow in K_1T_1 completes the verification of C.5.1.2 for γ.

Finally suppose $1 \neq R_0 \leq T_1$, with $R_0 \trianglelefteq H_0 := \langle Y, K_1T_1\rangle$. Notice that $t \in T \leq Y$, so that $H_0 \geq \langle K_1T_1, t\rangle = H$. Further as $R_0 \neq 1$, $H_0 \in \mathcal{H}$, so $H_0 \in \mathcal{H}^e$ by 1.1.4.6. Then by B.2.14, $\langle \Omega_1(Z(T))^{H_0}\rangle \leq \Omega_1(Z(O_2(H_0)))$. From C.5.3 there is $u \in U \cap \Omega_1(Z(T))$ with $U = \langle u^H\rangle$, so $U \leq O_2(H_0)$. But $x \in Y \leq H_0$, so $U_1^x \leq O_2(H_0) \cap H \leq O_2(H)$ as $H \leq H_0$, contrary to the original choice of x. So we have also verified C.5.2 for γ. □

Because of C.5.9, we can apply C.5.6 to γ. Now K_1 plays the role of K in γ, so the number "s" for γ is 1. Applying C.5.5 to (M_0, R, T_H, H) gave us an x with $[U_1^x, U_1] \neq 1$, and this same x works for γ. Recall $|T : T_1| = 2$, so that Y/T_1 is dihedral of order $2m$, for some odd integer m. It follows that $T_1O^2(Y) = T_1Y_0$ where Y_0 is a Hall $2'$-subgroup of $O^2(Y)$ of order m. Since x has even order by C.5.6.5, $U_1^{Y_0} \subseteq C_S(U_1) = Q_1$, so

$$V := \langle U_1^{T_1O^2(Y)}\rangle = \langle U_1^{Y_0}\rangle \leq Q_1.$$

Then as $Q_1 = U_1D_1$ by C.5.6.1,

$$V \trianglelefteq X := \langle O^2(Y), K_1T_1\rangle.$$

As $x \in T_0 - T_1$, $t \in T - T_1$, and Y/T_1 is dihedral of order $2m$, $xt \in O^2(Y)T_1$ and

$$V \geq [U_1^{xt}, U_2] = [U_1^x, U_1]^t = E_1^t = E_2,$$

using C.5.6.2. So V contains the diagonal involutions of E_1E_2 central in T, and hence by 1.1.4.3, $V \in \mathcal{S}_2^e(G)$ so that $N_G(V) \in \mathcal{H}^e$. Further $X \leq N_G(V)$, and $T_1 \leq X$ with $|T : T_1| = 2$, so by 1.1.4.7 we also have $X \in \mathcal{H}^e$. So we have shown:

LEMMA C.5.10. *$X \leq N_G(V)$, $V \leq Q_1$, and $X \in \mathcal{H}^e$.*

LEMMA C.5.11. *$T_1 \in Syl_2(X)$, so $x, t \notin X$.*

PROOF. If not, then T_1 is proper in a Sylow 2-group Z of $N_X(T_1) = X \cap Y$, and Z normalizes $O^2(Y)T_1$. Recall however that T_1 is Sylow in $O^2(Y)T_1 \leq X \cap Y$. Thus $O^2(Y)T_1 < O^2(Y)Z \leq X \cap Y \leq Y$. But $O^2(Y)T_1$ has index 2 in $Y = \langle x\rangle O^2(Y)T_1$, so $X \cap Y = Y$. Thus $x \in Y = X \cap Y \leq X$. But then as $V \trianglelefteq X$, $U_1^x \leq V \leq Q_1$ using C.5.10, contradicting the choice of x. □

To complete the proof of Theorem C.5.8 we will show:

LEMMA C.5.12. *$N_X(T_1) \leq N_X(U_1)$.*

We first check that this lemma is sufficient to complete the proof of the Theorem: By C.5.11 $x \in Y - X$, so

$$N_X(T_1) = X \cap Y = T_1O^2(Y) = (T_1O^2(Y))^t,$$

which by C.5.12 would normalize $U_1^t = U_2$ and hence also normalize $U = U_1U_2$; then $Y = O^2(Y)T$ acts on U. But now as $x \in Y$, $U_1^x \leq U \leq C_T(U_1)$, contradicting the choice of x.

Thus it remains to establish lemma C.5.12. Let $P_1 := O_2(T_1K_1)$. We first show:

LEMMA C.5.13. *Let* $X_1 := N_X(P_1)$ *and* $L := \langle K_1^{X_1} \rangle$. *Then* $N_{X_1}(T_1)$ *normalizes* U_1, *and either*

(1) $L = K_1$ *(that is,* $X_1 \leq N_X(K_1)$*), or*

(2) K_1 *is a block of type* A_5 *or* $L_2(4)$, *and* L *is an* A_7*-block or exceptional* A_7*-block, respectively.*

PROOF. Before addressing the conclusions of the lemma, we establish a few preliminaries. First T_1 is Sylow in X_1 by C.5.11. By C.5.10, $X \in \mathcal{H}^e$, so $X_1 \in \mathcal{H}^e$ using 1.1.4 and our QTKE-hypothesis. Also $U_1 \leq \langle \Omega_1(Z(T_1))^{K_1} \rangle$ from C.5.3, and I.2.3.1, so as $K_1 \leq X_1$, $U_1 \leq V_1 := \langle \Omega_1(Z(T_1))^{X_1} \rangle$, and $V_1 \in \mathcal{R}_2(X_1)$ by B.2.14. Now $P_1 = C_{T_1}(U_1)$ by C.5.4.4 with T_1 in the role of "R", and $P_1 \trianglelefteq X_1$, so $P_1 \leq C_{T_1}(V_1)$. Therefore as $U_1 \leq V_1$, $P_1 = C_{T_1}(V_1)$ is Sylow in $C_{X_1}(V_1)$. Let $X_1^* := X_1/C_{X_1}(V_1)$.

Recall from C.5.6.2 that

$$C_{U_1^x}(U_1) = U_1 \cap U_1^x = E_1 \leq C_{T_1}(U_1) = P_1 = C_{T_1}(V_1),$$

so $E_1 = C_{U_1^x}(V_1) = U_1^x \cap V_1$, and so $m(U_1^{x*}) = n$. As x normalizes T_1, we also have $U_1^x \trianglelefteq T_1$, so that $[U_1^x, V_1] \leq U_1^x \cap V_1 = E_1$, while $E_1 = [U_1^x, U_1]$ by C.5.6.2, so $[U_1^x, V_1] = E_1$. Then $m([V_1, U_1^x]) = m(E_1)$; and either $m(E_1) = n = m(U_1^{x*})$, or K_1 is an $L_2(2^n)$-block with $m(E_1) = n + m(C_{U_1}(K_1)) \leq 2n$. Therefore by B.4.7, either V_1 is a dual FF-module for U_1^{x*}, or K_1 is an $L_2(2^n)$-block and at least $q(X_1^*, W) \leq 2$, where W is the dual of V_1. In particular if K_1 is an A_3-block, then U_1^{x*} induces a transvection on V_1. Further T_1^* acts faithfully on K_1^*, as $P_1 = O_2(K_1T_1)$. We will make use of these facts a little later.

Now, with the preliminaries in place, we begin to establish that one of the two conclusions of the lemma holds. Notice first that if $L = K_1$ as in case (1), then X_1 acts on $[O_2(K_1), K_1] = U_1$, so the lemma holds. Thus we may assume that $K_1 < L$. Similarly if (2) holds, then in either of the cases listed there, the Sylow 2-subgroups of L are self-normalizing, so we have

$$N_{X_1}(T_1) \leq C_{X_1}(L)T_1 \leq N_{X_1}(K_1) \leq N_G(U_1),$$

again completing the proof. So it remains to show that (2) holds under our assumption that $K_1 < L$.

Suppose first that K_1 is an A_3-block. Then the Sylow group $T_1^* = T_1/P_1$ of X_1^* is of order 2, so by Thompson Transfer, $L^* = O(L^*)$ has odd order. As we just observed, $T_1^* = U_1^{x*}$ induces a transvection on V_1, so we conclude from Theorem B.5.6 that $L^* = K_1^*$ is of order 3; thus X_1 normalizes $O^2(L) = K_1$, contradicting $K_1 < L$.

Therefore K_1 is a block of type $L_2(2^n)$ or A_5, so $K_1 \in \mathcal{L}(X_1, T_1)$. Set $n := 2$ when K_1 is an A_5-block. By 1.2.4, $L \in \mathcal{C}(X_1)$ and the embedding of K_1 in L is described in A.3.12. Recall that $O_2(K_1) \leq P_1 \leq C_G(V_1)$, so that T_1^* acts faithfully on a subgroup $K_1^* \cong L_2(2^n)$ of L^*. We conclude from A.3.12 that $n = 2$ and $L^* \cong A_7$, $\hat{A}_7$, J_1, $L_2(25)$, or $L_2(p)$ for p an odd prime with $p \equiv \pm 1 \mod 5$. Let

$I := [V_1, L]$. As $U_1 = [P_1, K_1] \in Irr_+(K_1, V_1)$, $U_1 = [V_1, K_1] \leq I$, so $I = [P_1, L]$. Thus $I = [O_2(L), L]$. Then as we showed earlier that $q(L^*, W_I) \leq 2$, where W_I is the dual of I, it follows from B.4.2 and B.4.5 that $L^* \cong A_7$, and I is of dimension 4 or 6. Further L is a block with $I = U(L)$, so this completes the proof of (2), and hence of C.5.13. □

Finally we turn to the proof of C.5.12. First $X \in \mathcal{H}^e$ by C.5.10. As $V = \langle U_1^{O^2(Y)} \rangle$ with $O^2(Y) \leq X$, and X normalizes V by C.5.10, $V = \langle U_1^X \rangle$. Arguing as in the proof of C.5.13, $V \in \mathcal{R}_2(X)$. Set $X^* := X/C_X(V)$. As $U_1 \leq V$, $C_{T_1}(V) \leq C_{T_1}(U_1) = P_1$, so that $C_{T_1}(V) = C_{P_1}(V)$.

Suppose first that $P_1 \leq C_X(V)$, so that now $P_1 = C_{T_1}(V) \in Syl_2(C_X(V))$. Let $N := N_X(T_1)$. Then $P_1 \leq T_1 \leq N$, so also $P_1 \in Syl_2(C_N(V))$. Thus by a Frattini Argument, $N = C_N(V)N_N(P_1)$. Certainly $C_N(V) \leq C_N(U_1)$, while by C.5.13, $N_N(P_1) = N_{X_1}(T_1)$ normalizes U_1, so C.5.12 holds by the Frattini factorization above.

Thus we may assume that $P_1 \not\leq C_X(V)$, so that $P_1^* \neq 1$. Recalling $C_{P_1}(V) = C_{T_1}(V)$, a Frattini Argument gives

$$N_{X^*}(P_1^*) = N_X(P_1)^* = X_1^*,$$

with $L^* \trianglelefteq X_1^*$ from the definition of L in C.5.13. This puts us in a position to apply (4.4) or (5.1) of [**Asc81a**]; but we supply our own argument here to keep the treatment self contained.

If $L^* \trianglelefteq X^*$, then $P_1^* = C_{T^*}(L^*) \trianglelefteq N_{X^*}(T_1^*) = N_{X_1}(T_1)^*$, so C.5.12 follows from C.5.13. Thus we may assume that L^* is not normal in X^*, and similarly that $N_{X^*}(T_1^*)$ does not act on P_1^*.

Assume that L is not an A_3-block. Then $L \in \mathcal{L}(X, T_1)$ is T_1-invariant and $T_1 \in Syl_2(X)$ by C.5.11, so by 1.2.4, $L \leq L_X \in \mathcal{C}(X)$ with $L_X \trianglelefteq X$, and the embedding of L in L_X is described in A.3.12. As L^* is not normal in X^*, $L^* < L_X^*$. As L is irreducible on $U(L) = \langle U_1^L \rangle \leq V$, L_X has a unique noncentral 2-chief factor $U(L_X)$ and $U(L_X) \leq V$. Then as $L_X \trianglelefteq X$, $V = \langle U_1^{O^2(Y)} \rangle = U(L_X)$. If $L_X/O_2(L_X)$ is not quasisimple, then as $L/O_2(L)$ is $L_2(2^n)$ for some n, case (21) of A.3.12 holds with $L/O_{2,F}(L_X) \cong SL_2(5)$. So for suitable $g \in O_{2,F}(L_X)$, $D/O_2(D)$ is not quasisimple, where $D := \langle L, L^g \rangle$. This is impossible as $U' := U(L) + U(L^g)$ is of rank at most 8 and $C_D(U') \leq O_2(D)$, whereas $L_8(2)$ has no such subgroup. Thus L_X is a block. Next L^* is a component of $C_{L_X}(P_1^*)$ so by inspection of the embeddings listed in A.3.12, (L^*, L_X^*) is $(L_2(5), L_2(25))$, $(SL_2(5), L_3^\epsilon(5))$, (A_5, A_7), or (A_5, J_1). Except in the last case, we conclude from the structure of $Aut(L_X^*)$ that $N_{X^*}(T_1^*)$ acts on P_1^*, contrary to an earlier remark. In the last case, L_X^* is generated by two conjugates of L^*, whereas J_1 is not a subgroup of $L_8(2)$, contrary to an earlier argument.

Therefore we may assume that L is an A_3-block. An argument in the proof of C.5.13 shows that U_1^{x*} is generated by a transvection on V inverting L^*. It follows from G.6.4 that $L^* U_1^{x*} \leq L_H^* \leq X^*$ with $L_H^* \cong L_n(2)$ or S_n. As T_1^* acts on $L^* \cong \mathbf{Z}_3$, it follows that either $L_H^* \cong S_3$, or S_7 is normal in X^*. But now $N_X(T_1^*) \leq N_X(U_1)$, completing the proof of lemma C.5.12, and hence also the proof of Theorem C.5.8.

C.6. Some further pushing up theorems

In this section we provide some more detailed results on pushing up, to be used for example in section 4.1 of the proof of our Main Theorem.

We begin the section with a proof of the following result, which may be regarded as a specific subcase of the situation analyzed in C.5.6:

THEOREM C.6.1. *Assume H and Λ are subgroups of a group G such that:*

(i) $F^*(H) = O_2(H)$ and $O^2(H/O_2(H)) \cong L_2(2^n)$ or $H/O_2(H) \cong S_3$ or S_3 wr $\mathbf{Z}_2$.

(ii) Λ is a 2-group and $T_H := \Lambda \cap H \in Syl_2(H)$.

(iii) $T_H < \Lambda$.

(iv) If $1 \neq T_0 \leq T_H$ with $T_0 \trianglelefteq H$, then $N_\Lambda(T_0) = T_H$.

Let $Q := O_2(H)$, $L := O^2(H)$, and $\Sigma := N_\Lambda(T_H)$. Then

(1) $|\Sigma : T_H| = 2$ and $J(T_H) = Baum(T_H) = QQ^x$ for each $x \in \Sigma - T_H$.

(2) L is an $L_2(2^n)$-block or A_5-block or a product of $s \leq 2$ A_3-blocks.

(3) Assume that L is an $L_2(2^n)$-block or an A_3-block. Then $\mathcal{A}(T_H) = \{Q, Q^x\}$ and $m(C_{T_H}(L)) \leq n$, with $n := 1$ if L is an A_3-block.

(4) Assume that L is an A_5-block or a product of two A_3-blocks. Then $\mathcal{A}(T_H) = \{Q, Q^x, A_1, A_1^r\}$, where $|A_1 : (A_1 \cap Q)| = 2$ and $r \in T_H - J(T_H)$.

(5) $Q = C_{T_H}(L)O_2(L)$.

(6) One of the following holds:

(a) $|\Lambda : T_H| = 2$ and $J(T_H) = J(\Lambda)$. If L is an A_3-block, then $H \cong S_4 \times \mathbf{Z}_2$.

(b) L is an A_3-block, $H \cong S_4$, and Λ is dihedral or semidihedral.

(c) L is an A_5-block or the product of two A_3-blocks, with

$$J(T_H) = J(\Sigma), \ N_\Lambda(\Sigma)/J(T_H) \cong D_8, \quad \text{and} \quad |N_\Lambda(A_1Q)| \geq |T_H|.$$

PROOF. Set $R := T_H$, $S := \text{Baum}(R)$, and $D := C_S(L)$. By (iii) there is $M_0 \leq \Lambda$ with $|M_0 : R| = 2$; in particular, $R \trianglelefteq M_0$. Observe that Hypothesis C.5.1 is satisfied by the quadruple (M_0, H, R, T_H) with L, H playing the roles of "K, KR". Indeed by (iv), Hypothesis C.5.2 is also satisfied. Thus we can appeal to C.5.3 to see that L is a product of blocks—and hence (2) holds by our restrictions in (i). Let $U := U(L)$ and $D := C_S(L)$. By C.5.5 there is $x \in M_0 - R$ with $U^x \not\leq Q$. Further by C.5.6.7, $\Phi(D) = 1 = D \cap D^x$, $Q = DU$, $S = QU^x = J(R)$, and $D = C_R(L)$. Thus Q is elementary and $S = QQ^x$. Again by C.5.6.7:

(α) if L is an $L_2(2^n)$-block or an A_3-block, then $\mathcal{A}(R) = \{Q, Q^x\}$, while

(β) if L is either an A_5-block or the product of two A_3-blocks, then $\mathcal{A}(R) = \{Q, Q^x, A_1, A_1^r\}$ where $|A_1 : A_1 \cap Q| = 2$ and $r \in (R \cap L) - S$.

In particular (1)–(5) hold—modulo showing that $|\Sigma : R| = 2$ to finish (1), and that $m(D) \leq n$ in case (α) to finish (3). However in (α), using I.2.3.1 $Z(S) = D \times Z$, where $Z := [U, U^x]$ is of rank n. Thus as $D \cap D^x = 1$, while $DD^x \leq Z(S)$ since x normalizes R, we do get $m(D) \leq m(Z) = n$, finishing (3).

We first consider the case where L is an A_3-block, and further $Q = U$. Then $H \cong S_4$. Also by (iv), $U = C_\Lambda(U)$; thus as $U \cong E_4$, Λ is dihedral or semidihedral, by a lemma of Suzuki (cf. Exercise 8.6 in [**Asc86a**]). Therefore (6b) holds. But also since Λ is dihedral or semidihedral, $R = S \cong D_8$ and $|\Sigma : R| = 2$ completing the proof of (1); so we are done in this case.

So if L is an A_3-block, we may assume that $U < Q$. Hence as $m(D) \leq n = 1$ by (3), we have:

$$H = D \times LU^x \cong \mathbf{Z}_2 \times S_4,$$

establishing at least the final statement of (6a).

Suppose we can show that:

(A) $M_0 = \Lambda$, and
(B) $S = J(\Lambda)$.

Then (A) forces $M_0 = \Sigma$, completing the proof of (1) and the first assertion of (6a), while (B) establishes the second assertion of (6a), and we established the third assertion of (6a) above. So if we prove (A) and (B), we will have established (1)–(5) and shown (6a) holds. If either (A) or (B) fails, we'll see later that (1)–(5) and (6c) hold.

Next assume that $S \neq J(M_0)$. Then there is $A \in \mathcal{A}(M_0) - \mathcal{A}(R)$. Let $B := A \cap S$. As R/S is cyclic in (α) and (β), while $m(M_0/R) = 1$, $m(A/B) \leq 2$, and $m(A/B) = 1$ if $H \cong \mathbf{Z}_2 \times S_4$. Define $m := n$ in case (α) and $m := 2$ in case (β). In particular either $H \cong \mathbf{Z}_2 \times S_4$ and $m = 1$, or $m \geq 2$. Thus in any case, $m(A/B) \leq m$. Now $R = N_{M_0}(Q)$ by (iv), while $M_0 = R\langle x \rangle$ and $U^x \not\leq Q$, so $U^a \not\leq Q$ for $a \in A - R$, and hence we may take our element x to lie in A. Then as x interchanges the factors Q, Q^x of S, $B \leq C_S(x) \leq Q \cap Q^x$. Now as $S = QQ^x$, $m(Q/(Q \cap Q^x)) = m(S/Q) \geq m \geq m(A/B)$. Thus as $m(A) \geq m(Q)$ from the definition of $\mathcal{A}(M_0)$, we conclude $B = Q \cap Q^x$, $m = m(A/B) \leq 2$, and either $A = \langle x, t, B \rangle$ for some $t \in R - S$, or $H \cong \mathbf{Z}_2 \times S_4$ with $A = \langle x, B \rangle$. The latter is impossible, as x centralizes B and $1 \neq D \leq B$, whereas $D \cap D^x = 1$. The former is impossible as $S = C_R(Q \cap Q^x)$ (as we observed during the proof of C.5.6) so $[t, Q \cap Q^x] \neq 1$, contradicting A abelian.

This contradiction shows that $J(M_0) \leq R$ and hence $S = J(R) = J(M_0)$. Hence $\Delta := N_\Lambda(M_0)$ normalizes S and permutes $\mathcal{A}(R)$.

Assume that case (α) holds. Then M_0 is transitive on $\mathcal{A}(R) = \{Q, Q^x\}$, so by a Frattini Argument

$$\Delta = M_0 N_\Delta(Q) = M_0$$

since by (iv), $N_\Delta(Q) = R \leq M_0$. Therefore in fact $\Lambda = M_0$, establishing (A), and also (B), since we showed $S = J(M_0)$ in the previous paragraph. Therefore the treatment of case (α) is complete, since we showed earlier that it suffices to establish (A) and (B) to show that (1)–(5) and (6a) hold.

Therefore we may assume that case (β) holds. Here by (4), $\mathcal{A}(R)$ is of order 4, S is the kernel of the action of M_0 on $\mathcal{A}(R)$, and $M_0/S \cong E_4$ has two orbits of length 2 on $\mathcal{A}(R)$. If Δ preserves those orbits, then by a Frattini Argument as in the previous paragraph we obtain $\Lambda = M_0$—again giving (A) and (B), and completing the proof in this case.

Hence we may assume that Δ induces D_8 on $\mathcal{A}(R)$, with $S = J(R)$ the kernel of that action. But notice that Σ and Δ lie in $\Gamma := N_\Lambda(S)$, and Γ also permutes the 4-set $\mathcal{A}(R) = \mathcal{A}(S)$—again with kernel S. As the 2-subgroup Γ/S of S_4 has order at most 8, we have $M_0 < \Delta = \Gamma \geq \Sigma$. Further $M_0 = N_\Gamma(R)$ as R induces a transposition on $\mathcal{A}(R)$, so $\Sigma = M_0$. This gives $|\Sigma : R| = 2$ to complete the proof of (1), and also established the first two assertions of (6c), as we showed $J(R) = J(M_0)$ earlier, and now $\Delta = N_\Lambda(\Sigma)$. Thus it remains to verify the last assertion of (6c).

As Δ permutes the pairs $\{Q, Q^x\}$, $\{A_1, A_1^r\}$, there is $y \in \Delta$ with cycles (Q, A_1) and (Q^x, A_1^r) on $\mathcal{A}(R)$. Now y acts on A_1Q since y has cycle (Q, A_1). Also S acts on Q and A_1, and y acts on S, so

$$|N_\Lambda(A_1Q)| \geq |S\langle y\rangle| = 2|S| = |T_H|,$$

so that (6c) holds, completing the proof of Theorem C.6.1. □

In the remainder of the section, we establish a few more odds and ends involving pushing up. Roughly speaking, the first result deals with the case where we wish to push up a subgroup H with a block L, and the second specializes to the case where L is an A_7-block. We will assume:

HYPOTHESIS C.6.2. *L, R, T_H, Λ are finite subgroups of a group G such that*

(1) R acts on $L = O^2(L)$ with $O_2(LR) \leq R$ and $U := [O_2(L), L] \neq 1$ is elementary abelian.

(2) T_H is a 2-group with $R \trianglelefteq T_H$, T_H normalizes Λ, and L is subnormal in $\langle L, T_H\rangle =: H$.

(3) If $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq H$, then $N_\Lambda(R) \not\leq N_\Lambda(R_0)$.

In addition set $X := N_\Lambda(R)$, $Q := O_2(LR)$, $D := C_R(L)$, and $\overline{RL} := RL/D$.

Notice that R is not normal in H, or else we would obtain a contradiction from C.6.2.3 with $R = R_0$; so in particular $Q < R$.

LEMMA C.6.3. *Assume Hypothesis C.6.2. Then*

(1) There exists $x \in XT_H$ with $U^x \not\leq Q$.

(2) If $C_R(U) \leq Q$ then $m(U^x/C_{U^x}(U)) \geq m(R, U)$, where

$$m(R, U) := \min\{m(U/A) : A \leq U \text{ and } C_R(A) > C_R(U)\}.$$

(3) Assume

(a) $Q = C_R(U)$ and $\Phi(\bar{Q}) = 1$, and

(b) $R/Q \cong E_{2^n}$, where $n := m(R, U)$.

Then for $x \in XT_H$ with $U^x \not\leq Q$,

(i) $R = UU^xC_R(UU^x) = U^xQ$ and $Q = UC_Q(U^x)$.

(ii) If $C_{\bar{Q}}(\bar{R}) \leq \bar{U}$, then $\bar{U} = \bar{Q}$.

(iii) If in addition $|X : T_H \cap \Lambda| = 2$ and T_H normalizes L, then $\mathcal{A}(R) = \{Q, Q^x\}$, $R = QQ^x$, and $D \cap D^x = 1 = \Phi(D)$.

PROOF. By C.6.2.2, T_H normalizes R and Λ, and hence also X; then $XT_H \leq N_G(R)$. Let $U_0 := \langle U^{XT_H}\rangle$; as $U \leq O_2(L) \leq R \trianglelefteq XT_H$, $U_0 \leq R$. Next $[Q, L] \leq O_2(L)$, and hence $[Q, L] = [Q, L, L] \leq U$. Thus if $U_0 \leq Q$, then $[L, U_0] \leq U \leq U_0$, so $U_0 \trianglelefteq \langle L, T_H\rangle = H$ by C.6.2.2; since $N_\Lambda(R) = X$ normalizes U_0, this contradicts C.6.2.3. Thus $U_0 \not\leq Q$, and hence (1) holds.

Assume the hypotheses of (2) and set $\widetilde{LR} := LR/C_R(U)$. We make some arguments resembling those in the proof of C.5.6. Recall from the previous paragraph that $U^x \leq R$, so that $C_{U^x}(U) = U^x \cap C_R(U)$, and hence $\tilde{U}^x \cong U^x/C_{U^x}(U)$. Recall also that $x \in N_G(R)$, so $m(R, U) = m(R, U^x)$. Thus if $m(\tilde{U}^x) < n := m(R, U)$, then $C_R(C_{U^x}(U)) = C_R(U^x)$, so $U \leq C_R(C_{U^x}(U)) = C_R(U^x)$. Thus $U^x \leq C_R(U) \leq Q$, using the hypothesis of (2), contradicting our choice of x in (1). This shows $m(U^x/C_{U^x}(U)) \geq n$, proving (2).

Now assume the hypotheses (a) and (b) of (3). By (a), $C_{U^x}(U) = U^x \cap Q$, so (2) and (b) give

$$m(U^x/(U^x \cap Q)) \geq m(R,U) = n.$$

Then using (a) and (b), $|R : C_R(U)| = |R : Q| = 2^n$ and $R = U^x Q$. Now as $x \in N_G(R)$ we have

$$|R : C_R(U^x)| = |R : C_R(U)| = 2^n, \qquad (*)$$

while by (2) $m(U/C_U(U^x)) \geq n$, so $m(U/C_U(U^x)) = n$ and so $R = UC_R(U^x)$. Then as $Q = C_R(U)$,

$$Q = C_{UC_R(U^x)}(U) = UC_{C_R(U^x)}(U) = UC_R(UU^x) = UC_Q(U^x), \qquad (**)$$

using the Dedekind Modular Law. But we showed $R = U^x Q$ earlier, and $\bar{Q}$ is abelian by (a), so $C_{\bar{Q}}(\bar{U}^x) = C_{\bar{Q}}(\bar{R})$. If $C_{\bar{Q}}(\bar{R}) \leq \bar{U}$, then $\bar{Q} = \bar{U}C_{\bar{Q}}(\bar{U}^x) = \bar{U}$ by (**), establishing (ii). Further $R = U^x Q = UU^x C_R(UU^x)$ by (**), establishing (i).

Finally assume the hypotheses of (iii). Recall T_H acts on R and X, and $X = N_\Lambda(R)$, so $T_H \cap \Lambda = T_H \cap X$, and XT_H is a subgroup of G. But by hypothesis, $|X : (T_H \cap \Lambda)| = 2$, so $|XT_H : T_H| = 2$. Thus x normalizes T_H, and $x^2 \in T_H$. By hypothesis, T_H normalizes L and R, so it also must normalize U and D. Hence x^2 normalizes U and D, so x normalizes UU^x and $C_R(UU^x)$. Similarly x normalizes $D \cap D^x$. Further as U and D are normal in T_H, so are U^x and D^x, so UU^x and $D \cap D^x$ are normal in T_H as well. In particular $D \cap D^x$ is normalized by $LT_H = H$ and $(T_H \cap X)\langle x \rangle = X = N_\Lambda(R)$, so $D \cap D^x = 1$ by C.6.2.3.

By (a), $\overline{C_R(UU^x)} \leq \bar{Q}$ and $\Phi(\bar{Q}) = 1$, so $\Phi(C_R(UU^x)) \leq \Phi(Q) \leq D$ and hence $\Phi(C_R(UU^x)) \trianglelefteq LT_H = H$. Therefore $\Phi(C_R(UU^x)) = 1$ by C.6.2.3. Next $R = UU^x C_R(UU^x)$ by (i), so using (a),

$$Q = C_R(U) = C_{UU^xC_R(UU^x)}(U) = UC_R(UU^x)C_{U^x}(U) = UC_R(UU^x)$$

is elementary abelian. In particular $D = C_R(L) \leq C_R(U) = Q$, so $\Phi(D) = 1$, and $R = U^x Q = QQ^x$. Further it follows from the definition of $n = m(R,U)$, and R/Q of rank n in (b), that no proper subgroup of R/Q can be an FF*-offender on Q, so that $\mathcal{A}(R) = \{Q, Q^x\}$, completing the proof of (iii), and hence also the proof of C.6.3. □

In the next lemma, we will apply C.6.3 in the special case (used in section 4.1) of an A_7-block. We use the notational conventions of section B.3 in discussing the action of $H/Q \cong S_7$ on the natural module U for H/Q.

LEMMA C.6.4. *Assume Hypothesis C.6.2, and in addition assume*

(I) $L = O^2(H)$ is an A_7-block, $Q := O_2(H) \leq R$, and R/Q is generated by the transposition $(1,2)$ in H/Q.

(II) $N_\Lambda(R_0) = R$ for each $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq H$.

(III) Λ is a 2-group and $L_V := O^2(N_L(R)) \leq N_G(\Lambda)$.

Then

(1) $[U,R] = \langle e_{1,2} \rangle$.

(2) $Q = D \times U$, and $[Q,R] = \langle e_{1,2} \rangle$.

(3) $|X : R| = 2$ and $U^x \not\leq Q$ for $x \in X - R$.

(4) $R = QU^x$ and $\mathcal{A}(R) = \{Q, Q^x\}$.

(5) $D \cap D^x = 1$.

(6) $D \leq Z(R)$.

(7) $R \leq J(\Lambda) = Q_0 \times D_0$*, where* Q_0 *is of index* 2 *in* $Z(R)$ *and* D_0 *is dihedral of order at least* 8*.*

(8) $|\Lambda : J(\Lambda)| \leq 2$*.*

(9) $L_V = O^2(L_V\Lambda) \trianglelefteq L_V\Lambda$*, and* Λ *centralizes* $V := O_2(L_V)$*.*

(10) $e_{1,2} \in Z(T_H\Lambda)$*.*

PROOF. Let $H^* := H/Q$. By hypothesis, $R^* = \langle(1,2)\rangle$, so (1) follows. As L is an A_7-block, $QL = DL$ by C.1.13 and the fact that $H^1(L/U, U) = 0$ by B.3.3.1. In particular $Q = D \times U$, giving the first part of (2).

Next we note that $R < \Lambda$: for $R \leq \Lambda$ by (II)—and if $R = \Lambda$, then $R = N_\Lambda(R)$ normalizes $1 \neq U \leq R$ with $U \trianglelefteq H$, contrary to C.6.2.3. By (III), Λ is a 2-group, so $R < N_\Lambda(R) = X$; thus there is $x \in N_{XT_H}(T_H) - T_H$ with $x^2 \in T_H$. Set $X_1 := \langle x, R\rangle$. Notice that Hypothesis C.6.2 is satisfied, with R, R, X_1 in the roles of "R, T_H, Λ: First, C.6.2.1 is immediate from (I). Second, by construction X_1 contains R, while $H = LR$ as R contains Q and $R^* \not\leq L^*$, giving C.6.2.2. Finally if $x_1 \in X_1 - R$ acts on $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq H$, then $x_1 = t\lambda$ with $t \in T_H$ and $\lambda \in \Lambda$. Now $t \in H \leq N_G(R_0)$, so $\lambda \in N_\Lambda(R_0) = R$ by (II). But then $x_1 \in T_H$, so $X_1 = \langle x_1, R\rangle \leq T_H$, contradicting $x \in X_1 - T_H$. So C.6.2.3 holds.

Now X_1 plays the roles of "X" and "XT_H", so we apply C.6.3.1 to see that $U^x \not\leq Q$ for some $x \in X_1 - R$—and hence for any such x, as $|X_1 : R| = 2$. Next as $Q = D \times U$, hypothesis (a) of C.6.3.3 is satisfied, while hypothesis (b) follows (with $n = 1$) from the fact that R^* is generated by a transposition. Therefore $R = QU^x$ and $Q = UC_Q(U^x)$ by C.6.3.3.i, and then C.6.3.3.iii applied to X_1 in the role of "X" completes the proof of (4) and establishes (5). In particular Q and D are elementary abelian. Also as X permutes $\mathcal{A}(R)$, it follows from (4) that $|X : N_X(Q)| = 2$—while by (II), $N_X(Q) = R$, so $|XT_H : T_H| = 2 = |XR : R| = 2$, giving $X = X_1$, and completing the proof of (3). Since $R = QU^x$ and $Q = UC_Q(U^x)$ is abelian, we get $[Q, R] = [Q, U^x] = [U, U^x] = \langle e_{1,2}\rangle$, completing the proof of (2). Similarly $R = UC_R(U^x)$, so $[R, U^x] = [U, U^x] = \langle e_{1,2}\rangle$, and hence $[D, U^x] \leq \langle e_{1,2}\rangle \cap D = 1$. Then as Q is abelian, D centralizes $U^xQ = R$, so (6) holds.

Now observe that L_V is an A_5-block. Let $u \in U - C_U(U^x)$. Then $[u, u^x] = \langle e_{1,2}\rangle$ by (1), so $R = Q_0 \times \langle u, u^x\rangle$, where Q_0 is of index 2 in $Z(R)$ and $\langle u, u^x\rangle \cong D_8$ with $\langle e_{1,2}\rangle = Z(\langle u, u^x\rangle)$. We maximize with respect to these properties: Let $\mathcal{R}$ consist of those overgroups R_1 of R in Λ with $R_1 = Q_0 \times D_1$, D_1 dihedral, and $R_1 \leq N_\Lambda(L_V)$. Pick R_1 maximal in $\mathcal{R}$. If $R_1 = \Lambda$, then visibly (7) and (8) hold with $\Lambda = J(\Lambda)$—since the 4-groups of a dihedral group D_1 generate D_1. Further $\Lambda = R_1$ acts on L_V and hence must centralize $O_2(L_V) = V$, giving (9). Finally $e_{1,2}$ is central in R_1 and T_H, so (10) holds.

So we may assume that $R_1 < \Lambda$. Then $R_1 < R_0 := N_\Lambda(R_1)$. Now R_0 acts on

$$R_1/Z(R_1)\Phi(R_1) \cong E_4.$$

Let R_2 be the centralizer in R_0 of this section. Then for $r \in R_2$ we have $Q^r \leq QZ(R_1)\Phi(R_1)$, so that Q^r is conjugate to Q under the dihedral subgroup D_1 of R_1. Thus $Q^{R_2} = Q^{R_1}$, so by a Frattini Argument, $R_2 = R_1 N_{R_2}(Q) = R_1$, using $N_\Lambda(Q) = R$ from (II) again. Thus $R_2 = R_1 < R_0$; so as $|R_0 : R_2| \leq 2$ since the section is of rank 2, we conclude that $|R_0 : R_1| = 2$.

By (III), L_V normalizes the 2-group Λ, and by construction R_1 normalizes L_V, so

$$[L_V, R_1] \leq \Lambda \cap L_V = V \leq R_1.$$

Thus L_V normalizes R_1 and hence also normalizes $N_\Lambda(R_1) = R_0$. Then as $|R_0 : R_1| = 2$ and $L_V = O^2(L_V)$, $[L_V, R_0] \leq R_1$. We now get the assertions of (9) and (10) for R_0 in the role of "Λ": namely R_0 normalizes $L_V R_1$ and hence also normalizes $O^2(L_V R_1) = L_V$. Therefore $[L_V, R_0] = V$, so as $\mathrm{End}_{\mathbf{F}_2 L_V}(V) = \mathbf{F}_2$, R_0 centralizes V. Also R_0 normalizes $\Omega_1(\Phi(R_1)) = \langle e_{1,2} \rangle$.

Suppose now that $R_1 = J(R_0)$. Then $N_\Lambda(R_0) \leq N_\Lambda(R_1) = R_0$, so $R_0 = \Lambda$. Then (7) and (8) hold (this time with $|\Lambda : J(\Lambda)| = 2$) as do (9) and (10), since we verified them earlier for R_0 in the role of "Λ", and now $R_0 = \Lambda$.

Finally assume that $R_1 \neq J(R_0)$, so there is an involution $y \in R_0 - R_1$ with

$$m_2(C_{R_1}(y)) \geq m_2(R_1) - 1.$$

As D_1 is dihedral while y does not centralize the section $R_1/Z(R_1)\Phi(R_1)$, $R_1 = \langle u, u^y \rangle Z(R_1)$ and

$$m_2(C_{R_1}(y)) = m_2(C_{Z(R_1)}(y)) \leq m(Z(R_1)) = m_2(R_1) - 1. \qquad (*)$$

Our earlier inequality now forces equality in (*), so we conclude y centralizes $Z(R_1)$, and $R_0 = Q_0 \times D_0$, where $D_0 := \langle u, y \rangle$ is dihedral with center $\langle e_{1,2} \rangle$. In the previous paragraph we showed that R_0 normalizes L_V; thus $R_0 \in \mathcal{R}$, contradicting the maximal choice of R_1.

Thus the proof of C.6.4 is complete. $\square$

CHAPTER D

The qrc-lemma and modules with $\hat{q} \leq 2$

The previous chapter used results on FF-modules from chapter B to pin down the obstructions to pushing up, and to establish other related applications. The present chapter analyzes certain situations where weaker information than Failure of Factorization is available. In particular we study $\mathbf{F}_2G$-modules V where the parameters $q(G, V)$ and $\hat{q}(G, V)$ in Definitions B.1.1 and B.4.1 are at most 2—as opposed to FF-modules V, which satisfy $q(G, V) \leq 1$.

The study of such modules is motivated by Stellmacher's qrc-lemma. We develop a version D.1.5 of the qrc-lemma for QTKE-groups G in the first section of this chapter. Among its conclusions is the case where the parameter $q(Aut_H(V), V)$ is at most 2 for suitable internal modules V in 2-locals H of G.

Just as our description of general FF-modules in section B.5 was based on the irreducible case in Theorem B.4.2, in this chapter we provide a fairly complete description of the possibilities for faithful 2-reduced $\mathbf{F}_2X$-modules V with $\hat{q}(X, V) \leq 2$ under the SQTK-hypothesis, based on Theorem B.4.5, which gives the possibilities in the subcase where $F^*(X)$ is quasisimple and irreducible on V.

Since the machinery for analyzing modules with $\hat{q}(X, V) \leq 2$ is not available in the literature, we begin in section D.2 with a general discussion of the inheritance properties to subspaces of the parameters q and $\hat{q}$, and "offending" subgroups, somewhat in the spirit of B.1.5. Then in section D.3 we extend B.4.5 to a description of more general SQTK-groups G and modules V with $\hat{q}(G, V) \leq 2$. This discussion is the basis for our description of the Fundamental Setup FSU (3.2.1) in the proof of the Main Theorem on QTKE-groups.

D.1. Stellmacher's qrc-Lemma

One statement of Stellmacher's qrc-lemma appears in (3.4) of [**Ste92**]; roughly speaking, that result establishes numerical restrictions on certain parameters (most notably the parameter q as mentioned above) on internal modules for a pair G_1, G_2 of subgroups of a group G which share a common Sylow 2-subgroup, such that $O_2(\langle G_1, G_2\rangle) = 1$, and G_2 is a minimal parabolic in the sense of Definition B.6.1. We use the qrc-Lemma in implementing the Thompson strategy described in the outline of the proof of our Main Theorem given in the Introduction to Volume II.

The version of the lemma stated in this section was first shown to us by Ulrich Meierfrankenfeld in February 1997. Among other things, it provides a simplification of our original reduction to the list in section 3.2, of groups L and modules V appearing in the Fundamental Setup FSU in the proof of the Main Theorem.

Throughout this section we assume:

HYPOTHESIS D.1.1. *G_1 and G_2 are finite subgroups of a group G,*

$$T \in Syl_2(G_1) \cap Syl_2(G_2),\ R := O_2(G_1),$$

and $V \in \mathcal{R}_2(G_1)$, such that

(1) T is contained in a unique maximal subgroup M_2 of G_2.

(2) $R = C_T(V)$.

(3) No nontrivial subgroup of R is normal in $\langle G_1, G_2 \rangle$.

Notice that we do not assume that $F^*(G_i) = O_2(G_i)$, although in the proof of the Main Theorem, this condition will hold when T is Sylow in our QTKE-group G. We do however assume that $V \in \mathcal{R}_2(G_1)$, and so in particular $V \leq Z(R)$. Similarly, we do not require that G be quasithin, although at certain points we do appeal to lemmas proved in this work only under such restrictions, but which in fact hold more generally.

In the most important application of the qrc-Lemma in the proof of the Main Theorem, G_1 will be a uniqueness subgroup L_0T (cf. 1.4.1) contained in a maximal 2-local M, which acts faithfully on a 2-reduced module V. By Theorem 2.1.1, $\mathcal{M}(T) \neq \{M\}$, so we can choose G_2 in the set $\mathcal{H}_*(T, M)$ of the Introduction to Volume II (see also Definition 3.0.1). These choices guarantee that condition (3) of Hypothesis D.1.1 is satisfied, while condition (2) is a consequence of 1.4.1.4. As $G_2 \in \mathcal{H}_*(T, M)$, $G_2 \cap M$ is the unique maximal overgroup of T in G_2, so condition (1) holds (cf. 3.1.3.1).

In the remainder of the section, we adopt the following notation:

$$U := \langle V^{G_2} \rangle, R_2 := O_2(G_2), \text{ and } q := q(G_1/C_{G_1}(V), V).$$

When $V \not\leq R_2$ we show in E.2.13 that $\hat{q}(G_1/C_{G_1}(V), V) < 2$. Therefore in this section, we will concentrate on the case where $V \leq R_2$, so that also $U \leq R_2$. Similarly in earlier chapters we discussed techniques for handling the case when V is an FF-module for $G_1/C_{G_1}(V)$, so in this section we focus on the case $q > 1$. In lemmas D.1.2 through D.1.4, we establish various consequences of the assumptions $q > 1$ and $V \leq R_2$. Then we collect our conclusions in the qrc-Lemma D.1.5.

We begin with some elementary consequences of the assumption that V is not an FF-module for G_1:

LEMMA D.1.2. *Assume $q > 1$. Then*

(1) If $V \leq R_2$ then U is elementary abelian.

(2) $J(T) = J(R) = J(RR_2)$.

(3) T is not normal in G_2, and $J(R) \not\leq R_2$.

(4) G_2 is a minimal parabolic in the sense of Definition B.6.1, and so is described in B.6.8.

(5) If W is an $\mathbf{F}_2G_2$-module with $W \leq O_2(G_2) \leq C_{G_2}(W) \leq M_2$, then $W \in \mathcal{R}_2(G_2)$ and $q(G_2/C_{G_2}(W), W) \geq 1$.

PROOF. Assume that $V \leq R_2$. Then as $R_2 \leq T \leq G_1$, while $V \trianglelefteq G_1$ since $V \in \mathcal{R}_2(G_1)$ by Hypothesis D.1.1, also $V \trianglelefteq R_2$. Then for $g \in G_2$,

$$[V, V^g] \leq V \cap V^g \leq C_G(V) \cap C_G(V^g),$$

so that V and V^g act quadratically on each other. Interchanging the roles of V and V^g if necessary, we may assume that $m(V^g/C_{V^g}(V)) \geq m(V/C_V(V^g))$. Now if U is nonabelian then there is $g \in G_2$ with $[V, V^g] \neq 1$, so V^g is quadratic on V and $r_{Aut_{V^g}(V),V} \leq 1$, contradicting our hypothesis that $q > 1$. This establishes (1).

Observe next since $q > 1$ that $J(T) \leq C_T(V)$ by B.2.7, while $C_T(V) = R = O_2(G_1)$ by part (2) of Hypothesis D.1.1. Then by B.2.3.3, $J(T) = J(R) = J(RR_2)$, establishing (2). Similarly $J(R) \not\leq R_2$, since otherwise $J(R) = J(T) = J(R_2) \trianglelefteq \langle G_1, G_2 \rangle$ by B.2.3.3, contrary to part (3) of Hypothesis D.1.1. In particular T is not normal in G_2, completing the proof of (3).

As T is not normal in G_2, part (1) of Hypothesis D.1.1 establishes (4). Then (4) and B.6.8.6c say that each module W satisfying the hypotheses of (5) is in $\mathcal{R}_2(G_2)$. It also follows from (4) that $q(G_2/C_{G_2}(W), W) \geq 1$; we supplied a proof of this fact in B.6.9.1 when G_2 is a SQTK-group, which suffices for our applications. □

Under the assumption $q > 1$, our analysis of the case $V \leq R_2$ depends upon the number c of nontrivial G_2-composition factors on U. We see next that $q \leq 2$ when $c \geq 2$:

PROPOSITION D.1.3. *Assume $q > 1$, $V \leq R_2$, and G_2 has $c > 1$ noncentral 2-chief factors $U_1, \ldots, U_c$ on U. Then $\Phi(U) = 1$, $R_2 = O^{2'}(C_{G_2}(U))$, and:*

(1) $q \leq 2$.

(2) The set $\mathcal{A}^*$ of $A \in \mathcal{A}(R) = \mathcal{A}(T)$ with nontrivial quadratic action on U is nonempty, and includes those A with AR_2/R_2 minimal subject to being nontrivial.

(3) If $q = 2$, then $c = 2$ and for each $A \in \mathcal{A}^*$, setting $B := A \cap R_2$ we have:

$$2m(A/B) = m(U/C_U(A)) = 2m(B/C_B(U));$$

$$2m(B/C_B(V^h)) = m(V^h/C_{V^h}(B)) \quad \textit{for each } h \in G_2 \textit{ with } [V^h, B] \neq 1;$$

$m(A/B) = m(U_i/C_{U_i}(A))$; and $C_U(A) = C_U(B)$.

PROOF. By hypothesis $q > 1$, so we conclude from D.1.2.2 that $\mathcal{A}(T) = \mathcal{A}(R) = \mathcal{A}(RR_2)$, and from D.1.2.3 that there is $A \in \mathcal{A}(R)$ with $A \not\leq R_2$. Using the hypothesis $V \leq R_2$, U is elementary by D.1.2.1.

Notice if $O^2(G_2) \leq C_{G_2}(U)$, then V is normalized by G_1 and $O^2(G_2)T = G_2$, contradicting D.1.1.3; thus $O^2(G_2) \not\leq C_{G_2}(U)$. By D.1.2.4, we may apply B.6.8.6a to conclude that $C_{G_2}(U) \leq \ker_{M_2}(G_2)$; then by B.6.8.6b, $C_{R_2}(U) = O^{2'}(C_{G_2}(U))$, so that $C_A(U) \leq A \cap R_2 < A$. By the Thompson Replacement Lemma B.1.4.3, if AR_2/R_2 is minimal subject to being nontrivial, then A acts quadratically on U, establishing (2).

Let $A \in \mathcal{A}^*$ and set $B := A \cap R_2$. As noted above, $C_A(U) \leq R_2$, so $C_A(U) = C_B(U)$.

Suppose that $B \leq C_{G_2}(U)$, so that $C_A(U) = B$. As U_i is a non-trivial irreducible for G_2, $R_2 \leq C_{G_2}(U_i)$ and $O^2(G_2) \not\leq C_{G_2}(U_i)$, so $C_{G_2}(U_i) \leq M_2$ by B.6.8.6a. Thus using D.1.2.5, $q_i := q(G_2/C_{G_2}(U_i), U_i) \geq 1$. But from B.6.8.6b, $C_A(U_i) \leq A \cap R_2 = B = C_A(U)$, so $C_A(U_i) = B$, and hence A/B is faithful on U_i. Then as A acts quadratically on U_i and $q_i \geq 1$, we conclude that $m(U_i/C_{U_i}(A)) \geq m(A/B)$. Then the hypothesis $c \geq 2$ gives $m(U/C_U(A)) \geq 2m(A/B) > m(A/B)$. But as $A \in \mathcal{A}(T)$ and $B = C_A(U)$, $m(A/B) \geq m(U/C_U(A))$ by B.2.4.1. This contradiction shows that

$$C_A(U) < B \quad \text{and hence} \quad [U, B] \neq 1.$$

We now make some calculations based on the action of B on the generating set $V^{G_2} := \{V_1, \ldots, V_n\}$ of U. Set $Z_i := V_1 \cdots V_i$ and $B_i := C_B(Z_i)$. As we saw during the proof of D.1.2.1, V_i is normal in R_2 and hence is B-invariant, so Z_i is B-invariant. Observe

$$Z_i = Z_{i-1}V_i \quad \text{and so} \quad B_i = C_{B_{i-1}}(Z_i) = C_{B_{i-1}}(V_i). \qquad (a)$$

Since A is quadratic, by definition of q (applied to the conjugate V_i of V) we get

$$\frac{m(V_i/C_{V_i}(B_{i-1}))}{m(B_{i-1}/B_i)} \geq q \quad \text{whenever} \quad B_i < B_{i-1}. \tag{b}$$

Of course when $B_i = B_{i-i}$, both ranks in (b) are zero, so in any case we have

$$m(V_i/C_{V_i}(B_{i-1})) \geq m(B_{i-1}/B_i)q. \tag{c}$$

As $[U, B] \neq 1$, there is a first j with $B_j < B$. Then

$$Z_{j-1} \leq C_{Z_j}(B) \quad \text{and} \quad m(Z_j/C_{Z_j}(B)) = m(V_j/C_{V_j}(B)). \tag{d}$$

We claim:

$$m(Z_i/C_{Z_i}(B))/m(B/B_i) \geq q \quad \text{for all} \quad i \geq j, \tag{e}$$

$$\text{with equality only if} \quad \frac{m(V_k/C_{V_k}(B_{k-1}))}{m(B_{k-1}/C_{B_{k-1}}(V_k))} = q \quad \text{for all } j \leq k \leq i. \tag{f}$$

The proof is by induction on i. First consider the base step $i = j$: Since $B_j < B = B_{j-1}$ by our choice of j, (e) follows from (b) and (d). Moreover if (e) is an equality, so is the inequality in (b)—and using (a) to replace B_j by $C_{B_{j-1}}(V_j)$ in (b), we have the equality (f), establishing the claim when $i = j$.

Assume the claim holds at k for $j \leq k \leq i - 1$. We start with the relation:

$$m(Z_i/C_{Z_i}(B)) = m(Z_i/Z_{i-1}C_{Z_i}(B)) + m(Z_{i-1}C_{Z_i}(B)/C_{Z_i}(B)). \tag{g}$$

From (a) we have $Z_i = Z_{i-1}V_i$ and

$$Z_{i-1}C_{Z_i}(B) \leq Z_{i-1}C_{Z_i}(B_{i-1}) = Z_{i-1}C_{V_i}(B_{i-1}).$$

Using these facts and the standard isomorphism theorems, we obtain

$$m(Z_i/Z_{i-1}C_{Z_i}(B)) \geq m(Z_{i-1}V_i/Z_{i-1}C_{V_i}(B_{i-1})) = m(V_i/C_{V_i}(B_{i-1})) \tag{h}$$

and

$$m(Z_{i-1}C_{Z_i}(B)/C_{Z_i}(B)) \geq m(Z_{i-1}/C_{Z_{i-1}}(B)).$$

Next from (c) and our induction assumption we obtain

$$m(V_i/C_{V_i}(B_{i-1}) \geq m(B_{i-1}/B_i)q \quad \text{and} \quad m(Z_{i-1}/C_{Z_{i-1}}(B)) \geq m(B/B_{i-1})q. \tag{i}$$

Now combining (g), (h), and (i) we conclude

$$m(Z_i/C_{Z_i}(B)) \geq m(B_{i-1}/B_i)q + m(B/B_{i-1})q = m(B/B_i)q,$$

establishing (e). Furthermore if (e) is an equality, then the inequalities in (i) must be equalities. Then by (a), (f) holds when $k = i$. Finally by induction the inequalities in (e) must be equalities for all k between j and i, so (f) holds for all such k. The proof of the claim is complete.

Since $Z_n = U$ and $B_n = C_B(Z_n) = C_B(U)$, the claim at $i = n$ gives

$$\frac{m(U/C_U(B))}{m(B/C_B(U))} \geq q, \tag{j}$$

and in case of equality, $m(V^h/C_{V^h}(B))/m(B/C_B(V^h)) = q$ for each $h \in G_2$ with $[V^h, B] \neq 1$—since we can order the V_i so that $V^h = V_j$, and appeal to (f) using $B_{j-1} = B$.

As noted earlier in the proof,

$$m(A/C_A(U)) \geq m(U/C_U(A)) \geq m(U/C_U(B)) \geq m(B/C_B(U))q, \tag{k}$$

appealing to (j) for the final inequality. Next recalling that $C_A(U) = C_B(U)$:

$$m(B/C_B(U))q = [m(A/B) + m(B/C_B(U))]q - m(A/B)q$$
$$= m(A/C_A(U))q - m(A/B)q, \qquad (l)$$

and then combining (k) and (l) we obtain

$$m(A/B)q \geq m(A/C_A(U))(q-1) \geq m(U/C_U(A))(q-1). \qquad (m)$$

Let $q_i := m(U_i/C_{U_i}(A))/m(A/B)$. Then

$$m(U/C_U(A)) \geq \sum_{i=1}^{c} m(U_i/C_{U_i}(A)) = m(A/B)\sum_{i=1}^{c} q_i. \qquad (n)$$

Recall that that $q_i \geq 1$ for all i, so using these inequalities together with (m) and (n):

$$c \leq \sum_{i=1}^{c} q_i \leq \frac{m(U/C_U(A))}{m(A/B)} \leq q/(q-1) = 1 + \frac{1}{q-1}. \qquad (o)$$

But by hypothesis the integer c is at least 2, so $q \leq 2$, establishing conclusion (1) of our Proposition.

Now assume that $q = 2$. Then all inequalities in (j)–(o) are equalities. In particular $c = 2$ and $q_1 = 1 = q_2$. We verify the remaining statements in (3): As the inequalities in (k) are equalities, $C_U(A) = C_U(B)$ and $m(U/C_U(A)) = 2m(B/C_B(U))$. The equality $2m(A/B) = m(U/C_U(A))$ comes from (n), since $q_i = 1$ and $c = 2$. From the remark after (j), $m(V^h/C_{V^h}(B)) = 2m(B/C_B(V^h))$ for each $h \in G_2$ with $[V^h, B] \neq 1$. Finally $m(A/B) = m(U_i/C_{U_i}(A))$ as $q_i = 1$. Thus all parts of the Proposition are established. □

To complete our analysis of the case $V \leq R_2$, it remains to treat the case where G_2 has a unique nontrivial chief factor on U. In this case we do not need to assume $q > 1$.

Proposition D.1.4. *Assume that $V \leq R_2$ and that G_2 has a unique noncentral chief factor on U. Then either*

(1) The dual of V is an FF-module for $G_1/C_{G_1}(V)$, or

(2) $R \cap R_2 \trianglelefteq G_2$, U is elementary abelian, $[U, O^2(G_2)] \leq Z(R_2)$, $O^2(G_2) = [O^2(G_2), J(R)]$ with $J(T) = J(R)$, and U contains an FF-module for G_2 in $\mathcal{R}_2(G_2)$. If G_2 is solvable and $F^*(G_2) = O_2(G_2)$, then $O^2(G_2)$ is the product of A_3-blocks and $R \in Syl_2(O^2(G_2)R)$.

Proof. Let $I := [U, O^2(G_2)]$, $D := C_V(O^2(G_2))$, and $\tilde{I} := I/C_I(O^2(G_2))$. Note that if $X \leq I$ with $X \trianglelefteq G_2$ and $\tilde{I} \leq \tilde{X}$, then $I \leq XC_I(O^2(G_2))$, so

$$I = [I, O^2(G_2)] \leq [X, O^2(G_2)] \leq X \text{ and so } I = X. \qquad (*)$$

As $G_2 = O^2(G_2)T$ we have $U = \langle V^{G_2}\rangle = \langle V^{O^2(G_2)}\rangle$, so $U = [V, O^2(G_2)]V = IV$. By hypothesis G_2 has just one noncentral chief factor on U, so this irreducible is $\tilde{I}$. Notice also that $G_2 = O^2(G_2)T$ normalizes D. Set $Q := \langle (R \cap R_2)^{G_2}\rangle$. The hypothesis $V \leq R_2$ gives $V \leq R \cap R_2$, so $U = \langle V^{G_2}\rangle \leq Q \trianglelefteq G_2$. By Hypothesis D.1.1, $O_2(G_1/C_{G_1}(V)) = 1$, so $R = O_2(G_1)$ centralizes V. Therefore $R \cap R_2$ centralizes D, so as $D \trianglelefteq G_2$, in fact Q centralizes D.

We claim that $[V, Q] \leq D = C_V(O^2(G_2))$: For if not, $I_0 := [V, Q, O^2(G_2)] \neq 1$, so as G_2 is irreducible on $\tilde{I}$, we get $\tilde{I} \leq \widetilde{[U, Q]}$, and hence $I \leq [U, Q]$ by (*). Thus

$U = IV \leq [U,Q]V$, so $U/V = [U/V, Q]$, and hence $V = U \trianglelefteq G_2$. as Q is a 2-group. But then V is normal in both G_1 and G_2, contrary to D.1.1.3. This contradiction establishes the claim that $[V, Q] \leq D$.

Similarly $D < V$, since otherwise D is normal in G_1 and G_2. Thus there is $v \in V - D$ with $[v, T] \leq D$. Observe:

$$[v, Q] \text{ is normalized by } G_2. \qquad (**)$$

For $[v,Q] \leq [v,T] \leq D \leq C_V(O^2(G_2)Q)$, so T normalizes vD and hence also normalizes $[vD, Q] = [v, Q]$. So (**) is established.

Next as $v \notin D$ we have $[\tilde{v}, O^2(G)] \neq 1$, so $\tilde{I} = \langle \tilde{v}^{G_2} \rangle$ as G_2 is irreducible on $\tilde{I}$. Hence $I = \langle v^{G_2} \rangle$ by (*), so $U = IV \leq \langle v^{G_2} \rangle V$. Recalling that R centralizes V, $[U, R_2 \cap R] = [\langle v^{G_2} \rangle, R_2 \cap R]$, so

$$[U, Q] = [\langle v^{G_2} \rangle, Q] = \langle [v, Q]^{G_2} \rangle = [v, Q],$$

using (**). But we also showed that $[v, Q] \leq D \leq C_V(Q)$, so Q acts quadratically on U, and hence also on V. Thus $\Phi(Q) \leq C_Q(V)$. Also

$$m([V,Q]) \leq m([U,Q]) = m([v,Q]) \leq m(Q/C_Q(V)).$$

Therefore if $Q \not\leq C_{G_2}(V)$, then $Q/C_Q(V)$ is a dual-FF*-offender, so that conclusion (1) holds.

Thus we may assume that $[V, Q] = 1$, and so $Q \leq C_T(V) = R$ using D.1.1.2. Then

$$\langle (R_2 \cap R)^{G_2} \rangle = Q \leq R_2 \cap R,$$

so $R \cap R_2 = Q \trianglelefteq G_2$. As V centralizes R, $V \leq Z(Q)$, so that $U \leq Z(Q)$ and hence $\Phi(U) = 1$. By D.1.1.3, R is not normal in G_2, so as $R \cap R_2 \trianglelefteq G_2$, $R \not\leq R_2$. By D.1.1.1 and as T is not normal in G_2, we may apply B.6.8 to G_2: as R_2 is Sylow in $\ker_{M_2}(G_2)$ by B.6.8.5, it follows that $R \not\leq \ker_{M_2}(G_2)$, so by B.6.8.4, $O^2(G_2) \leq \langle R^{G_2} \rangle$, and hence

$$O^2(G_2) = [O^2(G_2), R]. \qquad (!)$$

We next show that $I \leq Z(R_2)$. Namely $[R_2, R] \leq Q$, so using (!) and $U \leq Z(Q)$, $[R_2, O^2(G_2)] \leq Q \leq C_{R_2}(U) \leq C_{R_2}(I)$. Set $G_2^* := G_2/C_{G_2}(I)$; then $O^2(G_2)^*$ commutes with R_2^*. Thus as $O^2(G_2)$ is nontrivial on I, $C_I(R_2) \not\leq C_I(O^2(G_2))$ by the Thompson $A \times B$ Lemma, so as $O^2(G_2)$ is irreducible on $\tilde{I}$, $\tilde{I} = \widetilde{C_I(R_2)}$. Hence $I = C_I(R_2) \leq Z(R_2)$ by (*).

As $O^2(G_2) \not\leq C_{G_2}(I)$, $C_{G_2}(I) \leq M_2$ by B.6.8.6a, so $I \in \mathcal{R}_2(G_2)$ by B.6.8.6c.

If $J(R) \leq R_2$, then $J(R) \leq R \cap R_2 = Q$, and then $J(R) = J(Q) \trianglelefteq \langle G_1, G_2 \rangle$, contrary to D.1.1.3. Thus $J(R) \not\leq R_2$, so arguing as above, $O^2(G_2) = [O^2(G_2), J(R)]$ and I is an FF-module. Thus we have completed those parts of the proof of (2) which do not depend upon G_2 being solvable of even characteristic.

Finally assume G_2 is solvable and $F^*(G_2) = O_2(G_2)$. As G_2 is solvable, B.6.8.2 says $O^2(G_2/O_2(G_2))$ is of odd order; so as $[R_2, O^2(G_2)] \leq Q \leq R$, $O_2(O^2(G_2)) \leq R$, and hence $R \in Syl_2(O^2(G_2)R)$. Then as $F^*(G_2) = O_2(G_2)$, by a theorem of Baumann (cf. C.1.29 when G_2 is strongly quasithin), either there is a nontrivial characteristic subgroup of R normal in G_2, or $O^2(G_2)$ is the product of A_3-blocks. The latter must hold in view of D.1.1.3, so the proof of D.1.4 is complete. □

We have essentially established our version of the qrc-lemma:

THEOREM D.1.5 (Stellmacher-Meierfrankenfeld qrc-Lemma). *One of the following holds:*

(1) $V \not\leq O_2(G_2)$.

(2) $q(G_1/C_{G_1}(V),V) \leq 1$, *so* V *is an FF-module for* $G_1/C_{G_1}(V)$.

(3) The dual of V *is an FF-module for* $G_1/C_{G_1}(V)$.

(4) $q(G_1/C_{G_1}(V),V) \leq 2$, U *is abelian, and* G_2 *has more than one noncentral chief factor on* U.

(5) $R \cap R_2 \trianglelefteq G_2$, U *is elementary abelian,* G_2 *has one noncentral chief factor on* U, $[U, O^2(G_2)] \leq Z(R_2)$, *and* $O^2(G_2) = [O^2(G_2), J(R)]$ *with* $J(R) = J(T)$.

PROOF. We may assume conclusions (1) and (2) fail, so $q(G_1/C_{G_1}(V),V) > 1$ and $V \leq R_2$. Then by D.1.2.1, U is elementary abelian. Let c be the number of noncentral chief factors for G_2 on U. If $c > 1$ then D.1.3.1 gives conclusion (4), while if $c = 1$ then D.1.4 says that either conclusion (3) or (5) holds. □

In section 3.1 we will show that when G is a QTKE-group, each of the conclusions of the qrc-Lemma implies that $\hat{q} := \hat{q}(G_1/C_{G_1}(V),V) \leq 2$. Thus in the remaining sections of this chapter, we investigate pairs (G,V) where V is a faithful $\mathbf{F}_2G$-module with $\hat{q}(G,V) \leq 2$.

We close this section with a brief overview of the reduction in section 3.1 just mentioned: First, in conclusions (2) and (4) of Theorem D.1.5, we already have the conclusion $q \leq 2$. In Conclusion (1), $\hat{q} \leq 2$ by E.2.13.2; this is the place where the parameter $\hat{q}$—as opposed to q—enters the picture. Conclusion (5) is eliminated using Theorem 3.1.1 (which is just a restatement of Theorem C.5.8), with a little extra work to show that $R \in Syl_2(O^2(G_2)R)$. For example this holds when G_2 is solvable by D.1.4.2. Finally $q \leq 2$ in conclusion (3) by B.5.13.

D.2. Properties of q and $\hat{q}$: $\mathcal{R}(G,V)$ and $\mathcal{Q}(G,V)$

Throughout this section G is a finite group and V is a faithful $\mathbf{F}_2G$-module. Recall from Definition B.1.1 that we denote the set of nontrivial elementary abelian 2-subgroups of G by $\mathcal{A}^2(G)$, and for $A \in \mathcal{A}^2(G)$ we define

$$r_{A,V} := \frac{m(V/C_V(A))}{m(A)}.$$

We have just seen, in the discussion in the final paragraph of the previous section, one instance of how the condition $\hat{q}(G,V) \leq 2$ arises in the proof of our Main Theorem. Earlier in B.4.5, we listed the cases where V is irreducible with $\hat{q}(G,V) \leq 2$ and G is an SQTK-group with $F^*(G)$ quasisimple. In order to describe more general representations satisfying that bound, it will be necessary to study the behavior of $q(Aut_G(U),U)$ and $\hat{q}(Aut_G(U),U)$ for suitable subspaces U of V, much as we did for FF-modules in B.1.5.

Since the parameters q and $\hat{q}$ were introduced only recently, as yet there are few results in the literature on the corresponding modules; thus our development here will be more detailed. We feel that development is best accomplished in the context of a system of more general but related parameters associated to faithful modules, which we now introduce.

The parameters q and $\hat{q}$ arise from the study of the ratios $r_{A,V}$ for $A \in \mathcal{A}^2(G)$. Recall that the parameters $q(G,V)$, $\hat{q}(G,V)$ denote the minimum value of $r_{A,V}$ as

A varies over the subsets of $\mathcal{A}^2(G)$ of members acting quadratically or cubically. We now introduce some more general notions:

DEFINITION D.2.1. For any $0 < r \in \mathbf{R}$, define

$$\mathcal{R}_r(G,V) := \{A \in \mathcal{A}^2(G) : r_{A,V} \leq r\}, \text{ and}$$

$$\mathcal{R}_{=r}(G,V) := \{A \in \mathcal{A}^2(G) : r_{A,V} = r\}.$$

Then define

$$\mathcal{R}^+(G,V) := \{A \in \mathcal{A}^2(G) : r_{A,V} \leq r_{B,V} \text{ for all } 1 < B \leq A\}.$$

For $A \leq G$ we obtain the commutator series for A on V by setting $L_A^0(V) := V$ and then defining $L_A^k(V) := [L_A^{k-1}(V), A]$ for $k > 0$ recursively. In particular A is quadratic on V when $L_A^2(V) = 0$, and cubic on V when $L_A^3(V) = 0$. Define

$$^k\mathcal{R}_r(G,V) := \{A \in \mathcal{R}_r(G,V) : L_A^k(V) = 0\},$$

$$\mathcal{R}_r^+(G,V) := \mathcal{R}_r(G,V) \cap \mathcal{R}^+(G,V),$$

$$^k\mathcal{R}_r^+(G,V) := {}^k\mathcal{R}_r(G,V) \cap \mathcal{R}^+(G,V),$$

and

$$^kq(G,V) := \min\{r \in \mathbf{R} : {}^k\mathcal{R}_r(G,V) \neq \emptyset\}.$$

In particular $q(G,V) = {}^2q(G,V)$ and $\hat{q}(G,V) = {}^3q(G,V)$.

Our main applications will be in D.1.5, E.2.13, and F.9.16.3, where these parameters are at most 2; this leads us to define

$$\mathcal{Q}(G,V) := {}^2\mathcal{R}_2(G,V) \text{ and } \hat{\mathcal{Q}}(G,V) := {}^3\mathcal{R}_2(G,V).$$

Notice that $\mathcal{Q}(G,V) \neq \emptyset$ or $\hat{\mathcal{Q}}(G,V) \neq \emptyset$ if and only if $q(G,V) \leq 2$ or $\hat{q}(G,V) \leq 2$, respectively. Finally if $\hat{q}(G,V) \leq 2$ define $\hat{\mathcal{Q}}_*(G,V)$ to consist of the members of ${}^3\mathcal{R}_{\hat{q}(G,V)}(G,V)$ which are minimal under inclusion, setting $\hat{\mathcal{Q}}_*(G,V) := \emptyset$ if $\hat{q}(G,V) > 2$.

D.2.1. General inheritance properties. The lemmas in this subsection discuss the relationship between $r_{A,V}$ and $r_{Aut_A(U),U}$, when $A \in \mathcal{A}^2(G)$ and U is an A-invariant subspace of V.

LEMMA D.2.2. *Assume that $A \in \mathcal{R}_r(G,V)$ and U is an $\mathbf{F}_2A$-submodule of V with $[U,A] \neq 0$. Set $B := C_A(U)$. Then one of the following holds:*

(1) $A/B \in \mathcal{R}_{r-\epsilon}(Aut_G(U),U)$ for some $\epsilon > 0$; that is $r_{A/B,U} < r_{A,V}$.

(2) $B \in \mathcal{R}_{r-\epsilon}(G,V)$ for some $\epsilon > 0$; that is $r_{B,V} < r_{A,V}$.

(3) A is faithful on U, $r_{A,V} = r_{A,U}$, and $V = U + C_V(A)$.

(4) $A \in \mathcal{R}_{=r}(G,V)$, $A/B \in \mathcal{R}_{=r}(Aut_G(U),U)$, $B \in \mathcal{R}_{=r}(G,V)$, and $C_V(B) = U + C_V(A)$.

PROOF. Suppose first that $B = 1$; then A is faithful on U. If $V = U + C_V(A)$, then we have conclusion (3). Otherwise $U + C_V(A) < V$, so $m(U/C_U(A)) < m(V/C_V(A))$ and hence $r_{A,U} < r_{A,V}$, so conclusion (1) holds.

Suppose instead that $B > 1$. We may assume that (1) and (2) fail, and hence that $m(U/C_U(A)) \geq r\ m(A/B)$ and $m(V/C_V(B)) \geq r\ m(B)$. Using these inequalities and $U + C_V(A) \leq C_V(B)$ we have:

$$\begin{aligned} m(V/C_V(A)) &= m(V/(U + C_V(A))) + m(U + C_V(A)/C_V(A)) \\ &\geq m(V/C_V(B)) + m(U/C_U(A)) \geq r\ [m(B) + m(A/B)] = r\ m(A). \end{aligned}$$

Since $r \geq r_{A,V}$, all inequalities in this paragraph must in fact be equalities, so that (4) holds. $\square$

LEMMA D.2.3. *(1) If $A \in \mathcal{R}^+(G,V)$ and $1 < B \leq A$ with $r_{A,V} = r_{B,V}$, then $B \in \mathcal{R}^+(G,V)$.*

(2) If $A \in \mathcal{A}^2(G)$ with $L^k_A(V) = 0$ and $r_{A,V} = {}^k q(G,V)$, then $A \in \mathcal{R}^+(G,V)$.

PROOF. Assume the hypotheses of (1). If $1 < C \leq B$ then as $A \in \mathcal{R}^+(G,V)$, $r_{C,V} \geq r_{A,V} = r_{B,V}$, so (1) is established. Assume the hypotheses of (2), and notice if $B \leq A$ then $L^k_B(V) \leq L^k_A(V) = 0$. Thus as $r_{A,V} = {}^k q(G,V)$, $r_{B,V} \geq r_{A,V}$, so that (2) holds. $\square$

Now we will see that if $A \in \mathcal{R}^+(G,V)$ then one possibility is eliminated from D.2.2:

LEMMA D.2.4. *Assume $A \in \mathcal{R}^+_r(G,V)$ and U is an $\mathbf{F}_2A$-submodule of V with $[U,A] \neq 0$. Set $B := C_A(U)$. Then one of the following holds:*

(1) $A/B \in \mathcal{R}_{r-\epsilon}(Aut_G(U),U)$ for some $\epsilon > 0$, and $r_{A/B,U} < r_{A,V}$.

(2) A is faithful on U, $V = U + C_V(A)$, and $m(U/C_U(D)) = m(V/C_V(D))$ for each $1 < D \leq A$, so $r_{D,U} = r_{D,V} \geq r$, $A \in \mathcal{R}^+_{=r}(G,V)$, and $A \in \mathcal{R}^+_{=r}(Aut_G(U),U)$.

(3) $A/B \in \mathcal{R}_{=r}(Aut_G(U),U)$, $B \in \mathcal{R}^+_{=r}(G,V)$, and $C_V(B) = U + C_V(A)$.

PROOF. We appeal to D.2.2. The hypothesis that $A \in \mathcal{R}^+(G,V)$ rules out case (2) of D.2.2. Now case (1) of D.2.2 is conclusion (1), and in case (4), D.2.3.1 says $B \in \mathcal{R}^+(G,V)$, so that conclusion (3) holds. Finally assume case (3) of D.2.2 holds, but (1) fails, so $r_{A,V} = r$ and hence $A \in \mathcal{R}_{=r}(G,V)$ and $A \in \mathcal{R}_{=r}(Aut_G(U),U)$. For $1 \neq D \leq A$ we have $C_V(A) \leq C_V(D)$ and hence $V = U + C_V(D)$, so that $r_{D,U} = r_{D,V} \geq r$ since $A \in \mathcal{R}^+(G,V)$; thus $A \in \mathcal{R}^+(Aut_G(U),U)$, so (2) holds. $\square$

LEMMA D.2.5. *Let $A \in \mathcal{R}^+_r(G,V)$. Then for each $\mathbf{F}_2A$-submodule U of V such that $[U,A] \neq 0$, $Aut_A(U) \in \mathcal{R}_r(Aut_G(U),U)$; that is $r_{Aut_A(U),U} \leq r$.*

PROOF. Let $B := C_A(U)$ and observe that $Aut_A(U) = A/B$, while in each of the conclusions of D.2.4, $r_{A/B,U} \leq r$. $\square$

LEMMA D.2.6. *Assume $A \in {}^k\mathcal{R}^+_r(G,V)$ and U is an $\mathbf{F}_2A$-submodule of V such that $[U,A] \neq 0$ and ${}^k q(Aut_G(U),U) \geq r$. Set $B := C_A(U)$. Then ${}^k q(Aut_G(U),U) = r$ and either*

(1) A is faithful on U, $V = U + C_V(A)$, and $A \in \mathcal{R}^+_{=r}(Aut_G(U),U)$, or

(2) $A/B \in \mathcal{R}_{=r}(Aut_G(U),U)$, $B \in \mathcal{R}^+_{=r}(G,V)$, and $C_V(B) = U + C_V(A)$.

PROOF. As $L^k_A(U) \leq L^k_A(V) = 0$, $r_{A/B,U} \geq {}^k q(Aut_G(U),U) \geq r$ by hypothesis, so equality holds by D.2.5. In particular case (1) of D.2.4 cannot occur. The remaining two conclusions of D.2.4 appear as (1) and (2). $\square$

D.2.2. The case $\hat{\mathbf{q}}(\mathbf{G},\mathbf{V}) \leq \mathbf{2}$. We next use results from the previous subsection to derive consequences in the case $\hat{q}(G,V) \leq 2$. One goal is to determine in D.2.17 all representations $\varphi : \hat{G} \to GL(V)$ of SQTK-groups $\hat{G}$ such that $\hat{q}(\hat{G}\varphi,V) \leq 2$ and $F^*(\hat{G}\varphi) = F(\hat{G}\varphi)$. The general case will be considered in the following section D.3 which concludes this chapter.

First, we will see how the minimality under inclusion of members of $\hat{\mathcal{Q}}_*$ restricts the possibilities from D.2.4, and then from D.2.6:

Lemma D.2.7. *Assume that $r := \hat{q}(G,V) \le 2$, $A \in \hat{\mathcal{Q}}_*(G,V)$, and U is an $\mathbf{F}_2A$-submodule of V with $[U,A] \neq 0$. Set $B := C_A(U)$. Then $A \in {}^3\mathcal{R}^+_{=r}(G,V)$ and either*

(1) $A/B \in \hat{\mathcal{Q}}_{r-\epsilon}(Aut_G(U),U)$ for some $\epsilon > 0$, or

(2) A is faithful on U and $V = U + C_V(A)$.

Proof. By definition of $\hat{\mathcal{Q}}_*(G,V)$, $r_{A,V} = r = \hat{q}(G,V)$ is minimal, so $A \in {}^3\mathcal{R}^+_{=r}(G,V)$ by D.2.3.2. Thus we have the hypotheses of D.2.4. Furthermore as A is also minimal under inclusion in ${}^3\mathcal{R}_r(G,V)$ by definition of $\hat{\mathcal{Q}}_*(G,V)$, case (3) of D.2.4 does not occur. Thus (1) or (2) holds. □

Lemma D.2.8. *Set $r := \hat{q}(G,V)$ and assume $r \le 2$, $A \in \hat{\mathcal{Q}}_*(G,V)$, and U is an $\mathbf{F}_2A$-submodule of V such that $[U,A] \neq 0$ and $\hat{q}(Aut_G(U),U) \ge r$. Then $\hat{q}(Aut_G(U),U) = r$, A is faithful on U, $V = U + C_V(A)$, and $A \in \hat{\mathcal{Q}}_*(Aut_G(U),U)$.*

Proof. By D.2.7, $A \in {}^3\mathcal{R}^+_{=r}(G,V)$ and one of the two conclusions of that lemma hold. By hypothesis $\hat{q}(U, Aut_G(U)) \ge r$, so the hypotheses of D.2.6 are satisfied. Case (2) of D.2.6 is not satisfied in either case of D.2.7, so case (1) of D.2.6 holds, giving the second and third assertions of D.2.8, and showing $A \in {}^3\mathcal{R}^+_{=r}(Aut_G(U),U)$. Then as $\hat{q}(Aut_G(U),U) \ge r$, $\hat{q}(Aut_G(U),U) = r$, so it remains to show that A is minimal under inclusion in ${}^3\mathcal{R}_r(Aut_G(U),U)$. Since A satisfies neither conclusion (1) nor (3) of D.2.4, it satisfies conclusion (2). Hence if $1 < D < A$ then $r_{D,U} = r_{D,V}$, while as $A \in \hat{\mathcal{Q}}_*(G,V)$, $r_{D,V} > r_{A,V}$. Therefore $r_{D,U} > r_{A,U}$, completing the proof. □

We sometimes need to know that V is primitive under $A \in \hat{\mathcal{Q}}(G,V)$. The following lemma says that this is usually the case.

Lemma D.2.9. *Let $A \in {}^3\mathcal{R}^+_2(G,V)$ and set $r := r_{A,V}$, so in particular $\hat{q}(G,V) \le r \le 2$. Assume A permutes the summands $\mathcal{I}$ of a vector-space decomposition $V = \bigoplus_{I \in \mathcal{I}} I$, with each summand of rank at least 2. Pick $I \in \mathcal{I}$ and set $B := N_A(I)$. Then*

(1) $|A : B| \le 2$.

(2) If $B < A$ then B is quadratic on I.

(3) If A is quadratic on V and $B < A$, then $m(I) = 2 = r$, and either $|A| = 2$ or $B \in {}^2\mathcal{R}^+_2(G,V)$.

(4) If $B < A$ and $m(I) > 2$, then $Aut_B(I) \in \mathcal{R}_{r-\epsilon}(Aut_G(I),I)$ for some $\epsilon > 0$.

(5) If $|\mathcal{I}| = 2$ and $B < A$, then either:

(i) B induces the full group of transvections on I with a fixed center. Further if $m(I) > 2$ and G permutes $\mathcal{I}$, then either $O_2(G) \neq 1$ or $r > q(G,V)$; or

(ii) $m(I/C_I(B)) = m(I) - 2 = m(B)$. Further if $m(I) > 2$, then $r_{B,V} = r = 2$, so $A \notin \hat{\mathcal{Q}}_*(G,V)$.

(6) Assume that G permutes $\mathcal{I}$, $|\mathcal{I}| = 2$, $m(I) > 2$, and $O_2(G) = 1$. Then $\hat{\mathcal{Q}}_*(G,V) \subseteq N_G(I)$.

Proof. If $B = A$, then (1) and (6) are immediate and (2)–(5) are vacuous, so we may assume that $B < A$.

Now (6) follows from (5), since under the hypotheses of (6), (5) says that $A \notin \hat{\mathcal{Q}}_*(G,V)$. So it remains to establish (1)–(5) when $B < A$.

Fix some $a \in A - B$ and set $U := I + I^a$; then $U = I \oplus [I, a]$, where $[I, a]$ is the diagonal $\{ii^a : i \in I\}$.

Assume as in (3) that A is quadratic on V. Since A permutes $\mathcal{I}$ and centralizes $0 \neq [I, a] \leq U$, it permutes the set $\{I, I^a\}$ of direct summands on which $[I, a]$ projects nontrivially; hence $|A : B| = 2$. Then as B centralizes the full diagonal $[I, a]$, it centralizes the projection i of ii^a on I for each $i \in I$, so that B centralizes U. Then $B = C_A(U)$, and hence $A/B = Aut_A(U)$ has rank 1, with $C_U(A) = [I, a]$. Thus

$$m(I) = m(U/C_U(A)) = r_{Aut_A(U),U} = {}^2q(Aut_A(U), U), \qquad (+)$$

with the last equality holding since $Aut_A(U)$ has no nontrivial subgroups. By hypothesis

$$r \leq 2 \leq m(I), \qquad (++)$$

so we have the hypotheses of D.2.6, with A, 2 in the roles of "G, k". Then that lemma says

$${}^2q(Aut_A(U), U) = r, \qquad (+++)$$

so by (+) and (+++), the inequalities in (++) are equalities, and hence $m(I) = r = 2$. Further one of the two conclusions of D.2.6 holds. If case (1) of D.2.6 holds, then $B = 1$ so that $|A| = 2$, while if case (2) holds, then $B \in {}^2\mathcal{R}_2^+(A, V)$, so since also $B < A$ and $r = 2$, $B \in {}^2\mathcal{R}_2^+(G, V)$. This completes the proof of (3).

We next prove (1) and (2). First consider the case where $[I, B] \neq 0$. Then as A is cubic on V we have

$$0 \neq [I, B, a] \leq C_U(A).$$

Since $[I, B, a] \leq [I, a]$, as in the previous paragraph, $\{I, I^a\}$ is an A-orbit, establishing (1). Similarly the cubic action of A gives $[I, B, B] \leq C_I(A) = 0$ since $I \cap I^a = 0$, giving quadratic action of B on I, and hence establishing (2) in this case.

Next we turn to the case where B centralizes I; in particular (2) holds trivially in this case. Now as A is abelian, $A^* := A/B$ is regular on I^A, so to prove (1), we assume that $|A^*| > 2$. Thus there is $c \in A$ with $c^* \notin \langle a^* \rangle$. Then

$$0 \neq [I, a, c] \leq U + U^c =: W,$$

and $[I, a, c]$ centralizes A by its cubic action, so arguing as above, $I^{\langle a,c \rangle}$ is an A-orbit of order 4. Thus $A^* \cong E_4$ and $B = C_A(W)$, so $A^* = Aut_A(W)$. Further $C_W(A)$ is a diagonal $\{\sum_{x \in A^*} i^x : i \in I\}$ over four conjugates of rank $m(I)$, so

$$r_{Aut_A(W),W} = \frac{m(W/C_W(A))}{m(A^*)} = \frac{3m(I)}{2}.$$

Now by hypothesis $A \in \mathcal{R}_r^+(G, V)$, so we may apply D.2.5 to conclude that

$$\frac{3m(I)}{2} = r_{Aut_A(W),W} \leq r \leq 2,$$

contrary to our hypothesis that $m(I) \geq 2$. This contradiction completes the proof of (1) and (2).

We turn to (4) and (5). Notice A acts on U by (1).

Assume first that B centralizes I and hence U. Then A is quadratic on U, so we may apply (3) to U in the role of "V" to conclude that $m(I) = 2$. Then (4) is vacuous. Further $|\mathcal{I}| = 2$ under the hypothesis of (5), so B centralizes $U = V$ and hence $B = 1$; then case (ii) of (5) holds, since its second sentence is also vacuous as $m(I) = 2$.

Therefore we may assume that B does not centralize I. Define $i := m(I)$ and $m := m(I/C_I(B))$, so $m(U/C_U(A)) = i + m$. Now $Aut_A(U) = A/C_A(U)$, and we have seen that $C_A(U) = C_B(U) = C_B(I)$, so setting $b := m(B/C_B(I))$ and using (1), we have $m(A/C_A(U)) = b + 1$. Since B does not centralize I, $b > 0$. By D.2.5, $r_{Aut_A(U),U} \leq r$, so

$$i + m \leq r(b + 1). \tag{$*$}$$

We rearrange (*) to obtain:

$$r_{Aut_B(I),I} = \frac{m}{b} \leq r + \frac{r - i}{b}. \tag{$**$}$$

Under the hypotheses of (4), $i > 2 \geq r$, so $r_{Aut_B(I),I} < r$, completing the proof of (4).

It remains to prove (5), so we may assume that $|\mathcal{I}| = 2$, and hence $V = U$. If $i = 2$ then B induces a transvection on I of rank 2, so case (i) of (5) holds; thus we may also assume that $i > 2$, When G permutes $\mathcal{I}$, $G = N_G(I)\langle a \rangle$ for $a \in A - B$, with $N_G(I)$ of index 2 and hence normal in G.

As A is faithful on $V = U$, B is faithful on I and $r_{Aut_A(U),U} = r_{A,V} = r$. Then (*) becomes $(i + m)/(b + 1) = r \leq 2$, so

$$\frac{i + m - 2}{2} \leq b, \tag{!}$$

and as $A \in \mathcal{R}^+(G, V)$,

$$\frac{i + m}{b + 1} = r \leq r_{B,V} = \frac{2m}{b}.$$

Thus $b(i + m) \leq 2m(b + 1)$ and then

$$b \leq \frac{2m}{i - m}. \tag{!!}$$

Combining (!) and (!!), we obtain $(i + m - 2)(i - m) \leq 4m$, which we rearrange to obtain $(i - 1)^2 \leq (m + 1)^2$, so $m \geq i - 2$. On the other hand $m < i$, so $m = i - 1$ or $i - 2$.

Suppose $m = i - 1$. Then by (!), $b \geq i - 3/2$, so as b is an integer, $b \geq i - 1$. So as $m = i - 1$, $C_I(B)$ is a point, while as B is quadratic on I by (2), $[I, B] \leq C_I(B)$. Therefore B induces a group of transvections on I with center $C_I(B)$. As the group T of all such transvections is of rank $i - 1$ and $b = m(B) \geq i - 1$, we conclude $b = i - 1$ and B induces the full group T on I.

Thus since $i > 2$, to show that case (i) of (5) holds, we may assume that G permutes $\mathcal{I}$, and it remains to show that either $O_2(G) \neq 1$ or $r > q(G, V)$.

Let $J := \langle C_I(B)^{N_G(I)} \rangle$ and $L := \langle B^{N_G(I)} \rangle$. Then $Aut_L(J) = GL(J)$ by B.4.10. As B centralizes $I/C_B(I)$, it centralizes $V/(J + J^a)$.

Assume first that $J < I$. Then $C_B(J) \neq 1$ since B induces T on I, and $C_B(J)$ centralizes $J + J^a$ and $V/(J + J^a)$. Now $N_G(I)$ is normal in G and normalizes $J + J^a$, so by Coprime Action $C_B(J) \leq O_2(N_G(I)) \leq O_2(G)$. Thus $O_2(G) \neq 1$, so (5i) holds.

Therefore we may assume that $J = I$. Then $Aut_L(I) = GL(I)$ and

$$N_G(I) \leq G_0 := N_{GL(V)}(I) \cap N_{GL(V)}(I^a) \cong GL(I) \times GL(I).$$

As $i > 2$, $GL(I)$ is simple, so either $L = N_G(I)$ is a full diagonal subgroup of G_0, or $N_G(I) = G_0$. Suppose the first case holds. Then as a centralizes B, and T is self-centralizing in $GL(I)$, a induces inner automorphisms on L. Thus as $G = L\langle a \rangle$,

$O_2(G) \neq 1$, so again (5i) holds. In the second case there is $A_0 \in \mathcal{Q}(G, V)$ with $r_{A_0,V} < r$, so once again conclusion (5i) holds: Namely if S is the group of all transvections in $C_G(I^a)$ with a fixed axis, and $A_0 := SS^a$, then

$$r_{A_0,V} = \frac{1}{i-1} < \frac{2i-1}{i} = r_{A,V} = r.$$

This leaves the case $m = i - 2$. Here by (!) and (!!), $b = i - 2$, so

$$r = r_{A,V} = \frac{2i-2}{i-1} = 2 = \frac{2(i-2)}{i-2} = r_{B,V};$$

thus A is not minimal under inclusion, so $A \notin \hat{\mathcal{Q}}_*(G, V)$, and conclusion (ii) of (5) holds.

This completes the proof of (5), and hence of the lemma. □

A special case where $q(G, V) \leq 2$ occurs when $A := \langle t \rangle$ is of order 2 with $m([V, t]) \leq 2$. Our next two lemmas are therefore devoted to the study of:

DEFINITION D.2.10. $\mathcal{T}$ is the set of involutions $t \in G$ such that $m([V, t]) \leq 2$.

These lemmas are largely independent of the other results in this section.

LEMMA D.2.11. *Let $t \in \mathcal{T}$, and set $m := m([V, t])$. Assume $x \in G^\#$ is of odd order and inverted by t. Then*

(1) If $m = 1$ then $|x| = 3$ and $m([V, x]) = 2$.
(2) If $m = 2$ then $|x| = 3$ or 5; and if $|x| = 5$, then $m([V, x]) = 4$.
(3) If $m([V, x]) = 2m$ then $[V, t] = [V, x, t]$, $[C_V(x), t] = 0$, and

$$[C_G(x) \cap C_G([V, x]), t] = 1.$$

(4) $\langle t \rangle \in \mathcal{Q}(G, V)$.

PROOF. Part (4) is an easy consequence of Definition D.2.1 where $\mathcal{Q}(G, V) = {}^2\mathcal{R}_2(G, V)$. Set $n := |x|$. Then t acts freely on $[V, x]$, so

$$\frac{m([V, x])}{2} = m([V, x, t]) \leq m, \qquad (*)$$

and hence $m([V, x]) \leq 2m$ and $\langle x, t \rangle \leq GL([V, x])$ is dihedral of order $2n$. If $m = 1$ then $m([V, x]) = 2$ by (*) and $|GL_2(2)| = 6$, so (1) holds. If $m = 2$ then $GL_4(2)$ contains D_{2n} with $1 < n$ odd only for $n = 3, 5$; and if $n = 5$, then $m([V, x]) = 4$, so (2) holds. In (3), $m([V, x]) = 2m$, so $m([V, x, t]) = m$ and $[V, t] = [V, x, t]$ by (*). Also $V = [V, x] \oplus C_V(x)$ by Coprime Action, so $t \in C_G(C_V(x))$. Thus if we set $Y := C_G(x) \cap C_G([V, x])$, then

$$[Y, t] \leq C_G([V, x]) \cap C_G(C_V(x)) = C_G(V) = 1,$$

completing the proof of (3). □

LEMMA D.2.12. *For $k := 1, 2$, let $\mathcal{X}_k$ be the set of subgroups X of $F(G)$ of order $p := 2k + 1$ such that $m([V, X]) = 2k$. Let $G_k := \langle \mathcal{X}_k \rangle$. Then*

(1) $G_k = X_1 \times \cdots \times X_s$ is the direct product of the members $X_1, \ldots, X_s$ of $\mathcal{X}_k$.
(2) $[V, G_k] = V_1 \oplus \cdots \oplus V_s$, where $V_i := [V, X_i]$.
(3) If $t \in \mathcal{T}$ with $m([V, t]) = k$, then either

(a) $[G_k, t] = 1$ and $[V, G_k, t] = 0$, or

(b) There exists a unique j such that t inverts X_j. Further $[V, t] = [V_j, t]$, and t centralizes X_i and V_i for $i \neq j$.

(4) If $t \in \mathcal{T}$ with $m([V,t]) = k$, then $t \in N_G(X_i)$ for each i.

(5) If $k = 1$ and $t \in \mathcal{T}$ with $m([V,t]) = 2$, then either:

(a) t acts on each X_i; or

(b) t induces a transposition $(X_j, X_{j'})$ on $\mathcal{X}_1$, $[V,t] = [V_j + V_{j'}, t]$, and t centralizes X_i and V_i for $i \neq j, j'$.

PROOF. Notice if $X_i \le N_G(X_1)$, then in fact $X_i \le C_G(X_1)$ since X_i and X_1 are of prime order p. Suppose there is some $X_2 \in \mathcal{X}_k \cap C_G(X_1) - \{X_1\}$. As X_1 is irreducible on V_1, $Aut_{X_2}(V_1) \le C_{GL(V_1)}(X_1)$ which is cyclic of order $2^{2k} - 1$. Thus either X_2 centralizes V_1, or $Aut_{X_2}(V_1) = Aut_{X_1}(V_1)$. In the latter case, X_1 is the projection of X_2 with respect to the decomposition $C_G([V,X_1]) \times C_G(C_V(X_1))$. Hence $[V,X_1] = V_1 \le [V,X_2] = V_2$, so that $V_1 = V_2$ as $\dim(V_i) = 2k$, and then $C_V(X_1) = C_V(X_2)$ as X_2 acts on $C_V(X_1)$. But then $X_2 = X_1$ as G is faithful on V, contradicting $X_2 \neq X_1$. So if X_2 centralizes X_1 it also centralizes V_1.

It follows that if $\Delta := \{X_1, \ldots, X_s\}$ is a maximal set of commuting members of $\mathcal{X}_k$, and if $D := \langle \Delta \rangle$, then $[V,D] = V_1 \oplus \cdots \oplus V_s$; so that $D = X_1 \times \cdots \times X_s$, and $\mathcal{X}_k \cap D = \Delta$ by maximality of Δ.

We claim in fact that $\Delta = \mathcal{X}_k$, and $D = G_k$. This is established using a standard weak-closure argument: For suppose $\Delta \subset \mathcal{X}_k \cap N_{G_k}(D)$. If $\langle x \rangle := X_i \in N_{\mathcal{X}_k}(D) - \Delta$ has a nontrivial cycle $(X_1, \ldots, X_{2k+1})$ on Δ, then

$$2k = m([V,x]) \ge 2k \cdot 2k = 4k^2 > 2k,$$

a contradiction. Therefore, X_i normalizes and hence centralizes all members of Δ, contradicting the maximal choice of Δ. We conclude $\mathcal{X}_k \cap N_{G_k}(D) = \Delta$, so that $N_{G_k}(N_{G_k}(D))$ normalizes Δ, and hence lies already in $N_{G_k}(D)$. As $G_k \le F(G)$, G_k is nilpotent, forcing $D = G_k$, and hence $\Delta = \mathcal{X}_k$. Thus (1) and (2) are established.

Let $t \in \mathcal{T}$ and set $m := m([V,t])$. By D.2.11.4, $A := \langle t \rangle \in \mathcal{Q}(G,V)$, with $r_{A,V} = m$. Now A permutes $\mathcal{X}_k$, so it permutes the summands V_i in the decomposition of (2), each of which has rank $2k \ge 2$. Hence the hypotheses of D.2.9 are satisfied, so we may apply D.2.9.3 to see that either t normalizes V_1, or $m(V_1) = 2$. The latter case forces $p = 3$ so $k = 1$, and $m = 2 > k$ since t interchanges V_1 and V_1^t. Hence under the hypothesis of (3) and (4), where $m = k$, t normalizes each V_i, and hence each X_i, establishing (4).

Next we prove (3). Assume first that t centralizes each X_i; then as X_i is irreducible on V_i, t centralizes each V_i. Thus t centralizes $[V, G_k]$, so that conclusion (a) of (3) holds. Therefore we may assume that t inverts X_1. As $m = k$, we have the hypotheses of D.2.11.3, with X_1 in the role of "$\langle x \rangle$." That result says that $[V,t] = [V,X_1,t] = [V_1,t]$, and that t centralizes $C_V(X_1)$ and $C_G(V_1X_1)$, and hence centralizes X_i and V_i for $i \neq 1$. Thus conclusion (b) of (3) holds in this case, so the proof of (3) is complete.

It remains to prove (5), so we may take $m = 2$ and $k = 1$. If $X_2 = X_1^t$, then $m([V_1 + V_2, t]) = 2 = m$, so $[V,t] = [V_1 + V_2, t]$. Therefore t centralizes V_n for $n \neq 1, 2$—and hence also X_n. Thus (5) holds, and the proof is complete. □

D.2.3. The case $\mathbf{F^*(G) = F(G)}$. In this subsection we analyze the case where $F^*(G) = F(G)$ and $\hat{q}(G,V) \le 2$.

LEMMA D.2.13. *Assume $r := \hat{q}(G,V) \le 2$ and $A \in \hat{\mathcal{Q}}_*(G,V)$ acts nontrivially on $O_p(G)$ for some odd prime p. Then*

(1) $p = 3$ or 5.

(2) If $p = 5$ then $r = 2$, $m(A) = 1$, $D := [O_5(G), A] \cong \mathbf{Z}_5$, and $V = [V,D] \oplus C_V(DA)$ with $m([V,D]) = 4$.

(3) If $m(A) = 1$ then $r = 1$ or 2, and if $r = 1$ then $D := [O_3(G), A] \cong \mathbf{Z}_3$, $V = [V,D] \oplus C_V(DA)$, and $m([V,D]) = 2$.

(4) If $r \leq 1$, then $r = 1$, $p = 3$, and $m(A) = 1$.

(5) If $m(A) > 1$ then $p = 3$, and for each hyperplane B of A such that $D_B := [C_{O_3(G)}(B), A] \neq 1$, we have $D_B \cong \mathbf{Z}_3$, $C_V(B) = [C_V(B), D_B] \oplus C_V(D_BA)$, and $m([C_V(B), D_B]) = 2$.

PROOF. First consider the case where $m(A) = 1$. Then $A = \langle t \rangle$ for some involution t, and as $r_{A,V} = \hat{q}(G,V) = r \leq 2$ and $A \in \hat{\mathcal{Q}}_*(G,V)$, we have

$$m([V,t]) = m(V/C_V(A)) = m(A)r = r \leq 2.$$

In particular, $m([V,t]) = r = 1$ or 2, which is the first assertion of (3). Furthermore $t \in \mathcal{T}$, and we may apply D.2.11 to elements $x \in O_p(G)^\#$ inverted by t. Parts (1) and (2) of D.2.11 establish (1). Further if $r = 1$ then $p = 3$ by D.2.11.1, so that (2) is vacuous. Thus to complete the proofs of (2) and (3), we may assume that $r = 1$ or 2, with $p = 2r+1$, and it remains to show $D := [O_p(G), t] \cong \mathbf{Z}_p$, $V = [V,D] \oplus C_V(DA)$, and $m([V,D]) = 2r$. By D.2.11, $m([V,x]) = 2r$, so $X_1 := \langle x \rangle \in \mathcal{X}_r$, in the notation of D.2.12. As t inverts X_1, it centralizes X_i for $i > 1$ by D.2.12.3. We may take $d \in D^\#$ inverted by t, and then we have symmetry between x and d, so $\langle d \rangle \in \mathcal{X}_r$ and hence $\langle d \rangle = X_1$ as $[d,t] \neq 1$. Therefore $D = X_1 \cong \mathbf{Z}_p$ as $m([V,D]) = 2r$, and using D.2.11.3, DA centralizes $C_V(D)$, completing the proof of (2) and (3). The proof in the case $m(A) = 1$ is now complete—since (5) is vacuous, and (1) and (2) imply (4) when $m(A) = 1$.

Therefore we may assume that $m(A) > 1$. Now (3) is vacuous. By Generation by Centralizers of Hyperplanes A.1.17, $D_B := [C_{O_p(G)}(B), A] \neq 1$ for some hyperplane B of A. Set $U_B := C_V(B)$. By the Thompson $A \times B$-Lemma, D_B is faithful on U_B, so as $D_B = [D_B, A]$, A is nontrivial on U_B. Now $B \neq 1$ as $m(A) > 1$, so by D.2.7:

$$A/B \cong Aut_A(U_B) \in \hat{\mathcal{Q}}_{r-\epsilon}(Aut_G(U_B), U_B) \text{ for some } \epsilon > 0.$$

Since $m(Aut_A(U_B)) = m(A/B) = 1$, and the pair $Aut_{D_BA}(U_B)$, U_B satisfy the hypotheses of D.2.13 in the roles of "G, V", we can appeal to our treatment of the case $m(A) = 1$. Since $m(A/B) = 1$ we have

$$\hat{q} := \hat{q}(Aut_{D_BA}(U_B), U_B) = r_{A/B,U_B} = r - \epsilon < 2, \qquad (*)$$

so (3) says $\hat{q} = 1$, $D_B \cong \mathbf{Z}_3$, $U_B = [U_B, D_B] \oplus C_{U_B}(D_BA)$, and $m([U_B, D_B]) = 2$. In particular this shows that $p = 3$ when $m(A) > 1$, completing the proof of (1), (2), and (5). Finally as $\hat{q} = 1$, (*) implies that $r > 1$, completing the proof of (4). □

LEMMA D.2.14. *Assume $r := \hat{q}(G,V) \leq 2$ and $A \in \hat{\mathcal{Q}}_*(G,V)$ is noncyclic and acts faithfully on $O(F(G))$. In addition assume $V = [V, O(F(G))]$ and $\hat{G}$ is a quotient of an SQTK-group. Then*

(1) $[O^{2,3}(F(G)), A] = 1$.

(2) $P := O_3(G) = P_1 \times \cdots \times P_s$ with $P_i \cong \mathbf{Z}_3$, $s = 2$ or 3, and $P = [P,A]$.

(3) $[V,P] = V_1 \oplus \cdots \oplus V_s$, where $V_i := [V,P_i]$ is of rank 2.

(4) $m(A) = 2$.

(5) $[C_V(P), A] = 0$.

(6) $r = 3/2$.

(7) Either

(i) $s = 2$ *and* $AP \cong S_3 \times S_3$ *with* P_1 *and* P_2 *diagonally embedded in the* S_3 *factors, so that* A *is transitive on* $\{P_1, P_2\}$*; or*

(ii) $s = 3$*,* A *acts on each* P_i*, and* $m([V_i, A]) = 1$ *for each* i*.*

PROOF. If $[O_p(G), A] \neq 1$, then as $m(A) > 1$ by hypothesis, D.2.13.5 forces $p = 3$; this establishes (1).

Next as A is faithful on $O(F(G))$ by hypothesis, in view of (1) it is faithful on $P = O_3(G)$. By hypothesis, G is a quotient of an SQTK-group, so as A is faithful on $O_3(G)$, it follows from A.1.31.1 that

$$C_P(a) \text{ is cyclic for all } \ a \in A^{\#}. \qquad (*)$$

Thus $m(A) \leq 2$ by A.1.5, giving (4), and A.1.5 together with (*) shows that

$$C_P(A) = 1.$$

By hypothesis $r_{A,V} = r \leq 2$, so it follows from (4) that

$$m(V/C_V(A)) = m(A)r = 2r \leq 4. \qquad (!)$$

If $r \leq 1$ then D.2.13.4 says $m(A) = 1$, contrary to our hypothesis. Thus $r > 1$, and hence by (!):

$$m(V/C_V(A)) = 4 \ \text{ or } 3, \text{ with } r = 2 \text{ or } 3/2, \text{ respectively.} \qquad (!!)$$

By Generation by Centralizers of Hyperplanes, there is a hyperplane B of A with $D_B := C_P(B) \neq 1$, and by the Thompson $A \times B$-Lemma, D_B is faithful on $U_B := C_V(B)$. We saw $C_P(A) = 1$, so $D_B = [D_B, A]$. Then by D.2.13.5, D_B is of order 3, $U_B = [U_B, D_B] \oplus C_V(D_B A)$, and $m([U_B, D_B]) = 2$. By (4), $m(B) = m(A) - 1 = 1$, so

$$2 = m([U_B, D_B]) = m(C_{[V,D_B]}(B)) \geq m([V, D_B])/2, \qquad (+)$$

and hence $m([V, D_B]) = 2$ or 4, with $[U_B, D_B] = C_{[V,D_B]}(B)$ of rank 2 if $m([V, D_B]) = 4$.

Let $\mathcal{Z}$ denote the set of A-invariant subgroups Z of $Z(P)$ of order 3; then $Z = [Z, A]$ as $C_P(A) = 1$. Thus as $m(A) = 2$, $B_Z := C_A(Z)$ is of order 2. The previous paragraph now applies to B_Z, $C_P(B_Z)$ in the roles of "B, D_B", so we conclude

$$U_Z := [V, Z] \leq [V, C_P(B_Z)], \ m(U_Z) = 2 \text{ or } 4,$$
$$\text{and } \ [U_{B_Z}, Z] = C_{U_Z}(B) \text{ is of rank 2 if } \ m(U_Z) = 4. \qquad (++)$$

Therefore $P/C_P(U_Z) \cong \mathbf{Z}_3$ or E_9 as $GL([V, Z]) \leq GL_4(2)$, and hence $\Phi(P) \leq C_P(U_Z)$. As Z is faithful on U_Z, this shows $Z \not\leq \Phi(P)$. Thus $\mathcal{Z} \cap \Phi(P) = \emptyset$, so $\Phi(P) = 1$.

Let $s := m_3(P)$, and note $s > 1$ since A is noncyclic and faithful on P. By (*), distinct involutions a and b invert subspaces of P of dimension at least $s - 1$, so the product ab centralizes a subspace of dimension at least $s - 2$. Thus $s - 2 \leq 1$, so $s = 2$ or 3. Since $C_P(A) = 1$, $P = [P, A]$, completing the proof of (2). Notice also that as P is abelian, *any* A-invariant subgroup X of order 3 is central in P, and hence is in $\mathcal{Z}$.

We next establish (5), so suppose that $[C_V(P), A] \neq 0$. First $O^{2,3}(F(G))$ centralizes A by (1), and hence normalizes $C_V(PA)$, and second $V = [V, O(F(G))]$ by hypothesis, so that $C_V(P) = [C_V(P), O^{2,3}(F(G))]$. Thus $m(C_V(P)/C_V(PA)) \geq$

3. On the other hand as A is noncyclic and faithful on P, $m([V,P]/C_{[V,P]}(A)) \geq 2$. Therefore $m(V/C_V(A)) \geq 5$, contrary to (!!). We conclude that (5) holds, so it remains to establish (3), (6), and (7).

By (5), PA satisfies the hypotheses of the lemma on $[V,P]$ with $r = r_{A,V} = r_{A,[V,P]}$, so if $[V,P] < V$, then the lemma holds by induction on $m(V)$. Therefore we may assume that $V = [V,P]$.

Suppose that $V = [V,Z] = U_Z$ for some $Z \in \mathcal{Z}$. By (++), $m(V) = m(U_Z) = 2$ or 4. Then as $s \geq 2$ by (2), $m(V) = 4$, which in turn forces $s = 2$ and $AP \cong S_3 \times S_3$, establishing (3) for suitable P_i. By (!!), $r = 3/2$ or 2, so A contain no transvection, and hence $N_A(P_1) = N_A(P_2)$ is the group of order 2 inducing a transvection on both $[V,P_1]$ and $[V,P_2]$, and A interchanges P_1 and P_2, Therefore conclusion (i) of (7) holds, and $m(C_V(A)) = 1$, establishing (6) and completing the proof in this case.

Thus we may assume that $[V,Z] < V$ for all $Z \in \mathcal{Z}$, and hence $W_Z := C_V(Z) \neq 0$. As $Z \leq P$ which is elementary abelian by (2), the decomposition $V = U_Z \oplus W_Z$ is PA-invariant.

By (++), $m(U_Z) = 2$ or 4. Suppose first that $m(U_Z) = 4$ for some $Z \in \mathcal{Z}$. In this case by (++), $[U_{B_Z}, Z] = C_{U_Z}(B_Z)$ is of rank 2. Then as $Z = [Z,A]$, $m(C_{U_Z}(A)) = 1$, so $m(U_Z/C_{U_Z}(A)) = 3$. By (!!) $m(V/C_V(A)) \leq 4$, so $m(W_Z/C_{W_Z}(A)) \leq 1$. As $V = [V,P]$, $W_Z = [W_Z,P]$, so as $P = [P,A]$ by (2) and $m(W_Z/C_{W_Z}(A)) \leq 1$, we conclude $m(W_Z) = 2$, $m(V/C_V(A)) = 4$, and $r = 2$. However, now $C := C_A(W_Z)$ is of order 2 with $r_{C,U_Z} = r_{C,V} = 2 = r$, so $C \in \hat{\mathcal{Q}}_r(G,V)$, contradicting the minimality under inclusion of $A \in \hat{\mathcal{Q}}_*(G,V)$.

We've shown that $m(U_Z) = 2$ for each $Z \in \mathcal{Z}$. Hence $P = Z \times R$ with $R := C_P(U_Z)$ faithful on W_Z. Set $P_1 := Z$ and $V_1 := U_Z = [V,P_1]$. Pick $P_2 \in \mathcal{Z} \cap R$; thus $V_2 := [V,P_2]$ is of rank 2. As $m_3(P) = 2$ or 3 from (2), either $R = P_2$, or $R = P_2 \times P_3$ with $P_3 \in \mathcal{Z}$. In the former case $V = [V,P] = V_1 \oplus V_2$ is of rank 4, with A centralizing a 2-subspace; this gives $r = 1$, contradicting (!!), so the latter case holds. Now (3) and conclusion (ii) of (7) are satisfied with respect to this decomposition of P. As $P = [P,A]$, $m(C_{V_i}(A)) = 1$ for each i, so $m(C_V(A)) = 3$ and hence $r = 3/2$ by (4), establishing (6) and completing the proof of the lemma. □

We are now ready to establish the main result of this section: Theorem D.2.17, which determines the modules V such that $\hat{q}(\hat{G}\varphi, V) \leq 2$ for representations $\varphi : \hat{G} \to GL(V)$ of SQTK-groups $\hat{G}$ with $F^*(\hat{G}\varphi) = O(F(\hat{G}\varphi))$.

In the spirit of the case of FF*-offenders, where we defined $J(G,V) := \langle \mathcal{P}(G,V) \rangle$, we set

Definition D.2.15. $\hat{J}(G,V) := \langle \hat{\mathcal{Q}}_*(G,V) \rangle$.

From now on we focus on the case where $G = \hat{J}(G,V)$.

In order to state and prove Theorem D.2.17 efficiently, we define an appropriate notion of decomposability:

Definition D.2.16. If $G = \hat{J}(G,V)$, we write $(G,V) = (G_1,V_1) + (G_2,V_2)$ if there is a proper partition $\hat{\mathcal{Q}}_*(G,V) = \mathcal{Q}_1 \cup \mathcal{Q}_2$ such that $G_i = \langle \mathcal{Q}_i \rangle$, $V = V_1 \oplus V_2$, $[G_i,V] \leq V_i$, and $[G_i, V_{3-i}] = 1$, for $i = 1,2$. We say (G,V) is *decomposable* if $(G,V) = (G_1,V_1) + (G_2,V_2)$ with $G_1 \neq 1 \neq G_2$; and we say (G,V) is *indecomposable* otherwise.

As G is finite, (G,V) can always be written as the sum of indecomposables.

Theorem D.2.17. *Assume* $r := \hat{q}(G,V) \leq 2$, $G = \hat{J}(G,V)$, *and* $V = [V, F(G)]$, *with* (G,V) *indecomposable. Assume further that* $F^*(G) = F(G)$, $O_2(G) = 1$, *and* G *is a quotient of an SQTK-group. Then* $\hat{\mathcal{Q}}_*(G,V) = Syl_2(G)$, *and one of the following holds:*

(1) $G \cong S_3$, $m(V) = 2r$, *and* $r = 1$ *or* 2.

(2) $G \cong D_{10}$, $m(V) = 4$, *and* $r = 2$.

(3) G *is* E_9 *extended by an involution inverting* $F(G)$, *with* $m(V) = 4$, *and* $r = 2$.

(4) G *is* 3^{1+2} *extended by an involution inverting* $F(G)/\Phi(F(G))$, $m(V) = 6$, *and* $r = 2$.

(5) $m(V) = 4$, $G = \Omega_4^+(V) \cong S_3 \times S_3$, *and* $r = 3/2$.

(6) G *is* E_{27} *extended by a 4-group,* $m(V) = 6$, *and* $r = 3/2$.

The proof of Theorem D.2.17 involves a series of reductions. In the remainder of this subsection, assume the hypotheses of the Theorem.

Consider any $A \in \hat{\mathcal{Q}}_*(G,V)$. By hypothesis, $F^*(G) = F(G)$ and $O_2(G) = 1$, so A is faithful on $F(G) = O(F(G))$. Then A is nontrivial on $O_p(G)$ for some odd p, so we have the hypotheses of D.2.13. In particular by D.2.13.3:

Lemma D.2.18. *If* $|A| = 2$ *then the hypotheses of D.2.13 are satisfied and* $r = 1$ *or* 2.

Also by hypothesis $V = [V, F(G)]$, so as $F(G) = O(F(G))$, we have $V = [V, O(F(G))]$.

Lemma D.2.19. *If* $m(A) > 1$ *then the hypotheses of D.2.14 are satisfied,*

(1) $r = 3/2$, *and*

(2) all *members of* $\hat{\mathcal{Q}}_*(G,V)$ *are noncyclic.*

Proof. If A is noncyclic, the hypotheses of D.2.14 are satisfied, so (1) follows from D.2.14.6. Then (2) follows from (1) and D.2.18. □

Note as $V = [V, O(F(G))] \oplus C_V(O(F(G))$ and $V = [V, O(F(G))]$, $C_V(O(F(G)) = 0$. In particular $C_V(G) = 0$, and hence as $G = \hat{J}(G,V)$:

Lemma D.2.20. $C_V(G) = 0$, *so no non-zero subspace of* V *is centralized by each member of* $\hat{\mathcal{Q}}_*(G,V)$.

Lemma D.2.21. *If* A *is faithful on* $O_3(G)$, *then* $m_3(C_G(a)) \leq 1$ *for each* $a \in A^\#$.

Proof. This follows from A.1.31.1 and our hypothesis that G is the image of an SQTK-group. □

Lemma D.2.22. *If* A *is noncyclic, then conclusion (5) or (6) of Theorem D.2.17 holds.*

Proof. Suppose that $m(A) > 1$. Then by D.2.19.2, *all* members of $\hat{\mathcal{Q}}_*(G,V)$ are noncyclic, and we may apply D.2.14. By D.2.14.1, $[O^3(F(G)), A] = 1$, and then $G = \hat{J}(G,V)$ centralizes $O^3(F(G))$, so that $F^*(G) = RZ(G)$, where $R := O_3(G)$. Now RA and its action on V are described in D.2.14. In particular, $R = [R, A]$ by D.2.14.2, and $s := m_3(R) = 2$ or 3. Further A centralizes $C_V(R)$ by D.2.14.5, so $G = \hat{J}(G,V)$ centralizes $C_V(R)$, and hence $C_V(R) = 0$ by D.2.20, so that $V = [V, R]$.

Suppose first that $s = 3$. Then case (ii) of D.2.14.7 holds, so that A acts on each P_i, and hence also on each of the commutator subspaces $V_i := [V, P_i]$ of V defined in D.2.14.3. Then as $G = \hat{J}(G,V)$, G acts on each V_i. As G acts faithfully on the sum $V = [V,R]$ of the V_i,

$$RA \le G \le G_0 := \bigcap_i N_{GL(V)}(V_i) \cong S_3 \times S_3 \times S_3,$$

with $|G_0 : G| \le 2$. But $G < G_0$ by D.2.21, so A is Sylow in $G = RA$, and case (6) of Theorem D.2.17 holds in this case.

Thus we may take $s = 2$, so from D.2.14.7,

$$AR \cong S_3 \times S_3 \le G \le G_0 := N_{GL(V)}(\{P_1, P_2\}) = O_4^+(2).$$

But if $G = G_0$, there are transvections in G, so $r = 1$, contradicting D.2.19. Thus $RA = G$ with A Sylow in G, so that case (5) of Theorem D.2.17 holds. □

By D.2.22, for the remainder of the proof we may assume that each $A \in \hat{\mathcal{Q}}_*(G,V)$ is of order 2. Thus by D.2.18, D.2.13 applies and $r = 1$ or 2. Now $A = \langle t \rangle$ with $r = r_{A,V} = m(V/C_V(A)) = m([V,t]) = 1$ or 2, so that t is in the set $\mathcal{T}$ of Definition D.2.10; thus

$$G = \langle \mathcal{T} \rangle$$

as $G = \langle \hat{\mathcal{Q}}_*(G,V) \rangle$. By D.2.13.1, $[O^{3,5}(F(G)), A] = 1$.

Lemma D.2.23. *If* $[O_5(G), A] \ne 1$ *then conclusion (2) of Theorem D.2.17 holds.*

Proof. Suppose $[O_5(G), A] \ne 1$. Then by D.2.13.2, $r = 2$, $D := [O_5(G), A] \cong \mathbf{Z}_5$, $V = [V,D] \oplus C_V(DA)$, and $m([V,D]) = 4$. Now D is a member of the set $\mathcal{X}_2$ of D.2.12, and as $r = 2$, $\mathcal{T}$ contains no transvections—that is $m([V,u]) = 2$ for each $u \in \mathcal{T}$. Thus $D \trianglelefteq \langle \mathcal{T} \rangle = G$ by D.2.12.4. Applying D.2.12.3 to $B \in \hat{\mathcal{Q}}_*(G,V)$, either B centralizes $D[V,D]$, or $D = [D,B]$ and $[C_V(D), B] = 1$. In the latter case B is faithful on $[V,D]$, and as $N_{GL([V,D])}(D)$ is the multiplicative group $F^\times$ of $F := \mathbf{F}_{16}$ extended by $Aut(F) \cong \mathbf{Z}_4$, $B \in A^D$; indeed $[F^\times, A] = D$ and hence $A^{F^\times} = A^D$. Therefore we have a partition $\hat{\mathcal{Q}}_*(G,V) = \mathcal{Q}_1 \cup \mathcal{Q}_2$, where $\mathcal{Q}_1 := A^D$, and $\mathcal{Q}_2$ consists of those B centralizing $D[V,D]$. Set $V_1 := [V,D]$ and $V_2 := C_V(D)$, and observe that members of $\mathcal{Q}_i$ normalize V_i and centralize V_{3-i}, so that $[V,B] \le V_i$ for each $B \in \mathcal{Q}_i$. Hence as (G,V) is indecomposable by hypothesis, we must have $\mathcal{Q}_2 = \emptyset$. Thus $G = \hat{J}(G,V) = \langle A^D \rangle \cong D_{10}$, and as A centralizes $C_V(D)$, so does $A^D = \hat{\mathcal{Q}}_*(G,V)$, so $C_V(D) = 0$ by D.2.20. Therefore case (2) of Theorem D.2.17 holds. □

Because of D.2.23 and the remark preceding it, in the remainder of the proof we may assume each $A \in \hat{\mathcal{Q}}_*(G,V)$ centralizes $O^3(F(G))$, so as $G = \hat{J}(G,V)$, $O^3(F(G)) \le Z(G)$. Set $R := O_3(G)$. Then $F^*(G) = F(G) = RZ(G)$, and A is faithful on R.

Lemma D.2.24. *If* $r = 1$, *then conclusion (1) of Theorem D.2.17 holds.*

Proof. Suppose $r = 1$. We argue as in the proof of D.2.23, replacing D.2.13.2 by D.2.13.3, and $\mathbf{F}_{16}$ by $\mathbf{F}_4$, to conclude $V = [V,D]$ is of rank 2, and $G \cong S_3$. Thus A is Sylow in G, and conclusion (1) of Theorem D.2.17 holds. □

Since D.2.22 reduced us to the case $|A| = 2$, by D.2.18 and D.2.24, we may assume for the remainder of the proof that $r = 2$. In particular, G contains no transvections on V, so all members of $\mathcal{T}$ satisfy

$$m([V,t]) = 2 = r. \qquad (*)$$

LEMMA D.2.25. *Let $\mathcal{X}_1$ denote the set of subgroups X of order 3 in R with $m([V,X]) = 2$. If $\mathcal{X}_1 \neq \emptyset$, then conclusion (3) of Theorem D.2.17 holds.*

PROOF. Assume $\mathcal{X}_1 = \{P_1, \dots, P_n\} \neq \emptyset$. Then by D.2.12, $Y := \langle \mathcal{X}_1 \rangle = P_1 \times \cdots \times P_n$ and $[V,Y] = V_1 \oplus \cdots \oplus V_n$, where $V_i := [V, P_i]$. As usual $V = [V,Y] \oplus C_V(Y)$ by Coprime Action. Set $\mathcal{T}_+ := C_{\mathcal{T}}(Y)$ and $\mathcal{T}_- := \mathcal{T} - \mathcal{T}_+$.

Suppose $\mathcal{T}_+ = \mathcal{T}$, so that $\mathcal{T} \subseteq C_G(Y)$. Then any $t \in \mathcal{T}$ centralizes P_i and hence also V_i. Thus $G = \langle \mathcal{T} \rangle$ centralizes $[V,Y] \neq 0$, contrary to D.2.20. Thus $\mathcal{T}_+ \neq \mathcal{T}$, and hence $\mathcal{T}_-$ is nonempty.

Let $t \in \mathcal{T}_-$. If t is not in the kernel K of the permutation representation of G on $\mathcal{X}_1$, then by D.2.12.5, t induces a transposition on $\mathcal{X}_1$, and centralizes those members of $\mathcal{X}_1$ which it fixes. Further as $m([V,t]) = 2$, $[V,t] \leq [V,Y]$, so t centralizes $C_V(Y)$.

Suppose on the other hand that $t \in K$. Then t normalizes each P_j, so as $t \in \mathcal{T}_-$, t inverts some P_i. Then $m([V_i,t]) = 1$, so as $m([V,t]) = 2$, $m([C_V(P_i),t]) = 1$. Suppose that $[C_V(Y),t] \neq 0$. Then we may apply D.2.11.1 to $C_V(Y)$ to conclude that t inverts some $x \in O_3(G)$ of order 3 with $m([C_V(Y),x]) = 2$. Let $X_t := \langle x \rangle$. As $[C_V(Y), X_t] \neq 0$, $X_t \notin \mathcal{X}_1$, so $m([V,X_t]) > 2$ and hence $[V,Y,X_t] \neq 0$. As $m([V,t]) = 2$, $[V_j,t] = 0$ for all $j \neq i$. As t is in the kernel of the action of G on $\mathcal{X}_1$, so is $X_t = [X_t,t]$, and hence also $[V_j,X_t] = 0$ for all $j \neq i$. Thus $m([V,X_t]) = 4$, and we see from the structure of $GL_4(2)$ that $P_iX_t = P_i \times X'$ for some $X' \in \mathcal{X}_1$ with $[V,X'] = [C_V(Y),X_t]$, contradicting $[V,X'] \leq [V,Y]$. Hence t centralizes $C_V(Y)$ when $t \in K$.

Thus the members of $\mathcal{T}_-$ centralize $C_V(Y)$ and normalize the complement $[V,Y]$, while members of $\mathcal{T}_+$ centralize $[V,Y]$ and normalize $C_V(Y)$. Therefore as (G,V) is indecomposable, and $\mathcal{T}_- \neq \emptyset$, $\mathcal{T}_- = \mathcal{T}$. In particular, $G = \langle \mathcal{T}_- \rangle$, so as $\mathcal{T}_-$ centralizes $C_V(Y)$, $C_V(Y) = 0$ by D.2.20, and hence $V = [V,Y]$. Now t centralizes all but two members of $\mathcal{X}_1$, so by D.2.21, $n := |\mathcal{X}_1| \leq 3$. Furthermore if $n = 3$, we claim t normalizes each P_k: for otherwise $1 \neq C_{P_1P_2}(t)$ while $[P_3,t] = 1$ by D.2.12.5, contrary to D.2.21. Note if $n = 1$, then $m(V) = 2$ and $m([V,t]) = 1$, contrary to (*); thus $n > 1$.

Suppose $n = 2$. Then $m(V) = 4$, so

$$P\langle t \rangle \leq G \leq N_{GL_4(2)}(P) = O_4^+(2).$$

Recall G contains no transvections, and each member of $\hat{\mathcal{Q}}_*(G,V)$ is of order 2, with $r = 2$ by (*); thus a Sylow 2-group of G is of order 2, so in fact $G = P\langle t \rangle$ has order 18 and conclusion (3) of Theorem D.2.17 holds.

Thus we may assume that $n = 3$. Recall t fixes each P_k, so as $\mathcal{T}_-$ generates G, each P_k is normal in G. We may assume t inverts P_1 and P_2 and centralizes P_3, and as $G = \langle \mathcal{T}_- \rangle$, there is also $t' \in \mathcal{T}_-$ inverting P_3 and one other P_k. But we may take $[t,t'] = 1$ since each P_k is normal, so that $A_0 := \langle t,t' \rangle \cong E_4$ with $r_{A_0,V} = 3/2$, contrary to $r = 2$ by (*). $\square$

Appealing to D.2.25, for the remainder of the proof we may assume that $\mathcal{X}_1$ is empty. Let $\mathcal{Y}$ consist of those subgroups Y of R of order 3 with $m([V,Y]) = 4$.

Let X be any nontrivial cyclic subgroup of $F(G) = F^*(G)$ inverted by $t \in \mathcal{T}$; such subgroups exist as $O_2(G) = 1$.

LEMMA D.2.26. *(1)* $X \in \mathcal{Y}$.
(2) $[V,t] = [V,X,t]$, $[C_V(X),t] = 0$, *and* $[C_G(X[V,X]),t] = 1$.
(3) $V = [V, \langle \mathcal{Y} \rangle]$.

PROOF. As $\mathcal{X}_1 = \emptyset$, $m([V,X_1]) \geq 4$ for each X_1 of order 3 in R. Recall we have reduced to the case where t centralizes $O^3(F(G))$, so $X \leq R$. By (*), $r = m([V,t]) = 2$, so $|X| = 3$ by D.2.11.2. As $m([V,X])/2 = m([V,X,t]) \leq m([V,t]) = 2$, $m([V,X]) = 4$, so (1) holds, and (2) follows from D.2.11.3. Since each $t' \in \mathcal{T}$ is faithful on $F(G)$, t' inverts some $Y \in \mathcal{Y}$, and hence centralizes $C_V(Y) \geq C_V(\langle \mathcal{Y} \rangle)$. Therefore $C_V(\langle \mathcal{Y} \rangle) = 0$ by D.2.20. Then as $\langle \mathcal{Y} \rangle \leq O_3(G)$ is a 3-group, (3) holds by Coprime Action. □

LEMMA D.2.27. *If* $|R| = 3$ *or* $\mathcal{Y} = \{X\}$, *then conclusion (1) of Theorem D.2.17 holds.*

PROOF. If $|R| = 3$ then $\mathcal{Y} = \{R\} = \{X\}$; thus in any case we may assume $\mathcal{Y} = \{X\}$. Thus $V = [V,X]$ has rank 4 by D.2.26.3, and then

$$G \leq N_{GL(V)}(X) \cong \Gamma L_2(4).$$

As we saw $F^*(G) = F(G) = RZ(G)$, and $X = [X,t]$, we conclude that either $G \cong S_3$ or $F(G) \cong E_9$. In the former case, conclusion (1) of Theorem D.2.17 holds with $r = 2$, while in the latter case $\mathcal{X}_1 \neq \emptyset$, contrary to an earlier reduction. □

By D.2.27 we may assume for the remainder of the proof that $|\mathcal{Y}| > 1$ and $X < R$; so in particular $X < N_R(X) = C_R(X) =: P$.

The remainder of the argument is devoted to reducing to conclusion (4) of Theorem D.2.17. It will take some further work to nail down all the details.

LEMMA D.2.28. *(1)* $P = X \times Z$ *where* $Z := C_P(t)$.
(2) $P \cong E_9$.
(3) $U := [V,P] = [V,Z]$ *is of rank* 6.
(4) $\mathcal{Y} \cap P$ *is the set of subgroups of* P *of order* 3 *distinct from* Z.

PROOF. Let $Z := C_P(t)$, and suppose (1) fails, so that $P > X \times Z$. Then t inverts some cyclic subgroup Y of P with $Y \not\leq X$, so by D.2.26.1, $Y \in \mathcal{Y} \cap P - \{X\}$. But by D.2.26.2, t centralizes $C_V(X)$, so $Y = [Y,t]$ does too, and then $[V,Y] \leq [V,X]$. Therefore $[V,Y] = [V,X]$ since $m([V,Y]) = 4$. But then $\mathcal{X}_1 \cap XY \neq \emptyset$, contrary to our earlier reduction to the case $\mathcal{X}_1$ empty. Thus (1) is established.

By D.2.21, Z is cyclic. If $\mathcal{Y} \cap P = \{X\}$, then as $\mathcal{Y}$ is G-invariant, X is weakly closed in $P = N_R(X)$, so $\mathcal{Y} = \{X\}$ by A.1.13, contrary to D.2.27. Thus $|\mathcal{Y} \cap P| > 1$.

Next set $P_0 := \Omega_1(P)$, and note that $\mathcal{Y} \cap P \subseteq P_0$ and $P_0 = X \times \Omega_1(Z)$. As Z is cyclic, $P_0 \cong E_9$.

Suppose $C_{P_0}([V,X]) \neq 1$. Then $P_0 = X \times C_{P_0}([V,X])$, and by D.2.26.2, t centralizes $C_{P_0}([V,X])$, forcing $C_{P_0}([V,X]) = C_{P_0}(t)$ since both have order 3. Then $\mathcal{Y} \cap P = \{X\} \cup (\mathcal{Y} \cap C_{P_0}([V,X]))$, since 3-elements w projecting on both factors have $m([V,w]) > 4$. Hence as Z is cyclic, $\mathcal{Y} \cap P = \{X, Y\}$ with $Y := C_{P_0}(t)$. Then as R is of odd order, $N_R(P)$ fixes $\mathcal{Y} \cap P$ pointwise, so $R = P = X \times C_R(t)$ and hence $\mathcal{Y} = \{X, Y\}$. Thus by D.2.26.3, $V = [V, XY] = [V,X] \oplus [V,Y]$, contradicting our hypothesis that (G,V) is indecomposable.

Therefore $C_{P_0}([V, X]) = 1$, and hence P_0 is faithful on $[V, X]$ of rank 4. Then $P = X \times Z$ is also faithful on $[V, X]$, and hence (2) holds by our usual appeal to the structure of $GL_4(2)$.

We have shown that there is $Y \in \mathcal{Y} \cap P - \{X\}$, and as $\mathcal{X}_1 = \emptyset$, $[V, X] \neq [V, Y]$. As P is faithful on $[V, X]$, $[V, X] \cap [V, Y] \neq 0$. Therefore $[V, X, Y]$ is of rank 2, and $U := [V, XY] = [V, X] + [V, Y]$ is of rank 6. In particular, P contains a unique subgroup Z_0 of order 3 with $[V, Z_0] = U$ of rank 6, and the remaining subgroups of order 3 constitute $\mathcal{Y} \cap P$. As Z_0 is unique, t normalizes Z_0. Then as $P_0 = X \times C_P(t)$, we must have $Z_0 = C_P(t) = Z$. This completes the proof of (3) and (4), and hence of the lemma. □

Lemma D.2.29. *$P < R$.*

Proof. Suppose $R = P$. Then $P \cong E_9$ by D.2.28.2, so $|\mathcal{Y}| = 3$ by D.2.28.4. Then each member of $\mathcal{T}$ induces a transposition on $\mathcal{Y}$ using D.2.28.1, and as $\langle \mathcal{T} \rangle = G$, we conclude G induces S_3 on $\mathcal{Y}$. By D.2.26.3, $V = [V, \langle \mathcal{Y} \rangle]$, so $V = U$ has rank 6 by D.2.28.3. Then the kernel K of the action of $GL(V) \cong GL_6(2)$ on $\mathcal{Y}$ is the direct product of 3 copies of S_3. As $r = 2$ by (*), $K \cap G$ contains no transvection, and as each member of $\mathcal{T}$ induces a transposition on $\mathcal{Y}$, $\mathcal{T} \cap K = \emptyset$. We conclude $|K \cap G : P| \leq 2$ (possibly $K \cap G$ might contain an involution which inverts P, and hence is central modulo P). But then as $G/K \cong S_3$, $P < R$, contrary to assumption. □

We now complete the proof of Theorem D.2.17 by showing that conclusion (4) of the Theorem holds. By D.2.29, $P < R$, and hence $P < Q := N_R(P)$. Note that Q normalizes the unique subgroup Z of order 3 in P with $m([V, Z]) = 6$ in view of D.2.28.3. Recall $P = C_R(X)$, so $P = C_R(P)$. Then as $P \cong E_9$, $Aut_Q(P) = Q/P$ is of order 3, so Q is extraspecial of order 27 with center Z. Further $Q\langle t \rangle$ induces S_3 on $\mathcal{Y} \cap P$, so t inverts Q/P as well as X, and therefore also Q/Z, so $Q \cong 3^{1+2}$ is of exponent 3. As $C_R(Q) \leq C_R(X) = P \leq Q$, $Z(R) = Z(Q) = Z$. Further t centralizes $C_V(X)$ by D.2.26.2, so as $[V, X] \leq U = [V, Z]$ by D.2.28.3, $C_V(Z) \leq C_V(X)$ and t centralizes $C_V(Z)$. Then as $V = [V, Z] \oplus C_V(Z)$,

$$[C_R(U), t] \leq C_G(U) \cap C_G(C_V(Z)) = C_G(V) = 1,$$

so $C_R(U) \leq C_R(t)$. But $Z \leq C_R(t)$ which is cyclic by D.2.21, so $\Omega_1(C_R(U)) \leq Z$ which is faithful on U. Thus $\Omega_1(C_R(U)) = 1$, and hence $C_R(U) = 1$; that is, R is faithful on U.

Next as $R\langle t \rangle$ centralizes Z, U has the structure of a 3-dimensional $\mathbf{F}_4$-space preserved by $R\langle t \rangle$, with Z defining the scalar multiplication; that is $R\langle t \rangle \leq GL_3(4)$. As $GL_3(4)$ has Sylow-3 group isomorphic to $\mathbf{Z}_3 \wr \mathbf{Z}_3$, $R = QC_R(t)$ with $C_R(t) \cong \mathbf{Z}_3$ or E_9. As $C_R(t)$ is cyclic by D.2.21, we have $C_R(t) = Z$ and hence $Q = R = O_3(G)$. Now $V = U$ is of rank 6 by D.2.26.3, and $N_{GL(V)}(Q)/Q \cong GL_2(3)$, so as $G = \langle \mathcal{T} \rangle$, it follows that $G = Q\langle t \rangle$. In particular, conclusion (4) of Theorem D.2.17 holds.

Thus the proof of D.2.17 is complete.

D.3. Modules with $\hat{\mathbf{q}} \leq 2$

In this section we extend our analysis (begun in B.4.5 for the case of V irreducible) of modules V for SQTK-groups G satisfying $\hat{q}(G, V) \leq 2$.

During much of this section we assume:

HYPOTHESIS D.3.1. *M is a quotient of of an SQTK-group, $T \in Syl_2(M)$, V_M is a faithful $\mathbf{F}_2M$-module, and $M_+ \trianglelefteq M$ satisfies either*

(i) $M_+ = \langle L^T \rangle$ for some component L of M, or

(ii) $M_+ = O_p(M_+)$ for some odd prime p, and T is irreducible on $M_+/\Phi(M_+)$.

Assume further that $V_M := \langle V^M \rangle$ for some $O_2(M_+T)$-invariant member V of $Irr_+(M_+, V_M)$. Set $\tilde{V}_M := V_M/C_{V_M}(M_+)$.

During the proof of our Main Theorem, we will apply the results of this section to certain internal modules of 2-locals in QTKE-groups. The following hypothesis, which holds for example in our Fundamental Setup FSU (3.2.1), is sufficient to achieve Hypothesis D.3.1 on the relevant internal modules:

HYPOTHESIS D.3.2. *$\dot{M}$ is an SQTK-group, $\dot{T} \in Syl_2(\dot{M})$, and $\dot{M}_+ = O^2(\dot{M}_+) \trianglelefteq \dot{M}$ such that either*

(i) $\dot{M}_+ = \langle \dot{L}^{\dot{T}} \rangle$ for some $\dot{L} \in \mathcal{C}(\dot{M})$ with $\dot{L}/O_2(\dot{L})$ quasisimple, or

(ii) $\dot{M}_+ = O_{2,p}(\dot{M}_+)$ for an odd prime p, and $\dot{T}$ is irreducible on $\dot{M}_+/O_{2,\Phi}(\dot{M}_+)$.

Further assume that $V_+ = Q_+/Q_-$ is an $\dot{M}$-invariant elementary abelian section of $O_2(\dot{M})$, with $O_2(\dot{M}_+/C_{\dot{M}_+}(V_+)) = 1$, and $V = Q_V/Q_-$ is an $O_2(\dot{M}_+\dot{T})$-invariant member of $Irr_+(\dot{M}_+, V_+)$.

Set $Q_M := \langle Q_V^{\dot{M}} \rangle$ and $V_M := Q_M/Q_-$.

LEMMA D.3.3. *Assume Hypothesis D.3.2 and let $M := Aut_{\dot{M}}(V_M)$, $M_+ := Aut_{\dot{M}_+}(V_M)$, and $T := Aut_{\dot{T}}(V_M)$. Then*

(1) M, M_+, V_M, V satisfy Hypothesis D.3.1.

(2) $O_2(\dot{M}_+) \leq C_{\dot{M}_+}(V_M) \leq O_{2,\Phi}(\dot{M}_+)$.

PROOF. By Hypothesis D.3.2, V_+ is an $\mathbf{F}_2\dot{M}$-module, $V \in Irr_+(\dot{M}_+, V_+)$, and $V_M = \langle V^{\dot{M}} \rangle$. Thus by definition of M and M_+, V_M is a faithful $\mathbf{F}_2M$-module, and $V \in Irr_+(M_+, V_M)$ with $V_M = \langle V^M \rangle$. Further $M = \dot{M}/C_{\dot{M}}(V_M)$ is a quotient of the SQTK-group $\dot{M}$.

By Hypothesis D.3.2, $O_2(\dot{M}_+)$ centralizes V_+, so it centralizes the submodule V_M. As $V \in Irr_+(\dot{M}_+, V_+)$, $0 \neq V = [V, \dot{M}_+]$. Further in both cases (i) and (ii) of Hypothesis D.3.2, each proper subgroup of M_+ normal in M_+T is contained in $O_{2,\Phi}(M_+)$; so $C_{M_+}(V_M) \leq O_{2,\Phi}(M_+)$, completing the proof of (2). Then (2), Hypothesis D.3.2, and the first paragraph of this proof imply (1). □

Here are some elementary consequences of Hypothesis D.3.1:

LEMMA D.3.4. *Assume Hypothesis D.3.1 and let $V_T := \langle V^T \rangle$. Then*

(1) $O_2(M) = 1 = O_2(M_+T/C_{M_+T}(V_T))$.

(2) $C_{M_+}(V_T) \leq \Phi(M_+)$.

(3) $V_M = [V_M, M_+]$, $V_T = [V_T, M_+]$, and $\tilde{V}_M$ and $\tilde{V}_T$ are semisimple M_+-modules, with M, T transitive on the M_+-homogeneous components of $\tilde{V}_M$, $\tilde{V}_T$, respectively.

(4) $C_{V_M}(M_+) = \langle C_V(M_+)^M \rangle$.

(5) $F^*(Aut_{M_+T}(V_T)) = Aut_{M_+}(V_T)$ and $F^*(Aut_{M_+T}(V)) = Aut_{M_+}(V)$ are semisimple in (i) and a p-group in (ii). [1]

(6) If $C_V(M_+) = 0$, then V is a TI-set under M.

[1]but notice we make no corresponding assertion for M and V_M.

PROOF. Let $X := M_+T$ or M and $V_X := \langle V^X \rangle$. Since V is M_+-invariant, $V_{M_+T} = V_T$. As $M_+ \trianglelefteq X$ and $V = [V, M_+]$, $V_X = [V_X, M_+]$. As $V \in Irr_+(M_+, V_+)$, $\tilde{V}$ is an irreducible M_+-module, so as $\tilde{V}_X = \langle \tilde{V}^X \rangle$, it follows that $\tilde{V}_X$ is a semisimple M_+-module, and X is transitive on its M_+-homogeneous components. Further as M_+T is transitive on the M_+-components of $V_{M_+T} = V_T$, T is also transitive on those components, completing the proof of (3). Indeed

$$C_V(M_+) \leq V_0 := \langle C_V(M_+)^M \rangle \leq C_{V_M}(M_+),$$

so the same argument applies to V_M/V_0; in particular $C_{V_M/V_0}(M_+) = 0$, so (4) holds.

By Hypothesis D.3.1, V is $O_2(M_+T)$-invariant, so $O_2(M_+T)$ centralizes V by A.1.41. Since $O_2(M) \leq O_2(M_+T)$ by A.1.6, we conclude $O_2(X)$ centralizes $V_X = \langle V^X \rangle$, establishing (1).

By (1), $O_2(Aut_{M_+T}(V_T)) = 1$, so from (1), and (i) and (ii) of D.3.1,

$$F^*(Aut_{M_+T}(V_T)) = Aut_{M_+}(V_T)$$

is semisimple or a p-group in the respective case. A similar argument works for V, so (5) holds.

As $V \in Irr_+(M_+, V_M)$, $1 \neq V = [V, M_+]$. Further in both cases (i) and (ii) of Hypothesis D.3.1, each proper subgroup of M_+ normal in M_+T is contained in $\Phi(M_+)$, so $C_{M_+}(V_T) \leq \Phi(M_+)$, establishing (2). If $C_V(M_+) = 0$ and $0 \neq v \in V \cap V^x$ for some $x \in M$, then V and V^x are M_+-irreducibles, so $V = \langle v^{M_+} \rangle = V^x$, establishing (6). □

The situation in case (ii) of D.3.1 can be handled using Theorem D.2.17 of the previous section, so in the remainder of this section we will concentrate on case (i). Therefore:

During the remainder of the section we assume that case (i) of Hypothesis D.3.1 holds, so that $M_+ = \langle L^T \rangle$ for some component L of M.

By (2) of Theorem A (A.2.1), the possibilities for $L/Z(L)$ are listed in Theorem C (A.2.3). Observe that $O_2(L) = 1$, since $O_2(M) = 1$ by D.3.4.1. Thus $Z(L)$ is of odd order, and is described in the list of Schur multipliers in I.1.3.

By Hypothesis D.3.1, $M = \hat{M}/\hat{K}_0$ for some SQTK-group $\hat{M}$ and normal subgroup $\hat{K}_0$ of $\hat{M}$. Let $\alpha : \hat{M} \to M$ be the natural surjection. By B.5.2 we may choose $\hat{M}$ and $\hat{K}_0$ to satisfy the conclusions of B.5.2. In particular, $\hat{K}_0$ is nilpotent of odd order, and $L = \hat{L}\alpha$ for some component $\hat{L}$ of $\hat{M}$. As $\hat{K}_0$ is of odd order and $O_2(M) = 1$, $O_2(\hat{M}) = 1$. Thus $\hat{M}$, $\hat{L}$, and $\hat{M}_+ := \langle \hat{L}^{\hat{M}} \rangle$ are described in section A.3, so M_+ and L are described in A.3.6 and in (1) and (3) of A.3.8; in particular, either $M_+ = L$, or $M_+ = LL^t$ for $t \in T - N_T(L)$.

In the rest of the section we set $V_T := \langle V^T \rangle$, and we adopt the notational convention

$$M_+^*T^* := M_+T/C_{M_+T}(V_T).$$

We also assume

$$\hat{q}(M_+^*T^*, V_T) \leq 2.$$

Finally for the rest of the section we assume that $V \in Irr_+(M_+, V_T, T)$.

Hence $\tilde{V}_T$ is the direct sum of the T-conjugates of $\tilde{V}$ by A.1.42.3, and these conjugates are not isomorphic as M_+^*-modules. In view of A.1.42.2, there is little loss of generality in this last assumption.

The restriction on $\hat{q}$ will hold in our FSU in view of 3.2.5.

Our next two results describe M_+^* and its action on V_T when $V < V_T$. Notice if $L = M_+$, then $C_{M_+}(V) \leq Z(M_+)$, so the hypothesis of the next lemma is satisfied.

LEMMA D.3.5. *Assume that $V < V_T$ and $C_{M_+}(V) \leq Z(M_+)$. Then*

(1) $M_+ = L$; that is, $T \leq N_M(L)$.

(2) V is an FF-module for $Aut_{M_+T}(V)$, and $Aut_L(V) = F^*(Aut_{M_+T}(V))$ is quasisimple.

(3) Each member of $\hat{\mathcal{Q}}_*(T^*, V_T)$ acts on V.

(4) $|T : N_T(V)| = 2$, so that $\tilde{V}_T = \tilde{V} \oplus \tilde{V}^t$, for $t \in T - N_T(V)$.

(5) Either $L^* \cong SL_3(2^n)$, $Sp_4(2^n)$, A_6, $L_4(2)$, or $L_5(2)$, and $\tilde{V}$ is a natural module for L^*; or V is a 4-dimensional module for $L^* \cong A_7$.

PROOF. Choose $A^* \in \hat{\mathcal{Q}}_*(T^*, V_T)$. By A.3.8.1, either $M_+ = L$ or $M_+ = LL^t > L$ for $t \in T - N_T(L)$. In either case by D.3.4.5, $F^*(M_+^*T^*) = M_+^* = E(M_+^*)$, and by D.3.4.2, $O_2(M_+^*T^*) = 1$, so A^* is faithful on M_+^*. By hypothesis, $C_{M_+}(V) \leq Z(M_+)$, so $C_{M_+^*}(V) \leq Z(M_+^*)$ and each component of M_+ is nontrivial on V.

By hypothesis $V \in Irr_+(M_+, V_T, T)$, so by A.1.42.3, $\tilde{V}_T$ is the direct sum of the T-conjugates of $\tilde{V}$. Let $V =: V_1, \ldots, V_r$ be representatives for the orbits of A^* on $Irr_+(M_+, V_T)$, and set $W_i := \langle V_i^{A^*} \rangle$. As T is transitive on $Irr_+(M_+, V_T)$, each component K of M_+ is nontrivial on each V_i by the previous paragraph. Thus $W_i = [W_i, K]$ and as A^* is faithful on M_+^*, A^* is faithful on W_i, so $Aut_{M_+}(W_i) = F^*(Aut_{M_+T}(W_i))$ and so $O_2(Aut_{M_+T}(W_i)) = 1$. Define

$$\hat{q}_i := r_{A^*,W_i} = \frac{m(W_i/C_{W_i}(A^*))}{m(A^*)}, \text{ for } 1 \leq i \leq r.$$

By definition $\hat{q} := r_{A^*,V_T}$, so as $\tilde{V}_T = \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_r$,

$$2 \geq \hat{q} = r_{A^*,V_T} = \frac{m(V_T/C_{V_T}(A^*))}{m(A^*)} \geq \frac{\sum_{i=1}^r m(W_i/C_{W_i}(A^*))}{m(A^*)} = \sum_{i=1}^r \hat{q}_i. \qquad (*)$$

Suppose that $r = 1$. Observe that the hypotheses of D.2.9 are satisfied with $M_+^*T^*$, $\tilde{V}_T^*$, $\tilde{V}^T$ in the roles of "G, V, $\mathcal{I}$": for $s := |\tilde{V}^T| > 1$ since $V < V_T$, while $m(\tilde{V}) > 2$ as M_+ is nonsolvable, and $\tilde{V}^T = \tilde{V}^{A^*}$ as $r = 1$. Then we conclude from D.2.9.1 that $s = 2$, so as $O_2(M_+^*T^*) = 1$, D.2.9.6 forces A^* to normalize each summand, contradicting our assumption that $V^T = V^{A^*}$.

Thus $r > 1$. Then $\hat{q}_i \leq 1$ for some i by (*), so by B.1.4.4, A^* contains an FF^*-offender B^* on the FF-module W_i. Set $M_0^* := [M_+^*, B]$, $G_0 := Aut_{M_0^*B^*}(W_i)$, and $L_0 := Aut_{M_0^*}(W_i)$. As $F^*(Aut_{M_+T}(W_i)) = Aut_{M_+}(W_i) = E(Aut_{M_+T}(W_i))$, $F^*(G_0) = L_0 = E(G_0)$; and as $W_i = [W_i, K]$ for each component K of M_+, $W_i = [W_i, L_0]$. Thus we may apply Theorem B.5.6 to the action of G_0 on W_i. Since $F^*(G_0) = E(G_0)$, cases (2) and (4) of B.5.6 are ruled out. We saw that each component of M_+ is nontrivial on V_i, so as $V_i \in Irr_+(M_+, W_i)$, cases (3) and (5) of B.5.6 are also ruled out. Thus case (1) of B.5.6 holds, so that $F^*(G_0) = L_0$ is quasisimple, and we can apply Theorem B.5.1 to the action of G_0 on W_i. In each case $C_{GL(W_i)}(L_0)$ contains no component K_0 with $K_0/Z(K_0) \cong L_0/Z(L_0)$; so as each component of M_+ is nontrivial on W_i, it follows that $M_+ = L$, establishing (1). As $\tilde{W}_i$ is the direct sum of A^*-conjugates of $\tilde{V}_i$ which are not M_+^*-isomorphic, B.5.1.1 says that either $W_i = V_i$ is an FF-module in $Irr_+(M_+^*, V_T)$, or V_i is a natural module for $M_+^* \cong L_n(2)$, with $n = 4$ or 5, and $|V_i^{A^*}| = 2$. In the latter case there is $a \in A^* - N_{A^*}(V_i)$ with $C_{L^*}(a^*) \cong Sp_4(2)$ or $S_4 \times \mathbf{Z}_2$, so $m(A^*) \leq m(C_{L^*A^*}(a^*)) = 4$. On the other hand, as $\tilde{W}_i = \tilde{V}_i \oplus \tilde{V}_i^a$,

$$m(W_i/C_{W_i}(A^*)) \geq m(V) + m(V_i/C_{V_i}(N_{A^*}(V_i))) > m(V) \geq 4,$$

contradicting $q_i \leq 1$. Thus A^* acts on V_i, and V_i is an FF-module for $Aut_{LT}(V_i)$. So (1)–(3) are established.

Now $\tilde{V}$ is an irreducible L-module, so by (2), $\tilde{V}$ is described in Theorem B.4.2. In each of the cases listed there, the stabilizer $N_T(V)$ of the equivalence class of $\tilde{V}$ is of index at most 2 in T, so as $|T : N_T(V)| = r > 1$ we conclude $r = 2$, establishing (4). As $V \neq V^t$, cases (1), (4), (5), and (10) of Theorem B.4.2 are ruled out, while cases (2), (3), (6), (7), and (9) appear in conclusion (5) of the lemma. Therefore it remains to eliminate cases (8) and (11) of Theorem B.4.2—namely $L^* \cong \hat{A}_6$ with $m(V) = 6$, and $L^* \cong L_5(2)$ with $m(V) = 10$. Here by (3), A^* acts on V_1 and V_2. But by B.4.2, $q(Aut_{LT}(V_i), V_i) \geq 1$; so as $\hat{q} \leq 2$, A^* must be an FF^*-offender on both V_1 and V_2. However by B.4.2, the unique FF^*-offender in T on V_1 is not an offender on the module V_2. This contradiction completes the proof. □

The next two results assume that $L < M_+$; observe in this case that the possibilities for L are listed in A.3.8.3.

LEMMA D.3.6. *Assume that $V < V_T$ and $M_+ = LL^t$ for some component L of M and $t \in T - N_T(L)$. Then*

(1) $\tilde{V}_T = \tilde{U} \oplus \tilde{U}^t$, where $U := [V_T, L] \leq C_{V_T}(L^t)$.

(2) Each member of $\hat{\mathcal{Q}}_*(T^*, V_T)$ acts on U, so $\hat{q}(Aut_{M_+T}(U), U) \leq 2$.

(3) Either $U = V$ or $L^* \cong L_3(2)$, $U = V \oplus V^s$ for $s \in N_T(L) - N_T(V)$, and $m(V) = 3$.

PROOF. As $L < M_+$, $C_{M_+}(V) \not\leq Z(M_+)$ by D.3.5.1. As L^{t*} is quasisimple, interchanging the roles of L and L^t if necessary, we may assume $[V, L^t] = 1$; hence $\tilde{V}$ is an irreducible $\mathbf{F}_2L$-module. Set $S := N_T(L)$ and $U := [V_T, L]$; thus S is of index 2 in T. By A.1.42.3, $\tilde{U}$ is the direct sum of the S-conjugates of $\tilde{V}$, and $\tilde{V}_T = \tilde{U} \oplus \tilde{U}^t$, with $U \leq C_{V_T}(L^t)$. That is, (1) is established.

Choose $A^* \in \hat{\mathcal{Q}}_*(T^*, V_T)$. As in the proof of the previous lemma, we have the hypotheses of D.2.9 for the decomposition $\tilde{V}_T = \tilde{U} \oplus \tilde{U}^t$ as the direct sum of just two summands. Again we may apply D.2.9.6 to see that A^* acts on U. By D.2.5,

$$r_{Aut_{A^*}(U),U} \leq r_{A^*,V_T} = \hat{q}(M_+^*T^*, V_T) \leq 2,$$

completing the proof of (2).

Next the quadruple $Aut_M(U)$, U, V, $Aut_S(U)$ satisfies Hypothesis D.3.1 in the roles of "M, V_M, V, T". Part (3) holds if $V = U$, so we may assume $V < U = V_S$. By (2), $\hat{q}(Aut_{LS}(U), U) \leq 2$, so the hypotheses of D.3.5 are also satisfied by this quadruple. Then comparing the possibilities for L^* in A.3.8.3 to the list of D.3.5.5, we conclude $L^* \cong L_3(2)$ and $\tilde{V}$ is a natural module for L^*. From D.3.5.4 we see $\tilde{U} = \tilde{V} \oplus \tilde{V}^s$ for $s \in S - N_S(V)$. However B.4.8.2 says that $\hat{q}(Aut_{LS}(U), U) > 2$ if $C_V(M_+) \neq 0$, so $m(V) = 3$. This completes the proof of (3), and the lemma is established. □

Having begun the case $L < M_+$ in D.3.6 under the assumption that $V < V_T$, we now continue that case when $V = V_T$:

LEMMA D.3.7. *Assume that $V = V_T$, and $M_+ = LL^t$ for some component L of M and $t \in T - N_T(L)$. Set $\hat{q} := \hat{q}(M_+^*T^*, V)$ and let $A^* \in \hat{\mathcal{Q}}_*(T^*, V_T)$. Then one of the following holds:*

(1) $L^* \cong L_2(2^n)$, V is the $\Omega_4^+(2^n)$-module for M_+^*, and either $\hat{q} = 3/2$ and $m(A^*) = 2n$ (with $m(A^*) = 4$ also allowed when $n = 3$), or $n = 2$, $m(A^*) = 3$, and $\hat{q} = 4/3$.

(2) $L^* \cong L_3(2)$, V is the tensor product of natural modules for L^* and L^{*t}, $\hat{q} = 5/4$, and $m(A^*) = 4$.

PROOF. Let $I \in Irr_+(L, V)$, $F := End_{\mathbf{F}_2 L}(\tilde{I}) = \mathbf{F}_{2^e}$, and $d := \dim_F(\tilde{I}) \geq 2$. Now $I^t \in Irr_+(L^t, V)$ and $\tilde{I}^t$ is quasi-equivalent (by t-conjugacy) to $\tilde{I}$, so also $F = End_{\mathbf{F}_2 L^t}(\tilde{I}^t)$. Then regarding $\tilde{I}$ and $\tilde{I}^t$ as F-modules for L^* and L^{t*}, and as $\tilde{V}$ is an irreducible M_+^*-module, $\tilde{V} = \tilde{I} \otimes \tilde{I}^t$ as an FM_+^*-module. Then by Theorem 3 in [**AS85**], $H^1(M_+^*, \tilde{V}) = 0$, so $C_V(M_+) = 0$ and $V = \tilde{V}$.

Pick an F-basis $x_1, \ldots, x_d$ for I and let $y_i := x_i^t$, so that $y_1, \ldots, y_d$ is a basis for I^t, and hence $x_i \otimes y_j$, $1 \leq i, j \leq d$, is an F-basis for V.

We first consider the case where $A^* \not\leq N_{T^*}(L^*)$. Then we may choose t^* to be an involution in A^*, so the linear span [2]

$$[V, t^*] = \langle v_J : J \in \Lambda \rangle$$

is of F-dimension $d(d-1)/2$, where Λ is the set of 2-subsets of $\{1, \ldots, d\}$, and $v_{i,j} := x_i \otimes y_j + x_j \otimes y_i$. Also

$$C_V(t^*) = [V, t^*] \oplus \langle x_i \otimes y_i : 1 \leq i \leq d \rangle$$

is of F-dimension $d(d+1)/2$.

Observe that

$$m(A^*) \leq 1 + m_2(N_T(L)^*) \leq 1 + m \qquad (*)$$

where $m := m_2(Aut(L^*))$.

Consider the subcase $d > 2$. Then

$$m(V/C_V(A^*)) \geq m(V/C_V(t^*)) = d(d-1)e/2 \geq de = k, \qquad (!)$$

where $k := \dim_{\mathbf{F}_2}(I)$. Indeed if $A^* > \langle t^* \rangle$, then $1 \neq N_{A^*}(L)$ acts faithfully on the diagonal $L_t^* := C_{M_+^*}(t^*)$ of L^*L^{t*}, and L_t^* is faithful on $C_V(t^*)$, so $m(V/C_V(A^*)) > m(V/C_V(t^*)) \geq k$. Thus:

$$2m(A^*) \geq \hat{q}\ m(A^*) = m(V/C_V(A^*) \geq k,$$

$$\text{and } 2m(A^*) > m(V/C_V(t^*)) \geq k \text{ if } m(A^*) > 1. \qquad (!!)$$

In particular if $m(A^*) = 1$, then $k \leq 2$, contrary to L not solvable. Thus $k > 2$ and $m(A^*) > 1$, so by (!!), $m(A^*) > k/2 > 1$. Then using (*),

$$m + 1 > k/2. \qquad (**)$$

We now examine the possibilities for L^* from A.3.8.3. If $L^* \cong Sz(2^n)$, then $m = n \geq 3$ is odd and $k \geq 4n$, so

$$m + 1 = n + 1 < 2n \leq k/2,$$

contradicting (**). If $L^* \cong J_1$, then $m = 3$, and $k \geq 18$ as L^* has an element of order 19, also contrary to (**). If $L^* \cong L_2(p)$ with $p > 5$ prime, then $m = 2$; thus

[2]Notational convention: We often use angle brackets to denote the vector subspace spanned by a set of vectors.

(**) gives $k < 2(m+1) = 6$, which forces $p = 7$ and $k = 3$. Therefore $L^* \cong L_3(2)$, so $e = 1$ and $d = k = 3$, so that I is the 9-dimensional module described in conclusion (2). However $m(A^*) \leq 3$ by (*), so $m(V/C_V(A^*)) \geq 6$ and hence $r_{A^*,V} \geq 2$— whereas the product B^* of transvection subgroups with a fixed axis from L^* and L^{t*} satisfies $r_{B^*,V} = 5/4 < r_{A^*,V}$, contradicting $A^* \in \hat{\mathcal{Q}}_*(T^*, V)$.

This leaves the possibility $L^* \cong L_2(2^n)$, so again $m = n$. Let $K := \mathbf{F}_{2^n}$. Then $F \leq K$ and we let $I_K := I \otimes_F K$. Then I_K is the tensor product of a set Δ of r Galois conjugates of the natural module, so

$$2^r = \dim_K(I_K) = \dim_F(I) = d,$$

as I is an absolutely irreducible FL-module. Now n/e is the order of the largest subgroup Γ of $Aut(K)$ acting on Δ, and all orbits of Γ on Δ are regular. Therefore n/e divides r, so (**) gives

$$2^r e = k < 2(n+1) = 2(e \cdot \frac{n}{e} + 1) \leq 2(er + 1),$$

so $2^{r-1} < r + 1/e$ and hence $r = 1$ or 2. However if $r = 1$ then I is the natural module, $F = K$, and $d = 2$, contrary to our assumption in this subcase that $d > 2$.

Thus $r = 2$, so that $I_K = N \otimes N^\sigma$ for N the natural module and σ a non-trivial automorphism of K. If σ has order other than 2, then Γ is trivial, so we have $n = e$ and as $n > 1$,

$$k = 4n > 2(n+1),$$

contradicting (**). Thus σ has order 2, and I is the orthogonal module of dimension $d = 4$ and minus type over the σ-fixed field F, and $e = n/2$. Now by (*), (!), and the strict inequality in (!!) as $m(A^*) > 1$,

$$2(n+1) \geq 2m(A^*) > m(V/C_V(t^*)) = \frac{d(d-1)e}{2} = 3n,$$

impossible as $n \geq 2$.

So when $A^* \not\leq N_{T^*}(L^*)$ we are reduced to the subcase $d = 2$. Then

$$L^{*t} \leq C_{GL(V)}(L^*) \cong GL_d(F) = GL_2(F),$$

so $L^* \cong L_2(2^n)$, I is the natural module for L^*, and $F = \mathbf{F}_{2^n}$. But now $V = I \otimes I^t$ is the orthogonal module of dimension 4 and plus type, as in conclusion (1). We argue much as in the earlier case $L^* \cong L_3(2)$, now with B^* the product of the Sylow groups of L^* and L^{t*}, giving $r_{B^*,V} = 3/2$, whereas $m(A^*) \leq n+1$ by (*) and $m(V/C_V(A^*)) \geq 2n$, so that $r_{A^*,V} \geq 2n/(n+1)$. Then as $A^* \in \hat{\mathcal{Q}}_*(T^*, V)$, $r_{A^*,V} \leq 3/2$, which forces $n \leq 3$, and $m(A^*) = 4$ when $n = 3$. When $n = 2$ we get $\hat{q} = 4/3$ and $m(A^*) = 3$. Thus conclusion (1) holds in these two cases, completing the treatment of the case $A^* \not\leq N_{T^*}(L^*)$.

Thus we have reduced to the case where A^* acts on L^*. As A^* is faithful on M_+^*, interchanging the roles of L and L^t if necessary, we may assume $L^* = [L^*, A^*]$. As M_+ is irreducible on V, V is a homogeneous L-module, so we may assume (cf. A.1.42.2) that A^* acts on I. As $L^* = [L^*, A^*]$, $[I, A^*] \neq 0$. Then by D.2.5, $\hat{q} \geq \hat{q}(Aut_{L^*A^*}(I), I)$, and in case of equality, using D.2.8 we have $V = I + C_V(A^*)$. But in the latter case as $L^* = [L^*, A^*]$, we have $I = [V, L]$; then as V is a homogeneous L-module, $V = I$, contradicting the fact that V is the sum of $d > 1$ copies of I. Thus $\hat{q}(Aut_{A^*L^*}(I), I) < \hat{q} \leq 2$. Applying this restriction to the groups in A.3.8.3, Theorems B.4.2 and B.4.5 show that one of the following holds:

(i) $L^* \cong L_2(2^n)$ and I is a natural module.
(ii) $L^* \cong L_2(2^n)$, n even, and I is an orthogonal module of minus type.
(iii)$L^* \cong L_3(2)$ and I is a natural module.

In (i), conclusion (1) holds—where A^* is the subgroup B^* described in our earlier discussion of the orthogonal module of plus type. Similarly conclusion (2) holds in (iii), so we may assume that (ii) holds. Then $m(V) = 4^2 \cdot n/2 = 8n$.

We claim that any involution j^* in M_+^* acts freely on V, so that $C_V(j^*) = [V, j^*]$ is of rank $4n$: For if $j^* \in L^*$ then $m(C_I(j^*)) = [I, j^*]$ is of rank $m(I)/2$, so as V is homogeneous under L^*, the same is true for V. On the other hand if j^* is not in L^* or L^{*t}, then $j^* = j_1^* j_2^*$ with $1 \ne j_1^* \in L^*$ and $1 \ne j_2^* \in L^{*t}$, and

$$m([C_V(j_1^*), j^*]) = m([C_V(j_1^*), j_2^*]) = m(C_V(j_1^*))/2,$$

and similarly $m([V/C_V(j_1^*), j^*]) = m(V/C_V(j_1^*))/2$, so $m(C_V(j^*)) = m(V)/2$, as claimed.

Thus if $j^* \in A^* \cap M_+^*$, then

$$4n \ge 2m(A^*) \ge m(V/C_V(A^*)) \ge m(V/C_V(j^*)) = m(V)/2 = 4n,$$

so all inequalities are equalities, and in particular $m(A^*) = 2n$ and $C_V(A^*) = C_V(j^*)$. But since $m(A^*) = 2n$, $A^* \in Syl_2(M_+^*)$, so that $C_V(A^*) < C_V(j^*)$, a contradiction. Hence there is no such involution j^*, so $A^* \cap M_+^* = 1$, and hence $m(A^*) \le 2$. But $m(V/C_V(A^*)) \ge 4$ for any such A^*, so as $r_{A^*,V} \le 2$, $m(V/C_V(A^*)) = 4$, which occurs only when $n = 2$ and $m(A^*) = 1$, contradicting $r_{A^*,V} \le 2$. This completes the proof. □

LEMMA D.3.8. *Assume $L/Z(L) \cong Sz(2^n)$ with $n > 1$. Then $\hat{q}(M_+^*T^*, V_T) = 2$, V is the natural module for L^*, $\hat{\mathcal{Q}}_*(T^*, V_T) \subseteq N_{T^*}(V)$, and either*

(1) $M_+ = L$ and $V_T = V$, or

(2) $M_+ = LL^t$ for $t \in T - N_T(L)$ and $V_T = V \oplus V^t$, with $V = [V_T, L] = C_{V_T}(L^t)$.

PROOF. As $\hat{q} := \hat{q}(M_+^*T^*, V_T) \le 2$ we must show that $\hat{q} = 2$.

Suppose first that $M_+ = L$. As we saw earlier that $Z(L)$ is of odd order, $Z(L) = 1$ by I.1.3, so that L is simple, and hence L is faithful on V by D.3.4.2. As Suzuki groups do not appear in D.3.5.5, we have $V_T = V$. Then as $\tilde{V}$ is irreducible, Theorem B.4.5 says $\tilde{V}$ is the natural module and $\hat{q} \ge 2$; thus $V = \tilde{V}$ is irreducible by I.1.6.11, so that $\hat{q} = 2$ and conclusion (1) holds in this case.

Thus we may assume that $M_+ = LL^t$ with $t \in T - N_T(L)$. Since Suzuki groups do not appear in D.3.7, we may assume $V < V_T$. Then by D.3.6, $\tilde{V}_T = \tilde{V} \oplus \tilde{V}^t$, with $V = [V_T, L]$, and $\hat{\mathcal{Q}}_*(T^*, V_T)$ acts on V. Then applying the result of the previous paragraph to $N_M(L)$, conclusion (2) holds in this case. □

The previous results provide a good description of V_T, so we turn next to V_M. In our proof of the Main Theorem (cf. 3.2.5) we will be able to apply the *qrc*-lemma to both M_+T on V_T, and M on V_M, so we are interested in the case where $\hat{q}(M, V_M) \le 2 \ge \hat{q}(M_+^*T^*, V_T))$.

LEMMA D.3.9. *Assume that*

$$\hat{q}(M, V_M) \le 2 \ge \hat{q}(M_+^*T^*, V_T).$$

Further assume that either $V = V_T$, or $V = [V, L] < [V_T, L]$. Let $A \in \hat{\mathcal{Q}}_*(M, V_M)$. Then

(1) If A is faithful on M_+, then either

(a) $V_T = V_M$, or

(b) $V = V_T$, $M_+ = L$, $\tilde{V}$ is an FF-module for $Aut_{LT}(\tilde{V})$, and L has at most $2/\hat{q}$ chief factors on $\tilde{V}_M$, where $\hat{q} := \hat{q}(Aut_{LT}(\tilde{V}), \tilde{V})$.

(2) If A is not faithful on M_+, then A acts on $X \leq C_M(M_+)$ such that $X = [X, C_A(M_+)]$, and either

(a) $X = \langle K^A \rangle$ for some component K of M with K not a Suzuki group, or

(b) $X \leq O_3(M)$ is of order 3.

Further $M_+ = L$ and A centralizes $O^3(F(M))$ and each Suzuki component of $C_M(L)$.

PROOF. We first prove (1), so suppose that A is faithful on M_+. Using D.3.4.3 and A.1.42.2, there is part of an M_+T-chief series

$$V_{M,0} < \cdots < V_{M,k} = V_M$$

with $V_{M,0} := C_{V_T}(M_+)$, $V_{M,1} := V_T$, and each $I_i := V_{M,i}/V_{M,i-1}$ is quasi-equivalent (M_+T-conjugate by Clifford's Theorem) to $\tilde{V}_T$ as an M_+T-module. Thus setting $\hat{q}_T := \hat{q}(Aut_M(\tilde{V}_T), \tilde{V}_T)$, by hypothesis

$$2 \geq \hat{q}(M, V_M) \geq k\hat{q}_T,$$

so in particular $k \leq 2/\hat{q}_T$.

We may assume that $V_T < V_M$, so that $k > 1$. Then $\hat{q}_T \leq 1$, so that $\tilde{V}_T$ is an FF-module for $Aut_{LT}(\tilde{V}_T)$. We claim then that $M_+ = L$: For by Theorem B.4.2, V_T is not an FF-module in either of the two cases of D.3.7, so the claim holds when $V = V_T$. On the other hand when $V < V_T$, by the hypotheses of the lemma, $V = [V, L] < [V_T, L]$; so if $L < M_+$, then by D.3.6.3, $[V_T, L]$ is the sum of the natural module and its dual for $L^* \cong L_3(2)$, and again V_T is not an FF-module. This completes the proof of the claim.

Therefore $M_+ = L$. Moreover the other assertions of conclusion (1b) were established above in the case where $V = V_T$, so to complete the proof of (1), we assume that $V < V_T$, and it remains to derive a contradiction. Then $F^*(L^*T^*) = L^*$ is quasisimple and V_T is an FF-module, so V_T is described in Theorem B.5.1. As $\tilde{V} < \tilde{V}_T$ and $\tilde{V}$ is an L-homogeneous component of $\tilde{V}_T$, it follows that B.5.1.1 holds with $L \cong L_n(2)$ for $n = 4$ or 5, and $\tilde{V}_T$ the sum of the natural and dual modules. Applying parts (iii) and (iv) of B.4.9.2 to $\tilde{V}_T$, we conclude that $\hat{q}_T \geq 5/6$. Then as $k \leq 2/\hat{q}_T$, we conclude that $k = 2$. Therefore as $\mathbf{F}_2$ is the splitting field of these modules,

$$C_M(L) \leq C_{GL(V_M)}(L) \cong L_2(2) \times L_2(2).$$

Then as $m_3(M) \leq 2$ and $Aut(L)$ is a 2-group, $M = LT$, contradicting the assumption that $V_T < V_M$. This completes the proof of (1).

Therefore it remains to prove (2), so we may assume that A is not faithful on M_+. Hence $1 \neq C_A(M_+)$ is faithful on $C_{F^*(M)}(M_+)$ as $O_2(M) = 1$ by D.3.4.1, so A acts on some $X \leq C_M(M_+)$ such that $X = [X, C_A(M_+)]$ and either $X = \langle K^A \rangle$ for some component K of M, or $X \leq O_p(M)$ is of order p for some odd prime p. To establish (2), we need to show that 3 divides the order of X, so we assume otherwise. Then if X is nonsolvable, $K \cong Sz(2^n)$. Set $B := C_A(X)$ and $W := C_{V_M}(B)$. By

D.2.5, $A/B \in \hat{\mathcal{Q}}(Aut_M(W), W)$, and by the Thompson $A \times B$-lemma, X is faithful on W.

Suppose first that $X \leq O_p(G)$. Then by D.2.13.1, X is of order $p = 3$ or 5, and since we are assuming X is a $3'$-group, $p = 5$. Now $m([V_M, X]) = 4$ by D.2.13.2, so $C_{GL([V_M,X])}(X)$ is cyclic of order 15. Then as $M_+ = M_+^\infty$, M_+ centralizes $[V_M, X]$—impossible as $C_{\tilde{V}_M}(M_+) = 0$, since $\tilde{V}_M$ is a direct sum of conjugates of $\tilde{V}$.

So $X = \langle K^A \rangle$ and $K \cong Sz(2^n)$. Now applying D.3.8 to X in the role of "M_+" and $I \in Irr_+(K, W)$ in the role of "V", we conclude A normalizes K and hence $X = K$. Further I is the natural module for K, and by A.1.42.2, we may choose I to be A-invariant. By Theorem B.4.5, $\hat{q}(Aut_{KA}(I), I) = 2$; so by D.2.8, $V_M = I + C_V(A)$. So as $K = [K, A]$, $I = [V_M, K]$ is irreducible, and we obtain a contradiction as in the previous paragraph since $C_{GL(I)}(I)$ is cyclic. □

The main results of this section are Theorems D.3.10 and D.3.21, describing the cases where $L = M_+$ and $L < M_+$, respectively.

THEOREM D.3.10. *Assume $L = M_+$ and*

$$\hat{q}(M, V_M) \leq 2 \geq \hat{q}(LT/C_{LT}(V_T), V_T).$$

Take $V \in Irr_+(L, V_T, T)$. Then one of the following holds:

(1) T acts on V, so $V = V_T$, and hence the action of LT on V is described in Theorems B.4.2and B.4.5.

(2) Either L is $SL_3(2^n)$, $Sp_4(2^n)$, A_6, $L_4(2)$, or $L_5(2)$, and $\tilde{V}$ is a natural module for L; or V is a 4-dimensional module for $L \cong A_7$. Further $\tilde{V}_M = \tilde{V} \oplus \tilde{V}^t$ with $t \in T - N_T(V)$, and $\tilde{V}^t$ is not $\mathbf{F}_2L$-isomorphic to $\tilde{V}$.

The proof of Theorem D.3.10 involves a series of reductions. Assume M, V is a counterexample. By our hypothesis that $V \in Irr_+(L, V_T, V)$, $\tilde{V}$ is a homogeneous component of $\tilde{V}_T$. If $V = V_T$, then (1) holds, so we may assume that

$$V < V_T.$$

As L is quasisimple by hypothesis, $C_L(V) \leq Z(L)$, so the hypotheses of D.3.5 are satisfied. Therefore by parts (4) and (5) of D.3.5:

LEMMA D.3.11. *V_T satisfies conclusion (2) of Theorem D.3.10.*

If $V_T = V_M$, then D.3.11 says that conclusion (2) of Theorem D.3.10 holds, contrary to the choice of M, V as a counterexample. Therefore

$$V_T < V_M.$$

By D.3.4.3, $\tilde{V}_M$ is a semisimple L-module, with M transitive on the homogeneous L-components of $\tilde{V}_M$. But by D.3.11, V_T is described in D.3.10.2; and in each of the cases listed there, the full stabilizer in $Aut(L)$ of the equivalence class of $\tilde{V}$ is of index 2 in $Aut(L)$. It follows that $\tilde{V}_M = \tilde{U} \oplus \tilde{U}^t$, where $\tilde{U}$ is the homogeneous L-component of $\tilde{V}$ on $\tilde{V}_M$, and $t \in T - N_T(V)$. Now $\tilde{U} = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k$, where $\tilde{U}_1, \ldots, \tilde{U}_k$ are copies of $\tilde{V}$, and as $V_T < V_M$, $k > 1$.

Set $\hat{q} := \hat{q}(M, V_M)$; by hypothesis, $\hat{q} \leq 2$. Let $A \in \hat{\mathcal{Q}}_*(T, V_M)$. As in earlier arguments, we may apply D.2.9.6 to the decomposition $V_M = U \oplus U^t$, to conclude:

LEMMA D.3.12. *$A \leq N_M(U)$.*

LEMMA D.3.13. *A is not faithful on L.*

PROOF. Since $V = [V, L] < [V_T, L]$, we have the hypotheses of D.3.9. But $V < V_T < V_M$, so that neither of the two conclusions of D.3.9.1 can hold. Thus A is not faithful by D.3.9.1. □

We just observed that the hypotheses of D.3.9 are satisfied, but A is not faithful on L. Therefore the hypotheses of D.3.9.2 are satisfied, so by D.3.9.2, A acts on some $X \leq C_M(L)$ such that

$$X = [X, C_A(L)],$$

where either $X = \langle K^A \rangle$ for some component K of M with K not a Suzuki group, or $X \leq O_3(M)$ is of order 3.

LEMMA D.3.14. *$m_3(C_M(L)) = 1$ and $L \cong L_3(2^m)$ with m odd.*

PROOF. By D.3.11, the possibilities for L are listed in D.3.10.2. In each case:

$$\text{there is a subgroup of } L \text{ of order 3 inverted in } L. \qquad (!)$$

Further

$$m_3(L) > 1 \text{ unless } L \cong L_3(2^m), \text{ with } m \text{ odd.} \qquad (!!)$$

Using (!), and applying A.1.31.1 to a section of $\hat{M}$ which is the product of an S_3-section of L with a Sylow 3-group of $C_M(L)$, $m_3(C_M(L)) = 1$. In particular $m_3(X) = 1$. Then we may assume that $m_3(L) = 2$ by (!!). If X has order 3, then as $X = [X, C_A(L)]$, an involution in $C_A(L)$ inverts X and commutes with an E_9-subgroup of L, contrary to A.1.31.1. So as $m_3(X) = 1$, we conclude from (2) of Theorem A (A.2.1) and Theorem C (A.2.3) that $X = K \cong L_2(2^n)$, $L_3^\epsilon(q)$, $L_2(p^e)$, or J_1. In each case some involution in X inverts an element of order 3 in X, so as before, $m_3(L) = 2$ contrary to A.1.31.1. □

LEMMA D.3.15. *$X = [F^*(C_M(L)), A]$ and either*

(1) $X = O_3(M) \cong \mathbf{Z}_3$, or

(2) X is a normal component of M.

PROOF. Set $Y := C_M(L)$. By D.3.9.2, A centralizes $O^3(F(Y))$ and each Suzuki component of Y, so $[F^*(Y)), A] \leq O_3(M)I$, where $I := O^{3'}(E(Y))$. As $m_3(Y) = 1$ by D.3.14, either $I = 1$ and $O_3(Y) \neq 1$ is cyclic, or $O_3(Y) = 1$ and I is a component of Y. In the second case (2) holds since $L = M_+$ is normal in M by Hypothesis D.3.1. In the first case $X = \Omega_1(O_3(Y))$, and it remains to show that $X = O_3(M)$. If not set $W := C_{V_M}(C_A(X))$, and argue as in the proof of D.3.9.2 using D.2.5 to conclude $A/C_A(X) \in \hat{\mathcal{Q}}(Aut_M(W), W)$, whereas $O_3(Y)$ is cyclic of order at least 9 and faithful on W, contrary to D.2.17. □

LEMMA D.3.16. *$C_{V_M}(L) = 0 = C_V(M)$.*

PROOF. If the lemma fails, then $C_V(L) \neq 0$ by D.3.4.4, so $H^1(L, \tilde{V}^*) \neq 0$, where $\tilde{V}^*$ is the dual of $\tilde{V}$. But by D.3.14 and D.3.11, $\tilde{V}$ is the natural module for $L \cong L_3(2^m)$, so we conclude from I.1.6 that $m = 1$. However by hypothesis, $\hat{q}(Aut_{LT}(V_T), V_T) \leq 2$, and this case is eliminated using B.4.8.2 as in the proof of D.3.6.3. So the lemma is established. □

Recall $\tilde{U}$ is the L-homogeneous submodule of $\tilde{V}_M$ with k summands defined after D.3.11. By D.3.16, $\tilde{V}_M = V_M$, so $\tilde{U} \cong U$ and $V_M = U \oplus U^t$.

LEMMA D.3.17. *(1)* $X \cong L_3(2^n)$ *with* n *odd.*

(2) There is $I \in Irr_+(X, U, N_T(U))$*, and each such* I *is a natural module for* X*.*

(3) A *induces inner automorphisms on* X*.*

PROOF. By D.3.12, we have $A \le N_M(U)$. Using A.1.42.2, we may choose $I \in Irr_+(X, U, N_T(U))$; then $I^t \le U^t$, so $I \ne I^t$—and hence the hypotheses of D.3.5 are satisfied with X, I in the roles of "L, V". So by D.3.5.2, I is an FF-module for $Aut_{N_T(U)X}(I)$. Therefore the sublist of simple groups of 3-rank 1 from Theorem C is reduced by applying Theorem B.4.2 to give $X \cong \mathbf{Z}_3$, $L_2(2^m)$, or $L_3(2^n)$ with n odd, and in the last case, A induces inner automorphisms on X. When $X \cong L_3(2^n)$ and $C_I(L) = 0$, I is natural by B.4.2.4, and we are done. Thus we may assume otherwise. Then from B.4.2.2 and B.4.8.2, $\hat{q}(Aut_{XN_T(U)}(I), I) = 1$, so that $\hat{q}(Aut_{XN_T(U)}(I + I^t), I + I^t) = 2$. Thus by D.2.8, we have $V_M = (I + I^t) + C_V(A)$, and hence $[V_M, X] = I + I^t$ as $X = [X, A]$. Then as X is irreducible on $I/C_I(X)$ and $[L, X] = 1$, $[I, L] = 0$ by A.1.41, contrary to D.3.16. This contradiction completes the proof. □

LEMMA D.3.18. $L = [L, A]$.

PROOF. Assume otherwise, so that A centralizes L. Then A is faithful on X by D.3.15. Choose I as in D.3.17.2. By D.3.17, I is a natural module for $X \cong L_3(2^n)$ and A induces inner automorphisms on X, so that $A \le XC_M(X)$. Then as $X = [F^*(C_M(L)), A]$, the projection of A on $C_M(X)$ centralizes $F^*(M)$ and so $A \le X$. Now $\hat{q}(X, I) = 1/2$ by Theorem B.4.2.2, so X has at most 4 chief factors on V_M, and hence at most 2 on U. Thus

$$L \le C_{GL(U)}(X) \cong GL_2(2^n),$$

whereas $L_3(2^n)$ has no such 2-dimensional representation. This contradiction completes the proof. □

Recall that $C_V(L) = 0$ by D.3.16, so $V_T = V \oplus V^t$. On the other hand, A acts on U by D.3.12, so A acts on $U \cap V_T$. Then as L is irreducible on V, $C_A(V_T) = C_A(V) = C_A(L)$, and therefore $Aut_A(V_T) \cong A/C_A(L)$.

LEMMA D.3.19.

$$r_{Aut_A(V_T), V_T} = \frac{m(V_T/C_{V_T}(A))}{m(A/C_A(L))} < 2.$$

PROOF. Otherwise we have the hypothesis of D.2.8 with V_T, A in the roles of "U, G"; then by D.2.8, A is faithful on V_T, and hence also on L since $C_A(V_T) = C_A(L)$, contradicting D.3.13. □

Now we work toward the final contradiction which will establish Theorem D.3.10. By D.3.14, $L \cong L_3(2^m)$, m odd, and by D.3.11 and D.3.16, $V_T = V \oplus V^t$ with V a natural module and V^t its dual. Therefore by D.3.19, interchanging the roles of V and V^t if necessary, $Aut_A(L)$ is contained in the subgroup $D \cong E_{2^{2m}}$ of L inducing the full group of transvections on V with fixed axis, and $C_{V_T}(A) = C_{V_T}(D)$ is of corank $3m$ in V_T. Now as U is L-homogeneous, using D.3.4.8, there is an LA-series

$$0 = V_0 < V_1 < \cdots V_k = V_M$$

with $W_i := V_i/V_{i-1} \cong V_T$. Indeed as $X \cong L_3(2^n)$, we must have $k \geq 3$ since $X \leq C_{GL(U)}(L) \cong GL_k(2^m)$. Therefore as $r_{A,V_M} \leq 2$,

$$2m(A) \geq m(V_M/C_{V_M}(A)) \geq 3km \geq 9m,$$

so $m(A) \geq 9m/2$. Therefore as $m(A/C_A(L)) \leq 2m$,

$$5m/2 \leq m(C_A(L)) \leq 2n$$

as $X \cong L_3(2^n)$ is of 2-rank $2n$. Therefore $n \geq 5m/4 > m$.

Pick I as in D.3.17.2, and let d be the number of composition factors of X on $U_I := \langle I^L \rangle$, and $B := C_A(L)$. As in the proof of D.3.18, using D.3.15 and D.3.17.3, $B \leq X$, so

$$m(A) \leq m(B) + m(A/B) \leq 2(m+n).$$

As

$$L_3(2^m) \cong L \leq C_{GL(U_I)}(X) \leq GL_d(2^n),$$

we have $d \geq 3$, so

$$m(U/C_U(B)) \geq dn \geq 3n,$$

and similarly $m(U^t/C_{U^t}(B)) \geq 3n$. As $V_T \leq C_{V_M}(B)$ and $m(V_T/C_{V_T}(A)) = 3m$, we conclude

$$m(V_M/C_{V_M}(A)) \geq 6n + 3m,$$

so as $n > m$,

$$\frac{m(V_M/C_{V_M}(A))}{m(A)} \geq \frac{6n+3m}{2(m+n)} > 2,$$

for our final contradiction to our hypothesis that $r_{A,V_M} \leq 2$. Thus the proof of Theorem D.3.10 is at last complete.

Notice we did not discuss V_M in case D.3.10.1. However during the proof of the Main Theorem, we will need the following information when V_M is an FF-module:

LEMMA D.3.20. *Assume $L = M_+$, $C_V(L) \neq 0$, and $L \leq J(M, V_M)$. Then M acts on V, so $V = V_T = V_M$.*

PROOF. By hypothesis V_M is an FF-module for M, so the action of $J(M, V_M)$ is described in Theorem B.5.6. By D.3.4.3, $V_M = [V_M, L]$, so cases (3)–(5) of B.5.6 do not hold, while case (2) does not hold as L is quasisimple. Therefore case (1) holds, so $F^*(M) = L$ and V_M is described in Theorem B.5.1.1. Then as $C_V(L) \neq 0$, M acts on V by B.5.1.1. □

We conclude the section with our main result in the case $L < M_+$. Notice in case (3) of Theorem D.3.21 that we can replace V by an $N_M(L)$-invariant member of $Irr_+(L, V_M)$, and hence reduce to case (1) of the Theorem after that replacement.

THEOREM D.3.21. *Assume $M_+ = LL^t$ for some component L of M and $t \in T - N_T(L)$, and*

$$\hat{q}(M, V_M) \leq 2 \geq \hat{q}(M_+T/C_{M_+T}(V_T), V_T).$$

Then one of the following holds:

(1) $V_T = V_M$.

(2) $L \cong L_3(2)$ and $V_M = U \oplus U^t$, where $U := [V_M, L] = C_{V_M}(L^t)$ is the sum of 4 isomorphic natural modules for L, each member of $\hat{\mathcal{Q}}_*(M, V_M)$ acts on U, and $O^2(C_M(M_+)) \cong \mathbf{Z}_5$ or E_{25}.

(3) $L \cong L_2(2^n)$, $\tilde{V}_M = \tilde{U} \oplus \tilde{U}^t$ where $U = [V_M, L] \leq C_{V_M}(L^t)$, $\tilde{U}$ is the sum of two natural modules for L, $C_M(L) = L^t$, and $N_M(L)$ acts on some member of $Irr_+(L, U)$.

PROOF. We may assume that $V_T < V_M$. Suppose first that either $V = V_T$ or $V = [V, L] < [V_T, L]$, so that we have the hypotheses of D.3.9. Case (1.a) of D.3.9 is ruled out by our assumption that $V_T < V_M$, and cases (1.b) and (2) do not hold since $L < M_+$.

Therefore $V < V_T$, and either $[V, L] < V$ or $V = [V, L] = [V_T, L]$. As $V < V_T$, we may apply D.3.6. The second case of D.3.6.3 is ruled out, since there $V = [V, L] < [V_T, L]$. Therefore $V = [V_T, L]$ by D.3.6.3. Then by D.3.6.1, $\tilde{V}_T = \tilde{V} \oplus \tilde{V}^t$ with $V \leq C_{V_T}(L^t)$. Now we get a situation analogous to that in D.3.10: By D.3.4.3, $\tilde{V}_M$ is a semisimple M_+-module, with M transitive on the homogeneous M_+-components of $\tilde{V}_M$. As $V = [V_T, L] \leq C_{V_T}(L^t)$, it follows that $\tilde{V}_M = \tilde{U} \oplus \tilde{U}^t$, where $U := [V_M, L] \leq C_{V_M}(L^t)$. Now $\tilde{U} = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_k$, where $U_1, \ldots, U_k$ are conjugates of V under $M_L := N_M(L)$, and as $V_T < V_M$, we have $k > 1$.

Set $\hat{q} := \hat{q}(M, V_M)$. By hypothesis, $\hat{q} \leq 2$. Let $D \in \hat{\mathcal{Q}}_*(M, V_M)$. As in the proof of D.3.10, we may apply D.2.9.6 to the decomposition $U \oplus U^t$, to conclude $D \leq N_M(U) = M_L$. Then by D.2.5, $\hat{q}(Aut_M(U), U) \leq \hat{q} \leq 2$, and similarly

$$\hat{q}(Aut_{LT}(V), V) \leq \hat{q}(LT/C_{LT}(V_T), V_T) \leq 2.$$

Thus we have the hypotheses of D.3.9 for M_L, L, $N_T(L)$, V, U in the roles of "M, M_+, T, V, V_M"; in particular, $V = V_{N_T(L)}$ plays the role of "V_T". Set $M_L^+ := M_L/C_{M_L}(U)$. Applying D.3.4.1 to M_L, U:

$$O_2(M_L^+) = 1.$$

Let $A^+ \in \hat{\mathcal{Q}}_*(M_L^+, U)$.

Observe that if L is not a Suzuki group, then $m_3(L) \geq 1$, so as $\hat{M}$ is an SQTK-group, $M_+ = O^{3'}(M)$; cf. the argument for 1.2.2.a, but use A.1.31.1 in place of the condition $m_p(H) \leq 2$. Similarly if $L \cong L_2(2^n)$, then $M_+ = O^{p'}(M)$ for each prime divisor p of $2^{2n} - 1$, while if $L \cong Sz(2^n)$, then $M_+ = O^{p'}(M)$ for $p = 5$, and also for each prime divisor p of $2^n - 1$. Finally if $L \cong L_3(2)$, then $M_+ = O^{p'}(M)$ for $p = 3$ and 7.

We first consider the case where A^+ is not faithful on L^+; in this case, we will derive a contradiction.

First D.3.9.2 supplies us with a subgroup $X = O^{3'}(X)$ of M such that $1 \neq X^+ \leq C_{M_L^+}(L^+)$. Thus if L is not a Suzuki group, $X \leq M_+$ by the previous paragraph, so $X \leq C_{M_+}(L^+) = L^t \leq C_M(U)$, contradicting $X^+ \neq 1$.

Therefore $L \cong Sz(2^n)$ for some odd $n \geq 3$. In particular by Theorem B.4.5, $\hat{q}(Aut_{LT}(V), V) = 2$. If in addition $L^+ = [L^+, A^+]$, then $[U, A^+] \neq 0$, establishing the hypotheses of D.2.8. Then by D.2.8, $U = V + C_U(A)$, so as $U = [U, L]$ we conclude $U = V$, contradicting $k > 1$.

Hence A^+ centralizes L^+. Set $B^+ := C_{A^+}(X^+)$. By the Thompson $A \times B$-lemma, X^+ is faithful on $C_U(B^+)$. We chose X as in D.3.9.2, so X^+ is either of order 3, or of the form $\langle K^{+A} \rangle$ for some component K^+ of M_L^+. Therefore X^+ is also faithful on $W := \langle I^{L^+ A^+} \rangle$ for some $I \in Irr_+(X^+, C_U(B^+))$. By D.2.5, $A/B \in \hat{\mathcal{Q}}(Aut_M(W), W)$. As $\tilde{U}$ is a homogeneous L^+-module, $\tilde{W}$ is the sum of copies of the natural module $\tilde{V}$ for the Suzuki group L^+. In particular if X^+ is of order 3, it is inverted by A/B of order 2, with $m(\tilde{W}) \geq m(V) = 4n \geq 12$, so

$$m(\tilde{W}/C_{\tilde{W}}(A^+)) \geq 6 > 2m(A/B),$$

contradicting $r_{Aut_A(W),W} \leq 2$. Therefore X^+ is a component of M_L^+. Also

$$L^+ \leq C_{GL(\tilde{W})}(X^+) \cong GL_d(F),$$

where $F := End_{\mathbf{F}_2 X^+}(\tilde{I})$ and $\tilde{W}$ is the sum of d copies of $\tilde{I}$. Therefore $d \geq 4$ as L^+ is a Suzuki group. By A.1.42.2, we may choose $I \in Irr_+(X, C_U(B^+), C_T(B^+))$. As $[L^+A^+, B^+] = 1$ and $[I, B^+] = 0$, we have $[W, B^+] = 0$; so as $B^+ = C_{A^+}(X^+)$ and X^+ is faithful on W, $Aut_{A^+}(W) \cong A^+/B^+$ is faithful on X^+. Hence as $r_{Aut_A(W),W} \leq 2$, the hypotheses of D.3.9.1 are satisfied with $Aut_{LXA}(W)$, $Aut_X(W)$, W, I in the roles of "M, M_+, V_M, V". Therefore by D.3.9.1,

$$\hat{q}_I := \hat{q}(Aut_{XA}(\tilde{I}), \tilde{I}) \leq 2/d \leq 1/2,$$

so by Theorem B.4.2, $X^+ \cong SL_3(2^m)$, $L_4(2)$, or $L_5(2)$. But recall we saw as L is a Suzuki group that $M_+ = O^{5'}(M)$, so $X^+ \cong SL_3(2^m)$ with m odd. Then by B.4.2, $\hat{q}_I = 1/2$, so $d = 4$ and $L^+ \leq GL_4(2^m)$. Therefore n divides m, impossible as we also saw that $M_+ = O^{p'}(M)$ for each prime divisor p of $2^n - 1$. This contradiction eliminates the case of A^+ not faithful on L^+.

So we turn to the case where A^+ is faithful on L^+. As V, U play the roles of "V_T, V_M" in D.3.9 and $V < U$, conclusion (b) of D.3.9.1 holds. Thus $\tilde{V}$ is an FF-module for $Aut_{LT}(\tilde{V})$, and $k \leq 2/r$, where $r := \hat{q}(Aut_{LT}(\tilde{V}), \tilde{V})$. In particular if $r = 1$ then $k = 2$. In any event

$$K^+ := C_{M_L^+}(L^+) \leq GL_k(F),$$

where $F := End_{\mathbf{F}_2 L^+}(\tilde{V})$. We observed earlier that $O_2(M_L^+) = 1$, so also $O_2(K^+) = 1$.

By A.3.8.3, L^+ is $L_2(2^n)$, $Sz(2^n)$, $L_2(p)$, or J_1, so as $\tilde{V}$ is an FF-module for $Aut_{LT}(\tilde{V})$, $L^+ \cong L_2(2^n)$ or $L_3(2)$ by B.4.2. In particular either $\tilde{V}$ is a natural module, or V is the A_5-module. Hence either $F = \mathbf{F}_2$, or $\tilde{V}$ is the natural module for $L^+ \cong L_2(2^n)$ with $F = \mathbf{F}_{2^n}$.

Suppose that $k = 2$ if L^+ is $L_2(2^n)$; we claim that conclusion (3) of the Theorem holds if $K^+ = 1$. Assume otherwise. As U is the sum of $k > 1$ M_L-conjugates of V and $LN_L(T)$ acts on V, $Out_M(L^+)$ is not a 2-group, so $L^+ \cong L_2(2^n)$, and thus $k = 2$ by hypothesis, so $|Irr_+(L, U)| = 2^n + 1$. Indeed $M_L^+ = L^+Y^+$ where Y^+ acts faithfully as a group of field automorphism on L^+. Let p be an odd prime divisor of $|Y^+|$ and $P^+ \in Syl_p(Y^+)$. By an earlier remark, p does not divide $2^n + 1$, so P^+ fixes two distinct members I_i, $i = 1, 2$, of $Irr_+(L, U)$, and hence induces a group of field automorphisms on $\tilde{I}_i$, and hence also on $\tilde{U}$. Thus P^+ fixes exactly $2^f + 1$ members of $Irr(L, U)$, where $f := n/|P^+|$. As Y permutes these fixed points, continuing in this fashion, $O^2(Y)$ fixes $2^s + 1$ members of $Irr(L, U)$, where $s := n/|O^2(Y^+)|$, and then Y fixes at least one member, so conclusion (3) of the Theorem holds, establishing the claim.

Suppose that $k = 2$. Then $K^+ \leq GL_2(F)$, and by the claim we may assume $K^+ \neq 1$. So as $O_2(K^+) = 1$, $O^2(K^+) \cong \mathbf{Z}_3$ if $F = \mathbf{F}_2$, and $|K^+|$ is divisible by a prime p dividing $2^{2n} - 1$ if $F = \mathbf{F}_{2^n}$ for $n > 1$. This is impossible, since we saw that $M_+ = O^{3'}(M)$, and $M_+ = O^{p'}(M)$ for each prime divisor p of $2^{2n} - 1$.

So $k > 2$, and hence $r < 1$; so by B.4.2, $L^+ \cong L_3(2)$ and $r = 1/2$. Therefore $k \leq 4$ since $k \leq 2/r$. Indeed if $C_V(L) \neq 0$, then we can pick an L^+A^+-series

$$0 =: V_0 < \cdots < V_k = U$$

such that $I_i := V_i/V_{i-1} = [I_i, L]$ is of rank 3 or 4 for each i, and $m(I_i) = 4$ for at least one i. Therefore by B.4.8.2, $m(U/C_U(A^+)) \geq k + 1$. But recall $r_{Aut_{A^+}(U),U} \leq 2$, so $m(U/C_U(A^+)) \leq 2\ m(A^+) \leq 4$, and hence $k \leq 3$ if $C_V(L) \neq 0$. However K^+ is of order coprime to 21, since we saw earlier that $O^{p'}(M) = M_+$ for $p = 3, 7$. Then as $K^+ \leq GL_k(2)$ and $K^+ \neq 1$ by the claim, we conclude that $k > 3$, so that $k = 4$. Hence $C_V(L) = 0$, and $O^2(K^+) \cong \mathbf{Z}_5$. Thus as $V_M = U \oplus U^t$, $O^2(C_{M_L}(M_+))$ is an elementary abelian 5-group of rank at most 2, so conclusion (2) of Theorem D.3.21 holds, since $O^2(C_{M_L}(M_+)) = O^2(C_M(M_+))$, completing the proof. $\square$

CHAPTER E

Generation and weak closure

In the first two sections of this chapter we collect material primarily focused on generation of a group by suitable subgroups, often chosen minimal subject to some property; the methods go back to Thompson, especially in the N-group paper. Then we develop the technique of weak closure, which, among other things, provides numerical restrictions on possible configurations in the local subgroups we analyze, and makes use of the theory of generation developed in the first two sections of this chapter.

The first two sections are devoted to our theory of generation. Often it is convenient to establish a result for a group G by reducing to minimal subgroups generating G. A particularly influential example of this approach is Thompson's argument [**Tho68**, 5.53] for Solvable Thompson Factorization (our B.2.16), where minimality is with respect to containing a fixed Sylow p-group. An axiomatic development of this method is provided in section E.1.

The subsequent section E.2 continues the study of minimal parabolics in the sense of Definition B.6.1. Again the focus is on generation by minimal subgroups; and in particular we investigate the structure of minimal parabolics under the hypothesis that G is an SQTK-group. These results are useful throughout the proof of the Main Theorem.

E.1. $\mathcal{E}$-generation and the parameter $\mathbf{n}(\mathbf{G})$

The theory recorded here was developed by Aschbacher in section 4 of [**Asc81c**] and section 4 of [**Asc82a**].

In this section, G is a finite group, Ω is a G-invariant collection of elementary abelian 2-subgroups of G, and $T \in Syl_2(G)$. As a tool primarily for doing weak closure, we consider a collection $\mathcal{E}$ of subgroups K of G, which are essentially minimal subject to $T \cap K \in Syl_2(K)$ and $\Omega \cap K \not\subseteq O_2(K)$; for example (E1) and (E2) below correspond roughly to the cases (2) and (3) for abstract minimal parabolics in B.6.8.

Note that in the version of (E2) appearing in definition E.1.2, the group $\tau(K, T\cap K, A)$ replaces the group $T_A(=\tau(G,T,A))$ which appears in (2) of 4.2 in [**Asc82a**]. That definition in [**Asc82a**] was not chosen with sufficient care to guarantee the inheritance properties in E.1.8 and E.1.10.

Definition E.1.1. Given an elementary abelian subgroup A of T, set

$$\tau(G,T,A) := C_T\big(AO_2(G)/O_2(G)\big).$$

Definition E.1.2. Given an elementary abelian subgroup A of T, let $\mathcal{E}(G,T,A)$ denote the set of subgroups K of G such that

(E0) $\tau(G,T,A) \leq T \cap K \in Syl_2(K)$,

and either

(E1)(a) $K = O_{2,p,2}(K)$ for some odd prime p, and

(b) for $\bar{K} := K/O_2(K)$ we have $1 \neq O_p(\bar{K}) = [O_p(\bar{K}), \bar{A}]$ and $m(\bar{A}) = 1$;

or

(E2)(a) $O_\infty(K)$ is 2-closed, $K = O_{\infty,E,2}(K)$, $O^2(K) = K^\infty$, and

(b) for $\bar{K} := K/O_\infty(K)$, $E(\bar{K})$ is the product of the $\tau(K, T \cap K, A)$-conjugates of a component $\bar{L}$ of $\bar{K}$ such that either

(1) $\bar{L} = [\bar{L}, \bar{A}]$, or

(2) $A \not\leq N_G(\bar{L})$ and $m(\bar{A}) = 1$.

REMARK E.1.3. Virtually the same theory holds for any prime p, although we restrict ourselves here to $p = 2$ since that is the only case we use.

LEMMA E.1.4. *For $K \in \mathcal{E}(G,T,A)$, $O^2(K) \leq \langle A^K \rangle$.*

PROOF. Let $X := \langle A^K \rangle$ and Y the kernel of the bar map in E.1.2. From E.1.2, $O^2(\bar{K}) = [O^2(\bar{K}), A]$, so $O^2(K) \leq XY$. But in (E1) Y is a 2-group, and in (E2) Y is solvable, so XY/X is a 2-group or solvable in the respective case. Hence as $O^2(K) = K^\infty$ in (E2), $O^2(K) \leq X$ as claimed. □

Now we refine our collection $\mathcal{E}(G,T,A)$ in definition E.1.2:

DEFINITION E.1.5. For i a positive integer, define $\mathcal{E}_i(G,T,A)$ to consist of those $K \in \mathcal{E}(G,T,A)$ such that for $\bar{K}$ as in definition E.1.2, $m(\bar{A}) \leq i$. Notice that, except possibly in (E2.1), $m(\bar{A}) = 1$, so $K \in \mathcal{E}_1(G,T,A)$. Define

$$E_i(G,T,\Omega) := \langle\ \mathcal{E}_i(G,T,A)\ : A \in \Omega \cap T\ \rangle.$$

For $X \leq G$, define

$$W(X,\Omega) = W(X) := \langle \Omega \cap X \rangle$$

to be the *weak closure* of Ω in X. Write $G \in \mathcal{E}_i$ if for *all* G-invariant collections Ω of elementary abelian 2-subgroups of G:

$$G = \langle\ E_i(G,T,\Omega)\ ,\ N_G\big(W(T,\Omega)\big)\ \rangle.$$

We focus on the following parameter:

DEFINITION E.1.6. If G is of odd order, set $n(G) := 0$, while if G is of even order, set:

$$n(G) := \min\{i > 0 : G \in \mathcal{E}_i\}.$$

REMARK E.1.7. As lemma E.1.14 suggests, if G is simple and not of Lie type and characteristic 2, then $n(G) = 1$—or very occasionally $n(G) = 2$. On the other hand, if G is of Lie type over $\mathbf{F}_{2^n}$, then roughly speaking, $n(G) = n$; more precisely, $n(G)$ is the maximum n such that for some minimal parabolic P of G, $P/O_2(P)$ is a rank 1 group of Lie type over $\mathbf{F}_{2^n}$. Observe in general from definition E.1.5 that $n(G) \leq m_2(G)$.

LEMMA E.1.8. *Let A be an elementary abelian subgroup of T and $H \leq G$ such that $\tau(G,T,A) \leq T \cap H \in Syl_2(H)$ and $K \in \mathcal{E}_i(H, T \cap H, A)$ for some i. Then*

(1) $AO_2(G) \leq \tau(G,T,A) \leq \tau(H, T \cap H, A) \leq \tau(K, T \cap K, A) \leq K$.

(2) $K \in \mathcal{E}_i(G,T,A)$.

(3) If Y is a normal subgroup of G with $O_2(G) \in Syl_2(Y)$, and $G^* := G/Y$, then $C_T(A^*) = \tau(G,T,A)$, so $\tau(G,T,A)^* \leq \tau(G^*, T^*, A^*)$.

PROOF. Observe $AO_2(G) \leq \tau(G,T,A)$, and by hypothesis $\tau(G,T,A) \leq T \cap H\ inSyl_2(H)$. Thus $O_2(G) \leq O_2(H)$, so $\tau(G,T,H) \leq \tau(H,T \cap H,A)$. Similarly $\tau(H,T \cap H,A) \leq \tau(K,T \cap K,A)$, establishing (1). Then (2) follows from (1) and the definition of $\mathcal{E}(G,T,A)$ in E.1.2.

Assume the hypothesis of (3). As $O_2(G) \leq Y$, $\tau(G,T,A) \leq C_T(A^*)$. On the other hand Y is 2-closed, so $[C_T(A^*),A] \leq T \cap Y = O_2(G)$, completing the proof of (3). □

DEFINITION E.1.9. Given a finite group U, a section U_1/U_2 of U with $U_i \trianglelefteq U$ for $i = 1,2$, and $U_0 \in Syl_2(U)$, write $\mathcal{S}(U,U_0,U_1,U_2)$ for the set of subgroups K of U_1U_0 minimal subject to $U_0 \leq K$ and $U_1U_0 = KU_2$. That is, setting $\bar{U} := U/U_2$, K is minimal subject to $U_0 \leq K$ and $\bar{K} = \bar{U}_1\bar{U}_0$.

LEMMA E.1.10. *Assume that X and Y are normal subgroups of G with $Y \leq X$, and let $K \in \mathcal{S}(G,T,X,Y)$. Set $K_X := K \cap X$, $K_Y := K \cap Y$, and $\bar{G} := G/Y$. Let A be an elementary abelian subgroup of T. Then*

(1) $K = K_XT$ and $X/Y \cong \bar{K}_X \cong K_X/K_Y$.

(2) The preimage in K of $O_2(\bar{K})$ is 2-closed.

(3) If $\bar{K}_X = O_{2,p}(\bar{K}_X)$ for some odd prime p, then $K = O_{2,p,2}(K)$.

(4) If $\bar{K}_X = O_{\infty,E}(\bar{K}_X)$, $O_\infty(\bar{K})$ is 2-closed, and $O^2(\bar{K}) = \bar{K}^\infty$, then also $K = O_{\infty,E,2}(K)$, $O_\infty(K)$ is 2-closed, and $O^2(K) = K^\infty$.

(5) If X/Y is a chief factor of G, then $\bar{K}_X$ is the direct product of copies of some composition factor of G.

(6) If X/Y is a p-group for some odd prime p, and $\bar{X} = [\bar{X},\bar{A}]$, then $X \leq \langle\ \mathcal{E}_1(G,T,A)\ \rangle Y$.

(7) If $\bar{I} \in \mathcal{E}_i(\bar{G},\bar{T},\bar{A})$, then $O^2(\bar{I}) = \overline{O^2(J)}$ for some $J \in \mathcal{E}_i(G,T,A)$.

(8) Assume $O^2(G) = G^\infty = [G^\infty,A]$, Y is 2-closed, and $T \leq J \leq G$ with $G = JY$. Then $G = \langle J,\mathcal{E}_1(G,T,A)\rangle$.

PROOF. As $K \in \mathcal{S}(G,T,X,Y)$, $T \leq K$ and $XT = KY$, so $K = XT \cap K = K_XT$ and $X = X \cap KY = K_XY$. Thus $X/Y = K_XY/Y = \bar{K}_X \cong K_X/K_Y$, establishing (1).

Let Z be the preimage in K of $O_2(\bar{K})$. As $T \in Syl_2(K)$ and $Z \trianglelefteq K$, $T \cap Z$ is Sylow in Z and normal in T. Then by a Frattini Argument, $K = N_K(T \cap Z)Z = N_K(T \cap Z)K_Y$. Thus $XT = KY = N_K(T \cap Z)Y$ and $T \leq N_K(T \cap K)$, so $K = N_K(T \cap Z)$ by minimality of K. That is (2) holds.

Assume the hypothesis of (3) holds. Since $\bar{K} = \bar{K}_X\bar{T}$, $\bar{K} = O_{2,p,2}(\bar{K})$, so $\bar{K}$ is solvable. By (2), K_Y is 2-closed and hence solvable, so K is solvable. Therefore by Hall's Theorem, T is contained in a Hall $\{2,p\}$-subgroup K_0 of K. Since $\bar{K}$ is a $\{2,p\}$-group, $\bar{K} = \bar{K}_0$, so $K = K_0$ by minimality of K. Then as Z is 2-closed and $K/Z = O_{p,2}(K/Z)$, (3) holds.

Assume that the hypothesis of (4) holds. We saw K_Y is solvable, so $O_\infty(K)$ is the preimage in K of $O_\infty(\bar{K})$. Then as Z and $O_\infty(\bar{K})$ are 2-closed, so is $O_\infty(K)$. Also as $\bar{K} = O_{\infty,E,2}(\bar{K})$, $K = O_{\infty,E,2}(K)$. As $O^2(\bar{K}) = \bar{K}^\infty = \overline{K^\infty}$, the minimality of K gives $K = K^\infty T$. Hence $O^2(K) = K^\infty$, completing the proof of (4).

Part (5) is well known; cf. 8.2 in [**Asc86a**].

Assume the hypothesis of (7), and choose $X := O^2(G)Y$, so that $O^2(\bar{I}) \leq \bar{X}$. Replacing G by K and appealing to E.1.8.2, we may assume $G = K$. Let $\bar{G}_1 := O^2(\bar{I})\tau(\bar{I},\bar{T} \cap \bar{I},\bar{A})$ and G_1 the preimage of $\bar{G}_1$. By A.1.16, $T \cap G_1 \in Syl_2(G_1)$, and by E.1.8.3, $\tau(G,T,A) \leq G_1$; so replacing G by G_1 and appealing to E.1.8.2, we may assume $G = G_1$. Set $G^* := G/O_2(G)$ if $\bar{G}$ is solvable, and set $G^* := G/O_\infty(G)$ otherwise. By (2) and E.1.8.3, $\tau(G, T, A)$ is the preimage in T of $\tau(\bar{G}, \bar{T}, \bar{A})$.

Suppose $\bar{G}$ is solvable. By (3), $G = O_{2,p,2}(G)$. By (2), $|A^*| = |\bar{A}| = 2$. As $\bar{G} = \bar{G}_1$, $\bar{T} = \tau(\bar{G}, \bar{T}, \bar{A})$, so as $\tau(G, T, A)$ is the preimage in T of $\tau(\bar{G}, \bar{T}, \bar{A})$ and $G^* = G/O_2(G)$, T centralizes A^*. As $O_p(\bar{G}) = [O_p(\bar{G}), \bar{A}]$, $O_p(G^*) = [O_p(G^*), A^*]O_p(Y^*)$, so as T centralizes A^*, T^* acts on $[O_p(G^*), A^*]$. Thus as $G = K$, $O_p(G^*) = [O_p(G^*), A^*]$, so $G \in \mathcal{E}_1(G, T, A)$, completing the proof of (7) in this case.

This leaves the case where G^* is nonsolvable, where $G \in \mathcal{E}_i(G, T, A)$ by (4) and the fact that $\tau(G, T, A)$ is the preimage in T of $\tau(\bar{G}, \bar{T}, \bar{A})$. Hence (7) holds.

Assume the hypothesis of (6). For B of index 2 in A, let $\bar{X}(B) := [C_{\bar{X}}(B), A]$. By Generation by Centralizers of Hyperplanes, $\bar{X} = \langle \bar{X}(B) : |A : B| = 2\rangle$. Further $\tau(\bar{G}, \bar{T}, \bar{A})$ acts on $\bar{X}(B)$, so $\bar{X}(B)\tau(\bar{G}, \bar{T}, \bar{A}) \in \mathcal{E}_1(\bar{G}, \bar{T}, \bar{A})$ for each hyperplane B of A. Then (7) completes the proof of (6).

Finally assume the hypotheses of (8). Let Y_1 be a minimal normal subgroup of G contained in Y, $\hat{G} := G/Y_1$, and $J_1 := \langle J, \mathcal{E}_1(G, T, A)\rangle$. Proceeding by induction on the order of G, $\hat{G} = \langle \hat{J}, \mathcal{E}_1(\hat{G}, \hat{T}, \hat{A})\rangle$, so $G = J_1 Y_1$ by (7). Thus replacing J, Y by J_1, Y_1, we may assume Y is a minimal normal subgroup of G, and it remains to show $Y \leq J_1$. As Y is 2-closed, Y is solvable, so Y is an elementary abelian p-group for some prime p and J is irreducible on Y. If $p = 2$ then $Y \leq T \leq J_1$, so we may take p odd. If $[Y, A] \neq 1$, then $[Y, A]\tau(G, T, A) \leq J_1$ by (6), so as J is irreducible on Y, $Y \leq J_1$. Thus we may assume $[Y, A] = 1$, so $G^\infty = [G^\infty, A]$ centralizes Y. Thus $J^\infty \trianglelefteq J^\infty Y \leq G^\infty$, so as Y is abelian, $Y \leq J^\infty \leq J_1$, completing the proof of (8). $\square$

Notice in case (4) of E.1.10 that K^∞ is a product of $\mathcal{C}$-components by A.3.3.5.

The following result is a weak form of lemma 4.5 from [**Asc81c**]. Since the proof given in [**Asc81c**] is difficult to read with many typos, and is the only result of any depth in that section of the paper, we include a slightly expanded proof here.

PROPOSITION E.1.11. *If $n(LAut_T(L)) \leq n$ for each composition factor L of G, then $n(G) \leq n$.*

PROOF. Let Ω be a G-invariant collection of elementary abelian 2-subgroups of G, $W := W(T, \Omega)$, $G_1 := \langle\ E_n(G, T)\ ,\ N_G(W)\ \rangle$, $X := O^2(G)O_2(G)$, and Y maximal subject to $O_2(G) \leq Y \trianglelefteq G$ and $Y < X$. By induction on the order of G, $n(YT) \leq n$, and therefore $YT \leq \langle\ \mathcal{E}_n(YT, T, \Omega \cap YT)\ ,\ N_G\big(W(T, \Omega \cap YT)\big)\ \rangle$; the first term lies in G_1 by E.1.8.2, as does the second since $(\Omega \cap YT) \cap T = \Omega \cap T$. Thus $Y \leq G_1$, so we may assume $X \not\leq G_1$. Set $G^* := G/Y$. Then $W^* = W(T^*, \Omega^*)$, and as W is weakly closed in T with respect to G, $N_{G^*}(W^*) = N_G(W)^* \leq G_1^*$. Further by E.1.10.7, $E(G^*, T^*, \Omega^*) \leq E(G, T, \Omega)^* \subseteq G_1^*$. Finally the composition factors of G^* are composition factors of G; so if $Y \neq 1$, then by induction on the order of G, $G^* = \langle\ E_n(G^*, T^*, \Omega^*)\ ,\ N_{G^*}(W^*)\ \rangle \leq G_1^*$, contradicting $X \not\leq G_1$. Hence $Y = 1$. By E.1.10.5, X is the direct product of subgroups isomorphic to some composition factor L of G. As $O_2(G) \leq Y = 1$, $|L| > 2$.

Next $X = [X, W]N_X(W)$, so as $G = XT$ and X is a chief section of G, $X = [X, W]$ or $N_X(W)$. As $N_G(W) \leq G_1$, $X = [X, W]$. Hence $X = \langle X_A : A \in \Omega \cap T\rangle$, where $X_A := [X, A]$. Therefore $X_A \not\leq G_1$ for some $A \in \Omega \cap T$. Notice X_A is subnormal in G, so $S_A := T \cap X_A \in Syl_2(X_A)$. Let $T_A := \tau(G, T, A)$. As $O_2(G) \leq Y = 1$, $T_A = C_T(A)$. Further T_A acts on X_A.

Suppose L is of prime order p; then p is odd since $|L| > 2$. As $X_A = [X_A, A]$ is a p-group, applying E.1.10.6 to X_AT_A, and recalling by E.1.8.2 that $\mathcal{E}_n(X_AT_A, S_AT_A, A) \subseteq \mathcal{E}_n(G,T,A) \subseteq G_1$, $X_AT_A \leq \langle \mathcal{E}_1(G,T,A)\rangle$ for each $A \in \Omega \cap T$, so $G \leq G_1$, contrary to the choice of G.

Therefore L is a nonabelian simple group. Let J be a component of X and set $S := N_T(J)$. As X is a chief section of G, $G = \langle J, T\rangle$, so $J \not\leq G_1$. Let $\Gamma := \{A \cap JS : A \in \Omega\}$, set $W_0 := W(S, \Gamma)$, $J_0 := N_J(W_0)$, and $\mathcal{P} := \mathcal{E}_n(JS, S, \Gamma)$. By hypothesis, $n(JS/O_2(JS)) \leq n$, so $JS = \langle \mathcal{P}, J_0S\rangle$; and hence as $J \not\leq G_1$:

$$\text{Either } J_0 \not\leq G_1 \text{ or } \mathcal{P} \not\subseteq G_1. \tag{!}$$

Let $A \in \Omega \cap T$, set $V := N_A(J)$, and pick a complement Z to V in A.

We claim that $J_0 \not\leq G_1$. Assume otherwise. Then $\mathcal{P} \not\subseteq G_1$ by (!), so we may pick A so that there is $P \in \mathcal{E}_n(JS, S, V)$ with $P \not\leq G_1$. Set $P_A := \langle O^2(P)^{T_A}\rangle$. Then P_A is the direct product of the T_A-conjugates of $O^2(P)$. Let $K_A := P_AT_A$ and $R_A := T \cap K_A$. Then $P \leq K_A$, so as $P \not\leq G_1$, $K_A \not\leq G_1$, and hence $K_A \not\leq \langle \mathcal{E}_n(G,T,A)\rangle$. Set $\bar{K}_A := K_A/O_2(K_A)$ if P is solvable, and $\bar{K}_A := K_A/O_\infty(K_A)$ if P is not solvable. Suppose first that P is solvable; then $O^2\big(P/O_2(P)\big)$ is a p-group, so $\bar{K}_A = O_p(\bar{K}_A)\bar{T}_A$ and $O_p(\bar{K}_A) = [O_p(\bar{K}_A), A]$. Thus by E.1.10.6, $K_A \leq \langle\ \mathcal{E}_1(K_A, R_A, A)\ \rangle O_2(K_A)$, contradicting $K_A \not\leq G_1$ in view of E.1.8.2. Thus P is not solvable, so $O^2(\bar{K}_A)$ is the direct product of the T_A-conjugates of simple components $\bar{L}_i$, $1 \leq i \leq r$, of $\bar{K}_A$ contained in $\bar{P}$. Further $O^2(P) = P^\infty$, so $P_A = P_A^\infty$. Recall Z is a complement to V in A, and define $\pi_i : \bar{L}_i \to C_{\bar{K}_A}(Z)$ by

$$x\pi_i := \prod_{z \in \bar{Z}} x^z \text{ for } \ x \in \bar{L}_i.$$

Then π_i is a $N_V(L_i)$-equivariant isomorphism of $\bar{L}_i$ with $\bar{I}_i$, where I_i is the $\mathcal{C}$-component in the preimage in K_A of $\bar{L}_i\pi_i$. Further $N_i := \langle I_i^{T_A}\rangle$ is T_A-invariant and $T \cap N_i \in Syl_2(N_i)$. Let $K_{Z,i} := N_iT_A$. Then $\bar{K}_A = \langle\ \bar{K}_{Z,i}\ ,\ \bar{R}_A : 1 \leq i \leq r\rangle$, so by E.1.10.8,

$$K_A = \langle\ K_{D,i}\ ,\ R_A\ ,\ \mathcal{E}_1(K_A, R_A, A) : 1 \leq i \leq r\rangle$$

for each $K_{D,i} \leq K_A$ with $O^2(\bar{K}_{Z,i}) \leq \bar{K}_{D,i}$. Thus as $K_A \not\leq G_1$, $K_{D,i} \not\leq G_1$ for some i using E.1.8.2. As $P \in \mathcal{E}_n(JS, S, V)$ either V acts on L_i or $|V : V \cap O_2(P)| = 2$; then the corresponding statement holds for $\bar{I}_i$ as π_i is $N_V(L_i)$-equivariant. Then as T_A centralizes $\bar{A}$, $\big(V \cap O_2(P)\big)Z$ centralizes $\bar{I}_i$, so $\bar{K}_{Z,i} \in \mathcal{E}_n(\bar{K}_{Z,i}, \overline{T \cap K_{Z,i}}, \bar{A})$. However $O^2(\bar{K}_{Z,i}) = O^2(\bar{K}_{D,i})$ for some $K_{D,i} \in \mathcal{E}_n(K_{Z,i}, T \cap K_{Z,i}, A)$ by E.1.10.7, while $K_{D,i} \in \mathcal{E}_n(G,T,A)$ by E.1.8.2, contrary to $K_{D,i} \not\leq G_1$. This completes the proof of the claim.

By the claim, $J_0 \not\leq G_1$. Let Y_0 be maximal subject to $Y_0 \trianglelefteq J_0$, Y_0 is S-invariant, and $Y_0 \leq G_1$. Hence there is a S-chief section X_0/Y_0 of J_0 with $X_0 \not\leq G_1$. Then X_0/Y_0 is a direct product of copies of a simple group L_0 and $S \cap X_0 \in Syl_2(X_0)$, so as $X_0 \not\leq G_1$, L_0 is not of order 2. Let $K_1 := O^2(X_0)$ and $H := \langle K_1^T\rangle$. As $S = N_T(J)$ acts on $X_0 \leq J$, and T is transitive on the components $\{J_1, \ldots, J_r\}$ of X, $r = |T : S|$ and $H = K_1 \times \cdots \times K_r$ where we choose notation so that $J =: J_1$ and $K_i := K_1^{t_i}$ for $t_i \in T$ with $J^{t_i} = J_i$. Let $Y_1 := Y_0 \cap K_1$, $Y_i := Y_1^{t_i}$, and $H_0 := Y_1 \cdots Y_r$; then $X_i/Y_i \cong X_0/Y_0$, so H/H_0 is a section of HT which is the direct product of copies of L_0. Let $K \in \mathcal{S}(HT, T, H, H_0)$.

Suppose L_0 is of order p for some prime p; then p is odd since L_0 is not of order 2. Then K is solvable by E.1.10.3, so $n(K) = 1$ by induction on the order of G; and using E.1.8.2 as in our argument on YT at the start of the proof, $K \leq G_1$, contrary to the choice of X_0. Therefore L_0 is a nonabelian simple group, so setting $K_Y := K \cap H_0$, $K_X := K \cap H$, and $K^+ := K/K_Y$, and using E.1.10.1, $K_X^+ = L_1^+ \times \cdots \times L_m^+$ is the direct product of copies of L_0 permuted transitively by T^+. Choose notation so that $L_1 \leq JK_Y$ and $\{L_1^+, \ldots, L_s^+\} = L_1^{+T_A}$.

If $W \leq S$ then $W = W_0$ and hence $J_0 \leq N_G(W) \leq G_1$, which is not the case. Thus we may choose $A \not\leq S$, so $Z \neq 1$. As $A \cap S \leq W_0 \leq O_2(J_0 S)$, $A \cap S$ centralizes L_1^+ and hence also centralizes $U^+ := L_1^+ \cdots L_s^+$, while the complement Z to V in A is semiregular on $\{L_1^+, \ldots, L_s^+\}$. As $Z \neq 1$, we may take C to be a hyperplane of Z, and define $\pi_0 : L_1^+ \to C_{U^+}(C)$ by

$$x^+\pi_0 := \prod_{c \in C} x^{+c}.$$

Then π_0 is an isomorphism of L_1^+ with $L_1^+\pi_0$. Finally let I_0 be the preimage in K_X of $L_1^+\pi_0$, $M := \langle I_0^{T_A} \rangle$, and $R := T \cap MT_A$. Then $M^+R^+ \in \mathcal{E}_1(M^+R^+, R^+, A^+)$, so $M \leq \langle \mathcal{E}_1(G, T, A) \rangle H_0$ by E.1.10.7 and E.1.8.2. Therefore $M \leq G_1$, and then $H \leq \langle\ M \cap H\ ,\ T\ \rangle H_0 \leq (G_1 \cap H)H_0$, so $X_0 \leq G_1$, contrary to the choice of X_0. This finally completes the proof. □

Proposition E.1.11 essentially reduces us to the simple case. Then one can reduce to a collection of overgroups of T which generates G, and hence to minimal parabolics of G via McBride's lemma B.6.3:

PROPOSITION E.1.12. *Let Δ be a collection of subgroups of G such that $G = \langle \Delta \rangle$, and for each $H \in \Delta$, $T \leq H$ and $n(H) \leq n$. Then $n(G) \leq n$.*

PROOF. This is 4.6 in [**Asc81c**]. □

COROLLARY E.1.13. *If G is solvable, then $n(G) \leq 1$.*

PROOF. This is a consequence of E.1.11 and the fact that $Aut(\mathbf{Z}_p)$ is cyclic. □

We will frequently make use of the following list of values of the parameter $n(G)$ for the simple SQTK-groups in Theorem C (A.2.3).

LEMMA E.1.14. *Let G be an SQTK-group with $L := F^*(G)$ simple and $G = LT$. Then*

(1) If $L \cong L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$, $Sp_4(2^n)'$, ${}^2F_4(2^n)'$, or $G_2(2^n)'$, then $n(G) = n$; and if $L \cong L_4(2)$ or $L_5(2)$, then $n(G) = 1$.
(2) If $L \cong {}^3D_4(2^n)$ then $n(G) = 3n$.
(3) If $L \cong A_7$ then $n(G) = 1$.
(4) If $L \cong M_{11}$, M_{12}, M_{24}, or He, then $n(G) = 1$.
(5) If $L \cong J_1$, J_2, J_4, M_{22}, M_{23}, HS, or Ru, then $n(G) \leq 2$.
(6) If $L \cong L_2(q)$, $q > 5$ odd, then $n(G) = 1$.
(7) If $L \cong L_3^\epsilon(p)$, p odd, then $n(G) = 1$.

PROOF. Recall Theorem C gives the list of possible L. Then the values for $n(G)$ are obtained from the following results in [**Asc82a**]: Lemma 4.7 for (1) and (2); 4.8 for (3); 4.9 for (4) and (5); 4.10 for (6); and 4.11 for (7). □

E.2. Minimal parabolics under the SQTK-hypothesis

Recall from Definition B.6.1 that a *minimal parabolic* of a finite group H (with respect to the prime 2) is a subgroup P of H such that some Sylow 2-subgroup T of H is contained in a unique maximal subgroup of P, but T is not normal in P. The terminology is suggested by the fact that in a group of Lie type of characteristic 2, the property holds for the parabolics of rank 1.

Recall also $\hat{\mathcal{U}}(T) = \hat{\mathcal{U}}_H(T)$ and $!\mathcal{N}(T) = !\mathcal{N}_H(T)$ from Definitions B.6.2 and B.6.7.

E.2.1. Structure of minimal parabolics in SQTK-groups.

In this subsection, H is a finite SQTK-group such that $H \in \hat{\mathcal{U}}_H(T)$ for some $T \in Syl_2(H)$. Set $M := !\mathcal{N}_H(T)$ and $J := \ker_M(H)$.

LEMMA E.2.1. *Assume that H is solvable, and that either $O_2(M) \not\leq O_2(H)$ or $C_T(J/O_2(J)) \not\leq O_2(H)$. Then $O^2(H/O_2(H)) \cong \mathbf{Z}_p$, E_{p^2}, or p^{1+2} for some odd prime p.*

PROOF. Let $H^* := H/O_2(H)$, $P^* := F(H^*)$ and $R := C_T(J/O_2(J)))$. By B.6.8.2, $O^2(H^*) = P^*$ is a p-group for some odd prime p, T^* is irreducible on $P^*/\Phi(P^*)$, and $\Phi(P^*) \leq J^*$, so R^* centralizes $\Phi(P)^*$. As $O_2(M) \leq R$, $R \not\leq O_2(H)$ by hypothesis, so that $R^* \neq 1$. Let P_0^* be a supercritical subgroup of P^*; as R^* is faithful on P^*, R^* is faithful on P_0^*, so as R^* centralizes $\Phi(P^*)$ and $R^* \neq 1$, $P_0^* \not\leq \Phi(P^*)$. Thus as T^* is irreducible on $P^*/\Phi(P^*)$, $P^* = P_0^*\Phi(P^*)$, so $P^* = P_0^*$. Then as H is an SQTK-group, the lemma follows from A.1.24. □

Next we refine B.6.8 using the results of section A.3.

LEMMA E.2.2. *Assume H is not solvable and set $H^* := H/J$. Then $H = \langle K, T\rangle$ for some $K \in \mathcal{C}(H)$ such that $K/O_2(K)$ is quasisimple, and setting $M_K := M \cap K$, one of the following holds:*

(1) $J = O_2(H)$, $K \neq K^t$ for some $t \in T$, $M = M_K M_K^t T$, and either

(a) $K^* \cong L_2(2^n)$ or $Sz(2^n)$ and M_K^* is a Borel subgroup of K^*, or

(b) $K^* \cong L_2(p)$, $p \equiv \pm 1 \mod 8$ an odd prime, $M_K^* = C_{K^*}(Z((T \cap K)^*))$, and $|Aut_T(K^*)| > 8$.

(2) $H = KT$, $J/O_2(H) = Z(KO_2(H)/O_2(H))$, and one of the following holds:

(a) K^* is a Bender group and M_K^* is a Borel group of K^*.

(b) $K^* \cong L_3(2^n)$ or $Sp_4(2^n)'$, T is nontrivial on the Dynkin diagram of K^*, and M_K^* is a Borel group of K^*.

(c) $K^* \cong L_2(p^e)$, $p > 3$ prime and $e \leq 2$, with $p^e \equiv \pm 1 \mod 8$, $M_K^* = C_{K^*}(Z((T \cap K)^*))$, and $|Aut_T(K^*)| > 8$. If $e = 2$, then $Aut_T(K^*)$ is not contained in the group of inner-field automorphisms of K^*.

(d) $K^* \cong L_3^{-\epsilon}(p)$, $p \equiv \epsilon \mod 4$ prime, T is nontrivial on the Dynkin diagram of K^* if $\epsilon = -1$, and $M_K^* = C_{K^*}(Z((T \cap K)^*))$.

In any case, H^* is also an SQTK-group, $n(H) = n$ in (1a), (2a), and (2b), and $n(H) = 1$ in all other cases.

PROOF. The lemma describes $H/O_2(H)$, so passing to that quotient, we may assume that $O_2(H) = 1$. We appeal to B.6.8 and the results of section A.3. As H is not solvable by hypothesis, B.6.8.3 says $H = K_0T$ where $K_0 := O^2(H)$, $J = F(H)$, and K_0^* is the direct product of the T-conjugates of a nonabelian simple group K_1^*. Let $K := K_1^{*\infty}$. Then $K \in \mathcal{C}(H)$ and $H = \langle K, T\rangle J$, so as $M = !\mathcal{N}(T)$, we conclude that $H = \langle K, T\rangle$. If K is not quasisimple, then case (3) or (4) of A.3.6 holds, so J contains an involution in $O_{2',Z}(K) - F(K)$, contrary to $J = F(H)$ nilpotent. Therefore K is quasisimple. In particular conclusion (ii) of B.6.8.3 holds, so $\langle K^T\rangle = F^*(H)$, and $J = O_\infty(H) = Z(K)$ is of odd order since $O_2(H) = 1$.

Suppose first that K is not normal in H. Then by A.3.8.1 $H = KK^tT$ for $t \in T - N_T(K)$. Further by A.3.8.3,

$$K \cong L_2(2^n),\ Sz(2^n),\ L_2(p),\ p \text{ an odd prime, or } J_1.$$

By I.1.3, the multiplier of each of these groups is a 2-group, so as $J = Z(K)$ is of odd order, we conclude that $J = 1$. Let $S := N_T(K)$ and $\overline{KS} := KS/O_2(KS)$. By B.6.8.3, $\bar{K}\bar{S} \in \hat{\mathcal{U}}_{\bar{L}\bar{S}}(\bar{S})$, so by induction on $|H|$, $\bar{K}\bar{S}$ satisfies conclusion (2) of the lemma. Then from the list of possibilities for K above, (1) holds.

Thus we may assume that $K \trianglelefteq H$, so that $H = KT$. We showed earlier that $J = Z(K)$; and to verify the remaining assertions we may pass to $H/Z(K)$, and so assume that K is simple and $J = 1$.

Now in view of (2) of Theorem A (A.2.1), K is described in Theorem C (A.2.3). If K is of Lie type and characteristic 2, the maximal T-invariant subgroups of K containing $T \cap K$ are parabolics, so $|\mathcal{N}(T)| = 1$ only if M_K is a Borel subgroup of K, where K is either of rank 1, or untwisted of rank 2 with T nontrivial on the diagram of K; thus (a) or (b) of (2) holds in this case. One checks directly (e.g. [**Asc86b**]) that none of the sporadic cases arise. Finally if K is of Lie type and odd characteristic, then as $M = !\mathcal{N}(T)$, we may apply Theorem A of [**Asc80**] to conclude that (c) or (d) of (2) holds, using Dickson's Theorem A.1.3 to verify the statements in (c) about $Aut_T(K^*)$ and the congruences on p^e. For example the condition $|Aut_T(K^*)| > 8$ is necessary to ensure that T^* is not contained in an A_4 or S_4 subgroup of H^*, and when $e = 2$, the condition that T^* does not induce inner-field automorphisms is necessary to ensure that T^* does not act on a $PGL_2(p)$ subgroup of K^*. Then the conditions on T imply that $p^e \equiv \pm 1 \mod 8$.

Finally we use E.1.14 to check the statements about $n(H)$. By (2) of Theorem A (A.2.1), H^* is an SQTK-group. This completes the proof. □

We will also need to know the action of H on members of $\mathcal{R}_2(H)$ when $F^*(H) = O_2(H)$ and $J(T)$ does not centralize $R_2(H)$. We obtained such information for more general minimal parabolics in B.6.9, so much of the following result is simply a restatement of that lemma for SQTK-groups:

LEMMA E.2.3. *Assume that $F^*(H) = O_2(H)$, $V \in \mathcal{R}_2(H)$, and $J(T) \not\leq C_T(V)$. Set $H^* := H/C_H(V)$, $\tilde{V} := V/C_V(O^2(H))$, and $S := Baum(T)$. Then*

(1) $O^2(H) \leq J(H)$, $J(H)^* = H_1^* \times \cdots H_s^*$, and $\tilde{V} = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_s$, where $s = 1$ or 2, T is transitive on $\{H_1, \cdots, H_s\}$, $V_i := [V, H_i]$, and $H_i^* \cong S_5$ or $L_2(2^n)$ (including $L_2(2) \cong S_3$).

(2) If $H_i^* \cong L_2(2^n)$ then $\tilde{V}_i$ is the natural module for H_i^* and $S^* = J(T)^*$ is Sylow in $J(H^*)$. Further S is Sylow in $O^2(H)S$, and setting $E := \Omega_1(Z(J(T)))$,

$$\langle E^{O^2(H)}\rangle = [V, O^2(H)]C_E(O^2(H)).$$

(3) If $H_i^* \cong S_5$ then V_i is the A_5-module and $J(T)^* = S^*$ is the product of the 4-subgroups $H_i^* \cap S^*$ generated by the transpositions in $H_i^* \cap T^*$.

(4) If no nontrivial characteristic subgroup of S is normal in H, then $H = \langle L, T\rangle$, where $L \in \mathcal{C}(H)$ is an $L_2(2^n)$-block or A_5-block.

PROOF. The structure of H^* and its action on $[V, H]$ follow from B.6.9. As H is an SQTK-group, $s \leq 2$ by A.1.31.1, and $m = 3$ or 5 if $H_i^* \cong S_m$. Set $F := \langle E^{O^2(H)}\rangle$ and $F_1 := \langle E^{O^2(H_1)}\rangle$.

Assume that $H_i^* \cong L_2(2^n)$. By B.6.8.6.b, $C_H(V)$ is 2-closed, so the hypotheses of Baumann's Lemma B.6.10 are satisfied. Hence S is Sylow in $\langle S^{O^2(H)}\rangle$, and $F_1 = V_1 C_E(O^2(H_1))$. Further $\langle S^{O^2(H)}\rangle = O^2(H)S$ by B.6.8.6.d. In particular if $s = 1$, then $F = F_1$ and (2) holds, so we may take $s = 2$. Then

$$F = \langle E^{O^2(H)}\rangle = \langle E^{O^2(H_1)O^2(H_2)}\rangle = \langle F_1^{O^2(H_2)}\rangle$$

$$= V_1\langle C_E(O^2(H_1))^{O^2(H_2)}\rangle = V_1V_2C_E(O^2(H)) = [V, O^2(H)]C_E(O^2(H)),$$

completing the proof of (2). Part (4) follows from the $C(G, T)$-Theorem C.1.29 in this case.

Finally assume $H_i^* \cong S_5$. Then we have the hypothesis of case (a) of C.1.24, and that result completes the proof of (3) and (4), and hence of E.2.3. □

E.2.2. Further results for the case b = 1.

In this subsection we continue the hypothesis and notation of the previous subsection.

We consider the situation where some elementary abelian normal 2-subgroup of T is not contained in $O_2(H)$. This situation arises in case (1) of the qrc-lemma D.1.5; in the amalgam literature, the situation corresponds to the case where the parameter "b" is equal to 1; cf. Definition F.7.8 and remark F.7.12.

In particular, the situation arises in the proof of our Main Theorem as follows: G is a QTKE-group, $T \in Syl_2(G)$, $M_0 = !\mathcal{M}(L_0T)$, for a suitable uniqueness subgroup L_0T, $V \in \mathcal{R}_2(L_0T)$ is a TI-set under M_0, $H \in \mathcal{H}_*(T, M_0)$ (cf. Definition 3.0.1), and $M = !\mathcal{N}(T) = M_0 \cap H$. Recall from B.6.8.5 that J is 2-closed, so that $V \leq J$ iff $V \leq O_2(H)$. Thus when $V \not\leq J$, the parameter b in Definition F.7.8 is equal to 1 by F.7.9.3, for the amalgam defined by L_0T and H.

To deal with this situation, we consider certain subgroups I of H, which are essentially minimal subject to $V \not\leq O_2(I)$. The theory of such subgroups is useful in many places in the proof of the Main Theorem; we also use it in the next subsection to show that then $\hat{q}(Aut_{M_0}(V), V) < 2$. We are led to the following definition:

DEFINITION E.2.4. For V a nontrivial elementary abelian 2-subgroup of T, define $\mathcal{I}(H, T, V)$ to consist of those subgroups I of H such that $I = \langle V, V^g\rangle$ for some $g \in I$, $T \cap I \in Syl_2(I)$, $\ker_{M\cap I}(I)$ is 2-closed, and setting $I^* := I/\ker_{M\cap I}(I)$, one of the following holds:

(1) $I^* \cong L_2(2^k)$ or $Sz(2^k)$ for some $k > 1$, and $(M \cap I)^*$ is a Borel subgroup of I^*.

(2) I^* is dihedral of order $2p^a$, p an odd prime and $a \geq 1$; in this case set $k := 1$.

(3) $O^2(H/J) \cong Sp_4(2^n)'$, $I^* \cong Sp_4(2^k)$ for some $k \geq 1$, $(M \cap I)^*$ is a Borel subgroup of I^*, and $V^* \leq Z(T^* \cap I^*)$ but V^* is contained in neither root subgroup of $Z(T^* \cap I^*)$.

Moreover we require that k divides $n(H)$ in each case.

REMARK E.2.5. When $k = 1$ in case (3), notice $I^* \cong S_6$ rather than A_6.

DEFINITION E.2.6. When discussing involutions t in subgroups of $Sp(V)$ for V a symplectic space of dimension $2m$ over F of characteristic 2, we follow the terminology of *Suzuki type* defined on pages 16-17 of [**AS76a**]. Briefly, t can be of Suzuki type a_i, b_i, or c_i, where $i = \dim_F([V,t])$ and in the respective cases: i is even and $v \perp v^t$ for all $v \in V$; i is odd; i is even and v is not perpendicular to v^t for some $v \in V$. In particular, if $\dim_F V = 4$, then root involutions have type b_1 and a_2, and non-root involutions have type c_2.

LEMMA E.2.7. *Assume that $J = 1$, and V is a nontrivial elementary abelian 2-group normal in T, with $V \leq O_2(M)$. Then $\mathcal{I}(H,T,V) \neq \emptyset$.*

PROOF. By B.6.8.1, $O_2(H) \leq J$, so $O_2(H) = 1$ since $J = 1$ by hypothesis. In particular as $V \neq 1$, $N_H(V) < H$. Then as $T \leq N_H(V)$, $N_H(V) \leq M$ as $M = !\mathcal{N}(T)$.

If V is of order 2, then by the Baer-Suzuki Theorem we can choose $g \in H$ with $I := \langle V, V^g \rangle$ not a 2-group. As I is generated by two involutions, I is dihedral; then replacing I by a suitable subgroup, we may assume $|I| = 2p^a$ for some odd prime p. Then V is conjugate to V^g in I, so we may take $g \in I$. Also $V \in Syl_2(I)$, so as $V \leq O_2(M)$ by hypothesis, $V = M \cap I = N_I(V)$, and hence $\ker_{M \cap I}(I) = 1$. Thus I is in case (2) of $\mathcal{I}(H,T,V)$.

Therefore we may assume that $m(V) > 1$. Set $K := O^2(H)$, $T_K := T \cap K$, and $M_K := M \cap K$.

Suppose that H is solvable. Then by B.6.8.2, and as $O_2(H) = 1$, K is a p-group for some prime p, and $K = F^*(H)$. Hence using Generation by Centralizers of Hyperplanes A.1.17, there is a hyperplane U of V with $[C_K(U), V] \neq 1$. Choose $g \in [V, C_K(U)]$ inverted by $v \in V - U$ and define $I := \langle V, V^g \rangle$. Then I is the direct product of U and a dihedral group of order of $2|g|$, and as K is a p-group, $|g|$ is a power of p. Arguing as in paragraph two, $I \in \mathcal{I}(H,T,V)$.

Thus we may suppose that H is not solvable. So as $J = 1$, K is a product of at most two simple groups by E.2.2.

Suppose first that K is a Bender group $X(2^n)$. Then by E.2.2, M_K is a Borel group of K. As $V \leq O_2(M)$, V induces inner automorphisms on K, so that $V \leq K$ as $F^*(H) = K$. Then as V is elementary abelian and K is a Bender group, $V \leq Z(T_K)$. Hence for $g \in K - M$, $I := \langle V, V^g \rangle \cong X(2^k)$ for some k dividing n, and the Sylow 2-group of I containing V is $T \cap I$, so that $k \geq m(V) > 1$. Moreover we may take $g \in I - M$, and $M \cap I$ is a Borel group of I, so that $\ker_{M \cap I}(I) = 1$. Hence $I \in \mathcal{I}(H,T,V)$ as I is described in case (1) of Definition E.2.4.

Similarly if $K = K_1 K_1^t$ with K_1 a Bender group, then $V \leq K$. Let V_i denote the projection of V on K_i; we may assume V_1 is nontrivial, and choosing $g \in K_1 - M$, $I_1 := \langle V_1, V_1^g \rangle \times V_2$. If $m(V_1) = 1$, our usual argument produces $I \in \mathcal{I}(H,T,V)$ satisfying case (2) of Definition E.2.4, while if $m(V_1) > 1$, then $\langle V_1, V_1^g \rangle$ is a Bender group and again $I_1 \in \mathcal{I}(H,T,V)$.

Suppose next that case (2c) of E.2.2 holds. Then $M = C_H(Z)$, where $Z := Z(T_K)$, and this centralizer possesses no noncyclic elementary abelian 2-subgroup V contained in $O_2(M)$ with $V \trianglelefteq T$—unless $p^e = 9$ and $H \cong Aut(K)$, a case we will consider later in treating the case $K \cong Sp_4(2^n)'$. Similarly if case (1b) of E.2.2 holds we can again choose the projection V_1 of V on K_1 nontrivial, and arguing as in the previous paragraph, either $m(V_1) = 1$, leading to a member of $\mathcal{I}(H,T,V)$, or $m(V_1) > 1$, leading to a contradiction.

Next suppose we are in case (2d) of E.2.2, where $K \cong L_3^{\epsilon}(p)$ for a suitable odd prime p. Here $M = C_H(Z(T_K))$ and $O_2(M_K)$ is cyclic, contradicting $V \leq O_2(M)$ since $m(V) > 1$.

This leaves case (2b) of E.2.2. Suppose first that $K \cong L_3(2^n)$. We have treated the case $L_2(7) \cong L_3(2)$ already, so we may take $n > 1$. Now $T_K = AA^t$, where $t \in T$ is nontrivial on the Dynkin diagram of K, and A, A^t are the maximal elementary abelian subgroups of T_K. Hence as $\Phi(V) = 1$ and $V \trianglelefteq T$, $V \leq A \cap A^t =: Z$. But Z is a long root group of K, so Z is Sylow in some $L \leq K$ with $L \cong L_2(2^n)$ for $n > 1$, and $M \cap L$ is Borel in L. As $V \leq L$, we can construct $I \in \mathcal{I}(H, T, V)$ contained in L as in an earlier treatment.

Finally assume $K \cong Sp_4(2^n)'$; this case causes the most difficulties. Set $F := \mathbf{F}_{2^n}$. If $n = 1$ then as T is nontrivial on the Dynkin diagram of K by E.2.2, while V is a noncyclic elementary abelian normal subgroup of T, it follows that $H = Aut(K)$. In this case we let K_S denote the subgroup of H isomorphic to $Sp_4(2)$. If $n > 1$ let $K_S := K$. In any case set $S := T \cap K_S$.

As before $S = AA^t$, where now $A \cong E_{2^{3n}}$, and $V \leq A \cap A^t =: Z$. This time $Z = Z_l \times Z_s$, where Z_l and Z_s are long and short root groups of K_S, respectively. Let U be the natural module for K_S. Then Z_l is the group of transvections with center U_l for some point U_l, and $[U, Z_s] = C_U(Z_s) =: U_s$ is a totally singular line containing U_l. Let W_l, W_s be a flag opposite to U_l, U_s in the building for K_S. We can use these flags to define a symplectic basis $X := \{x_1, x_2, x_3, x_4\}$ for U, with $U_l = \langle x_1 \rangle$, $U_s = \langle x_1, x_2 \rangle$, $W_l = \langle x_4 \rangle$, $W_s = \langle x_3, x_4 \rangle$, and $(x_1, x_4) = 1 = (x_2, x_3)$. With respect to this basis, the action of root elements on U is given by

$$Z_l = \{z_l(\lambda) : \lambda \in F\} \text{ and } Z_s = \{z_s(\lambda) : \lambda \in F\},$$

where $z_l(\lambda)$ centralizes $U_l^{\perp}$, and $z_l(\lambda) : x_4 \mapsto x_4 + \lambda x_1$; while $z_s(\lambda)$ centralizes U_s, and

$$z_s(\lambda) : x_3 \mapsto x_3 + \lambda x_1, \quad \text{and} \quad z_s(\lambda) : x_4 \mapsto x_4 + \lambda x_2.$$

Next for $\mu \in F^{\#}$, define

$$R(\mu) := \{z_l(\lambda) z_s(\lambda\mu) : \lambda \in F\}.$$

Then the groups $R(\mu)$, $\mu \in F^{\#}$, form a partition of the diagonal involutions of Z, and those involutions are of Suzuki type c_2 (recall Definition E.2.6); in particular they are not root involutions. Further the groups $R(\mu)$ are characterized by the property that for each $r_\mu \in R(\mu)^{\#}$:

$$R(\mu)^{\#} \text{ is the set of involutions } r \in K \text{ such that } [r, Fu] = [r_\mu, Fu] \text{ for all } u \in U. \tag{$*$}$$

We claim that $R(\mu)$ is a Sylow group of an $L_2(2^n)$-subgroup L of K, with Borel subgroup $N_L(R(\mu)) \leq N_G(R(\mu)) = M_K$: Namely the map $\varphi : x_1 \to x_2$ and $\varphi : x_4 \mapsto x_3$ induces an isometry $\varphi : \langle x_1, x_4 \rangle \to \langle x_2, x_3 \rangle$ and hence an isomorphism $\varphi^* : Sp(\langle x_1, x_4 \rangle) \to Sp(\langle x_2, x_3 \rangle)$. Further a Sylow 2-subgroup D of the diagonal

$$L := \{y \varphi^*(y) : y \in Sp(\langle x_1, x_4 \rangle)\}$$

satisfies (*), so D is conjugate to $R(\mu)$, completing the proof of the claim. Consequently if $V \leq R(\mu)$, we can find a member of $\mathcal{I}(H, T, V)$ in L. Hence we may assume for each μ that $V \not\leq R(\mu)$.

Observe also that if $Z_l \cap V =: R_l \neq 1$, then as t acts on V with $Z_l^t = Z_s$, $R_s := V \cap Z_s = R_l^t \neq 1$. In particular V is contained in neither root group.

Let $g \in K$ map U_l, U_s to W_l, W_s; and set $I := \langle V, V^g \rangle$. We first show that I is irreducible on U as F-module. For suppose $W \neq 0$ is an FI-submodule of U. Then $W \not\leq U_s$, since $U_s^g = W_s$ and $U_s \cap W_s = 0$, so there is $w \in W - U_s$. We claim that $U_l \leq W$: Suppose first that $w \in U_l^{\perp}$, so that $w = u_s + \nu x_3$ for $u_s \in U_s$ and $\nu \neq 0$. We saw $V \cap Z_s \neq 1$, so from the action of $z_s(\lambda)$ on V, $U_l \leq [V, w] \leq W$, as claimed. Suppose on the other hand that $w \notin U_l^{\perp}$. Then the coefficient of x_4 in the expression for w as an F-linear combination of members of X is nonzero; in this case we make the stronger claim that $U_s \leq W$. If $R_l \neq 1$, then also $R_s \neq 1$, so $U_s \leq [V, w]$ from action of Z_l and Z_s on V. Otherwise $R_l = 1 = R_s$, and hence all elements $v \in V^{\#}$ are involutions of Suzuki type c_2. Then for each such v, $[v, Fw]$ is a 1-dimensional F-subspace of the 2-space U_s spanned by x_1 and x_2. Since $V \not\leq R(\mu)$ for any μ, $[Fw, v] \neq [Fw, v']$ for some $v, v' \in V^{\#}$, and hence $U_s \leq [w, V] \leq W$, completing the proof of the claims. Thus in any case we have $U_l \leq W$. But now by symmetry between V, U_l, U_s and V^g, W_l, W_s, $W_s = [U_l, V^g] \leq W$, and then as $W_s \not\leq U_l^{\perp}$, $U_s = [W_s, V] \leq W$ by the stronger claim. Hence $U = U_s + W_s \leq W$, establishing irreducibility of I on U as F-module.

So indeed I is irreducible on U, and hence also on the quasiequivalent FI-module U^t, since our argument was independent of the representative of the quasiequivalence class.

Assume first that I contains involutions not of Suzuki type c_2—that is, I contains long or short root involutions. Then replacing U by U^t if necessary, we may assume that I contains transvections on U. Hence as I is irreducible on U, we conclude from G.4.1 that either I is $Sp_4(2^k)$, or I is $O_4^{\epsilon}(2^k)$, or I preserves a decomposition of U as the direct sum of two 2-dimensional subspaces. Assume the second or third case holds; then as $I = \langle V, V^g \rangle$, V contains an involution of Suzuki type b_1 or a_2: This is because all involutions in $I - \Omega_4^{\epsilon}(2^k)$ are of these types in the second case, and all involutions moving the two factors are of Suzuki type a_2 in the third case. Then as $R_l \cong R_s$, V contains an involution of Suzuki type b_1 commuting with one of type a_2, whereas such a pair does not exist in cases two and three. Therefore $I \cong Sp_4(2^k)$, and as $T_K = C_K(v)$ for $v \in V$ of Suzuki type c_2, $T \cap I \in Syl_2(I)$. Then as V contains elements of Suzuki type c_2 in $Z(T \cap I)$, while $Z(T \cap I) \leq Z(T_K)$, the Borel subgroup $N_I(Z(T \cap I))$ of I is contained in $M \cap I$. Further $\ker_{M \cap I}(I) = 1$ and we can choose $g \in I$. Thus I satisfies case (3) for $\mathcal{I}(H, T, V)$ in Definition E.2.4.

Thus we may assume that all involutions in I are of Suzuki type c_2, so

$$C_K(i) \text{ is a 2-group for each involution } i \in I. \qquad (!)$$

By (!), I has at most one component. Also $O(I)$ is generated by centralizers of hyperplanes of V, so as V is noncyclic, we conclude from (!) that $O(I) = 1$. As I is irreducible and faithful on U, $O_2(I) = 1$. Therefore $F^*(I)$ is simple. Then inspecting the centralizers of involutions in the automorphism groups of the simple groups listed in Theorem C satisfying (!) (see [**GLS98**], or 16.1.4 and 16.1.5), we conclude that I is $L_2(2^k)$, $Sz(2^k)$, A_6, or $L_2(p)$, where $p > 5$ is a Fermat or Mersenne prime. The last case is impossible as $Sp_4(2^n)$ contains no Frobenius group of order $p(p-1)/2$ with $p > 5$ prime (e.g. over a suitable extension field of $\mathbf{F}_{2^n}$, an element of order p can be taken to be diagonal, and so is acted on nontrivially only by elements of the Weyl group D_8). Further if I is A_6, then $V \cong E_4$ and each 4-subgroup of I is contained in an $L_2(4)$-subgroup of I, so such an overgroup of V is in $\mathcal{I}(H, T, V)$. Thus we may assume that I is $L_2(2^k)$ or $Sz(2^k)$.

Let $E := End_{FI}(U)$. If $E \neq F$ then as $\dim_F(U) = 4$, $|E : F| = 2$ and U is the natural EI-module for $I \cong L_2(2^k)$. But then involutions in I are of type a_2 on I, a contradiction. Thus $E = F$ so $U = F \otimes_{\mathbf{F}_{2^e}} \bar{U}$ for some divisor e of n and some 4-dimensional absolutely irreducible $\mathbf{F}_{2^e}I$-module $\bar{U}$ such that $\mathbf{F}_{2^e} \leq F$ is a splitting field for $\bar{U}$. Invoking the theory of small dimensional representations of I (cf. pages 26-27 and 77-78 of [**GLS98**]) we conclude that either $e = k/2$ and $\bar{U}$ is the orthogonal module for $I \cong \Omega_4^-(2^e)$, or $e = k$ and $\bar{U}$ is the natural module for $I \cong Sz(2^e)$ (i.e., the module obtained by restriction from the embedding of $Sz(2^e)$ in $Sp_4(2^e)$).

As each involution in $V^\#$ is of type c_2, $C_U(v) = U_s = C_U(V)$ for each $v \in V^\#$, so $V \leq V_0 := \{v_0 \in I : C_U(v_0) = C_U(V)\}$. When $I \cong \Omega_4^-(2^e)$, V_0 is Sylow in a subgroup $I_0 \cong \Omega_3(2^e)$. When $I \cong Sz(2^e)$, let $I_0 := I$. Then in either case I_0 is a Bender group such that the Borel subgroup B_0 of I_0 over V is contained in $N_K(V_0) = N_K(U_s)$. Now $B_0 = R_0H_0$ where R_0 is the unipotent radical of B_0 and $H_0 \cong \mathbf{Z}_{2^e-1}$ is a Cartan subgroup. If I_0 is a Suzuki group, then $R_0 \not\leq C_K(U_s)$ and $R_0 \leq C_K(V)$, so $B_0 \leq M_K$. If I_0 is $\Omega_3(2^e)$ then $C_{U_s}(H_0) \neq 0$, so $H_0 \cap O_{2,2'}(N_K(U_s)) = 1$, and hence M_K is the unique Borel subgroup of $N_K(U_s)$ containing each $h \in H_0^\#$ with $V \cap V^h \neq 1$, and again $B_0 \leq M_K$. Then as before we can find a member of $\mathcal{I}(H,T,V)$ inside I_0, completing the proof of the lemma. □

Lemma E.2.7 supplies subgroups of H/J which, under suitable hypotheses on V, can be "pulled back" to members of $\mathcal{I}(H,T,V)$. The necessary hypotheses are:

HYPOTHESIS E.2.8. *Assume that $H \in \hat{\mathcal{U}}(T)$ with $M := !\mathcal{N}(T)$, and H is an SQTK-group. Assume further that V is an elementary abelian 2-group normal in T, with $V \leq O_2(M)$ and $V \not\leq J := \ker_M(H)$.*

LEMMA E.2.9. *Assume Hypothesis E.2.8. Then $N_H(V) \leq M$. Furthermore $\mathcal{I}(H,T,V) \neq \emptyset$, and for each $I \in \mathcal{I}(H,T,V)$:*

(1) $[J,I] \leq O_2(H)$ and $O_2(H) \leq N_G(I)$, so $J \leq N_G(O^2(I))$.
(2) If $F^*(H) = O_2(H)$ then $F^*(I) = O_2(I)$.

PROOF. If $V \trianglelefteq H$, then $V \leq O_2(H) \leq J$ by B.6.8.1, contradicting hypothesis E.2.8. Then $T \leq N_H(V) < H$, so as $M = !\mathcal{N}(T)$ we get $N_H(V) \leq M$.

Let $H^* := H/J$. If H is not solvable then H^* is an SQTK-group by E.2.2. If H is solvable, then as $V \leq O_2(M)$ but $V \not\leq O_2(H)$, $O^2(H^*) \cong \mathbf{Z}_p$ or E_{p^2} by E.2.1 and B.6.8.2, so again H^* is an SQTK-group. Thus in any event H^* is an SQTK-group. Then since $J \leq M$ by definition, Hypothesis E.2.8 is satisfied by the tuple H^*, T^*, M^*, V^*, and $\ker_{M^*}(H^*) = J^* = 1$. Hence by E.2.7, there is $I_0^* \in \mathcal{I}(H^*,T^*,V^*)$. Let I_0 be the preimage of I_0^* in H, and pick $g \in I_0$ with $I_0^* = \langle V^*, V^{*g}\rangle$ and $I := \langle V, V^g\rangle$ minimal subject to this constraint. We will show that $I \in \mathcal{I}(H,T,V)$.

First $I_0^* = I^* \cong I/(I \cap J)$, and by minimality of I, there is $g_1 \in I$ with $V^{*g} = V^{*g_1}$, so without loss $g \in I$. By B.6.8.5, J is 2-closed, so $T \cap J = O_2(H) \in Syl_2(J)$; hence as $(T \cap I_0)^* \in Syl_2(I_0^*)$, also $T \cap I_0 \in Syl_2(I_0)$.

Now set $\bar{H} := H/O_2(H)$. As $J \leq M$ and $V \leq O_2(M)$ by hypothesis, $[\bar{V}, \bar{J}] \leq O_2(\bar{J}) = 1$, so as $I = \langle V, V^g\rangle$, $\bar{I}$ centralizes $\bar{J}$. Thus $\bar{I} \trianglelefteq \bar{I}\bar{J} = \bar{I}_0$, so $\overline{T \cap I} \in Syl_2(\bar{I})$; then as $O_2(H) \leq T$, also $T \cap I \in Syl_2(I)$. As $\bar{I}$ centralizes $\bar{J}$, $\bar{J} \cap \ker_{\overline{M \cap I}}(\bar{I}) \leq Z(\ker_{\overline{M \cap I}}(\bar{I}))$, so as $\ker_{(M \cap I)^*}(I^*) \cong \ker_{(M \cap I_0)^*}(I_0^*)$ is 2-closed by Definition E.2.4,

$\ker_{\overline{M\cap I}}(\bar{I})$ is 2-closed. Hence the preimage $\ker_{M\cap I}(I)$ is 2-closed. This completes the verification that $I \in \mathcal{I}(H,T,V)$.

Now let I be any member of $\mathcal{I}(H,T,V)$. The argument above shows that $\bar{I}$ centralizes $\bar{J}$, so $[J,I] \leq O_2(H)$, and in particular, J normalizes $IO_2(H)$. As

$$O_2(H) \leq T \cap T^g \leq N_G(V) \cap N_G(V^g),$$

$O_2(H)$ acts on $\langle V, V^g\rangle = I$. Then $O^2(I) = O^2(O_2(H)I)$ is J-invariant, completing the proof of (1). By (1), $O_2(H)$ centralizes $O^2(F^*(I))$, so (2) holds as well. □

In the next lemma E.2.10 we study the structure of groups like those in $\mathcal{I}(H,T,V)$, under a weaker version of Hypothesis E.2.8, in which we replace the condition that $H \in \hat{\mathcal{U}}(T)$ by the assumption that $N_H(V) \leq M$.

Lemma E.2.10 is often applied in the proof of the Main Theorem in the following way: We are given $L \in \mathcal{L}_f^*(G,T)$; set $M_0 := N_G(\langle L^T\rangle)$, and suppose $V \in \mathcal{R}_2(\langle L,T\rangle)$ and $H \in \mathcal{H}_*(T, M_0)$ such that $V \not\leq \ker_M(H)$. Set $M := M_0 \cap H$. Then E.2.10 and later lemmas supply very precise information on the structure of $I \in \mathcal{I}(H,T,V)$. If G is an example or shadow, then M is usually an end-node parabolic in a diagram geometry, and H is the rank 1 parabolic not contained in M. Often $H = IT$, and the special 2-group P constructed in E.2.10 is $O_2(N_G(V \cap V^g))$.

Proposition E.2.10. *Assume X is a finite group, $T \in Syl_2(X)$, $V \trianglelefteq T$ is an elementary abelian 2-group, and $N_X(V) \leq M \leq X$ with $V \leq O_2(M)$. Set $L := O^2(X)$, $J := \ker_M(X)$, and $X^* := X/J$. Assume J is 2-closed and $X = \langle V, V^g\rangle$ for some $g \in X$. Then*

(1) Set $B := J \cap V$, $P := \langle B, B^g\rangle$, and $Z := V \cap V^g$. Then $P := BB^g \trianglelefteq X$, and $Z \leq P \cap Z(X)$.

(2) Set $\tilde{X} := X/Z$. Then $\tilde{P} = \tilde{B} \oplus \tilde{B}^g$ is elementary abelian, V is quadratic on $\tilde{P}$, and $C_{\tilde{P}}(V) = \tilde{B}$.

(3) $O_2(X^*) = 1$.

(4) X centralizes J/P.

(5) $A := B^g$ is cubic on V. That is $[V,A,A,A] = 1$.

(6) If V^* is of order 2, then $P = O_2(X) = J$ and $X/P \cong X^* \cong D_{2m}$ for some odd integer m.

(7) If $X^* \cong L_2(2^n)$ or $Sz(2^n)$, with $n > 1$, then $J = O_2(X)$ and X/P is the direct product of $O^2(X/P)$ with an elementary abelian 2-group. Further either $P = O_2(LP)$ and $LP/P \cong X^*$, or LP/P is a perfect central extension of a 2-group by $Sz(8)$.

(8) If $X^* \cong Sp_4(2^n)$, then $O_2(X) = J$ and X/P is the direct product of a copy of X^* and an elementary abelian 2-group. Further $O_2(LP) = P$ and $LP/P \cong [X^*, X^*]$.

(9) If $X > V$ and $F^*(X) = O_2(X)$, then $[V,A] \neq 1$.

Proof. As V and V^g are abelian, $Z = V \cap V^g$ is in the center of $\langle V, V^g\rangle = X$, so $Z \leq O_2(X)$, and hence $Z \leq P$. Next $B, B^g \leq J$ which is 2-closed, so $B = V \cap O_2(X)$ and $P \leq O_2(X) \leq T \leq N_X(V)$, so

$$[V,P] \leq V \cap O_2(X) = B \leq P,$$

and similarly $[V^g, P] \leq P$. Therefore $X = \langle V, V^g\rangle$ normalizes P. Then as $P \leq N_X(V)$,

$$B = V \cap O_2(X) \trianglelefteq P,$$

and similarly B^g is normal in P, so that $P = BB^g$, completing the proof of (1). Now as B and $A := B^g$ are normal in P and elementary abelian, $\tilde{P} = \tilde{B} \oplus \tilde{B}^g = \tilde{B} \oplus \tilde{A}$ is elementary abelian. As $[V,P] \leq V \cap P$ and V is abelian, V is quadratic on $\tilde{P}$. Further as $\tilde{P} = \tilde{B} \oplus \tilde{A}$ with $\tilde{B} \leq C_{\tilde{P}}(V)$, $C_{\tilde{P}}(V) = \tilde{B} \oplus C_{\tilde{A}}(V)$, with $C_{\tilde{A}}(V)$ centralized by $\langle V, V^g \rangle = X$. Then the preimage C in X of $C_{\tilde{A}}(V)$ is normalized by X and contained in V^g, so $C \leq V \cap V^g = Z$. That is $C_{\tilde{P}}(V) = \tilde{B}$, completing the proof of (2).

Next $O_2(X^*) \leq T^*$ and $JT \leq M$, so $O_2(X^*) \leq \ker_{M^*}(X^*) = J^* = 1$, so (3) holds.

Set $\bar{X} := X/P$. As $O_2(X) \leq N_X(V)$, $[O_2(X), V] \leq V \cap O_2(X) = B \leq P$, so $X = \langle V, V^g \rangle$ centralizes $\overline{O_2(X)}$. Then as J is 2-closed, $\bar{J} = O(\bar{J}) \times \overline{O_2(X)}$. As $V \leq O_2(M)$ by hypothesis, $[J,V] \leq O_2(M) \cap J \leq O_2(X)$, so $X = \langle V, V^g \rangle$ also centralizes $O(\bar{J})$, which with the previous observation completes the proof of (4).

We have seen that $A \leq P \trianglelefteq X$. Then $[V,A] \leq P$, so $[V,A,A] \leq [P,A] \leq Z$ as $\tilde{P}$ is abelian. Then as $Z \leq Z(X)$, $[V,A,A,A] = 1$, and (5) holds.

Notice that $\bar{V} = V/(V \cap P) = V/(V \cap J) \cong V^*$, and in particular we have $\bar{V} \cap O_2(\bar{X}) = 1$. Also by (4), $\bar{J} \leq Z(\bar{X})$.

Assume $|V^*| = 2$; then $\bar{V} \cong V^*$ also has order 2. Thus $X^* = \langle V^*, V^{*g} \rangle \cong D_{2m}$ and $\bar{X}$ are dihedral. By (3), m is odd. Then as $\bar{J} \leq Z(\bar{X})$, we conclude that $|\bar{J}| \leq 2$. But if $\bar{J} \neq 1$, then $\bar{V}$ and $\bar{V}^g$ are not conjugate in the dihedral group $\bar{X}$, so V is not conjugate to V^g in X, contradicting our hypothesis. Thus $J = P$, completing the proof of (6).

Assume that $X^* \cong L_2(2^n)$ or $Sz(2^n)$. Thus $X = JL$, so as $\bar{J} \leq Z(\bar{X})$, $\bar{X} = \bar{L}Z(\bar{X})$. Then as $X = \langle V^X \rangle$, also $\bar{X} = \bar{L}\bar{V}$, and hence $X = PLV$. Then since $\bar{X} = Z(\bar{X})\bar{L}$, $Z(\bar{X})/Z(\bar{L})$ is an elementary abelian 2-group. By I.1.3, the Schur multiplier Σ of L^* is a 2-group, and Σ is trivial unless $L^* \cong L_2(4)$ or $Sz(8)$. Therefore $Z(\bar{X}) = O_2(\bar{X}) = \overline{O_2(X)}$, and $Z(\bar{L}) = 1$ unless $L^* \cong L_2(4)$ or $Sz(8)$. Thus $J = O_2(X)$, and if $Z(\bar{L}) = 1$ then $\bar{X} = O_2(\bar{X}) \times \bar{L}$ with $O_2(\bar{X})$ elementary abelian, so that $P = O_2(LP)$ and (7) holds. Thus we may assume that $Z(\bar{L}) \neq 1$, so that $L^* \cong L_2(4)$ or $Sz(8)$. In the latter case by I.2.2.4, every involution in L^* lifts to an involution of $\bar{L}$, so $O_2(\bar{X})$ is elementary abelian and we are in the exceptional case allowed in (7). Finally if $L^* \cong L_2(4)$, then $\bar{L} \cong SL_2(5)$ by I.1.3. But then as $V \trianglelefteq T$, $Z(\bar{L}) \leq \bar{V} \cap O_2(\bar{X}) = 1$ as we saw earlier, contradicting our assumption that $Z(\bar{L}) \neq 1$. So the proof of (7) is complete.

Assume the hypotheses of (8). Using I.1.3 as in the previous paragraph, $X = PLV$, $\bar{L}$ is quasisimple, and either $\bar{X} = \bar{L}Z(\bar{X})$ or $X^* \cong Sp_4(2) \cong S_6$. Further when $n > 1$, the multiplier of X^* is trivial by I.1.3, and arguments in the previous paragraph complete the proof. Thus we may assume that $X^* \cong S_6$. As $Z(\bar{L}) \leq \bar{J} \leq Z(\bar{X})$, $|Z(\bar{L})| \leq 2$, since the center of the triple cover $\hat{A}_6$ is not centralized by any element of $S_6 - A_6$ (cf. the proof of I.2.2—the normalizer in S_6 of a Sylow 3-subgroup contains D_8, rather than the Sylow 2-group Q_8 of $SL_2(3)$ centralizing that center). If $Z(\bar{L}) \neq 1$, then as $V \trianglelefteq T$, again $Z(\bar{L}) \leq \bar{V} \cap O_2(\bar{X}) = 1$, a contradiction. Therefore $\bar{L} \cong A_6$, so that (8) holds.

Assume that $F^*(X) = O_2(X)$ and $X > V$. If $V \leq O_2(X)$ then $X = \langle V, V^g \rangle \leq O_2(X)$, so X is a 2-group. Then since $V \trianglelefteq T$, $X = \langle V, V^g \rangle = V$, contradicting the hypothesis of (9). Therefore $V \not\leq O_2(X)$. But if V centralizes A, then V centralizes $AB = P$, while by (4), V centralizes $O_2(X)/P$. Then as $F^*(X) = O_2(X)$, $V \leq$

$C_X(P) \cap C_X(O_2(X)/P) \leq O_2(X)$ by Coprime Action, contrary to our previous reduction. This completes the proof. □

LEMMA E.2.11. *Assume Hypothesis E.2.8 and let* $I := \langle V, V^g \rangle \in \mathcal{I}(H,T,V)$, *and set* $M_I := M \cap I$, $J_I := \ker_{M_I}(I)$, $I^* := I/J_I$, *and* $T_I := T \cap I$. *Then*

(1) I, M_I, T_I, V *satisfy the hypotheses of E.2.10 in the roles of "X, M, T, and V".*

(2) $V^* \leq Z(T_I^*)$.

(3) $T_I = O^{2'}(M_I)$*; that is,* M_I *is 2-closed.*

(4) $V^g \cap O_2(I) = N_{V^g}(V)$.

(5) Define $P := (O_2(I) \cap V)(O_2(I) \cap V^g)$. *Then* $O_2(I) = J_I$, *and* $O_2(I)/P = Z(I/P)$ *is elementary abelian.*

PROOF. Part (1) is immediate, once we observe that $N_H(V) \leq M$ by E.2.9, so that $N_I(V) \leq M_I$.

Next the possiblities for I^* in (1)–(3) of Definition E.2.4 are those in (6)–(8) of E.2.10. Hence by E.2.10, $J_I = O_2(I)$ and $O_2(I)/P = Z(I/P)$ is elementary abelian, establishing (5). Thus to prove (3), it suffices to check that M_I^* is 2-closed. We verify this claim, and also that $V^* \leq Z(T_I^*)$ in each of the three cases: If I^* is dihedral, $M_I^* = T_I^* = V^*$, so our claim holds. If I^* is a Bender group or a symplectic group, then M_I^* is a Borel subgroup of I^*, so $T_I^* = O_2(M_I^*)$. Further when I^* is Bender, $V^* \leq \Omega_1(T_I^*) = Z(T_I^*)$, while if I^* is symplectic, then $V^* \leq Z(T_I^*)$ from Definition E.2.4. Thus our claim is established, completing the proof of (2) and (3).

It remains to prove (4). As $O_2(I) \leq T \leq N_I(V)$, $V^g \cap O_2(I) \leq N_{V^g}(V)$, so it suffices to show that $U^* := N_{V^{g*}}(V^*) = 1$. But using (2) and (3) we see that $U^* \leq O^{2'}(M_I^*) = T_I^* \leq C_{I^*}(V^*)$; so $I^* = \langle V^*, V^{g*} \rangle$ centralizes U^*, and hence $U^* \leq O_2(I^*) = 1$. □

E.2.3. Modules for H when $\mathbf{b = 1}$.

Using the lemmas in the previous subsection, we can establish a key result which shows that in conclusion (1) of the *qrc*-lemma D.1.5, $\hat{q} \leq 2$. In that case of the *qrc*-lemma, $V \not\leq O_2(H)$, which is equivalent to $V \not\leq J$ in our setup.

DEFINITION E.2.12. Given an odd prime p, define $d(p^a)$ to be the order of 2 in the group of units of the ring of integers modulo p^a, set $d'(p^a) := d(p^a)/2$ if $d(p^a)$ is even, and $d'(p^a) := d(p^a)$ if $d(p^a)$ is odd.

PROPOSITION E.2.13. *Assume Hypothesis E.2.8 with* $F^*(H) = O_2(H)$. *Then*

(1) $\mathcal{I}(H,T,V) \neq \emptyset$.

Pick $I = \langle V, V^g \rangle \in \mathcal{I}(H,T,V)$, *and set* $A := V^g \cap O_2(I)$. *Then:*

(2) $A = N_{V^g}(V)$, *A is cubic on* V, $[V, A] \neq 1$, $C_A(V) = V \cap V^g$, *and*

$$m(V/C_V(A)) \leq 2\ m(A/C_A(V)).$$

(3) $\hat{q}(Aut_H(V), V) \leq 2$.

(4) $m(Aut_A(V)) \geq d'(p^a)$, k, $2k$, $2k$ *for* $I/\ker_{M\cap I}(I) \cong D_{2p^a}$, $L_2(2^k)$, $Sz(2^k)$, $Sp_4(2^k)$, *respectively.*

(5) If $V \cap V^g = 1$ *then* $q(Aut_H(V), V) \leq 1$.

(6) If $V^I \cap T \subseteq C_I(V)$ *then* $I/\ker_{M\cap I}(I)$ *is not* $Sp_4(2^k)$.

(7) $C_I(P/(V \cap V^g)) \leq O_2(I) = \ker_{M\cap I}(I)$.

PROOF. Part (1) follows from E.2.9. By E.2.11.1, I, $M_I := M \cap I$, $T_I := T \cap I$, and V satisfy the hypotheses of E.2.10, so we can also appeal to that lemma. By E.2.9.2, $F^*(I) = O_2(I)$, so by E.2.10.9, $[V, A] \neq 1$. By E.2.10.5, A is cubic on V, and by E.2.11.4, $A = N_{V^g}(V)$. Thus to complete the proof of (2) and (3), we must show that $C_A(V) = V \cap V^g$ and $m(V/C_V(A)) \leq 2\, m(A/C_A(V))$.

Adopt the notation of E.2.11, and set $B := V \cap J_I$, $P := BA$, $Z := V \cap V^g$, and $\tilde{P} := P/Z$. As $I \in \mathcal{I}(H, T, V)$, $I^* \cong D_{2p^a}$, p odd, $L_2(2^k)$, $Sz(2^k)$, or $Sp_4(2^k)$. By E.2.10.2,

$$C_{\tilde{P}}(V) = \tilde{B},$$

and $\tilde{P} = \tilde{B} \oplus \tilde{A}$. Therefore $C_A(V) = A \cap B = Z$ so $Aut_A(V) = A/Z$, and then

$$m(A/C_A(V)) = m(A/Z) = m(B/Z)$$

$$= m(V/Z) - m(V/B) \geq m(V/C_V(A)) - m(V/B) \qquad (*)$$

We will show later that

$$m(V/B) = m(V^*) \leq m(B/Z) = m(A/C_A(V)), \qquad (**)$$

and then (2) and (3) will follow from (*) and (**).

We first establish (7). Let $Y := C_I(\tilde{P})$. Then Y centralizes $O_2(I)/P$ and $O_2(I) = \ker_{M \cap I}(I)$ by E.2.11.5, so Y centralizes the quotients in the normal series $1 \leq Z \leq P \leq O_2(I)$. Thus $Y \leq O_2(I)$ as $F^*(I) = O_2(I)$, so (7) holds.

Now (cf. the discussion and references in the proof of Theorem G.9.3) the minimal degree d of a faithful $\mathbf{F}_2 I^*$-module is $2d'(p^a)$, $2k$, $4k$, $4k$ for I^* isomorphic to D_{2p^a}, $L_2(2^k)$, $Sz(2^k)$, $Sp_4(2^k)$, respectively. Therefore (4) is established, as $m(\tilde{P}) = 2\, m(A/Z)$. Further $m(V^*) \leq m_2(I^*) = 1$, k, k, $3k$, in the respective cases, and indeed when I^* is $Sp_4(2^k)$, $V^* \leq Z(T_I^*)$ by Definition E.2.4, so that $m(V^*) \leq 2k$. Thus $m(V^*) \leq d/2$, so as $d/2 \leq m(A/C_A(V))$ by (4), we have established (**) and hence also (2) and (3).

Next assume that $Z = 1$. Then $P = A \times B$ by (2) and (5) of E.2.10 (where the definition of A is given). Also by E.2.10.2, $C_P(V) = B$. Since $A \leq P$, $C_A(V) \leq C_P(V) = B$, so $C_A(V) \leq A \cap B = 1$, and hence A acts faithfully on V. It follows that $B = C_V(A)$, and now (5) is a consequence of (**).

Finally assume that $W := \langle V^I \cap T_I \rangle \leq C_I(V)$ and $I^* \cong Sp_4(2^k)$. By Definition E.2.4, $V^* \leq Z(T_I^*)$ and V^* is contained in neither of the root groups in $Z(T_I^*)$. This implies that $T_I^* = W^* = C_{I^*}(V^*)$; for example T_I^* is the product of the unipotent radicals R_i^* of the two parabolics I_i^*, $i = 1, 2$ of I^* above T_I^*, and $R_i^* = \langle V^{*I_i^*} \rangle$ as V^* is not contained in the root group $Z(O^{2'}(I_i^*))$. Indeed as $R_i^* = \langle V^{*I_i^*} \rangle$ and $V^* \leq Z(T_I^*)$, it follows from the standard connectivity (cf. p. 208 and Exer. 14.5 in [**Asc86a**]) of the building for I^* that the commuting graph on V^{*I} (where commuting pairs define edges) is connected. By hypothesis, W centralizes V and hence also $\tilde{B}$. Therefore as $\tilde{B} = C_{\tilde{P}}(V)$, also $\tilde{B} = C_{\tilde{P}}(W) = C_{\tilde{P}}(V^i)$ for each $i \in I$ with $V^{i*} \leq T_I^*$, since i normalizes P and so $\tilde{V}^i \leq C_{\tilde{P}}(V) = \tilde{B}$. Then as the commuting graph on V^{*I} is connected, $\tilde{B} = C_{\tilde{P}}(V^g)$, whereas $\tilde{A} = C_{\tilde{P}}(V^g)$ by E.2.10.2. This contradiction completes the proof of (5), and hence of the proposition. □

In the proof of the Main Theorem we often encounter a subgroup H satisfying Hypothesis E.2.8 with $F^*(H) = O_2(H)$ and V a TI-set under the action of M.

Further in many such situations we will know that $V \cap V^g \neq 1$ for each $g \in H$. The next lemma gives us extra information under these hypotheses:

LEMMA E.2.14. *Assume Hypothesis E.2.8. Assume also that* $F^*(H) = O_2(H)$, V *is a TI-set under the action of* M, *and* $I = \langle V, V^g \rangle \in \mathcal{I}(H,T,V)$ *such that* $Z := V \cap V^g \neq 1$. *Set* $M_I := M \cap I$, $T_I := T \cap I$, $J_I := \ker_{M_I}(I)$, $B := V \cap O_2(I)$, $A := V^g \cap O_2(I)$, $P := AB$, $\tilde{I} := I/Z$, $I^* := I/J_I$, *and* $k := n(I^*)$. *Then*

(1) $M_I = N_I(V)$.

(2) *One of the following holds:*

(i) $P = J_I$ *and* $V^* = Z(T_I^*)$.

(ii) I/P *is quasisimple with* $I^* \cong Sz(8)$, $J_I/P = Z(I/P)$, *and* $VP/P = [\Omega_1(T_I/P), M_I]$. *Further* M_I^* *is a Borel subgroup of* I^*, *and* $V^* = Z(T_I^*)$.

(iii) $I/P \cong S_6 \times \mathbf{Z}_2$ *and* $J_I = Z(I/P)$.

(3) *If* $I^* \cong L_2(2^k)$ *then* $\tilde{P}$ *is a sum of natural modules for* I^*.

(4) *If* $I^* \cong Sz(2^k)$ *or* $Sp_4(2^k)$ *then each chief factor for* I *on* $\tilde{P}$ *is of rank at least* $4k$.

(5) *If* $I^* \cong L_2(2^k)$ *or* $Sp_4(2^k)$ *and* $\tilde{P}$ *is a natural module for* I^*, *then* P *is abelian and* $q(Aut_I(V), V) \leq 1$.

(6) $Aut_A(V) \trianglelefteq Aut_I(V)$.

(7) *If* $m(Aut_A(V)) \leq 2k$ *and* $q(Aut_I(V), V) > 1$, *then* $I^* \cong L_2(2^k)$, $Sz(2^k)$, *or* D_{10}, $m(Aut_A(V)) = 2k$, $m(V/Z) = 3k$, *and* $\tilde{P}$ *is either the sum of two natural modules for* $L_2(2^k)$, *or one natural module for* $Sz(2^k)$ *or* D_{10}.

(8) *If* $k = 2$, $m(Aut_A(V)) \leq 6$, *and* $q(Aut_I(V), V) > 1$, *then* $I^* \cong L_2(4)$, $m(Aut_A(V)) = 2s$, $s = 2$ *or* 3, *and* $m(V/Z) = 2(s+1)$.

(9) *If* $I^* \cong L_2(2^k)$ *or* $Sz(2^k)$ *with* $k > 1$, *then* $Z = C_V(X^*)$ *for* X^* *of order* $2^k - 1$ *in* M_I^*.

(10) *If* $I^* \cong D_{2p}$ *then* $\tilde{P}$ *is the direct sum of faithful irreducible modules of dimension* $2d'(p)$.

(11) $[V, \tilde{P}] = \tilde{B}$.

PROOF. Observe that we may apply E.2.13, and by E.2.11.1 we may apply E.2.10. By the latter result, $N_I(V) \leq M$ and $Z \leq Z(I)$, so as $Z \neq 1$ and V is a TI-set under M, $M_I \leq N_M(Z) \leq N_M(V)$. Hence (1) is established.

As $P \trianglelefteq I$ and $P = AB$ from E.2.10, $Aut_A(V) = Aut_P(V) \trianglelefteq Aut_I(V)$, establishing (6).

By E.2.10.2 and E.2.13.2, $\tilde{P} = \tilde{A} \oplus \tilde{B}$ and $Z = C_A(V)$, so that

$$\tilde{A} \cong A/C_A(V) \text{ and } \tilde{B} = C_{\tilde{P}}(V) \text{ are of rank } m(\tilde{P})/2. \qquad (*)$$

We now consider each of the three cases in Definition E.2.4, establishing the appropriate parts of (2)–(4) and (9)–(11) in each case. Let $L := O^2(I)$ and $\bar{I} := I/P$. By E.2.11.5, $J_I = O_2(I)$ and $\bar{J}_I = Z(\bar{I})$.

Suppose that I^* is dihedral. Then $J_I = P$ by E.2.10.6, so that $I^* = \bar{I}$, and in particular $V^* = \bar{V} = \bar{T}_I = Z(\bar{T}_I)$ has order 2—and hence (2i) holds in this case. Further by E.2.13.7, $\bar{I}$ is faithful on $\tilde{P}$, and by (*), $\bar{V}$ is free on $\tilde{P}$ with $\tilde{B} = C_{\tilde{P}}(V) = [\tilde{P}, V]$, establishing (11) in this case. Thus $\tilde{P}$ is the direct sum of faithful irreducible $\mathbf{F}_2\bar{I}$-modules of degree $2d'(p)$, proving (10).

Suppose next that $I^* \cong L_2(2^k)$ or $Sz(2^k)$, with $k > 1$. From Definition E.2.4, M_I^* is the Borel subgroup over T_I^*, so M_I^* is irreducible on $\Omega_1(T_I^*) = Z(T_I^*)$. Then as $V \trianglelefteq M_I$, we conclude $V^* = Z(T_I^*)$. Also as $V^* = [V^*, M_I^*]$ and $V \trianglelefteq M_I$

with $V^* \cong \bar{V}$, $\bar{V} = [\bar{V}, \bar{M}_I]$. But by E.2.10.7, $\bar{I} = Z(\bar{I})\bar{L}$ with $\bar{L}$ quasisimple, so $\bar{V} = [\bar{V}, \bar{M}_I \cap \bar{L}] \le \bar{L}$. Thus $\bar{I} = \langle \bar{V}, \bar{V}^g \rangle \le \bar{L}$, and hence $\bar{I} = \bar{L}$ is quasisimple. Then (2i) or (2ii) holds by E.2.10.7 in this case. But as $V^* = Z(T_I^*)$, $I^* = \langle V^*, e^* \rangle$ for each $1 \neq e^* \in V^{*g}$, so

$$C_{\tilde{B}}(e^*) \le C_{\tilde{P}}(I^*) = C_{\tilde{P}}(V^*) \cap C_{\tilde{P}}(V^{*g}) = \tilde{B} \cap \tilde{A} = 0,$$

and hence $\tilde{A} = C_{\tilde{P}}(e^*) = [\tilde{P}, e]$, establishing (11) in this case. Hence if $I^* \cong L_2(2^k)$, $\tilde{P}$ is the sum of natural modules for I^* by G.1.6, establishing (3) and also (9) in this case. Further if $I^* \cong Sz(2^k)$, then the minimum rank of a nontrivial module for I^* is $4k$; so to establish (4) in this case, it remains to verify that I^* has no nonzero central chief factors on $\tilde{P}$. This holds since if $1 \neq v^* \in V^*$, then $1 \neq x^* := v^* v^{g*}$ is of odd order, while as

$$C_{\tilde{P}}(v^*) \cap C_{\tilde{P}}(v^{g*}) = \tilde{B} \cap \tilde{A} = 0,$$

we have $C_{\tilde{P}}(x^*) = 0$. Notice this also completes the proof of (9), since an element of order $2^k - 1$ is inverted in I^*.

Suppose finally that $I^* \cong Sp_4(2^k)$. Then $Z(T_I^*) = Z_l^* \times Z_s^*$ is the product of a long and short root group, and by Definition E.2.4, $V^* \le Z(T_I^*)$ and $V^* \not\le Z_r^* \neq 1$ for $r = l$ and s. As I^* is generated by two conjugates of V^*, V^* is noncyclic; thus $V^* = Z(T_I^*)$ if $k = 1$. On the other hand if $k > 1$, then Z_l^* and Z_s^* are the only nontrivial proper M_I^*-submodules of $Z(T_I^*)$, so again $V^* = Z(T_I^*)$. Now we appeal to E.2.10.8 and argue as in the case where I^* was $L_2(2^k)$ or $Sz(2^k)$, to establish (2i)—except possibly when $k = 1$, where $H^* \cong S_6$ is not quasisimple. In that case, $\bar{V} = \langle \bar{v}, \bar{u} \rangle$, where $\bar{v}$ induces an inner automorphism on $\bar{L}$, $\bar{u}$ induces a transposition, and $\bar{w}^g \in \bar{w}\bar{L}$ for each $\bar{w} \in \bar{V}$. Thus $\bar{H} = \langle \bar{V}, \bar{V}^g \rangle = \langle \bar{u} \rangle \bar{L} \times \langle \bar{c} \rangle$, where $\bar{c}$ is the projection of $\bar{v}$ on $C_{\bar{I}}(\bar{L})$; hence (2i) or (2iii) holds. To establish (4), we can use the argument we used when I^* was $Sz(2^k)$—once we observe that $I^* = I_{e^*}^*$ for $1 \neq e^* \in V^{g*}$ of Suzuki type c_2 (recall Definition E.2.6, where $I_{e^*} := \langle V^*, e^* \rangle$). This observation is established as follows: We will show that if $v^* \in V^*$ is of Suzuki type c_2, then $v^* e^*$ is of odd order. Then v^* is conjugate to e^* in $\langle v^*, e^* \rangle \le I_{e^*}$, so V^* is conjugate to V^{*g} in I_{e^*} and hence $I^* = \langle V^*, V^{*g} \rangle = I_{e^*}^*$. Finally if $|v^* e^*|$ is even, then there is an involution

$$i^* \in C_{I^*}(v^*) \cap C_{I^*}(e^*) = T_I^* \cap T_I^{*g},$$

whereas M_I^* and M_I^{*g} are opposite Borel subgroups, so that $T_I^* \cap T_I^{*g} = 1$. This contradiction completes the proof of (4) and (11) in this final case; and hence completes the proof of (2)–(4) and (9)–(11).

Next assume that $\tilde{P}$ is a natural module for $I^* \cong L_2(2^k)$ or $Sp_4(2^k)$. Then I^* is transitive on $\tilde{P}^{\#}$. Therefore as $\Phi(A) = 1$, all elements in P are involutions, and hence also $\Phi(P) = 1$. Then $B = C_V(A)$ by (*), and hence (5) follows from (**) in the proof of E.2.13.

Assume the hypotheses of (7). Suppose first that I^* is $Sz(2^k)$ or $Sp_4(2^k)$. Then by E.2.13.4, $m(Aut_A(V)) = 2k$, so from (*) and (4), $\tilde{P}$ is irreducible of rank $4k$. From the representation theory of these groups, the only $\mathbf{F}_2$-irreducibles of this rank are quasiequivalent to the natural module. Then as $q(Aut_I(V), V) > 1$ by the hypotheses of (7), (5) forces $I^* \cong Sz(2^k)$. Hence $m(V^*) = m(Z(T_I^*)) = k$ by (2), so $m(V/Z) = m(V^*) + m(\tilde{B}) = 3k$, and (7) holds in this case. Suppose next that $I^* \cong L_2(2^k)$ with $k > 1$. Then $\tilde{P}$ is a sum of natural modules by (3), so using (*) and the hypothesis $m(Aut_A(V)) \le 2k$, $\tilde{P}$ has at most two summands. The case of a single irreducible is ruled out using (5), so we have two summands, and hence $m(Aut_A(V)) = 2k$. By (2) $m(V^*) = k$, so that $m(V/Z) = 3k$, and hence (7) holds in this case as well. Finally if $I^* \cong D_{2p^a}$ then by E.2.13.4, $d'(p^a) \leq 2$, so $p^a = 3$ or 5 and $\tilde{P}$ is a sum of at most 2 or 1 "natural" modules of dimension $2d'(p)$ respectively. The case of a single summand for $D_6 \cong L_2(2)$ is again ruled out using (5). This completes the proof of (7) in all cases.

Finally assume the hypotheses of (8). As $k = 2$, $I^* \cong L_2(4)$ or $Sp_4(4)$. In the latter case as $m(\tilde{P}) = 2m(\tilde{A}) \leq 12$ by hypothesis and (*), so we conclude from (4) that $\tilde{P}$ is an irreducible of rank 8 for I^*. Therefore from the representation theory of $Sp_4(4)$, $\tilde{P}$ is a natural module for I^*, and then (5) contradicts our assumption that $q(Aut_I(V), V) > 1$. Thus $I^* \cong L_2(4)$, and now (3) and (5) complete the proof of (8). □

In our next result we see that when V is a TI-set in M, we can improve E.2.13 to obtain the strict inequality $\hat{q} < 2$. We will use this in 3.1.6.2 and 3.1.8.2 in the proof of the Main Theorem.

LEMMA E.2.15. *Assume Hypothesis E.2.8. Assume also that* $F^*(H) = O_2(H)$ *and* V *is a TI-set under the action of* M. *Then* $\hat{q}(Aut_H(V), V) < 2$. *Indeed for* $I \in \mathcal{I}(H, T, V)$ *and* $A := V^g \cap O_2(H)$, $r_{Aut_A(V),V} < 2$.

PROOF. By E.2.13 we can choose $I \in \mathcal{I}(H, T, V)$, and adopt the notation of that lemma and its proof. By E.2.13.3, $r := \hat{q}(Aut_H(V), V) \leq 2$, so we may assume that $r = 2$. Therefore $Z := V \cap V^g \neq 1$ by E.2.13.5, so we can also appeal to E.2.14. Now $I^* \cong D_{2p^a}$, $L_2(2^k)$, $Sz(2^k)$, $Sp_4(2^k)$, and by E.2.14.2, $m(V^*) = 1$, k, k, $2k$, respectively. By E.2.13.2, $Z = C_A(V)$; and by E.2.10.2, $m(\tilde{P}) = 2\, m(A/Z)$, so as $r_{Aut_A(V),V} \geq r = 2$,

$$m(\tilde{P}) = 2\, m(A/C_A(V)) \leq m(V/C_V(A)) \leq m(V/Z) = m(V^*) + m(\tilde{B})$$
$$= m(V^*) + m(\tilde{P})/2.$$

Thus $m(\tilde{P}) \leq 2\, m(V^*) = 2$, $2k$, $2k$, $4k$. It follows from E.2.14.4 that I^* is not $Sz(2^k)$, and if $I^* \cong Sp_4(2^k)$ then $\tilde{P}$ is an irreducible of rank $4k$, and hence a natural module. Similarly if $I^* \cong L_2(2^k)$, then $\tilde{P}$ is a natural module by E.2.14.3. Finally if $I^* \cong D_{2p^a}$, then $m(\tilde{P}) \leq 2$, so $d'(p) = 1$ by E.2.13.4; we conclude that $\tilde{P}$ is the natural module for $I^* \cong L_2(2)$.

So in any case, $\tilde{P}$ is a natural module for $I^* \cong L_2(2^k)$ or $Sp_4(2^k)$. But then by E.2.14.5, $r \leq 1$, contradicting our assumption that $r = 2$. This contradiction completes the proof of the lemma. □

We use the following convention in many places in our work:

NOTATION E.2.16. Recall that a 2-group X is *extraspecial* if $X' = \Phi(X) = Z(X)$ is of order 2. The standard classification (e.g., p. 111 in [**Asc86a**]) says that X has structure $Q_8^m D_8^n$ by which we denote the central product of m copies of Q_8 and n copies of D_8, with centers identified.

The following technical lemma is needed in 15.1.11 in the proof of the Main Theorem.

LEMMA E.2.17. *Assume Hypothesis E.2.8. Further assume* $F^*(H) = O_2(H)$, V *is a TI-set under the action of* M, $\hat{q}(Aut_H(V), V) = 3/2$, $O^2(M)$ *centralizes* V, *and* $m(V) = 4$. *Then* $\langle V^H \rangle \cong S_3/Q_8^2$, $L_3(2)/D_8^3$, *or* $(\mathbf{Z}_2 \times L_3(2))/D_8^3$, *and in the latter two cases,* $|Aut_T(V)| \geq 8$.

PROOF. By E.2.13 we can choose $I \in \mathcal{I}(H,T,V)$, and adopt the notation of that lemma and its proof. As $r := \hat{q}(Aut_H(V), V) = 3/2$, $Z := V \cap V^g \neq 1$ by E.2.13.5, so we can appeal to E.2.14. Further as $\tilde{B}, Z \neq 1$ we have

$$m(V^*) = m(V) - m(Z) - m(\tilde{B}) \leq 4 - 1 - 1 = 2;$$

and if $m(V^*) = 2$, then $m(\tilde{B}) = 1$, so $m(\tilde{P}) = 2$ and $I^* \leq GL_2(2)$, contradicting the assumption that $m(V^*) = 2$. Therefore $m(V^*) = 1$, so $I^* \cong D_{2p^a}$ and

$$4 = m(V) = 1 + m(\tilde{B}) + m(Z) \geq m(\tilde{B}) + 2,$$

so that $m(\tilde{B}) = 1$ or 2. If $m(\tilde{B}) = 1$, then $\tilde{P}$ is a natural module for $I^* \cong L_2(2)$ by E.2.14.10, and hence $r \leq 1$ by E.2.14.5, contradicting our hypothesis that $r = 3/2$. Therefore $m(\tilde{B}) = 2$, and hence $m(Z) = 1$. If P is abelian, then $B = C_V(A)$, so $\tilde{A}$ induces transvections on V, giving $r \leq 1/2$, again a contradiction. So P is nonabelian. Next $m(\tilde{P}) = 2\ m(\tilde{B}) = 4$ and hence $p^a = 3$ or 5. But if $p^a = 5$, the five conjugates of the elementary abelian group B cover $\tilde{P}$, contradicting P nonabelian. Thus $p^a = 3$. As the chief factors of I on $\tilde{P}$ are of rank 2, either $Z(P) = Z$ or $Z(P) \cong E_8$, and in the latter case $P/Z(P)$ is covered by the 3 conjugates of $BZ(P)/Z(P)$, again contradicting P nonabelian. Thus $Z(P) = Z$, and P is extraspecial. Then as $m(B) = 3$, $P \cong Q_8^2$. In particular $I \cong S_3/Q_8^2$, using E.2.10.6 to see that $P = O_2(I) = J_I$.

Thus it remains to show that either $\langle V^H \rangle = I$, or $\langle V^H \rangle$ is D_8^3 extended by $L_3(2)$ or $L_3(2) \times \mathbf{Z}_2$. This part of the argument is more complicated. Let $\bar{H} := H/O_2(H)$ and $K := O^2(H)$.

We first consider the case where H is solvable. As $V \leq O_2(M)$ but $V \not\leq O_2(H)$, $\bar{K} \cong \mathbf{Z}_r$, E_{r^2} or r^{1+2} for some odd prime r by E.2.1. Then as $I \leq H$, $r = 3$. Hence using E.2.10.6, $P = [O_2(I), O^2(I)] \leq O_2(H)$. Thus $B = V \cap O_2(I) = V \cap O_2(H)$ and so $\bar{V} \cong V^*$ is of order 2. By hypothesis $V \trianglelefteq T$, so as T is irreducible on $\bar{K}/\Phi(\bar{K})$ by B.6.8.2, $\bar{V}$ inverts $\bar{K}/\Phi(\bar{K})$. Next $Z = C_I(O_2(I))$ is of order 2 and hence $Z = V \cap Z(O_2(H))$, since we saw that $O_2(I) \leq O_2(H)$. Since $\bar{K}$ is a 3-group and $\bar{V}$ inverts $\bar{K}/\Phi(\bar{K})$, for $h \in K - M$ we obtain $I_h := \langle V, V^h \rangle \in \mathcal{I}(H,T,V)$, with $h \in I_h$ (cf. the proof of E.2.9). So by symmetry between I and I_h, also $V \cap V^h = V \cap Z(O_2(H)) = Z$. Thus $Z \leq Z(H)$.

Let $\tilde{H} := H/Z$ and $U := \langle B^H \rangle = \langle B^K \rangle$. Again by symmetry between I and I_h, $[B, B^h] \leq Z^h = Z$ for any $h \in K - M$, and hence $\Phi(\tilde{U}) = 1$. Now $O_2(H) \leq T \leq N_H(V)$, so $[O_2(H), V] \leq O_2(H) \cap V = B$. Thus V centralizes $O_2(H)/U$, so also $K = [K, V]$ centralizes $O_2(H)/U$, and hence $O_2(K) = [O_2(K), K] \leq [O_2(H), K] \leq U$. On the other hand $B \leq [O_2(I), O^2(I)] \leq [O_2(K), K]$, so in fact $U = O_2(K)$.

As $[O_2(H), V] \leq B$, while $[A, V] = B$ by E.2.14.11, $[\tilde{U}, V] = \tilde{B}$ is of rank 2, and hence $r_{\bar{V},\tilde{U}} = 2$. Let $H^+ := H/C_H(\tilde{U})$ and apply D.2.17 to the action of K^+V^+ on $\tilde{U}$. As V^+ is of order 2 and Sylow in K^+V^+, the pair $(V^+K^+, \tilde{U})$ is indecomposable in the sense of Definition D.2.16, and as $\tilde{B} \leq [\tilde{U}, K]$ and $U = \langle B^K \rangle$, $[\tilde{U}, K] = \tilde{U}$. As $\hat{q}(K^+V^+, \tilde{U}) = 2$, conclusions (5) and (6) of D.2.17, as well as conclusion (1) with $r = 1$, are eliminated. As $p = 3$, conclusion (2) is eliminated. This leaves the cases $K^+ = \mathbf{Z}_3$ or E_9 and $m(\tilde{U}) = 4$, or $K^+ \cong 3^{1+2}$ and $m(\tilde{U}) = 6$. As $O_2(K) = U$ and

$Q_8^2 \cong O_2(I) \le U$, we conclude that $U = O_2(I)$ in the cases with $m(\tilde{U}) = 4$. Further as T is irreducible on $\bar{K}/\Phi(\bar{K})$, if $K^+ \cong E_9$, then as H^+ lies in the stabilizer $O_4^+(2)$ in $GL_4(2)$ of the set of two quaternion subgroups of U, $H^+ \cong S_3$ wr $\mathbf{Z}_2$ or $\mathbf{Z}_4/E_9$; so that $V \trianglelefteq T$ forces $[U, V] \cong \mathbf{Z}_4 \times \mathbf{Z}_2$, contrary to $B = [U, V]$. Therefore $K^+ \cong \mathbf{Z}_3$, where $\langle V^H \rangle = I$, which is one of the conclusions in (2). On the other hand if $K^+ \cong 3^{1+2}$, then $J^+ = Z(K^+)$ and $\tilde{B} = [\tilde{B}, J]$, contradicting the hypothesis that $O^2(M)$ centralizes V. This completes the treatment of the case H solvable.

We turn to the case where H is not solvable, so $H^! := H/J$ is described in E.2.2. We recall from B.6.8.4 that J is 2-closed and $\bar{V} \cong V^! \neq 1$ in view of the hypothesis that $V \not\le J$. Also by hypothesis $[V, O^2(M)] = 1$ and $V \trianglelefteq T$, so $V \trianglelefteq M = O^2(M)T$ and $[V^!, O^2(M^!)] = 1$. As $V^! \trianglelefteq T^!$ we have $Z(M^!) \neq 1$, so inspecting the list of E.2.2, we conclude either that one of cases (1b), (2c), or (2d) of E.2.2 hold, or that case (2b) holds and $K^! \cong A_6$. Further $\bar{V} \cong V^!$ is an elementary abelian normal 2-subgroup of $\bar{M}$ centralizing $O^2(\bar{M})$, so either $\bar{V}$ is of order 2; or case (1b) of E.2.2 holds, $\bar{K} = \bar{K}_1 \times \bar{K}_2$, $\bar{K}_i \cong L_2(p)$, and $\bar{V} = \langle \bar{v}_1, \bar{v}_2 \rangle \cong E_4$ with $\bar{v}_i \in \bar{K}_i$. Define $\bar{v} := \bar{v}_1\bar{v}_2$ in the exceptional case where $\bar{V}$ is noncyclic. When $\bar{V}$ is cyclic, let $\bar{v}$ be the generator for $\bar{V}$.

We claim next that $|\bar{V}| = 2$, and also that for each $h \in H$ such that $1 \neq n = |\bar{v}\bar{v}^h|$ is odd, we have $n = 3$. Notice it suffices to prove this for $n = p^a$ an odd prime power, as we can always pass to dihedral subgroups of order $2p^a$. If $|\bar{V}| = 2$ but h is a counterexample to the latter statement, let $I_0 := \langle V, V^h \rangle$, so that $\bar{I}_0 = \langle \bar{v}, \bar{v}^h \rangle \cong D_{2n}$, with $n = p^a > 3$. If $|\bar{V}| > 2$, choose $h \in K_1$ with $1 \neq n = |\bar{v}_1\bar{v}_1^h|$ odd, and let $I_0 := \langle V, V^h \rangle$, so that this time $\bar{I}_0 \cong D_{2n} \times \mathbf{Z}_2$. Then by construction (cf. E.2.9), $I_0 \in \mathcal{I}(H, T, V)$; thus $I_0 \cong S_3/Q_8^2$ by the first paragraph of the proof. Hence $I_0/[O_2(I_0), I_0] \cong S_3$, contrary to the construction of I_0 in either case. Thus the claim is established.

By the claim, $|\bar{V}| = 2$, so $\bar{V} \le Z(\bar{M})$. Further $|\bar{v}\bar{v}^h| = 3$ whenever $|\bar{v}\bar{v}^h| \neq 1$ is odd, so it follows—from the existence of dihedral subgroups of order $p^e \pm 1$ in $L_2(p^e)$ and of order $2p$ in $L_3^{-\epsilon}(p)$—that $\bar{K} \cong L_2(7)$, $L_2(7) \times L_2(7)$, or $L_3(3)$. In the last two cases, there is a subgroup H_0 of H satisfying $T_0 := T \cap H_0 \in Syl_2(H_0)$, $T_0 = !\mathcal{N}_{H_0}(T_0)$, $V \not\le O_2(H_0)$, and $H_0/O_2(H_0) \cong D_8/E_9$. Now applying our treatment of the solvable case to H_0 in place of H, we have a contradiction. So $\bar{K} \cong L_2(7)$ and hence by E.2.2, $\bar{H} \cong PGL_2(7)$. Further arguing as in the solvable case, $O_2(I) \le O_2(H)$ and $B = V \cap O_2(H)$. Again set $U := \langle B^H \rangle$; our earlier arguments give $[O_2(H), K] = U = O_2(K)$ and $[U, V] = B$. Indeed $\bar{K}$ is generated by the elements of order 3 inverted by $\bar{V}$, so $K \le \langle \mathcal{I}(H, T, V) \rangle$, and hence an earlier argument shows $Z \le Z(H)$. Then setting $\tilde{H} := H/Z$, $[\tilde{U}, V] = \tilde{B} \cong E_4$, so from the representation theory of $PGL_2(7)$, $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$, with $\tilde{U}_1$ the natural module for $\bar{K} \cong L_3(2)$, and $U_2 = U_1^t$ for $t \in H - O_2(H)K$. It follows that U is extraspecial, and hence $U \cong D_8^3$ as it admits $L_3(2)$; that is, $K \cong L_3(2)/D_8^3$. As $V \trianglelefteq T$ and $H = KT$, $V^H = V^K$; so setting $K_0 := \langle V^H \rangle$, we conclude that either $K_0 = K$, or $K_0/U \cong \mathbf{Z}_2 \times L_3(2)$ (since $\bar{V}$ has rank 1). These are the remaining possibilities for K_0 listed in the lemma, so the proof of the lemma is at last complete. □

E.3. Weak Closure

Weak closure arguments are used heavily in parts of our proof of our Main Theorem classifying simple QTKE-groups. Here is a rough overview of one part of the proof: Let L, V, and $M := !\mathcal{M}(\langle L, T\rangle)$ be as in the Fundamental Setup FSU (3.2.1). Weak closure studies the embedding in T of G-conjugates of subgroups of V. We define various numerical parameters, some determined locally by the action of $N_M(V) = N_G(V)$ on the "internal module" V, others determined globally by the fusion under G of conjugates of subgroups of V and by the centralizers of such subgroups. In section E.3, we reproduce some elementary lemmas from [**Asc81c**] which show how the values of these parameters restrict the value of $n(H)$ (see Definition E.1.6) for subgroups H of G not contained in M. This provides useful information on the possible structure of many of the subgroups considered in the proof of our Main Theorem.

In the initial subsection, we introduce the basic parameters and give some sense of their significance for weak closure. The second subsection gives a number of basic results, primarily taken from [**Asc81c**]. The final subsection provides a more detailed outline of the weak closure arguments in this work, while at the same time establishing some of the machinery necessary to implement that outline. (The reader unfamiliar with the background and the methods might wish to start with the final subsection, to understand the motivation better, referring back as needed to the earlier subsections).

E.3.1. The basic weak closure parameters. In this section G is a finite group and V is a nontrivial elementary abelian 2-subgroup of G.

We begin by recalling the definition of a module parameter used in the work of Thompson, Aschbacher, and Mason on weak closure:

DEFINITION E.3.1. Assume a group X acts faithfully on an $\mathbf{F}_2$-module W. If X is of even order, set:

$$m(X, W) := \min\{m(W/C_W(t)) : t \text{ an involution of } X\},$$

and if X is of odd order set $m(X, W) := m(W)$.

Notice that if $W \neq 0$ then as X is faithful on W, $m(X, W) \geq 1$.

In this section we often use the abbreviation $m := m(Aut_G(V), V)$. To acclimate the reader to one standard use of the parameter, we record the following elementary consequence of Definition E.3.1:

LEMMA E.3.2. *If A is a nontrivial elementary abelian 2-subgroup of $Aut_G(V)$, then $m(V/C_V(A)) \geq m(Aut_G(V), V)$.*

Notice that our next parameter is "global" in the sense that is determined by G, whereas the parameter m is "local" in that it is determined by the local subgroup $N_G(V)$.

DEFINITION E.3.3. $r(G, V) := \min\{m(V/U) : U \leq V \text{ and } C_G(U) \not\leq N_G(V)\}$.

Note that as $V \neq 1$ and $C_G(V) \leq N_G(V)$, we have $r(G, V) \geq 1$.

Again we often use the abbreviation $r := r(G, V)$, and record easy consequences of the definition:

LEMMA E.3.4. *(1) If $B \leq N_{V^g}(V)$ satisfies $C_V(B) \not\leq N_G(V^g)$, then $m(V^g/B) \geq r(G, V)$.*

(2) If $V \not\leq N_G(V^g)$, then $m(V^g/C_{V^g}(V)) \geq r(G, V)$.

Next we combine the parameters m, r in:

DEFINITION E.3.5. $s(G,V) := \min\{r(G,V), m(Aut_G(V),V)\}$

From previous remarks, $r \geq 1 \leq m$, so also $s(G,V) \geq 1$. In this section we typically use the abbreviation $s := s(G,V)$. Here is one elementary property of the parameter:

LEMMA E.3.6. *Let $B \leq A \leq V$, with $m(V/B) < s(G,V)$, and E an A-invariant 2-subgroup of G. Then $C_E(A) = C_E(B) = C_E(V)$.*

PROOF. As $m(V/B) < r(G,V)$, we have $C_E(B) \leq N_G(V)$. Therefore since $m(V/C_V(C_E(B))) \leq m(V/B) < m$, $C_E(B) \leq C_G(V)$ by E.3.2. Thus as $B \leq A \leq V$, $C_E(B) = C_E(A) = C_E(V)$. □

Applying E.3.6 to V^g, V in the roles of "V, E":

LEMMA E.3.7. *If $B \leq N_{V^g}(V)$ with $m(V^g/B) < s(G,V)$, then $C_V(B) = C_V(V^g)$.*

DEFINITION E.3.8. Assume X is a finite group, W is a faithful $\mathbf{F}_2X$-module, and k is a positive integer. Define $\mathcal{A}_k(X,W)$ to consist of those nontrivial elementary 2-subgroups A of X such that $C_W(A) = C_W(B)$ for all $B \leq A$ with $m(A/B) < k$. [1]

Notice that $\mathcal{A}_1(X,W)$ is the set $\mathcal{A}^2(X)$ of all nontrivial elementary abelian 2-subgroups of X. We define another local parameter:

DEFINITION E.3.9. $a(X,W) := \max\{k : \mathcal{A}_k(X,W) \neq \emptyset\}$ if X is of even order, and $a(X,W) := 0$ otherwise.

Following our usual convention, write $a := a(Aut_G(V),V)$. The following variant of E.3.6 establishes a relation between s and a:

LEMMA E.3.10. *Suppose $A := N_{V^g}(V)$ satisfies $b := m(V^g/A) < s := s(G,V)$ and $[V,V^g] \neq 1$. Then $C_V(A) = C_V(V^g)$ and $Aut_A(V) \in \mathcal{A}_{s-b}(Aut_G(V),V)$, so $s - a \leq b < s$.*

PROOF. For $B \leq A$ with $m(A/B) < s - b$, we have $m(V^g/B) < s$, so by E.3.7, $C_V(V^g) = C_V(B)$. Then as $B \leq A \leq V^g$, $C_V(A) = C_V(V^g)$, so as $[V,V^g] \neq 1$, $Aut_A(V) \neq 1$. Hence as $C_V(A) = C_V(B)$, the lemma holds. □

LEMMA E.3.11. *If V is not an FF-module for $Aut_G(V)$ and $V^g \leq N_G(V)$ with $[V,V^g] \neq 1$, then $V \not\leq N_G(V^g)$.*

PROOF. Suppose $V^g \leq N_G(V)$. Then interchanging V and V^g if necessary, we may assume that $m(V^g/C_{V^g}(V)) \geq m(V/C_V(V^g))$. Thus $r_{Aut_{V^g}(V),V} \leq 1$, contrary to the assumption that V is not an FF-module for $Aut_G(V)$. □

E.3.2. Basic results on weak closure. We now begin to establish the basic machinery of weak closure. In particular we begin to see how restrictions on the weak closure parameters of the previous subsection, and on $n(H)$ (cf. Definition E.1.6) for various 2-locals H, lead to restrictions on the 2-local structure of G. The material in this section comes from section 6 of [**Asc81c**].

[1] Don't confuse this notation with the notation $\mathcal{A}_k(G)$ in Definition B.2.2 for subgroups of corank k in a maximal elementary 2-subgroup of a group G, used elsewhere in this work in contexts of FF-modules.

In this subsection, $H \le G$ and $S \in Syl_2(H)$.

First we establish a property of the set $\mathcal{E}(H,S,A)$, defined in Definition E.1.2; observe that the minimal parabolics have this same property by B.6.8.5:

LEMMA E.3.12. *Let $A \le V \cap S$, $K \in \mathcal{E}(H,S,A)$, and $Y := \ker_{N_K(V)}(K)$. Then Y is 2-closed and $A \not\le Y$.*

PROOF. Replacing H by K, we may assume $H = K$. We adopt the bar conventions in the definition E.1.2 of $\mathcal{E}(H,S,A)$; that is $\bar{H} := H/O_2(H)$ in (E1) where H is solvable, while $\bar{H} := H/O_\infty(H)$ in the other case (E2). As $A \le V \cap S$ and Y acts on V, $[\bar{A}, \bar{Y}]$ is a subgroup of $\overline{V \cap H}$ and hence is a 2-group.

In case (E1), $\bar{H} = O_{p,2}(\bar{H})$ for some odd prime p, and $O_p(\bar{H})/\Phi(O_p(\bar{H}))$ is inverted by $\bar{A}$ of order 2. Therefore every normal subgroup $\bar{Y}_0$ of $\bar{H}$ with $[\bar{A}, \bar{Y}_0]$ a 2-group is contained in $\bar{P} := \Phi(O_p(\bar{H}))$. Thus $\bar{Y} \le \bar{P}$, and hence $Y \le P$ which is 2-closed, so that Y is also 2-closed. But $A \not\le O_2(K)$ as $K \in \mathcal{E}(H,S,A)$, so $A \not\le Y$, completing the proof in this case.

Similarly in case (E2), $F^*(\bar{H})$ is the product of the $\bar{T}_A$-conjugates of a simple component $\bar{L}$ of $\bar{H}$ with $[\bar{L}, \bar{A}] \ne 1$, so $[\bar{A}, \bar{Y}_0]$ is not a 2-group for any nontrivial normal subgroup $\bar{Y}_0$ of $\bar{H}$. Therefore $Y \le O_\infty(H)$ which is 2-closed, so also Y is 2-closed, and hence $A \not\le Y$. □

DEFINITION E.3.13. For $X \le G$ and $i < m(V)$, let $\Gamma_i(X) = \Gamma_i(X,V)$ be the set of subgroups A of X such that $A \le V^g$ and $m(V^g/A) = i$ for some $g \in G$. Define

$$\begin{aligned} W_i(X) = W_i(X,V) &:= \langle \Gamma_i(X,V) \rangle \\ C_i(X) = C_i(X,V) &:= C_X(W_i(X,V)). \end{aligned}$$

REMARK E.3.14. Weak closure methods are founded on elementary properties of weakly closed subgroups such as $W_i(X)$. In particular, $W_i(X) \trianglelefteq N_G(X)$, and then also $C_i(X) = C_X(W_i(X)) \trianglelefteq N_G(X)$. Usually X is taken to be a 2-group; indeed in this section, it is often the Sylow 2-subgroup S of H. In that case a standard application of Sylow's Theorem extends these properties to any 2-overgroup of $W_i(X)$ in H:

LEMMA E.3.15. *If $W_i(S)$ is contained in a 2-subgroup Y of H, then $W_i(S) = W_i(Y)$. In particular $W_i(S)$ is also weakly closed in Y, and hence normal in $N_G(Y)$.*

As we will see in E.3.21 and later lemmas, this property can be exploited in conjunction with the existence of uniqueness subgroups such as those appearing in the proof of the Main Theorem in chapter 1. Such applications are based on:

LEMMA E.3.16. *Assume $W_i(S)$ is contained in a 2-subgroup Y of H, and in addition assume $N = !\mathcal{M}(N_G(Y))$. Then*

(1) $N_G(Y) \le N_G(W_i(S)) \le N$.
(2) $N = !\mathcal{M}(N_G(W_i(S)))$.
(3) $C_G(C_i(S)) \le N$.

PROOF. By E.3.15, $N_G(Y) \le N_G(W_i(S))$, so as $N = !\mathcal{M}(N_G(Y))$, (1) and (2) hold. Similarly $N_G(Z(W_i(S)) \le N$, so $C_G(C_i(S)) \le C_G(Z(W_i(S))) \le N$, establishing (3). □

The next two lemmas E.3.17 and E.3.18 are important technical results, which serve as the basis of the proofs of most of the results in this section.

LEMMA E.3.17. *Suppose nonnegative integers* i, j, k *satisfy* $i+j \leq k < r(G,V)$, *and that* $A \in \Gamma_i(S)$ *and* $K \in \mathcal{E}_j(H,S,A)$ *satisfy* $[O^2(K), C_k(S)] \neq 1$. *Then*

(1) $C_k(S) \leq C_k(O_2(K))$, *so* $[O^2(K), C_k(O_2(K))] \neq 1 \neq [A, C_k(O_2(K))]$.

(2) If $A \leq V$, *then* $V \cap Z(W_k(O_2(K))) \neq 1$.

(3) $m(Aut_G(V), V) \leq k$.

PROOF. As $A \in \Gamma_i(S,V)$, conjugating in G we may take $A \leq V$. Set $Q := O_2(K)$ and $B := A \cap Q$. As $K \in \mathcal{E}_j(H,S,A)$, $m(A/B) \leq j$ by Definition E.1.5. As $A \in \Gamma_i(S)$, $m(V/A) \leq i$. Therefore $m(V/B) \leq i + j \leq k$, so $B \leq W_k(Q) := W$. Then as $k < r(G,V)$, $C_G(B) \leq N_G(V)$, so $C_G(W) \leq C_G(B) \leq N_G(V)$.

Set $Y := \ker_{N_K(V)}(K)$. By E.3.15, $W \trianglelefteq N_G(Q)$, so $C_Q(W)$ and $C_K(W)$ are normal in K. Then as $C_G(W) \leq N_G(V)$, $C_K(W) \leq Y$. As $A \in \Gamma_i(S)$ and $i \leq k$, $C_k(S) \leq C_S(A) \leq S \cap K$ by Definition E.1.2. As $Q \leq S$, $W = W_k(Q) \leq W_k(S)$, and then $C_k(S) = C_S(W_k(S)) \leq C_S(W)$, so $C_k(S) \leq C_K(W) \leq Y$. By E.3.12, Y is 2-closed; therefore $C_k(S) \leq O_2(Y) \leq Q$, so $C_k(S) \leq C_Q(W) = C_k(Q)$. By hypothesis, $[O^2(K), C_k(S)] \neq 1$, so $[O^2(K), C_k(Q)] \neq 1$. Furthermore by E.1.4, $O^2(K) \leq \langle A^K \rangle$, so $[A, C_k(Q)] \neq 1$, completing the proof of (1).

Assume that $A \leq V$. Recall that $C_k(Q) \leq C_H(B) \leq N_G(V)$ and $C_k(Q) = C_Q(W) \trianglelefteq K$. So using (1), we have $1 \neq [A, C_k(Q)] \leq V \cap C_k(Q)$. But $B \leq V \cap Q$, so $m(V/V \cap Q) \leq m(V/B) \leq k$, and hence $V \cap Q \leq W$. Therefore as $C_k(Q) = C_Q(W)$, we have $V \cap C_k(Q) \leq Z(W)$, so $1 \neq [A, C_k(Q)] \leq V \cap Z(W)$, establishing (2).

Assume that $m > k$. By hypothesis, $r > k$, so $s > k \geq m(V/B)$. Thus by E.3.6, $C_k(Q) = C_{C_k(Q)}(B) = C_{C_k(Q)}(A) \leq C_K(A)$, which contradicts (1). This establishes (3) and completes the proof. □

LEMMA E.3.18. *Assume* U *is a normal elementary abelian* 2*-subgroup of* H, *and* $0 \leq i < s(G,V) - a(H/C_H(U), U)$ *with* $W_i(S) \neq 1$. *Then*

$$[U, W_i(S)] = 1, \text{ so } H = C_H(U)N_H(W_i(S)).$$

PROOF. Set $H^* := H/C_H(U)$. By hypothesis $W_i(S) \neq 1$, so there exists $A \in \Gamma_i(S)$. Also by hypothesis $i + a(H^*, U) < s$, so for any $B \leq A$ with $m(A/B) \leq a(H^*, U)$, E.3.6 says $C_U(A) = C_U(B)$. Thus if $A^* \neq 1$ then $A^* \in \mathcal{A}_{a(H^*,U)+1}(H^*, U)$, contrary to the maximality of $a(H^*, U)$ in Definition E.3.9. Hence $A \leq C_H(U)$ for all $A \in \Gamma_i(S)$, so $W_i(S) \leq C_S(U)$, and hence $W_i(S) \trianglelefteq N_H(C_S(U))$ by E.3.15. Then a Frattini Argument gives $H = C_H(U)N_H(C_S(U)) = C_H(U)N_H(W_i(S))$. □

We now come to our first main result, which shows that if $n(H)$ is small relative to $s(G,V)$, then the structure of H is controlled by normalizers and centralizers of certain weakly closed subgroups.

PROPOSITION E.3.19. *Let* $n(H) = j$ *and suppose* i *is a nonnegative integer such that* $i + j < s(G,V)$. *Then* $H = \langle C_H(C_{i+j}(S)), N_H(W_i(S)) \rangle$.

PROOF. Set $\Omega := \Gamma_i(H)$. Recall from Definition E.1.6 that as $n(H) = j$,

$$H = \langle E_j(H,S,\Omega), N_H(W_i(S)) \rangle.$$

Thus it suffices to prove that $K = C_K(C_{i+j}(S))N_K(W_i(S))$ for all $A \in \Gamma_i(S)$ and all $K \in \mathcal{E}_j(H,S,A)$. Set $B := A \cap O_2(K)$, and recall by Definition E.1.5 that $m(A/B) \leq j$. Let $A \leq V^g$ so that

$$m(V^g/B) = m(V^g/A) + m(A/B) \leq i + j < s(G,V).$$

Assume that $[O^2(K), C_{i+j}(S)] \neq 1$. Then as $s \leq r$, we can apply E.3.17.1 with $i + j$ in the role of "k" to conclude that $E := C_{i+j}(O_2(K))$ centralizes B, but does not centralize A. Hence E.3.6 supplies a contradiction as $[E, B] = 1 \neq [E, A]$. We conclude that $O^2(K)$ centralizes $C_{i+j}(S)$. Certainly S normalizes the weakly closed subgroup $W_i(S)$, so $K = O^2(K)(S \cap K) = C_K(C_{i+j}(S))N_K(W_i(S))$. This completes the proof. □

We sometimes apply E.3.19 during the proof of the Main Theorem in the following form with T in the role of the Sylow group "S" of H:

LEMMA E.3.20. *Assume the Fundamental Setup FSU (3.2.1), $H \in \mathcal{H}_*(T, M)$, $[Z, H] \neq 1$ and $W_0(T) \not\leq O_2(H)$. Then $n(H) \geq s(G, V)$.*

PROOF. First T normalizes $W_0(T)$, but since $W_0(T) \not\leq O_2(H)$, H does not normalize $W_0(T)$ by E.3.15. Thus $N_H(W_0(T))$ is contained in the unique maximal subgroup $H \cap M$ of H containing T (cf. 3.1.3.1), so $N_H(W_0(T)) \leq M$. Similarly as $[H, Z] \neq 1$, $C_H(Z) \leq M$, and as $Z \leq C_m(T)$ for any m, $C_H(C_m(T)) \leq M$. Thus $n(H) \geq s(G, V)$ by E.3.19 with $i = 0$. □

We now use E.3.18 and E.3.19 to establish another important result under the hypothesis that $N_G(C_S(V))$ is a uniqueness subgroup. This hypothesis holds when V is a module in the Fundamental Setup FSU in the proof of the Main Theorem. The lemma says that $H \leq M$ when $n(H)$ is sufficiently small. Later in the section, the result will be applied to obtain lower bounds for $n(H)$ when $H \not\leq M$.

PROPOSITION E.3.21. *Assume that $s > a + i$ for some nonnegative integer i, where $s := s(G, V)$ and $a := a(Aut_G(V), V)$. Assume further that $V \trianglelefteq S$ and that $M = !\mathcal{M}(N_G(Q))$ where $Q := C_S(V)$. Then*

(1) $[V, W_i(S)] = 1$, so $W_{s-a-1}(S) \leq C_S(V) = Q$ and $N_G(W_i(S)) \leq M \geq C_G(C_i(S))$.

(2) If $n(H) = i$, then $H \leq M$.

(3) If $i \geq 1$ and H is solvable, then $H \leq M$.

PROOF. We verify the hypotheses of E.3.18 with S, V in roles of "H, U": As $V \leq S$, $1 \neq V \leq W_i(S)$, and by hypothesis $V \trianglelefteq S$ and $i < s - a(\bar{S}, V)$. Thus by E.3.18, $[V, W_i(S)] = 1$. In particular, $W_i(S) \leq C_S(V) = Q$—giving the first part of (1), and then E.3.16 implies the second part.

Recall E.1.13 says that if H is solvable, then $n(H) = 1$; thus (3) follows from (2). It remains to prove (2), so we assume $n(H) = i$, and we must show that $H \leq M$. As $a \geq 0$, $i < s$ by hypothesis, so we may apply E.3.19 with 0, i in the roles of "i, j" to conclude that $H = \langle C_H(C_i(S)), N_H(W_0(S))\rangle$. Therefore $H \leq M$ by (1), so the proof is complete. □

E.3.3. Applying weak closure. The purpose of this subsection is to provide an overview of applications of the basic theory of weak closure just developed, and to put in place the machinery necessary to implement these applications. The reader may find this helpful, since the methods are fairly technical, and probably not well known to a very wide audience.

We begin with a brief overview of the role played by weak closure in the proof of our Main Theorem. There G is a QTKE-group, and the pair M, V arises in the Fundamental Setup FSU. Because the action of $Aut_G(V)$ on V is highly restricted in the FSU, we will be able to compute or estimate the values of the parameters

m, r, s, a; then weak closure arguments provide some initial restrictions on $n(H)$ for $H \not\leq M$—particularly when V is not an FF-module. For example, we will see in chapter 7 that when a pair $Aut_G(V)$, V does not correspond to an example or a shadow, usually the elementary numerical restrictions on $n(H)$ and the weak closure parameters already suffice to eliminate the configuration. We will also see (for example in chapter 8) that when $Aut_G(V)$, V actually occurs in an example or shadow G^*, weak closure can typically pin down the structure of H: namely H will resemble $C_{G^*}(U)$ for a suitable subgroup U of V. In effect, H determines another maximal 2-local $N_G(U)$, corresponding in a diagram geometry for G^* to a node adjacent to that for M. This provides a route either toward a contradiction eliminating the shadow, or toward the eventual identification of G with the example G^*.

In the remainder of the subsection, we develop more weak closure machinery. The main goal is to produce numerical restrictions on $n(H)$ for $H \not\leq M$, in terms of a parameter w to be defined below. The case where H contains a Sylow 2-subgroup of G will be of particular importance, so throughout the remainder of this subsection we assume:

HYPOTHESIS E.3.22. *$T \in Syl_2(G)$ and $V \trianglelefteq T$. Set $Q := C_T(V)$.*

Abbreviate

$$W_i := W_i(T) \text{ and } C_i := C_i(T).$$

We begin to focus on those i for which W_i centralizes V, so we introduce yet another weak closure parameter:

DEFINITION E.3.23. $w(G,V) := \min\{j : W_j \not\leq C_T(V) = Q\}$,

We adopt the abbreviations $w := w(G,V)$, $m := m(Aut_G(V),V)$, $r := r(G,V)$, $s := s(G,V)$, and $a := a(Aut_G(V),V)$.

In the examples and shadows, we usually find there is $H \not\leq M$ such that $n(H)$ achieves the minimal value w. Thus it makes sense to try to establish bounds forcing the values of the two parameters w and $n(H)$ to be close together. We sometimes need the following hypothesis:

HYPOTHESIS E.3.24. *Either*

(1) $w > 0$, or

(2) V is not an FF-module for $Aut_G(V)$.

LEMMA E.3.25. *If $w > 0$, then*

$$V^g \leq N_G(V) \text{ iff } V^g \leq C_G(V) \text{ iff } V \leq C_G(V^g) \text{ iff } V \leq N_G(V^g).$$

LEMMA E.3.26. *Assume Hypothesis E.3.24 and $V^g \leq N_G(V)$ with $[V,V^g] \neq 1$. Then $V \not\leq N_G(V^g)$.*

PROOF. If $w > 0$, this is a consequence of E.3.25. If V is not an FF-module for $Aut_G(V)$, it follows from E.3.11. □

We next focus attention on a conjugate realizing the minimum codimension w, and define:

DEFINITION E.3.27. A *w-offender* is a subgroup $N_{V^g}(V)$ for $g \in G$ satisfying $[V,V^g] \neq 1$ and $w = m(V^g/N_{V^g}(V))$.

The next few results develop some basic properties of w-offenders.

LEMMA E.3.28. *Assume Hypothesis E.3.24 and $A := N_{V^g}(V)$ is a w-offender. Then*

(1) $m(V^g/C_A(V)) \geq r$.
(2) $r \leq w + m(Aut_A(V)) \leq w + m_2(Aut_G(V))$.
(3) $m(Aut_A(V)) \geq r - w$.
(4) $V \not\leq N_G(V^g)$, and if $w > 0$ then $V^g \not\leq N_G(V)$.

PROOF. If $w > 0$, $V^g \not\leq N_G(V)$ and $V \not\leq N_G(V^g)$ by E.3.25. On the other hand if $w = 0$, then $A = V^g$ by Definition E.3.27, so $V \not\leq N_G(V^g)$ by E.3.26. This establishes (4).

By (4), $V \not\leq N_G(V^g)$, so (1) follows from E.3.4.2. It follows from (1) that

$$r \leq m(V^g/C_A(V)) = m(V^g/A) + m(Aut_A(V)) = w + m(Aut_A(V)),$$

and of course $m(Aut_A(V)) \leq m_2(Aut_G(V))$, so (2) holds. Then (2) implies (3). □

REMARK E.3.29 (Fundamental Weak Closure Inequality).
Notice that if Hypothesis E.3.24 holds, then by E.3.28.2:

$$m_2(Aut_G(V)) + w \geq r \qquad (\textbf{FWCI}).$$

which (as the name is intended to suggest) is an important tool for obtaining contradictions or restrictions on structure.

In examples and shadows, the FWCI is often an equality. We see next that when the FWCI *is* an equality, we nearly have symmetry between V and V^g for $N_{V^g}(V)$ a w-offender, and that we are close to achieving the hypotheses of E.2.10.

To this end, it is convenient to define a notation roughly dual to the usual notation of the k-generated core $\Gamma_{P,k}(H)$ (cf. page 246 of [**Asc86a**] and Definition F.4.41):

DEFINITION E.3.30. If an elementary abelian 2-group P acts on a group H, and k is a positive integer, define

$$\check{\Gamma}_{k,P}(H) := \langle C_H(X) : X \leq P,\ m(P/X) \leq k \rangle.$$

LEMMA E.3.31. *Assume Hypothesis E.3.24, that $A := N_{V^g}(V)$ is a w-offender, and that the Fundamental Weak Closure Inequality E.3.29 is an equality. Define $\bar{A} := A/C_A(V)$, and the subspace generated by fixed points of involutions in $\bar{A}$:*

$$W := \check{\Gamma}_{m(\bar{A})-1,\bar{A}}(V).$$

Then

(1) $m(Aut_A(V)) = m_2(Aut_G(V))$, $r = m(V^g/C_A(V))$, and $W \leq N_V(V^g)$.

(2) $m(V/W) \geq w$—and in case of equality, $w > 0$ and $W = N_V(V^g)$ is a w-offender on V^g, so also $m(Aut_W(V^g)) = m_2(Aut_G(V))$.

PROOF. By E.3.28.2,

$$m_2(Aut_G(V)) + w \geq m(V^g/C_A(V)) = m(Aut_A(V)) + w \geq r,$$

so since the FWCI is an equality, we have $m(Aut_A(V)) = m_2(Aut_G(V))$ and $r = m(V^g/C_A(V))$. Then for $C_A(V) \leq B \leq A$ with $m(B/C_A(V)) = 1$ we see $m(V^g/B) = m(V^g/C_A(V)) - 1 = r - 1 < r$, so $C_G(B) \leq N_G(V^g)$ and hence $W \leq N_V(V^g)$, completing the proof of (1). Next by Definition E.3.23,

$m(V/W) \geq m(V/N_V(V^g)) \geq w$, and in case of equality $W = N_V(V^g)$ is a w-offender on V^g and $w > 0$ since $V \not\leq N_G(V^g)$ by E.3.28.4. Then by (1) and symmetry between V and V^g, $m(Aut_W(V^g)) = m_2(Aut_G(V))$, completing the proof. □

Observe that an argument in the proof of the previous lemma also shows:

LEMMA E.3.32. *Assume $w < r$ and $A := N_{V^g}(V)$ is a w-offender. Set $\bar{A} := A/C_A(V)$. Then $\check{\Gamma}_{r-w-1,\bar{A}}(V) \leq N_V(V^g)$.*

LEMMA E.3.33. *Assume $A := N_{V^g}(V)$ is a w-offender, and $0 < w < s$. Then*
(1) $U := C_V(A) = C_V(V^g) < V$.
(2) $V < \langle V, V^g \rangle \leq C_G(U) \not\leq N_G(V)$.
(3) $m(V/U) \geq r$.
(4) $Aut_A(V) \in \mathcal{A}_{s-w}(Aut_G(V), V)$ so $m(Aut_A(V)) \geq s - w \geq 1$.

PROOF. By hypothesis, $m(V^g/A) = w < s$, so (1) and (4) follow from E.3.10. By E.3.25, $V^g \not\leq N_G(V)$, so (1) implies (2). Then (2) and E.3.4.2 (with the roles of V and V^g reversed) imply (3). □

The remaining results of the section discuss other useful bounds and relations on the parameters, often involving $n(H)$ for $H \not\leq M$, and usually under the assumption of the existence of a suitable uniqueness group.

The most elementary way to show that $w > 0$ is to prove that $s > a$, and then apply some variant of E.3.21, as in the proof of E.3.34.1 below. To see that this approach might be feasible in the proof of our Main Theorem, notice that for most of the pairs $\bar{L}$, V arising in the Fundamental Setup FSU, we have recorded (in section E.4 and chapter H) the values of the parameters m and a determined by those pairs. Furthermore in section E.6, we show that if $m > 2$, then $r \geq m$, so that $s = m$. Thus we do have some initial knowledge of m, r, s, a. So when we can show $w > 0$, then under the assumption that $N_G(Q)$ is a uniqueness subgroup, E.3.34.2 provides control over normalizers and centralizers of some of our weakly closed subgroups.

LEMMA E.3.34. *(1) If $s > a$, then $w \geq s - a > 0$.*
(2) If $M = !\mathcal{M}(N_G(Q))$, then for each $0 \leq i < w$, $N_G(W_i) \leq M \geq C_G(C_i)$.

PROOF. Assume the hypotheses of (1); then the hypotheses of E.3.18 are satisfied with V, $N_G(V)$, T in the roles of "U, H, S" for each $0 \leq i < s-a$. Therefore by E.3.18, W_i centralizes V, so (1) follows.

Next assume the hypothesis of (2), and suppose that $0 \leq i < w$. This time $W_i \leq Q$ by Definition E.3.23, so (2) follows from E.3.16 and our hypothesis that $M = !\mathcal{M}(N_G(Q))$. □

In the remainder of this subsection, let H denote some subgroup of G, and $S \in Syl_2(H)$.

Our next lemma begins to provide lower bounds on $n(H)$ for subgroups H such that $Q \leq H$ and $H \not\leq M$.

LEMMA E.3.35. *Assume $M = !\mathcal{M}(N_G(Q))$, $C_H(V) \leq M$, and $Q = C_T(V) \leq S$. Then*

(1) If $n(H) < \min\{w, r\}$, then $H \leq M$. In particular if H is solvable and $\min\{w, r\} > 1$, then $H \leq M$.

(2) If $s > a$, $r \geq w$, and $H \not\leq M$, then $n(H) \geq w \geq s - a > 0$.

PROOF. Suppose $j := n(H) < \min\{w, r\}$. To prove (1), we establish the analogues of certain steps in the proofs of E.3.21 and E.3.19, but now using our hypothesis on w since we have no restriction on s. First $j < w$ and $Q \leq S$ by hypothesis, so that $W_j \leq S$. Therefore $W_i = W_i(S, V)$ for all $i \leq j$, by E.3.15 applied with T, S, G in the roles of "S, Y, H". Also $V \leq W_j$, so $C_j = C_T(W_j) \leq C_T(V) \leq S$, and hence $C_j = C_j(S, V)$. Then by E.3.34.2, $N_G(W_0) \leq M \geq C_G(C_j)$; in particular $S \leq M$. Next as $n(H) = j$, Definition E.1.6 says that

$$H = \langle\ N_H(W_0), K : K \in \mathcal{E}_j(H, S, A) \text{ and } \ A \in \Gamma_0(S)\ \rangle,$$

so it suffices to show that $K \leq M$ for each $A := V^g \in \Gamma_0(S)$ and $K \in \mathcal{E}_j(H, S, A)$. Further $K = O^2(K)(S \cap K)$ and $O^2(K) \leq \langle A^K \rangle$ by E.1.4, so since $C_H(V) \leq M$ by hypothesis, it suffices to show that $A^k \leq C_H(V)$ for each $k \in K$. Thus as $w > n(H)$ by hypothesis and $n(H) \geq 0$, we may assume that $A^k \not\leq N_G(V)$ by E.3.25. However as $K \in \mathcal{E}_j(H, S, A)$, $m(A^k/(A^k \cap O_2(K))) \leq j$ by Definition E.1.5, so as $j < w$, $A^k \cap O_2(K)$ centralizes V. Therefore $m(A^k/C_{A^k}(V)) \leq j < r$, contrary to E.3.4.2. This completes the proof of (1), since $n(H) = 1$ when H is solvable by E.1.13.

Now assume that $s > a$. Then $w \geq s - a > 0$ by E.3.34.1. If in addition $w \leq r$ and $H \not\leq M$, then $n(H) \geq w$ by (1), completing the proof of (2). □

When H contains T, we can take $S = T$, so certainly the hypothesis $Q \leq S$ of E.3.35 is satisfied; in particular this holds for $H \in \mathcal{H}_*(T, M)$ in the proof of the Main Theorem. Then if in addition $C_G(V) \leq M = !\mathcal{M}(N_G(Q))$, $s > a$, and $r \geq w$, E.3.35.2 says that w is a lower bound on $n(H)$ for $H \in \mathcal{H}_*(T, M)$. We see next that under more restrictive hypotheses, we also get useful upper bounds on $n(H)$, and hence also on w.

Indeed we can expect roughly that $n(H) \leq n(Aut_G(V))$. The method goes back (at least) to [**Asc78b**], based on the idea of finding a "Cartan subgroup" B of H inside $H \cap M$, and showing that that Cartan subgroup embeds faithfully in $Aut_G(V)$. In the proof of the Main Theorem, we use results like Theorem 4.4.14 (depending on the applications of pushing up in QTKE-groups) to establish such an embedding under suitable hypotheses related to the Fundamental Setup. Rather than recalling those details of the FSU, we instead axiomatize the setup which emerges from that method in the following hypothesis:

HYPOTHESIS E.3.36. *Assume Hypothesis E.3.22 with G a QTKE-group and*

(1) $M = !\mathcal{M}(N_G(Q))$.

(2) If $H \in \mathcal{H}_*(T, M)$ with $n(H) > 1$, then

$$H \cap M = N_H(V) \ \ \textit{and} \ \ C_{H\cap M}(V) \leq O_2(H \cap M).$$

Before going further we define another parameter which, given a T-invariant section X of G, roughly gives an upper bound on the exponent c of the field of definition $\mathbf{F}_{2^c}$ of any minimal parabolic H over T such that a Cartan subgroup of H is embedded in X.

DEFINITION E.3.37. If X is a finite group, define $n'(X)$ to be the maximum power c such that X contains a cyclic subgroup Y of order $2^c - 1$ and an abelian overgroup of Y of odd order permuting with a Sylow 2-subgroup of X.

As usual we often abbreviate $n' := n'(Aut_G(V))$. Notice $n' \geq 1$.

LEMMA E.3.38. *Assume Hypothesis E.3.36 holds. Then* $\min\{w, r\} \leq n(H) \leq n'(Aut_G(V))$ *for each* $H \in \mathcal{H}_*(T, M)$.

PROOF. The first inequality is a consequence of E.3.35.1, so it remains to establish the second. Let $H \in \mathcal{H}_*(T, M)$ (see Definition 3.0.1) and set $H^* := H/O_2(H)$. As $n' \geq 1$, we may assume $n := n(H) > 1$. Recall (cf. 3.1.3.1) that $H \cap M$ is the unique maximal subgroup of H containing T. As $n(H) > 1$, H is nonsolvable by E.1.13, so H^* is described in E.2.2. By that result, $O^2(H^*)$ is of Lie type and characteristic 2, and if B is a Hall $2'$-subgroup of $H \cap M$, then B is abelian, $BT = TB$, and B contains a cyclic subgroup Y of order $2^n - 1$. By Hypothesis E.3.36.2, B is faithful on V. As B permutes with T, $B \cong Aut_B(V)$ permutes with $Aut_T(V) \in Syl_2(Aut_G(T))$, so $n' \geq n = n(H)$ by Definition E.3.37. □

LEMMA E.3.39. *Assume Hypothesis E.3.36, and also* $n'(Aut_G(V)) < r$. *Assume* $H \in \mathcal{H}_*(T, M)$. *Then*

(1) $w \leq n(H) \leq n'(\bar{M}) < r$.

(2) If $s > a$ *then* $0 < s - a \leq w \leq n(H) \leq n'(\bar{M}) < r$.

PROOF. Let $n := n(H)$. By E.3.38, $\min\{w, r\} \leq n \leq n'$, while by hypothesis, $n' < r$—so (1) holds. Then (1) and E.3.35.2 imply (2). □

LEMMA E.3.40. *Let* W *be a faithful* $\mathbf{F}_2X$*-module and* $A \in \mathcal{A}_2(X, W)$. *Then* A *centralizes* $O(X)$.

PROOF. If the lemma fails, then using Generation by Centralizers of Hyperplanes A.1.17, there is some hyperplane B of A such that $Y := [C_{O(X)}(B), A] \neq 1$. By the Thompson $A \times B$ Lemma A.1.18.2, Y acts faithfully on $C_W(B)$. But as $A \in \mathcal{A}_2(G, W)$, $C_W(A) = C_W(B)$, so $Y = [Y, A] \leq C_G(C_W(A)) = C_G(C_W(B))$, contrary to the previous sentence. □

E.4. Values of a for $\mathbf{F}_2$-representations of SQTK-groups.

In this section we establish upper bounds on the parameter $a(G, V)$, when $F^*(G) =: L$ is quasisimple with $L/Z(L)$ a Mathieu group, and V is a faithful $\mathbf{F}_2G$-module. These bounds are required in Part 3 of our proof of the Main Theorem. The original calculations are from [**Asc82a**], but to keep our treatment self-contained, we reproduce the proofs here.

Throughout this section, we assume that G is a finite group such that $O_2(G) = 1$, $T \in Syl_2(G)$, and V is a faithful $\mathbf{F}_2G$-module.

Recall from Definition E.3.8 that given a positive integer k, $\mathcal{A}_k(G, V)$ is the set of nontrivial elementary abelian 2-subgroups A of G such that $C_V(A) = C_V(B)$ for all $B \leq A$ with $m(A/B) < k$. Further from Definition E.3.9

$$a(G, V) = \max\{k : \mathcal{A}_k(G, V) \neq \emptyset\},$$

if G is of even order, and $a(G, V) = 0$ if G is of odd order.

LEMMA E.4.1. *If* G *is solvable then* $a(G, V) = 1$.

PROOF. As G is solvable and $O_2(G) = 1$, $F^*(G) = O(F(G)) =: F$, and any elementary abelian 2-subgroup A of G is faithful on F. Therefore by G.8.8, AF contains a direct product AF_1 of dihedral groups of order $2p_i$ for suitable odd primes p_i. In particular A contains a hyperplane A_0 centralizing a dihedral factor

$\langle a\rangle F_2 \cong D_{2p}$ with $F_2 \leq F$ and $a \in A - A_0$. By the Thompson $A \times B$ Lemma, F_2 acts nontrivially on $C_V(A_0)$, and then a is also nontrivial on $C_V(A_0)$, so that $C_V(A_0) > C_{C_V(A_0)}(a) = C_V(A)$, completing the proof. □

The next lemma is a weak version of 4.16 in [**Asc82a**].

LEMMA E.4.2. *Assume $F^*(G) =: L$ is quasisimple, $G = LT$, and Δ is a set of overgroups of T in G such that $G = \langle\Delta\rangle$, and for each $H \in \Delta$, $F^*(H)/O_2(H)$ is quasisimple or of prime order. For $H \in \Delta$, set $i(H) := m_2(H/O_2(H))$, and for $\Gamma \subseteq \Delta$, define*

$$k(\Gamma) := \sum_{H\in\Gamma} i(H),$$

with $k(\emptyset) := 0$. Assume $[C_V(T), L] \neq 0$. Then $a(G, V) \leq k(\Delta)$.

PROOF. Let $k := k(\Delta)$, $\Omega := \mathcal{A}_{k+1}(G, V)$, and for $j \leq k$, define

$$W_j = W_j(T, \Omega) := \langle B : m(A/B) \leq j \text{ for some } A \in \Omega\rangle,$$

and $V_j := C_V(W_j)$. We may assume $a(G, V) > k$ so $\Omega \neq \emptyset$. For $A \in \Omega$, and $B \leq A$ with $m(A/B) \leq k$, $C_V(A) = C_V(B)$; thus $B \neq 1$ as A is faithful on V. In particular $W_k \neq 1$, and hence $W_j \neq 1$ for $j \leq k$, so $L = [L, W_j]$ as $L = F^*(G)$ is quasisimple and $O_2(G) = 1$.

Let $\mathcal{S}$ be the set of subsets Γ of Δ such that $[V_{k(\Gamma)}, O^2(H)] = 0$ for each $H \in \Gamma$; for example $\emptyset \in \mathcal{S}$ vacuously. Let Γ be a maximal member of $\mathcal{S}$, partially ordered by inclusion, and set $j := k(\Gamma)$. Observe that for $H \in \Gamma$, $H = O^2(H)T$ acts on V_j. Also $j \leq k$, so $L = [L, W_j]$ by the previous paragraph.

Suppose $W_j \trianglelefteq K$ for each $K \in \Delta - \Gamma$. Then K acts on V_j, so $G = \langle\Delta\rangle$ acts on V_j. Therefore as W_j centralizes V_j, so does $L = [L, W_j]$. This is a contradiction, as $C_V(T) \leq V_j$, and $[C_V(T), L] \neq 0$ by hypothesis.

Thus we can choose $K \in \Delta - \Gamma$, $A \in \Omega \cap T$, and $B \leq A$ with $m(A/B) = j$, such that $B \not\leq O_2(K)$. By hypothesis, $F^*(K/O_2(K))$ is quasisimple or of prime order, so $O^2(K) = [O^2(K), B]$. Let $\Theta := \Gamma \cup \{K\}$, $m := k(\Theta)$, $W := W_m(O_2(K))$, and $D := B \cap O_2(K)$. Then $m(B/D) \leq i(K)$, so $m(A/D) \leq j + i(K) = m \leq k$. Thus $C_V(A) = C_V(D)$ and $D \leq W \leq W_m$, so $V_m \leq C_V(W) \leq C_V(D) = C_V(A)$. Hence $O^2(K) = [O^2(K), B]$ centralizes $C_V(W)$, so $O^2(H)$ centralizes V_m for each $H \in \Theta$. That is, $\Theta \in \mathcal{S}$, contrary to the maximality of Γ. □

LEMMA E.4.3. *Assume $C_V(G) < C_V(T)$ and $F^*(G) = L$ is quasisimple. Then*

(1) If $L/Z(L) \cong M_{12}$, then $a(G, V) \leq 2$.

(2) If $L/Z(L) \cong M_{22}$ or M_{24}, then $a(G, V) \leq 3$.

(3) If V is the code or cocode module for $G \cong M_{23}$, then $a(G, V) \leq 3$.

PROOF. Recall that [**Asc86b**] describes the maximal subgroups of L containing a Sylow 2-subgroup.

As the code and cocode modules for M_{23} are obtained by restriction from the corresponding modules for M_{24}, (2) implies (3). To prove (1) and (2), we appeal to E.4.2. To apply that lemma, we need a set Δ of overgroups H of T in G such that $G = \langle\Delta\rangle$, $F^*(H/O_2(H))$ is quasisimple or of prime order, and $k := \sum_{H\in\Delta} i(H)$ is 2 if $L/Z(L)$ is M_{12} and 3 otherwise.

When $L/Z(L)$ is M_{12}, let Δ be the set of maximal subgroups of G containing T; thus Δ is of order 2 and $H/O_2(H) \cong S_3$ for both members H of Δ, so that $i(H) = 1$.

When $L/Z(L) \cong M_{22}$, we take Δ to consist of $H_1 \cong S_5/E_{16}$, and H_2 the rank 1 parabolic in $M \cong A_6/E_{16}$ not contained in H_1. Thus $i(H_1) = 2$ and $H_2/O_2(H_2) \cong S_3$, so $i(H_2) = 1$ and $k = 3$. Finally when $L \cong M_{24}$, we take Δ to be the set of three minimal proper overgroups of T which generate G; for each such H, $H/O_2(H) \cong S_3$, so $i(H) = 1$. □

E.5. Weak closure and higher Thompson subgroups

In this section we present some elementary lemmas related to failure of factorization, which we will use in conjunction with weak closure. The main result is Proposition E.5.2 involving higher Thompson groups J_j for suitable j, which we will most often apply in the case $j = 1$—particularly in E.6.26 in the section after this one.

We assume in this section that P is a Sylow 2-subgroup of G. The results are elementary and hold for any prime p, although we only state them here for the case $p = 2$, since that is the only case where we need them.

Recall Definition B.2.2: For $0 \le i \le m_2(G)$ and $H \le G$, $\mathcal{A}_i(H)$ consists of the elementary abelian 2-subgroups of H of rank $m_2(H) - i$, and $J_i(H) = \langle \mathcal{A}_i(H) \rangle$. Also $\mathcal{A}(H) = \mathcal{A}_0(H)$ and $J(H) = J_0(H)$.

Assume in this section that $i \ge 0$ and $j > 0$, so that $0 \le i < i + j$. Set $m := m_2(G)$.

The following elementary result E.5.1 has probably long been known, at least for the original case of $i = 0, j = 1$—and indeed that is the case of most interest to us. The argument in that case appears essentially in Thompson's treatment of factorizations of solvable groups in the N-group paper ([**Tho68**, 5.53], after "Suppose $i = 1$" on page 424). For general i, j it is contained (partly implicitly) in Aschbacher's $GF(2)$-representations paper [**Asc82a**, 14.9]. One advantage of the formulation given here is that the original hypothesis of $F^*(K) = O_2(K)$ can be replaced by the weaker hypothesis below on the centralizer of $O_2(K)$.

Proposition E.5.1. *Let $A \in \mathcal{A}_i(P)$, and $K \in \mathcal{E}_j(G, P, A)$, and assume that $C_{O^2(K)}(O_2(K)) \le R$, for $R := O_{2,\Phi(F^*)}(K)$ in case (E1), or $R := O_\infty(K)$ in case (E2), of Definition E.1.2. Set $W := \langle B \in \mathcal{A}_{i+j}(P) : B \le O_2(K) \rangle$. Then $D := \Omega_1(Z(J_{i+j}(P))) \le \Omega_1(Z(W))$ and $W \trianglelefteq K$.*

Proof. If $i + j \ge m$ then $J_{i+j}(P) = W = 1$, so there is nothing to prove. Thus we may assume $i + j < m$. Set $Q := O_2(K)$, and observe that W is weakly closed in Q, so that $W \trianglelefteq K$. From B.2.3.2, $D \le C_P(A)$, while Definition E.1.2 says that $C_P(AO_2(G)/O_2(G)) \le K$, so $D \le K$. By definition D centralizes $J_{i+j}(P)$ and hence W, so $D \le C_{P \cap K}(W)$.

Next $Q \le P \le N_G(D)$ and $Q \trianglelefteq K$, so that $[D, Q] \le D \cap Q \le \Omega_1(C_Q(W)) \le W$ by B.2.3.2. Thus D centralizes Q/W and W. By hypothesis $C_{O^2(K)}(Q) \le R$, while by Coprime Action, $(C_K(Q/W) \cap C_K(W))/C_K(Q)$ is a 2-group. However by Definition E.1.2, $O_2(K/R) = 1$ and R is 2-closed, so $D \le Q$. Thus $D \le C_Q(W)$, so $D \le \Omega_1(Z(W))$ by B.2.3.2. □

The next result extends a statement in [**Mas**, 1.5.11] for the case $i = 0, j = 1$; we also remove his hypotheses that $F^*(G) = O_2(G)$ and G is quasithin. The result could be stated for any prime p. Mason's result was for solvable groups; we replace that hypothesis by a restriction on $n(G)$; that is, when G is solvable, $n(G) = 1$ by E.1.13, giving rise to Corollary E.5.3. As a proof, Mason just refers to Thompson's original proof of factorizations for solvable groups, presumably meaning [**Tho68**, 5.53].

PROPOSITION E.5.2. *Let $n(G) = j$ and assume $i \geq 0$ with $i + j < m_2(G)$. Set $D := \Omega_1(Z(J_{i+j}(P)))$. Then $G = \langle N_G(J_i(P)), J_i(C_G(D_1)) : m(D/D_1) \leq i + j\rangle$.*

PROOF. Define

$$\mathcal{D} := \{D_1 : m(D/D_1) \leq i + j\} \text{ and } G_0 := \langle N_G(J_i(P), J_i(C_G(D_1)) : D_1 \in \mathcal{D}\rangle;$$

we must show that $G \leq G_0$. We first claim that

$$C_G(D_1) \leq G_0 \text{ for each } D_1 \in \mathcal{D}. \qquad (*)$$

For let $D_1 \in \mathcal{D}$ and $G_1 := C_G(D_1)$. By B.2.3.1, $J_i(P) \leq J_{i+j}(P) \leq C_G(D) \leq C_G(D_1) = G_1$, so $J_i(P) = J_i(P_1) \leq J_i(G_1)$ for some $P_1 \in Syl_2(G_1)$. Thus by a Frattini Argument,

$$G_1 = J_i(G_1)N_{G_1}(P_1 \cap J_i(G_1)) \leq J_i(G_1)N_{G_1}(J_i(P)) \leq G_0$$

using the definition of G_0, establishing the claim.

At this point, the machinery in section E.1 and our hypothesis that $n(G) = j$ takes the place of the reductions in Thompson's proof in [**Tho68**, 5.53]. In the notation of that section, choose:

$$\Omega := \mathcal{A}_i(G),$$

so that

$$\Omega \cap P = \mathcal{A}_i(P) \text{ and } W(P, \Omega) = J_i(P).$$

As $n(G) = j$, Definition E.1.6 says that G is generated by $N_G(J_i(P))$ together with the subgroups $K \in \mathcal{E}_j(G, P, A)$ as A varies over $\mathcal{A}_i(P)$. Thus it will suffice to show that each such K is contained in G_0. Indeed $K = O^2(K)(P \cap K)$ with $P \cap K \leq P \leq N_G(J_i(P)) \leq G_0$, so it suffices to show that $H := O^2(K) \leq G_0$. As in the proof of E.5.1, $D \leq C_P(A) \leq K$. In discussing K, adopt the notation of Definition E.1.2 such as $\bar{K} := K/O_2(K)$.

Assume first that $H \leq C_K(O_2(K))$. Since $A \in \mathcal{A}_i(P)$ and DA is elementary abelian, $m(DA/A) \leq i$. As $K \in \mathcal{E}_j(G, P, A)$, $m_2(\bar{K}) \leq j$, so $m(\bar{D}) \leq m(\bar{D}\bar{A}) \leq i + j$. Set $D_1 := D \cap O_2(K)$; then $m(D/D_1) \leq i + j$, and so $H \leq C_G(D_1) \leq G_0$ by (*), completing the proof in this case.

Therefore we may assume that $C_H(O_2(K)) < H$. Define R as in E.5.1, and let $K^* := K/R$. In case (E2) of Definition E.1.2, H^* is the product of simple components permuted transitively by K, so $C_H(O_2(K)) \leq R$. In case (E1), choosing K minimal we may assume that K is irreducible on H^*, so again $C_H(O_2(K)) \leq R$. Thus in either case we have the hypothesis of E.5.1, so we conclude from that result that $D \leq E := \Omega_1(Z(W))$, where $W := \langle B \in \mathcal{A}_{i+j}(P) : B \leq O_2(K)\rangle$. As W is weakly closed in $O_2(K)$, $E \trianglelefteq K$.

Now define $A_1 := A \cap O_2(K)$ and $E_1 := E \cap A_1$. As $m(A/A_1) \leq j$, $A_1 \leq W$, and of course $W \leq C_K(E)$. Then A_1E is an elementary abelian subgroup of $O_2(K) \leq P$ of rank $m_2(G) - (i + j) + m(E/E_1)$, so $m(E/E_1) \leq i + j$. Now A centralizes its subgroup E_1, so for $k \in K$, A^k centralizes $D_1 := D \cap E_1^k$. As $D \leq E \trianglelefteq K$, $m(D/D_1) \leq m(E/E_1^k) = m(E/E_1) \leq i + j$, so $D_1 \in \mathcal{D}$. Thus each member of A^K centralizes some $D_1 \in \mathcal{D}$, so $H = O^2(K) \leq \langle A^K\rangle \leq G_0$ using E.1.4, and the proof is complete. □

Corollary E.5.3. *Assume G is solvable and set $D := \Omega_1(Z(J_1(P)))$. Then*

$$G = \langle N_G(J(P)), J(C_G(D_1)) : m(D/D_1) \le 1\rangle.$$

Lemma E.5.4. *Assume that $F^*(G) = O_2(G)$, $F^*(G/O_2(G))$ is quasisimple, $m_2(G/O_2(G)) \le j$, and G/X has no FF-modules for any $X \trianglelefteq G$ with $O_2(G) \le X \le O_\infty(G)$. Then either*

(1) $J(P) \trianglelefteq G$, or

(2) $O^2(G)$ centralizes $D := \Omega_1(Z(J_j(P)))$, so $D \trianglelefteq G$.

Proof. Assume that (1) fails; then $J(P) \not\le Q := O_2(G)$, so there is $A \in \mathcal{A}(P)$ with $A \not\le Q$. We proceed as in the proof of E.5.2. Namely let $\Omega := \mathcal{A}(G)$, and observe that our hypotheses imply that $G \in \mathcal{E}_j(G, P, A)$. As $F^*(G) = O_2(G)$, E.5.1 says $D \le E := \Omega_1(Z(W)) \trianglelefteq G$, where $W := \langle B \in \mathcal{A}_j(P) : B \le O_2(G)\rangle$. Thus if $K := O^2(G) \le C_G(E)$ then $[K, D] = 1$, so $D \trianglelefteq KP = G$, and hence (2) holds. Therefore we may assume that $[K, E] \ne 1$. Let $G^* := G/Q$, V a G-chief section of E with $[V, K] \ne 1$, and $\bar{G} := G/C_G(V)$. As $F^*(G^*) = K^*$ is quasisimple, $C_{K^*}(V) \le Z(K^*)$, so $F^*(\bar{G}) = \bar{K}$ and $m(A^*) = m(\bar{A})$. Since $Z(K^*) \le O_\infty(G)^*$, by hypothesis V is not an FF-module for $\bar{G}$.

Arguing as in the last paragraph of the proof of E.5.2 with G in the role of "K" and $i = 0$, $m(E/C_E(A)) \le m(A^*)$. Thus $m(V/C_V(\bar{A})) \le m(A^*) = m(\bar{A})$, contradicting our observation that V is not an FF-module. □

Lemma E.5.5. *Let G be a 2-group, $Q \trianglelefteq G$, $\bar{G} := G/Q$, and $V \le \Omega_1(Z(Q))$ with $V \trianglelefteq G$. Assume that j is a nonnegative integer, and that $m(V/C_V(\bar{A})) > m(\bar{A}) + j$ for each $\bar{A} \in \mathcal{A}^2(\bar{G})$. Then $J_j(G) \le Q$.*

Proof. This is by now a familiar argument. If $A \in \mathcal{A}_j(G)$, then $m(\bar{A}) + j \ge m(V/C_V(\bar{A}))$ by B.2.4.1, and hence $\bar{A} = 1$ by hypothesis, so $A \le Q$. □

E.6. Lower bounds on $\mathbf{r(G, V)}$

Let V be a nontrivial elementary abelian 2-subgroup of a finite group G. We begin by recalling the weak closure parameter from Definition E.3.3:

$$r(G, V) := \min\{m(V/U) : U \le V \text{ and } C_G(U) \not\le N_G(V)\}.$$

Since G will be fixed, we will typically use the abbreviation $r := r(G, V)$ as in section E.3. Recall that r is a global parameter depending on G, rather then simply on the local subgroup $N_G(V)$. We saw in section E.3 that in order to do weak closure, we need lower bounds on r.

We are primarily interested in establishing a lower bound on r in our Fundamental Setup (3.2.1). So in this section, we will assume a hypothesis which holds in the FSU and implies the hypotheses used for weak closure in section E.3 such as Hypothesis E.3.22.

Hypothesis E.6.1.

(1) G is a simple QTKE-group, $T \in Syl_2(G)$, and $M \in \mathcal{M}(T)$.

(2) $V \ne 1$ is a normal elementary abelian 2-subgroup of T, and is a TI-set under M.

Set $M_V := N_M(V)$, $\bar{M}_V := M_V/C_M(V)$, $Q := C_T(V)$, and $Z := C_V(T)$.

(3) $M = !\mathcal{M}(N_{M_V}(Q))$.

Note that by (2), $V \trianglelefteq T$, and by (1), T is Sylow in G; thus T is also Sylow in M_V and $Q \in Syl_2(C_M(V))$. Also by (3), $M = !\mathcal{M}(N_G(Q)) = !\mathcal{M}(N_G(V))$.

Often in applications the TI-set assumption in (2) is strengthened to $V \trianglelefteq M$, so that $M_V = M$ and $N_{M_V}(Q) = N_M(Q)$. The uniqueness condition $M = !\mathcal{M}(N_{M_V}(Q))$ typically comes from the fact that L_0T is a uniqueness subgroup in the FSU.

Notice that as $M = !\mathcal{M}(N_G(V))$, we have $M_V = N_M(V) = N_G(V)$, and hence $C_G(V) \leq M$; as a consequence:

Lemma E.6.2. *If $L \leq G$ and $L \not\leq M$, then $[V, L] \neq 1$.*

E.6.1. The main result: $\mathbf{m > 2}$ implies $\mathbf{s = r \geq m > 2}$. We begin with the statement of the main result of the section; its proof will require most of the section, but along the way we will also establish other results of independent interest. The primary virtue of the following theorem is that it is easy to establish its hypothesis without detailed knowledge of the action of $N_G(V)$ on V.

Theorem E.6.3. *Assume Hypothesis E.6.1, and that $m(\bar{M}_V, V) > 2$. Then $r(G,V) \geq m(\bar{M}_V, V)$, so $s(G,V) = m(\bar{M}_V, V) > 2$.*

In this subsection we let $m := m(\bar{M}_V, V)$. To prove Theorem E.6.3, must show that if $m > 2$ then $C_G(U) \leq M$ for each $U \leq V$ with $m(V/U) < m$. However we will also prove that $C_G(U) \leq M$ under weaker hypotheses on the subspace U of V. In particular we will concentrate on the set

Definition E.6.4.

$$\Gamma := \{U \leq V : 1 \neq U,\ C_G(U) \not\leq M, \text{ and } O^{2'}(C_M(U)) \leq C_M(V)\}.$$

Notice that our hypothesis E.6.1.2 that V is a TI-subgroup of M shows that for $1 \neq U \leq V$, $C_M(U) \leq N_M(V) = M_V$. Also the condition $O^{2'}(C_M(U)) \leq C_M(V)$ is equivalent to the assertion that $C_{\bar{M}_V}(U)$ is of odd order.

By Definition E.3.3, there is $U \leq V$ with $m(V/U) = r$ and $C_G(U) \not\leq M$; so if $r < m$ then $U \in \Gamma$. Thus if Theorem E.6.3 fails, Γ contains the set of obstructions to Theorem E.6.3.

But in general there are subspaces U of V with $C_{\bar{M}_V}(U)$ of odd order but $m(V/U) \geq m$, and often we also want to know that $C_G(U) \leq M$ for such subspaces.

In the remainder of this subsection, we let U denote some member of Γ. Set $H := C_G(U)$ and $M_H := H \cap M$.

In the results through E.6.13, we establish some useful restrictions on U and H without making use of the hypothesis $m > 2$ of Theorem E.6.3.

We begin by observing the condition that $|C_{\bar{M}_V}(U)|$ is odd leads to the pushing up hypothesis (CPU) of Definition C.1.6, in the optimal case where Q is Sylow in H:

Lemma E.6.5. *(1) $Q \in Syl_2(H)$.*
(2) $C(H, Q) \leq M_H < H$.
(3) $V \trianglelefteq M_H$.

Proof. As Q is Sylow in $C_M(V)$ and $O^{2'}(C_M(U)) \leq C_M(V)$, Q is also Sylow in $C_M(U) = M_H$. Next $M = !\mathcal{M}(N_G(Q))$ from Hypothesis E.6.1.3, so for $1 \neq C$ char Q, $N_H(C) \leq N_H(Q) \leq M_H$. Then as $Q \in Syl_2(M_H)$ and $N_H(Q) \leq M_H$,

(1) holds. Further $H \not\leq M$ by Definition E.6.4, so $M_H < H$, completing the proof of (2). Finally we observed earlier that $M_H = C_M(U)$ normalizes V because of the TI-hypothesis, so (3) holds. Since $Q \in Syl_2(M_H)$, trivially $Q \in \mathcal{B}_2(M_H)$ and $Q \in Syl_2(\langle Q^{M_H}\rangle)$, so (CPU) is satisfied with H, Q, M_H in the roles of "G, R, M". $\square$

Our analysis will depend upon the fact that certain subgroups are in $\mathcal{H}^e$ (cf. Definition 1.1.1), so we need:

LEMMA E.6.6. *(1)* $F^*(M_H) = O_2(M_H)$.
(2) $F^*(C_M(A)) = O_2(C_M(A))$ *for each nontrivial* 2*-subgroup* A *of* M.
(3) $F^*(C_H(z)) = O_2(C_H(z)) \geq O_2(C_G(z)) \cap C_G(O_2(C_H(z)))$ *for each* $z \in Z^{\#}$.
(4) If $U \cap Z \neq 1$ *then* $F^*(H) = O_2(H)$.

PROOF. As M contains $T \in Syl_2(G)$, $M \in \mathcal{H}^e$ by 1.1.4.6, and then (2) follows from 1.1.3.2. Part (1) is the special case of (2) with $A = U$. Part (4) follows using 1.1.4.3. If $z \in Z^{\#}$, then $C_G(z) \in \mathcal{H}(T)$, so $C_G(z) \in \mathcal{H}^e$ by 1.1.4.6, and then $C_H(z) \in \mathcal{H}^e$ by 1.1.3.2. Then since $U \leq O_2(C_H(z))$ we have

$$O_2(C_G(z)) \cap C_G(O_2(C_H(z))) \leq O_2(C_G(z)) \cap C_G(U) \leq O_2(C_H(z)),$$

and hence (3) holds. $\square$

LEMMA E.6.7. *If* $Q \leq K \leq H$ *and* $F^*(K) = O_2(K)$, *then*

(1) $K = (K \cap M)L$, *where* $L = L_1 \cdots L_s$ *is the central product of* $s \leq 2$ *blocks of type* $L_2(2^{n_i})$, A_3, *or* A_5, *such that* $L_i = [L_i, J(Q)]$ *for each* i.

(2) If L_i *is an* A_n*-block, then* $m(V/C_V(L_i)) = 1$, $|L_i : L_i \cap M| = n$, *and* $r(G, V) = 1$.

(3) If L_i *is an* $L_2(2^n)$*-block then* $V = C_V(L_i) \times [V, X_i]$, *where* $X_i \cong \mathbf{Z}_{2^n-1}$, $m([V, X_i]) = n$, *and* $L_i \cap M = (L_i \cap Q)X_i$ *with* $(L_i \cap M)/O_2(L_i)$ *a Borel subgroup of* $L_i/O_2(L_i)$. *In particular* $m(V/U) \geq n$.

PROOF. Observe that the hypotheses of C.1.29 are satisfied with K, Q, $M \cap K$ in the roles of G, T, M: By hypothesis $F^*(K) = O_2(K)$ and $Q \leq K \leq H$, so Q is Sylow in K and $C(K, Q) \leq M_K := K \cap M$ by E.6.5. Finally K is an SQTK-group by Hypothesis E.6.1.1. Thus (1) follows from C.1.29, which also says that $(L_i \cap M)/O_2(L_i)$ is a Borel subgroup of $L_i/O_2(L_i)$, so in particular $L_i \cap M$ is of index n in L_i when L_i is of type A_n.

As $Q \in Syl_2(K)$, and $V \leq Z(Q)$, V centralizes the Sylow group $L_i \cap Q$ of L_i, so V acts on L_i and induces inner automorphisms on L_i (cf. 16.1.6). Thus $Aut_V(L_i) \leq Aut_{Z_i}(L_i)$, where $Z_i := \Omega_1(Z(L_i \cap Q))$. As $L_i \not\leq M$, $[V, L_i] \neq 1$ by E.6.2, and it follows from Definition E.3.3 that

$$r \leq m(V/C_V(L_i)) \leq m_i := m(Z_i/(Z_i \cap Z(L_i))).$$

Now if L_i is an A_n-block, we calculate that $m_i = 1$. This gives $r = 1 = m(V/C_V(L_i))$ since $[V, L_i] \neq 1$, and completes the proof of (2).

So assume L_i is an $L_2(2^n)$-block. Then since $(L_i \cap M)/O_2(L_i)$ is a Borel subgroup of $L_i/O_2(L_i)$, $L_i \cap M = (L_i \cap Q)X_i$, where X_i is cyclic of order $2^n - 1$. Further by I.2.3, $Z_i = [Z_i, X_i] \times C_{Z_i}(L_i)$, with $m([Z_i, X_i]) = n$. But by E.6.5.3, V is normal in M_H, and hence $[V, X_i]$ is normal in $L_i \cap M$. Since $1 \neq Aut_V(L_i) \leq Aut_{Z_i}(L_i)$ and X_i is irreducible on $[Z_i, X_i]$, it follows that $[Z_i, X_i] = [V, X_i]$ and hence $V = C_V(L_i) \times [V, X_i]$. This completes the proof of (3), once we observe that $m(V/U) \geq m(V/C_V(L_i)) = n$, since $U \leq C_V(L_i)$. $\square$

When $F^*(H) = O_2(H)$, E.6.7 gives a good description of H. In the next few lemmas we obtain some restrictions on $E(H)$ and $O(H)$.

LEMMA E.6.8. *If L is a component of H, then $L \not\leq M$ and one of the following holds:*

(1) $L \cong L_2(2^n)$, $Sz(2^n)$, $L_3^\epsilon(2^n)$, or a covering of $L_3(4)$ (where we set $n := 2$) with $Z(L)$ a 2-group; $L \cap M$ is the preimage of a Borel subgroup of $L/Z(L)$, and $\overline{L \cap M} \cong \mathbf{Z}_{2^n-1}$ with $n = m(V/C_V(L)) \leq m(V/U)$. Further if $n > 1$, then $V = [V, L \cap M] \times C_V(L)$ and $n = m([V, L \cap M])$.

(2) $L \cong Sp_4(2^n)'$, and $m(V/U) \geq m(V/C_V(L))$. Further if $n > 1$, then $V = [V, L \cap M] \times C_V(L)$. Further either:

(a) $L \cap M$ is a Borel subgroup of L—and either $m(V/C_V(L)) = 2n$, with $C_L(A) \not\leq M$ for some $U < A < V$ with $m(V/A) = n$, or $L \cong A_6$ with $m(V/C_V(L)) = 1$.

(b) $L \cap M$ is a maximal parabolic of L, $Aut_V(L)$ is a root subgroup of $Aut_{LV}(L)$, and $m(V/C_V(L)) = n$.

(3) $L \cong L_4(2)$, $LQ/O_2(LQ) \cong S_8$, $U = C_V(L)$, and $m(V/C_V(L)) = r(G, V) = 1$.

(4) $L \cong A_7$, $\hat{A}_6$, or $\hat{A}_7$ and $C_L(A) \not\leq M$ for some hyperplane A of V containing U, so that $r(G, V) = 1$. Further $m(V/C_V(L)) \leq 2$.

(5) $L \cong L_2(p)$, p a Fermat or Mersenne prime, $L_3(3)$, or M_{11}, $U = C_V(L)$, and $m(V/C_V(L)) = r(G, V) = 1$.

PROOF. Let $z \in Z^\#$. By E.6.6.3, $C_{O_2(C_G(z))}(O_2(C_H(z))) \leq H$, so the hypotheses of 1.1.5 are satisfied with Q, $C_G(z)$ in the roles of "S, M". Therefore we can appeal to that lemma, and conclude that $L = [L, z]$ is described in 1.1.5.3.

Further as $Q \in Syl_2(H)$ and $V \leq Z(Q)$, V centralizes the Sylow group $Q \cap L$ of L and hence acts on L. By E.6.6.1, $L \not\leq M_H$ and hence $L \not\leq M$; so by E.6.2, V acts nontrivially on L. Set $Q_L := N_Q(L)$ and $(LQ_L)^* := LQ_L/O_2(LQ_L)$. Recall from 1.2.1.3 that $Q_L = Q$ unless $L \cong L_2(2^n)$, $Sz(2^n)$, $L_2(p)$ for p odd, since J_1 does not appear in 1.1.5.3. As in the the proof of the previous lemma, $V \leq Q_L \in Syl_2(LQ_L)$ and

$$V^* \text{ is central in } Q_L^* \text{ and normal in } (L \cap M)^* Q_L^*. \tag{!}$$

In particular,

$$r \leq m(V/C_V(L))) \leq m(Z(Q_L^*)). \tag{$*$}$$

Further $O^{2'}(L \cap M) \leq C_L(V)$ from Definition E.6.4.

Suppose first that L is either of Lie type and characteristic 2 (including $A_6 \cong Sp_4(2)'$), or a covering of $L_3(4)$ or $G_2(4)$. Then case (a), (b), or (c) of 1.1.5.3 holds and Z induces inner automorphisms on L, except possibly in case (c). Recall (cf. pp. 220 and 257–258 in [**Asc86a**]) that the maximal overgroups of Q_L^* in $L^* Q_L^*$ are of form $P^* Q_L^*$, for P^* a maximal Q_L^*-invariant parabolic of L^*. Let P be the preimage of P^* in LQ_L. Then $F^*(\langle P, Q \rangle) = O_2(\langle P, Q \rangle)$, so E.6.7 says that:

Either $P \leq M$, or P has a χ_0-block.

Now if L is $L_5(2)$, or L^* is of Lie rank 2 other than $L_3(q)$ or $Sp_4(q)$, then $Q_L = Q$ and L^* is generated by Q^*-invariant parabolics P^* such that PQ has no χ_0-block; hence all such P lie in M, contradicting $L \not\leq M$. Similarly if L is $L_4(2)$ then the same argument supplies a contradiction unless $LQ \cong S_8$, in which case $Z(Q^*)$ is of order 2, so as in the proof of E.6.7, the inequality (*) gives $m(V/C_V(L)) = r = 1$, and hence conclusion (3) holds. Therefore:

$$\text{Either } L \text{ is of Lie rank 1, or } L/Z(L) \text{ is } L_3(q) \text{ or } Sp_4(q)'.$$

Now the preimage B of a Borel subgroup over $(L \cap Q)^*$ is solvable, and unless $L \cong L_2(4)$, B has no A_3-block, so that $B \leq L \cap M$. Thus either L is $L_2(4)$, or $(L \cap M)^*$ is a Q_L-invariant proper parabolic P^* of L^* over B^*.

Assume first that L is of Lie rank 1. Then B is the only proper parabolic of L over $L \cap Q$, so by the previous paragraph, either $L \cong L_2(4)$ or $L \cap M = B$. In the latter case arguing as in the proof of E.6.7 using (!) and (*), (1) holds. We consider the former case below by regarding $L_2(4)$ as $L_2(5)$.

If $L/Z(L)$ is $L_3(q)$ then since $O^{2'}(L \cap M) \leq C_L(V)$, we conclude that $L \cap M = B$, and arguing as above, (1) holds. Finally if $L \cong Sp_4(q)'$ then $Q_L = Q$, and $L \cap M$ is B or maximal parabolic over B, and the argument above leads to conclusion (2).

We turn to the cases remaining in (c) or (d) of 1.1.5.3, where $L/Z(L) \cong A_6$ or A_7, $Z(L)$ is of order 1 or 3, and some $z \in Z$ induces a transposition on $L/Z(L) \cong A_6$, or an involution of cycle type 2^3 on $L/Z(L) \cong A_7$. In either case $C_L(z) \cong S_4$. We have already treated the case where $L \cong Sp_4(2)' \cong A_6$, so when $L/Z(L) \cong A_6$, we may take $L \cong \hat{A}_6$; thus $Z = \langle z \rangle$ in this case by 1.1.5.3. Now in any case $m(Z(Q^*)) \leq 2$, so by (*), $m(V^*) \leq 2$, and $r = 1$ if $m(V^*) = 1$. In the latter case, conclusion (4) holds, so we may assume that $m(V^*) = 2$. Then

$$L = \langle C_L(A^*) : 1 \neq A^* < V^* \rangle,$$

so $C_L(A) \not\leq M$ for some hyperplane of V containing U, completing the proof of (4).

In case (e) of 1.1.5.3, $L \cong L_3(3)$ or $L_2(p)$, p a Fermat or Mersenne prime. We also consider the subcase $L \cong M_{11}$ from case (f) of 1.1.5.3. Here $m(Z(Q^*)) = 1$, so using (*) as before we see that (5) holds.

This leaves the sporadic groups in case (f) of 1.1.5.3 other than M_{11}. In each case L^* possesses a Q_L^*-invariant subgroup P^* with no χ_0-block, such that $F^*(PQ_L) = O_2(PQ_L)$ and $C_{Z(Q_L^*)}(O^{2'}(P^*)) = 1$. As before using E.6.7 we have $P \leq M$, contradicting $O^{2'}(L \cap M) \leq C_L(V)$. This completes the proof. □

Lemma E.6.9. *(1) $O(H) \cap M = 1$.*

(2) If $O(H) \neq 1$, then $C_{O(H)}(A) \not\leq M$ for some hyperplane A of V containing U, so $r(G, V) = 1$.

Proof. Note $O(H) \cap M \leq O(H \cap M) = 1$ by E.6.6.1, giving (1). Further if $r > 1$, then using A.1.17,

$$O(H) = \langle C_{O(H)}(A) : A \leq V \text{ and } m(V/A) = 1 \rangle \leq M$$

in view of Definition E.3.3, so that (2) holds. □

In various places during the proof of the Main Theorem (including but not limited to the proof of Theorem E.6.3 in this section) we will want to prove $r > 1$. The next lemma provides some restrictions on H when $r = 1$.

Lemma E.6.10. *Assume that $m(V/U) = 1$, and that either $Z \cap U \neq 1$ or $F^*(H) = O_2(H)$. Then*

(1) $F^*(H) = O_2(H)$.

(2) $H = (H \cap M)L$ where $L = L_1 \cdots L_s$, $s \leq 2$, is the central product of A_{n_i}-blocks L_i with $|L_i : L_i \cap M| = n_i$ and $n_i = 3$ or 5.

(3) There exists no $W \le U$ such that an E_9-subgroup of G acts faithfully on W.

(4) $\Omega_1(Z(N_T(U))) \not\le U$.

PROOF. If $Z \cap U \neq 1$, then (1) follows from E.6.6.4; otherwise it holds by hypothesis. Then as $m(V/U) = 1$, E.6.7.3 says that H has no $L_2(2^n)$-blocks for $n \ge 2$, so (2) follows from (1) and (2) of E.6.7. In particular if $W \le U$, then $C_G(U) \le C_G(W)$, so 3 divides the order of $C_G(W)$; then (3) holds as $m_3(N_G(W)) \le 2$ since G is a QTKE-group.

Finally let $S := N_T(U)$, $L_0 := \langle L_1^S \rangle$, $U_1 := O_2(L_1)$, and $U_0 := \langle U_1^S \rangle$. Now $N_M(U)$ permutes the blocks $L_1, \dots, L_s$, $s \le 2$, and $C_{U_i}(L_i) = 1$ by C.1.13.c. Thus either $L_0 = L_1$ with $U_0 = U_1$, or $L_0 = L_1 \times L_1^t$ for $t \in S - N_S(L_1)$ and $U_0 = U_1 \times U_1^t$. In any event S acts on U_0, so $1 \neq C_{U_0}(S) =: Z_0 \le \Omega_1(Z(S))$. Then $C_{U_0}(L_0) = 1$ as $C_{U_i}(L_i) = 1$, so as $U \le Z(H)$, $U \cap U_0 = 1$. Thus $Z_0 \not\le U$, so (4) holds. □

When $m(V/U) > 1$, we can get some further information by choosing U maximal in Γ; since the condition $O^{2'}(C_G(U)) \le C_G(V)$ is inherited by overgroups of U in V, this amounts to considering U maximal subject to $C_G(U) \not\le M$.

LEMMA E.6.11. *Assume that $m(V/U) =: n > 1$ and U is maximal in Γ; that is, maximal subject to $C_G(U) \not\le M$. Then*

(1) $H = (H \cap M)L$ for a suitable $L \trianglelefteq H$, described in (3) below.

(2) $V = U \times [V, X]$, where $1 \neq \mathbf{Z}_{2^n-1} \cong X \le L \cap M$ is regular on $[V,X]^\#$, and $1 \neq \bar{X} \trianglelefteq N_{\bar{M}}(U)$.

(3) One of the following holds:

(a) $F^*(H) = O_2(H)$, L is an $L_2(2^n)$-block, and $|L : L \cap M| = 2^n + 1$.

(b) $F^*(H) = O_2(H)L$, $L/Z(L) \cong L_2(2^n)$, $Sz(2^n)$, or $L_3^\epsilon(2^n)$, $M \cap L$ is a Borel subgroup of L, and $O_2(L)[V,X] = Z(L \cap Q)$.

(c) $F^*(H) = O_2(H)L$, $L \cong Sp_4(2^n)$, $L \cap M$ is a maximal parabolic of L with $(L \cap M)^\infty = C_L(V)$, and $[V,X]$ is a root subgroup of L.

(4) $C_H(L) \le C_M(V)$.

PROOF. By hypothesis $m(V/U) = n > 1$, so by maximality of U, $C_G(W) \le M$ for each hyperplane W with $U < W < V$. Therefore $O(H) = 1$ by E.6.9. If $F^*(H) = O_2(H)$, then by E.6.7, H has a χ_0-block L with $L \not\le M$. On the other hand if $F^*(H) > O_2(H)$, then since $O(H) = 1$, H has a component L—and then by E.6.8, $L \not\le M$. In any event by maximality of U,

$$C_G(W) \le M \text{ for } W \text{ with } U < W \le V, \qquad (*)$$

so

$$U = C_V(L) \text{ and } m(V/C_V(L)) = n > 1.$$

We will first establish (1)–(3).

Suppose first that $F^*(H) = O_2(H)$, so that L is a χ_0-block. As $m(V/C_V(L)) > 1$, E.6.7.2 says L is not of type A_3 or A_5. Therefore L is an $L_2(2^n)$-block. As $U = C_V(L)$ while $V \cap L_i \not\le Z(L_i)$ for each of the blocks L_i in E.6.7.1, the parameter s of that lemma is 1. Thus $H = (H \cap M)L$, and L is the unique block of H not contained in M, so $H \cap L$ acts on L and hence $L \trianglelefteq H$, establishing (1). By E.6.7.3, (2) and (3a) hold, where X is a Hall $2'$-subgroup of $L \cap M$; note $\bar{X} = \overline{L \cap M}$, and as L is the unique block of $H = C_G(U)$ not in M, $L \cap M \trianglelefteq N_M(U)$.

Thus we may suppose instead that L is a component of H, so that L is described in E.6.8. As $m(V/C_V(L)) = n > 1$, cases (3) and (5) of E.6.8 are ruled out, as are case (1) and case (2b) when $n = 1$, and the exceptional subcase of (2a) where $m(V/C_V(L)) = 1$. Case (4) and the remaining subcase of (2a) are ruled out, since then $C_G(W) \not\leq M$ for some W with $U < W < V$, contrary to (*).

Assume that case (2b) of E.6.8 holds. By the previous paragraph $n > 1$. Further $L \trianglelefteq H$ by 1.2.1.3. In this case $L \cap M$ is a maximal parabolic of L, and as $n > 1$, $[V, L \cap M] \cong Aut_V(L)$ is the root group of L normalized by $L \cap M$, and $C_L(V) = C_L([V, L \cap M]) = (L \cap M)^\infty$, with a complement X to $C_L(V)$ in $L \cap M$ where X is cyclic and regular on $[V, L \cap M]^\#$. As $V = U \times [V, L \cap M]$, by a Frattini Argument, $H = LN_H([V, L \cap M]) = LM_H$ with $C_H(L) \leq C_H([V, L \cap M]) \leq M_H$, so as $F^*(M_H) = O_2(M_H)$, $F^*(H) = O_2(H)L$. Thus (1), (2) and (3c) hold in this case.

This leaves case (1) of E.6.8 with $n > 1$, where $L \cap M$ is the Borel subgroup of L over $L \cap Q$, $L/Z(L)$ is a Bender group over $\mathbf{F}_{2^n}$ or $L_3(2^n)$, and $V = V_L \times U$, where $V_L := [V, L \cap M] = [V_L, L \cap M] = Z(L \cap Q)$. Arguing as in the previous paragraph, $H = LM_H$ and $F^*(H) = O_2(H)L$, so that (1) and (3b) hold. There is a cyclic complement X to $C_L(V_L)$ regular on $V_L^\#$, so (2) holds. Thus we have now established (1), (2), and (3) in all cases.

Finally (4) holds, since we may use (2) in each case to see that $[V, L \cap M] = [V, X]$ and hence $C_H(L) \leq C_H([V, X]) = C_H(U[V, X]) = C_H(V) \leq M$. □

We obtain a corollary when U satisfies a somewhat stronger hypothesis than $U \in \Gamma$:

LEMMA E.6.12. *Assume $1 \neq U \leq V$, with $C_M(U) = C_M(V)$ and $C_G(U) \not\leq M$. Then $C_G(U_1) \not\leq M$ for some hyperplane U_1 of V containing U, so $r(G, V) = 1$.*

PROOF. As $C_M(U) = C_M(V)$ and $C_G(U) \not\leq M$, $U \in \Gamma$. Furthermore for $U \leq U_1 \leq V$, $C_M(U) = C_M(V) \leq C_M(U_1) \leq C_M(U)$, so $C_M(U_1) = C_M(V)$. Hence, replacing U by U_1 if necessary, we may assume that U is maximal in V subject to $C_G(U) \not\leq M$. Now assume that U is not a hyperplane of V, so that $m(V/U) > 1$. Then the hypotheses of E.6.11 are satisified, and by E.6.11.2, there is $\bar{X} \neq 1$ in $Aut_{C_G(U)}(V)$, contrary to our hypothesis that $C_M(U) = C_M(V)$. Thus $m(V/U) = 1$, so as $r(G, V) \leq m(V/U)$, $r(G, V) = 1$. □

Our next result shows that members of Γ are not normal in T; it depends on Theorem 3.1.1, which we recall is a version of Theorem C.5.8. We use this result later to establish contradictions in many situations.

PROPOSITION E.6.13. *Assume $1 \neq W \leq V$ with $O^{2'}(C_M(W)) \leq C_M(V)$, and $W \trianglelefteq T$. Then $C_G(W) \leq M$, so $W \notin \Gamma$.*

PROOF. Assume that $C_G(W) \not\leq M$, so that in fact $W \in \Gamma$, and hence we may take $U = W$. In particular Q is Sylow in H by E.6.5.1. As $U \trianglelefteq T$ by hypothesis, T normalizes $C_G(U) = H$, so we may choose $H_1 \in \mathcal{H}_*(T, M)$ (cf. Definition 3.0.1) with $H_1 \leq HT$, and $H_1 \cap M$ is the unique maximal subgroup of H_1 over T by 3.1.3.1. We verify the hypotheses of Theorem 3.1.1 with Q, $N_G(Q)$, H_1 in the roles of "R, M_0, H": First $O^2(H_1) \leq O^2(H)$, so as Q is Sylow in H, Q is Sylow in $O^2(H_1)Q$; by construction $N_G(Q) \in \mathcal{H}(T)$ and $Q \trianglelefteq N_G(Q)$. Now Theorem 3.1.1 shows that $O_2(\langle N_G(Q), H_1 \rangle) \neq 1$. Then as $M = !\mathcal{M}(N_G(Q))$ by Hypothesis E.6.1.3, we conclude $H_1 \leq M$, contrary to the choice of H_1. □

LEMMA E.6.14. *Assume* $1 \neq W \leq V$ *such that*

(i) $O^{2'}(C_M(W)) \leq C_M(V)$.

(ii) $F^*(C_G(W)) = O_2(C_G(W))$, *and either* $m_3(C_G(W)) \leq 1$ *or* $N_G(W)$ *contains a subgroup of order* 3 *faithful on* W.

(iii) $T \leq M_0 \leq N_G(Q)$ *with* $M = !\mathcal{M}(M_0)$, *and* $N_{M_0}(W)$ *is a maximal subgroup of* M_0, *but not of index* 2 *in* M_0.

Then $C_G(W) \leq M$, *so* $W \notin \Gamma$.

PROOF. Assume that $C_G(W) \not\leq M$, so that $W \in \Gamma$, and hence we may take $U = W$. Thus Q is Sylow in $C_G(W) = H$ by E.6.5.1. By (ii), $F^*(H) = O_2(H)$, and either $m_3(H) \leq 1$ or $N_G(W)$ contains an element of order 3 faithful on W. Since $N_G(W)$ is an SQTK-group, we conclude from E.6.7 that $H = (H \cap M)L$ where $L \not\leq M$ is a block of H and $L = O^{3'}(H) \trianglelefteq N_G(W)$.

We will apply C.5.7, so we need to check that Hypotheses C.5.1 and C.5.2 hold, with Q, $H_1 := LQ$ in the roles of "R, H". As $F^*(H) = O_2(H)$ by (ii) and $Q \leq H_1$, $F^*(H_1) = O_2(H_1)$. From the list of groups in E.6.7, $H_1 \cap M$ is the unique maximal subgroup of H_1 containing Q, and Q is Sylow in H_1. Thus Hypothesis C.5.1 is satisfied. By (iii), $M =!\mathcal{M}(M_0)$, so as $H_1 \not\leq M$, Hypothesis C.5.2 is also satisfied.

As $L = O^2(H_1)$ is a block, the parameter s of C.5.7 is indeed 1. By (iii), $N_{M_0}(W)$ is maximal in M_0, so as $L \trianglelefteq N_G(W)$, we have $N_{M_0}(L) = N_{M_0}(W)$ or M_0. In the latter case $W_L := C_V(L)$ is T-invariant and $C_G(W_L) \not\leq M$, contrary to E.6.13. Thus the hypotheses of C.5.7 are satisfied, so we conclude from that lemma that $|M_0 : N_{M_0}(W)| = 2$, contrary to (iii). □

As mentioned earlier, it is useful in various situations to show that $r > 1$; in particular in E.6.23 we will establish this fact when $m > 2$. The next few results impose some restrictions on $C_G(W)$ for W a hyperplane of V—culminating in E.6.22 showing that $F^*(C_G(W)) = O_2(C_G(W))$.

These intermediate results can be useful in other contexts as well, and in particular we will also establish them when $r > 1$. Thus for the remainder of this subsection, we will assume:

HYPOTHESIS E.6.15. *Either* $r(G, V) > 1$ *or* $m := m(\bar{M}_V, V) > 2$.

Let $\mathcal{U}$ denote the set of hyperplanes of V.

LEMMA E.6.16. *Let* $W \in \mathcal{U}$. *Then*

(1) If $C_G(W) \not\leq M$, *then* $r = 1$, $m > 2$, *and* $W \in \Gamma$.

(2) If $O^2(F^*(C_G(W))) \neq 1$ *then* $C_G(W) \not\leq M$.

PROOF. Assume first that $C_G(W) \not\leq M$. Then $r = 1$, so by Hypothesis E.6.15, $m > 2$; then $O^{2'}(C_M(W)) \leq C_M(V)$, and hence $W \in \Gamma$, establishing (1). On the other hand if $O^2(F^*(C_G(W)) \neq 1$, then $C_G(W) \not\leq M$ by E.6.6.2, establishing (2). □

We continue the convention that U denotes a member of Γ and $H = C_G(U)$.

LEMMA E.6.17. *If* $r = 1$ *then* $m > 2$ *and* $m(V) \geq 6$.

PROOF. As $r = 1$, $m > 2$ by Hypothesis E.6.15. Now if there is an involution $\bar{t}$ in $\bar{M}_V$, then $m(V/C_V(\bar{t})) \leq m(V)/2$, so that $m(V) \geq 6$. Thus if the lemma fails, $\bar{M}_V$ is of odd order, so that $T = C_T(V) = Q$. Then any $W \in \mathcal{U}$ is normal in T, and as $m > 2$, $O^{2'}(C_M(W)) \leq C_M(V)$. Hence $C_G(W) \leq M$ for each $W \in \mathcal{U}$ by E.6.13, so that $r > 1$, contrary to the hypothesis of the lemma. □

LEMMA E.6.18. *If $m(V/U) \leq 2$ then H has no component isomorphic to A_7 or $\hat{A}_7$.*

PROOF. Assume that L is a component of H isomorphic to A_7 or $\hat{A}_7$. Let $U_L := C_V(L)$, so that $U \leq U_L \in \Gamma$ since $U \in \Gamma$. Now $L \leq L_U \in \mathcal{C}(C_G(U_L))$ by 1.2.4, and L is Q-invariant by 1.2.1.3, so L_U centralizes $O_2(C_G(U_L))$, and hence L_U is a component of $C_G(U_L)$. Then by comparing the lists in E.6.8 and A.3.12, $L = L_U$, so without loss, $U = U_L = C_V(L)$.

By E.6.8, $r = 1$, so by Hypothesis E.6.15, $m > 2$—and then $m(V) \geq 6$ by E.6.17. Let $g \in M_V$; then $U_1 := U \cap U^g \leq V$, and we set $H_1 := C_G(U_1)$. As $U = C_V(L)$, $m(V/U) \leq 2$ by E.6.8.4, so that $m(V/U_1) \leq 4$. Therefore as $m(V) \geq 6$, $U_1 \neq 1$.

Next $H = C_G(U) = C_{H_1}(U)$, so L is a component of $C_{H_1}(U)$. As $m_3(L) = 2$, I.3.1.3 says $L \leq L_1$ for some normal component L_1 of H_1, and L is a component of $C_{L_1}(U)$. But by inspection of the choices for L_1 in Theorem C (A.2.3), no nontrivial 2-subgroup of $Aut(L_1)$ has a centralizer with a component isomorphic to A_7 or $\hat{A}_7$. Thus $L_1 = L$ is a component of H_1, and by symmetry, so is L^g. Then by A.3.18, $L = O^{3'}(E(H_1)) = L^g$.

As this holds for all $g \in M_V$, it follows that M_V acts on L and thus also on $C_V(L) = U$, contradicting E.6.13. This completes the proof. □

LEMMA E.6.19. *If $W, W_0 \in \mathcal{U}$ then $O(C_G(W)) \leq O(C_G(W \cap W_0))$.*

PROOF. We may assume that $O(C_G(W)) \neq 1$. Therefore by E.6.16, $r = 1$, $m > 2$, and $W \in \Gamma$, so we may take $U = W$. There is nothing to prove if $U = W_0$, so $U_1 := U \cap W_0$ is of corank $2 < m$ in V. Hence $U_1 \in \Gamma$, so we can apply the results of this section to $H_1 := C_G(U_1)$. As $H = C_G(U) = C_{H_1}(U)$, $O(H) = O(C_{H_1}(U))$. We may assume that E.6.19 fails, so $O(C_{H_1}(U)) \not\leq O(H_1)$. Thus setting $\tilde{H}_1 := H_1/O(H_1)U_1$, $\tilde{U}$ is of order 2 and $O(C_{\tilde{H}_1}(\tilde{U})) \neq 1$. Therefore by 31.14.1 and 31.18 in [**Asc86a**], there is a component $\tilde{L} = [\tilde{L}, \tilde{U}] = [\tilde{L}, O(C_{\tilde{H}_1}(\tilde{U}))]$ of $\tilde{H}_1$. But $U \leq V \leq Z(Q)$ and $Q \in Syl_2(H_1)$, while L is on the list of E.6.8, with $O(C_{Aut_{H_1}(L)}(Aut_U(L))) \neq 1$. This forces L to be A_7 or $\hat{A}_7$, contradicting E.6.18 applied to U_1 in the role of "U". This completes the proof. □

LEMMA E.6.20. *$O(C_G(U_+)) = 1$ for each $U_+ \in \mathcal{U}$.*

PROOF. Assume $U_+ \in \mathcal{U}$ with $O(C_G(U_+)) \neq 1$. By E.6.16, $C_G(U_+) \not\leq M$, $r = 1$, $m > 2$, and $U_+ \in \Gamma$, so we may take $U = U_+$.

Let $U_0 \leq U \leq V$ with $m(V/U_0) = 3$; notice $U_0 \neq 1$ since $m(V) \geq 6$ by E.6.17. Set $H_0 := C_G(U_0)$ and $H_0^* := H_0/U_0$. Thus $V^* \cong E_8$. Let $U_0 \leq U_i \leq V$, $i = 1, 2$, with U_1^* and U_2^* distinct subgroups of V^* of order 2. Then $W_1 := U_1U_2 \in \mathcal{U}$, and we may choose $W_0 \in \mathcal{U}$ with $W_1 \cap W_0 = U_1$. We now apply E.6.19 with W_1 in the role of "W" to see that $O(C_G(W_1)) \leq O(C_G(U_1)$, and hence

$$O(C_G(U_1U_2)) = O(C_G(W_1)) = O(C_G(U_1)) \cap C_G(W_1) = O(C_G(U_1)) \cap C_G(U_2).$$

All these centralizers lie in H_0, and $O^2(C_{H_0^*}(U_i^*)) = O^2(C_{H_0}(U_i))^*$, so

$$O(C_{H_0^*}(U_1^*)) \cap C_{H_0^*}(U_2^*) = (O(C_G(U_1)) \cap C_G(U_2))^* = O(C_G(U_1U_2))^*.$$

Thus if we set $\theta(U_i^*) := O(C_{H_0^*}(U_i^*))$, we have

$$\theta(U_1^*) \cap C_{H_0^*}(U_2^*) = O(C_G(U_1U_2))^* = \theta(U_2^*) \cap C_{H_0^*}(U_1^*),$$

with the last equality holding by symmetry between U_1 and U_2. That is (using also the Odd Order Theorem), θ is a solvable V^*-signalizer functor (cf. chapter 15 (section 44) in [**Asc86a**]); hence by the Solvable Signalizer Functor Theorem

$$\langle O(C_G(A^*)) : 1 \neq A^* \leq V^* \rangle = \langle \theta(A^*) : 1 \neq A^* \leq V^* \rangle = \theta(H_0^*)$$

is of odd order. Now U_0 is central in H_0, so the preimage is the direct product of U_0 with

$$X := \langle O(C_G(A)) : U_0 < A \leq V \rangle$$

of odd order. In particular as $O(C_G(U)) \neq 1$, $X \neq 1$.

Recall that $Z = C_V(T)$, let $z \in Z^\#$, and set $G_z := C_G(z)$. As $O(H) \neq 1$, in view of E.6.6.4, $z \notin U$. Hence $z \notin U_0$, so $U_0 < A := \langle U_0, z \rangle < V$. By 1.1.4.6, $F^*(G_z) = O_2(G_z)$, so by 1.1.3.2,

$$1 = O(C_{G_z}(A)) = O(C_G(A)) = X \cap C_G(A) = X \cap G_z,$$

using completeness of the functor (again see chapter 15 (section 44) of [**Asc86a**]). Therefore z inverts X, so X is abelian. Thus if p is a prime divisor of $|X|$,

$$1 \neq Y := O_p(X) = \langle O_p(C_G(W)) : U_0 \leq W \in \mathcal{U} \rangle. \qquad (*)$$

Next as $Y \leq H_0 \in \mathcal{H}$, $m_p(Y) \leq 2$ since G is QTKE-group. Thus $m(V/C_V(Y)) \leq 2$ by A.1.29, and hence $Y = O_p(C_G(U_4))$ for some $U_0 \leq U_4 \leq V$ with $m(V/U_4) \leq 2$.

Now choose $U_4 \geq U_0$ with $m(V/U_4) \leq 2$, so that $Y_1 := O_p(C_G(U_4))$ is of maximal order; notice $Y_1 \neq 1$ as $Y \neq 1$. Now we make a new choice of U: that is, we choose U so that $U_4 \leq U \in \mathcal{U}$ with $C_{Y_1}(U) \neq 1$; this choice is possible since Y_1 is generated by centralizers of such hyperplanes using A.1.17. As $C_{Y_1}(U) \neq 1$, also $O(C_G(U)) \neq 1$. We claim that

$$Y_1 = \langle O_p(C_G(W)) : W \in \mathcal{U} \rangle. \qquad (**)$$

For if not, there is $U_3 \in \mathcal{U}$ with $O_p(C_G(U_3)) \not\leq Y_1$. Then we may choose a new U_0 such that $U_0 \leq U_4 \cap U_3$ and $m(V/U_0) = 3$, and obtain a contradiction from (*) to our choice of U_3. So the claim is proved.

As M_V permutes $\mathcal{U}$, (**) says M_V acts on Y_1, and hence also on $C_V(Y_1)$. But $U_4 \leq C_G(Y_1)$ and $m(V/U_4) \leq 2 < m$, so $C_V(Y_1) \in \Gamma$, contrary to E.6.13. This completes the proof. □

As an easy corollary to E.6.20 we obtain:

Corollary E.6.21. *$O(C_G(A)) = 1$ for all $1 \neq A \leq V$.*

Proof. By Generation by Centralizers of Hyperplanes A.1.17,

$$O(C_G(A)) = \langle O(C_G(A)) \cap C_G(U) : A \leq U \in \mathcal{U} \rangle,$$

and $O(C_G(A)) \cap C_G(U) \leq O(C_G(U))$ for $A \leq U \in \mathcal{U}$, while $O(C_G(U)) = 1$ by E.6.20. □

Lemma E.6.22. *$F^*(C_G(U)) = O_2(C_G(U))$ for each $U \in \mathcal{U}$.*

PROOF. Assume otherwise. Then by E.6.21, there is $U_+ \in \mathcal{U}$ such that $C_G(U_+)$ has a component L. By E.6.16, $r = 1$, $m > 2$, and $U_+ \in \Gamma$, so we may take $U_+ = U$. As $m(V/C_V(L)) = m(V/U) = 1$, E.6.8 says $L/Z(L)$ is A_n, $5 \leq n \leq 8$, $L_2(p)$ for p a Fermat or Mersenne prime, $L_3(3)$, or M_{11}. Let $g \in M_V$ and set $U_0 := U \cap U^g$ and $H_0 := C_G(U_0)$. Then $m(U/U_0) \leq 2 < m$ and $C_G(U) \leq C_G(U_0) \not\leq M$, so as $m(U/U_0) < m$, $U_0 \in \Gamma$. Therefore by E.6.5.1, $Q \in Syl_2(H_0)$ with $U \leq Z(Q)$, so U acts on each component of H_0. Further $O(H_0) = 1$ by E.6.21. Thus I.3.1 shows that L is contained in a component L_0 of H_0. Then L is a component of $C_{L_0}(U)$, so from the list of possibilities for L_0 in E.6.8, and keeping in mind that $U \leq Z(Q)$ so that the centralizer of a 2-central involution has a component, we see that either $L = L_0$, or $L \cong A_5$ and $L_0/Z(L_0) \cong A_7$. The last case is out by E.6.18, so $L = L_0$ is a component of H_0.

By symmetry L^g is a component of H_0, so either $L = L^g$ or $[L, L^g] = 1$. Now we arrive at a contradiction much as we did in the last paragraph of the proof of E.6.18: Namely L can have at most two conjugates under M_V by 1.2.1.3, and these centralize $U_0 \neq 1$. Then M_V normalizes $C_V(\langle L^{M_V}\rangle) \in \Gamma$, so we obtain our usual contradiction from E.6.13. □

We now obtain the main preliminary result involved in the proof of Theorem E.6.3:

PROPOSITION E.6.23. *If* $m(\bar{M}_V, V) > 2$, *then* $r(G, V) > 1$.

PROOF. As $m > 2$, Hypothesis E.6.15 holds. Assume that also $r = 1$. Then there is $W \in \mathcal{U}$ with $C_G(W) \not\leq M$, and by E.6.16.1, $W \in \Gamma$, so we take $U = W$. By E.6.22, $H \in \mathcal{H}^e$, so by E.6.10, $H = M_H L$, where L is the central product of A_{n_i}-blocks $L_1, \ldots, L_s$, with $s \leq 2$, $L_i \not\leq M$, and $n_i = 3$ or 5. Let $U_0 \leq U$ with $m(V/U_0) = 2$ and set $H_0 := C_G(U_0)$. Again as $m > 2$ and $H_0 \not\leq M$, $U_0 \in \Gamma$, so $Q \in Syl_2(H_0)$ by E.6.5.1. Now $L_i \in \mathcal{L}(H_0, Q)$, so by 1.2.4, $L_i \leq X_i$ with $X_i \in \mathcal{C}(H_0)$. As $L_i \not\leq M$, also $X_i \not\leq M$. Fix $i \in \{1, \ldots, s\}$ and set $X := X_i$ and $K := \langle Q, X\rangle$. By E.6.21, $O(H_0) = 1$, so $O(K) = 1$. Thus if $E(K) \neq 1$ then X is a component of H_0, and hence is described in E.6.8, while if $E(K) = 1$, then $F^*(K) = O_2(K)$, and we apply E.6.7 to see that X is a χ_0-block of H_0.

Assume $L_i < X$. Then $U_0 = C_V(X)$ is of corank 2 in V. If X is a χ_0-block, then E.6.7 shows that X is an $L_2(4)$-block, which is impossible since $L_i \leq C_X(U)$ and U is a hyperplane of V. Therefore X is a component of H_0, so by E.6.8, $X/Z(X) \cong L_2(4)$, $L_3^\epsilon(4)$, $Sp_4(4)$, A_6, or A_7. However $X/Z(X)$ is not A_7 by E.6.18, and as $O(H_0) = 1$, X is not $\hat{A}_6$. As the A_{n_i}-block L_i is normal in $C_X(U)$ and U is a hyperplane of V, we conclude that $X \cong A_6$.

Claim $X = \langle L_i^{M_V}\rangle$. For suppose $L_i^g \not\leq X$ for some $g \in M_V$, and set $U_1 := U_0 \cap U^g$ and $H_1 := C_G(U_1)$. By E.6.17, $m(V) \geq 6$, so $U_1 \neq 1$. Further $H_0 \leq H_1$, so by I.3.1.3, X is contained in a component Y of H_1 which is normal in H_1. Next Y appears in Theorem C, and $X \cong A_6$ is a component of the centralizer in $Aut_G(Y)$ of the subgroup $Aut_{U_0}(Y)$ of order 2, so $Y \cong L_4(2)$ or $L_5(2)$, and U_0 induces an involutory outer automorphism on Y. But we can also take $U_1 \leq U_2 \leq V$ with $m(V/U_2) = 2$ and U_2 inducing a 2-central involutory automorphism on Y. Therefore $U_2 \leq Z(R)$ for some $R \in Syl_2(YV)$, so $R \leq O^{2'}(C_G(U_2)) \leq C_G(V)$ since $m > 2$, impossible as U_0 centralizes no Sylow 2-subgroup of Y. This completes the proof of the claim.

We have shown that $X = \langle L_i^{M_V} \rangle$, so that X is M_V-invariant. Then as in earlier arguments, E.6.13 supplies a contradiction.

This contradiction shows that $L_i = X_i$, so X_i is a χ_0-block of H_0. But if $g \in M_V$, we can take U_0 to be $U \cap U^g$, and conclude L_i and L_i^g are blocks of H_0, so that $L_i = L_i^g$ or $[L_i, L_i^g] = 1$ since distinct blocks commute by C.1.9. We complete the proof just as we completed the proof of E.6.22: $Y := \langle L_i^{M_V} \rangle$ is the central product of the M_V-conjugates of L_i, which lie in $N_G(L_i)$, and there are at most two factors by 1.2.1.3. Therefore $m(V/C_V(Y)) \leq 2 < m$, so that $C_V(Y) \in \Gamma$ is T-invariant, contradicting E.6.13. This completes the proof. □

In the remainder of the subsection, we complete the proof of Theorem E.6.3. As we pointed out just after Definition E.6.4, we may choose $U \in \Gamma$ with $H = C_G(U) \not\leq M$ and $m(V/U) = r < m$. As $m > 2$, Hypothesis E.6.15 holds, and $r > 1$ by E.6.23. As $m(V/U) = r$, E.6.11 says $H = M_H L$, where L is either a normal component of H, or an $L_2(2^n)$-block of H for $n > 1$.

We begin by obtaining restrictions on the structure of $\bar{M}_V$. As $m(V/U) = r$, $U = C_V(L)$; and by E.6.11.2, $V = U \times [V, X]$ where $\mathbf{Z}_{2^r-1} \cong X \leq L \cap M$ with $X \cong \bar{X} \trianglelefteq N_{\bar{M}}(U)$, and X regular on $[V, X]^\#$. These are the hypotheses of 10.4 in [**Asc81c**, 10.4] (which is a slight extension of O'Nan's Lemma 14.2 in [**GLS96**, 14.2]), and that result says:

LEMMA E.6.24. *(1)* $\bar{Y} := \langle \bar{X}^{M_V} \rangle = \bar{Y}_1 \times \cdots \times \bar{Y}_s$ *and* $V = C_V(\bar{Y}) \times [V, \bar{Y}]$ *with* $[V, \bar{Y}] = V_1 \times \cdots \times V_s$*, where* $V_i = [V_i, \bar{Y}_i]$*; and* M_V *permutes the* $\bar{Y}_i$ *transitively.*

(2) Either we may take $\bar{X} = \bar{Y}_1$*; or* $\bar{X} \cong \mathbf{Z}_3$, $\bar{Y}_1 = \langle \bar{X}^{\bar{Y}_1} \rangle$ *is the Frobenius group* $Frob_{21}$ *of order* 21*, and* $m(V_i) = 3$.

We can then reduce to a very restricted situation:

PROPOSITION E.6.25. *(1)* $\bar{M}_V = \bar{Y}_1$ *is a Frobenius group* $Frob_{21}$ *of order* 21*.*

(2) $m(V) = 3$ *and* U *is of order* 2*.*

(3) $T = Q$*.*

PROOF. Note that $C_V(\bar{Y}) \leq C_V(\bar{X}) = U$ as $V = U \times [V, X]$. Suppose $C_V(\bar{Y}) \neq 1$. Then as $Y M_V \leq N_G(C_V(\bar{Y}))$, with $M = !\mathcal{M}(M_V)$ in view of Hypothesis E.6.1, we have $C_G(U) \leq C_G(C_V(\bar{Y})) \leq M$, contradicting our choice of $U \in \Gamma$. So $C_V(\bar{Y}) = 1$, and hence $V = [V, \bar{Y}] = V_1 \times \cdots \times V_s$ by E.6.24.1.

Assume next that $s = |\bar{Y}_1^{M_V}| > 1$. Then $Y_0 := Y_2 \cdots Y_s$ centralizes $\bar{Y}_1$, and hence also $\bar{X}$ by E.6.24.2, so Y_0 normalizes $C_V(X) = U$. Now as $L \in \mathcal{C}(H)$, $Y_0 \leq O^2(N_G(U)) \leq N_G(L)$ by 1.2.1.3. As $[\bar{Y}_0, \bar{X}] = 1$, $[Y_0, X] \leq O_2(L \cap M)$. Thus by symmetry, $Y_0 X$ contains the direct product of s copies of X, so as $m_p(N_G(U)) \leq 2$ for p a prime divisor of $2^r - 1$, $s = 2$, and hence $V = V_1 \times V_2$.

Suppose that $\bar{X} = \bar{Y}_1$. If $s = 2$, then $V = V_1 V_2$ with $V_2 = C_V(\bar{Y}_1) = C_V(\bar{X}) = U$; thus

$$m(U) = m(V/U) = m(V)/2 \geq m,$$

so $r = m(V/U) \geq m$, contrary to our assumption that $r < m$. Thus $s = 1$, so $U = C_V(\bar{X}) = C_V(\bar{Y}_1) = 1$, contradicting $U \in \Gamma$.

Therefore by E.6.24.2, $\bar{Y}_1 \cong Frob_{21}$, with $m(V_1) = 3$, and $r = m(V/U) = m([V, X]) = 2$. As $\bar{Y}_1$ is maximal in $GL(V_1) \cong L_3(2)$ and subnormal in $\bar{M}_V$, $\bar{Y}_1 = Aut_{M_V}(V_1)$. If $s = 1$, then $m(V) = 3$ and $m(U) = 1$, with $\bar{M}_V = \bar{Y} \cong Frob_{21}$; then $T = Q = C_T(V)$ and the lemma holds. Thus we may assume that $s = 2$, and it remains to derive a contradiction.

As $m(V/U) = 2$, E.6.11 says that L is either an $L_2(4)$-block of H, or a component of H with $L/Z(L) \cong L_2(4)$, $L_3^\epsilon(4)$, or $Sp_4(4)$.

Set $H_2 := C_G(V_2)$. Then V_2 is a hyperplane of U. Thus $H \leq H_2$, and $O(H_2) = 1$ by E.6.21. Furthermore we saw $\bar{Y}_1 = Aut_{M_V}(V_1)$ is of odd order, so as $C_M(V_2) \leq C_{M_V}(V_2)$ since V is a TI-subgroup of M, we see $O^{2'}(C_M(V_2)) \leq C_M(V)$ and so $V_2 \in \Gamma$; in particular, Q is Sylow in H_2 by E.6.5.1. Then $L \in \mathcal{L}(H_2, Q)$, so by 1.2.4, $L \leq L_2 \in \mathcal{C}(H_2)$. Now if L is a component of H, then as $O(H_2) = 1$, I.3.1.2 says that L_2 is a component of H_2. Next $U \leq V \leq Z(Q)$, and inspecting the list in E.6.8 for a group admitting a 2-central involution whose centralizer contains a component isomorphic to L, we conclude that if L is a component then either $L = L_2$ or L is $L_2(4)$ and $L_2/Z(L_2)$ is A_7. On the other hand if L is an $L_2(4)$-block, then a similar inspection of the list in E.6.8 shows that either $L_2 = L$ or $L_2 \cong Sp_4(4)$. Now $X \leq L$ and $Y_1 \leq C_G(V_2) = H_2$, so $Y_1 = O^2(Y_1)$ normalizes L_2 by 1.2.1.3. Hence $Y_1 = \langle X^{Y_1} \rangle \leq L_2$, so $C_V(L_2) \leq C_V(Y_1) = V_2$, and then $C_V(L_2) = V_2$ with $m(V/V_2) = 3$. Thus $L \neq L_2$, so as $L_2/Z(L_2) \cong Sp_4(4)$ or A_7, $m(V/C_V(L_2)) \leq 2$ or $m(V/C_V(L_2)) = 4$ by E.6.8. This contradiction completes the proof of E.6.25. □

We are now in a position to derive the final contradiction which establishes Theorem E.6.3. Namely by E.6.25, $T = Q = C_T(V)$, so $U \trianglelefteq T$, contrary to E.6.13. So the proof is complete.

E.6.2. Bounding r via fixed points of $(\mathbf{F-1})$-offenders or odd-order elements. Notice that when $m > 2$, we have $r \geq m > 2$ by Theorem E.6.3. Even when $m \leq 2$, we may be able to show that $r > 1$ by other means. In the results in this subsection, we will assume that $r > 1$, and prove some results which increase that bound in certain circumstances.

We begin with an easy corollary of E.5.2, relating r to $n(H)$ for suitable subgroups H of G, and to $(F-j)$-modules (ie. modules V with $J_j(T) \not\leq C_T(V)$), usually in the case $j = 1$.

LEMMA E.6.26. *Assume G is a finite group, $T \in Syl_2(G)$, V is a nontrivial normal elementary abelian subgroup of T, $Q := C_T(V)$, and $N_G(V) \leq M = !\mathcal{M}(N_G(Q))$. Further assume j is a positive integer, $H \leq G$ with $n(H) \leq j$, $S \in Syl_2(H)$ with $Q \leq S \leq T$, and*

(1) $r(G, V) > j$.

(2) $J_j(S) \leq C_S(V)$.

Then $H \leq M \geq N_G(S)$.

PROOF. Applying E.5.2 to H, 0 in the roles of "G,i":

$$H = \langle N_H(J(S)), J(C_H(D_1)) : D_1 \in \mathcal{D} \rangle, \qquad (*)$$

where $\mathcal{D}$ consists of the subgroups D_1 of corank at most j in $D := \Omega_1(Z(J_j(S)))$. By B.2.3.1, $J(S) \leq J_j(S)$, and by hypothesis (2), $J_j(S) \leq C_S(V) \leq Q$. Hence $J(S) = J(Q)$ (e.g., B.2.3.3), so $N_G(Q) \leq N_G(J(S))$. Then as $M = !\mathcal{M}(N_G(Q))$, $N_H(J(S)) \leq M \geq N_G(S)$, so by (*), it remains to show that $J(C_H(D_1)) \leq M$ for each $D_1 \in \mathcal{D}$.

Next by hypothesis (2), $V \leq C_S(J_j(S))$, so by B.2.3.2, $V \leq \Omega_1(Z(J_j(S))) = D$. Then $J(C_H(D_1))$ centralizes $V_1 := V \cap D_1$ with $m(V/V_1) \leq m(D/D_1) = j$, so by hypothesis (1), $J(C_H(D_1)) \leq C_G(V_1) \leq N_G(V) \leq M$. This completes the proof. □

Our next lemma extends a result for the case $j = 1$ in Mason (2.2.10 in [**Mas**])—where the proof in effect made a delicate analysis of the possibilities for a quasithin minimal-parabolic H (cf. B.6.8 and E.2.2). [2] However the concepts and machinery in section E.1 make possible a much more natural statement and proof. Finally a suitable result should hold for arbitrary p in place of 2.

PROPOSITION E.6.27. *Assume Hypothesis E.6.1, let $1 \neq U \leq V$, and assume:*
(1) $r(G,V) > j \geq 1$.
(2) $J_j(C_M(U)) \leq C_M(V)$.
(3) $n'(C_{\bar{M}_V}(U)) \leq j$.
(4) If $1 \neq X$ is of odd order in $C_M(V)$, then $N_G(X) \leq M$.
Then $C_G(U) \leq M$.

PROOF. Recall $C_M(U) \leq M_V$ as V is a TI-subgroup of M by Hypothesis E.6.1.2. Thus replacing U by an M_V-conjugate if necessary, we may assume that $S := C_T(U)$ is Sylow in $C_M(U) = C_{M_V}(U)$. By hypothesis (2), $J_j(S) \leq C_S(V) =: Q$, so an argument in the proof of E.6.26 shows that $N_G(S) \leq N_G(J(S)) \leq M$, and hence S is Sylow in $C_G(U)$.

Set $K := C_G(U)$ and $K_1 := O^{2'}(K) = \langle S^K \rangle$. By a Frattini Argument, $K = K_1 N_K(S)$, so as $N_G(S) \leq M$, it suffices to show that $K_1 \leq S$. Now $K_1 = O^{2'}(K_1)$, so by B.6.5, it suffices to show that each minimal parabolic H of K_1 is contained in M.

If $n(H) \leq j$, then the hypotheses of E.6.26 are satisfied, and hence $H \leq M$ by that lemma. Thus we may assume that $n(H) > j$, so in particular $n(H) \geq 2$. Hence H is nonsolvable by E.1.13. Then E.2.2 says $H/O_2(H)$ is an extension of a group $L/O_2(H)$ of Lie type (possibly a product of two Bender groups) over $\mathbf{F}_{2^{n(H)}}$. In particular, $M \cap H/O_2(H)$ contains a Borel subgroup of $L/O_2(H)$, so $H \cap M$ contains a cyclic subgroup B of odd order $2^{n(H)} - 1$. If B is faithful on V, then by Definition E.3.37 $n'(C_{\bar{M}_V}(U)) \geq n(H) > j$, contrary to hypothesis (3). Therefore $C_B(V) \neq 1$. Let Y be a Hall $2'$-subgroup of $H \cap M$ containing B; then $C_B(V) \leq C_Y(V) =: X$, so $X \neq 1$. By hypothesis (4), $N_H(X) \leq M$. But this contradicts 4.4.13, so the proof is complete. □

COROLLARY E.6.28. *Assume Hypothesis E.6.1, and:*
(1) $r(G,V) > 1$.
(2) $J(T) \leq Q$.
(3) If $1 \neq X$ is of odd order in $C_M(V)$, then $N_G(X) \leq M$.
Let

$$\alpha := \min\{m(V/U) : U \leq V \quad and \quad J_1(C_{M_V}(U)) \not\leq C_M(V)\},$$

and

$$\beta := \min\{m(V/U) : n'(C_{M_V}(U)) > 1\}.$$

Then $r(G,V) \geq \min\{\alpha, \beta\}$.

PROOF. Let $U \leq V$ with $m(V/U) = r$ and $C_G(U) \not\leq M$. Then by E.6.27, either $J_1(C_{M_V}(U)) \not\leq C_M(V)$ or $n'(C_{\bar{M}_V}(U)) > 1$. In the first case, $r \geq \alpha$, and in the second, $r \geq \beta$. Hence the lemma holds. □

[2] The result is stated for quasithin groups, but presumably holds in general; one would need to check that non-quasithin minimal parabolics with $n(H) = j > 1$ are of Lie type over an extension of $\mathbf{F}_{2^j}$.

CHAPTER F

Weak BN-pairs and amalgams

In this chapter we first review basic definitions and results from the literature on weak BN-pairs, amalgams, and the "amalgam method". Then we prove a number of theorems involving these notions.

In particular we establish sufficient conditions for a completion of a weak BN-pair of rank 2 to be a group of Lie type. We use such theorems to identify groups during the proof of the Main Theorem. For example we use them to identify the groups arising in the Generic Case. Since these recognition theorems are of independent interest, we develop them in appropriate generality.

In the final three sections of the chapter we record the basic lemmas used in our version of the amalgam method. The results in section F.7 are either well known or slight variations on well known results, but sections F.8 and F.9 are more specialized; and the approach in F.9.16 and F.9.18, which produces an internal module with $q \leq 2$, is probably new.

Sections 28 and 29 of [**GLS96**] contain more discussion of amalgams and the amalgam method, and section 36 in [**Asc94**] discusses amalgams.

F.1. Weak BN-pairs of rank 2

At various points, particularly in the treatment of the Generic Case, we require results on weak BN-pairs of rank 2 from the "Green Book" of Delgado-Goldschmidt-Stellmacher [**DGS85**]; so in section F.1 we review some of their theory.

The definition of a weak BN-pair of rank 2 is given in Hypothesis A in [**DGS85**]. One example of a weak BN-pair is the triple of parabolics over a fixed Borel subgroup in a finite group of Lie type and Lie rank 2. We will begin instead by stating hypotheses better suited for applications in our work. Then we will that verify our hypotheses lead to Hypothesis A of [**DGS85**].

Thus throughout this section, we will assume the following hypothesis:

Hypothesis F.1.1. *G is a group, and L_1, L_2, S are subgroups of G; set $G_0 := \langle L_1, L_2, S\rangle$, and for $i = 1$ and 2, and $j := 3 - i$ assume:*

(a) S is a finite 2-subgroup of $N_G(L_i)$.
(b) $S_i := S \cap L_i \in Syl_2(L_i)$.
(c) $L_i/O_2(L_i) \cong L_2(2^{n_i})$, $Sz(2^{n_i})$, $(S)U_3(2^{n_i})$, or D_{10}.
(d) $B_j := N_{L_j}(S_j) \leq N_G(L_i)$.
(e) $O_2(G_0) = 1$.
(f) $F^*(L_iSB_j) = O_2(L_iSB_j)$.

Remark F.1.2. Notice by (c) we may write $B_j = S_jD_j$ where D_j is a *Cartan subgroup* of L_j; that is, D_j is a Hall $2'$-subgroup of B_j. Then in (f) we may write

$L_iSB_j = L_iSD_j$, and to check condition (d), it suffices to verify $D_j \leq N_G(L_i)$ in view of (a).

LEMMA F.1.3. *For $i = 1$ and 2, and $j := 3 - i$:*

(1) B_i is the unique maximal subgroup of L_i containing S_i, and $B_i/O_2(L_i)$ is a Borel subgroup of $L_i/O_2(L_i)$. In particular, $O_2(B_i) = S_i$.

(2) $B_j = N_{L_j}(L_i)$.

(3) B_j acts on B_i, and hence on S_i.

(4) $B := B_1B_2S = N_{L_iSB_j}(S_i)$ is solvable.

(5) We may choose D_i and D_j so that D_i normalizes D_j. Then $B = SD_1D_2$ with D_1D_2 an abelian Hall $2'$-subgroup of B.

(6) $L_1B \cap L_2B = B$.

PROOF. By F.1.1.b, $S_i \in Syl_2(L_i)$. Then since the groups $L_i/O_2(L_i)$ listed in F.1.1.c are groups of Lie type in characteristic 2 of Lie rank 1 (including D_{10} regarded as truncated version of $Sz(2)$), the statements in (1) follow.

If (2) fails then as B_j is maximal in L_j, L_j acts on L_i, so $O_2(L_i) \trianglelefteq G_0$. But by (f), $O_2(L_i) \neq 1$, so (e) supplies a contradiction. This establishes (2).

By (a) and (d) of F.1.1, SB_j acts on L_i, so SB_j acts on $SB_j \cap L_i$ and of course $S_i = S \cap L_i \leq SB_j \cap L_i$. By (2), $SB_j \cap L_i \leq N_{L_i}(L_j) = B_i$, so by (1), B_i is the unique maximal subgroup of L_i containing $SB_j \cap L_i$. Thus B_j acts on B_i, so (3) holds.

By (c), B_i is 2-closed and D_i is abelian. Thus B_i is solvable and by (3), B_2 acts on B_1, so B_1B_2 is solvable. Finally S acts on B_i by (1) and hence on B_1B_2, so B is a solvable group. By construction, B acts on S_i, so as $B_i = N_{L_i}(S_i)$, $B = N_{L_iB}(S_i) = N_{L_iSB_j}(S_i)$, completing the proof of (4).

Visibly $B \leq BL_1 \cap BL_2$ and by (1), B is maximal in L_1B, while by (2), $BL_1 \cap BL_2$ is proper in BL_1, so (6) holds.

By Hall's Theorem D_2 is contained in a Hall $2'$-subgroup D of B, and as $B_1 \trianglelefteq B$, $D \cap B_1 = D_1$ is Hall in B_1 and of course $D_1 \trianglelefteq D$. Similarly $D_2 \trianglelefteq D$ and as B/B_1B_2 is a 2-group, $D = D_1D_2$. As $D_2 \trianglelefteq D$ and D_2 is abelian, D_2 centralizes $[D_1, D_2]$. But we may apply Coprime Action to the field automorphisms of $Out(L_1/O_2(L_1))$ acting on D_1: since for an odd prime p, p does not divide $2^p - 1$ which does divide $2^{pr} - 1$—hence we conclude that $[D_1, D_2] = 1$. Thus D is abelian and (5) holds. □

For the remainder of the section, we continue from F.1.3 the abbreviation

$$B := B_1B_2S.$$

DEFINITION F.1.4. A *rank 2 amalgam* of groups is a pair $\alpha := (\alpha_1, \alpha_2)$ of group homomorphisms $\alpha_i : G_{1,2} \to G_i$, $i = 1, 2$. A *morphism* $\phi : \alpha \to \alpha'$ of amalgams is a triple $\phi = (\phi_1, \phi_{1,2}, \phi_2)$ of group homomorphisms $\phi_J : G_J \to G'_J$ such that $\alpha_i\phi_i = \phi_{1,2}\alpha'_i$ for $i = 1, 2$. As usual $Aut(\alpha)$ denotes the group of automorphisms of α.

REMARK F.1.5. See section 36 in [**Asc94**] for a discussion of amalgams. In particular by Example 36.2 in [**Asc94**], if X_1 and X_2 are subgroups of a group X, then the inclusion maps $\iota_i : X_{1,2} := X_1 \cap X_2 \to X_i$ define a rank 2 amalgam which we denote by $(X_1, X_{1,2}, X_2)$, and call a *subgroup amalgam* (see also Definition F.2.9). For example under Hypothesis F.1.1, $\alpha = (L_1B, B, L_2B)$ is a rank 2 subgroup amalgam.

DEFINITION F.1.6. A *completion* of an amalgam α is a group X together with homomorphisms $\phi_J : G_J \to X$, such that $(G_1\phi_1, G_{1,2}\phi_{1,2}, G_2\phi_2)$ is a subgroup amalgam, $\phi := (\phi_1, \phi_{1,2}, \phi_2)$ is a morphism from α to this subgroup amalgam, and $X = \langle G_1\phi_1, G_2\phi_2\rangle$. The completion is *faithful* if each ϕ_J is injective.

By Lemma 36.4 in [**Asc94**], there is a *universal completion* of α (see also Definition F.2.2). For example under Hypothesis F.1.1, G_0 together with the inclusion maps is a faithful completion of the amalgam $\alpha = (L_1B, B, L_2B)$, so G_0 is a homomorphic image of the universal completion of the amalgam α. We often refer to the group G_0 as a "completion" of α, when there is no danger in suppressing the associated mappings.

The notion of "amalgam" in Definition F.1.4 is more general than that used on page 61 of [**DGS85**], since we do not assume condition (A3) in (2.0) of [**DGS85**] that $\ker_{X_{1,2}}(X) = 1$. Of course we do include that assumption in part (e) of Hypothesis F.1.1. Parts (c), (e), and (f) of Hypothesis F.1.1 say that the amalgam (L_1B, B, L_2B) resembles the amalgam of parabolics in a group of Lie type of Lie rank 2. As we will see in F.1.9, they ensure that the amalgam is a weak BN-pair of rank 2 in the sense of the following definition:

DEFINITION F.1.7. A *weak BN-pair of rank 2* is a triple $\alpha = (P_1, B, P_2)$ of subgroups of a group X, such that $X = \langle P_1, P_2\rangle$, $B = P_1 \cap P_2$, $\ker_B(X) = 1$, and the triple satisfies Hypothesis A on page 94 in the Green Book [**DGS85**].

For us, the prime appearing in Hypothesis A will always be 2.

REMARK F.1.8. Notice that the condition $\ker_B(X) = 1$ is included in the remark following Hypothesis A on page 94 of [**DGS85**]—since this condition is part of the definition of "amalgam" used in [**DGS85**].

PROPOSITION F.1.9. $\alpha := (L_1B, B, L_2B) = (L_1SD_2, D_1SD_2, D_1SL_2)$ *is a weak BN-pair of rank* 2. *Further* G_0 *is a completion of* α.

PROOF. Set $P_i := L_iB$. By F.1.3.6, $B = P_1 \cap P_2$. Suppose Y is a nontrivial normal subgroup of G_0 contained in B. Then $Y \trianglelefteq P_i$, so $O_2(Y) \neq 1$ by part (f) of Hypothesis F.1.1, contrary to part (e) of that Hypothesis. Thus $\ker_B(G_0) = 1$, so it remains to show that α satisfies Hypothesis A of the Green Book.

Set $P_i^* := O_2(P_i)L_i$; by (a) and (d) of F.1.1, $P_i^* \trianglelefteq P_i = L_iSB_j$. By construction, $O_2(P_i) = O_2(P_i^*)$ and $P_i = P_i^*B$, establishing condition (i) of Hypothesis A. Condition (ii) is a restatement of F.1.1.f. Next by construction $S_i^* := O_2(P_i)S_i \in Syl_2(P_i^*)$ with $S_i = S_i^* \cap L_i$, so

$$P_i^* \cap B = O_2(P_i)L_i \cap B = O_2(P_i)(L_i \cap B) = O_2(P_i)B_i = N_{P_i^*}(S_i) = N_{P_i^*}(S_i^*)$$

by F.1.3.4, and

$$P_i^*/O_2(P_i^*) = L_iO_2(P_i)/O_2(P_i) \cong L_i/O_2(L_i),$$

so condition (iii) follows from F.1.1.c. This completes the verification of the requirements from [**DGS85**]. □

As indicated in section 4 of [**DGS85**], Hypothesis A of [**DGS85**] is maintained if the amalgam (P_1, B, P_2) is replaced by $(P_1^*B_0, B_0, P_2^*B_0)$, where $B_0 := (P_1^* \cap B)(P_2^* \cap B)O_2(B)$. In our setup, this amounts to replacing B by $B_1B_2O_2(B)$ and S by $S \cap B_1B_2O_2(B)$. This minimal situation corresponds essentially to the amalgam of parabolics in a *simple* group $\bar{L}$ of Lie type of characteristic 2 and Lie rank 2.

However Hypothesis A is also satisfied by amalgams of extensions of $\bar{L}$ by outer automorphisms. (For example, see the discussion of "Λ^0" on page 99 of [**DGS85**], involving autormphisms trivial on the Dynkin diagram of a rank-2 group). We refer to such an amalgam as an *extension* of the amalgam of $\bar{L}$. Hypothesis B of [**DGS85**] imposes the constraint that $B = B_0$. The following lemma gives some sufficient conditions for Hypothesis B to be satisfied.

LEMMA F.1.10. *(1) If $S = O_2(L_iS)(S \cap L_i)$ for $i = 1$ and 2, then $S \trianglelefteq SB_i$ for each i.*

(2) Assume either

(a) $S \trianglelefteq SB_i$ for $i = 1$ and 2, or

(b) $S \leq L_j$ for $j = 1$ or 2.

Then $S = O_2(B)$, and Hypothesis B of section 4 of the Green Book [**DGS85**] *is satisfied; that is, $B = B_1B_2O_2(B)$.*

PROOF. By definition $B = B_1B_2S$ and $B_i = N_{L_i}(S_i)$. Suppose first that $S = O_2(L_iS)S_i$. Then as B_i acts on S_i and $B_i \leq L_i \leq N_G(O_2(L_iS))$, B_i acts on S, establishing (1).

Assume next that either $S \trianglelefteq SB_i$ for $i = 1$ and 2, or $S \leq L_j$ for $j = 1$ or 2. In the first case, S is normal in $B_1B_2S = B$. In the second case, $S = S_j \trianglelefteq B$ by F.1.3.4. Thus in either case $S = O_2(B)$, since $S \in Syl_2(B)$. Then as $B_i \leq L_i \cap B$, $B = B_1B_2S = B_1B_2O_2(B)$. Since $P_i^* = O_2(P_i)B_i$, this verifies the form of Hypothesis B stated in [**DGS85**]. Thus (2) also holds. □

NOTATION F.1.11. Given a power q of a prime p, q^{1+2w} denotes a special p-group of order q^{1+2w} with center of order q.

The amalgams of weak BN-pairs are isomorphic as amalgams if and only if the weak BN-pairs are "locally isomorphic" in the sense of [**DGS85**]. Define two amalgams α and α' to be *parabolic isomorphic* if there exist group isomorphisms $\phi_J : G_J \to G'_J$ for $J = \{1\}, \{2\}, \{1, 2\}$. In particular, isomorphic amalgams are parabolic isomorphic, but the converse need not hold. Theorem A of [**DGS85**] does not determine each weak BN-pair up to isomorphism of amalgams; rather in several cases the amalgam is determined only up to a weaker equivalence relation such as parabolic isomorphism. Fortunately all but two of the cases left open by Theorem A of [**DGS85**] and relevant to us (cases (10) and (11) below) are shown to be unique up to isomorphism of amalgams in [**Gol80**] and [**Fan86**]. Collecting these results we obtain:

PROPOSITION F.1.12. *Assume $S \trianglelefteq SB_j$ for $j = 1$ or 2, and G_0 is quasithin. Let $\alpha := (L_1B, B, L_2B)$. Then*

(I) One of the following holds, where we choose notation so that $[L_1, Z(S)] = 1$ whenever $Z(S)$ centralizes L_1 or L_2:

(1) α is the $L_3(q)$-amalgam and either L_1 and L_2 are $L_2(q)$-blocks, or $q = 2$ and $L_1 \cong L_2 \cong S_4$.

(2) α is the $Sp_4(q)$-amalgam and either L_1 and L_2 are $L_2(q)$-blocks, or $q = 2$ and $L_1S \cong L_2S \cong \mathbf{Z}_2 \times S_4$.

(3) α is the $G_2(q)$-amalgam, $L_i/O_2(L_i) \cong L_2(q)$, $i = 1, 2$, $O_2(L_1S) \cong q^{1+4}$, and $|O_2(L_2S)| = q^5$.

(4) α is the ${}^3D_4(q)$-amalgam, $L_1/O_2(L_1) \cong L_2(q^3)$, $O_2(L_1) \cong q^{1+8}$, $L_2/O_2(L_2) \cong L_2(q)$, and $|O_2(L_2)| = q^{11}$.

(5) α is the ${}^2F_4(q)$-amalgam, $L_1/O_2(L_1) \cong Sz(q)$, $|O_2(L_1)| = q^{10}$, $L_2/O_2(L_2) \cong L_2(q)$, and $|O_2(L_2)| = q^{11}$.

(6) α is the $U_4(q)$-amalgam or its extension of degree 2, $L_1/O_2(L_1) \cong L_2(q)$, $O_2(L_1) \cong q^{1+4}$, and L_2 is an $\Omega_4^-(q)$-block.

(7) α is the $U_5(4)$-amalgam, $L_1/O_2(L_1) \cong SU_3(4)$, $O_2(L_1) \cong 4^{1+6}$, $L_2/O_2(L_2) \cong L_2(16)$, and $|O_2(L_2)| = 2^{16}$.

(8) α is the $G_2(2)'$-amalgam, $L_i/O_2(L_i) \cong L_2(2)$, $O_2(L_1S) \cong \mathbf{Z}_4 * Q_8$, and $O_2(L_2S) \cong \mathbf{Z}_4^2$.

(9) α is the ${}^2F_4(2)'$-amalgam, $L_1/O_2(L_1) \cong D_{10}$, $|O_2(L_1)| = 2^{10}$, $L_2/O_2(L_2) \cong L_2(2)$, and $|O_2(L_2)| = 2^{10}$.

(10) α is parabolic-isomorphic to the J_2-amalgam, $L_1/O_2(L_1) \cong L_2(4)$, $O_2(L_1) \cong Q_8D_8$, $L_2/O_2(L_2) \cong L_2(2)$, and $|O_2(L_2)| = 2^6$.

(11) α is parabolic-isomorphic to the $Aut(J_2)$-amalgam, $L_1S/O_2(L_1S) \cong S_5$, $O_2(L_1) \cong Q_8D_8$, $L_2/O_2(L_2) \cong S_3$, and $|O_2(L_2)| = 2^6$.

(12) α is the M_{12}-amalgam, $L_1/O_2(L_2) \cong L_2(2)$, $O_2(L_2) \cong Q_8^2$, $L_2/O_2(L_2) \cong L_2(2)$, and $|O_2(L_2)| = 2^5$.

(13) α is the $Aut(M_{12})$-amalgam, $L_1/O_2(L_1) \cong L_2(2)$, $O_2(L_1) \cong \mathbf{Z}_4 * Q_8^2$, $L_2/O_2(L_2) \cong L_2(2)$, and $|O_2(L_2)| = 2^6$.

(II) Either

(i) $L_i = L_i^\infty$ and $S \leq L_i$ for $i = 1$ and 2, or

(ii) α is the amalgam of $L_3(2)$, $Sp_4(2)$, $G_2(2)$, ${}^3D_4(2)$, ${}^2F_4(2)$, $U_4(2)$, $G_2(2)'$, J_2, $Aut(J_2)$, M_{12}, $Aut(M_{12})$, or the extension of the $U_4(q)$-amalgam of degree 2.

PROOF. By F.1.9, α is a weak BN-pair of rank 2. Then Theorem A in the Green Book [**DGS85**] says that α resembles the amalgam of the extension of a simple group $\bar{L}$. Work of Fan [**Fan86**] and Goldschmidt [**Gol80**] shows that the weaker conclusions in the cases where $\bar{L}$ is $G_2(2)'$, M_{12}, or ${}^2F_4(2)'$ in (b) and (c) of that result, can be improved to an isomorphism of amalgams. Thus in our terminology, the three papers say that either α is parabolic isomorphic to the amalgam of J_2 or $Aut(J_2)$, or α is isomorphic to an extension of the amalgam of $\bar{L}$, where one of the following holds: $\bar{L}$ is listed in (1)–(6); as in (7), $\bar{L}$ is $U_5(q)$, but possibly with $q \neq 4$; $\bar{L}$ is $G_2(2)'$, M_{12}, or ${}^2F_4(2)'$, and in particular is listed in (8)–(13).

Notice that in cases (8)–(13), $\bar{L}$ is $G_2(2)'$, ${}^2F_4(2)'$, J_2 or M_{12}, and the amalgams of $Aut(\bar{L}) = G_2(2)$, ${}^2F_4(2)$, $Aut(J_2)$, and $Aut(M_{12})$ are contained in (3), (5), (11), and (13) of our list. Indeed in (8)–(13) we check that all assertions in (I) hold, as does alternative (ii) of (II).

Thus we may assume that we are in one of the first seven cases; in particular α is an extension of a *Lie amalgam* (that is, the amalgam of a simple group $\bar{L}$ of Lie type) by a 2-group, since $B = B_1B_2S$ so that B/B_1B_2 is a 2-group. We will show next that Hypothesis B of the Green Book is satisfied; then we will show that α is actually a Lie amalgam or an extension of the $U_4(q)$-amalgam of degree 2.

Let $\hat{G}$ be the universal completion of α. Then by Theorem A in [**DGS85**], there is a quotient $\bar{G}$ of $\hat{G}$ which is an extension of a rank 2 group $\bar{L} = F^*(\bar{G})$ of Lie type by a 2-group of outer automorphisms. There are isomorphic copies of L_iB in $\bar{G}$, which we denote by $\bar{L}_i\bar{B}$. We may also assume in cases (1)–(6) that $q > 2$, since if $q = 2$ then $\bar{L}$ is the subgroup of $Aut(\bar{L})$ trivial on the Dynkin diagram, so the amalgam of $\bar{L}$ has no proper extensions, and hence α is that amalgam; indeed one of cases (1)–(6) of (I) holds, as well as alternative (ii) of (II).

Assume first that $\bar{G}$ does not contain $L \cong U_4(q)$ extended by an involutory outer automorphism, and that α is not an extension of the $U_5(2)$-amalgam. Because of these hypotheses and our reduction to the case $q > 2$ in (1)–(6) in the previous paragraph, $L_i = L_i^\infty$ and $O_2(\bar{L}_i\bar{S}) = O_2(\bar{L}_i)$ for $i = 1$ and 2, Hence $O_2(L_iS) = O_2(L_i)$. By hypothesis $S \trianglelefteq SB_j$ for some j; thus S induces inner automorphisms on $L_j/O_2(L_j)$, so $S = O_2(L_jS)S_j = O_2(L_j)S_j = S_j$. Thus $S \leq L_j$, and hence $B = B_1B_2O_2(B)$ by F.1.10.2, so that $\bar{B} \leq \langle \bar{L}_1, \bar{L}_2 \rangle \leq \bar{L}$. Then $\bar{G} = \langle \bar{L}_1\bar{B}, \bar{L}_2\bar{B} \rangle = \bar{L}$, so that α is a Lie amalgam. This implies in turn that $\bar{S} \leq \bar{L}_i$ for $i = 1$ and 2, so $S \leq L_i$ and hence alternative (i) of (II) holds. If α is of type $U_5(q)$ for $q > 2$, then (7) holds when $q = 4$, so it remains to eliminate the case $q > 4$. For the other types, one of cases (1)–(6) of (I) holds.

Suppose next that $\bar{G}$ does contain the extension of $\bar{L} \cong U_4(q)$ by an involutory outer automorphism. Here $O_2(\bar{B}) = \bar{S}_1\langle\sigma\rangle$, where σ induces a graph automorphism on $\bar{L}$ (in the convention of Notation 16.1.3), and a field automorphism on $\bar{L}_2/O_2(\bar{L}_2)$. Thus case (6) of (I) holds, and this extension is explicitly allowed in alternative (ii) of (II).

It remains to eliminate the cases where α is an extension of the $U_5(2)$-amalgam, or is the $U_5(q)$-amalgam for $q > 4$. But (as we saw in the proof of Theorem B (A.2.2)) when $q \neq 4$ there is an odd prime divisor p of $q + 1$ such that $m_p(L_1B) = 3$, contradicting our hypothesis that G is quasithin. Thus the deduction of F.1.12 is complete. □

F.2. Amalgams, equivalences, and automorphisms

In the Generic Case of the proof of the Main Theorem (and in several other places), we will produce a weak BN-pair of rank 2 in our group G. Then we apply Theorem A of the Green Book [**DGS85**], via appeals to section F.1, to identify the corresponding amalgam $(G_1, G_{1,2}, G_2)$ as isomorphic to the amalgam of parabolics over a fixed Sylow 2-subgroup of some group of Lie type of rank 2 and characteristic 2. However it then remains to identify $G_0 := \langle G_1, G_2 \rangle$ and G; in the next few sections we develop some machinery to help implement that identification.

In this section we begin by reviewing some notions from the literature on amalgams of groups; then we put in place some formalism not in the literature, but which proceeds along familar lines. This provides a language suitable for discussing completions, equivalences, and automorphisms of amalgams. These concepts and results in turn will allow us (in section F.4) to describe effective sufficient conditions to ensure that a "small" completion G_0 of an amalgam α defined by a group $\bar{G}$ of Lie type is isomorphic to $\bar{G}$. The literature contains some geometric results of this nature, but our principal result Theorem F.4.31 has natural local-group-theoretic hypotheses.

In most of our work, we will be dealing with the rank-2 amalgams from Definitions F.1.4, but in this section we work in a more general setting:

Definition F.2.1. Let D be a poset with partial ordering $\leq$. The reader is referred to Definition 28.1 in [**GLS96**] for the definitions of an amalgam α based on D, morphisms of amalgams, $Aut(\alpha)$, and completions of amalgams.

Notice Definition F.1.4 constitutes the special case of Definition F.2.1 where D is the the poset of nonempty subsets of $\{1,2\}$ ordered by inclusion, and in F.1.6 the maps $X_{1,2} \to X_i$, $i = 1, 2$, defining α are injective.

DEFINITION F.2.2. Let $\xi : \alpha \to gp(\alpha)$ be the *universal completion* of α (see 28.2 in [**GLS96**]). By the universal property of this completion, if $\mu : \alpha \to G$ is any completion of α, then there exists a unique group homomorphism $\hat{\mu} : gp(\alpha) \to G$ such that

$$\hat{\mu} \circ \xi = \mu;$$

Here $\xi = (\xi_d : X_d \to gp(\alpha) : d \in D)$ and $\mu = \hat{\mu} \circ \xi = (\hat{\mu} \circ \xi_d : d \in D)$; we may write μ_d for $\hat{\mu} \circ \xi_d$.

We begin with the formal notions of equivalence and quasiequivalence common to all representation theories. Then eventually we prove in section F.4 that there is a unique quasiequivalence class of "small" completions. Indeed in in most cases all such completions appear to be equivalent.

In section 3 of [**Asc86a**] and section 1 of [**Asc94**], a representation of a group G on an object X in a category $\mathcal{C}$ is defined to be a homomorphism of G into $Aut(X)$. Here our "representations" are representations of amalgams, not groups: that is, they are the completions of the amalgam. An amalgam is a family of groups, and a completion is family of homomorphisms, so we are dealing here with a "higher dimensional" representation theory; but the notions in section 3 of [**Asc86a**] and section 1 of [**Asc94**] are formal and adapt without change to this setting.

Automorphisms of amalgams α and their completions are important in the proof of our Main Theorem in at least two ways: First, in the Generic Case, the Fundamental Setup provides subgroups M_i which may be proper extensions of the parabolics subgroups in a group of Lie type. Second, in Theorem F.4.8, we treat the cases where α is the amalgam of $G_2(2)'$ or ${}^2F_4(2)'$, groups which are of index 2 in groups of Lie type, but are not themselves of Lie type. Using the results in this section, we can show that the automorphisms induced on α in the automorphism group of a group of Lie type must extend to the "small" completions of α; then we use this fact to identify those completions.

We first define the relevant notion of equivalence:

DEFINITION F.2.3. For a fixed amalgam α, an *equivalence* of completions $\mu : \alpha \to G$ and $\eta : \alpha \to H$ is an isomorphism $\varphi : G \to H$ such that $\varphi \circ \mu = \eta$.

LEMMA F.2.4. *Let $\mu : \alpha \to G$ and $\eta : \alpha \to H$ be completions of α. Then*
(1) An isomorphism $\varphi : G \to H$ is an equivalence of μ with η iff $\varphi \circ \hat{\mu} = \hat{\eta}$.
(2) μ is equivalent to η iff $\ker(\hat{\mu}) = \ker(\hat{\eta})$.

PROOF. For the most part this is formal, so we omit proofs. We do however indicate how to construct the isomorphism φ for the converse in (2): Namely assume $K := \ker(\hat{\mu}) = \ker(\hat{\eta})$. We have induced isomorphisms $\dot{\mu} : gp(\alpha)/K \to G$ and $\dot{\eta} : gp(\alpha)/K \to H$, and the canonical surjection $p : gp(\alpha) \to gp(\alpha)/K$. Set $\varphi := \dot{\eta} \circ \dot{\mu}^{-1}$. Then $\dot{\mu} \circ p \circ \xi = \mu$ and $\dot{\eta} \circ p \circ \xi = \eta$, so $\varphi \circ \mu = \eta$. □

Next (as in the "1-dimensional case" for a single group in section 1 of [**Asc94**]) associated to each $a \in Aut(\alpha)$ is a permutation $\pi(a)$ of the set of completions μ of α, defined by

DEFINITION F.2.5.

$$\pi(a)(\mu) = \mu \circ a^{-1} \text{ for } a \in Aut(\alpha).$$

Notice in particular if $\mu : \alpha \to G$, then the completion $\pi(a)(\mu) : \alpha \to G$ possesses the same completion group G. Further π affords a permutation representation of $Aut(\alpha)$ on the set of completions of α, and $\pi(a)$ permutes the equivalence classes of completions, so π induces a permutation representation of $Aut(\alpha)$ on the set of equivalence classes of completions of α.

The action of $Aut(\alpha)$ leads in a standard way to the weaker notion of quasiequivalence:

DEFINITION F.2.6. Given (possibly distinct) amalgams α, γ, define completions $\mu : \alpha \to G$ and $\eta : \gamma \to H$ to be *quasiequivalent* (via φ, ψ) if there exist isomorphisms $\varphi : G \to H$ and $\psi : \alpha \to \gamma$ such that $\varphi \circ \mu = \eta \circ \psi$.

In the case $\alpha = \gamma$ considered above, the isomorphism ψ is in fact an automorphism of α, and μ is quasiequivalent to $\pi(\psi)(\mu)$ via $1, \psi$. Conversely if μ is quasiequivalent to η via φ, ψ, then $\eta = \varphi \circ (\mu \circ \psi^{-1})$, so that η is equivalent to $\pi(\psi)(\mu)$. It follows that the quasiequivalence classes of completions of α are precisely the orbits of $\pi(Aut(\alpha))$ on the classes under equivalence of completions of α.

We observe next that $Aut(\alpha)$ lifts to a group of automorphisms of the universal completion group $gp(\alpha)$. Notice that $\xi \circ a : \alpha \to gp(\alpha)$ is a completion, so:

REMARK F.2.7. There exists a unique homomorphism $\hat{\pi}(a) := \widehat{\xi \circ a} : gp(\alpha) \to gp(\alpha)$ satisfying

$$\xi \circ a = \hat{\pi}(a) \circ \xi.$$

Further $\hat{\pi}$ preserves multiplication in $Aut(\alpha)$, and $\hat{\pi}(a^{-1})$ is an inverse for $\hat{\pi}(a)$, so the map $\hat{\pi} : Aut(\alpha) \to Aut(gp(\alpha))$ is a representation of $Aut(\alpha)$ on $gp(\alpha)$.

The properties in the next lemma follow in a largely formal way from the definitions; see for example Lemma 1.1 in [**Asc94**]. Let $\mu : \alpha \to G$ be a completion of α. Part (4) of lemma F.2.8 tells us that elements of the stabilizer $Aut(\alpha)_\mu$ lift to elements of $Aut(G)$; and part (5) shows that if μ is faithful, then this lift induces an isomorphism of $Aut(\alpha)_\mu$ with the stablizer in $Aut(G)$ of $\mu(\alpha)$.

LEMMA F.2.8. *Let $\mu : \alpha \to G$ be a completion, $a \in Aut(\alpha)$, and $A := gp(\alpha)$. Then*

(1) $\widehat{\pi(a)(\mu)} = \hat{\mu} \circ \hat{\pi}(a^{-1})$.

(2) $\ker(\widehat{\pi(a)(\mu)}) = \hat{\pi}(a)(\ker(\hat{\mu}))$.

(3) $\pi(a)(\mu)$ is equivalent to μ iff $\hat{\pi}(a)$ normalizes $\ker(\hat{\mu})$.

(4) Assume $\pi(a)(\mu)$ is equivalent to μ. Set $A^+ := A/\ker(\hat{\mu})$. Define $\mu^+ : A^+ \to G$ by $\mu^+(g^+) := \hat{\mu}(g)$, where $g \in A$ and $g^+ := g\ker(\hat{\mu})$ is its image under the canonical surjection on A^+. Then

(i) a induces an automorphism a^+ on A^+ defined by $a^+ : g^+ \mapsto [\hat{\pi}(a)(g)]^+$.

(ii) For $x \in X_d$, $a^+(\xi(x)^+) = \xi(a(x))^+$.

(iii) a induces an automorphism a^- on G defined by $a^- := \mu^+ \circ a^+ \circ (\mu^+)^{-1}$.

(iv) For $x \in X_d$, $a^-(\mu(x)) = \mu(a(x))$.

(v) Let $Aut(\alpha)_\mu$ be the stabilizer in $Aut(\alpha)$ of the equivalence class of μ. Then $\pi^- : Aut(\alpha)_\mu \to Aut(G)$ is a representation of $Aut(\alpha)_\mu$ on G, where $\pi^-(a) := a^-$.

(5) Assume μ is faithful, and set

$$B := N_{Aut(G)}(\mu(\alpha)) := \bigcap_{d\in D} N_{Aut(G)}(\mu_d(X_d)).$$

For $b \in B$, define $\sigma_d(b) : X_d \to X_d$ by $\sigma_d(b) := \mu_d^{-1} \circ b \circ \mu_d$. Then

$$\sigma(b) := (\sigma_d(b) : d \in D) \in Aut(\alpha),$$

and $\sigma : B \to Aut(\alpha)$ is a representation of B on α, such that $\pi(\sigma(b))(\mu)$ is equivalent to μ for each $b \in B$. Indeed π^- and σ are inverse mappings, so that $\sigma(B) = Aut(\alpha)_\mu$.

Proof. From the definitions of $\hat{\mu}$, $\hat{\pi}(a)$, and $\pi(a)$ we see

$$\hat{\mu} \circ \hat{\pi}(a) \circ \xi = \hat{\mu} \circ \xi \circ a = \mu \circ a = \pi(a^{-1})(\mu),$$

so that $\widehat{\pi(a^{-1})(\mu)} = \hat{\mu} \circ \hat{\pi}(a)$, and hence (1) holds. Then (1) implies (2). By F.2.4.2, μ is equivalent to $\pi(a)(\mu)$ iff $\ker(\hat{\mu}) = \ker(\widehat{\pi(a)(\mu)})$, so (2) implies (3).

Assume $\pi(a)(\mu)$ is equivalent to μ, and let $K := \ker(\hat{\mu})$ and $p : A \to A^+$ the canonical surjection. Thus $\hat{\mu} = \mu^+ \circ p$. By (3), $\hat{\pi}(a)$ acts on K, and hence induces the automorphism a^+ on $A/K = A^+$ via

$$a^+ \circ p = p \circ \hat{\pi}(a),$$

establishing conclusion (i) of (4). As $\xi \circ a = \hat{\pi}(a) \circ \xi$, for $x \in X_d$, $\hat{\pi}(a)(\xi(x)) = \xi(a(x))$; then applying p and the definition of a^+ establishes (ii). Part (iii) is immediate from (i). We saw $\mu^+ \circ p = \hat{\mu}$, so $\mu^+ \circ p \circ \xi = \hat{\mu} \circ \xi = \mu$. By (ii), $a^+ \circ p \circ \xi = p \circ \xi \circ a$, so

$$a^- \circ \mu = \mu^+ \circ a^+ \circ (\mu^+)^{-1} \circ \mu = \mu^+ \circ a^+ \circ p \circ \xi = \mu^+ \circ p \circ \xi \circ a = \mu \circ a,$$

proving (iv). Part (v) is an easy consequence of (iii), as we vary over all a satisying the hypothesis of part (4).

Assume the hypotheses and notation of (5). As μ is faithful, $\mu_d : X_d \to G$ is injective for each $d \in D$, and by hypothesis, b induces an automorphism of $\mu(X_d) := \mu_d(X_d)$, so $\sigma_d(b) \in Aut(X_d)$. As μ is a completion, for $d \le e \in D$ we have $\mu_d(X_d) \le \mu_e(X_e)$, so as these subgroups are normalized by b, it follows that $\sigma(b) \in Aut(\alpha)$. Visibly σ preserves composition, and hence affords a representation of B on α. Further

$$b \circ \hat{\mu} \circ \xi = b \circ \mu = \mu \circ \mu^{-1} \circ b \circ \mu = \mu \circ \sigma(b) = \pi(\sigma(b^{-1}))(\mu),$$

so $\hat{\theta} = b \circ \hat{\mu}$, where $\theta := \pi(\sigma(b^{-1}))(\mu)$. Therefore as $\ker(\hat{\mu}) = \ker(b \circ \hat{\mu})$ since b is an automorphism, it follows from (2) and (3) that $\pi(\sigma(b^{-1})(\mu)$ is equivalent to μ. Thus $\sigma(b^{-1}) \in Aut(\alpha)_\mu$. Further as μ is faithful, so is π^-. The remaining statements in (5) follow. □

Recall from Definition 28.1 in [**GLS96**]:

Definition F.2.9. Given a group Y, a Y-*amalgam* (based on D) is an amalgam $\gamma = (Y_d : d \in D)$ of subgroups Y_d of Y such that $Y = \langle Y_d : d \in D\rangle$, and for $d \le e$ in D, $Y_d \le Y_e$ and the morphism $Y_d \to Y_e$ is inclusion.

Of course the inclusions $\eta_d : Y_d \to Y$ define a faithful completion η of γ. This terminology allows us to emphasize the group Y in the completion $\eta : \gamma \to Y$, without specifying the map η.

LEMMA F.2.10. *Assume*

(a) D has a least member e.

(b) γ is a Y-amalgam on D for some group Y.

(c) α is an X-amalgam on D where $X \leq Y$, with $X_d \trianglelefteq Y_d$ for each $d \in D$.

Then

(1) Define the conjugation map $c : Y_e \to Aut(\alpha)$ by $c(y)(x) := {}^y x = yxy^{-1}$ for $y \in Y_e$ and $x \in X_d$. Then c affords a representation of Y_e on α.

(2) $\ker(c) = \bigcap_{d \in D} C_{Y_e}(X_d)$.

(3) $\hat{\pi}(c(x)) = c'(\xi(x))$ for each $x \in X_e$, where $c' : gp(\alpha) \to Aut(gp(\alpha))$ is the corresponding conjugation map.

(4) $\hat{\pi}(c(X_e))$ stabilizes the equivalence class of each completion $\mu : \alpha \to G$ of α.

(5) Define $\pi^- : Aut(\alpha)_\mu \to Aut(G)$ as in F.2.8.4.v, and assume μ is faithful. Then $\pi^-(c(x)) = c^-(\mu(x))$ for each $x \in X_e$, where $c^- : G \to Aut(G)$ is the conjugation map.

PROOF. As e is the least element in D by (a), we have $Y_e \leq Y_d$ for all $d \in D$. Thus $Y_e \leq Y_d \leq N_Y(X_d)$ by (c), so $c(y)$ restricts to an automorphism of α, and then the conjugation map c affords a representation of Y_e on α. Thus (1) is established, and then (2) is straightforward. Next for $y \in X_d$ and $x \in X_e$, $c(x)(y) = {}^x y = xyx^{-1}$ and

$$(c'(\xi(x)) \circ \xi)(y) = {}^{\xi(x)}\xi(y) = \xi({}^x y) = (\xi \circ c(x))(y);$$

since $y \in X_d$ and d are arbitrary, we conclude $c'(\xi(x)) \circ \xi = \xi \circ c(x) = \hat{\pi}(c(x)) \circ \xi$, and hence (3) holds. Now $c'(\xi(X_e))$ induces inner automorphisms of $gp(\alpha)$, and hence by (3), $\hat{\pi}(c(X_e))$ preserves the normal subgroup $\ker(\hat{\mu})$, so that (4) follows from F.2.8.3.

Assume the hypotheses and notation of (5), and adopt the notation of F.2.8.4. By (4), for $x \in X_e$, $c(x) \in Aut(\alpha)_\mu$, so $c(x)$ satisfies the hypothesis of F.2.8.4 in the role of "a". Then

$$[c^-(\mu(x)) \circ \mu](y) = {}^{\mu(x)}\mu(y) = \mu({}^x y) = (\mu \circ c(x))(y),$$

so again as $y \in X_d$ and d are arbitrary, we conclude $c^-(\mu(x)) \circ \mu = \mu \circ c(x)$. By F.2.8.4.iv, $\mu \circ c(x) = c(x)^- \circ \mu = \pi^-(c(x)) \circ \mu$, so $c^-(\mu(x)) \circ \mu = \pi^-(c(x)) \circ \mu$. Therefore (5) holds as μ is faithful by hypothesis. □

We remark that condition (f) in F.2.11 below can be expected to hold for the extensions of Lie amalgams which we will study in section F.4.

LEMMA F.2.11. *Assume hypotheses (a), (b), and (c) of F.2.10, and in addition assume:*

(d) $\mu : \alpha \to G$ is a faithful completion of α, such that $c(Y_e)$ stabilizes the equivalence class of μ.

(e) $\bigcap_{d \in D} C_{Y_e}(X_d) = 1$.

(f) $Y_d = Y_e X_d$ and $Y_e \cap X_d = X_e$, for each $d \in D$.

Then

(1) $\eta : \gamma \to G_0$ is a faithful completion of γ, where $\eta(yx) := \pi^-(c(y)) \circ c^-(\mu(x))$ for $y \in Y_e$ and $x \in X_d$, and $G_0 := Inn(G)\pi^-(c(Y_e)) \leq Aut(G)$.

(2) $\eta_{|\alpha} = c^- \circ \mu$.

(3) $\eta_{|\alpha} : \alpha \to Inn(G)$ is a faithful completion.

(4) If $Z(G) = 1$, then $\eta_{|\alpha}$ is equivalent to μ.

Proof. By (d), $c(Y_e) \leq Aut(\alpha)_\mu$, so π^- is defined on $c(Y_e)$ as in F.2.8.4.v. Let $y, y_1 \in Y_e$ and $x, x_1 \in X_d$. Then

$$\pi^-(c(y)) \circ c^-(\mu(x)) = \pi^-(c(y_1)) \circ c^-(\mu(x_1)) \ \text{ iff } \ \pi^-(c(y_1^{-1}y)) = c^-(\mu(x_1 x^{-1})).$$

But if $yx = y_1 x_1$, then $y_1^{-1}y = x_1 x^{-1} \in Y_e \cap X_d = X_e$ using (f); then using F.2.10.5, $\pi^-(c(y_1^{-1}y)) = c^-(\mu(y_1^{-1}y)) = c^-(\mu(xx_1^{-1}))$, as required. That is, η is well-defined on $Y_d = Y_e X_d$ using (f).

By definition, $\eta(x) = c^-(\mu(x)) = (c^- \circ \mu)(x)$, so (2) holds. Thus $\eta(X_d) = c^-(\mu(X_d)) \leq Inn(G)$. As $G = \langle \mu(X_d) : d \in D \rangle$,

$$Inn(G) = \langle c^-(\mu(X_d)) : d \in D \rangle = \langle \eta(X_d) : d \in D \rangle,$$

so using (f),

$$\langle \eta(Y_d) : d \in D \rangle = \langle \eta(X_d), \eta(Y_e) : d \in D \rangle = Inn(G)\pi^-(c(Y_e)).$$

Thus $\eta : \gamma \to G_0$ and $\eta_{|\alpha} : \alpha \to Inn(G)$ are completions.

Suppose $xy \in \ker(\eta)$. Then $\pi^-(c(y)) = c^-(\mu(x^{-1}))$, so for each $d \in D$ and each $z \in X_d$, $\pi^-(c(y))(\mu(z)) = {}^{\mu(x^{-1})}(\mu(z)) = \mu(z^x)$. Thus using F.2.8.4.iv,

$$\mu(z^x) = \pi^-(c(y))(\mu(z)) = c(y)^-(\mu(z)) = \mu(c(y)(z)) = \mu({}^y z)$$

Therefore as μ is faithful, ${}^{x^{-1}}z = z^x = {}^y z$, and hence $xy = 1$ by (e). Thus η is faithful, completing the proofs of (1) and (3). If $Z(G) = 1$, then $c^- : G \to Inn(G)$ is an isomorphism. Now by (2), $\eta_{|\alpha} = c^- \circ \mu$, which has the same kernel as μ as both are faithful by (d) and (3), so the completions are equivalent by F.2.4.2, proving (4). □

F.3. Paths in rank-2 amalgams

In this short section, we review some standard facts about paths in the coset graph of a completion of a rank-2 amalgam. This material will be used to recognize generalized polygons in the following section F.4.

So in this section G is a group, G_1 and G_2 are finite subgroups of G, $G_{1,2} := G_1 \cap G_2$, and $G = \langle G_1, G_2 \rangle$. Thus, in the language of Definitions F.2.1, F.1.4, and F.1.6, $\alpha := (G_1, G_{1,2}, G_2)$ is a rank-2 G-amalgam, and the inclusion $\alpha \to G$ is a faithful completion of α.

Definition F.3.1. Let $\mathcal{F} := \{G_1, G_2\}$, and form the coset complex $\Gamma := \Gamma(G, \mathcal{F})$ as in section 4 of [**Asc94**].

Thus Γ is a rank-2 geometry. Set $\Gamma_i := G_0/G_i$ for $i = 1, 2$; we call Γ_1 the set of *points* of the geometry Γ, and Γ_2 the set of *lines* of Γ. Write x for G_1 regarded as a point and l for G_2 regarded as a line. Thus the points and lines are the orbits under G of x and l, and $G_1 = G_x$ and $G_2 = G_l$ are the stabilizers in G of x and l, respectively. More generally for $y \in \Gamma$, G_y denotes the stabilizer in G of y, and $Q_y := O_2(G_y)$. Write $\Gamma^i(y)$ for the set of objects at distance i from y in the graph Γ, and abbreviate $\Gamma^1(y)$ by $\Gamma(y)$. As $G = \langle G_1, G_2 \rangle$, Γ is connected.

As in Definition F.2.2, let $\xi : \alpha \to gp(\alpha)$ be the universal completion of α. Write $\hat{G}$ for $gp(\alpha)$, and let $\hat{G}_J := \xi(G_J)$. Let $\hat{\Gamma}$ be the geometry of $\hat{G}$, $\hat{x} := \hat{G}_1$, etc. Let $\theta : \hat{G} \to G$ be the map $\hat{\iota}$ induced by the completion $\iota : \alpha \to G$ as in Definition F.2.2. Then θ induces a map $\theta : \hat{\Gamma} \to \Gamma$ of coset geometries via

$\theta(\hat{G}_i g) := G_i\theta(g)$, or equivalently $\theta(\hat{u}g) := u\theta(g)$ for $u := x, l$. Both maps θ are *coverings* in the appropriate category; that is the maps are surjective local isomorphisms. For example $\theta : \hat{\Gamma} \to \Gamma$ induces an isomorphism of $\hat{\Gamma}(\hat{u})$ with $\Gamma(u)$.

DEFINITION F.3.2. For $u \in \Gamma$, let $P(u)$ be the set of *paths* in Γ with origin u; here a path is a finite sequence of vertices in which consecutive vertices are adjacent, but which might contain circuits. A path $x_0 \cdots x_n$ is said to be *without backtracks* if it contains no circuits of length 2: that is, for each $0 \le i \le n-2$, $x_i \ne x_{i+2}$. Write $\Pi(u)$ for the paths in $P(u)$ without backtracks, let $P_n(u)$ be the paths in $P(u)$ of length n, and $\Pi_n(u) := P_n(u) \cap \Pi(u)$.

We recall the standard way in which such paths in Γ are covered by paths in $\hat{\Gamma}$:

LEMMA F.3.3. *Let $u \in \Gamma$, and choose some corresponding basepoint $\hat{u} \in \theta^{-1}(u)$. Then*

(1) θ induces a bijection $\varphi : P(\hat{u}) \to P(u)$ via $\varphi(x_0 \cdots x_n) := \theta(x_0) \cdots \theta(x_n)$. The bijection φ restricts to bijections $\varphi : P_n(\hat{u}) \to P_n(u)$, $\varphi : \Pi(\hat{u}) \to \Pi(u)$, and $\varphi : \Pi_n(\hat{u}) \to \Pi_n(u)$.

(2) φ and θ define a quasiequivalence of the permutation representations of $\hat{G}_{\hat{u}}$ on $P(\hat{u})$ and G_u on $P(u)$.

(3) For $p \in P(\hat{u})$, $\theta(\hat{G}_p) = G_{\varphi(p)}$ and $\theta : \hat{G}_p \to G_{\varphi(p)}$ is an isomorphism.

(4) If $\hat{G}_u$ is transitive on $\Pi_n(\hat{u})$, then G_u is transitive on $\Pi_n(u)$.

PROOF. The assertions of (1) for paths are straightforward; they follow from the fact that $\theta : \hat{\Gamma} \to \Gamma$ is a local isomorphism. In particular any backtrack in Γ occurs in some $\Gamma(x_i)$, so the restriction to paths without backtracks is also bijective. Part (2) follows from (1), using the facts that $\theta(vg) = \theta(v)\theta(g)$ for $v \in \hat{\Gamma}$ and $g \in \hat{G}$, and that $\theta : \hat{G} \to G$ is a local isomorphism. Then (1) and (2) imply (3) and (4). □

Now let $\beta : \alpha \to \dot{G}$ be any faithful completion of α, with $\dot{\theta} := \hat{\beta} : \hat{G} \to \dot{G}$ and $\dot{\theta} : \hat{\Gamma} \to \dot{\Gamma}$ the corresponding coverings. Set $\dot{G}_J = \beta(G_J)$. Notice that $\beta = \dot{\theta} \circ \theta^{-1}$ as a map of amalgams, since the completion $\iota : \alpha \to G$ is inclusion. Using this fact together with F.3.3, we can establish some relations among paths and their stabilizers for G and $\dot{G}$:

LEMMA F.3.4. *Let $u := x$ or l, and $\varphi : P(\hat{u}) \to P(u)$ and $\dot{\varphi} : P(\hat{u}) \to P(\dot{u})$ be the maps defined as in F.3.3. Set $\psi := \dot{\varphi} \circ \varphi^{-1} : P(u) \to P(\dot{u})$. Then*

(1) ψ and $\beta : G_u \to \dot{G}_{\dot{u}}$ define a quasiequivalence of the actions of G_u on $P(u)$ and $\dot{G}_{\dot{u}}$ on $P(\dot{u})$.

(2) For $p \in P(u)$, $\beta(G_p) = \dot{G}_{\psi(p)}$ and $\beta : G_p \to G_{\psi(p)}$ is an isomorphism.

(3) If G_u is transitive on $\Pi_n(u)$, then $\dot{G}_{\dot{u}}$ is transitive on $\Pi_n(\dot{u})$.

PROOF. Lemma F.3.4 follows from F.3.3, recalling that $\beta = \dot{\theta} \circ \theta^{-1} : G_u \to \dot{G}_{\dot{u}}$ is an isomorphism. □

Recall that a *geodesic* between points $u, v \in \Gamma$ is a path of minimal length. The next lemma characterizing thick generalized polygons in terms of path-transitivity and uniqueness of geodesics is well known (e.g., Proposition 1.1 in [**Wei90**]), but we include a proof here for completeness. Recall a *generalized m-gon* is a connected bipartite graph of diameter m in which the minimal length of a cycle is $2m$ and each vertex has valence at least 2.

Lemma F.3.5. *Assume $|\Gamma(x)| > 2 < |\Gamma(l)|$, and m is a positive integer such that G_u is transitive on $\Pi_{m+1}(u)$. Pick $p := x_0 \cdots x_{m+1} \in \Pi_{m+1}(u)$. Then*

(1) For each $i \leq m$, $x_i \in \Gamma^i(u)$. Thus $\Pi_i(u)$ consists of geodesics from u to members of $\Gamma^i(u)$ and G_u is transitive on $\Gamma^i(u)$.

(2) For each $i < m$, $x_0 \cdots x_i$ is the unique geodesic from u to x_i. Thus there is a bijection between paths in $\Pi_i(u)$ and their endpoints in $\Gamma^i(u)$.

(3) Either

(i) For each $p \in \Pi_{m+1}(u)$, $x_{m+1} \in \Gamma^{m+1}(u)$ and $x_0 \cdots x_m$ is the unique geodesic from u to x_m, or

(ii) Γ is a generalized m-gon, $\Gamma(x_m) \subseteq \Gamma^{m-1}(u)$, each member of $\Gamma(x_m)$ is on a unique geodesic from u to x_m, and G_{u,x_m} is 2-transitive on $\Gamma(x_m)$ as well as on the geodesics from u to x_m.

(4) If G_p fixes no member of $\Gamma(x_{m+1}) - \{x_m\}$, then $x_{m+1} \in \Gamma^{m+1}(u)$.

Proof. Since $|\Gamma(v)| \geq 2$ for any vertex v by hypothesis, any path without backtracks can be extended to a longer path without backtracks; thus our hypothesis that G_u is transitive on $\Pi_{m+1}(u)$ implies transitivity of G_u on $\Pi_n(u)$ for $n \leq m+1$. Hence the assertion of transitivity in (1) will follow from the assertion preceding it.

Pick $0 \leq n \leq m+1$ maximal subject to $x_n \in \Gamma^n(u)$. As p has no backtracks and $m \geq 1$, $n > 1$. By definition of n, $x_0 \cdots x_j$ is a geodesic from u to x_j for each $j \leq n$. Suppose there is $i < n$ with $y_0 \cdots y_i$ a second geodesic from u to x_i, and choose i minimal subject to this constraint. Then $i > 1$ and $q := x_0 \cdots x_i y_{i-1} \in \Pi_{i+1}(u)$, since $y_{i-1} \neq x_{i-1}$ by minimality of i. This is impossible as $i+1 \leq n \leq m+1$, so $x_0 \cdots x_{i+1}$ is conjugate to q under G_u by hypothesis, whereas $x_{i+1} \in \Gamma^{i+1}(u)$ while $y_{i-1} \in \Gamma^{i-1}(u)$. This contradiction shows that for each $i < n$, $x_0 \cdots x_i$ is the unique geodesic from u to x_i. Thus if $n \geq m$ then (1) and (2) hold, and if $n = m+1$, then conclusion (i) of (3) holds. Thus in the remainder of the proof of (1)–(3), we may assume that $n \leq m$.

Therefore $x_{n+1} \in \Gamma^{n-1}(u)$. For $i \leq m$, $G_{x_0,\cdots x_i}$ is transitive on $\Gamma(x_i) - \{x_{i-1}\}$ again by hypothesis, so as $n \leq m$ and $x_{n+1} \in \Gamma^{n-1}(u)$, it follows that $\Gamma(x_n) \subseteq \Gamma^{n-1}(u)$. Similarly as $x_n \in \Gamma^n(u)$, $\Gamma(x_{n-1}) - \{x_{n-2}\} \subseteq \Gamma^n(u)$. Moreover G_u is transitive on $\Gamma^{n-1}(u)$, so x_{n+1} is conjugate to x_{n-1} under G_u. Thus as $n > 1$, by the previous paragraph there is a unique $v \in \Gamma(x_{n+1}) \cap \Gamma^{n-2}(u)$ and $\Gamma(x_{n+1}) - \{v\} \subseteq \Gamma^n(u)$. By hypothesis $|\Gamma(x)| > 2 < |\Gamma(l)|$, so there exists $w \in \Gamma(x_{n+1}) - \{x_n, v\}$. Thus $r := x_0 \cdots x_n x_{n+1} v$ and $s := x_0 \cdots x_n x_{n+1} w$ are in $\Pi_{n+2}(u)$. But if $n < m$, then $n+2 \leq m+1$, so r and s are conjugate under G_u, impossible as $v \in \Gamma^{n-2}(u)$ while $w \in \Gamma^n(u)$.

Therefore $n = m$, and hence (1) and (2) hold by an earlier observation. As G_u is transitive on $\Gamma^m(u)$ and $\Gamma(x_m) \subseteq \Gamma^{m-1}(u)$, it follows that Γ is of diameter m. By (2) there is a unique geodesic from u to y for each $y \in \Gamma^{m-1}(u)$, so there is a unique geodesic from u to x_m through each member of $\Gamma(x_m)$. Hence a the minimal length of a cycle is $2m$, so Γ is a generalized m-gon.

We saw that $G_{x_0 \cdots x_m}$ is transitive on $\Gamma(x_m) - \{x_{m-1}\}$. By hypothesis, G_u is transitive on $\Pi_m(u)$, so G_t is transitive on $\Gamma(x_m) - \{x_{m+1}\}$, where t is the unique geodesic from u to x_m through x_{m+1}. Thus G_{u,x_m} is 2-transitive on $\Gamma(x_m)$, and hence also on the corresponding geodesics from u to x_m. So conclusion (ii) of (3) holds, completing the proof of (3).

Finally suppose G_p fixes no member of $\Gamma(x_{m+1}) - \{x_m\}$. Then it is not the case that $x_{m+1} \in \Gamma^{m-1}(u)$ and there is a unique geodesic from u to x_{m+1}, since in that event G_p would fix the unique member of $\Gamma(x_{m+1}) \cap \Gamma^{m-2}(u)$. Therefore conclusion (ii) of (3) does not hold, so conclusion (i) holds, establishing (4). □

We close the section with two lemmas showing how the existence of suitable involutions in the subgroups G_i affords a construction of $2m$-gon subgeometries of Γ. The first will be applied when α is the amalgam of a group of Lie type. The second and more complicated construction is required when α is the amalgam of $G_2(2)'$, where the automizer in G of an apartment in the geometry of a "small" completion G is D_6—rather than D_{12}, as is the case for the amalgam of the $G_2(2)$-extension of α.

LEMMA F.3.6. *Let $m > 1$ be an integer, and assume for each $u \in \Gamma$ and $v \in \Gamma^{m-1}(u)$ that there is a unique geodesic from u to v. Assume $n \leq m$ is an integer, and $s \in G_1 - G_{1,2}$ and $t \in G_2 - G_{1,2}$ are involutions with $(st)^n \in G_{1,2}$. Then $n = m$, $\Sigma := x^{\langle s,t\rangle} \cup l^{\langle s,t\rangle}$ is a $2m$-gon in Γ, and $\langle s,t\rangle/\langle (st)^m\rangle$ acts faithfully as the group of automorphisms D_{2m} of Σ.*

PROOF. Let $W := \langle s,t\rangle$ and $\Sigma := x^W \cup l^W$, regarded as a subgraph of Γ. Let $K := \langle st\rangle \cap G_{1,2}$ and $\bar{W} := W/K$. Then K is the kernel of the action of $\langle st\rangle$ on Σ, and $N := |\langle st\rangle : K|$ divides n. As $\bar{W} \cong D_{2N}$, $\Delta := \Gamma(\bar{W}, \{\langle \bar{s}\rangle, \langle \bar{t}\rangle\})$ is a $2N$-gon. Define $\phi : \Delta \to \Sigma$ by $\phi(\langle \bar{s}\rangle \bar{w}) := xw$ and $\phi(\langle \bar{t}\rangle \bar{w}) := lw$. As $s \in G_1$ and $t \in G_2$, ϕ is well-defined. As x and l are adjacent in Γ, ϕ is a morphism of geometries. As $s \notin G_2$ and $t \notin G_1$, if $y_0y_1y_2$ is a geodesic in Δ, then $\phi(y_0)\phi(y_1)\phi(y_2)$ is a geodesic in Γ and $\phi(y_0) \neq \phi(y_2)$.

We claim that if $p := x_0 \cdots x_k$ is a path without backtracks in $\phi(\Delta)$ with $k \leq m$, then p is a geodesic in Γ. The proof is by induction on k and is trivial if $k = 0$. By induction $x_0 \cdots x_{k-1}$ is a geodesic, so $d(x_0, x_k) = k - 2$ or k. In the former case as $k \leq m$, there is a unique geodesic from x_0 to x_{k-1} in Γ by hypothesis, so $x_k = x_{k-2}$, impossible as we saw in the previous paragraph that $x_{k-2}x_{k-1}x_k$ is a geodesic. This contradiction shows that $d(x_0, x_k) = k$, establishing the claim.

By the claim, $N \geq m$, since there are two geodesics between opposites in the $2N$-gon Δ. Thus as $N \leq n \leq m$ by hypothesis, we have $n = m$. Therefore Σ is a $2m$-gon. Visibly the last remark in the lemma holds, so the proof is complete. □

LEMMA F.3.7. *Let $m \geq 4$ be an even integer, and assume for each $u \in \Gamma$ and $v \in \Gamma^{m-1}(u)$ that there is a unique geodesic from u to v. Assume $n \leq m/2$ is an integer, $c \in \Gamma(x) - \{l\}$, and $s \in G_c - G_{x,c}$ and $t \in G_2 - G_{1,2}$ are involutions with $(st)^n \in G_{x,l,c}$. Then $n = m/2$, $\Sigma := x^{\langle s,t\rangle} \cup l^{\langle s,t\rangle} \cup c^{\langle s,t\rangle}$ is a $2m$-gon in Γ, and $\langle s,t\rangle/\langle (st)^n\rangle$ acts faithfully as D_m on Σ.*

PROOF. The proof is much like that of F.3.6. Again let $W := \langle s,t\rangle$, and this time let $\Sigma := x^W \cup l^W \cup c^W$ regarded as a subgraph of Γ. Again defining K to be the kernel of the action of $\langle st\rangle$ on Σ, $N := |\langle st\rangle : K|$, and $\bar{W} := W/K$, we have $\bar{W} \cong D_{2N}$ and N divides n. Let D be the dual of the poset $\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$ under inclusion, and let $\bar{W}_1 := \bar{W}_{1,2} = \bar{W}_{1,3} = 1$, $\bar{W}_2 := \langle \bar{s}\rangle$, $\bar{W}_3 := \langle \bar{t}\rangle$, and $\mathcal{F} := \{\bar{W}_d : d \in D\}$. This time $\Delta := \Gamma(\bar{W}, \mathcal{F})$ is a $4N$-gon, since for $i = 2, 3$, vertices of type i are adjacent to two vertices of type $1, i$, vertices of type $1, i$ are adjacent to a unique vertex of type 1 and type i, and vertices of type 1 are adjacent to a unique vertex of type $1, i$. Define $\phi : \Delta \to \Sigma$ by $\phi(\bar{w}) := xw$, $\phi(\bar{W}_2\bar{w}) := cw$ and $\phi(\bar{W}_3\bar{w}) := lw$. Then argue as in the proof of the previous lemma. □

F.4. Controlling completions of Lie amalgams

This section is devoted to results establishing sufficient conditions for a completion G_0 of the amalgam of a weak BN-pair (as in section F.1) to be isomorphic to the extension of a group of Lie type defining that weak BN-pair.

The literature contains various results of this type—notably the theorem of Tits, proved as Theorem 8 of section II in [**Ser80**], where groups of Lie type are characterized in terms of the rank 3 amalgam obtained by adjoining the stabilizer of an apartment to the parabolics. (Compare with the viewpoint in 4.2 of [**Asc93**]). In contrast, our ultimate goal in this section is to provide conditions more in the spirit of local group theory; thus in our main result Theorem F.4.31 in the second subsection, the crucial condition (b) is that the centralizer in G_0 of a 2-central involution be contained in one of the subgroups G_i defining the amalgam. This result is in fact based primarily on geometric analysis, carried out in the first subsection, that culminates in Theorem F.4.8—which characterizes groups of Lie type as those completions in which there exist at least two geodesics between vertices at distance m in the coset graph (see (ii)–(iv) of F.4.6.8), where m is the diameter of the building. The work of the first subsection uses material on paths from the previous section F.2.

One further advantage of our approach is that it allows us later to identify groups defined over $\mathbf{F}_2$, in addition to those defined over larger fields. Indeed our treatment includes results for $G_2(2)'$ and the Tits group ${}^2F_4(2)'$, which are of index 2 in the corresponding groups of Lie type, and hence are not covered by many standard treatments in the literature. The difficulty with most groups over $\mathbf{F}_2$ is that the Cartan group is trivial, and hence one cannot retrieve the stabilizer of an apartment within the normalizer of the Cartan group.

DEFINITION F.4.1. In this section the term *Lie group* means an adjoint group G of Lie type of Lie rank 2 over a finite field. In particular in the first subsection we do not require G to be of characteristic 2. Thus a Lie group is one of $L_3(q)$, $Sp_4(q)$, $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, $U_4(q)$, or $U_5(q)$ for q a power of a prime.

We use the term *Lie amalgam* to mean a rank-2 G-amalgam $\alpha := (G_1, G_{1,2}, G_2)$ in the sense of Definitions F.1.4 and F.2.9, where G is a Lie group, $G_{1,2}$ is a Borel subgroup of G, and G_1 and G_2 are the maximal parabolics over $G_{1,2}$. When we wish to emphasize the role of G, we say α is the Lie amalgam *defined by* the Lie group G.

We also consider two of the cases in which a Lie group possesses a subgroup of index 2. We define a *generalized Lie group* to be $G_2(2)' \cong U_3(3)$ or the Tits group ${}^2F_4(2)'$. We say α is a *generalized Lie amalgam* if α is the amalgam defined by a generalized Lie group. [1]

NOTATION F.4.2. Recall the notion of the coset geometry of a completion from Definition F.3.1. We adopt the convention (except in the case of $L_3(q)$, where

[1]Since $A_6 \cong Sp_4(2)'$ has the same amalgam as $L_3(2)$, we do not consider A_6 as a generalized Lie group. Further since both $L_3(2)$ and A_6 are completions of this amalgam, and the centralizer of an involution in both groups is D_8, we exclude the amalgam of $L_3(2)$ in Theorem F.4.31.

it makes no sense) that the points in the geometry are the cosets of the maximal parabolic normalizing a long root group, and we choose G_1 to denote this parabolic.

Recall also from Definition F.1.4 the definition of an automorphism of amalgams, and the discussion in section F.2, particularly in F.2.8.5, of those automorphisms of a generalized Lie amalgam induced by a suitable subgroup of $Aut(G)$, where G is a generalized Lie group.

Definition F.4.3. Define an *extension* of a generalized Lie group G to be a group M satisfying $G \trianglelefteq M \leq Aut(G)$, with M trivial on the Dynkin diagram of G. An *extension* of a generalized Lie amalgam α defined by the generalized Lie group G, is an M-amalgam $\gamma := (M_1, M_{1,2}, M_2)$, where M is an extension of G such that $\bar{G}_J = M_J \cap \bar{G}$ for each J, and $M = \bar{G}M_{1,2}$. When we wish to emphasize the role of the group M, we will say γ an M*-extension* of α.

Notice that the automorphism group of $G_2(2)'$ or the Tits group is $G_2(2)$ or ${}^2F_4(2)$, respectively, and these extension are Lie groups. Consequently each extension of a generalized Lie amalgam is either an extension of a Lie amalgam, or the amalgam of $G_2(2)'$ or the Tits group.

F.4.1. Small faithful completions.

Notation F.4.4. Througout this subsection $\alpha := (\bar{G}_1, \bar{G}_{1,2}, \bar{G}_2)$ is an extension of a generalized Lie amalgam over a finite field of characteristic p. More precisely, α is a $\bar{G}$-extension of a generalized Lie amalgam defined by a generalized Lie group $\bar{G}_+$. Let $\beta : \alpha \to G$ be a faithful completion of α. Let $G_J := \beta(\bar{G}_J)$, for $J = \{1\}$, $\{2\}$, $\{1,2\}$, and let $\iota : \alpha \to \bar{G}$ be the Lie completion determined by the inclusion maps.

Write $\hat{G}$ for the universal completion $gp(\alpha)$ (cf. Definition F.2.2), and $\xi : \alpha \to \hat{G}$ for the universal completion. Let $\theta := \theta_G = \hat{\beta} : \hat{G} \to G$ be the map from the universal completion determined by β.

Form the coset geometries Γ, $\bar{\Gamma}$ and $\hat{\Gamma}$ (as in Definition F.3.1) of G, $\bar{G}$, and $\hat{G}$, respectively. As in section F.3, write $x := G_1$ and $l := G_2$ for these cosets regarded as members of Γ, and write $\bar{x}$, $\hat{l}$, etc. for the members $\bar{G}_1$, $\hat{G}_2$ of $\bar{\Gamma}$ and $\hat{\Gamma}$.

The *Weyl group* of a Lie amalgam α is the Weyl group of the corresponding Lie group. The Weyl group of the amalgam of $G_2(2)'$ or the Tits group is the Weyl group of the Lie group $G_2(2)$ or ${}^2F_4(2)$, respectively. Let $2m$ denote the order of the Weyl group of α.

Define an *apartment* of Γ to be a subgraph of Γ which is a $2m$-gon. Write $\mathcal{A}$ for the set of apartments of Γ; it is possible that $\mathcal{A}$ is empty. Indeed it is a consequence of Theorem F.4.8 that apartments exist precisely when G is the extension of a generalized Lie group.

Define q_i by $q_i + 1 := |G_i : G_{1,2}|$. For example when α is the amalgam of $G_2(2)'$ or the Tits group, then $q_i = 2$—except when α is the amalgam of the Tits group and $i = 1$, where $q_1 = 4$. For $u \in \Gamma_i$ let $q(u) := q_i$.

For $\bar{u} \in \bar{\Gamma}$, let $\bar{Q}_{\bar{u}} := O_p(\bar{G}_{+,\bar{u}})$ be the unipotent radical of $\bar{G}_{+,\bar{u}}$, and $\bar{S}$ the unipotent radical of $\bar{G}_{+,1,2}$. Thus $\bar{S} \in Syl_p(\bar{G}_+)$. Set $Q_u := \beta(\bar{Q}_{\bar{u}})$ for $u = x, l$, and $S := \beta(\bar{S})$. For $g \in G$, let $Q_{ug} := Q_u^g$.

We first record some basic properties of the building $\bar{\Gamma}$ of our Lie completion $\bar{G}$. Much of the following material is in the literature, going back for example to the work of Tits (e.g., [**Tit74**]).

Recall from Definition F.3.2 the notation of $\Pi(\Gamma)$ for paths without backtrack.

LEMMA F.4.5. *Assume $G = \bar{G}$, so that $\Gamma = \bar{\Gamma}$.*

(1) Γ is a generalized m-gon.

(2) If α is the amalgam of $G_2(2)'$ or the Tits group, then G is of index 2 in $G^* \cong G_2(2)$ or ${}^2F_4(2)$, G_i is of index 2 in G_i^*, and the map $G_ig \mapsto G_i^*g$ is an isomorphism of Γ with the corresponding geometry Γ^* for G^*.

(3) If $u, v \in \Gamma$ with $d(u, v) < m$, then there is a unique geodesic in Γ from u to v.

(4) If u and v are opposites (that is, $d(u, v) = m$), then there are $|\Gamma(u)|$ geodesics from u to v in Γ. Moreover the apartments of Γ containing u and v are precisely the unions of pairs of these geodesics.

(5) If α is an extension of a Lie amalgam and u and v are opposites, then $G_{u,v}$ is a complement to Q_u in G_u, and to Q_v in G_v, containing a Levi complement of each of G_u and G_v, and $G_{u,v}$ is 2-transitive on the geodesics from u to v.

(6) If (u, v) is an edge in Γ and α is a generalized Lie amalgam (so that $\bar{G} = \bar{G}_+$), then $O^{p'}(G_{u,v})/Q_u$ is regular on $\Gamma(u) - \{v\}$ of order $q(u)$.

(7) If α is an extension of a Lie amalgam then G_u is transitive on $\Pi_{m+1}(u)$.

(8) If α is the amalgam of the Tits group or $G_2(2)'$, then

(a) G_x is transitive on $\Pi_m(x)$, and

(b) G_l is transitive on $\Pi_{m-1}(l)$, has two orbits of equal length on $\Pi_m(l)$, and $G_{l,v}$ is 2-transitive on paths in $\Pi_m(l)$ from l to v, for each $v \in \Gamma^m(l)$.

(9) Assume that α is the amalgam of the Tits group or $G_2(2)'$, and u and v are opposites. If u is a line, then $G_{u,v}$ is a complement to Q_u. If u is a point, then $G_{u,v} \cong D_{10}$ or $\mathbf{Z}_3$.

PROOF. When $\bar{G}$ is a Lie group, these are well-known properties of rank 2 buildings; see for example Tits-Weiss [**TW02**]. To obtain (7), notice that a path in Π_{m+1} consists of a geodesic p from u to an opposite v through some $a \in \Gamma^{m-1}(u)$, followed by an edge from v to some $w \in \Gamma^{m-1}(u) - \{a\}$, and that the root group in G_p is transitive on such w.

The properties are presumably less well known in the cases of $G_2(2)'$ and the Tits group, so we provide details here. Thus we assume $G = \bar{G}_+$ is $G_2(2)'$ or the Tits group.

We first prove (2). Certainly G is of index 2 in G^*, and G_i is of index 2 in maximal parabolics G_i^* of G^* containing a Sylow 2-group S^* of G^*, with $G_i^* = G_iS^*$ and $G_{1,2} = S = S^* \cap G$. Define $\varphi : \Gamma \to \Gamma^*$ by $\varphi : G_ig \mapsto G_i^*g$ for $g \in G$. As $G_i \leq G_i^*$, φ is well-defined, and as $G_i = G \cap G_i^*$, φ is an injection. As G^* is transitive on Γ_i^* and $G^* = S^*G$ with $S^* \leq G_i^*$, G is transitive on Γ_i^*, and hence φ is surjective. Thus $\varphi : \Gamma \to \Gamma^*$ is a bijection. Finally $G_1^*G_2^* = S^*G_1S^*G_2 = S^*G_1G_2$, so $G_1^*G_2^* \cap G = G_1G_2$. Then G_1g is incident with G_2h in Γ iff $gh^{-1} \in G_1G_2$ iff $gh^{-1} \in G_1^*G_2^*$ iff G_1^*g is incident with G_2^*h in Γ^*, so φ is an isomorphism. This proves (2).

As (1), (3), and (4) are statements about Γ which hold for Γ^*, they also hold for Γ by (2). Further $G_i^* = O_2(G_i^*)G_i$, so again as (6) holds for G^*, it also holds for G. Conclusions (5) and (7) are vacuous for the Tits group and $G_2(2)'$, so it remains to prove (8) and (9).

Let v be an opposite to u. By (5), $G_{u,v}^*$ is a Levi complement in G_u^* and G_v^*.

Suppose first that u is a point. We may take u to be the point x defined earlier by the coset G_1. Then $G_{x,v} = G_{x,v}^* \cap G$ is of index 2 in $G_{x,v}^*$, since the root element generating a Sylow 2-subgroup S_v^* of the Levi complement $G_{x,v}^*$ is not in G. This establishes (9) in the case where u is a point. In particular, $S_v^* \not\leq S$, so as $|S^* : S| = 2$, $S^* = SS_v^*$. By (7), G_u^* is transitive on $\Gamma^m(u)$; so as $G_u^* = Q_u^* G_{u,v}^*$ by (5), Q_u^* is also transitive on $\Gamma^m(u)$. As $Q_u^* \leq S^* = SS_v^*$, we conclude that S is transitive on $\Gamma^m(u)$. Moreover $|G_{x,v} : S_v| = |\Gamma(u)|$ and $S_v = G_{x,v} \cap G_l$, where l is the line defined earlier by the coset G_2. Further by (3) and (4), there is a unique geodesic from u to v through each $w \in \Gamma(x)$. Thus $G_{x,v}$ is transitive on the geodesics from x to v, establishing (8a).

So assume instead that u is our standard line l. This time the Levi complement $G_{l,v}^*$ is contained in G, so $G_{l,v}^* = G_{l,v}$, completing the proof of (9). Further as G_l is of index 2 in G_l^*, it also shows that G_l has two orbits of equal length on opposites to u. Then appealing to (3) and (4) as in the previous paragraph, we conclude G_l is transitive on $\Pi_{m-1}(l)$. As the root group S_v^* is contained in $G_{l,v}$, $G_{l,v}$ is 2-transitive on paths in $\Pi_m(l)$ from l to v, completing the proof of (8b). □

Next we use Lemma F.3.4 to transfer certain properties of the building $\bar{\Gamma}$, appearing in F.4.5, to the geometry Γ of an arbitrary faithful completion G of α. We cannot expect all properties to transfer, since our results must hold for the universal completion $\hat{G}$, where $\hat{\Gamma}$ is an infinite tree.

LEMMA F.4.6. *Let $n \leq m$, $a \in \Gamma$, and $p := x_0 \cdots x_n \in \Pi_n(a)$. Then*

(1) If α is an extension of a Lie amalgam, then G_a is transitive on $\Pi_{m+1}(a)$.

(2) If α is the amalgam of $G_2(2)'$ or the Tits group, then

(a) if a is a point, G_a is transitive on $\Pi_m(a)$; while

(b) if a is a line, then G_a is transitive on $\Pi_{m-1}(a)$, G_a has two orbits of equal length on $\Pi_m(a)$, and G_{a,x_m} is transitive on paths in $\Pi_m(a)$ to x_m.

(3) For $a = x, l$, there exists a bijection $\psi : P(\bar{a}) \to P(a)$ such that β, ψ is a quasiequivalence of the actions of $\bar{G}_{\bar{a}}$ and G_a on $P(\bar{a})$ and $P(a)$, restricting to a bijection of $\Pi_k(\bar{a})$ and $\Pi_k(a)$, for each integer k.

(4) $|G_a : G_p| = (q_i+1)(q_1q_2)^j$ or $(q_i+1)(q_1q_2)^j q_{3-i}$ for $a \in \Gamma_i$ and $0 < n := 2j+1$ or $2j+2$, respectively, unless α is the amalgam for $G_2(2)'$ or the Tits group, a is a line, and $n = m$.

(5) If $n = m$ then either G_p is a complement to Q_{x_m} in G_{x_{m-1},x_m}; or α is the amalgam of $G_2(2)'$ or the Tits group, a is a point, $G_p \cap Q_{x_m} = 1$, and $|G_{x_{m-1},x_m} : Q_{x_m} G_p| = 2$.

(6) If $n < m$, then G_a is transitive on $\Gamma^n(a)$, $x_n \in \Gamma^n(a)$, and p is the unique geodesic from a to x_n. In particular, Γ contains no $2n$-gons for $n < m$.

(7) If $n < m$, then $|\Gamma^n(a)| = |G_a : G_p|$.

(8) If $n = m$ and $b := x_n = x_m$, then $b \in \Gamma^m(a)$, so $\Pi_m(a)$ consists of geodesics from a to members of $\Gamma^m(a)$. Further one of the following holds:

(i) p is the unique geodesic from a to b.

(ii) $G_{a,b}$ is a complement to Q_a in G_a, and acts 2-transitively on the $q(a)+1$ geodesics from a to b.

(iii) α is the amalgam of the Tits group, a is a point, and $G_{a,b} \cong D_{10}$ is transitive on the 5 geodesics from a to b.

(iv) α is the amalgam for $G_2(2)'$, a is point, and $G_{a,b} \cong \mathbf{Z}_3$ is transitive on the 3 geodesics from a to b.

In cases (ii)–(iv), the map $x_0 \cdots x_m \mapsto x_1$ is a bijection between geodesics from a to b and the members of $\Gamma(a)$.

PROOF. Recall that G_a is G-conjugate to either G_1 or G_2; as our conclusions describe G_a and its subgroups, we may assume $a = x$ or l.

We will apply lemma F.3.4 with $\bar{G}$, G in the roles of "G, $\dot{G}$". The hypotheses of section F.3 hold by Notation F.4.4. Thus F.3.4 implies that (3) holds. Then apply (3) to obtain a path $\bar{p} := \psi^{-1}(p) = \bar{x}_0 \cdots \bar{x}_n$ in $\Pi_n(\bar{\Gamma})$ corresponding to p. Applying the quasiequivalence in (3) and appealing to the transitivity of $\bar{G}_{\bar{a}}$ on $\Pi_{m+1}(\bar{a})$ in F.4.5.7, we conclude that (1) holds. Similarly (3) and F.4.5.8 imply (2).

We next prove (4) by induction on n: Namely by transitivity of G_a on $\Gamma(a)$,

$$|G_a : G_{a,b}| = |\Gamma(a)| = q(a) + 1$$

for each $a \in \Gamma$ and $b \in \Gamma(a)$. Thus (4) holds when $n = 1$. So assume the result at $n - 1$, and let $r := x_0 \cdots x_{n-1}$. By (1) and (2), G_r is transitive on $\Gamma(x_{n-1}) - \{x_{n-2}\}$—except possibly when $\bar{G}$ is $G_2(2)'$ or the Tits group, $n = m$, and a is a line, where (4) is vacuously true, as it explicitly excludes this case. Thus

$$|G_a : G_p| = |G_a : G_r||G_r : G_p| = |G_a : G_r|q(x_{n-1}),$$

and now (4) follows using the inductive value for $|G_a : G_r|$.

Let $n = m$, and now choose notation so that our standard edge xl occurs at the end of the m-path path p: that is, $\{x_{m-1}, x_m\} = \{x, l\}$. By F.4.5.3, $\bar{G}_{\bar{a},\bar{x}_{m-1},\bar{x}_m} = \bar{G}_{\bar{p}}$. Thus by F.4.5.5 and F.4.5.9, either $\bar{G}_{\bar{p}} = \bar{L}_{\bar{x}_{n-1}}$, where $\bar{L}$ is a complement to $\bar{Q}_{x_n}$ in $\bar{G}_{\bar{x}_n}$; or α is the amalgam of $G_2(2)'$ or the Tits group, a is a point, and $\bar{G}_{\bar{p}} \cap \bar{Q}_{\bar{x}_n} = 1$ with $|\bar{G}_{\bar{x}_{n-1},\bar{x}_n} : \bar{Q}_{\bar{x}_n}\bar{G}_{\bar{p}}| = 2$. Applying β and appealing to (3), we conclude (5) holds.

We next prove (6) and (8). Let $b := x_m$. Recall that Γ is thick since $q_i \geq 2$ for $i = 1$ and 2; hence we may apply F.3.5 whenever the transitivity hypothesis of that result holds.

Suppose first that α is an extension of a Lie amalgam. Then by (1), G_a is transitive on $\Pi_{m+1}(a)$, so we conclude from (1) and (2) of F.3.5 that (6) holds, as well as the initial statement in (8) that $b \in \Gamma^m(a)$. By F.3.5.3, either conclusion (i) of (8) holds, or Γ is a generalized m-gon, and $G_{a,b}$ is 2-transitive on the $q(a) + 1$ geodesics from a to b. Suppose the latter case holds. Then F.3.5.3 also shows that $G_{a,b}$ is transitive on $\Gamma(a)$, so that $G_a = G_{a,x_1}G_{a,b}$. Then by (5), $G_a = Q_aG_pG_{a,b} = Q_aG_{a,b}$ and $G_p \cap Q_a = 1$. But $G_{a,b} \cap Q_a \leq G_{a,b,x_1}$ and $G_{a,b,x_1} \leq G_p$ by (6), so $G_{a,b} \cap Q_a \leq G_p \cap Q_a = 1$, and hence $G_{a,b}$ is a complement to Q_a in G_a. Thus conclusion (ii) of (8) holds in this case, so (8) is established when α is an extension of a Lie amalgam.

Now assume instead that α is the amalgam of $G_2(2)'$ or the Tits group. We will first establish (6). Suppose for the moment that $a = x$ is a point. Then G_a is transitive on $\Pi_m(a)$ by (2.a), and hence (as mentioned at the start of the proof of F.3.5) also on $\Pi_n(a)$ for $n < m$. Thus we conclude from parts (1) and (2) of F.3.5 that for $n < m$, $x_n \in \Gamma^n(a)$ and G_a is transitive on $\Gamma^n(a)$, and for $n < m - 1$, there is a unique geodesic from a to x_n. If $x_0 \cdots x_{m-1}$ is the unique geodesic from a to x_{m-1}, then all parts of (6) hold in the case where a is a point. But since m is even, x_{m-1} is a line, so reversing the direction of geodesics from a to x_{m-1}, (6) also holds in the case where a is a line. Thus if (6) fails, we may assume that there is more than one geodesic from a to x_{m-1}. Then by F.3.5.3, Γ is a generalized $(m-1)$-gon. We now obtain a contradiction as in F.3.5.4: Namely let $y_0 := l$ and $q := y_0 \cdots y_m \in \Pi_m(l)$, so that $y_m \in \Gamma^{m-2}(l)$. Since $y_0 = l$ is a line, we conclude from (5) that G_q is a complement to Q_{y_m} in $G_{y_m,y_{m-1}}$, so G_q is transitive on $\Gamma(y_m) - \{y_{m-1}\}$ since $G_{y_m,y_{m-1}}$ is transitive on that set. However as Γ is a generalized $(m-1)$-gon, $\Gamma^{m-3}(l) \cap \Gamma(y_m) =: \{y\}$ is of order 1, so G_q fixes $y \neq y_{m-1}$, contrary to the transitivity just obtained. This contradiction completes the proof of (6).

It follows from (6) that $b \in \Gamma^m(a)$: for otherwise $b \in \Gamma^{m-2}(a)$, so there is a geodesic p' of length $m-2$ from a to b; and then $p'x_{m-1}$ and $x_0 \cdots x_{m-1}$ are distinct geodesics from a to x_{m-1}, contrary to (6). Thus the first statement of (8) is established.

To complete the proof of (8), we may assume that conclusion (i) of (8) does not hold, so that there is a second geodesic $q := y_0 \cdots y_n$ from a to b. Notice $y_1 \neq x_1$, since by (6) there is a unique geodesic from x_1 to b.

Let $H := G_{a,b}$. Then $K := G_b \cap Q_a$ fixes x_1, so by (6), $K \leq G_p \cap Q_a$, and hence $K = 1$ by (5). In particular, $H \cap Q_a = 1$. Indeed by (5), either G_p is a complement to Q_a in G_{a,x_1}, or a is a point and $|G_{a,x_1} : G_pQ_a| = 2$.

Assume that the second case does not hold, so that the first case holds for both p and q, and hence G_q is a complement to Q_a in G_{a,y_1}. But as G_a is 2-transitive on $\Gamma(a)$ and $x_1 \neq y_1$, $G_a = \langle G_{a,x_1}, G_{a,y_1}\rangle$, so $G_a = \langle G_p, G_q\rangle Q_a = HQ_a$. We saw $H \cap Q_a = 1$, so H is a complement to Q_a in G_a. Thus H is 2-transitive on $\Gamma(a)$, so (8.ii) holds, as $x_1 \cdots x_m$ is the unique geodesic from x_1 to b by (6).

Thus we may assume that the second case holds, and in particular a is a point. If α is the amalgam of the Tits group then $|G_p| = 2 = |G_q|$ by (5), so $H = \langle G_p, G_q\rangle \cong D_{10}$ as $G_a/Q_a \cong Sz(2)$ and $H \cap Q_a = 1$. In particular H is transitive on $\Gamma(a)$, and hence on the 5 geodesics from a to b, so that (8.iii) holds. Finally if α is the amalgam for $G_2(2)'$, then $G_p = 1$ by (5). Thus as $H \cap Q_a = 1$ and $G_a/Q_a \cong S_3$, either $H = 1$ or $H \cong \mathbf{Z}_3$. In the latter case, G_b is transitive on $\Gamma(a)$, and hence transitive on the 3 geodesics from a to b, so that (8.iv) holds. The former case is impossible, since we are assuming there are at least two geodesics from a to b, and H is transitive on these geodesics by (2.a). This completes the proof of (8).

It remains to prove (7), so assume $n < m$. Then by (6), G_a is transitive on $\Gamma^n(a)$, so for $b \in \Gamma^n(a)$, $|\Gamma^n(a)| = |G_a : G_{a,b}|$. Also by (6), $G_{a,b} = G_p$, so (7) is established. Thus the proof of the F.4.6 is complete. □

DEFINITION F.4.7. Define a faithful completion $\alpha \to G$ of an extension of a generalized Lie amalgam α to be *large* if there is a unique geodesic in Γ between any pair of objects at distance m in Γ. A faithful completion is *small* if it is not large.

THEOREM F.4.8. *Assume that α is an extension of a generalized Lie amalgam, defined by an extension $\bar{G}$ of a generalized Lie group $\bar{G}_+$. Then*

(1) If $\beta : \alpha \to G$ is a small completion of α, then β is equivalent to a completion $\dot{\beta} : \alpha \to \dot{G}$ for some extension $\dot{G}$ of $\bar{G}_+$. In particular, $G \cong \dot{G}$.

(2) If α is a generalized Lie amalgam, so that $\bar{G} = \bar{G}_+$, then there is a unique quasiequivalence class of small completions $\beta : \alpha \to G$ of α. In particular, $G \cong \bar{G} = \bar{G}_+$.

REMARK F.4.9. This result can probably be extended as follows: While we have not attempted to write out a proof, it appears that under the hypotheses of Theorem F.4.8, $G \cong \bar{G}$, and, with the exception of a few groups over $\mathbf{F}_2$, there is even a unique *equivalence* class of small completions of any given extension of a generalized Lie amalgam α. That is (cf. Lemma F.4.11), for almost all extensions α of Lie amalgams, $Aut(\alpha)$ is realized in $Aut(\bar{G})$. But for example if $\bar{G}$ is $L_3(2)$, then $Aut(\alpha) = \bar{S} \times \langle\tau\rangle$, where τ centralizes $\bar{G}_1$ and induces the involutory inner automorphism generating $Z(\bar{S})$ on G_2. Thus in this case there are two equivalence classes of small completions.

REMARK F.4.10. Our proof of Theorem 5.3.4 in the Generic Case originally made use of the fact that over fields of order at least 4, a Cartan subgroup is sufficiently large to determine the completion. While we supplied our own proof, we could have appealed to Theorem 2 of Bennett and Shpectorov in [**BS01**] in that case. Since Cartan subgroups are usually trivial in groups over $\mathbf{F}_2$, and since we need to identify those groups too, we were led more recently to our present more uniform argument applicable in all cases. This approach uses Theorem F.4.8.

Bennett and Shpectorov also prove a result in [**BS01**] related to Theorem F.4.8, and it is probably worthwhile to briefly discuss the difference between that result and ours. Theorems 1 and 3 in [**BS01**] do not include the case of $G_2(2)'$; but that aside, those results essentially constitute a weak version of Corollary F.4.26. Using some of the results in F.4.6, one can give a quick proof of F.4.8 by assuming F.4.26; in this sense, F.4.8 and F.4.26 are essentially equivalent. However the quantification in Theorems 1 and 3 of [**BS01**] is more restrictive than that of F.4.26, so that those theorems do not imply Theorem F.4.8. More precisely, the hypothesis of [**BS01**] amounts to restricting the subgroup of G generated by at least one pair of involutions from a certain set of pairs, which are the image under the completion defining G of pairs in the group $\bar{G}_+$ of Lie type which normalize some apartment in the building of $\bar{G}_+$. However in the hypothesis of F.4.26, the restriction is on a pair allowed to range over a larger set of pairs, and that latitude is necessary in proving Theorem F.4.8. In particular in a small completion G apartments exist, but pairs of involutions acting on those apartments need not a priori be images of pairs acting on apartments of the building of $\bar{G}_+$.

Over most fields, a Cartan subgroup of $\bar{G}_+$ acts without fixed points on $\bar{S}$; when that happens, the above problem with quantification can be avoided—since as in F.4.16 below (compare Theorem 2 in [**BS01**]), apartments for both G and $\bar{G}_+$ are then determined by Cartan subgroups. But over small fields this argument is not available.

Because it presupposes no constraints on the Cartan subgroup, Theorem F.4.8 is very useful over small fields. For example, it makes possible the identification of ${}^2F_4(2)$ as a subgroup of the Rudvalis group in section 14.7. Also the form of the theorem and its proof should lend itself to further generalization.

We prove Theorem F.4.8 in a series of lemmas. Until the proof is complete, assume that $\beta : \alpha \to G$ is small.

LEMMA F.4.11. *(1) A faithful completion $\mu : \alpha \to \dot{G}$ is quasiequivalent to ι iff $\bar{G} \cong \dot{G}$.*

(2) The number of equivalence classes in the quasiequivalence class of ι is $|Aut(\alpha) : Aut_{Aut(\bar{G})}(\alpha)|$.

PROOF. If ι is quasiequivalent to μ, then by definition $\bar{G} \cong \dot{G}$. Conversely suppose $\varphi : \bar{G} \to \dot{G}$ is an isomorphism. Let $\bar{T}$ and $\dot{T}$ be Sylow p-subgroups of $\bar{G}_{1,2}$ and $\dot{G}_{1,2}$, respectively. By hypothesis μ is faithful, so $|\dot{T}| = |\bar{T}|$. Then as $\bar{G} \cong \dot{G}$, $\dot{T} \in Syl_p(\dot{G})$ and $\varphi^{-1}(\dot{T}) \in Syl_p(\bar{G})$. Therefore $\varphi^{-1}(\dot{G}_i)$, $i = 1, 2$, are the normalizers in $\bar{G}$ of maximal parabolics over $\bar{S}_+ := \varphi^{-1}(\dot{T}) \cap \bar{G}_+$, and $\varphi^{-1}(\dot{G}_{1,2})$ is the normalizer in $\bar{G}$ of a Borel subgroup over $\bar{S}_+$. Thus composing φ with an inner automorphism of $\bar{G}$ mapping $\bar{S}$ to $\bar{S}_+$, we may assume $\varphi(\bar{G}_J) = \mu(\bar{G}_J)$. As μ is faithful, we can form $\mu^{-1} : \mu(\alpha) \to \alpha$. Therefore $\mu^{-1} \circ \varphi \circ \iota =: \sigma \in Aut(\alpha)$, and $\mu \circ \sigma = \varphi \circ \iota$, so μ and ι are quasiequivalent. Thus (1) holds.

By definition, the number of equivalence classes in the quasiequivalence class of ι is $|Aut(\alpha) : Aut(\alpha)_\iota|$. Further by parts (4.v) and (5) of F.2.8, $Aut(\alpha)_\iota = Aut_{Aut(\bar{G})}(\alpha)$, so (2) holds. □

LEMMA F.4.12. *For all $a \in \Gamma$ and $b \in \Gamma^m(a)$, there are at least two geodesics from a to b in Γ.*

PROOF. By hypothesis G is small, so there exists some pair of *opposites* c and d (meaning vertices at distance m in Γ) with distinct geodesics p and q from c to d. Then for each pair u and v of vertices opposite in the cycle $\Sigma := pq^{-1}$, there are two geodesics from u to v in Σ. But by F.4.6.8, $\Pi_m(a)$ is the set of geodesics from a to opposites of a, for each $a \in \Gamma$. Further by F.4.6.1 and F.4.6.2.a, G is transitive on pairs of opposite points. Hence pq^{-1} contains a conjugate of each such pair, so the result holds when a is a point. So suppose instead a, b is a pair of opposite lines. There is a geodesic $r := y_0 \cdots y_m$ from a to b. Further each $y \in \Gamma(b) - \{y_{m-1}\}$ is an opposite to the point y_1, since $t := y_1 \cdots y_m y \in \Pi_m(y_1)$. As the result holds for points, there is a second geodesic s from y_1 to y. Then the lines a and b are on two geodesics in ts^{-1}. The proof is complete. □

The next result shows that various numerical parameters associated to G and Γ are the same as those of $\bar{G}$ and $\bar{\Gamma}$, including the order of G. This is not enough, however, to conclude immediately that we have a group isomorphism.

LEMMA F.4.13. *(1) Γ is a generalized m-gon.*
(2) $|G| = |\bar{G}|$.

PROOF. We remark that when α is a Lie amalgam, (1) follows from F.3.5.3, in view of F.4.6.1 and our hypothesis that the completion is small. But the following easy argument also works in the case of $G_2(2)'$ and the Tits group: Let $u \in \Gamma$. By F.4.6.6, there is a unique geodesic from u to each v of distance less than m in Γ. Further for each opposite v of u, F.4.12 and F.4.6.8 say that $G_{u,v}$ is transitive on $\Gamma(v)$, and hence $\Gamma(v) \subseteq \Gamma^{m-1}(u)$. Then (1) holds just as in the proof of F.3.5.3, so in particular Γ has diameter m.

By (1), the set Γ_1 of points is the disjoint union of the sets $\Gamma^{2n}(x)$, $0 \leq n \leq m/2$. Further for $n < m/2$, by (7) and (3) of F.4.6,

$$|\Gamma^{2n}(x)| = |G_x : G_p| = |\bar{G}_{\bar{x}} : \bar{G}_{\bar{p}}| = |\bar{\Gamma}^{2n}(\bar{x})|.$$

Similarly for $b \in \Gamma^m(x)$ and $\bar{b} \in \bar{\Gamma}^m(\bar{x})$, by (1) and (2.a) of F.4.6, and the transitivity of $G_{x,b}$ on $\Gamma(x)$ in (ii)–(iv) of F.4.6.8,

$$|\Gamma^m(x)| = |G_x : G_{x,b}| = |\bar{G}_{\bar{x}} : \bar{G}_{\bar{x},\bar{b}}| = |\bar{\Gamma}^m(\bar{x})|.$$

Thus $|\Gamma_1| = |\bar{\Gamma}_1|$, so

$$|G| = |\Gamma_1||G_x| = |\bar{\Gamma}_1||\bar{G}_{\bar{x}}| = |\bar{G}|.$$

□

LEMMA F.4.14. *Let $a \in \Gamma$ and $b \in \Gamma^m(a)$. Then*

(1) The apartments through a and b are the subgraphs of the form pq^{-1}, where p and q are distinct geodesics from a to b.

(2) There is a unique geodesic from a to b through each $c \in \Gamma(a)$.

(3) There are $\binom{q(a)}{2}$ apartments containing a and b.

(4) Each pair of simplices in Γ is contained in an apartment.

PROOF. Part (1) is a straightforward consequence of the definition of an apartment. We established (2) during the proof of F.4.13; then (1) and (2) imply (3). Let s and t be simplices in Γ; thus s is a point, line, or incident point-line pair. Pick $a \in s$ and $u \in t$ with $d(a, u) =: n$ maximal, and let $a := x_0 \cdots x_n := u$ be a geodesic. By F.4.6 we can extend this geodesic to a geodesic $p := x_0 \cdots x_m$ of length m; without loss $x_m = b$. If $|s| = |t| = 1$ we are done, so we may assume that $s = \{a, c\}$. Thus $d(c, u) < n$ by our maximal choice, so $d(c, u) = n - 1$. Similarly either $t = \{u\}$, or $t = \{u, v\}$ with $d(a, v) = n - 1$. Now if $n < m$ then by F.4.6.6, there is a unique geodesic from a to u, so $c = x_1$, and $v = x_{n-1}$ if $|t| = 2$. Thus we may assume that $n = m$. Here by (2) we may choose p so that $x_1 = c$, so we are done unless $|t| = 2$ and $v \neq x_{m-1}$. In this final case, by (2) there is a geodesic q from u to a through v, and by (1) , $pq \in \mathcal{A}$, completing the proof. □

We use the term *Tits amalgam* to refer to the amalgam for the Tits group ${}^2F_4(2)'$.

LEMMA F.4.15. *Let $\Sigma \in \mathcal{A}$. Then*

(1) If α is not the Tits amalgam then G is transitive on $\mathcal{A}$.

(2) If α is not the $G_2(2)'$-amalgam then there exist reflections $s_u \in G_u$ on Σ for $u := x, l$, with $W := \langle s_x, s_l \rangle$ acting as D_{2m} on Σ, and transitively on $\Gamma_i \cap \Sigma$ for $i = 1, 2$. If α is the Tits amalgam, then W is faithful on Σ.

(3) If α is the Tits amalgam then G has two orbits of equal length on $\mathcal{A}$.

(4) If α is the $G_2(2)'$-amalgam then $W := N_G(\Sigma) = \langle s_l, s_c \rangle \cong S_3$ is regular on the 6 points of Σ, and has two orbits lW and cW of length 3 on the lines of Σ, where $c \in \Gamma(x) \cap \Sigma$, and s_l and s_c act as reflections on Σ through l and c, respectively.

PROOF. Let a, b be opposites in Γ, $H := G_{a,b}$, and Λ the set of geodesics from a to b. Pick distinct $p, q \in \Lambda$; by F.4.14.1, $\Sigma := pq^{-1} \in \mathcal{A}$.

Assume first that α is not the amalgam of $G_2(2)'$. By F.4.6.8, either H is a complement to Q_a in G_a and H is 2-transitive on Λ, or α is the Tits amalgam, a is a point, and $H \cong D_{10}$ is transitive on Λ of order 5. In either case there is an involution $s_a \in H$ interchanging p and q. Thus s_a induces a reflection on the $2m$-gon Σ, fixing the opposites a and b. Similarly for $c \in \Gamma(a) \cap \Sigma$, there is an involution s_c in G_c inducing a reflection on Σ. Therefore $W := \langle s_a, s_c \rangle$ induces D_{2m} on Σ, establishing the first sentence of (2).

Suppose α is the Tits amalgam, and let K denote the kernel of W on Σ. Then $K \leq G_{a,b,c,cs_a} = H_{c,cs_a} = 1$ from the action of D_{10}, so W is faithful on Σ. This completes the proof of (2). Also G has two orbits on opposites (a, b) with a a line by F.4.6.2.b, while by (2), $N_G(\Sigma)$ is transitive on the lines in Σ, so that (3) holds.

Suppose instead that α is an extension of a Lie amalgam. Then as H is 2-transitive on Λ by F.4.6.8, H is transitive on the pairs of geodesics from a to b, and hence transitive on the apartments through a and b by F.4.14.1. Thus as G is transitive on pairs (a, b) of opposites with $a \in \Gamma_i$ by F.4.6.1, (1) holds when α is an extension of a Lie amalgam.

Therefore we may assume α is the $G_2(2)'$-amalgam; thus $m = 6$ and it remains to prove (1) and (4). By F.4.6.2.a, G_x is transitive on $\Gamma^6(x)$. If $b \in \Gamma^6(x)$ then by F.4.6.8, $G_{x,b} \cong \mathbf{Z}_3$ is transitive on 2-subsets of the set of three geodesics from x to b, and hence on pairs of these geodesics. We conclude that $G_{x,\Sigma} = 1$, so that G_x is regular on the set of apartments of Γ containing x, with representative $\Sigma = pq^{-1}$. This implies that G is transitive on $\mathcal{A}$, completing the proof of (1). As $G_{x,\Sigma} = 1$, $W := N_G(\Sigma)$ is regular on the 6 points of Σ, so that $|W| = 6$. On the other hand, we may choose $l \in \Sigma$. Then if k is the opposite to l in Σ, F.4.6.8 says that $G_{l,k}$ acts 2-transitively as S_3 on the three geodesics from l to k, so $W_l = W_{l,k} = W_k$ is of order 2, and is generated by the reflection s_l on Σ through l. Thus as $|W| = 6$ it follows that $W \cong S_3$ has two orbits on the lines of Σ, with representatives l and k. Also $W = \langle s_l, s_c\rangle$, where $c \in \Gamma(x) - \{l\}$ and $W_c = \langle s_c\rangle$. Thus (4) holds. □

Recall from Notation F.4.4 that $\bar{S} \in Syl_p(\bar{G}_+)$ and $S = \beta(\bar{S})$.

We will supply two proofs of Proposition F.4.17. The following lemma is used in the second of those proofs, but not elsewhere; thus the reader may safely skip this lemma.

LEMMA F.4.16. *Assume α is an extension of a Lie amalgam, and let $x, l \in \Sigma \in \mathcal{A}$. Then*

(1) $S \trianglelefteq G_{1,2}$, and the kernel K of the action of $N_G(\Sigma)$ on Σ is a complement to S in $G_{1,2}$.

(2) $N_G(\Sigma) = KW$, where W is the subgroup defined in F.4.15.

PROOF. As $\bar{S} \trianglelefteq \bar{G}_{1,2}$, applying β we conclude $S \trianglelefteq G_{1,2}$. Let b be the opposite to x in Σ, $c \neq l$ the second member of $\Gamma(x) \cap \Sigma$, and $H := G_{x,b}$. By F.4.14.1, $\Sigma = pq^{-1}$, where p, q are the geodesics from x to b through l and c, respectively. As α is an extension of a Lie amalgam, conclusion (ii) of F.4.6.8 holds, so H is a complement to Q_x in G_x. Therefore $G_{1,2} = Q_x H_l$ and Q_x is regular on $\Gamma^m(x)$. Then as S/Q_x is regular on $\Gamma(x) - \{l\}$, S is regular on $\Gamma^m(x) \times (\Gamma(x) - \{l\})$, so $H_{l,c}$ is a complement to S in $G_{1,2}$. Now $K \leq H_{l,c}$, and by F.4.6.6, H_l and H_c fix p and q pointwise, so $H_{l,c} = K$. This establishes (1). Further from F.4.15, W acts on Σ as $Aut(\Sigma)$, so (2) follows from (1). □

Next in Proposition F.4.17 we complete the proof of Theorem F.4.8 in the case where α is an extension of a Lie amalgam, by quoting well-known results from the literature. Indeed with an eye to possible future revisions after the publication of this work, we are providing two proofs: One uses the Tits-Weiss classification of (finite) Moufang generalized m-gons, and the other uses the Fong-Seitz classification of finite split BN-pairs. We regard the former as our primary proof, so we have included the Tits-Weiss work in our Background References. Our discussion of weak BN-pairs relies on the Green Book [**DGS85**], and the Green Book makes use of the Fong-Seitz Theorem, which in turn is equivalent to the finite case of the Tits-Weiss Theorem. Thus at the moment no purpose would be served by avoiding an appeal to one of these results. On the other hand if the proof in the Green Book were to be modified in some future revision, one might wish to revisit the issue of appeals to Tits-Weiss or Fong-Seitz, to consider the desirability of a more self-contained proof of F.4.17.

PROPOSITION F.4.17. *If α is an extension of a Lie amalgam, then Theorem F.4.8 holds.*

PROOF. By F.4.11, it suffices to show G is isomorphic to an extension of $\bar{G}_+$, and that $G \cong \bar{G}$ if $\bar{G} = \bar{G}_+$.

We first give a proof based on Tits-Weiss: Recall Γ is a generalized m-gon by F.4.13.1. We begin with the Moufang condition (cf. [**TW02**]): For $p := x_0 \cdots x_m \in \Pi_m(a)$, we claim that $X := \bigcap_{i=0}^{m-2} G_{\Gamma(x_i)}$ is transitive on $\Gamma(x_{m-1}) - \{x_{m-2}\}$. To see this, we may take $x_0 := x$ or l, and by F.4.6.3, we may pick a path $\bar{p} := \bar{x}_0 \cdots \bar{x}_m$ in $\bar{\Gamma}$ with $\psi(\bar{p}) = p$. Let $\bar{H} := \bar{G}_+$, so that $\bar{G}$ is an extension of the Lie group $\bar{H}$. Then F.4.5.6 applies to $\bar{H}$, so we conclude $O^{p'}(\bar{H}_{\bar{x}_{i-1},\bar{x}_i,\bar{x}_{i+1}}) = \bar{Q}_{\bar{x}_i}$ for each $0 < i < m$, so $O^{p'}(\bar{H}_{\bar{x}_0 \cdots \bar{x}_{m-1}}) \leq \bar{X} := \beta^{-1}(X)$ by F.4.6.3. Then as $\bar{H}_{\bar{x}_0 \cdots \bar{x}_{m-1}}$ is transitive on $\bar{\Gamma}(\bar{x}_{m-1}) - \{\bar{x}_{m-2}\}$ of order a power of p, $O^{p'}(\bar{H}_{\bar{x}_0 \cdots \bar{x}_{m-1}})$ is transitive on $\bar{\Gamma}(\bar{x}_{m-1}) - \{\bar{x}_{m-2}\}$. So applying β and appealing to F.4.6.3, X is transitive on $\Gamma(x_{m-1}) - \{x_{m-2}\}$, establishing the claim. Then as Γ is a Moufang generalized m-gon, we can appeal to Tits and Weiss [**TW02**] to conclude that $\Gamma \cong \bar{\Gamma}$. This isomorphism of geometries induces an isomorphism of groups $\varphi : Aut(\bar{\Gamma}) \to Aut(\Gamma)$, which maps the subgroup $\bar{G}_+$ of $Aut(\bar{\Gamma})$ generated by all elations of $\bar{\Gamma}$ (that is, automorphisms fixing either all points on some line, or all lines on some point) to the subgroup G_+ of $Aut(\Gamma)$ generated by all elations of Γ. The image G_+ is contained in G, since G_1 and G_2 contain all elations centered at x and l, respectively, and G is transitive on Γ_i. Thus $\dot{G} := \varphi^{-1}(G)$ is an extension of $G_+ \cong \bar{G}_+$, with $G \cong \dot{G}$. Further $G \cong \bar{G}$ if $\bar{G} = \bar{G}_+$.

Next we give a second proof, based on Fong-Seitz: It will suffice to show that

$$\mathcal{T} := (G, G_{1,2}, N_G(\Sigma), \{s_x, s_l\})$$

is a Tits system, or BN-pair (cf. section 43 in [**Asc86a**]). For observe that by F.4.16, S is a normal complement in $G_{1,2}$ to the kernel K of the action of $N_G(\Sigma)$ on Σ, so that the BN-pair is split [**FS73**], (cf. (L2) in section 47 of [**Asc86a**]). Thus G is isomorphic to an extension $\dot{G}$ of a Lie group by the classification of finite split BN-pairs of rank 2 and $G \cong \bar{G}$ if $\bar{G} = \bar{G}_+$ just as in the previous paragraph.

To show that $\mathcal{T}$ is a Tits system, we prove Γ is a building (cf. section 42 of [**Asc86a**]), and then appeal to 43.1 in [**Asc86a**].

We first check that Γ is a building: We have seen Γ is a generalized m-gon, and that as $|\Gamma(u)| = q(u) + 1 > 2$, Γ is thick. The members of $\mathcal{A}$ are ordinary n-gons and hence thin. By F.4.14.4, each pair of chambers is contained in an apartment. Further by F.4.15.2, G is transitive on the set of pairs (Φ, C), where $\Phi \in \mathcal{A}$ and C is a chamber in Φ. Thus if s and t are simplices contained in apartments Σ and Φ, then there is $g \in G_s$ with $\Phi g = \Sigma$. By F.4.15.2, $N := N_G(\Sigma)$ induces $Aut(\Sigma)$ on Σ. Thus as $t, tg \in \Sigma$ with $d(s,t) = d(s,t^g)$, there is $w \in N_s$ with $gw \in G_t$. Now $gw : \Phi \to \Sigma$ is an isomorphism fixing s and t. This completes the verification of the conditions in section 42 of [**Asc86a**], so that Γ is a building.

We have verified the hypotheses of 43.1 in [**Asc86a**], so by (3) of 43.1, $\mathcal{T}$ is indeed a Tits system. Hence the second proof of Proposition F.4.17 is complete. □

By Proposition F.4.17, we may assume during the remainder of the proof of Theorem F.4.8 that α is the amalgam of $G_2(2)'$ or the Tits group. No characterization of generalized m-gons which are not Moufang is available in the literature, so we provide a self-contained treatment in the remainder of this subsection. We will use results from section F.2 to show that the automorphisms of α defined by the extension $F \cong G_2(2)$ or ${}^2F_4(2)$ of $\bar{G}$ in fact lift to G, at which point we can appeal to Proposition F.4.17 to complete the proof.

We begin by parametrizing small completions in terms of certain pairs of involutions from our amalgam.

DEFINITION F.4.18. Pick $c \in \Gamma(x) - \{l\}$ and $y \in \Gamma(l) - \{x\}$.

If α is the Tits amalgam, let $u_1 := x$, $u_2 := l$, $v_1 := y$, $v_2 := c$, and $\mathcal{N}_i$ the set of involutions in G_{u_i} with cycle (u_{3-i}, v_{3-i}), and such that each $s \in \mathcal{N}_1$ generates G_{p_s} for some geodesic p_s of length $m = 8$ and origin u_1. Define $m_0 := 8$ in this case.

If α is the $G_2(2)'$-amalgam, let $u_1 := l$, $u_2 := c$, and $v_1 := y$; pick $y' \in \Gamma(c) - \{x\}$ and let $v_2 := y'$. In this case let $\mathcal{N}_i$ be the set of involutions in G_{u_i} with cycle (x, v_i), and define $m_0 := 3$.

In either case, given $(s,t) \in \mathcal{N}_1 \times \mathcal{N}_2$, define $W(s,t) := \langle s, t \rangle$ and $m_0(s,t) := |st|$. If α is the Tits amalgam let $\Sigma(s,t) := x^{W(s,t)} \cup l^{W(s,t)}$, and if α is the $G_2(2)'$-amalgam let $\Sigma(s,t) := x^{W(s,t)} \cup l^{W(s,t)} \cup c^{W(s,t)}$.

LEMMA F.4.19. *Let $\mathcal{P}$ be the set of pairs $(s,t) \in \mathcal{N}_1 \times \mathcal{N}_2$ with $m_0(s,t) = m_0$. Then the map $\phi : (s,t) \mapsto \Sigma(s,t)$ is a bijection of $\mathcal{P}$ with the set $\mathcal{B}$ of apartments of Γ containing $cxly$ or $y'cxly$, for α the Tits amalgam or $G_2(2)'$-amalgam, respectively.*

PROOF. For $\Sigma \in \mathcal{B}$, F.4.15 says that $N_G(\Sigma)$ contains reflections s_x and s_l of Σ through x and l if α is the Tits amalgam, and reflections s_l and s_c if α is the $G_2(2)'$-amalgam. Define a map ψ on $\mathcal{B}$ by $\psi(\Sigma) := (s_x, s_l)$ or $\psi(\Sigma) := (s_l, s_c)$ in the respective cases. Then by (2) and (4) of F.4.15, $\psi(\Sigma) \in \mathcal{P}$, and $\phi(\psi(\Sigma)) = \Sigma(\psi(\Sigma))) = \Sigma$.

Conversely let $(s,t) \in \mathcal{P}$. Then $m_0(s,t) = m_0$, so $W := W(s,t) \cong D_{2m_0}$. If α is the Tits amalgam, then as $|st| = m_0 = 8 = m$, $\Sigma := \Sigma(s,t) \in \mathcal{A}$, $s = s_x$, and $t = s_l$ by F.3.6. So by construction of ψ, $\psi(\phi(s,t)) = \psi(\Sigma) = (s,t)$. Thus ψ and ϕ are inverses of each other, and the lemma holds in this case. Furthermore our definitions are designed so that a similar argument works in the $G_2(2)'$-case: As $m_0 = 3 = m/2$ and c and l are distinct members of $\Gamma(x)$, $\Sigma := \Sigma(s,t) \in \mathcal{A}$ with $s = s_l$ and $t = s_c$ by F.3.7. Thus $\psi(\Sigma) = (s_l, s_c) = (s,t)$. □

LEMMA F.4.20. *(1) If α is the Tits amaglam, then $|\mathcal{N}_1| = 2^4$, $|\mathcal{N}_2| = 2^6$, $|\mathcal{P}| = 2^9$, and $Q_x \cap Q_l$ is transitive on $\mathcal{N}_1$.*

(2) If α is the $G_2(2)'$-amalgam, then $|\mathcal{N}_i| = 4$, $|\mathcal{P}| = 8$, and $Q_l \cap Q_c$ is transitive on $\mathcal{N}_1$.

PROOF. By F.4.19, $|\mathcal{P}| = |\mathcal{B}|$, where $\mathcal{B}$ is the set of apartments through $cxly$ or $y'cxly$.

Suppose first that α is the Tits amalgam. By parts (2) and (3) of F.4.15, G_x has two orbits on apartments through x, each of length $|G_x|/2 = 5 \cdot 2^{10}$. Thus there are $5 \cdot 2^{11}$ apartments through x, and each such apartment has two paths $x_0x_1x_2x_3$ with $x_1 = x$. Since x_1 has 5 neighbors and x_2 has 3 neighbors, there are $\binom{5}{2} \cdot 2 \cdot 2 = 40$ such paths, which G_x also permutes transitively, so $|\mathcal{B}| = 5 \cdot 2^{12}/40 = 2^9$.

First consider $s \in \mathcal{N}_1$. Then $s \in G_{x,b}$ for some $b \in \Gamma^8(x)$, and $G_{x,b} \cong D_{10}$ by F.4.6.8, so $G_{x,b} = \langle s \rangle X$ where s inverts X of order 5. Next calculating in $\bar{G}$, $\bar{s}\bar{z} \notin \bar{s}^{\bar{G}_{\bar{x}}}$ (cf. 3.2.13 in [**Asc82b**]), so applying β, $sz \notin s^{G_x}$. Thus $C_{\tilde{G}_x}(\tilde{s}) = \widetilde{C_{G_x}(s)}$. But there are two G_x-chief sections I on $Q_x/\langle z \rangle$, both of rank 4, and $[I,s] = C_I(s)$ for each section, so by Exercise 2.8 in [**Asc94**], $|C_{Q_x}(s)| = 2^5$ and Q_x is transitive on $s^{G_x} \cap sQ_x$. Furthermore as G_x is transitive on geodesics p_s of length m and origin x, if $p_{s'}$ is a second such geodesic, then $s^g = s'$ for some $g \in G_x$. If in addition s' has cycle (l, c), then $s' \in sQ_x$, so as $C_{G_x}(s)Q_x = N_{G_x}(sQ_x)$, we may choose $g \in Q_x$. Thus $s^{G_x} \cap sQ_x = \mathcal{N}_1$. Therefore $|\mathcal{N}_1| = 2^4$. Further $Q_x \cap Q_l$ is of index 2 in Q_x, with $N_{G_x}(Q_x \cap Q_l) = G_{x,l}$. So as s has cycle (l, c), $[Q_x, s] \not\leq Q_x \cap Q_l$, and hence $C_{Q_x}(s) \not\leq Q_x \cap Q_l$, so $Q_x = (Q_x \cap Q_l)C_{Q_x}(s)$. Therefore $Q_x \cap Q_l$ is also transitive on $\mathcal{N}_1$.

So consider $t \in \mathcal{N}_2$. Now t inverts Y of order 3 in G_l, $N_{G_l}(Y)$ is the product of Y with $S \cong D_8$, and t inverts $\langle y \rangle = C_S(Y) = S \cap Q_l \cong \mathbf{Z}_4$. This time $\mathcal{N}_2$ is the union of the Q_l-orbits of t and ty, $|Q_l| = 2^{10}$, and $|C_{Q_l}(t)| = 2^5$, so $|\mathcal{N}_2| = 2|Q_l : C_{Q_l}(t)| = 2^6$, completing the proof of (1).

Suppose finally that α is the $G_2(2)'$-amalgam. Calculating as above but with $G_p = 1$, there are $|G_x| = 3 \cdot 2^5$ apartments through x, each containing two geodesics $x_0 \cdots x_4$ with $x_2 = x$; and G_x is transitive on the 24 such geodesics, so $|\mathcal{P}| = 8$. Let $s \in \mathcal{N}_1$. Then G_l is $\mathbf{Z}_4^2$ extended by S_3 with $s \in G_l - Q_l$, so this time $\mathcal{N}_1 = s^{Q_l}$ is of order 4. By symmetry, $\mathcal{N}_2 = t^{Q_c}$ is also of order 4. Also $Q_c \cap Q_l \cong \mathbf{Z}_4$ is a complement to $C_{Q_l}(s)$, and hence is transitive on s^{Q_l}. □

Let $G_3 := G_c$, $I := \{1, 2, 3\}$, and $\mathcal{E} := (G_i : i \in I)$. Form the amalgam $\lambda := \mathcal{A}(\mathcal{E})$ as in Example 36.1 in [**Asc94**]. Notice that $G_2 \cap G_3 \leq G_1$ by F.4.6.6, so $G_{2,3} = G_{1,2,3}$.

We now begin to consider arbitrary faithful completions $\dot{G}$ of α, which are not necessarily small. Recall that α is the $\bar{G}$-amalgam $(\bar{G}_1, \bar{G}_{1,2}, \bar{G}_2)$; however for notational purposes, it will be more convenient to identify α with the G-amalgam $(G_1, G_{1,2}, G_2)$ via β. Observe that given any faithful completion $\rho : \alpha \to \dot{G}$, ρ defines an isomorphism $\rho : \alpha \to \dot{\alpha}$ of amalgams, where $\dot{\alpha} := (\dot{G}_1, \dot{G}_{1,2}, \dot{G}_2)$ with $\dot{G}_J := \rho(G_J)$. Further we can form the corresponding amalgam $\dot{\lambda}$ in $\dot{G}$ with respect to $\dot{c} \in \dot{\Gamma}(\dot{x})$ defined by $\dot{G}_{\dot{x},\dot{c}} = \rho(G_{x,c})$.

LEMMA F.4.21. *Assume $\rho : \alpha \to \dot{G}$ is a faithful completion of α. Then ρ extends to an isomorphism $\rho : \lambda \to \dot{\lambda}$.*

PROOF. There is $g \in G_x$ with $lg = c$. Let $\dot{g} := \rho(g)$, and define $\varphi : G_c \to \dot{G}_{\dot{c}}$ by $\varphi = c(\dot{g}) \circ \rho \circ c(g^{-1})$, where c maps an element to conjugation by that element. Then for $h \in G_{x,c}$,

$$\varphi(h) = \rho(h^{g^{-1}})^{\dot{g}} = (\rho(h)^{\rho(g)^{-1}})^{\dot{g}} = (\rho(h)^{\dot{g}^{-1}})^{\dot{g}} = \rho(h),$$

so φ extends ρ to $G_c = G_3$, since $G_2 \cap G_3 \leq G_1$. □

The next result will provide the hypothesis for F.2.11.4.

LEMMA F.4.22. $Z(G) = 1$.

PROOF. Observe that for $u \in \Gamma_1$, $Z(G_u)$ is generated by an involution $z(u)$. Let $N := m/2$; we check in $\bar{G}$ that $z(\bar{x}) \in \bar{G}_{\bar{y}}$ for $\bar{y} \in \bar{\Gamma}^N(\bar{x})$; so applying β, $z := z(x) \in G_y$. Thus as there is a unique geodesic from x to y, z fixes each member of that geodesic, and hence by F.4.6.6, z fixes each member of $\Gamma^i(x)$ for $i \leq N$.

Let $x_0 \cdots x_m$ be a geodesic between opposite points x_0 and x_m with $x_N \in \{x, l\}$. By the previous paragraph, $z(x_{2i}) \in G_{x_N}$ for $1 \leq i \leq N$. Furthermore as $\bar{G}_{\bar{x}} = C_{\bar{G}}(\bar{z})$, $\bar{z}(x_{2i}) \neq z(\bar{x}_{2j})$ for $i \neq j$; so applying β, $z(x_{2i}) \neq z(x_{2j})$. Thus as $\Gamma^{2i}(x)$ are orbits of G_x on Γ_1, we conclude that the map $u \mapsto z(u)$ is an injection on Γ_1. Thus $|z^G| \geq |\Gamma_1|$ and hence as $|z^G||C_G(z)| = |G| = |\Gamma_1||G_x|$, $|C_G(z)| =$

$|\Gamma_1||G_x|/|z^G| \leq |G_x|$, so as $G_x \leq C_G(z)$ we conclude that $G_x = C_G(z)$. Then as $Z(G_x) = \langle z\rangle$ and $z \notin Z(G_l)$, the lemma holds. □

Given any faithful completion $\rho : \alpha \to \dot{G}$ of α, we have seen in F.4.21 that the isomorphism $\rho : \alpha \to \dot{\alpha}$ extends to an isomorphism $\rho : \lambda \to \dot{\lambda}$. This isomorphism therefore takes $\mathcal{N}_i$ to $\dot{\mathcal{N}_i}$, and hence extends to a bijection $\rho : \mathcal{N}_1 \times \mathcal{N}_2 \to \dot{\mathcal{N}_1} \times \dot{\mathcal{N}_2}$; we abuse notation and identify the two sets via this bijection. Next we can define $\dot{\mathcal{P}} = \mathcal{P}(\dot{G})$ as in F.4.19, now viewed as a subset of $\mathcal{N}_1 \times \mathcal{N}_2$, subject to our identification; of course $\mathcal{P}$ is also a subset of $\mathcal{N}_1 \times \mathcal{N}_2$. We will soon see that ρ is small iff $\dot{\mathcal{P}} \neq \emptyset$, and in that event $\dot{\mathcal{P}}$ determines the equivalence class of the small completion ρ.

For $(s,t) \in \dot{\mathcal{P}}$, define $M(s,t)$ to be the normal subgroup of $\hat{G}$ generated by $(\hat{s}\hat{t})^{m_0}$, where $\hat{r} := \xi(r)$ is the image of $r := s$ or t under the universal completion ξ of α.

Let $\gamma := (F_1, F_{1,2}, F_2)$ be the amalgam of the Lie group $F \cong G_2(2)$ or ${}^2F_4(2)$, extending our generalized Lie group $\bar{G} = E(F)$. Recall we identify α with $\bar{\alpha}$ via the isomorphism β, so that $G_J = \bar{G}_J$; choose notation so that $G_J \leq F_J$. As we saw during the proof of F.4.5.2, $G_J \trianglelefteq F_J$ and $F_J = F_{1,2}G_J$, with $F_{1,2}$ a Sylow 2-group of F. In particular, hypotheses (a), (b), and (c) of F.2.10 are satisfied with F, $\bar{G}$ in the roles of "Y, X", and $\{1,2\}$ is the least member "e" of our poset D. Moreover hypotheses (e) and (f) of F.2.11 are also satisfied. By F.2.10.1, the conjugation map $c : F_{1,2} \to Aut(\alpha)$ is a representation of $F_{1,2}$ on α.

We will show next that the sets $\dot{\mathcal{P}}$ parametrize the equivalence classes of small completions of α, and then use this fact to lift the automorphisms of α induced in the completion F to G. Then we can complete the proof of Theorem F.4.8 using Proposition F.4.17.

Recall from Notation F.4.4 that θ_G is the induced map from the universal completion $\hat{G}$ to G, and $\iota : \alpha \to \bar{G}$ is the Lie completion.

LEMMA F.4.23. *Assume $\rho : \alpha \to \dot{G}$ is a faithful completion of α. Then*
(1) $\rho : \alpha \to \dot{\Gamma}$ is small iff $\dot{\mathcal{P}} \neq \emptyset$.
(2) If $(s,t) \in \dot{\mathcal{P}} \cap \mathcal{P}$ then $\ker(\theta_G) = M(s,t) = \ker(\theta_{\dot{G}})$ and β is equivalent to ρ.
(3) If $\dot{\mathcal{P}} \cap \mathcal{P} \neq \emptyset$, then β is equivalent to ρ.
(4) There are at most two equivalence classes of small completions of α.
(5) If the equivalence class of β is invariant under the image $c(F_{1,2})$ of conjugation by elements in $F_{1,2}$, then β is quasiequivalent to ι.

PROOF. If $\dot{\Gamma}$ is small then $\dot{\mathcal{P}} \neq \emptyset$ by F.4.20. Conversely if $(s,t) \in \dot{\mathcal{P}}$ then an argument in the proof of F.4.19 (based on F.3.6 and F.3.7) shows that $\Sigma(s,t)$ is a $2m$-gon in $\dot{\Gamma}$, so $\dot{G}$ is small. Thus (1) is established.

Now assume the hypotheses of (2). Let $M := M(s,t)$, $G^* := \hat{G}/M$, and $\theta_{\dot{G}} := \hat{\rho} : \hat{G} \to \dot{G}$ the surjection induced by the completion $\rho : \alpha \to \dot{G}$ as in Definition F.2.2. As $(s,t) \in \dot{\mathcal{P}}$, $(\dot{s}\dot{t})^{m_0} = 1$; then $(\dot{s}\dot{t})^{m_0} \in \ker\theta_{\dot{G}}$, so that $\theta_{\dot{G}}$ induces a surjection $\varphi : G^* \to \dot{G}$. Thus as ρ is faithful, φ is injective on the image G_i of $\hat{G}_i$ in G^*, so G^* is a faithful completion of α. By construction $(s,t) \in \mathcal{P}^* := \mathcal{P}(G^*)$, so G^* is small by (1). Therefore $|G^*| = |\bar{G}| = |\dot{G}|$ by F.4.13.2, so $M(s,t) = \ker(\theta_{\dot{G}})$. We have symmetry as we are also assuming $(s,t) \in \mathcal{P}$, so similarly $M(s,t) = \ker(\theta_G)$. The final assertion of equivalence in (2) now follows from F.2.4.2. Of course (2) implies (3).

For $s \in \mathcal{N}_1$ let $\mathcal{Q}(s) := \{t \in \mathcal{N}_2 : (s,t) \in \mathcal{P}\}$, and let $P := Q_x \cap Q_l$ or $Q_c \cap Q_l$ for α the Tits amalgam or the $G_2(2)'$-amalgam, respectively. Notice P acts on the $\mathcal{N}_1$ and $\mathcal{N}_2$, and hence on $\mathcal{P}$. By F.4.20, P is transitive on $\mathcal{N}_1$. Thus $|\mathcal{P}| = |\mathcal{N}_1||\mathcal{Q}(s)|$. But by F.4.20, $|\mathcal{P}| = |\mathcal{N}_1 \times \mathcal{N}_2|/2$, so $|\mathcal{Q}(s)| = |\mathcal{N}_2|/2$. Further if the completions β and ρ are not equivalent, then by (3), $\mathcal{Q}(s) \cap \dot{\mathcal{Q}}(s) = \emptyset$, so $\mathcal{N}_2$ is the disjoint union of $\mathcal{Q}(s)$ and $\dot{\mathcal{Q}}(s)$. Each subset determines an equivalence class by (3), completing the proof of (4).

It remains to prove (5), so assume $c(F_{1,2})$ stabilizes the equivalence class of β. We have already observed that hypotheses (a), (b), and (c) of F.2.10 are satisfied, as are hypotheses (e) and (f) of F.2.11. Further we are now assuming hypothesis (d) of F.2.11, so we can appeal to F.2.11. Let $\tilde{G} := Inn(G)$, and in the notation of F.2.10 and F.2.11, let $\tilde{F} := \tilde{G}\ \pi^-(c(F_{1,2}))$. By F.2.11.1, we have a faithful completion $\eta : \gamma \to \tilde{F}$. By F.4.22, $Z(G) = 1$, so $c^- : G \to \tilde{G}$ is an isomorphism, and by F.2.11.4, β is equivalent to $\eta_{|\alpha}$. Let Φ be the geometry of $\tilde{F}$. Then for $f \in F$, the map $\tilde{G}_i f \mapsto \tilde{F}_i f$ is an injection of Γ into Φ, so as Γ possesses $2m$-subgons, so does Φ. Thus $\tilde{F}$ is small, so by Proposition F.4.17, η is quasiequivalent to the inclusion $\iota_F : \gamma \to F$. Thus $\tilde{F} \cong F$, so $\tilde{G} = E(\tilde{F}) \cong E(F) = \bar{G}$. We saw $G \cong \tilde{G}$, so β is quasiequivalent to ι by F.4.11.1. Thus (5) is established. □

We are now in a position to complete the proof of Theorem F.4.8 for our two remaining generalized Lie amalgams. As we observed earlier, $c(F_{1,2}) \leq Aut(\alpha)$. Therefore $c(F_{1,2})$ permutes the equivalence classes of small completions of α via the representation π of Definition F.2.5. By F.4.23.4, there are at most two such equivalence classes. Assume that β is not quasiequivalent to ι; then in particular β is not equivalent to ι, so that there are exactly two equivalence classes. Applying F.2.8.5 with ι in the role of "μ", $c(F_{1,2})$ acts on the equivalence class $[\iota]$, so it must also act on the second class $[\beta]$. But then by F.4.23.5, ι and β are quasiequivalent. This contradicts our assumption, and so completes the proof of Theorem F.4.8.

We have the following two corollaries to Theorem F.4.8:

Corollary F.4.24. *Suppose $\beta : \alpha \to G$ is a faithful completion of an extension of a generalized Lie amalgam α, defined by an extension $\bar{G}$ of a generalized Lie group $\bar{G}_+$. Assume also that $|G| \leq |\bar{G}|$. Then $|G| = |\bar{G}|$, and G is isomorphic to an extension of $\bar{G}_+$.*

Proof. By Theorem F.4.8 and Lemma F.4.13.2, it suffices to show that β is small. The argument establishing F.4.13.2 shows that $|\bar{\Gamma}^m(\bar{x})| = |Q_x|$. Then the number of pairs $(\bar{p}, \bar{b})$ where $\bar{b} \in \bar{\Gamma}^m(\bar{x})$ and $\bar{p}$ is a geodesic from $\bar{x}$ to $\bar{b}$ is $|\bar{G}_{\bar{x}} : \bar{G}_{\bar{p}}| = (q_1+1)|Q_{\bar{x}}|$, since $\bar{b}$ is on exactly $q_1 + 1$ such geodesics. By F.4.6.3, this value is also the number of corresponding pairs (p, b) from Γ. Assume that β is large. Then each b determines a unique p, so that in fact

$$|\Gamma^m(x)| = |G_x : G_p| = (q_1 + 1)|Q_x|.$$

Define $\Phi(x) := \{u \in \Gamma_1 : d(x,u) \leq m\}$. If $\bar{G}$ is not an extension of $L_3(q)$ then m is even, so the counting argument in the proof of F.4.13, applied now just to the points $\Phi(x)$ at distance at most m (which uses only F.4.6, and hence does not depend on the hypothesis for Theorem F.4.8 that the completion is small) shows that $|\Phi(x)| = |\bar{\Gamma}_1| + q_1|Q_x|$, so that

$$|G| = |\Gamma_1||G_x| \geq |\Phi(x)||G_x| > |\bar{\Gamma}_1||\bar{G}_{\bar{x}}| = |\bar{G}|,$$

contrary to our hypothesis, establishing the corollary in this case. Similarly if $\bar{G}$ is an extension of $L_3(q)$, the members b of $\Gamma^3(x)$ are lines, and as there is a unique geodesic from x to b, there are q points in $\Gamma(b) - \Phi(x)$. Then $|\Gamma_1| > |\Phi(x)| = |\bar{\Gamma}_1|$, so we obtain the same contradiction. □

REMARK F.4.25. Notice that when $\bar{G}$ is $G_2(2)'$, the following Corollary holds vacuously, as there are no involutions in $G_1 - Q_1$.

COROLLARY F.4.26. *If $\beta : \alpha \to G$ is a faithful completion of a generalized Lie amalgam α defined by an extension $\bar{G}$ of a generalized Lie group $\bar{G}_+$, and s_i is an involution in $G_i - G_{1,2}$ for $i = 1, 2$, then $|s_1 s_2| \geq m$; and in case of equality, G is isomorphic to an extension of $\bar{G}_+$.*

PROOF. By F.3.6, $|s_1 s_2| \geq m$, and in case of equality, the completion is small. In the latter case, G is isomorphic to an extension of $\bar{G}_+$ by Theorem F.4.8. □

F.4.2. Sufficient local-group-theoretic conditions for uniqueness of completions. In this subsection, we assume that our Lie groups are defined over fields of characteristic 2. We continue the convention that if $\bar{G}$ is the generalized Lie group defining a generalized Lie amalgam, and $\bar{G}$ is not $L_3(q)$, then $\bar{G}_1$ is the maximal parabolic normalizing a long root subgroup of $\bar{G}$. Subject to that convention, the following facts are well known; see for example section 18 in [**AS76a**].

LEMMA F.4.27. *Let $G = \bar{G}$ be the generalized Lie group in characteristic 2 defining a generalized Lie amalgam α, with Sylow 2-group S as in Notation F.4.4. Then*

(1) If $G \cong L_3(q)$ or $G_2(2)'$, then G has one conjugacy class of involutions, the long root involutions.

(2) If $G \cong Sp_4(q)$ then G has three classes of involutions: the long and short root involutions and the involutions of type c_2, all fused into $Z(S)$.

(3) If G is not $L_3(q)$, $Sp_4(q)$, or $G_2(2)'$, then G has two classes of involutions, the long and short root involutions. The long root involutions are the involutions fused into $Z(S)$.

(4) Assume G is not $L_3(q)$, and let $Z_1 := Z(O^{2'}(G_1))$ and $z \in Z_1^{\#}$. Then

(i) $O^{2'}(G_1) = C_G(z)$ and z is a long root involution, and

(ii) unless $G \cong G_2(2)'$, $C_G(t) \leq G_2$, for some short root involution t in $O_2(G_2)$.

(5) If G is $L_3(q)$ then $S = C_G(z)$ for each $z \in Z(S)^{\#}$.

Recall from Definition F.4.3 the definition of an $\bar{M}$-extension γ of a generalized Lie amalgam α, defined by an extension $\bar{M}$ of a generalized Lie group $\bar{G}$, in which $\bar{G} = F^*(\bar{M})$. As in Notation F.4.4, we adopt some notational conventions:

NOTATION F.4.28. Let γ be an $\bar{M}$-extension of a generalized Lie amalgam α, defined by the generalized Lie group $\bar{G}$. Let $\beta : \gamma \to M_0$ be a faithful completion of γ; we also write $\beta : \alpha \to G_0$ for the restriction of β to α. Let $\bar{S} := O_2(\bar{G}_{1,2})$ and $S := \beta(\bar{S})$. Typically T will denote a Sylow 2-group of $M_{1,2}$, so that T is also Sylow in each M_i. Also q denotes the order of the field of definition of $\bar{G}$.

LEMMA F.4.29. *Let γ be an $\bar{M}$-extension of the Lie amalgam α defined by a Lie group $\bar{G}$, $\beta : \gamma \to M_0$ a faithful completion, and $T \in Syl_2(M_{1,2})$. Let $L_i := O^{2'}(G_i)$. Then*

(1) If $q = 2$, then either $\alpha = \gamma$, or $\bar{G} \cong U_4(2)$ or $U_5(2)$ is of index 2 in $M = Aut(\bar{G})$.

(2) T/S is cyclic and T splits over S. If t is an involution in $\bar{T} - \bar{S}$, then one of the following holds:

(i) t induces a field automorphism on $\bar{G}$.

(ii) t induces a graph automorphism on $\bar{G} \cong U_4(q)$ or $U_5(q)$, and $C_{\bar{G}}(t) \cong Sp_4(q)$.

(iii) $\bar{G} \cong U_4(q)$, $t = sz$, where s induces a graph automorphism, z is a long root involution in $C_{\bar{G}}(s)$, and $C_{\bar{M}}(t) = C_{\bar{M}}(s) \cap C_{\bar{M}}(z)$.

(3) If $q > 2$, then $L_i = G_i^\infty$.

(4) $S = O_2(M_{1,2})$ and $O_2(M_i) = O_2(G_i)$ for $i = 1, 2$.

(5) Either

(i) $O_2(L_iT) = O_2(L_i)$, or

(ii) $\bar{G} \cong U_4(q)$, $\alpha \leq \delta \leq \gamma$, with δ a $\mathbf{Z}_2/U_4(q)$-amalgam, $|O_2(L_iT) : O_2(L_i)| = 2$, $i = 1, 2$, and $O_2(L_iT) = O_2(L_i)\langle t \rangle$, where $\beta^{-1}(t)$ is an involution inducing a graph automorphism on $\bar{G}$.

(6) Either

(i) $J(T) = J(S)$ and $Baum(T) = Baum(S)$, or

(ii) $\bar{G} \cong U_4(2)$ or $U_5(2)$.

(7) $Z(T) = C_{Z(S)}(T)$ and one of the following holds:

(i) $Z(S) = Z(L_1)$.

(ii) $\bar{G} \cong L_3(q)$ and $Z(\bar{S})$ is a long root subgroup of $\bar{G}$.

(iii) $\bar{G} \cong Sp_4(q)$ and $Z(\bar{S})$ is the direct product of a long and a short root subgroup of $\bar{G}$.

(8) $\beta(\bar{S} \cap O^2(\bar{G})) \leq O^2(G_0)$. Thus either $S \leq O^2(G_0)$, or $\bar{G} \cong Sp_4(2)$, $G_2(2)$, or ${}^2F_4(2)$ and $|S : S \cap O^2(G_0)| = 2$.

PROOF. As the first seven statements are about the amalgam γ, we may work in the extension $\bar{M}$ of $\bar{G}$ in proving those statements, and we take $G = \bar{G}$ and $M = \bar{M}$ to simplify the notation. If $q = 2$ then either the subgroup of $Out(G)$ trivial on the Dynkin diagram of G is of odd order, or G is $U_4(2)$ or $U_5(2)$ and $|Out(G)| = 2$. Thus (1) holds. Similarly the subgroup of $Out(G)$ trivial on the Dynkin diagram has cyclic Sylow 2-groups, so T/S is cyclic. Indeed the remainder of (2) also holds; cf. [**AS76a**]. Similarly (3) is a well-known fact about the structure of the maximal parabolic subgroups of G.

Now $S = O_2(G_{1,2}) \leq O_2(M_{1,2})$. Suppose first that $S = O_2(M_{1,2})$. Then as $T \leq M_{1,2}$ is Sylow in M_i, A.1.6 says that $O_2(M_i) \leq O_2(M_{1,2}) = S \leq G_i$, so as $G_i \trianglelefteq M_i$ we have $O_2(M_i) = O_2(G_i)$, and (4) holds. So suppose that $S < O_2(M_{1,2})$. Then by (2) there is an involution $t \in O_2(M_{1,2}) - S$, and t induces a field or graph automorphism on G. In particular t is nontrival on $B \cap L_i$ for $i = 1$ or 2, contradicting $t \in O_2(M_{1,2})$. Thus (4) holds.

The proof of (5) is similar: If (5) fails, then there is $u \in O_2(L_iT) - O_2(L_i)$, and either u induces a nontrivial field automorphism on $L_i/O_2(L_i)$ or conclusion (ii) of (5) holds.

Assume $J(T) \not\leq S$. Then there is $A \in \mathcal{A}(T)$ with $A \not\leq G$. Let $V := R_2(L_2T)$ and $(L_2T)^* = L_2T/C_{L_2T}(V)$. By our convention that G_1 is the normalizer of a long root group when $\bar{G}$ is not $L_3(q)$, $L_2^* \neq 1$. From the structure of M, V is the natural module for $L_2^* \cong L_2(q)$, the maximal nonsplit extension extension of that module with a trivial submodule in I.2.3 if G is $Sp_4(q)$, or the orthogonal module for $L_2^* \cong L_2(q^2) \cong \Omega_4^-(q)$ if G is unitary. When G is unitary and $q = 2$, no further assertion is made in (6.ii), so we may assume that case does not hold. Thus from B.4.2.1, all offenders in T^* are in S^*, contradicting $A \not\leq S$. Therefore $J(T) = J(S)$. In each case $Z(J(S))$ contains a long root group, so that $C_T(Z(J(S))) \leq S$ from the structure of $Aut(\bar{G})$, and hence $\mathrm{Baum}(T) = \mathrm{Baum}(S)$. Thus (6) is established.

Next we prove (7). As $F^*(M_i) = O_2(M_i)$ for each extension M of the generalized Lie group G, $Z(T) \leq O_2(M_i)$. Thus $Z(T) \leq O_2(G_i) \leq S$ by (4). The structure of $Z(S)$ is well known (cf. [**AS76a**]), so (7) holds.

Finally we prove (8). Let $\bar{S}_0 := \bar{S} \cap O^2(\bar{G})$ and $S_0 := \beta(\bar{S}_0)$. If $L_i = G_i^\infty$ for $i = 1$ or 2, then as $S \leq L_i$, $S \leq O^2(G_0)$. Thus we may assume that L_i is solvable for $i = 1$ and 2, so $\bar{G}$ is $L_3(2)$, $Sp_4(2)$, $G_2(2)$, ${}^2F_4(2)$, $G_2(2)'$, or the Tits group. Now $S_i := O_2(O^2(L_i)) \leq O^2(G_0)$, so $S_1 S_2 \leq O^2(G_0)$. Let $\bar{S}_i := \beta^{-1}(S_i)$; we calculate in $\bar{G}$ that $\bar{S}_0 = \bar{S}_1 \bar{S}_2$, Further $\bar{S} = \bar{S}_0$ when $\bar{G}$ is $L_3(2)$, $G_2(2)'$, or the Tits group. In the remaining possibilities, $|\bar{S} : \bar{S}_0| = 2$, completing the proof of (8). □

LEMMA F.4.30. *Let γ be an $\bar{M}$-extension of the generalized Lie amalgam α, $\bar{T} \in Syl_2(\bar{M}_{1,2})$, $\beta : \gamma \to M_0$ a faithful completion of γ, $T := \beta(\bar{T})$, and $M_0 \leq M$. Then if $U, U' \subseteq \bar{T}$ are conjugate in $\bar{M}$, then $\beta(U)$ is conjugate to $\beta(U')$ in M_0.*

PROOF. Recall from Definition 2.2.1 the Alperin-Goldschmidt conjugation family $\mathcal{D}$ for $\bar{T}$ in $\bar{M}$.

We claim that $\bar{M}_{\bar{D}} := N_{\bar{M}}(\bar{D}) \leq \bar{M}_{j(\bar{D})}$ for each $\bar{D} \in \mathcal{D}$ and some $j(\bar{D})$. Notice for $\bar{D} \in \mathcal{D}$ that $\bar{D} = O_2(\bar{M}_{\bar{D}})$ by condition (c) of Definition 2.2.1. From this it follows (e.g. C.2.1.2) that:

$$\text{If } \bar{M}_{\bar{D}} \leq \bar{K} \leq \bar{M}, \text{ then } O_2(\bar{K}) \leq \bar{D}. \qquad (*)$$

Next let $\bar{E} := \bar{D} \cap \bar{G}$. If $\bar{E} = 1$, then $\bar{D}$ induces outer automorphisms of $\bar{G}$, and then from F.4.29.2, $O^{2'}(C_{\bar{G}}(\bar{D})) \neq 1$. But from conditions (b) and (c) of Definition 2.2.1, $O^{2'}(C_{\bar{M}_{\bar{D}}}(\bar{D})) \leq \bar{D}$, contradicting that observation. Thus $\bar{E} \neq 1$. Now $\bar{M}_{\bar{D}} \leq \bar{M}_{\bar{E}}$. Applying (*) with $\bar{M}_{\bar{E}}$ in the role of "$\bar{K}$", we conclude

$$\bar{E} \leq O_2(N_{\bar{G}}(\bar{E})) \leq O_2(\bar{M}_{\bar{E}}) \cap \bar{G} \leq \bar{D} \cap \bar{G} = \bar{E}.$$

Thus $\bar{E} = O_2(N_{\bar{G}}(\bar{E}))$, so by 3.1.5 in [**GLS98**], $\bar{M}_{\bar{E}} = O_2(\bar{M}_J)$ for some $\emptyset \neq J \subseteq \{1, 2\}$, and hence $\bar{M}_{\bar{E}} = \bar{M}_J$. Therefore $\bar{M}_{\bar{D}} \leq \bar{M}_i$ for some i. Thus the claim is established.

By the Alperin-Goldschmidt Fusion Theorem 16.1 in [**GLS96**], $U, U' \subseteq \bar{T}$ are conjugate in $\bar{M}$ iff there exists a sequence $U := U_0, \ldots, U_n =: U'$ of subsets of $\bar{T}$, $\bar{D}_i \in \mathcal{D}$, and $g_i \in \bar{M}_{\bar{D}_i}$, with $U_i, U_{i+1} \subseteq \bar{Q}_i := O_2(\bar{M}_{\bar{D}_i})$, and $U_i^{g_i} = U_{i+1}$. Then

$$W_i := \beta(U_i) \subseteq \beta(\bar{Q}_i) \leq \beta(\bar{T}) = T$$

and

$$h_i := \beta(g_i) \in \beta(\bar{M}_{\bar{D}_i}) \leq \beta(\bar{M}_{j(\bar{D}_i)}) = M_{j(\bar{D}_i)} \leq M_0$$

with $W_i^{h_i} = W_{i+1}$. Thus $\beta(U)$ is conjugate to $\beta(U')$ in M_0. □

We can now state our main result, giving sufficient local-group-theoretic conditions for the uniqueness of the completion M_0.

THEOREM F.4.31. *Let γ be an $\bar{M}$-extension of the Lie amalgam or Tits amalgam α, defined by $\bar{G}$, a Lie group in characteristic 2 or the Tits group. Assume $\bar{G}$ is not $L_3(2)$. Let $\beta : \gamma \to M_0$ be a faithful completion of γ, and $M_0 \leq M$ a* *finite group. Let z be an involution in $Z(O^{2'}(G_1))$ if $\bar{G}$ is not $L_3(q)$, while we take $z \in Z(S)$ if $\bar{G} \cong L_3(q)$. Assume further:*

(a) If $\bar{G} \cong Sp_4(q)$ or $U_n(q)$, then $N_M(Z(O_2(G_i))) = M_i$.

(b) $C_M(z) \leq M_1$.

(c) If $\bar{G} \cong Sp_4(q)$, then $C_M(t) \leq M_2$ for $t \in Z(O^{2'}(G_2))^{\#}$, and $F^*(C_M(u)) = O_2(C_M(u))$ for each involution $u \in Z(S)$.

(d) If $\bar{G} \cong Sp_4(q)$ then $|M : M_{1,2}|$ is odd.

(e) Either $M = O^2(M)$, or $\bar{G} \cong Sp_4(2)$, $G_2(2)$, or ${}^2F_4(2)$.

Then either

(1) $\bar{G} = O^{2'}(\bar{M})$ and M is isomorphic to an extension of $\bar{G}$ of odd degree, or

(2) $\bar{M} \cong M_0 \cong Aut(U_4(2)) \cong PO_5(3)$, and $M \cong L_4(3) \cong \Omega_6^+(3)$.

REMARK F.4.32. The hypothesis in (e) that $M = O^2(M)$ is not essential; it could be eliminated, of course at the expense of modifying conclusion (1) by removing the statement that $\bar{G} = O^{2'}(\bar{M})$ and allowing any extension of $\bar{G}$: In the proof, simply pass to $O^2(M)$ and the subamalgam of γ obtained by intersecting M_J with $O^2(M)$; use F.4.29.8 to ensure that the resulting amalgam is still an extension of the amalgam of $\bar{G}$.

We exclude the case where α is the amalgam of $G_2(2)' \cong U_3(3)$, primarily because we cannot use either the Thompson order formula (as we do in F.4.43) or Suzuki's trick from F.4.49. We exclude the amalgam of $L_3(2)$ (which is the same as the amalgam of $A_6 = Sp_4(2)'$) for similar reasons.

We prove Theorem F.4.31 in a series of lemmas. Let $L_i := O^{2'}(G_i)$, $Q_i := O_2(L_i)$, $Z_i := Z(L_i)$, $Z_S := Z(S)$, $T \in Syl_2(M_{1,2})$, and $Z := Z(T)$. For $u \in M$, let $M_u := C_M(u)$.

Embed $T \leq T_+ \in Syl_2(M)$. If $\bar{G} \cong Sp_4(q)$, then $T = T_+$ by hypothesis (d) of Theorem F.4.31. Otherwise $Z(T) \leq Z_1$, where Z_1 is the root group of z, so $Z(T)^{\#} \subseteq z^{M_{1,2}}$ from the structure of $\bar{M}$. Hence $T_- := N_{T_+}(N_{T_+}(T))$ centralizes some $M_{1,2}$-conjugate of z, which we may take to be z. Therefore $T_- \leq C_M(z) \leq M_1$ by hypothesis (b), so that $T_- = T$, and hence $T_+ = T$ in this case as well. Therefore:

LEMMA F.4.33. *$T \in Syl_2(M)$.*

LEMMA F.4.34. *(1) If $\bar{G}$ is not $L_3(q)$, then Z_1 is a TI-set in M and $M_1 = N_G(Z_1)$.*

(2) If $\bar{G} \cong Sp_4(q)$, then Z_2 is a TI-set in M and $M_2 = N_M(Z_2)$.

(3) If $\bar{G} \cong L_3(q)$, then Z_S is a TI-set in M and $M_{1,2} = N_M(Z_S)$.

PROOF. Suppose first that $\bar{G}$ is not $L_3(q)$. By hypothesis (b), $C_M(z) \leq M_1$ for $z \in Z_1^{\#}$, while from the structure of $\bar{G}_1$, G_1 is transitive on $Z_1^{\#}$. So (1) follows from I.6.1.1. The proofs of (2) and (3) are similar; for (2), the appeal to (b) is replaced by an appeal to (c). □

LEMMA F.4.35. *Involutions $i, j \in \bar{S}$ are fused in $\bar{G}$ iff $\beta(i)$ and $\beta(j)$ are fused in M.*

PROOF. By F.4.30, if i and j are fused in $\bar{G}$, then $u := \beta(i)$ and $v := \beta(j)$ are fused in M. Thus we may assume that u and v are fused in M, but i is not fused to j in $\bar{G}$. Then $\bar{G}$ has at least two classes of involutions, so by F.4.27.1, $\bar{G}$ is not $L_3(q)$.

Suppose first that $\bar{G} \cong Sp_4(q)$. Then by F.4.29.6 and the structure of S, $\mathcal{A}(T) = \{Q_1, Q_2\}$. Further as $Q_i \trianglelefteq T$, if $Q_1 \in Q_2^M$ then $Q_1 \in Q_2^{N_M(T)}$ using Burnside's Fusion Lemma A.1.35, a contradiction as $|\mathcal{A}(T)| = 2$ while $T \in Syl_2(M)$ by F.4.33. Thus Q_1 is a weakly closed abelian subgroup of T, so by Burnside's Fusion Lemma A.1.35, $N_G(Q_1)$ controls fusion in Q_1. However by hypothesis (a), $N_M(Q_1) = M_1$. Now all three classes of involutions in $\bar{G}$ are fused into $\bar{Q}_1$, and there is no further fusion in M_1, contrary to our assumption that there is further fusion in M.

Therefore $\bar{G}$ is not $Sp_4(q)$, so by F.4.27.3 we may take $u := z$ and j a short root involution. By hypothesis (b), $M_z \leq M_1$.

Thus if $\bar{G} \cong {}^2F_4(q)$ and $q > 2$, then M_z^∞ is a $3'$-group, and M_z is a $3'$-group if $\bar{G} \cong {}^2F_4(2)$ or the Tits group. However $C_{\bar{M}_1}(j)^\infty \cong C_{M_1}(v)^\infty$ is not a $3'$-group if $q > 2$, and $C_{M_1}(v)$ is not a $3'$-group when $q = 2$. This contradiction shows that $\bar{G}$ is not ${}^2F_4(q)$ or the Tits group.

Suppose next that $\bar{G}$ is unitary. If $q > 2$, then by F.4.29.6 and the structure of $\bar{G}_2$, $J(T) =: V = Z(Q_2)$. Again by Burnside's Fusion Lemma A.1.35, $N_M(V)$ controls fusion in V, so we obtain a contradiction as in the case of $Sp_4(q)$. Therefore $q = 2$, and using the same argument, we may assume V is not weakly closed in T with respect to M. Thus $V^h \leq T$ but $V^h \not\leq S$ for some $h \in M$. Also all involutions in V are fused to z or v under L_2, so by our assumption $V^\# \subseteq z^G$. However as $V^h \in \mathcal{A}(T)$ and $V^h \not\leq S$, we conclude from F.4.29.2 that V^h contains a graph automorphism r; thus $r = z^y$ for some $y \in M$. Further we may choose r so that $X_i := C_{G_i}(r) \cong \mathbf{Z}_2 \times S_4$ and $V_i := O_2(O^2(X_i)) \not\leq O_2(X_{3-i})$. However $M_r \leq M_1^y$ and $O^2(M_1)$ is 2-closed, so $V_i \leq O_2(M_1^y)$, contradicting $V_i \not\leq O_2(X_{3-i})$. Notice this argument shows:

LEMMA F.4.36. *If $\bar{G} \cong U_4(2)$ or $U_5(2)$, and $\bar{f}$ is an involution in $\bar{T}$ inducing a graph automorphism on $\bar{G}$, then $\beta(\bar{f}) \notin z^M$.*

Returning to the proof of F.4.35, we are left with the cases where $\bar{G} \cong G_2(q)$ or ${}^3D_4(q)$. Here Q_1 is a special group with center Z_1, and j is fused into $\bar{Q}_1$ under $\bar{G}$, so we may assume $v \in Q_1$ by F.4.30. By F.4.29.4, $Q_1 = O_2(M_1)$. We could apply the standard theory of large special groups (cf. [**Smi81**]) to obtain a contradiction, but as the arguments are easy in our case, we go through the details: Let $z^g = v$ for $g \in M$. From the structure of $\bar{M}_1$ and F.4.30, v centralizes at least one pair of conjugates Z_1^r and Z_1^s of Z_1 in Q_1 with $\Phi(U) = Z_1$, where $U := \langle Z_1^r, Z_1^s \rangle$. As Z_1 is a TI-set in M by F.4.34.1, U centralizes Z_1^g by I.6.2.1. Now $C_T(Z_1) = S$, so conjugating in M_v, we may assume $U \leq S^g$. Then as S/Q_1 is abelian, $Z_1 = \Phi(U) \leq Q_1^g$. Now conjugating in $C_{M_z}(Z_1)$, we may assume $R := C_{Q_1^g}(Z_1) \leq C_T(Z_1) = S$. As

$$\Phi(Q_1 \cap Q_1^g) \leq \Phi(Q_1) \cap \Phi(Q_1^g) = Z_1 \cap Z_1^g = 1,$$

$R \cap Q_1$ is elementary abelian; hence $|R \cap Q_1| \leq q^{w+1}$, where $w = 2$ or 4, for $\bar{G} \cong G_2(q)$ or ${}^3D_4(q)$, respectively. Thus as $|R| = q^{2w}$, $|S : Q_1| \geq |R : R \cap Q_1| \geq q^{w-1}$. As $|S : Q_1| = q^{w-1}$, we conclude $S = RQ_1$; so S centralizes the image of v in Q_1/Z_1, which is not the case. This contradiction completes the proof of F.4.35. □

We wish to reduce to the case where $T = S$, so we consider involutions in T which are the image under β of outer automorphisms of $\bar{G}$. The branch of the argument that will lead to the exceptional conclusion (2) of Theorem F.4.31 arises in case (ii) of F.4.37 when such involutions exist.

LEMMA F.4.37. *Suppose* $\bar{f} \in \bar{T}$ *induces a field or graph automorphism on* $\bar{G}$ *such that* $C_{\bar{S}}(\bar{f}) \in Syl_2(C_{\bar{G}}(\bar{f}))$, *and set* $f := \beta(\bar{f})$. *Then*

(1) Either $q = r^2$ *and* $\bar{G}$ *is of Lie type* $X(r^2)$, *or* $\bar{G} \cong U_n(q)$. *Further* $\bar{D} := C_{\bar{G}}(\bar{f}) \cong X(r)$ *or* $Sp_4(q)$, *respectively.*

(2) Let $\delta := (\bar{D}_1, \bar{D}_{1,2}, \bar{D}_2)$, *where* $\bar{D}_J := C_{\bar{G}_J}(\bar{f})$. *Then* δ *is a* $\bar{D}$*-amalgam, and the restriction* $\beta : \delta \to D$ *is a faithful completion, where* $D_J := \beta(\bar{D}_J)$ *and* $D := \langle D_1, D_2 \rangle$.

(3) Let $\bar{E}_J := C_{\bar{M}_J}(\bar{f})$, $\bar{E} := C_{\bar{M}}(\bar{f})$, *and* $E^* := \bar{E}/\langle \bar{f} \rangle$. *Then* $\epsilon = (E_1^*, E_{1,2}^*, E_2^*)$ *is an* E^**-extension of* δ.

(4) Either

(i) $C_T(f) \in Syl_2(C_M(f))$, *or*

(ii) $\bar{M} \cong M_0 \cong Aut(U_4(2))$ *and* $M \cong L_4(3)$.

PROOF. Part (1) is standard (cf. F.4.29.2), and (2) and (3) follow easily from (1). Thus it remains to establish (4). Therefore we assume that conclusion (i) of (4) does not hold, and we must establish conclusion (ii).

Let $T_f := C_T(f)$ and $S_f := C_S(f)$, so that $T_f \leq C_{M_i}(f)$ since $T \leq M_{1,2}$. Define $M_f := C_M(f)$, and embed $T_f \leq R \in Syl_2(M_f)$; by assumption, $T_f < R$. From (3) and F.4.29.7, $Z(T_f) \leq Z(S_f) \times \langle f \rangle$. Further $[T_f, T_f] \leq S_f$ by F.4.29.2, and $[S_f, S_f] \neq 1$. Thus $Z_f := Z(T_f) \cap [T_f, T_f] \leq Z(S_f)$, and is normalized by $N_R(T_f)$. Indeed by F.4.29.7 we obtain one of three cases: either $\bar{G} \cong L_3(q)$ and $Z_f \leq Z$; or $Z_{f,1} := Z_f \cap Z_1 \neq 1$; or $\bar{G} \cong Sp_4(4)$, $U_4(2)$, or $U_5(2)$, with $\bar{Z}_f$ generated by an involution of type c_2 in $\bar{D}' \cong A_6$. Further by F.4.35, $Z_{f,1}$ is weakly closed in Z_f.

Let $Z_R := C_{Z_f}(N_R(T_f))$ and $E_J := \beta(\bar{E}_J)$. If $C_M(Z_R) \leq M_J$ for some J, then as $T_f \in Syl_2(E_J)$, we conclude $N_R(T_f) = T_f$ and hence $T_f = R$, contary to our assumption. Thus $C_M(Z_R) \not\leq M_J$. First assume we are in the case where $Z_{f,1} \neq 1$. Then as $Z_{f,1}$ is weakly closed in Z_f, $N_R(T_f)$ normalizes Z_f and hence also $Z_{f,1}$, so that $1 \neq Z_R \cap Z_{f,1}$. Then $C_G(Z_R) \leq M_1$ by F.4.34.1, contradicting our previous observation. In case $\bar{G} \cong L_3(q)$, $N_R(T_f)$ normalizes $1 \neq Z_f \leq Z \leq Z_S$, so we obtain a similar contradiction using F.4.34.3.

This leaves the case where $\bar{G}$ is $Sp_4(4)$, $U_4(2)$, or $U_5(2)$. Hence $\bar{E} = \langle \bar{f} \rangle \times \bar{D}$ with $\bar{D} \cong Sp_4(2)$. Thus $Z(T_f) = \langle f \rangle \times Z(S_f) \cong E_8$, and z is weakly closed in $Z(S_f)$. But z is not weakly closed in $Z(T_f)$, arguing as in the previous paragraph with $N_R(T_f)$ but using hypothesis (b) of Theorem F.4.31. Thus $z^M \cap T \not\subseteq S$. Suppose first $\bar{G} \cong Sp_4(4)$. As z is fused into $T - S$, $f = z^g$ for some $g \in G$ by F.4.29.2. Then by hypothesis (b) of Theorem F.4.31, $A_4 \cong O^2(D_i) \leq O^2(M_1^g) = L_1^g$. Further as D_i is irreducible on $V_i := O_2(D_i)$ and $V_i \not\leq O_2(D_{3-i})$, $V_i \cap Q_1^g = 1$, so $D_8 \cong V_1V_2$ is embedded in L_1^g/Q_1^g. This is impossible as $L_1/Q_1 \cong L_2(4)$.

Therefore $\bar{G}$ is $U_4(2)$ or $U_5(2)$. Then by F.4.36, $f \notin z^M$. Now if $\bar{G} \cong U_5(2)$, then by F.4.29.2 all involutions in $T - S$ are in f^M, whereas we saw $z^M \cap T \not\subseteq S$. Therefore $\bar{G} \cong U_4(2)$. Here by F.4.29.2 there are two classes $\bar{f}^{\bar{G}}$ and $\bar{e}^{\bar{G}}$ of involutions in $\bar{M} - \bar{G}$, with $\bar{e} = \bar{f}\bar{z}$. Observe $Z(\bar{T}_f)$ contains two conjugates each of $\bar{f}$, $\bar{e}$, and $\bar{t}$ the short root involution, in addition to $\bar{z}$. We have seen that z is not weakly closed in $Z(T_f)$, and that $t, f \notin z^M$, so we conclude $z^M \cap Z(T_f) = \{z\} \cup \beta(\bar{e}^{\bar{M}} \cap Z(\bar{T}_t))$ is of order 3. Also $T_f < R < T$ as $f \notin z^G$, so as $|T_f| = 2^5$ and $|T| = 2^7$, we conclude $|R| = 2^6$. Thus f is fused to some involution f' in T with $|C_T(f')| = 2^6$, so $f \in t^M$. Thus M has two classes of involutions, with representatives z and f, with $t \in f^M$ and $fz \in z^M$.

At this point, we put aside the remainder of the analysis of this case, relegating it to the next section F.5. That analysis identifies M as $L_4(3)$ and is independent of the rest of this section, aside from an inductive appeal to that final identification. Thus we may regard the lemma as established. □

During the remainder of the proof of Theorem F.4.31, we assume that M is not $L_4(3)$.

Suppose that $S < T$. Then we may choose f as in F.4.37. Hence $\bar{G}$ is not $Sp_4(2)$, $G_2(2)$ or ${}^2F_4(2)$, and $T_f \in Syl_2(M_f)$ by F.4.37.4. Now $|T_f| < |C_{\bar{M}}(i)|_2$ for each involution $i \in \bar{S}$ and each $\bar{G}$. By F.4.30, each M-class of involutions fused into S has a representative of the form $\beta(i)$ for some $i \in \bar{S}$ with $|C_T(\beta(i))| = |C_{\bar{M}}(i)|_2$, so we conclude that $f^M \cap S = \emptyset$. However by F.4.29.2, T/S is cyclic, so that $f^M \cap T \subseteq fS$. Thus by Generalized Thompson Transfer A.1.37.2, $f \notin O^2(M)$, since we saw in F.4.33 that T is Sylow in M. However we just observed that $\bar{G}$ is not $Sp_4(2)$, $G_2(2)$, or ${}^2F_4(2)$, so $M = O^2(M)$ by hypothesis (e) of Theorem F.4.31. This contradiction establishes:

Lemma F.4.38. *$T = S$. In particular $\bar{G} = O^{2'}(\bar{M})$, so $\bar{M}$ is an extension of $\bar{G}$ of odd degree.*

Thus it remains to show that M is isomorphic to an extension of $\bar{G}$ of odd degree.

Lemma F.4.39. *If $\bar{G} \cong Sp_4(q)$ and $\bar{u}$ is an involution of type c_2 in $Z(\bar{S})$, then $C_M(\beta(\bar{u})) \leq M_{1,2}$.*

Proof. Let $u := \beta(\bar{u})$. By F.4.38, $S = T$, so $Z = Z_S$. By hypothesis (c) of Theorem F.4.31, $F^*(M_u) = O_2(M_u) =: Q_u$. As $\bar{u} \in Z(\bar{S})$, $u \in Z$. Thus $Z \leq E_u := \Omega_1(Z(Q_u))$. As $Z = Z_S$, $Z = Z_1 \times Z_2$, and Z_i is strongly closed in Z with respect to M by F.4.35. Hence if $Z = E_u$, then $M_u \leq N_M(Z_1) \cap N_M(Z_2) = M_{1,2}$ by (1) and (2) of F.4.34. Thus we may assume $Z < E_u$. Then as Q_1 and Q_2 are the maximal elementary abelian subgroups of $S = T$, $E_u \leq Q_i$ for $i = 1$ or 2. Thus $Q_u \leq C_S(E_u) = Q_i$ as $E_u > Z$. Hence $Q_i \leq \Omega_1(C_{M_u}(Q_u)) = E_u$, so $Q_u = Q_i$. Thus $M_u \leq N_G(Q_u) = N_G(Q_i) \leq M_i$ by hypothesis (a). Therefore $M_u = C_{M_i}(u) \leq M_{1,2}$. □

Lemma F.4.40. *If $\bar{v}$ is an involution in $\bar{S}$ with $C_{\bar{S}}(\bar{v}) \in Syl_2(C_{\bar{G}}(\bar{v}))$, then $C_S(\beta(\bar{v})) \in Syl_2(C_M(\beta(\bar{v}))$.*

Proof. Let $v := \beta(\bar{v})$. If $\bar{v} \in Z(\bar{S})$, the result is clear as $S = T \in Syl_2(M)$. If not, then by F.4.27, $\bar{v}$ is a short root involution in $\bar{G}$, and $\bar{G}$ is not $L_3(q)$ or $Sp_4(q)$. By F.4.35, $v \notin z^M$, and indeed z^M and v^M are the two conjugacy classes of involutions of M. Hence v is *extremal* in S; that is, $C_S(v) \in Syl_2(M_v)$.

□

We recall (cf. page 246 of [**Asc86a**]) the definition of the k-generated p-core:

Definition F.4.41. Given a p-group P acting on a group H, and a positive integer k, define:

$$\Gamma_{k,P}(H) := \langle N_H(X) : X \leq P, m_p(X) \geq k \rangle.$$

Lemma F.4.42. *If $\bar{G}$ is not $L_3(q)$, there is a short root involution $\bar{t} \in O_2(\bar{G}_2)$ with $C_M(\beta(\bar{t})) \leq M_2$.*

PROOF. By F.4.27.4.ii, there is a short root involution $\bar{t} \in \bar{Q}_2$ with $C_{\bar{M}}(\bar{t})) \leq \bar{M}_2$. For $\bar{u} \in \bar{M}_1 \cup \bar{M}_2$, let $\bar{X}_{\bar{u}} := C_{\bar{M}}(\bar{u})$, and set $u := \beta(\bar{u})$. Whenever $\bar{X}_{\bar{u}} \leq \bar{M}_i$ for $i = 1$ or 2, set $X_u := \beta(\bar{X}_{\bar{u}})$, so that $X_u = C_{M_i}(u)$. For example $M_z = C_M(z) = X_z$ by hypothesis (b). Further $X_t = C_{M_2}(t)$ by the choice of $\bar{t}$.

The lemma holds by hypothesis (c) if $\bar{G} \cong Sp_4(q)$, so we may assume that $\bar{G}$ is not $Sp_4(q)$.

Next let r denote the number of orbits of $\bar{X}_{\bar{z}}$ on $\bar{t}^{\bar{M}} \cap \bar{X}_{\bar{z}}$. Since $\bar{X}_{\bar{z}} = C_{\bar{M}}(\bar{z})$, $\bar{X}_{\bar{t}}$ has r orbits on $\bar{z}^{\bar{M}} \cap \bar{X}_{\bar{t}}$. Choose representatives $\bar{z}^{\bar{g}_j}$, $1 \leq j \leq r$, with $\bar{g}_j \in \bar{M}$ for the latter orbits; then $\bar{t}^{\bar{g}_j^{-1}}$, $1 \leq j \leq r$ are representatives for the former orbits, and $|C_{\bar{X}_{\bar{t}}}(\bar{z}^{\bar{g}_j}))| = |C_{\bar{X}_{\bar{z}}}(\bar{t}^{\bar{g}_j^{-1}})|$.

By F.4.35, $\beta(\bar{t}^{\bar{g}_j^{-1}})$, $1 \leq j \leq r$, are representatives for the distinct orbits of X_z on $t^M \cap X_z$ and $\beta(\bar{z}^{\bar{g}_j})$, $1 \leq j \leq r$, are representatives for the orbits of X_t on $z^M \cap X_t$. Therefore as $M_z = X_z$, M_z has r orbits on $t^M \cap M_z$, and hence M_t has r orbits on $z^M \cap M_t$. Now $C_{\bar{S}}(\bar{t}) \in Syl_2(C_{\bar{M}}(\bar{t}))$, so by F.4.40 $C_S(t) \in Syl_2(M_t)$; thus each orbit has a representative in $C_S(t) \leq X_t$. Therefore the elements $\beta(\bar{z}^{\bar{g}_j})$, $1 \leq j \leq r$, are representatives for the M_t-orbits. Let $g_j \in M$ with $z^{g_j} = \beta(\bar{z}^{\bar{g}_j})$. As $M_z = X_z$,

$$|C_{M_t}(z^{g_j})| = |C_{M_z}(t^{g_j^{-1}})| = |C_{\bar{X}_{\bar{z}}}(\bar{t}^{\bar{g}_j^{-1}})| = |C_{\bar{X}_{\bar{t}}}(\bar{z}^{\bar{g}_j})| = |C_{X_t}(z^{g_j})|,$$

so we conclude that $C_{X_t}(z^{g_j}) = C_{M_t}(z^{g_j})$.

We have shown that X_t controls M_t-fusion in $z^M \cap X_t$, and that $C_{M_t}(u) \leq X_t$ for each $u \in z^M \cap X_t$. Therefore by 7.3 in [**Asc94**], each member of $z^M \cap M_t$ fixes a unique point of M_t/X_t. Then by 7.4 in [**Asc94**], X_t controls M_t-fusion of 2-elements in X_t.

At this point we assume $X_t < M_t$, and derive a contradiction. Set $V := \langle z^{X_t} \rangle$, $K_t := O^{2'}(X_t)$, and $L_t := \langle z^{M_t} \rangle$. Note that $F^*(X_t) = O_2(X_t)$ using β, so that $V \leq \Omega_1(Z(O_2(X_t))$ by B.2.14.

Suppose first that $\bar{G}$ is not unitary; then as $\bar{G}$ is not $L_3(q)$ or $Sp_4(q)$, $\bar{G}$ is $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, or ${}^2F_4(2)'$. In these cases V is the natural module for $K_t/C_{K_t}(V) \cong L_2(q)$, so V is noncyclic and $V^\# = z^{M_t}$. Thus we have the hypotheses of lemma I.8.5, and part (1) of that lemma says that $X_t \cap L_t$ is a Borel subgroup of the Bender group L_t with $V = \Omega_1(T_L)$ for $T_L \in Syl_2(X_t \cap L_t)$. Further by I.8.5.2, $Aut_{Aut(X_t)}(V)$ is solvable, so as X_t induces $L_2(q)$ on V, it follows that $q = 2$. Hence $|V| = 4$, so as $V = \Omega_1(T_L)$, $L_t \cong L_2(4)$ or $U_3(4)$. As $q = 2$, X_t is a $5'$-group for each choice of $\bar{G}$, so L_t is not $U_3(4)$, and hence $L_t \cong L_2(4)$. Therefore as $O_2(X_t)$ centralizes V, but V is self-centralizing in $Aut(L_t)$, $O_2(X_t) = V \times C_{O_2(X_t)}(L_t)$. This contradicts the fact that $O_2(X_t)$ does not split over V from the structure of $\bar{X}_t$.

Therefore $\bar{G} \cong U_4(q)$ or $U_5(q)$. Thus V is the 3-dimensional orthogonal $\mathbf{F}_q$-module for $K_t/C_{K_t}(V) \cong L_2(q)$, with $t \in U := C_V(K_t)$. Suppose $X_t < N_{M_t}(U) =: M_U$, and let $M_U^* := M_U/U$ and $L_U := \langle z^{M_U} \rangle$. Then V^* is the natural module for $K_t^*/C_{K_t^*}(V^*)$, and as z fixes a unique point of M_t/X_t, z^* fixes a unique point of M_U^*/X_t^*, so we can argue as in the previous paragraph, with V^*, M_U^* in the roles of "V, M_t", and conclude that $q = 2$ and $L_U^* \cong L_2(4)$. Again $O_2(X_t) = (V \cap L_U) \times C_{O_2(X_t)}(L_U)$, so $O_2(X_t)$ splits over $V \cap L_U = [V, O^2(X_t)]$. This reduces us to the case $\bar{G} \cong U_4(2)$, with $N_{M_t}(U)$ the extension of $E_4 \times L_2(4)$ by an involutory outer automorphism. But in that case, $O_2(N_{M_t}(U)) = C_{Q_2}(K_t)$ contains an element u

of z^{M_2}. Then M_u contains $O^2(L_U) \cong L_2(4)$—impossible, since by hypothesis (b), $M_z \leq M_1$ is solvable. This contradiction shows that $X_t = N_{M_t}(U)$. In particular, $\langle t \rangle < U$ since we are assuming $X_t < M_t$; hence $q > 2$.

Now choose W maximal subject to $\langle t \rangle \leq W \leq U$ and $M_W := N_{M_t}(W) > X_W := N_{X_t}(W)$. By the previous paragraph, $W < U$. Observe first by applying 1.1.3.2 in X_t that $F^*(X_W) = O_2(X_W)$, so in particular $O(X_W) = 1$. Further as $q > 2$, the long root group $\bar{R}$ in $\bar{Q}_2$ containing $\bar{z}$ is noncyclic, so as $M_r \leq M_1$ for each $r \in R^{\#}$ by hypothesis (b), using Generation by Centralizers of Hyperplanes A.1.17,

$$O(M_W) = \langle C_{O(M_W)}(r) : 1 \neq r \in R^{\#} \rangle \leq X_W,$$

and hence $O(M_W) = 1$ as $O(X_W) = 1$. As z fixes a unique point of M_t/X_t, z fixes a unique point of M_W/X_W, so X_W controls 2-fusion in X_W with respect to M_W by 7.4 in [**Asc94**].

Set $\tilde{M}_W := M_W/W$ and $L_W := \langle U^{M_W} \rangle$. By maximality of W, in the notation of Definition F.4.41, $\Gamma_{1,\tilde{U}}(\tilde{M}_W) \leq \tilde{X}_W$, and as X_W controls fusion in X_W with respect to M_W, $\tilde{u}^{M_W} \cap \tilde{X}_W = \tilde{u}^{X_W}$ for each $\tilde{u} \in \tilde{U}^{\#}$. Therefore by 7.3 in [**Asc94**], $\tilde{u}$ fixes a unique point of $\tilde{M}_W/\tilde{X}_W$. Hence the hypotheses of I.8.5 are satisfied by $\tilde{M}_W$, $\tilde{X}_W$, $\tilde{U}$. As $M_W > X_W$ and $\tilde{X}_W = N_{\tilde{M}_W}(\tilde{U})$, $\tilde{U} \neq \tilde{L}_W$. Thus as $O(M_W) = 1$, $\tilde{U}$ is noncyclic by I.8.5.1. Thus I.8.5.1 says that $\tilde{X}_W \cap L_W$ is a Borel subgroup of the Bender group $\tilde{L}_W$. As K_t centralizes U, it centralizes W; thus $K_t \leq M_W \leq N_G(L_W)$. As $q > 2$, $K_t = K_t^{\infty}$, so as $C_{Aut(\tilde{L}_W)}(\tilde{U})$ is solvable, K_t centralizes $\tilde{L}_W$. This is impossible as $U \leq K_t$ and $Z(\tilde{L}_W) = 1$. This contradiction completes the proof of F.4.42. □

LEMMA F.4.43. *If $\bar{G}$ is not $L_3(q)$, then $|M| = |\bar{M}|$.*

PROOF. Assume that $\bar{G}$ is not $L_3(q)$. Then by F.4.27, $\bar{M}$ has at least two classes of involutions.

By F.4.38, $S = T$. Then by F.4.35 we may choose sets $\bar{I}$ and I of representatives for the conjugacy classes of involutions of $\bar{M}$ and M, such that $\bar{I} \subseteq \bar{S}$ and $\beta : \bar{I} \to I$ is a bijection. By F.4.27, $C_{\bar{M}}(\bar{i}) \leq \bar{M}_{j(\bar{i})}$ for $j(\bar{i}) = 1$ or 2, while by Hypothesis (b) of Theorem F.4.31, together with F.4.39 and F.4.42, $\beta(C_{\bar{M}}(\bar{i})) = C_M(\beta(\bar{i})) \leq M_{j(i)}$. Then by another application of F.4.35, for involutions $\bar{u}, \bar{v} \in C_{\bar{M}}(\bar{i})$, $\bar{v} \in \bar{u}^{\bar{M}}$ iff $\beta(\bar{v}) \in \beta(\bar{u})^M$. Finally as $\bar{G}$ is not $L_3(q)$, $\bar{M}$ has at least two classes of involutions, so applying the Thompson Order Formula 45.6 in [**Asc86a**] we conclude that $|\bar{M}| = |M|$. This establishes the result. □

LEMMA F.4.44. *If $\bar{G}$ is not $L_3(q)$, then M is isomorphic to an extension of $\bar{G}$ of odd degree.*

PROOF. By F.4.43, $|\bar{M}| = |M|$. Therefore $|\bar{M}| = |M_0|$, and M_0 is isomorphic to an extension of $\bar{G}$ by F.4.24, while that extension is of odd degree by F.4.38. Thus $|M_0| = |\bar{M}| = |M|$, so $M = M_0$, and the lemma holds. □

By F.4.38, $\bar{G} = O^{2'}(\bar{M})$. Then by Lemma F.4.44, we have established conclusion (1) of Theorem F.4.31 when $\bar{G}$ is not $L_3(q)$. Thus we may assume in the remainder of the section that $\bar{G} \cong L_3(q)$. We will show that the coset geometry Γ of M is small, and then apply Theorem F.4.8; to do so, we will make use of an observation of Suzuki.

Recall by the hypotheses of Theorem F.4.31 that $\bar{G}$ is not $L_3(2)$, so that $q > 2$.

Lemma F.4.45. $M = M_0$.

Proof. Assume that $M_0 < M$. Then by induction on the order of G, M_0 is an extension of $\bar{G} \cong L_3(q)$. By hypothesis (b), $C_M(z) \leq M_1 \leq M_0$. By F.4.38, $S = T$, so that $O^{2'}(M_0) \cong L_3(q)$, and hence M_0 has one class of involutions. Thus $C_M(j) \leq M_0$ for each involution $j \in M_0$. Also $N_G(S) \leq N_G(Z) \leq M_0$ by F.4.34.3, so M_0 is strongly embedded in M by I.8.1.3. Therefore by 7.6 in [**Asc94**], M_0 has a subgroup of odd order which acts transitively on the involutions in M_0. But M_0 has $i := (q^3 - 1)(q + 1)$ involutions, while no subgroup X of M_0 of odd order is divisible by i. (E.g., argue as in the proof of A.1.12 to show that X has a normal subgroup of order r, where r is a Zsigmondy prime divisor of $q^3 - 1$ if $q > 4$, and $r = 7$ if $q = 4$. Then $|N_{M_0}(Y)| \leq (q^2 + q + 1)3 \log_2(q) < i$, since $q > 2$.) □

For convenience we next collect some facts established at various earlier points. Recall $L_i = O^{2'}(G_i)$.

Lemma F.4.46. *(1) M has one class z^M of involutions.*
(2) For each involution $j \in M$, $O_2(C_M(j)) = O^{2'}(C_M(j)) =: T(j) \in Syl_2(M)$.
(3) Q_i is weakly closed in M_i with respect to M.
(4) $M_i = N_M(Q_i)$ and $Z^M \cap Q_i = Z^{L_i}$ is of order $q + 1$.
(5) $N_M(T) = M_{1,2}$.

Proof. By F.4.38, $T = S$; so by F.4.35, M has one class z^M of involutions, proving (1). As $M_{1,2} \leq N_M(T) \leq N_M(Z) = M_{1,2}$, (5) holds. By hypothesis (b), $C_M(z)$ lies in M_1 and hence in $M_{1,2}$. Then $S = O_2(M_{1,2}) = O^{2'}(C_M(z))$, and hence (2) holds. Next $\mathcal{A}(T) = \{Q_1, Q_2\}$ and $Q_i \trianglelefteq T$, so if the normal subgroups Q_1 and Q_2 of T are conjugate in M, they are conjugate in $N_M(T)$ by Burnside's Fusion Lemma A.1.35. Then Q_1 and Q_2 are conjugate in $M_{1,2}$ by (5), whereas $M_{1,2}$ normalizes both Q_1 and Q_2. Thus (3) holds. By F.4.34.3, Z is a TI-set in M with $N_M(Z) = M_{1,2} \leq M_i$, so as Z^{L_i} partitions Q_i, $Z^M \cap Q_i = Z^{L_i}$ and hence $N_M(Q_i) \leq L_i N_M(Z) \leq M_i$. Finally $M_i \leq N_M(Q_i)$, so that (4) holds. □

Let $\bar{B}$ be a Hall $2'$-subgroup of $\bar{M}_{1,2}$, and $B = \beta(\bar{B})$. Thus $N_M(T) = M_{1,2} = TB$ using F.4.46.5. Let $\bar{Y}_i := C_{\bar{B}}(\bar{L}_i)$, $\bar{M}_+$ the subgroup of $\bar{M}$ inducing inner-diagonal automorphisms on $\bar{G}$, and $\bar{B}_0 := \bar{B} \cap \bar{M}_+$. Set $Y_i := \beta(\bar{Y}_i)$ and $B_0 := \beta(\bar{B}_0)$. Notice that since field automorphisms do not centralize L_i, $Y_i \leq B_0$ is abelian.

Lemma F.4.47. *(1) Either*
(i) $Y_2 \neq 1$, or
(ii) $q = 4$, $M_i = L_i$, $B_0 = B$ is of order 3, and $C_M(B)$ is of odd order.
(2) If $Y_2 \neq 1$, then $B_0 = C_M(B_0)$ and Y_2 is not inverted in M.
(3) $B_0 = F(B)$.

Proof. By construction, $\bar{B}_0 \bar{L}_i$ is a maximal parabolic in $L_3(q)$ or $PGL_3(q)$, with $\bar{M}_i$ an extension of $\bar{B}_0 \bar{L}_i$ by field automorphisms, which are of odd order by F.4.38. Therefore applying β to the structure of $P\Gamma L_3(q)$, (3) and all parts of (1) hold, except possibly the statement in (1.ii) that $C_M(B)$ is of odd order. In particular if $\bar{M}$ is $L_3(4)$, then $M_i = L_i$, $M_{1,2} = TB$, and $C_Z(B) = 1$, so $M_z = T$ using hypothesis (b). Then by F.4.46.1, the centralizer of each involution in M is a 2-group, so $C_M(B)$ is of odd order. Thus (1) is established.

It remains to prove (2), so assume that $Y_2 \neq 1$. From the structure of $\bar{M}$ (cf. 18.7 in [**AS76a**]),

$$C_{\bar{M}}(\bar{Y}_2) = C_{\bar{M}_2}(\bar{Y}_2) = \bar{Y}_2 \times \bar{I},$$

where $\bar{I}$ is a Levi complement to $\bar{Q}_2$ in $\bar{L}_2$, and $N_{\bar{M}}(Y_2) = \bar{I}\bar{B}$. Then applying β, $C_{M_2}(Y_2) = Y_2 \times I$, $N_{M_1}(Y_2) \leq BI$, and Y_2 is not inverted in M_2. Similarly $C_{\bar{M}}(\langle \bar{i}, \bar{Y}_2 \rangle) \leq \bar{I}\bar{Y}_2$ for each involution $\bar{i} \in \bar{I} \cap T$, and $\bar{I} \cap \bar{T} = \bar{Z}^{\bar{x}}$ for some $\bar{x} \in \bar{M}_1$. Set $i := \beta(\bar{i})$; then $i \in Z^{M_1}$, so $C_M(i) \leq M_1$ by hypothesis (b), and Z^x is a TI-set in M with normalizer $M_{1,2}^x$ by F.4.34.3. Then applying β, $C_M(\langle i, Y_2 \rangle) \leq IY_2$ and $N_M(Z^x Y_2) \leq IY_2$. Thus if $C_M(Y_2) > Y_2 I$, then by I.8.1.3, $Y_2 I$ is strongly embedded in $C_M(Y_2)$, so by 7.6 in [**Asc94**], there is a subgroup of odd order in I transitive on the involutions in I. But $L_2(q)$ has no subgroup of order $q^2 - 1$.

This contradiction shows that $C_M(Y_2) = Y_2 I$; hence as $Y_2 \leq B_0$, we conclude $C_M(B_0) = Y_2 C_I(B_0) = B_0$. Also by a Frattini Argument,

$$N_M(Y_2) = C_M(Y_2)(N_M(Y_2) \cap N_M(Z^x)) \leq Y_2 I N_{M_1}(Y_2) \leq IB,$$

so Y_2 is not inverted in M. □

LEMMA F.4.48. *Let $g \in M - M_{1,2}$. Then one of the following holds:*
(1) $T \cap T^g = 1$ and zz^g is of odd order.
(2) $T \cap T^g = ZZ^g = Q_1$ or Q_2, and $|zz^g| = 2$.
(3) $Z, Z^g \neq T \cap T^g \in Z^G$ and $|zz^g| = 4$.

PROOF. Suppose $1 \neq T \cap T^g$, and let $j \in T \cap T^g$ be an involution. Then $j \in Q_i$ for $i = 1$ or 2, and

$$Z \leq Q_i \leq C_T(j) = T \cap O^{2'}(C_M(j)) = T \cap T(j) \qquad (*)$$

by F.4.46.2. Similarly $Z^g \leq T(j)$. Now if $z^g \in Q_i$ then we may take $j = z^g$, so $Q_i = ZZ^g$ by (*) as $g \notin M_{1,2}$, so (2) holds. On the other hand, if $z^g \notin Q_i$, then $\langle Z, Z^g \rangle = T(j)$ and (3) holds.

We have shown that if $T \cap T^g \neq 1$, then zz^g has even order, and (2) or (3) holds. Conversely if $|zz^g|$ is even, then there is an involution $j \in \langle zz^g \rangle$, and

$$j \in O^{2'}(C_M(z)) \cap O^{2'}(C_M(z^g)) = T \cap T^g;$$

thus $T \cap T^g = 1$ iff $|zz^g|$ is odd, completing the proof of the lemma. □

Let Γ be the geometry of M defined in Notation F.4.4. By F.4.46.4, $M_i = N_M(Q_i)$, so we may identify Γ_i with Q_i^M, and Q_i^g is incident with Q_{3-i}^h iff $[Q_i^g, Q_{3-i}^h] = Q_i^g \cap Q_{3-i}^h \in Z^M$. Recall that we regard Γ_1 as the points of the geometry, and Γ_2 as the lines of the geometry.

We now invoke a clever argument of Suzuki from [**Suz65**], which will allow us to construct an apartment, and then we can finish things off via an appeal to Theorem F.4.8.

Using β we may choose involutions $s_i \in N_{M_i}(B) - M_{1,2}$.

LEMMA F.4.49. *Let $W := N_M(B)$ and $W^* := W/B$. Let $W_+ := \langle s_1, s_2 \rangle B$. Then*

(1) Z^{s_i} is the unique B-invariant member of $Z^M \cap Q_i - \{Z\}$.

(2) If $g \in W$, then either Bg contains an involution, or $T \cap T^g = T$, Z^{s_1}, or Z^{s_2}.

(3) $s_1^* s_2^*$ is of order 3.

(4) $\Sigma = Q_1^{W_+} \cup Q_2^{W_+}$ is an apartment in Γ.

PROOF. As M_i/Q_i is an extension of $L_2(q)$, B acts on exactly two Borel subgroups of L_i/Q_i, and hence on exactly two members Z and Z^{s_i} of $Z^M \cap Q_i$ by F.4.46.4. Hence (1) holds.

Let $g \in W$. First $N_T(B) = C_T(B) = 1$ as $q > 2$, so

$$N_W(T) = W \cap N_M(T) = W \cap BT = BN_T(B) = B.$$

Thus B is the stabilizer in W of T, T^g, and hence if $w \in W$ with $T^g = T^w$ then

$$w \in \{u \in W : T^u = T^g\} = Bg.$$

Next if $T \cap T^g = 1$, then

$$N_M(T) \cap N_M(T^g) = TB \cap T^gB = (T \cap T^g)B = B,$$

and $|zz^g|$ is odd by F.4.48; so there is an involution $w \in \langle z, z^g\rangle$ with $z^w = z^g$. Then

$$T^w = T(z)^w = T(z^w) = T(z^g) = T^g,$$

and w interchanges T and T^g, so w acts on $N_M(T) \cap N_M(T^g) = B$. Thus $w \in W$, and then $w \in Bg$ by the previous paragraph.

So assume $1 < T \cap T^g < T$. Then by F.4.48, either $T \cap T^g = Q_i$, or $T \cap T^g \in Z^G$. In the first case, from the structure of M_i, there is an involution $w \in N_{M_i}(B)$ with $T^g = T^w$, so $w \in Bg$ as in the previous paragraph. In the second, $T \cap T^g = Z^{s_i}$ by (1). Thus (2) is established.

Suppose $T \cap T^g = Z^{s_1} = T \cap T^h$ for some $h \in W$. Then $Z^g \neq Z^{s_1}$ by F.4.48. Hence as Q_1 is the unique member of Q_1^M through Z^{s_1}, $Q_0 := Z^g Z^{s_1}$ is the unique member of Q_2^M through Z^{s_1}. Similarly $Q_0 = Z^h Z^{s_1}$. Thus applying (1) to Q_0, Z^{s_1} in the roles of "Q_i, Z", $Z^g = Z^h$ is the unique B-invariant member of $Z^M \cap Q_0$ distinct from Z^{s_1}. Thus $Bg = Bh$.

In view of (2) and the previous paragraph, we have shown that there are at most 3 cosets B,Bg_1, Bg_2 of B in W which do not contain involutions, where $T \cap T^{g_i} = Z^{s_i}$. Thus there are at most 3 members of W^* which are not involutions. Therefore the dihedral group W_+^* must be of order 4, 6, or 8. Now if $Y_2 = 1$, then by F.4.47.1, $B_0 = B$ is of order 3; then s_1 and s_2 invert B, so that $s_1 s_2 \in C_M(B)$, which by F.4.47.1 is of odd order. Therefore W_+^* is not of order 4 or 8, so that (3) holds. Thus we may assume that $Y_2 \neq 1$, so $|B_0| > 3$, W acts on B_0 by F.4.47.3, and B_0 is the kernel of this action by F.4.47.2. Then as B_0 is of rank 2, if $s_1^* s_2^*$ is of even order, then $\langle s_1 s_2\rangle$ contains an element inverting B_0, which contradicts F.4.47.2. Thus (3) is established.

As $Q_1, Q_2 \in Fix(B)$ and $|s_1^* s_2^*| = 3$, F.3.6 shows that Σ is an apartment in Γ. □

We are now in a position to complete the proof of Theorem F.4.31. By F.4.45, $M = M_0$. By F.4.49.4, Γ is small. Thus by part (1) of Theorem F.4.8, M is isomorphic to an extension of $\bar{G}$, and that extension is of odd degree by F.4.38.

F.5. Identifying $\mathbf{L_4(3)}$ via its $\mathbf{U_4(2)}$-amalgam

In this section we complete the proof of Lemma F.4.37.4 begun in the previous section. Thus it remains to use the information produced at that point to identify M as $L_4(3)$.

In our first two lemmas, we record information about the target group $L_4(3)$. Let $F := \mathbf{F}_3$, and V the natural 6-dimensional F-module for $\dot{M} := \Omega_6^+(3) \cong L_4(3)$. Thus $\dot{M}$ preserves a quadratic form q and the associated bilinear form $(\ ,\ )$ on V.

Pick an orthogonal basis $X := (x_i : 1 \leq i \leq 6)$ for V, with $q(x_i) = 1$ for $1 \leq i \leq 5$ and $q(x_6) = -1$.

For $J \subseteq I := \{1, 2, 3, 4, 5\}$ of even order, write e_J for the involution in $O(V, q)$ with $[V, e_J] = \langle x_j : j \in J\rangle$; thus $C_V(e_J) = \langle x_6, x_i : i \in I - J\rangle$.

Let $\dot{M}_0$ be the stabilizer in $\dot{M}$ of Fx_6. The following facts are easily established:

LEMMA F.5.1. *(1)* $\dot{M}_0 = \dot{G}\langle \dot{f}\rangle$, *where* $\dot{G} := C_{\dot{M}}(x_6) \cong \Omega_5(3) \cong U_4(2)$, *and* $\dot{f}$ *is the involution in* $\dot{M}$ *fixing* x_i, $i = 3, 4, 5$, *with* $\dot{f}(x_6) = -x_6$, *and with* $\dot{f}(x_1) = x_2$.

(2) $\dot{M}_0 \cong PO_5(3) \cong Aut(U_4(2))$.

(3) $V_2^- := [V, \dot{f}] = \langle x_1 - x_2, x_6\rangle$ *is a projective line of sign* -1, *and* $V_4^- := \langle x_1 + x_2, x_3, x_4, x_5\rangle = C_V(\dot{f})$ *is a linear 4-space of sign* -1.

(4) $\dot{D} := C_{\dot{G}}(\dot{f}) = C_{\dot{G}}(x_1 - x_2)\langle \dot{z}\rangle$, *with* $C_{\dot{G}}(x_1 - x_2)$ *acting faithfully as* $\Omega_4^-(3)$ *on* V_4^-, *and* $\dot{z} := e_{1,2,3,4}$ *the involution with* $[V, \dot{z}] = V_4^+ := \langle x_j : 1 \leq j \leq 4\rangle$ *of dimension* 4 *and sign* $+1$. *Further* $-\dot{z}$ *induces a reflection on* V_4^-, *so* $\dot{D} \cong S_6$.

Let $\dot{M}_1 := C_{\dot{M}}(\dot{z}) = N_{\dot{M}}(V_4^+)$, and $\dot{M}_2$ the "monomial" subgroup which is the stabilizer in $\dot{M}$ of $\Lambda := \{Fx_i : 1 \leq i \leq 6\}$. Observe that:

LEMMA F.5.2. *(1)* $\dot{M}_2$ *is the split extension of* $\{e_J : J \subseteq I$ *and* $|J|$ *is even* $\}$ $=: \dot{Q}_2 \cong E_{16}$ *by* S_5. $\dot{M}_1$ *is the split extension of* $\dot{Q}_1 \cong Q_8^2$ *by* $S_3 \times S_3$.

(2) $\dot{M}_1$ *and* $\dot{M}_2$ *are the maximal parabolics of* $\dot{M}_0 \cong Aut(U_4(2))$ *over* $\dot{T} := \dot{M}_{1,2} \in Syl_2(\dot{M})$, *where* $\dot{T} = \dot{S}\langle \dot{f}\rangle$ *and* $\dot{S} := O_2(N_{\dot{M}_2}(Fx_5))$.

Recall we had reduced the proof of F.4.37.4 to the case where γ is the $\bar{M}$-amalgam for $\bar{M} \cong Aut(U_4(2))$. By F.5.1 and F.5.2, there is a faithful completion $\mu : \gamma \to \dot{M}_0$ of the $\bar{M}$-extension γ of α, with $\mu(\bar{M}_J) = \dot{M}_J$ for $J = \{1\}$, $\{2\}$, and $\{1, 2\}$, $\mu(\bar{z}) = \dot{z}$, and $\mu(\bar{f}) = \dot{f}$. Thus if we write $\gamma_M := (M_1, M_{1,2}, M_2)$ and $\dot{\gamma} := (\dot{M}_1, \dot{M}_{1,2}, \dot{M}_2)$, then

$$\sigma := \beta \circ \mu^{-1} : \dot{\gamma} \to \gamma_M$$

is an isomorphism of amalgams.

For $\bar{X}$ a subset of $\bar{M}_i$ for $i = 1$ or 2, let $\dot{X} := \mu(\bar{X})$ and $X := \beta(\bar{X})$. Thus $\sigma(\dot{X}) = X$. Recall the definition of $\bar{E}_J = C_{\bar{M}_J}(\bar{f})$ from F.4.37.3; applying μ, we obtain $\dot{E}_i = C_{\dot{M}_i}(\dot{f}) \cong E_4 \times S_4$. Set $\dot{A}_i := O_2(\dot{E}_i) \cong E_{16}$.

LEMMA F.5.3. *(1)* $\dot{A}_i = \langle \dot{f}\rangle \times C_{\dot{Q}_i}(\dot{f}) \cong E_{16}$.

(2) $C_{\dot{Q}_2}(\dot{f}) = \{e_J : 1, 2 \in J \subseteq I$ *and* $|J|$ *is even* $\}$.

(3) $C_{\dot{Q}_1}(\dot{f})^\#$ *consists of* $\dot{z}$ *and the involutions* $\dot{u}$ *with* $[V, \dot{u}]$ *a projective line in* V_4^+ *of sign* -1.

(4) $\dot{M}$ *has two classes* $\dot{z}^{\dot{M}}$ *and* $\dot{f}^{\dot{M}}$ *of involutions: the involutions* $\dot{w}$ *with* $[V, \dot{w}]$ *a 4-space of sign* $+1$, *or a projective line of sign* -1, *respectively.*

(5) $\sigma(\dot{u}^{\dot{M}} \cap \dot{T}) = u^M \cap T$ *for* $\dot{u} := \dot{z}$ *and* $\dot{f}$.

(6) $\dot{t} := e_{1,2}$ *generates* $[\dot{Q}_2, \dot{f}]$.

(7) $ftz \in z^M$.

(8) $|z^M \cap A_1| = |z^M \cap Q_2| = 5$ *and* $|z^M \cap A_2| = 9$.

PROOF. Parts (1)–(4) and (6) are either well-known or easy calculations. Recall we showed in the proof of F.4.37.4 that M has two classes of involutions, with representatives f and z; and $\bar{M}$ has four classes, with representatives $\bar{f}$, $\bar{f}\bar{z}$, $\bar{z}$, and

$\bar{t}$, where $fz \in z^M$ and $t \in f^M$. As $\dot{M}$ and M satisfy the same hypothesis, by symmetry $\dot{f}\dot{z} \in \dot{z}^{\dot{M}}$ and $\dot{t} \in \dot{f}^{\dot{M}}$; or this is easy to calculate directly using (4). Then (5) follows using F.4.35. Finally (7) and (8) are easy calculations in $\dot{M}$; given (5), they follow in M also. □

Now recall from the proof of F.4.37.4 that $M_t := C_M(t)$, $T_t := C_T(t) \in Syl_2(M_t)$, $T_f := C_T(f) \in Syl_2(C_{M_i}(f))$, and $T_f < R$ with R Sylow in $M_f := C_M(f)$.

LEMMA F.5.4. *(1)* $A_i = C_M(A_i)$.
(2) $R \cong T_t \cong D_8 \times D_8$.
(3) R acts on E_i and $E_iR \cong D_8 \times S_4$.
(4) $\mathcal{A}(T) = \{A_1, A_2, A_2^y, Q_2\}$ *for* $y \in T - T_t$, $N_M(Q_2)/Q_2 \cong N_M(A_1)/A_1 \cong S_5$, *and* $N_M(A_2)/A_2 \cong S_3 \times S_3$.
(5) $Aut_M(A_i)$ *contains a transvection.*

PROOF. As $z \in A_i$, $C_M(A_i) = C_{M_1}(A_i)$ using hypothesis (b) of Theorem F.4.31; then (1) follows as $C_{\dot{M}_1}(\dot{A}_i) = \dot{A}_i$. Since $T_t \cong D_8 \times D_8$ is Sylow in M_t, and $f \in t^M$, (2) follows. In particular by (2), R acts on each of the 4 members of $\mathcal{A}(R)$, so R acts on $A_i = O_2(E_i)$. Indeed calculating in $\dot{M}$ and applying μ, we conclude $\mathcal{A}(T) = \{A_1, A_2, A_2^y, Q_2\}$ for $y \in T - T_t$. Thus A_1 and Q_2 are the members of $\mathcal{A}(T)$ normal in T, so A_1 and Q_2 are not conjugate in $N_M(T)$, since if they were then $\{A_1, Q_2\}$ would be an orbit of length 2. Therefore by Burnside's Fusion Lemma A.1.35, $A_1 \notin Q_2^M$. Also $A_2 \notin A_1^M \cup Q_2^M$, as A_2 has a different fusion pattern by F.5.3.8. Thus M_z is transitive on $\{A^g : z \in A^g\}$, for each $A \in \mathcal{A}(T)$, so $N_M(A_i)$ is transitive on $z^M \cap A_i$. Further $Aut_{M_1}(A_1) \cong S_4$ and $N_{M_1}(A_2) = T_t$, so we conclude from F.5.3.8 and (1) that $N_M(A_1)/A_1$ acts faithfully as S_5 on $z^M \cap A_1$, and that $|N_M(A_2) : A_2| = 36$. Then from the structure of $GL_4(2)$, (4) holds. An element of T_t induces a transvection on A_i, so (5) holds. We saw R acts on A_i, so R acts on $O^2(N_{M_f}(A_i))$. Calculating in $\dot{M}_i$ and applying σ, $O^2(E_i) \cong A_4$. But by (4), $O^2(N_{M_f}(A_i)) \cong A_4$, so $O^2(E_i) = O^2(N_{M_f}(A_i))$ is R-invariant. As $|T_f| = 2^5$ and $|R| = 2^6$, $T_f \trianglelefteq R$. Thus R acts on $O^2(E_i)T_f = E_i$, so (3) holds. □

Recall from F.4.37 that $\bar{D} := C_{\bar{f}} \cong Sp_4(2)$, $\bar{D}_J := C_{\bar{G}_J}(\bar{f})$, and $D_J := \beta(\bar{D}_J)$. Thus $D_i \cong \mathbf{Z}_2 \times S_4$.

LEMMA F.5.5. *(1)* $D := \langle D_1, D_2 \rangle \cong S_6$.
(2) $R_0 := C_R(O^2(D)) \cong \mathbf{Z}_4$.
(3) $RD = M_f = C_M(f)$.
(4) $M_0 \cong \bar{M} \cong \dot{M}_0 \cong Aut(U_4(2))$.

PROOF. Let $M_f^* := M_f/\langle f \rangle$. Now $R \cong D_8 \times D_8$ by F.5.4.2, and $u := ftz$ is the diagonal element in the center of R, while $u \in z^M$ by F.5.3.7. Thus f is central in a D_8 factor of R, so $R^* \cong E_4 \times D_8$. Now from F.5.4, t and tz are the involutions i in T with $C_T(i) = T_t = J(T)$, and $tz \in t^T$. We saw in the proof of F.5.3 that $t \in f^M$, so $f = t^m$ for some $m \in M$, with $R^m = T_t$, and it follows that $(z, t)^m = (u, f)$. Thus as $T_t = C_M(\langle z, t \rangle)$, $R = C_{M_f}(u)$. Therefore $R^* = C_{M_f^*}(u^*)$ as $u \in z^M$ is not M-conjugate to uf since $fu = tz \in f^M$. Further as $\langle u^* \rangle = \Phi(R^*)$, $N_{M_f^*}(R^*) \leq C_{M_f^*}(u^*) = R^*$. Thus by Burnside's Fusion Lemma A.1.35, there is no fusion among the involutions in $Z(R^*)$. Then as $U := O_2(O^2(D_1))O_2(O^2(D_2)) \cong D_8$ is contained in $O^2(M_f)$, it follows from successive applications of Thompson Transfer that $U^* \in Syl_2(O^2(M_f^*))$, and $O^2(M_f^*)$ has one conjugacy class of involutions. As $R^* = C_{M_f^*}(u^*)$ is a 2-group, we conclude from I.4.1.2 that $O^2(M_f^*) \cong A_6$ or $L_3(2)$. The latter is impossible as z^* acts faithfully on $O^2(M_f^*)$, with $A_4 \cong O^2(D_1^*) \leq C_{O^2(M_f^*)}(z^*)$. Thus $O^2(M_f^*) \cong A_6$ and z^* induces a transposition on $O^2(M_f^*)$. Then as $R^* \cong E_4 \times D_8$, we conclude from the structure of the centralizers of involutions in $Aut(A_6)$ that $O^2(M_f^*)R^* \cong \mathbf{Z}_2 \times S_6$. As $D_8 \cong U \leq O^2(M_f)$, $O^2(M_f) \cong A_6$, so $O^2(M_f) = O^2(D)$. Then $R_0 = C_R(O^2(D)) = C_R(O^2(M_f)) = O_2(M_f)$ is of order 4, and $M_f/R_0 \cong S_6$. This establishes (1) and (3).

Suppose that $R_0 \cong E_4$ rather than $\mathbf{Z}_4$. As f and z are representatives for the classes of involutions in M, and $C_M(z)$ is solvable, $R_0^\# \subseteq f^M$. Then $O^2(M_f) = O^2(M_r)$ for each $r \in R_0^\#$, so by I.6.1.1, R_0 is a TI-set in M. Then $Aut_M(R_0)$ contains $\mathbf{Z}_3$ as $R_0^\# \subseteq f^M$, so as $R \cong D_8 \times D_8$, $Aut_M(R_0) \cong S_3$. Then $X := N_M(R_0) \cap C_M(O^2(M_f)) \cong A_4$, so for $A \in \mathcal{A}(R_0 \times U)$, $Aut_{M_f}(A) \cong S_3 \times S_3$. Then by F.5.4.4, up to conjugacy in M, $\mathcal{A}(R_0 \times U) = \{A_2, A_2^y\}$. However $A_2 \cap A_2^y = \langle z, t\rangle$ is of order 4, whereas the members of $\mathcal{A}(R_0 \times U)$ intersect in an E_8-subgroup. This contradiction completes the proof of (2). By (1), there are involutions $s_i \in D_i - D_{1,2}$ with $|s_1 s_2| = 4$. Now (4) follows from F.4.26. □

LEMMA F.5.6. $M = \langle M_0, N_M(A_2)\rangle$.

PROOF. By F.5.4.4, $N_M(A_2)$ contains $R \not\leq M_0$; thus $M_0 < M_+ := \langle M_0, N_G(A_2)\rangle$. Suppose that $M_+ < M$. Then M_+ satisfies the hypotheses of Theorem F.4.31, and F.4.37.4.i does not hold as $T_t < R \in Syl_2(C_{M_+}(f))$. Thus by induction on $|M|$, we may apply F.4.37.4 to M_+ in the role of "M", to conclude $M_+ \cong L_4(3)$. Therefore M_+ has two classes of involutions, so $u^{M_+} = u^M \cap M_+$ for $u \in \{z, f\}$. As $D \leq M_0$ and $R \leq N_M(A_2)$, $M_f = C_M(f) = RD \leq M_+$ by F.5.5.3. Also we have seen that $N_M(T) \leq M_z = M_1 \leq M_+$, so M_+ is strongly embedded in M by I.8.1.3. This is impossible by 7.6 in [**Asc94**], as M_+ has two conjugacy classes of involutions. □

We next construct two uniqueness systems in the sense of section 37 of [**Asc94**].

Let Δ be the graph with vertex set $\Delta := M/M_0$, and M_0 adjacent to $M_0 g$ iff $g \in M_0 r M_0$ for $r \in R - T_f$. From F.5.5, $\langle f\rangle D = C_{M_0}(f)$ is of index 2 in M_f, so the orbital $M_0 r M_0$ is independent of the choice of $r \in R - T_f$ and self-paired; hence the relation on Δ defined by the orbital is symmetric. Define $H := N_M(A_2)$. Write x for the coset M_0, and y for the coset $M_0 r$. Then $M_x = M_0$, so $M = \langle H, M_x\rangle$ by F.5.6. By F.5.4.4, $H/A_2 \cong S_3 \times S_3$, while from the structure of M_0, $H_x = N_{M_0}(A_2) \cong D_8 \times S_4$. Thus $|H : H_x| = 3$ is prime, so H_x is maximal in $H \cong S_4 \times S_4$, and H induces S_3 on the coset space H/H_x. As $\langle r\rangle D \trianglelefteq M_f$ and $\langle f\rangle D \leq M_0$, $\langle f\rangle D \leq M_0 \cap M_0^r = M_{x,y}$. Then as $\langle f\rangle D$ is maximal in M_0, $M_{x,y} = \langle f\rangle D$. As $M_{x,y}$ is maximal in $M_x = M_0$, but $H_x \not\leq \langle f\rangle D = M_{x,y}$, $M_x = \langle M_{x,y}, H_x\rangle$. Finally r lies in the setwise stabilizer $H(\{x, y\})$ in H of the pair $\{x, y\}$ but not in H_x and H_x is maximal in H, so $H = \langle H_x, H(\{x, y\})\rangle$.

Set $\mathcal{U} := (M, H, \Delta, \Delta_H)$, where $\Delta_H := \{xh : h \in H\}$. We saw that H induces S_3 on the three cosets H/H_x, and $y = xr \in \Delta_H$, so Δ_H is a clique of order 3 with edge set $(x, y)H$. In the previous paragraph, we checked the other conditions in display (U) on page 198 of [**Asc94**]; so $\mathcal{U}$ is a uniqueness system.

Define $\dot{\mathcal{U}} := (\dot{M}, \dot{H}, \dot{\Delta}, \dot{\Delta}_H)$, where: $\dot{\Delta}$ is the graph on the points Fv of V with $q(v) = -1$ (i.e., on the points of discriminant 1), with Fv adjacent to Fu

if u and v are orthogonal; $\dot{H} := N_{\dot{M}}(V_3^+) = N_{\dot{M}}(\dot{A}_2)$, where $V_3^+ := \langle x_1, x_2, x_6 \rangle$ is a nondegenerate 3-space of discriminant $+1$, containing exactly three points: $\dot{x} := F\ x_6$, $\dot{y} := F\ (x_1 - x_2)$, and $F\ (x_1 + x_2)$ of Δ; and $\dot{\Delta}_H$ is this clique $\dot{\Delta} \cap V_3$. Again one can check that $\dot{\mathcal{U}}$ is a uniqueness system; indeed one can use the following lemma, and the discussion of $\mathcal{U}$ in the previous paragraph:

LEMMA F.5.7. *(1) There is a group isomorphism $\sigma : \dot{M}_0 \to M_0$ extending the amalgam isomorphism $\sigma : \dot{\gamma} \to \gamma_M$.*

(2) $\sigma_{|\dot{H}_{\dot{x}}}$ extends to $\zeta : \dot{H} \to H$, with $\zeta(\dot{H}(\{\dot{x}, \dot{y}\})) = H(\{x, y\})$.

(3) σ and ζ define an equivalence of the uniqueness systems $\dot{\mathcal{U}}$ and $\mathcal{U}$.

PROOF. By F.5.5.4, there is an isomorphism $\psi : \dot{M}_0 \to M_0$ with $\psi(\dot{\gamma}) = \gamma_M$. As in F.4.11.1, μ is quasiequivalent to $\psi^{-1} \circ \beta$, so there are $\tau \in Aut(\gamma)$ and $\varphi \in Aut(\dot{M}_0)$ with $\mu \circ \tau = \varphi \circ \psi^{-1} \circ \beta$. Then replacing μ by $\mu \circ \tau$ and ψ by $\psi \circ \varphi^{-1}$, $\psi = \beta \circ \mu^{-1} = \sigma$, so that the new ψ is the extension σ required for (1).

By F.5.4.4, there is an isomorphism $\rho : \dot{H} \to H$ with $\rho(\dot{H}_{\dot{x}}) = H_x$. Then $\sigma^{-1} \circ \rho \in Aut(\dot{H}_{\dot{x}})$ acts on $\dot{A}_2$; so as $Inn(\dot{H}_{\dot{x}})$ is the group of all such automorphisms, adjusting ρ by an inner automorphism, we obtain our extension $\zeta : \dot{H} \to H$ of $\sigma_{|\dot{H}_{\dot{x}}}$. If $\zeta(\dot{H}_{\dot{y}}) \neq H_y$, then replace $\dot{y}$ by $\dot{h}(\dot{y})$ for $\dot{h} \in \dot{H}_{\dot{x}} - \dot{H}_{\dot{y}}$, to obtain (2). Then by (1) and (2), and our construction, $\zeta(\dot{H}_{\dot{x}}) = H_x$ and $\sigma(\dot{M}_{\dot{x}\dot{y}}) = M_{xy}$, so σ and ζ define a similarity of $\dot{\mathcal{U}}$ and $\mathcal{U}$, as defined on page 199 of [**Asc94**].

Finally $\zeta(O^2(\dot{D}_2)) = \sigma(O^2(\dot{D}_2)) = O^2(D_2)$ and $\zeta(\dot{f}) = \sigma(\dot{f}) = f$, so using F.5.4.4,

$$\zeta(C_{\dot{H}}(\langle \dot{f} \rangle O^2(\dot{D}_2)) = C_H(\langle f \rangle O^2(D_2)) \cong D_8.$$

Therefore as $\dot{R}_0$ and R_0 are the unique cyclic subgroups of index 2 in the respective D_8-subgroups, $\zeta(\dot{R}_0) = R_0$. Pick a generator $\dot{r}$ of $\dot{R}_0$ and set $r := \zeta(\dot{r})$. Then $\dot{r}$ centralizes $O^2(\dot{D})\dot{R}_0 = F^*(\dot{D}\dot{R})$, and for $\dot{s} \in \dot{R} - \dot{F}^*(\dot{D}\dot{R})$, $[\dot{s}, \dot{r}] = \dot{f}$. Similarly r satisfies the analogous conditions; thus $\sigma(\dot{b}^{\dot{r}}) = \sigma(b)^r$ for all $b \in \dot{M}_{\dot{x}\dot{y}} = \dot{D}\langle \dot{f} \rangle$, so (3) follows from the definition of equivalence on page 199 of [**Asc94**]. □

The equivalence in F.5.7.3 reduces the identification of M to the following calculations in the graph Δ:

LEMMA F.5.8. *(1) All triangles of $\dot{\Delta}$ are conjugate under $\dot{M}$ to $\dot{\Delta}_H$.*

(2) $\dot{\Delta}$ is simply connected.

(3) $\dot{\Delta}_H$ is a base for $\dot{\mathcal{U}}$.

PROOF. By definition (cf. p.182 of [**Asc94**]) $\dot{\Delta}$ is triangulable iff the closure $\mathcal{C}_3$ of the triangles of $\dot{\Delta}$ is the set of all cycles of $\dot{\Delta}$. By 35.14 in [**Asc94**], $\dot{\Delta}$ is simply connected iff $\dot{\Delta}$ is triangulable. Thus (1) and (2) will imply (3) by definition of "base" (cf. p.200 of [**Asc94**]).

If $\theta := \{Fu_1, Fu_2, Fu_3\}$ is a triangle, then $U := \langle \theta \rangle$ is the orthogonal direct sum of the points Fu_i, so U is of rank 3 and discriminant $+1$; thus by Witt's Lemma (e.g., section 20 of [**Asc86a**]), U is $\dot{M}$-conjugate to V_3^+, and therefore θ is conjugate to the triple $\dot{\Delta}_H = \dot{\Delta} \cap V_3^+$. Hence (1) holds.

Next $\dot{\Delta}$ is a rank-3 graph under $\dot{M}$, and the three orbits of $\dot{M}_0$ are $\{\dot{x}\}$, $\dot{\Delta}(\dot{x})$, and $\Sigma(\dot{x})$, where $\Sigma(\dot{x})$ consists of the points $Fw \in \dot{\Delta}$ with $\langle w, x_6 \rangle$ a degenerate 2-space containing a point of discriminant 1. In particular $\dot{\Delta}$ is of diameter 2, so to prove (2), it suffices by 34.5 in [**Asc94**] to show that each n-gon in $\dot{\Delta}$ is in $\mathcal{C}_3$ for $n \leq 5$. By definition of $\mathcal{C}_3$, this holds if $n \leq 3$.

Let Fv be at distance 2 from $\dot{x} = Fx_6$ in $\dot{\Delta}$; thus $U := \langle v, x_6 \rangle$ is a degenerate line. Let Fu be the radical of U; then $\dot{\Delta}(\dot{x}, Fv)$ is the set of points of discriminant 1 in $U^\perp$. But $u^\perp = Fu \perp W$, where W is a 4-subspace of sign $+1$ containing x_6, and $U^\perp = Fu \perp Y$ where $Y := W \cap x_6^\perp$ is a 3-space of discriminant 1. Thus $\dot{\Delta} \cap U^\perp$ is the union of the sets $\dot{\Delta} \cap \langle u, y_i \rangle$, $1 \leq i \leq 3$, where Fy_i are the three points of discriminant 1 in Y. Hence $\dot{\Delta}(\dot{x}, Fv)$ is connected. Therefore each 4-gon in $\dot{\Delta}$ is in $\mathcal{C}_3$ by 34.6 in [**Asc94**].

Let $Fw \in \dot{\Delta}(\dot{x})$, and suppose Fv is at distance 2 from both $\dot{x}$ and Fw in $\dot{\Delta}$. Set $P := \langle x_6, w, v \rangle$. The line $U := \langle x_6, v \rangle$ contains no points of discriminant 1 orthogonal to $\dot{x}$, so $\dim(P) = 3$. Therefore $\dim(P^\perp) = 3$. Now P contains the nondegenerate line $X := \langle x_6, w \rangle$, so $\dim(Rad(P^\perp)) = \dim(P \cap P^\perp) \leq 1$, and hence $P^\perp$ contains some $Fa \in \dot{\Delta}$. Thus $Fa \in \dot{\Delta}(\dot{x}, Fw, Fv)$, so all 5-gons are in the closure $\mathcal{C}_4$ of 4-gons by 34.8 in [**Asc94**]. But we just saw that $\mathcal{C}_4 \subseteq \mathcal{C}_3$. This completes the verification of (2), and hence the proof of the lemma. $\square$

We can now complete the proof of Lemma F.4.37.4, and hence also of Theorem F.4.31. Namely F.5.7.3 and F.5.8.3, together with simplicity of $\dot{M} \cong L_4(3)$, supply the hypotheses of Exercise 13.1 in [**Asc94**]. Therefore $M \cong \dot{M}$ by that Exercise.

F.6. Goldschmidt triples

The pioneering work of Goldschmidt [**Gol80**] (extending earlier results of Tutte and Sims) considered rank-2 amalgams corresponding to trivalent graphs; these are the amalgams $(G_1, G_{1,2}, G_2)$ in which $G_{1,2}$ is a 2-group of index 3 in G_1 and G_2. The case in which each $G_i/O_2(G_i) \cong L_2(2)$ also plays an important role at various places in this work, particularly in the treatment of groups over $\mathbf{F}_2$, and of those groups in which $\mathcal{L}_f(G,T)$ is empty. So in this section, we consider such amalgams from a somewhat more general viewpoint than in the previous sections: essentially we relax parts (e) and (f) of Hypothesis F.1.1. See Remark F.6.4 below for a discussion of the relation between the Goldschmidt triples treated in this section and the weak BN-pairs considered in earlier sections.

DEFINITION F.6.1. A *Goldschmidt triple* is a 3-tuple (G, G_1, G_2) such that G is a finite group and G_1 and G_2 are subgroups of G with $G = \langle G_1, G_2 \rangle$, $G_i/O_2(G_i) \cong S_3$, and $G_1 \cap G_2 \in Syl_2(G)$. A *Goldschmidt amalgam* is a triple $(G_1, G_1 \cap G_2, G_2)$ such that (G, G_1, G_2) is a Goldschmidt triple with $O_2(G) = 1$.

Throughout this section, we assume:

HYPOTHESIS F.6.2. *(G, G_1, G_2) is a Goldschmidt triple. Set $T := G_1 \cap G_2$, $Q_i := O_2(G_i)$, $L_i := O^2(G_i)$, and $D_i \in Syl_3(G_i)$. Also set $L := O^2(G)$ and $Q := O_2(G)$.*

In the first part of the section, we establish some elementary consequences of Hypothesis F.6.2; later we apply those results to the quotient $G/O_{3'}(G)$ to obtain very detailed information.

First, it is immediate from the definitions that:

LEMMA F.6.3. *Q is the largest subgroup of T normal in G_1 and G_2.*

REMARK F.6.4. We digress briefly to discuss the relationship of Goldschmidt triples with the weak BN-pairs of Definition F.1.7 Let $\beta := (G_1, T, G_2)$ and $\alpha := (G_1/Q, T/Q, G_2/Q)$. Observe that α and β are rank 2 amalgams and α is a Goldschmidt amalgam. It is straightforward to check that the hypothesis $G_i/Q_i \cong S_3$ implies that α and β satisfy parts (a) through (d) of Hypothesis F.1.1, with G_i, G_i/Q in the role of "L_i", and T, T/Q in the roles of "S, S_i, B_i", for β, α, respectively.

Further part (e) is satisfied in α by F.6.3, but in general not in β. Thus if in addition part (f) of Hypothesis F.1.1 holds in α, then α is a weak BN-pair of rank 2 by F.1.9. Moreover the hypothesis of F.1.12 holds, so as $G_i/Q_i \cong S_3$, α is one of the six weak BN-pairs listed below in F.6.5.2.

Appealing to F.6.3, Remark F.6.4, and the Corollary to Theorem A in [**Gol80**], and inspecting the amalgams listed in Table 1 of [**Gol80**], we obtain the following result:

LEMMA F.6.5. *(1) $\alpha := (G_1/Q, T/Q, G_2/Q)$ is a Goldschmidt amalgam and $(G/Q, G_1/Q, G_2/Q)$ is a Goldschmidt triple.*

(2) α is described in those 11 of the 15 cases of Table 1 of [**Gol80**] *in which $G_i/Q_i \cong S_3$ for $i = 1$ and 2. The possible pairs $(G_1/Q, T/Q, G_2/Q)$ are:*

(i) $(S_3, \mathbf{Z}_2, S_3)$
(ii) (D_{12}, E_4, D_{12})
(iii) (D_{24}, D_8, S_4)
(iv) $(S_4, D_8, \mathbf{Z}_2/(\mathbf{Z}_3 \times E_4)$
(v) $(\mathbf{Z}_2 \times S_4, \mathbf{Z}_2 \times D_8, S_3 \times D_8)$
(vi) one of cases (1), (2), (3), (8), (12), or (13) of F.1.12.

(3) In each case in (vi), $F^*(G_i/Q) = O_2(G_i/Q)$ for $i = 1$ and 2, so α is a weak BN-pair of rank 2, while in cases (i)–(v), $F^*(G_j/Q)$ is not a 2-group for one of $j = 1$ or 2.

LEMMA F.6.6. $L = O^2(G) = \langle L_1, L_2 \rangle$.

PROOF. First T acts on L_1 and L_2, so T acts on $L_0 := \langle L_1, L_2 \rangle$, and hence

$$G = \langle G_1, G_2 \rangle = \langle T, L_0 \rangle = L_0 T,$$

so that $L = O^2(G) \leq L_0$. Conversely $L_i = O^2(L_i)$, so $L_0 \leq L$. □

LEMMA F.6.7. *Assume $X \trianglelefteq \trianglelefteq G$. Then the following are equivalent:*
(1) $G_i \cap X \leq Q_i$
(2) $G_i \cap X = Q_i \cap X = T \cap X \in Syl_2(X)$.
(3) $L_i \not\leq X$.

PROOF. As $G_i/Q_i \cong S_3$ and $L_i = O^2(G_i)$, $L_i \not\leq X$ iff $L_i \cap X \leq O_2(L_i)$ iff $G_i \cap X \leq Q_i$ iff $G_i \cap X = Q_i \cap X = T \cap X \in Syl_2(X)$, since X is subnormal in G. □

LEMMA F.6.8. *Assume $X \trianglelefteq G$ with $G_i \cap X \leq Q_i$ for $i = 1$ and 2. Then $T \cap X \leq Q$, so XQ/Q is of odd order and in particular XQ/Q is solvable.*

PROOF. By F.6.7, $G_1 \cap X = T \cap X = G_2 \cap X$, so that $T \cap X$ is normal in G_1 and G_2, and hence $T \cap X \leq O_2(G) = Q$. As $Q \leq T$, $Q \cap X = T \cap X \in Syl_2(X)$, and the lemma follows, using the Odd Order Theorem for the final assertion of solvability. □

LEMMA F.6.9. *Assume G is a solvable quotient of an SQTK-group and L_i centralizes $O^3(F(G))$ for $i = 1$ or 2. Then G is a $\{2,3\}$-group.*

PROOF. By hypothesis $L_i \leq X := C_G(O^3(F(G)))$. Observe that $F(X) = Z(X)O_3(G)$.

Next by hypothesis there is an SQTK-group $\hat{G}$ and $\hat{M} \trianglelefteq \hat{G}$ with $G = \hat{G}/\hat{M}$. Let P be a Sylow 3-subgroup of the preimage in $\hat{G}$ of $O_3(G)$; by B.5.2.1, we may assume that $P \trianglelefteq \hat{G}$. By A.1.25.3 $Aut_{\hat{G}}(P)$ is a $\{2,3\}$-group, so as $C_{\hat{G}}(P)\hat{M}/\hat{M} \leq C_G(O_3(G))$, $G/C_G(O_3(G))$ is a $\{2,3\}$-group. Since $C_X(O_3(G))$ centralizes $Z(X)O_3(G) = F(X)$, $X/Z(X)$ is a $\{2,3\}$-group by Fitting's Theorem 31.10 in [**Asc86a**]. Therefore $X = KZ(X)$ where K is a Hall $\{2,3\}$-subgroup of X, so $K = O^{\{2,3\}'}(X)$ is normal in G. Now G_{3-i} is a $\{2,3\}$-group acting on the $\{2,3\}$-group K, so $Y := G_{3-i}K$ is a $\{2,3\}$-group. Finally $G = \langle G_{3-i}, L_i\rangle \leq Y$, so G is a $\{2,3\}$-group. □

LEMMA F.6.10. *Assume G is a nonsolvable quotient of a quasithin $\mathcal{K}$-group and K is a component of G. Then $L_i \leq K$ for $i = 1$ or 2, $K \trianglelefteq G$, and $G = KG_{3-i}$. In particular, $K \not\leq O_{3'}(G)$.*

PROOF. By hypothesis G is a quotient of a quasisimple $\mathcal{K}$-group, so as $K_0 := \langle K^G\rangle$ is a semisimple subgroup of G, (1) of Theorem A (A.2.1) says that either K_0 is also a quasithin $\mathcal{K}$-group; or $K_0 = K_1K_2$ with $K_i \cong U_3(2^{n_i})$ where $(n_1, n_2) = 1$ or 3—but $n_1 = n_2$ as $K_1 \cong K_2$, so $n_1 = n_2 = 3$ and $K_i \cong U_3(8)$. In the first case applying A.3.8.1 to K_0 in the role of "H", $K_0 = \langle K^G\rangle = K$ or KK^t for $t \in T - N_T(K)$, and in the second case this clearly holds. In each case $L_i = O^2(L_i) \leq N_G(K)$.

Suppose that $L_i \not\leq K$ for $i = 1$ or 2. As $K \trianglelefteq\trianglelefteq G$, $G_i \cap K = T \cap K$ by F.6.7. As KQ/Q is not solvable, K is not normal in G by F.6.8, so $K_0 = KK^t$ by paragraph one. Also $T \cap K = G_i \cap K = \trianglelefteq L_iN_T(K)$, so as $t \in T$ normalizes L_i and $N_T(K) = N_T(K^t)$, $R := (T \cap K)(T \cap K)^t$ is normal in $L_iN_T(K)\langle t\rangle = L_iT = G_i$ for each i, so $R \leq Q$ by F.6.3. But $T \cap K \in Syl_2(K)$, so K is 2-closed, contradicting K nonsolvable.

Thus we may assume $L_1 \leq K$, so from paragraph one $T \leq N_G(L_1) \leq N_G(K)$ and $L_i \leq N_G(K)$, so $K \trianglelefteq \langle L_1, L_2, T\rangle = G$. Then $G = \langle L_1, L_2, T\rangle \leq KL_2T = KG_2$. □

LEMMA F.6.11. *(1) $Q = O_2(G) \in Syl_2(O_{3'}(G))$, so $O_{3'}(G)$ is solvable.*

(2) Let $G^* := G/O_{3'}(G)$. Then either

(i) $G_1^* \cap G_2^* = T^*$, (G^*, G_1^*, G_2^*) is a Goldschmidt triple, and $(G_1^*, T^*, G_2^*) \cong (G_1/Q, T/Q, G_2/Q)$ is a Goldschmidt amalgam, or

(ii) $Q_1 = Q_2 = Q$, $G^* \cong S_3$, and $G = G_1O_{3'}(G)$. Further $O_{3'}(G)/Q$ is not cyclic.

PROOF. Let $X := O_{3'}(G)$ and $\bar{G} := G/Q$. Then

$$Q \leq X \cap G_i \leq O_{3'}(G_i) = Q_i,$$

so by F.6.8, $T \cap X \leq Q$ and hence $Q = T \cap X = G_i \cap X$, establishing (1). As $Q = G_i \cap X$, the map $\bar{u} \mapsto u^*$ is an isomorphism of $(\bar{G}_1, \bar{T}, \bar{G}_2)$ with $\gamma := (G_1^*, T^*, G_2^*)$.

As T^* is maximal in G_i^*, either $G_1^* = G_2^* \cong S_3$, or $G_1^* \cap G_2^* = T^* \in Syl_2(G^*)$. In the latter case by F.6.5.1, (G^*, G_1^*, G_2^*) is a Goldschmidt triple and γ is a Goldschmidt amalgam, so that (i) holds. In the former case $G = G_iO_{3'}(G)$ for $i = 1, 2$, so to show that (ii) holds it remains to assume that $\bar{X}$ is cyclic and to exhibit a contradiction. Passing to $\bar{G}$ and appealing to F.6.5.1, we may take $Q = 1$. Then X is cyclic, so $G_iX = G$ is solvable and quasithin, and $L_i = [L_i, T]$ centralizes X. Hence $X = 1$ by F.6.9, so $G = G_iX = G_i$ for each i, contradicting $T = G_1 \cap G_2$. □

LEMMA F.6.12. *Assume that $O_2(G) = 1$. Then for $i = 1$ and 2,*

(1) There is no nontrivial D_i-invariant subgroup S of Q_i with $S \trianglelefteq Q_iL_{3-i}$.

(2) There is no nontrivial characteristic subgroup S of Q_i with S normal in Q_iL_{3-i}.

PROOF. Notice (1) implies (2), so we assume that S is a counterexample to (1), and without loss take $i = 2$. By a Frattini Argument there is $t \in N_T(D_2) - Q_2$. Since $T = Q_2\langle t\rangle$ normalizes Q_2D_2 and L_1, $SS^t \trianglelefteq \langle Q_2D_2, t, L_1\rangle \leq \langle G_2, L_1\rangle = G$, contrary to our hypothesis that $O_2(G) = 1$. □

LEMMA F.6.13. *Assume that $O_2(G) = 1$ and $L_1 \leq X \trianglelefteq G$, but $L_2 \not\leq X$. Then*

(1) $G = XG_2$, $XQ_2 \trianglelefteq G$, and $Q_2 \in Syl_2(XQ_2)$.

(2) $L_1 \cong \mathbf{Z}_3$.

(3) $Q_2 \cong E_{2^n}$, $n \leq 3$. If $Q_2 \trianglelefteq Q_2L_1$, then $Q_2 = 1$, and if $O_2(L_2) \trianglelefteq Q_2L_1$, then $O_2(L_2) = 1$.

(4) If $X = O(X)$ then $[Q_1, L_1] = 1$.

PROOF. As $L_1 \leq X \trianglelefteq G$, $G = \langle L_1, G_2\rangle = XG_2$. Hence $XQ_2 \trianglelefteq G$. Next $L_2 \not\leq X$ by hypothesis, so by F.6.7, $X \cap Q_2 \in Syl_2(X)$. Therefore $Q_2 \in Syl_2(Q_2X)$, completing the proof of (1).

Next as $Q_2 \in Syl_2(Q_2X)$, Q_2 is Sylow in $Y := Q_2L_1$. By F.6.12.2 and maximality of Q_2 in Y, $Q_2 = N_Y(C)$ for each $1 \neq C$ char Q_2.

Assume that $L_1 \not\cong \mathbf{Z}_3$. Then $F^*(Y) = O_2(Y)$, so as $L_1 \trianglelefteq Y$ we conclude from the C(G,T)-Theorem C.1.29 that L_1 is an A_3-block; that is $L_1 \cong A_4$. Further as $Q = 1$ by hypothesis, the amalgam $\alpha = (G_1, T, G_2)$ appears in the list of F.6.5. As $L_1 \cong A_4$, either case (iii), (iv), or (v) holds, or case (1) or (2) of F.1.12 holds. In each case $Y \cong S_4$ or $S_4 \times \mathbf{Z}_2$, so $A := O_2(Y) \in \mathcal{A}(Q_2)$ and $|\mathcal{A}(Q_2)| = 2$. Thus $D_2 = O^2(D_2)$ acts on A, contrary to F.6.12.2. This contradiction establishes (2).

As L_1 is cyclic by (2), $\Phi(Q_2)$ centralizes L_1. Then $\Phi(Q_2) \trianglelefteq Y$ is D_2-invariant, so $\Phi(Q_2) = 1$ by F.6.12.2. Then as $L_1 \cong \mathbf{Z}_3$, $A = C_{Q_2}(L_1)$ is of index at most 2 in Q_2. Let d be a generator of D_2; as d normalizes Q_2 and A centralizes L_1,

$$B := A \cap A^d \cap A^{d^2} \trianglelefteq Q_2L_1,$$

and B is D_2-invariant, so $B = 1$ by F.6.12.1. Thus $m(Q_2) = m(Q_2/B) \leq 3$ as $m(Q_2/A) \leq 1$; indeed if $Q_2 = A$, then $A = Q_2 = B = 1$. Similarly if $A_2 := O_2(L_2) \trianglelefteq Q_2L_1$, then $A_2 \trianglelefteq Y$, so $A_2 = 1$ by F.6.12.1. This completes the proof of (3).

Finally if $X = O(X)$, then $[L_1, Q_1] \leq X \cap Q_1 = 1$, so (4) holds. □

LEMMA F.6.14. *If $L = O(G)$ and $O_2(G) = 1$, then either*

(1) $T \cong \mathbf{Z}_2$, or

(2) $Q_i \cong \mathbf{Z}_2$ and $T \cong E_4$.

PROOF. Observe $[Q_i, L_i] \leq Q_i \cap O(G) = 1$, so $Q_1 \cap Q_2$ is centralized by L and normal in T, and hence $Q_1 \cap Q_2 \leq Q = 1$. Thus $|T| = |Q_1Q_2| \leq 4$, and examining the cases in F.6.5.2 for this condition, we find that only cases (i) or (ii) can occur. □

REMARK F.6.15. In the remainder of this section we assume that $G = \hat{G}/O_{3'}(\hat{G})$, where $\hat{G}$ is either an SQTK-group or a quasithin $\mathcal{K}$-group. Observe that because of this hypothesis, if K is a component of G, then by (a) or (b) of (1) in Theorem A, $K/Z(K)$ is described in Theorem B (A.2.2) or Theorem C (A.2.3), for $\hat{G}$ quasithin or strongly quasithin, respectively. Notice also that this hypothesis implies that $O_{3'}(G) = 1$. In particular $O_2(G) = 1$, so by F.6.5, $\alpha := (G_1, T, G_2)$ is a Goldschmidt amalgam described in (i)-(vi) of part (2) of F.6.5.

LEMMA F.6.16. *Assume that $G = \hat{G}/O_{3'}(\hat{G})$ for some SQTK-group $\hat{G}$, and that $O_3(G)$ is noncyclic. Then L is a 3-group, and either*
(1) $T \cong \mathbf{Z}_2$ and $Q_1 = Q_2 = 1$, or
(2) $G \cong S_3 \times S_3$ or $E_4/3^{1+2}$.

PROOF. Suppose first that L is a 3-group. Then $L = O(G) = F^*(G)$ since $G = LT$ and $O_2(G) = 1$. In particular, F.6.14 applies. If $T \cong \mathbf{Z}_2$ then (1) holds, so by F.6.14, we may assume that $T = Q_1 \times Q_2 \cong E_4$. Next $L_i \cong \mathbf{Z}_3$ as L is a 3-group, so as $L = \langle L_1, L_2 \rangle$ by F.6.6, $L/\Phi(L) \cong E_9$, and hence $L = L_1 L_2 \Phi(L)$. Now as $G = LT$ and T normalizes L_i, $X_i := L_i \Phi(L)$ is a normal subgroup of G. If $\Phi(L) = 1$, then $G \cong S_3 \times S_3$ and conclusion (2) holds, so we may assume that $\Phi(L) \neq 1$. Let $L^i(L)$ denote the i-th member of the descending central series for the 3-group L, starting at $L^1(L) = L$, and set $V := L_3(L)$. Then $1 \neq \Phi(L) = [L, L] = L^2(L)$, and hence $L/V \cong 3^{1+2}$ by A.1.24, with $t_i \in Q_{3-i} - Q_i$ inverting X_i/V and centralizing L_{3-i}. In particular if $V = 1$ then conclusion (2) holds, so we may assume $V \neq 1$. Set $\bar{G} := G/L^4(L)$. Then $\bar{T} \cong T \cong E_4$ acts on some subgroup $\bar{U}$ of index 3 in $\bar{V}$, so as $\bar{V} \leq Z(\bar{L})$, $\bar{U} \trianglelefteq \bar{L}\bar{T}$. Set $G^* := \bar{G}/\bar{U}$. As G^* is a quotient of $\hat{G}$, A.1.31.1 says that $C_{L^*}(t)$ is cyclic for each $t \in T^{\#}$. As t_i inverts X_i^*/V^* and V^* is of order 3, either t_i centralizes V^*, or X_i^* is inverted by t_i and in particular is abelian. As t_i centralizes L_{3-i} and $C_{L^*}(t_i)$ is cyclic but $\bar{X}_i\bar{V}$ is noncyclic, the latter case must hold for each i. But then V^* centralizes $\langle X_1^*, X_2^* \rangle = L^*$, contradicting L^* of class 3.

We have reduced to the case where L is not a 3-group; and again it remains to derive a contradiction. By hypothesis, $O_3(G)$ is noncyclic and $G = \hat{G}/O_{3'}(\hat{G})$ with $\hat{G}$ strongly quasithin, so $m_3(O_3(G)) = m_3(G) = 2$. By A.1.25.1 there is a supercritical subgroup X of $O_3(G)$ with $X \cong E_9$ or 3^{1+2}, such that $O^3(H) \leq C_G(O_3(G))$, where $H := C_G(X/\Phi(X))$.

Suppose first that $L_1 \leq X$. Then $L = XL_2$ and as L is not a 3-group, $L_2 \not\leq X$. Thus by F.6.13.3, $Q_2 = O_2(G_2)$ is elementary abelian, so $O_2(L_2)$ is elementary abelian. Therefore since A_4 is not involved in $SL_2(3)$, it follows that $O_2(L_2)$ centralizes $X/\Phi(X)$, so that $O_2(L_2) \leq O^3(H) \leq C_G(O_3(G))$ by an earlier remark. In particular $O_2(L_2)$ centralizes L_1, so $O_2(L_2) = 1$ by F.6.13.3, and hence $L = XL_2$ is a 3-group, contrary to assumption.

Thus we may assume that neither L_1 nor L_2 is contained in X. Now for $i = 1, 2$, $m_3(XL_i) \leq m_3(G) = 2$ by an earlier remark, so as $X \cong E_9$ or 3^{1+2} is of 3-rank 2, L_i does not induce inner automorphisms on X, and hence $[X/\Phi(X), L_i] \neq 1$. Further $H \trianglelefteq G$ with $G_i \cap H \leq Q_i$ and $Q = 1$, so H is solvable by F.6.8. Thus as $O_{3'}(G) = 1$, $F^*(H) = O_3(H)$, so as $O^3(H) \leq C_H(O_3(G))$, $H = O_3(H)$. If $L_1 H = L_2 H$, then $L_1 H = L_2 H = L$ is a 3-group, contrary to assumption. Thus $L_1 H \neq L_2 H$, so as $L_i = O^2(L_i)$, L induces $SL_2(3)$ on $X/\Phi(X)$. Let W denote the preimage in G of $O_2(G/H)$; then $W \cap G_i \leq Q_i$, so again using F.6.8 and $Q = 1$,

W is of odd order, contrary to $Q_8 \cong O_2(G/H)$. This contradiction completes the proof of F.6.16. □

LEMMA F.6.17. *Assume that $G = \hat{G}/O_{3'}(\hat{G})$ for some quasithin $\mathcal{K}$-group $\hat{G}$, and $O_3(G)$ is cyclic. Then either*

(1) $L = F^*(G)$ is quasisimple, or

(2) $L = K \times D$, where $K \cong L_2(q)$ for some prime power $q \equiv \pm 11 \mod 24$, $D \cong \mathbf{Z}_3$, $T \cong D_8$, $G_1 \cong D_{24}$, $G_2 \cong S_4$, and $t \in T - K$ induces an outer automorphism on K and D.

PROOF. If $F^*(G) = O_3(G)$ then as $O_3(G)$ is cyclic, $L = O^2(G) \leq C_G(O_3(G)) \leq O_3(G)$, so that $L = O_3(G)$; but then $L_1 = \Omega_1(O_3(G)) = L_2$, contrary to $G_1 \cap G_2 = T$. Thus $F^*(G) > O_3(G)$, so as $O_{3'}(G) = 1$ there is a component K of G. Hence by F.6.10 we may take $L_1 \leq K \trianglelefteq G = KG_2$. Now if $L_2 \leq K$ then $O^2(G) = L \leq K \leq O^2(G)$, so that (1) holds; thus we may assume that $L_2 \not\leq K$. Hence by F.6.13, $L_1 \cong \mathbf{Z}_3$, $Q_2 \in Syl_2(Q_2K)$, and $Q_2 \cong E_{2^n}$, $n \leq 3$. By Remark F.6.15, $K/Z(K)$ appears in Theorem B, so we conclude that $K \cong L_2(8)$, J_1, or $L_2(q)$ for some prime power $q \equiv \pm 3 \mod 8$; notice K is simple using the list of Schur multipliers in I.1.3 since $O_2(G) = 1$. In the first two cases a Sylow 2-group $Q_2 \cap K$ of K does not act on a subgroup L_1 of K of order 3, a contradiction. Similarly in the remaining case (cf. A.1.3) we must have $L_1(Q_2 \cap K) \cong D_{12}$. As $G_2/Q_2 \cong S_3$ and $Out(K)$ is cyclic, L_2 induces inner automorphisms on K. Then as $L_2 \not\leq K$, $L = K \times D$ with $D \cong \mathbf{Z}_3$, and L_2 is diagonally embedded in $K \times D$. As Q_2 centralizes $L_2/O_2(L_2)$, it centralizes the projection D of L_2. But as $O_2(G) = 1$, $F^*(G) = DK$, so Q_2 is faithful on K. Then as G_2 acts on $Q_2 \cap K$, and $N_{Aut(K)}(Q_2 \cap K) \cong S_4$, it follows that $Q_2 \leq K$, $G_2 \cong S_4$, and $G/D \cong PGL_2(q)$. As D_2 is inverted by some $t \in T$, so is its projection D, so t induces outer automorphisms on D and K. Further for z the generator of $Z(T)$, $G_1 \leq C_G(z)$ so $G_1 \cong D_{24}$. Let $q \equiv \epsilon \mod 4$, with $\epsilon = \pm 1$; we conclude that 3 divides $q - \epsilon$, so as $q \equiv \pm 3 \mod 8$, it follows that $q \cong \pm 11 \mod 24$. Thus (2) holds. □

THEOREM F.6.18. *Assume that $G = \hat{G}/O_{3'}(\hat{G})$ for some SQTK-group $\hat{G}$. Then one of the following holds:*

(1) $T \cong \mathbf{Z}_2$, $Q_1 = Q_2 = 1$, and $O_3(G)$ is noncyclic.

(2) $G \cong S_3 \times S_3$ or $E_4/3^{1+2}$.

(3) $L = K \times D$ where $K \cong L_2(q)$, for some prime $q \equiv \pm 11 \mod 24$, $D \cong \mathbf{Z}_3$, $T \cong D_8$, $G_1 \cong D_{24}$, $G_2 \cong S_4$, and $t \in T - K$ induces an outer automorphism on K and D.

(4) $G \cong L_2(p)$, for some prime $p \equiv \pm 11 \mod 24$, $T \cong E_4$, and $G_1 \cong G_2 \cong D_{12}$.

(5) $G \cong PGL_2(q)$, for a prime $q \equiv \pm 11 \mod 24$, $T \cong D_8$, $G_1 \cong D_{24}$, and $G_2 \cong S_4$.

(6) $G \cong \hat{A}_6$ or $L_2(q)$ for some prime power $p^e \equiv \pm 7 \mod 16$ and $e \leq 2$, $T \cong D_8$, and $G_1 \cong G_2 \cong S_4$.

(7) $G \cong A_7$ or $\hat{A}_7$, $T \cong D_8$, $G_1 \cong S_4$, and $G_2 \cong \mathbf{Z}_2/(\mathbf{Z}_3 \times E_4)$.

(8) G is $\hat{S}_6$ or $L_2(p^2)$ extended by a field automorphism, for some prime $p \equiv 3 \mod 8$, $T \cong \mathbf{Z}_2 \times D_8$, and $G_1 \cong G_2 \cong \mathbf{Z}_2 \times S_4$.

(9) $G \cong S_7$ or $\hat{S}_7$, $T \cong \mathbf{Z}_2 \times D_8$, $G_1 \cong \mathbf{Z}_2 \times S_4$, and $G_2 \cong S_3 \times D_8$.

(10) $G \cong (S)L_3^\epsilon(p)$, for a prime $p \equiv \pm 3 \mod 8$, $\epsilon \equiv p \mod 4$, $T \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$, $G_1 \cong S_3/(\mathbf{Z}_4 * Q_8)$, and $G_2 \cong S_3/\mathbf{Z}_4^2$.

(11) G is $(S)L_3^{\epsilon}(p)$ extended by a graph automorphism, for some prime $p \equiv \pm 3 \mod 8$ and $\epsilon \equiv p \mod 4$, $T \cong E_4/\mathbf{Z}_4^2$, $G_1 \cong S_3/Q_8^2$, and $G_2 \cong D_{12}/\mathbf{Z}_4^2$.

(12) $G \cong M_{12}$, $T \cong E_4/\mathbf{Z}_4^2$, $G_1 \cong S_3/Q_8^2$, and $G_2 \cong D_{12}/\mathbf{Z}_4^2$.

(13) $G \cong Aut(M_{12})$, $T \cong D_8/\mathbf{Z}_4^2$, $G_1 \cong D_{12}/Q_8^2$, and $G_2 \cong \mathbf{Z}_2/(\mathbf{Z}_3 \times E_4)/\mathbf{Z}_4^2$.

PROOF. Recall from Remark F.6.15 that $O_{3'}(G) = 1 = O_2(G)$, and if K is a component of G then $K/Z(K)$ appears in Theorem C. In particular if $K/Z(K) \cong L_2(q)$ for q odd, then $q = p^e$ for some prime p and $e \leq 2$, and $K \cong L_2(q)$ or $\hat{A}_6$ from the list of Schur multipliers in I.1.3 since $O_2(G) = 1$. Further if $q \equiv \pm 3 \mod 8$, then q is not a square, so $e = 1$ and $q = p$ is prime. Finally if $T \cong E_4$, then $K \cong L_2(q)$ with $q \equiv \pm 3 \mod 8$ by Theorem C.

Next if $O_3(G)$ is noncyclic, then F.6.16 says that conclusion (1) or (2) holds; so we may assume that $O_3(G)$ is cyclic. Then by F.6.17, either $F^*(G) = L$ is quasisimple or conclusion (3) holds—since the prime power q in that result must now be prime using the previous paragraph.

Thus we may assume that $F^*(G) = L$ is quasisimple. By Remark F.6.15, the possibilities for the Goldschmidt amalgam $\alpha = (G_1, T, G_2)$ are listed in F.6.5.2. As $F^*(G)$ is quasisimple, case (i) where a Sylow 2-subgroup of G is of order 2 is excluded by Cyclic Sylow 2-Subgroups A.1.38.

In case (ii), $T \cong E_4$, so L is $L_2(q)$ for some prime $q \equiv \pm 3 \mod 8$ by the first paragraph. Thus conclusion (4) holds, with the congruence mod 24 following as in the proof of F.6.17.

In cases (iii) and (iv) of F.6.5.2 as well as case (1) of F.1.12, $T \cong D_8$. This time applying Theorem C together with the list of Schur multipliers in I.1.3, G is $L_2(q)$ for some prime power $p^e \equiv \pm 7 \mod 16$ and $e \leq 2$, $\hat{A}_6$, $PGL_2(q)$ for $q \equiv \pm 3 \mod 8$ (with q prime by the first paragraph), A_7, or $\hat{A}_7$. Inspecting the 2-locals of these groups for subgroups isomorphic to G_1 and G_2, we conclude that (6), (5), or (7) holds; in case (5) the congruence mod 24 follows as in earlier cases. In fact there are amalgams of type F.1.12.1 inside A_7 and $\hat{A}_7$, but those amalgams generate a proper subgroup A_6 or $L_3(2)$.

In case (v) of F.6.5.2 as well as case (2) of F.1.12, $T \cong \mathbf{Z}_2 \times D_8$. This time the possibilities for G are $L_2(p^2)$ extended by a field automorphism for some prime $p \equiv 3 \mod 8$, $\hat{S}_6$, S_7, or $\hat{S}_7$. As before we inspect for G_1 and G_2 to conclude that (8) or (9) holds.

In case (8) of F.1.12, $T \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$, so $G \cong (S)L_3^{\epsilon}(p)$ for some prime $p \equiv \pm 3 \mod 8$ and $\epsilon \equiv p \mod 4$, so (10) holds.

Similarly cases (3), (12), and (13) of F.1.12 lead to conclusion (11), (12), and (13). This completes the proof of Theorem F.6.18. □

F.7. Coset geometries and amalgam methodology

In section F.1 we dealt with a rank 2 amalgam $(L_1, L_{1,2}, L_2)$ with $L_i/O_2(L_i)$ a group of Lie type of Lie rank 1 for $i = 1$ and 2. In our work, this represents a special case of the more general setup of the Thompson strategy described in the Introduction to Volume II, where $M \in \mathcal{M}(T)$ and $H \in \mathcal{H}_*(T, M)$, so that (in view of E.2.2) typically only $H/O_2(H)$ can be expected to behave like a rank 1 group. Indeed usually we will have $M = !\mathcal{M}(\langle L, T \rangle)$ for some L, V in the Fundamental Setup (3.2.1).

However, the work of Meierfrankenfeld, Stellmacher, Stroth, and others has shown that more general situations can be profitably analyzed using the the Tutte-Sims graph methods in conjunction with techniques from local group theory. This circle of ideas has come to be known as the "amalgam method". In the next few sections, we put in place some machinery, which draws upon portions of the amalgam method and some of our own variations on that method, to use in the proof of our Main Theorem.

So in this section, we assume:

HYPOTHESIS F.7.1. *G is a group, G_1 and G_2 are finite subgroups of G with $F^*(G_1) = O_2(G_1)$, $G_{1,2} := G_1 \cap G_2$, $T \in Syl_2(G_{1,2})$ is Sylow in G_1 and G_2, $G_0 := \langle G_1, G_2 \rangle$, and $O_2(G_0) = 1$.*

Notice in particular that Hypothesis F.7.1 will be satisfied whenever G is a QTKE-group, $G_1 = \langle L, T \rangle$ for some $L \in \mathcal{L}^*(G,T)$ and $G_2 \in \mathcal{H}_*(T, M)$, where $M = !\mathcal{M}(LT)$. This is the setup appearing in 1.4.1.

In this section we establish some immediate consequences of Hypothesis F.7.1, and of an extension Hypothesis F.7.6; the following two sections will then develop results in the more specialized situation described in section G.2.

We next define notation analogous to that introduced earlier, such as in F.3.1:

DEFINITION F.7.2. Let $\Gamma := \Gamma(G_0; G_1, G_2)$ be the coset geometry determined by G_1 and G_2 in G_0: That is, Γ is the rank-2 geometry with object set $\Gamma := \Gamma_0 \cup \Gamma_1$, where $\Gamma_{i-1} := G_0/G_i$, and G_1x is adjacent to G_2y in Γ if $G_1x \cap G_1y \neq \emptyset$.

For $\gamma \in \Gamma$, let $\Gamma^i(\gamma)$ be the set of vertices at distance i from γ in Γ; we will abbreviate $\Gamma(\gamma) := \Gamma^1(\gamma)$. Let

$$\Gamma^{<i}(\gamma) := \bigcup_{j<i} \Gamma^j(\gamma) \quad \text{and} \quad \Gamma^{\leq i}(\gamma) := \bigcup_{j \leq i} \Gamma^j(\gamma).$$

For $\delta = G_i$ $(i = 1,2)$ regarded as a point of Γ and $g \in G_0$, write $G_{\delta g}$ for the conjugates G_i^g stabilizing the points defined by the cosets $\delta_i g = G_i g$. Similarly for $\alpha_1, \ldots, \alpha_n \in \Gamma$, define the pointwise stabilizer:

$$G_{\alpha_1,\ldots,\alpha_n} := G_{\alpha_1} \cap \cdots \cap G_{\alpha_n}.$$

Observe that G is represented as a group of automorphisms of Γ via right multiplication. We will use the abbreviation $G_\gamma^{(n)}$ for the subgroup $G_{\Gamma^{\leq n}(\gamma)}$ of G_γ fixing all points of Γ at distance at most n from γ in Γ.

LEMMA F.7.3. *Let $\gamma \in \Gamma$. Then*

(1) Γ_0 and Γ_1 are the orbits of G_0 on Γ, with representatives $\gamma_0 := G_1 \in \Gamma_0$ and $\gamma_1 := G_2 \in \Gamma_1$.

(2) G_0 is transitive on unordered edges of Γ, with representative $\{\gamma_0, \gamma_1\}$ and stabilizer $G_{1,2}$.

(3) Γ is connected.

(4) G_γ is the stabilizer in G_0 of γ, and $G_{\gamma_i} = G_{i+1}$, for $i = 0, 1$.

(5) G_γ is transitive on $\Gamma(\gamma)$.

(6) $G_\gamma^{(n)}$ is a normal subgroup of G_γ.

(7) For $i, i+1 \in \{0,1\} \mod 2$, $G_{\gamma_i}^{(1)} = \ker_{G_{1,2}}(G_{i+1})$ is the largest normal subgroup of G_{i+1} contained in $G_{1,2}$.

(8) There exists a positive integer n with $G_\gamma^{(n)} = 1$ for all $\gamma \in \Gamma$.

PROOF. Parts (1) and (2) and the second statement in (4) follow from the definition of Γ and the action of G on Γ. Then (4) and (5) follow from (1) and (2). As $G_0 = \langle G_1, G_2 \rangle$ by hypothesis, Γ is connected, giving (3). As G_γ permutes $\Gamma^j(\gamma)$ for each $j \leq n$, (6) holds. Part (7) follows from (2) and (5).

As G_γ is finite in view of the hypothesis on G_1 and G_2, and since G_0 has just two orbits Γ_0 and Γ_1 on Γ, there exists n with $G_\gamma^{(n)} = G_\gamma^{(m)}$ for all $\gamma \in \Gamma$ and all $m \geq n$. Let $K := G_{\gamma_0}^{(n)}$. Then as Γ is connected by (3),

$$K = G_{\gamma_0}^{(n+1)} \leq G_{\gamma_1}^{(n)} = G_{\gamma_1}^{(n+1)} \leq G_{\gamma_0}^{(n)} = K.$$

Therefore $K = G_{\gamma_0}^{(n)} = G_{\gamma_1}^{(n)}$ is normal in G_1 and G_2 by (6), and hence also in $G_0 = \langle G_1, G_2 \rangle$. Thus as $O_2(G_0) = 1$, $O_2(K) = 1$. Then as $K \trianglelefteq G_1$ and $F^*(G_1) = O_2(G_1)$, $K = 1$, completing the proof of (8). □

LEMMA F.7.4. *For each $\beta \in \Gamma$ and $\gamma \in \Gamma(\beta)$, $O_2(G_\beta) \cap G_\gamma^{(1)} \leq O_2(G_{\beta,\gamma}) \cap G_\gamma^{(1)} \leq O_2(G_\gamma^{(1)})$.*

PROOF. The first containment is immediate. As $G_\gamma^{(1)} \leq G_{\beta,\gamma}$, $G_\gamma^{(1)}$ acts on $X := O_2(G_{\beta,\gamma}) \cap G_\gamma^{(1)}$, so $X \leq O_2(G_\gamma^{(1)})$. □

LEMMA F.7.5. *Let $\alpha \in \Gamma$, n a positive integer, $\gamma \in \Gamma^{<n}(\alpha)$, and $\delta \in \Gamma(\gamma)$. Then*

(1) $O_2(G_\alpha^{(n)}) \leq O_2(G_\gamma^{(1)})$, and

(2) $O_2(G_\alpha^{(n)}) \leq O_2(G_{\gamma,\delta})$.

PROOF. The proof of (1) is by induction on $d := d(\alpha, \gamma)$. If $d = 0$, (1) is trivial, so take $d > 0$ and let $\beta \in \Gamma(\gamma)$ with $d(\alpha, \beta) = d - 1$. Then by induction on d, $X := O_2(G_\alpha^{(n)}) \leq O_2(G_\beta^{(1)}) \leq O_2(G_\beta)$ using F.7.3.6. As $d < n$ by hypothesis, $X \leq G_\gamma^{(1)}$. Thus F.7.4 completes the proof of (1). Further $G_\gamma^{(1)} \leq G_\delta$, so (1) implies (2), again using F.7.3.6. □

In the remainder of this section, we assume:

HYPOTHESIS F.7.6. *(1) Hypothesis F.7.1 holds.*

(2) There is $V \in \mathcal{R}_2(G_1)$ such that $C_T(V) = O_2(C_{G_1}(V))$. Set $\bar{G}_1 := G_1/C_{G_1}(V)$.

This hypothesis will for example be satisfied in the setup of 1.4.1.4—notably in the Fundamental Setup (3.2.1).

LEMMA F.7.7. *(1) $O^{2'}(C_{G_1}(V)) = O_2(G_{\gamma_0}^{(1)}) = O_2(G_1)$.*

(2) For each $\alpha \in \Gamma$ and $\beta \in \Gamma(\alpha)$, $O_2(G_\alpha) = O_2(G_{\alpha,\beta}) \cap G_\alpha^{(1)}$.

PROOF. Let $Q := C_T(V)$. By Hypothesis F.7.6, $Q = O_2(C_{G_1}(V)) \trianglelefteq G_1$, and V is 2-reduced so that $Q = O_2(G_1)$; but also $Q = C_T(V)$ is Sylow in $C_{G_1}(V)$, so that $Q = O^{2'}(C_{G_1}(V))$. Next $Q \leq T \leq G_{1,2}$; so as G_1 is transitive on $\Gamma(\gamma_0)$ by F.7.3.5 and $Q \trianglelefteq G_1$, $Q \leq G_1^{(1)}$. Thus

$$O_2(G_1) = Q \leq O_2(G_1^{(1)}) \leq O_2(G_1),$$

completing the proof of (1). As $G_{\alpha,\beta}$ contains a Sylow 2-subgroup of G_α, $Q_\alpha := O_2(G_\alpha) \leq O_2(G_{\alpha,\beta})$ by A.1.6. Then as $G_\alpha^{(1)} = \ker_{G_{\alpha,\beta}}(G_\alpha)$, $Q_\alpha \leq G_\alpha^{(1)}$, so $Q_\alpha \leq P := O_2(G_{\alpha,\beta}) \cap G_\alpha^{(1)}$. On the other hand, $G_\alpha^{(1)} \leq G_{\alpha,\beta} \leq N_G(P)$, so $P \leq O_2(G_\alpha^{(1)}) \leq Q_\alpha$. Thus (2) holds. □

DEFINITION F.7.8. We introduce the "track" parameter of the amalgam method: By F.7.3.8, there is greatest positive integer $b := b(\Gamma, V)$ such that $V \leq G_{\gamma_0}^{(b)}$.

Notice in particular that we define b with respect to γ_0, rather than minimal with respect to both types; the latter is perhaps more common in the literature.

LEMMA F.7.9. *(1) For each $\gamma \in \Gamma^{<b}(\gamma_0)$ and each $\delta \in \Gamma(\gamma)$, $V \leq O_2(G_\gamma^{(1)})$ and $V \leq O_2(G_{\gamma,\delta})$.*

(2) There exists $\alpha \in \Gamma^b(\gamma_0)$ with $V \not\leq G_\alpha^{(1)}$.

(3) b is the least n such that $V \not\leq G_\gamma^{(1)}$ for some $\gamma \in \Gamma^n(\gamma_0)$.

(4) If $W_0(T, V) \leq O_2(G_2)$, then b is even.

PROOF. By definition of b, $V \leq G_{\gamma_0}^{(b)}$, so as $V \leq O_2(G_{\gamma_0})$, also $V \leq O_2(G_{\gamma_0}^{(b)})$. Hence (1) follows from F.7.5. On the other hand by definition of b, $V \not\leq G_{\gamma_0}^{(b+1)}$, so there is $\beta \in \Gamma^{b+1}(\gamma_0)$ not fixed by V. Then there is $\alpha \in \Gamma(\beta) \cap \Gamma^b(\gamma_0)$, and by construction $V \not\leq G_\alpha^{(1)}$. This establishes (2), and (1) and (2) imply (3). Finally $O_2(G_2) \leq G_{\gamma_1}^{(1)}$ by F.7.7. But for γ at odd distance from γ_0, if $V \leq G_\gamma$ then $V \leq T^g$ for suitable $g \in G$ with $\gamma_1 g = \gamma$, so if $W_0(T, V) \leq O_2(G_2)$ then $V \leq W_0(T^g, V) \leq O_2(G_\gamma) \leq G_\gamma^{(1)}$. Hence b is not odd, establishing (4). □

DEFINITION F.7.10. If $\gamma \in \Gamma_0$, then $\gamma = \gamma_0 g$ for some $g \in G_0$, and we define $V_\gamma := V^g$. Then for $\delta \in \Gamma_1$ we define

$$V_\delta := \langle V_\gamma : \gamma \in \Gamma(\delta) \rangle.$$

Notice using F.7.9.1 that when $b \geq 2$, we have $V_{\gamma_1} \leq O_2(G_1)$, so in particular V_{γ_1} is a 2-group in this case.

LEMMA F.7.11. *Let $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b(\Gamma, V) =: b$ and $V \not\leq G_\gamma^{(1)}$. Then*

(1) If $n \leq b$ then V fixes each n-path $\alpha := \gamma_0, \ldots, \gamma_n$ in Γ pointwise.

(2) If b is even, then $1 \neq [V, V_\gamma] \leq V \cap V_\gamma$, $V_\gamma \leq G_1$, and $1 \neq \bar{V}_\gamma$ is quadratic on V.

(3) $[V, V_\delta] = 1$ for each $\delta \in \Gamma_0$ with $d(\gamma_0, \delta) < b$.

(4) If $b > 2$ then $\Phi(V_{\gamma_1}) = 1$.

(5) If γ_1 and $\gamma_{b-1} \in \Gamma(\gamma)$ are on a geodesic from γ_0 to γ, then $V_{\gamma_1} \leq O_2(G_{\gamma_{b-1},\gamma})$.

(6) If $b \geq 3$ is odd then $V_\gamma \leq O_2(G_{\gamma_1,\gamma_2})$ for each $\gamma_2 \in \Gamma(\gamma_1) \cap \Gamma^{b-2}(\gamma)$. Further $[V_{\gamma_1}, V_\gamma] \leq V_{\gamma_1} \cap V_\gamma$, so the action of V_{γ_1} and V_γ on each other is quadratic.

(7) If V is not an FF-module for $\bar{G}_1$, then b is odd.

(8) If $V \leq O_2(G_2)$ and V is not an FF-module for $\bar{G}_1$, then $\langle V^{G_2} \rangle$ is elementary abelian.

PROOF. Part (1) is immediate from the definition of b. In particular, $V \leq G_\gamma \leq N_{G_0}(V_\gamma)$. Suppose b is even. Then by symmetry, $V_\gamma \leq G_{\gamma_0} = G_1 \leq N_{G_0}(V)$, so that $[V, V_\gamma] \leq V \cap V_\gamma$. Then as $V \cong V_\gamma$ is abelian, V_γ is quadratic on V. If $[V_\gamma, V] = 1$, then by F.7.7.1 applied to γ in the role of "γ_0", $V \leq G_\gamma^{(1)}$, contrary to the choice of γ. So (2) is established.

Suppose $\delta \in \Gamma_0$ with $d := d(\gamma_0, \delta) < b$. Then as $d < b$, $V_\delta \leq O_2(G_{\gamma_0}^{(1)})$ by F.7.9.1 with δ, γ_0 in the roles of "γ_0, γ", so V_δ centralizes V by F.7.7.1. That is, (3) holds. Further if $b > 2$ and $\nu, \mu \in \Gamma(\gamma_1)$ are distinct, then $d(\nu, \mu) = 2 < b$, so V_ν and V_μ commute by (3). Thus (4) holds.

Assume the hypotheses of (5). For $\gamma' \in \Gamma(\gamma_1)$, we have $d(\gamma', \gamma_{b-1}) < b$; thus applying F.7.9.1 to γ', γ_{b-1}, γ in the roles of "γ_0, γ, δ", $V_{\gamma'} \leq O_2(G_{\gamma_{b-1},\gamma})$. Therefore by Definition F.7.10, $V_{\gamma_1} \leq O_2(G_{\gamma_{b-1},\gamma})$, so that (5) is established. A similar argument establishes the first assertion of (6). Then V_{γ_1} and V_γ normalize each other, so $[V_{\gamma_1}, V_\gamma] \leq V_{\gamma_1} \cap V_\gamma$. Further each is elementary by (4), so the actions are quadratic.

Assume V is not an FF-module for $\bar{G}_1$. If b is even, then $1 \neq [V, V_\gamma] \leq V \cap V_\gamma$ by (2), so interchanging V and V_γ if necessary, we may assume that $m(\bar{V}_\gamma) \geq m(V/C_V(V_\gamma))$, contrary to our assumption that V is not an FF-module for $\bar{G}_1$. This establishes (7). Assume in addition that $V \leq O_2(G_2)$. Then $b > 1$ by F.7.7.2, so $b \geq 3$ as b is odd by (7), and then (8) follows from (4). □

REMARK F.7.12. For G_1 and G_2 determined by M, L, V in the Fundamental Setup (3.2.1), and $H \in \mathcal{H}_*(T, M)$ as discussed at the start of the section, we will develop further methods depending on the value of b: When $V \not\leq O_2(H)$, we have $b = 1$ by F.7.9.3; and techniques for this situation are developed in the subsections starting at E.2.2, particularly under the corresponding Hypothesis E.2.8. When $V \leq O_2(H)$ but V_{γ_1} is nonabelian, we obtain $b = 2$ from F.7.11.4; for this case, see the methods developed for "$U^* \neq 1$" and certain L in the latter part of section G.2. Finally when $V \leq O_2(H)$ and $b \geq 3$ so that V_{γ_1} is abelian by F.7.11.4, often we can show that b is odd (for example via F.7.11.6). We develop methods for this situation in F.7.11.3, F.7.13, and under the corresponding Hypotheses F.8.1 and F.9.8 of the following two sections; see F.8.5.1 and F.9.11.1 for the deduction that b is odd.

LEMMA F.7.13. *Let $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$, and $V \not\leq G_\gamma^{(1)}$. Assume γ_1 is on a geodesic from γ_0 to γ and*

(a) $b \geq 3$ is odd.

(b) $N_{G_2}(V)$ is the unique maximal subgroup of G_2 containing T.

(c) $n(G_2) = 1$.

Let $A \leq V_{\gamma_1}$ with $A \not\leq G_\gamma^{(1)}$. Then

(1) There exists $\alpha \in \Gamma(\gamma)$ such that $|A : N_A(V_\alpha)| = 2$.

(2) If G_2 is an SQTK-group, then we can choose α and $h \in G_\gamma$ such that $A^h \leq G_\alpha$ and $\langle A, A^h \rangle$ is not a 2-group.

PROOF. By (a), $b \geq 3$, so by F.7.11.4, $\Phi(V_{\gamma_1}) = 1$. Thus $\Phi(A) = 1$. Also by (a), b is odd, so $\gamma = \gamma_1 g$ for some $g \in G_0$. Let $\Omega := A^{G_\gamma}$, $\beta \in \Gamma(\gamma)$ on the geodesic from γ_0 to γ, $M := N_{G_\gamma}(V_\beta)$, $A \leq T_\gamma \in Syl_2(M)$, and $W_0 := W_0(T_\gamma, \Omega)$. As $A \not\leq G_\gamma^{(1)}$ by hypothesis, $A \not\leq O_2(G_\gamma)$ by F.7.7.2, so $G_\gamma \not\leq N_G(W_0)$. Therefore as $T_\gamma \leq N_{G_\gamma}(W_0)$, (b) implies that $N_{G_\gamma}(W_0) \leq M$. By (c) and Definition E.1.6,

$$G_\gamma = \langle E_1(G_\gamma, T_\gamma, \Omega), N_{G_\gamma}(W_0) \rangle \leq \langle E_1(G_\gamma, T_\gamma, \Omega), M \rangle,$$

so as $M < G_\gamma$, there exists $A_0 \in T_\gamma \cap \Omega$ and $H_0 \in \mathcal{E}_1(G_\gamma, T_\gamma, A_0)$ with $H_0 \not\leq M$. Now $A_0^x = A$ for some $x \in G_\gamma$; set $H := H_0^x$ and $S := T_\gamma^x$. Then $H \in \mathcal{E}_1(G_\gamma, S, A)$ and $H \not\leq N_{G_\gamma}(V_{\beta x})$. Thus as S acts on $V_{\beta x}$ and $H = \langle A^H \rangle S$ by E.1.4, $A \not\leq N_{G_\gamma}(V_\alpha)$ for some $\alpha \in \beta x H$. Then by the definition in E.1.2 of $H \in \mathcal{E}_1(G_\gamma, S, A)$, $B := A \cap O_2(H)$ is of index 2 in A. Further S fixes βx, so $O_2(H)$ fixes α, and thus $B = N_A(V_\alpha)$, completing the proof of (1).

It remains to prove (2), so we must show that we can choose α and $h \in H \leq G_\gamma$ such that $A^h \leq G_\alpha$ and $\langle A, A^h \rangle$ is not a 2-group. Following the convention in Definition E.1.2, set $H^* := H/O_2(H)$ if H is solvable and $H^* := H/O_\infty(H)$ otherwise. As $|A : B| = 2$, A^* is generated by an involution a^*, so we want h with $|a^* a^{h*}|$ not a power of 2. Let $\Delta := a^{*H}$, and for $u \in \Delta$, let $\Delta(u) := \{v \in \Delta : |uv| \text{ is not a power of } 2\}$ and $Fix(u)$ the fixed point set of u on H/H_α. As $1 \neq O^2(K^*) \leq \langle A^{*H} \rangle$ by E.1.4, $\Delta(u) \neq \emptyset$ by the Baer-Suzuki Theorem. If $v \in \Delta(a^*)$ with $Fix(v) \not\subseteq Fix(a^*)$ then (2) holds for any α in $Fix(v) - Fix(a^*)$. Thus as members of Δ are conjugate, we may assume that $Fix(v) = Fix(u)$ for all $u \in \Delta$ and all $v \in \Delta(u)$.

Regard Δ has a graph with u adjacent to the vertices in $\Delta(u)$, and let $\Sigma(u)$ be the connected component of u. Then $Fix(u) = Fix(v)$ for all $v \in \Sigma(u)$. As $H \in \mathcal{E}_1(G_\gamma, S, A)$, E.1.4 says $H^* = K^* S^*$ where $K^* := [O^2(H^*), a^*]$, and replacing H^* by its subgroup $K^* C_{S^*}(a^*)$, and the Sylow group S^* by its subgroup $(S \cap K)^* C_{S^*}(a^*)$, we may assume $\langle \Delta \rangle = K^* \langle a^* \rangle$. Thus as S^* fixes βx but H^* does not, $\Sigma(a^*) \neq \Delta$; so since H^* is transitive on Δ, H^* does not act on $\Sigma(a^*)$. Now $C_{H^*}(a^*)$ and $\Sigma(a^*)$ act on $\Sigma(a^*)$, so $H^* > \langle \Sigma(a^*), C_{H^*}(a*) \rangle =: J^*$. Since $\emptyset \neq \Delta(a^*) \subseteq \Sigma(a^*)$, $J > C_{H^*}(a^*)$, so $C_{H^*}(a^*)$ is not maximal in H^*.

If K^* is a p-group for some odd prime p, then $\Delta = \{a^*\} \cup \Delta(a^*) = \Sigma(a^*)$, contrary to the previous paragraph. Similarly if $K^* = L^* \times L^{*a}$ with L^* simple, then $C_{H^*}(a^*)$ is maximal in H^*, again contrary to the previous paragraph. Therefore by the definition E.1.2 of $H \in \mathcal{E}_1(G_\gamma, S, A)$, K^* is the direct product of simple components $L^* = [L^*, a^*]$, and replacing H^* by $L^* \langle a^* \rangle$, we may assume $H^* = L^* \langle a^* \rangle$.

As H is not solvable, G_2 is not solvable, and so in particular T is not normal in G_2; then $G_2 \in \hat{\mathcal{U}}(T)$ by hypothesis (b). Now for the first time, we use the hypothesis in (2) that G_2 is an SQTK-group: as $n(G_2) = 1$, we conclude from E.2.2 that $(G_2/O_\infty(G_2))^\infty$ is $L_2(p^e)$ or $L_3^\epsilon(p)$ for some odd prime p. Thus the simple section K^* is also such a group. But then as $C_{H^*}(a^*)$ is not maximal in H^*, K^* is $L_2(5)$ or $L_2(7)$, where one checks directly that $H^* = \langle \Sigma(a^*) \rangle$, a contradiction completing the proof. □

Lemma F.7.14. *If $W_0(T, V) \leq O_2(G_i)$ for $i = 0$ or 1, then $b \equiv i + 1 \mod 2$.*

Proof. By F.7.9.2, there is $\gamma \in \Gamma^b(\gamma_0)$ with $V \not\leq G_\gamma^{(1)}$. Now $\gamma = \gamma_i g$ for some $g \in G$ where $i = 0, 1$; thus $i \equiv b \mod 2$. But if $W_0(T, V) \leq O_2(G_i)$, then $W_0(T^g, V) \leq O_2(G_i^g) = O_2(G_\gamma) \leq G_\gamma^{(1)}$ by F.7.7.2, so $V^G \cap G_\gamma \subseteq G_\gamma^{(1)}$. This contradicts $V \not\leq G_\gamma^{(1)}$, and establishes the lemma. □

F.8. Coset geometries with $\mathbf{b} > \mathbf{2}$

In this section we establish results under a variant of Hypothesis F.7.6 with $b > 2$, where we assume V contains a subgroup V_1 normal in G_2 leading to a configuration satisfying Hypothesis G.2.1. In applications in the Fundamental Setup (3.2.1), V often admits a vector space structure ${}_F V$ over $F := \mathbf{F}_{2^n}$ preserved by a group L with $L/C_L(V) \cong SL({}_F V)$, and V_1 is a 1-dimensional F-subspace of ${}_F V$.

As mentioned in Remark F.7.12, this situation will correspond to the case where the amalgam parameter b is odd and at least 3; cf. F.8.5.1.

Thus in this section we assume:

HYPOTHESIS F.8.1. *G is a finite group, $T \in Syl_2(G)$, V is an elementary abelian 2-subgroup of G, $L = O^2(L)$ is a T-invariant subgroup of G with $F^*(L) = O_2(L)$, and $V \in \mathcal{R}_2(LT)$ with $C_T(V) = O_2(LT)$. Further $1 \neq V_1$ is a T-invariant subgroup of V satisfying:*

(a) If $g \in G$ with $1 \neq V_1 \cap V^g$ then $[V, V^g] = 1$.

(b) Set $G_{V_1} := N_G(V_1)$, $\tilde{G}_{V_1} := G_{V_1}/V_1$, and $L_1 := O^2(N_L(V_1))$. Then L_1T is irreducible on $\tilde{V}$.

(c) H is a subgroup of G_{V_1} containing L_1T, such that $Q_H := O_2(H) = F^*(H)$ and $\ker_{C_H(\tilde{V})}(H) \leq Q_H$. Set $U_H := \langle V^H \rangle$ and $H^* := H/Q_H$.

(d) $O_2(\langle LT, H \rangle) = 1$.

LEMMA F.8.2. *Hypothesis G.2.1 is satisfied, with G_{V_1}, U_H in the roles of "G_1, U". Further:*

(1) $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$ and $O_2(H/C_H(\tilde{U}_H)) = 1$. In particular U_H is a 2-group.

(2) $\Phi(U_H) \leq V_1$.

(3) $Q_H = C_H(\tilde{U}_H)$.

PROOF. It is an easy exercise to verify that Hypothesis F.8.1 implies Hypothesis G.2.1. Then (1) and (2) follow from G.2.2. By (1) and F.8.1.c,

$$Q_H \leq C_H(\tilde{U}_H) \leq \ker_{C_H(\tilde{V})}(H) \leq Q_H,$$

so (3) holds. □

LEMMA F.8.3. *Hypothesis F.7.6 is satisfied with LT, H, V in the roles of "G_1, G_2, V". In particular, $L_1T = LT \cap H$ plays the role of "$G_{1,2}$".*

PROOF. Hypothesis F.8.1 includes Hypothesis F.7.1, since $LT \in \mathcal{H}^e$ as $L \in \mathcal{H}^e$. Further $O_2(LT) = C_T(V)$ and $V \in \mathcal{R}_2(LT)$ by F.8.1, so that $O_2(LT) = O_2(C_{LT}(V))$, and hence part (2) of Hypothesis F.7.6 holds. As $L_1 = O^2(N_L(V_1))$, $L_1T = N_{LT}(V_1)$, so that the final statement of F.8.3 holds. □

DEFINITION F.8.4. By F.8.3 we may adopt the notation of section F.7. In particular define Γ as in Definition F.7.2 and $b := b(\Gamma, V)$ as in Definition F.7.8. Observe $U_H = \langle V^H \rangle$ plays the role of "V_{γ_1}" in Definition F.7.10. Choose $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$ and $V \not\leq G_\gamma^{(1)}$. Without loss, γ_1 is on a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_{b-1}, \gamma_b := \gamma$$

from γ_0 to γ.

LEMMA F.8.5. *(1) $b \geq 3$ is odd.*

(2) U_H is elementary abelian.

PROOF. By F.8.2.1 and F.7.7.2, $V \leq O_2(H) \leq O_2(G_2^{(1)})$, so that $b > 1$. Hence (1) will follow once we prove b is odd. Then once (1) holds, $U_H = V_{\gamma_1}$ is elementary abelian by F.7.11.4, so that (2) will also follow.

So assume b is even. Then there is $g \in G_0 := \langle LT, H \rangle$ with $\gamma_0 g = \gamma$; indeed by F.7.3.2 we may assume g takes the edge γ_0, γ_1 to the edge γ, γ_{b-1}. Now by F.7.9.1 and F.8.3,

$$V \leq O_2(G_{\gamma_{b-1},\gamma}) = O_2((L_1T)^g),$$

so $[V, V^g] \leq [O_2((L_1T)^g), V^g] \leq V_1^g$, since $(L_1T)^g$ is irreducible on V^g/V_1^g by F.8.1.b. Now by F.7.11.2, $1 \neq [V, V^g] \leq V \cap V^g$, so in particular $1 \neq V \cap V_1^g$.

Hence by Hypothesis F.8.1.a applied to V_1^g, V^g in the roles of "V_1, V", $[V, V^g] = 1$, a contradiction completing the proof. $\square$

As b is odd by F.8.5.1, by F.7.3.2 we may choose $g_b \in G_0$ to take the edge γ_0, γ_1 to γ_{b-1}, γ. Let $L_\gamma := L_1^{g_b}$ and $U_\gamma := U_H^{g_b}$. Notice that $L_\gamma = O^2(G_{\gamma_{b-1},\gamma})$ using F.8.3.

We will exploit the symmetry between the edges γ_0, γ_1 and γ_{b-1}, γ, and the groups U_H and U_γ defined at the vertices γ_1 and γ. Notice however we do not have complete symmetry, since by choice of γ, $V \not\leq G_\gamma^{(1)}$ so that $U_H \not\leq G_\gamma^{(1)}$; but we do not know that $U_\gamma \not\leq G_{\gamma_1}^{(1)}$.

Definition F.8.6. Set $A_1 := V_1^{g_b}$,$D_H := C_{U_H}(U_\gamma/A_1)$, and $D_\gamma := C_{U_\gamma}(\tilde{U}_H)$.

Lemma F.8.2 gives information about the conjugate U_γ of U_H, with A_1 playing the role of "V_1". We see next that U_H and U_γ normalize each other, and later we analyze those actions.

Lemma F.8.7. *(1)* $G_{\gamma_0,\gamma_1} = L_1 T$.
(2) $U_H \leq O_2(G_{\gamma_{b-1},\gamma})$ *and* $U_\gamma \leq O_2(G_{\gamma_1,\gamma_2})$.
(3) $[U_H, U_\gamma]$ *leq*$U_H \cap U_\gamma$.
(4) U_γ *acts on* V_{γ_1}.
(5) $[U_\gamma, V] \not\leq A_1$*, so* $V_1 \cap A_1 = 1$.
(6) $[U_\gamma, D_H] \leq A_1$ *and* $[U_H, D_\gamma] \leq V_1$.
(7) $[D_H, D_\gamma] = 1$.

Proof. Part (1) follows from F.8.3, and parts (2) and (3) follow from F.8.5.1 and parts (5) and (6) of F.7.11. Part (4) follows from (2).

Suppose $[U_\gamma/A_1, V] = 1$. Then by F.8.2.3 with G_γ, U_γ in the roles of "H, U_H", $V \leq O_2(G_\gamma)$, so $V \leq G_\gamma^{(1)}$ by F.7.7.2. This contradicts the choice of γ, proving the first part of (5). Suppose $V_1 \cap A_1 \neq 1$. As $H \leq G_{V_1}$, $A_1 \leq V_\beta$ for all $\beta \in \Gamma(\gamma)$, so $[V, V_\beta] = 1$ by Hypothesis F.8.1.a. Then V centralizes U_γ, contrary to the previous reduction. This establishes the remaining part of (5).

Part (6) follows from the definitions in F.8.6; then $[D_H, D_\gamma] \leq A_1 \cap V_1 = 1$ by (5) and (6), so that (7) holds. $\square$

F.9. Coset geometries with $\mathbf{b > 2}$ and $\mathbf{m(V_1) = 1}$

In the most difficult cases in the Fundamental Setup (3.2.1), $End_{\mathbf{F}_2 L}(V) = \mathbf{F}_2$, and we need to control the centralizer of a 1-dimensional $\mathbf{F}_2$-subspace V_1 of V. Further $N_{LT}(V_1)$ is not always irreducible on V/V_1. Thus in this section we assume a slight modification of the hypotheses of section F.8: on the one hand we add the restriction $m(V_1) = 1$, while on the other we add some flexibility by letting $L_1 T$ be irreducible on V_+/V_1 for some V_+ with $V_1 < V_+ \leq V$. Also we replace F.8.1.a by the weaker condition F.9.1.e. Many of the results and proofs will be similar to those in section F.8.

Specifically in this section we assume:

Hypothesis F.9.1. *G is a finite group, $T \in Syl_2(G)$, V is an elementary abelian 2-subgroup of G, $L = O^2(L)$ is a T-invariant subgroup of G with $F^*(L) = O_2(L)$, $V \in \mathcal{R}_2(LT)$ with $C_T(V) = O_2(LT)$, and $V_1 = \langle z \rangle$ is a T-invariant subgroup of V of order 2. Set $G_z := C_G(z)$, $\tilde{G}_z := G_z/V_1$, $L_1 := O^2(C_L(z))$, $U_H := \langle V_+^H \rangle$, $V_H := \langle V^H \rangle$, and $H^* := H/C_H(\tilde{U}_H)$. Assume $V_1 < V_+ \leq V$ and:*

(a) H is a subgroup of G_z containing L_1T such that $Q_H := O_2(H) = F^*(H)$ and $\ker_{C_H(\tilde{V}_+)}(H) \leq Q_H$.

(b) L_1T is irreducible on $\tilde{V}_+$.

(c) $N_H(V_+) \leq N_H(V)$.

(d) $O_2(\langle LT, H\rangle) = 1$.

(e) If $g \in H$ with $[V, V^g] \leq V \cap V^g$ and $[\tilde{V}_+, V^g] = 1 = [\tilde{V}_+^g, V]$, then $[V_+, V^g] = 1 = [V_+^g, V]$.

Lemma F.9.2. *Hypothesis G.2.1 is satisfied with G_z, V_+, U_H in the roles of "G_1, V, U". Further*

(1) $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$ and $O_2(H/C_H(\tilde{U}_H)) = 1$.

(2) $\Phi(U_H) \leq V_1$.

(3) $Q_H = C_H(\tilde{U}_H) \leq C_H(\tilde{V}_+) \leq N_H(V)$.

(4) Hypothesis F.7.6 of section F.7 is satisfied with LT, H, V in the roles of "G_1 , G_2, V". In particular, $L_1T = LT \cap H$ plays the role of "$G_{1,2}$".

Proof. Again it is an easy exercise to verify that Hypothesis F.9.1 implies Hypothesis G.2.1. Then, just as in the proof of F.8.2, (1) and (2) follow from G.2.2. Similarly as in the proof of F.8.3, (4) follows from Hypothesis F.9.1. By F.9.1.c, $C_H(\tilde{V}_+) \leq N_H(V)$, and by (1) and F.9.1.a,

$$Q_H \leq C_H(\tilde{U}_H) \leq \ker_{C_H(\tilde{V}_+)}(H) \leq Q_H,$$

so (3) holds. □

Lemma F.9.3. *The following are equivalent:*

(1) $V \leq Q_H$.

(2) $V \leq C_H(U_H)$.

(3) $V \leq C_H(\tilde{U}_H)$.

Proof. It is immediate that (2) implies (3). By F.9.2.3, $Q_H = C_H(\tilde{U}_H))$, so (1) and (3) are equivalent. If $V \leq Q_H$ then for each $h \in H$, $[\tilde{V}_+^h, V] \leq [\tilde{U}_H, V] = 1$ by F.9.2.1. Further $V^h \leq Q_H$, so similarly $[\tilde{V}_+, V^h] = 1$. Also $Q_H \leq T \leq N_G(V)$, so $[V, V^h] \leq V \cap V^h$. Hence by F.9.1.e, $[V_+^h, V] = 1$, so V centralizes $\langle V_+^H\rangle = U_H$. That is (1) implies (2). Hence (1)–(3) are equivalent. □

Lemma F.9.4. *(1) If $V \leq Q_H$, then U_H is elementary abelian.*

(2) If $[\tilde{V}, C_H(V_+) \cap N_H(V)] = 1$ and U_H is abelian, then $V \leq Q_H$.

(3) If $C_H(V_+) \cap N_H(V) = C_H(V)$, then the following are equivalent:

(i) $V \leq Q_H$.

(ii) U_H is elementary abelian.

(iii) V_H is elementary abelian.

Proof. If $V \leq Q_H$, then $V_+ \leq V \leq C_H(U_H)$ by F.9.3, so $U_H = \langle V_+^H\rangle$ is abelian. Hence (1) holds. Assume the hypotheses of (2). Then $U_H \leq C_H(V_+) \cap N_H(V) \leq C_H(\tilde{V})$, so $V \leq C_H(\tilde{U}_H)$, and hence $V \leq Q_H$ by F.9.3. Thus (2) is established.

Assume the hypotheses of (3). Trivially (iii) implies (i), and (i) implies (ii) by (1). By F.9.1.c, $N_H(V_+) = N_H(V_+) \cap N_H(V)$, so that $C_H(V_+) = C_H(V)$ by the hypotheses of (3). Therefore if (ii) holds then for each $h \in H$, $V_+^h \leq C_H(V_+) = C_H(V)$, so $V \leq C_H(V_+^h) = C_H(V^h)$; that is, (ii) implies (iii). □

In two difficult cases in the Fundamental Setup, $V/C_V(L)$ is the natural module for $L/C_L(V) \cong Sp_4(2)'$ or $\Omega_4^-(2)$. In these cases we take $V_+ := V_1^\perp$, so that $|V : V_+| = 2$, and we appeal to the next lemma:

LEMMA F.9.5. *Assume $|V : V_+| = 2$ and $V^* \neq 1$. Then*
(1) V^* is of order 2.
(2) $[\tilde{Q}_H, V^*] = [\tilde{U}_H, V^*] = \tilde{V}_+$.
(3) $C_{H^*}(V^*) = N_H(V_+)^* = N_H(V)^*$.
(4) V is weakly closed in $C_H(\tilde{U}_H)V$ with respect to H.
(5) Assume $\langle V, V^g\rangle$ is a 2-group for each $g \in H$ with $V_1 < V \cap V^g$. Then for each $h \in H$ with $I^* := \langle V^*, V^{*h}\rangle$ not a 2-group, I^* acts faithfully on $U_I := V_+V_+^h$, $\tilde{U}_H = \tilde{U}_I \oplus C_{\tilde{U}_H}(I^*)$, and $\tilde{U}_I = \tilde{V}_+ \oplus \tilde{V}_+^h$ with $\tilde{V}_+ = [\tilde{U}_I, V^*] = C_{\tilde{U}_I}(V^*)$. Further $U_I = I \cap Q_H$.
(6) Assume the hypotheses of (5), and in addition assume $[V, C_H(V_+)] \leq V_1$. Then
(i) If $m(V) = 4$, then $U_I = V_+V_+^h \cong Q_8^2$, and $I^* \cong S_3$.
(ii) If $m(V) = 6$ and $C_H(V_+) = C_H(V)$, then $U_I = V_+V_+^h \cong Q_8^4$, and $I^* \cong D_6$, D_{10}, or D_{12}.

PROOF. As $|V : V_+| = 2$ and $V^* \neq 1$ by hypothesis, while $V_+ \leq V \cap Q_H = C_V(\tilde{U}_H)$ by F.9.2.3, $V_+ = V \cap Q_H$ and (1) holds. Then $[Q_H, V] \leq V \cap Q_H = V_+$. Now $L_1^*T^*$ centralizes V^* by (1), and so normalizes $[\tilde{U}_H, V^*] \neq 1$. Therefore as L_1T is irreducible on $\tilde{V}_+$ by F.9.1.b, (2) holds. Then by (2) and F.9.1.c,

$$C_{H^*}(V^*) \leq N_H(\tilde{V}_+)^* \leq N_H(V)^* \leq C_{H^*}(V^*),$$

so (3) holds.

If $V^h \leq C_H(\tilde{U}_H)V$ for $h \in H$, then $V^* = V^{*h}$, and so by (2),

$$\tilde{V}_+^h = [\tilde{U}_H, V^{*h}] = [\tilde{U}_H, V^*] = \tilde{V}_+,$$

so $V_+ = V_+^h$. Hence $V = V^h$ by F.9.1.c; that is, (4) holds.

Assume the hypotheses of (5), and suppose $h \in H$ such that $I^* = \langle V^*, V^{*h}\rangle$ is not a 2-group. Thus by the hypotheses of (5), $V_1 = V \cap V^h$. By (2), I acts on $V_+V_+^h = U_I$. Next let D be the preimage in V_+ of $C_{\tilde{V}_+}(V^{*h})$. Then $\tilde{D} \leq Z(\tilde{I})$ and $D \leq V_+$, so $D \leq V \cap V^g$ for each $g \in I$. Hence if $V_1 < D$, then by the hypotheses of (5), each $\langle V, V^g\rangle$ is a 2-group; it then follows from the Baer-Suzuki Theorem that $V \leq O_2(I)$, contradicting $I = \langle V, V^h\rangle$ not a 2-group. Therefore $D = V_1$ and so $\tilde{D} = 1$. Thus $C_{\tilde{U}_I}(V^*) = \tilde{V}_+$ and $\tilde{U}_I = \tilde{V}_+ \oplus \tilde{V}_+^h$. Then by (2), $m(\tilde{U}_H/C_{\tilde{U}_H}(V^*)) = m(\tilde{U}_I/C_{\tilde{U}_I}(V^*))$, and similarly $m(\tilde{U}_H/C_{\tilde{U}_H}(V^{*h})) = m(\tilde{U}_I/C_{\tilde{U}_I}(V^{*h}))$, so $\tilde{U}_H = \tilde{U}_I \oplus C_{\tilde{U}_H}(I^*)$. In particular, I^* is faithful on $\tilde{U}_I$. Finally I/U_I is dihedral and generated by VU_I and V^hU_I by (1). Thus if $U_I < I \cap Q_H$, then $1 \neq X/U_I := Z(I/U_I) \leq (I \cap Q_H)/U_I$, and $X = VV^jU_I$, where either $j \in I$ or $j = hi$ for some $i \in I$. In either case $V^j \neq V$ but $V^* = V^{j*}$, contradicting (4). Therefore the proof of (5) is complete.

Assume the hypotheses of (6). Then by hypothesis, $[V, C_H(V_+)] \leq V_1$, so as $V^* \neq 1$, U_H is nonabelian by F.9.4.2. Let $V^* = \langle v^*\rangle$ and $V^{*h} = \langle i^*\rangle$. As V^* is of order 2, I^* is dihedral. Let $P := U_I$; by (5), I^* is faithful on $\tilde{P}$.

We first prove (6.ii), so assume that $m(V) = 6$ and $C_H(V_+) = C_H(V)$. Then $C_{V_+^h}(V_+) = C_{V_+^h}(V) = V_1$, since $C_{\tilde{U}_I}(V^*) = \tilde{V}_+$ by (5). Thus as $P = V_+V_+^h$, P is extraspecial of order 2^9. Then as $V_+ \cong E_{32}$, $P \cong Q_8^4$, and as $V_+ = [P, V]$, v^* is of Suzuki type a_4 (cf. definition E.2.6) on the orthogonal space $\tilde{P}$. Similarly i^* is of type a_4 on $\tilde{P}$. Let x^* be of odd order $m > 1$ in I^*. Applying (5) to $\langle V^*, V^{*x}\rangle$ in the role of "I^*", we conclude $[\tilde{U}_H, x] = \tilde{V}_+ \oplus \tilde{V}_+^x$, so $[U_H, x] = P$ and hence $C_{\tilde{P}}(x^*) = 0$. Therefore the irreducibles for $\langle x^*\rangle$ on $\tilde{P}$ are of dimension $d(m)$, where $d(m)$ is the order of 2 modulo m. Thus $d(m)$ divides 8, so either m is 3, 5, or 15, or 17 divides m. As 17 does not divide the order of $O_8^+(2) \cong Out(\tilde{P}) := O_P$, the last case is out.

Suppose $m = 5$. Now $O^2(O_P) \cong \Omega_8^+(2)$ has 3 classes of elements of order 5, permuted transitively by the triality outer automorphism. Let $y \in O_P$ be of order 5 with $W_1 := C_{\tilde{P}}(y) \neq 0$. Then W_1 and $W_2 := W_1^\perp$ are 4-dimensional of sign -1, so $O := N_{O_P}(\langle y\rangle) = O_1 \times O_2$, where $O_1 \cong Sz(2)$ centralizes W_1, and $O_2 \cong S_5$ centralizes W_2. The involutions in O inverting y are the involutions in O_1 of type c_2 and the diagonals are of type c_4 and b_3. But x^* is conjugate to y under triality, and only c_2 and a_4 involutions are conjugate under triality to a_4 involutions. Thus as v^* is of type a_4, v^* is not diagonal, so that $m \neq 15$; similarly $|I^*| \neq 20$ as a pair of elements of type c_2 in O do not generate a dihedral subgroup of order 20. We conclude that if 5 divides $|I^*|$, then $I^* \cong D_{10}$, in which case (6.ii) holds.

Thus we may assume $|I^*| = 2^e \cdot 3$ and $m = 3$. As x^* is fixed-point-free on $\tilde{P}$, $O := N_{O_P}(\langle x^*\rangle)$ is $GU_4(2)$ extended by a graph-field automorphism, and there are two classes of involutions under O inverting x^*: involutions j^* of type a_4 with $C_O(j^*) \cong \mathbf{Z}_2 \times S_6$, and involutions of type c_4. Thus v^* and i^* are in the first class, which projects on a class of 3-transpositions in $O/\langle x^*\rangle$. Therefore $|v^*i^*| = 3$ or 6, so $I^* \cong D_6$ or D_{12}, and again (6.ii) holds.

Finally we prove (6.i), so we assume that $m(V) = 4$. If P is extraspecial then as $V_+ \leq P$, $P \cong Q_8^2$; then from the structure of $Out(P) \cong O_4^+(2)$, $I^* \cong S_3$ or D_{12}. But in the latter case, one of the two classes of involutions not in $Z(I^*)$ is of type c_2, whereas v^* and i^* are of type a_2 on $\tilde{P}$. So suppose that P is not extraspecial. Then $P \cong D_8 \times E_4$, so $|\mathcal{A}(P)| = 2$. Therefore $O^2(I)$ acts on both members of $\mathcal{A}(P) = \{V_+, V_+^h\}$, contrary to the faithful action of $O^2(I^*)$ on $\tilde{P}$ in (5). Thus the proof of the lemma is complete. □

LEMMA F.9.6. *Let $g \in L$ with $V_1 \neq V_1^g$, and define $D_H := U_H \cap Q_H^g$, $D_{H^g} := U_H^g \cap Q_H$, $E_H := V_H \cap Q_H^g$, and $E_{H^g} := V_H^g \cap Q_H$. Then*

(1) For each $y \in LT$ with $V_1^g = V_1^y$, we have $X^g = X^y$ for each $X \in \{H, Q_H, U_H, V_H\}$.

(2) If there is $g_0 \in L$ with $V_1^{g_0} = V_1^g$ and $g_0^2 \in L_1T$, then $(D_H)^g = D_{H^g}$ and $(E_H)^g = E_{H^g}$.

(3) $[E_H, U_H^g] \leq V_1^g$ and $[D_H, D_{H^g}] = 1$.

PROOF. Recall $N_{LT}(V_1) = L_1T \leq H$. Thus if $y \in L$ with $V_1^g = V_1^y$, then $Hg = Hy$; so if X is an H-invariant subgroup of G, then $X^g = X^{Hg} = X^{Hy} = X^y$. Hence (1) holds. In proving (2), appealing to (1) we may assume $g_0 = g$, so that $g \in H$. Then $g^2 \in H$, so g^2 acts on each H-invariant subgroup of G, and hence

$$D_H^g = (U_H \cap Q_H^g)^g = U_H^g \cap Q_H = D_{H^g}.$$

Similarly $E_H^g = E_{H^g}$, so (2) holds. Next

$$[E_H, U_H^g] \leq [Q_H^g, U_H^g] \leq V_1^g$$

by F.9.2. Similarly as $D_H \leq E_H$ and $D_H^g \leq U_H^g$, $[D_H, D_{H^g}] \leq V_1 \cap V_1^g = 1$, as $V_1 \neq V_1^g$ and $m(V_1) = 1$, so (3) holds. □

LEMMA F.9.7. *Assume U_H is elementary abelian, and set $Q_C := C_{Q_H}(U_H)$ and $Z_H := C_{U_H}(Q_H)$. Then there is an H-isomorphism φ from Q_H/Q_C to the dual space of U_H/Z_H, defined by $\varphi(xQ_C) : uZ_H \mapsto [x, u]$ for all $u \in U_H$.*

PROOF. As U_H is elementary abelian, we may regard it as a vector space over $F := \mathbf{F}_2$. Let D denote the dual space $Hom_F(U_H/Z_H, F)$. By F.9.2.1,

$$[U_H, Q_H] \leq V_1 \leq Z(H), \qquad (*)$$

so if we identify V_1 with F, for each $xQ_C \in Q_H/Q_C$, the mapping $\varphi_{xQ_C} : uZ_H \mapsto [x, u] \in F$ on U_H/Z_H is well defined and F-linear by (*) and the standard commutator relation 8.5.4 in [**Asc86a**]. Next define $\varphi : Q_H/Q_C \to D$ by $\varphi : xQ_C \mapsto \varphi_{xQ_C}$. Using (*) and the commutator relations, we conclude that φ is an H-equivariant homomorphism. By construction φ is injective, and $C_{U_H/Z_H}(\varphi(Q_H/Q_C)) = 0$, so φ is a surjection. □

In the remainder of this section we essentially add Hypothesis F.8.1.a, as well as a hypothesis we used in F.9.5.6, and in a stronger form in F.9.4.3; namely we assume:

HYPOTHESIS F.9.8. *Hypothesis F.9.1 holds and*
(f) If $g \in G$ with $V_1 \leq V^g$ then $[V, V^g] = 1$.
(g) One of the following holds:
(i) $[V, C_H(V_+)] \leq V_1$.
(ii) If $x \in C_H(V_+)$ and $[V, x] \neq 1$, then $V^G \cap N_G([V, x]) \subseteq C_G(V)$.

REMARK F.9.9. Observe that the following two hypotheses are equivalent:
(1) Hypothesis F.8.1 with $|V_1| = 2$.
(2) Hypothesis F.9.8 with $V_+ = V$.

PROOF. When $|V_1| = 2$, F.8.1.2 and F.9.8.f are equivalent. Similarly when $V_+ = V$, F.8.1.b is equivalent to F.9.1.b, and F.8.1.c is equivalent to F.9.1.a. Of course F.8.1.d and F.9.1.d are the same. Thus when $V_+ = V$, Hypothesis F.9.8 implies Hypothesis F.8.1 with $|V_1| = 2$. Also when $V_+ = V$, F.9.1.e is a consequence of F.9.8, while F.9.8.g.1 and F.9.1.c are trivial. Thus if $|V_1| = 2$, then Hypothesis F.8.1 implies Hypothesis F.9.8 with $V_+ = V$. □

DEFINITION F.9.10. By F.9.2.4, we may adopt the notation of section F.7. In particular define Γ as in Definition F.7.2, and $b := b(\Gamma, V)$ as in Definition F.7.8. Observe $V_H = \langle V_+^H \rangle$ plays the role of "V_{γ_1}" in Definition F.7.10. Choose $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$ and $V \not\leq G_\gamma^{(1)}$. Without loss, γ_1 is on a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b = \gamma$$

from γ_0 to γ.

We next establish the analogues of F.8.5 and F.8.7; the proofs will of course be similar and in many cases can be repeated essentially verbatim.

LEMMA F.9.11. *(1) $b \geq 3$ is odd.*
(2) V_H is abelian.
(3) U_H is abelian.

PROOF. Part (2) follows from Hypothesis F.9.8.f, as $H \leq C_G(V_1)$. Then (3) follows from (2) as $U_H \leq V_H$. By (2), $V \leq Q_H$ so $V \leq O_2(G_1^{(1)})$ by F.7.7.2. Thus $b > 1$. Hence (1) will follow once we prove b is odd.

So we assume that b is even, and derive a contradiction. Then there is $g \in G_0$ with $\gamma_0 g = \gamma$. Indeed by F.7.3.2, we may assume g takes the edge γ_0, γ_1 to the edge γ, γ_{b-1}. By F.7.9.1 and F.9.2.4,

$$V \leq O_2(G_{\gamma_{b-1},\gamma}) = O_2((L_1T)^g).$$

Therefore

$$[V, V_+^g] \leq [O_2((L_1T)^g), V_+^g] \leq V_1^g, \tag{$*$}$$

since $(L_1T)^g$ is irreducible on V_+^g/V_1^g by F.9.1.b. Further by F.7.11.2,

$$1 \neq [V, V^g] \leq V \cap V^g. \tag{$**$}$$

We claim that $V_1^g \leq [V, V^g] \leq V$: For by (*), $[V, V_+^g] \leq V_1^g$. Thus if $V_1^g = [V, V_+^g]$, then $V_1^g \leq [V, V^g] \leq V$ by (**), and the claim holds. Otherwise $[V, V_+^g] = 1$, so that $V \leq C_{H^g}(V_+^g)$. In case (i) of F.9.8.g, $[V, V^g] \leq [C_{H^g}(V_+^g), V^g] \leq V_1^g$ by hypothesis, so $[V, V^g] = V_1^g$ by (**), and again the claim holds. So suppose case (ii) of F.9.8.g holds. By (**) there is $x \in V$ with $1 \neq [V^g, x] \leq V$, so $V \leq C_G([V^g, x]) \leq C_G(V^g)$ as (ii) holds, contrary to (**). This contradiction completes the proof of the claim.

Now $V_1^g \leq V$ by the claim, so we may apply Hypothesis F.9.8.f with the roles of V and V^g reversed, to conclude that $[V, V^g] = 1$ by that hypothesis, contrary to (**). This contradiction shows b is odd, completing the proof of F.9.11. □

As b is odd by F.9.11.1, by F.7.3.2 we may choose $g_b \in G_0$ to take the edge γ_0, γ_1 to γ_{b-1}, γ. Let $L_\gamma := L_1^{g_b}$ and $U_\gamma := U_H^{g_b}$. Notice that $L_\gamma = O^2(G_{\gamma_{b-1},\gamma})$ using F.9.2.4.

As in Definition F.8.6, we will exploit the symmetry between the edges γ_0, γ_1 and γ_{b-1}, γ, and the subgroups U_H and U_γ defined at the vertices γ_1 and γ.

DEFINITION F.9.12. Set $A_1 := V_1^{g_b}$, $D_H := C_{U_H}(U_\gamma/A_1)$, $D_\gamma := C_{U_\gamma}(\tilde{U}_H)$, $E_H := C_{V_H}(U_\gamma/A_1)$, and $E_\gamma := C_{V_\gamma}(\tilde{U}_H)$.

LEMMA F.9.13. *(1)* $G_{\gamma_0,\gamma_1} = L_1T$.
(2) $V_H \leq O_2(G_{\gamma_{b-1},\gamma})$ *and* $V_\gamma \leq O_2(G_{\gamma_1,\gamma_2})$.
(3) $[X_H, Y_\gamma] \leq X_H \cap Y_\gamma$ *for each choice of* $X, Y \in \{U, V\}$.
(4) V_γ *acts on* V_H *and* U_H.
(5) $[U_\gamma, V] \not\leq A_1$ *but* $V_1 \cap A_1 = 1$.
(6) $[U_\gamma, E_H] \leq A_1$ *and* $[U_H, E_\gamma] \leq V_1$.
(7) $[D_H, D_\gamma] = 1$.

PROOF. Part (1) follows by construction, as we saw in F.9.2.4, and (2) follows from F.7.11.5. By (2), X_H and Y_γ normalize each other, so (3) and (4) hold.

The remainder of the proof is exactly like that of the corresponding statements in F.8.7, whose proof can be repeated essentally verbatim. □

LEMMA F.9.14. *Set* $U_L := \langle U_H^L \rangle$. *Then:*
(1) U_L *is abelian iff* $b > 3$.
(2) If $b = 3$ *then* $A_1 \leq V^h$ *for some* $h \in H$.
(3) Assume that L *is transitive on* $V^\#$ *and* $H = G_z$. *If* $A_1^h \leq V$ *for some* $h \in H$, *then* $b = 3$ *and* $U_{\gamma h} \in U_H^L$.

PROOF. For $g \in L - H$, $U_H^g = U_\beta$ for some $\beta \in \Gamma(\gamma_0)$ at distance 2 from γ_1. Thus if $b > 3$, then $[U_H, U_H^g] = 1$ by F.7.11.3, so U_L is abelian. So suppose instead that $b = 3$. By edge-transitivity in F.7.3.2, we can pick $g \in L$ and $\delta \in \Gamma(\gamma_1)$ so that $d(\delta, \gamma_1 g) = 3$ and $V_\delta \not\leq G_{\gamma_1^g}^{(1)}$; hence $V_\delta \not\leq O_2(G_{\gamma_1}^g)$ by F.7.7.2, so $[U_H^g/Z^g, V_\delta] \neq 1$ in view of F.9.2.3. Then as $V_\delta \leq U_H$, $[U_H^g, U_H] \neq 1$, so U_L is nonabelian. Thus (1) holds.

Next assume that $b = 3$. Then for some $h \in H$, $\gamma_0 h = \gamma_2 \in \Gamma(\gamma)$. Thus setting $y := g_b h^{-1}$, $\gamma_1 y = \gamma h^{-1}$ is adjacent to $\gamma_2 h^{-1} = \gamma_0$. Therefore $\gamma_1 y = \gamma_1 x$ for some $x \in L$, so $xy^{-1} \in H$ and hence $V_1 = V_1^{xy^{-1}} \leq V^{xy^{-1}} = V^{y^{-1}} = V^{hg_b^{-1}}$, so $A_1 = V_1^{g_b} \leq V^h$. Thus (2) holds.

Finally assume that $H = G_z$ and $A_1^h \leq V$ for some $h \in H$. Then $V_1 A_1 \leq V^{h^{-1}}$, so since L is transitive on $V^\#$, there is $x \in L^{h^{-1}}$ with $V_1^x = A_1 = V_1^{g_b}$. Thus $g_b x^{-1} \in C_G(V_1) = H$, so as the subgroup H is the stabilizer of the vertex γ_1 of Γ, $\gamma_1 x = \gamma_1 g_b = \gamma$. Now $x^h \in L \leq G_{\gamma_0}$, so that $\gamma h = \gamma_1 x h = \gamma_1 x^h \in \Gamma(\gamma_0)$; hence applying h^{-1}, $d(\gamma, \gamma_1) = 2$, so that $b = 3$. It also follows that $U_{\gamma h} = U_H^{x^h} \in U_H^L$, completing the proof of (3). □

LEMMA F.9.15. *$V_H \cap G_\gamma^{(1)} = E_H$ and $V_\gamma \cap G_{\gamma_1}^{(1)} = E_\gamma$.*

PROOF. By F.9.13.2, $V_H \cap G_\gamma^{(1)} = V_H \cap O_2(G_{\gamma_{b-1},\gamma}) \cap G_\gamma^{(1)}$, so $V_H \cap G_\gamma^{(1)} = V_H \cap O_2(G_\gamma)$ by F.7.7.2. But as $Q_H = C_H(\tilde{U}_H)$ by F.9.2.3, $V_H \cap O_2(G_\gamma) = E_H$, so the first statement in the lemma holds. Similarly the second holds. □

LEMMA F.9.16. *(1) If $U_\gamma = D_\gamma$, then either*

(i) $D_H < U_H$ and U_H induces a nontrivial group of transvections with center V_1 on U_γ, or

(ii) $D_H = U_H$, so $[U_H, U_\gamma] = 1$, and V induces a nontrivial group of transvections with center V_1 on U_γ.

(2) If $0 < m(U_\gamma/D_\gamma) \geq m(U_H/D_H)$, then $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$; and in case

$$2\ m(U_\gamma^*) = m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*)),$$

then also $m(U_\gamma/D_\gamma) = m(U_H/D_H)$ and U_γ^* acts faithfully on $\tilde{D}_H$ as a group of transvections with center $\tilde{A}_1$.

(3) $q(H^*, \tilde{U}_H) \leq 2$. Indeed there is $B \leq O_2(G_{\gamma_0,\gamma_1})$ with $B^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

(4) If there is γ with $U_\gamma > D_\gamma$, then we can choose γ with $0 < m(U_\gamma/D_\gamma) \geq m(U_H/D_H)$, in which case $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

PROOF. By F.9.13.5, V does not centralize U_γ/A_1.

Assume that $U_\gamma = D_\gamma$, so that $[U_\gamma, U_H] \leq V_1$. Therefore if U_H acts nontrivially on U_γ, then $[U_\gamma, U_H] = V_1$, so that conclusion (i) of (1) holds; $D_H < U_H$ since U_H is nontrivial on U_γ/A_1. Thus we may assume that $[U_H, U_\gamma] = 1$, and hence $D_H = U_H$ and $U_\gamma \leq C_H(V_+)$. Suppose first that case (i) of F.9.8.g holds. Then $[U_\gamma, V] \leq [C_H(V_+), V] \leq V_1$ by (i), so since V does not centralize U_γ by F.9.13.5, $[U_\gamma, V] = V_1$ and hence conclusion (ii) of (1) holds. Suppose instead that case (ii) of F.9.8.g holds. Since b is odd by F.9.11.1, for $\gamma_{b+1} \in \Gamma(\gamma) - \{\gamma_{b-1}\}$, F.7.3.2 says we may choose $g \in G_0$ to take the edge γ_0, γ_1 to the edge γ_{b+1}, γ, and thus $V_{b+1} = V^g$. Since V does not centralize U_γ, it does not centralize V_γ; then we may choose g so that V does not centralize $V^g = V_{b+1}$. Thus there is $x \in V^g$ with $[V, x] \neq 1$. Since

V^g centralizes x and normalizes V, $V^g \leq N_G([V,x])$, so that V^g centralizes V by (ii), contrary to $[V,x] \neq 1$. This contradiction completes the proof of (1).

Next by F.9.2.3, $U_\gamma/D_\gamma = U_\gamma^*$. Assume that $0 < m(U_\gamma^*) \geq m(U_H/D_H)$. By F.9.13.6 and as $|A_1| = 2$, D_H induces a group of transvections on U_γ with center A_1, so $m(D_H/C_{D_H}(U_\gamma)) = m(U_\gamma/C_{U_\gamma}(D_H))$. Also by F.9.13.7, $[D_H, D_\gamma] = 1$, so $m(D_H/C_{D_H}(U_\gamma)) \leq m(U_\gamma/D_\gamma) = m(U_\gamma^*)$, with equality only if $D_\gamma = C_{U_\gamma}(D_H)$. Therefore

$$m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*)) \leq m(U_H/C_{D_H}(U_\gamma^*)) \leq m(U_H/D_H) + m(D_H/C_{D_H}(U_\gamma))$$

$$\leq m(U_H/D_H) + m(U_\gamma^*) \leq 2\ m(U_\gamma^*),$$

where the last inequality uses the hypothesis $m(U_\gamma^*) \geq m(U_H/D_H)$. This establishes the first statement of (2): $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. Moreover if $2\ m(U_\gamma^*) = m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*))$, then $D_\gamma = C_{U_\gamma}(D_H)$ and all the inequalities in the display are equalities. In particular $m(U_\gamma^*) = m(U_H/D_H)$, and $D_\gamma < U_\gamma$. By F.9.13.6, U_γ induces a group of transvections on D_H with center A_1, so as $D_\gamma = C_{U_\gamma}(D_H)$, U_γ^* acts faithfully on $\tilde{D}_H$ as a group of transvections with center $\tilde{A}_1$. This completes the proof of (2).

We turn to the proof of (3) and (4). First suppose $U_\gamma = D_\gamma$. Then as (γ, γ_{b-1}) is conjugate to (γ_1, γ_0), if (1i) holds, then some conjugate U_H^g of U_H induces a group of transvections on $\tilde{U}_H$ with center $\tilde{V}_1^g$. Indeed by F.9.13.2, $U_H \leq V_H \leq O_2(G_{\gamma_{b-1},\gamma})$, so we can pick g with $U_H^g \leq O_2(G_{\gamma_0,\gamma_1})$, so that (3) holds in this case. Similarly in case (1ii), V induces a group of transvections on $\tilde{U}_\gamma$ with center $\tilde{V}_1$ and $V \leq O_2(G_{\gamma_{b-1},\gamma})$, so for some $g \in G_0$ with $V^g \leq O_2(G_{\gamma_0,\gamma_1})$, V^g induces a group of transvections on $\tilde{U}_H$, and hence (3) holds in this case also.

Thus we may assume that $U_\gamma > D_\gamma$, so $V_{\gamma_{b+1}} \not\leq G_{\gamma_1}^{(1)}$ for some $\gamma_{b+1} \in \Gamma(\gamma)$. Thus we have symmetry between the geodesics $p := \gamma_0, \gamma_1, \ldots, \gamma, \gamma_{b+1}$ and $p' := \gamma_{b+1}, \gamma \ldots, \gamma_1, \gamma_0$, so if necessary replacing p by $p'g$ for some $g \in G_0$ with $(\gamma_{b+1}, \gamma)g = (\gamma_0, \gamma_1)$, we may assume that $0 < m(U_\gamma^*) \geq m(U_H/D_H)$. Now $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ by (2), so $q(H^*, \tilde{U}_H) \leq 2$, and hence (3) and (4) hold. □

REMARK F.9.17. We expand on the use of symmetry such as in the last paragraph of the proof of F.9.13.4, as we use similar arguments elsewhere.

Suppose that $m(U_\gamma^*) > 0$. Then $V_\gamma \not\leq G_{\gamma_1}^{(1)}$ by F.9.15, so there is $\beta \in \Gamma(\gamma)$ with $V_\beta \not\leq G_{\gamma_1}^{(1)}$. Thus by transitivity of G on Γ_0, minimality of b, and F.7.9.3, $d(\beta, \gamma_1) = b$. Thus we have the symmetry between the geodesics $\gamma_0, \gamma_1, \cdots, \gamma$ and $\beta, \gamma, \cdots, \gamma_1$, and hence we also have symmetry between γ_1 and γ.

Further $U_\gamma > D_\gamma$, so by F.9.16.4, we can choose γ with $m(U_\gamma^*) \geq m(U_H/D_H)$. Indeed the proof of that result showed that for suitable $\gamma_{b+1} \in \Gamma(\gamma) - \{\gamma_{b-1}\}$, we have symmetry between the geodesics $\gamma_0, \gamma_1, \ldots, \gamma, \gamma_{b+1}$ and $\gamma_{b+1}, \gamma, \ldots, \gamma_1, \gamma_0$.

Then if we can show that $m(U_\gamma) = m(U_H/D_H)$, we also have $m(U_H/D_H) \geq m(U_\gamma/D_\gamma)$, so that this additional hypothesis is also symmetric in γ and γ_1. We can use this symmetry to conclude that all results of the form $S(\gamma_1, \gamma)$ (that is, involving only γ_1 and γ), proved under the choice $m(U_\gamma^*) \geq m(U_H/D_H)$, also hold with the roles of γ and γ_1 reversed: that is, $S(\gamma, \gamma_1)$ also holds.

Later we will invoke such arguments using the phrase *symmetry between* γ_1 *and* γ. Similarly a statement of the form $S(\gamma_0, \gamma_1, \gamma)$ implies the statement $S(\beta, \gamma, \gamma_1)$.

LEMMA F.9.18. *Assume H is an SQTK-group and there is $K \in \mathcal{C}(H)$ with $K/O_2(K)$ quasisimple and $H = K_0(LT \cap H)$, where $K_0 := \langle K^T \rangle$. Let $\hat{U}_H := \tilde{U}_H/C_{\tilde{U}_H}(K_0^*)$. Then*

(1) Case (i) of Hypothesis D.3.2 is satisfied with H, K, U_H, V_1, I in the roles of "$\dot{M}$, $\dot{L}$, Q_+, Q_-, Q_V", respectively, where $\tilde{I} \in Irr_+(K_0, \tilde{U}_H, T)$.

(2) There is $B \leq O_2(G_{\gamma_0,\gamma_1})$ with $B^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, and B^* faithful on K_0^*.

(3) Let $I_T := \langle I^T \rangle$ and $I_H := \langle I^H \rangle$. Then for $X := T$ and H, $C_{K_0^*}(\tilde{I}_T) \leq Z(K_0^*)$ and $B^* \cong Aut_{B^*}(\tilde{I}_X) \in \mathcal{Q}(Aut_{K_0^*T^*}(\tilde{I}_X), \tilde{I}_X)$, so $q(Aut_{KT}(\tilde{I}_X), \tilde{I}_X) \leq 2$.

(4) If $K = K_0$, then $K^*/Z(K^*)$ is a Bender group, $L_3(2^n)$, $Sp_4(2^n)'$, $G_2(2^n)'$, $L_4(2)$, $L_5(2)$, or A_7 (but K^* is not $\hat{A}_7$), or $K^* \cong M_{22}$ or $\hat{M}_{22}$. Moreover one of the following holds:

(i) $I \trianglelefteq H$.

(ii) $\tilde{I} = \tilde{I}_T$ is an FF-module for $Aut_{KT}(\tilde{I})$.

(iii) Either $\hat{I}$ is the natural module for $K^* \cong SL_3(2^n)$, $Sp_4(2^n)$, A_6, $L_4(2)$, or $L_5(2)$, or $\tilde{I}$ is a 4-dimensional module for $K^* \cong A_7$. Further $\hat{I}_H = \hat{I} \oplus \hat{I}^t$ for $t \in T - N_T(I)$, and $\hat{I}^t$ is not $\mathbf{F}_2K$-isomorphic to $\hat{I}$.

(5) If $K < K_0$, then $K^* \cong L_2(2^n)$, $Sz(2^n)$, A_5, or $L_3(2)$, and one of the following holds:

(i) $I \trianglelefteq H$, $K^* \cong L_2(2^n)$, and $\tilde{I}$ is the $\Omega_4^+(2^n)$-module for K_0^*.

(ii) $I \trianglelefteq H$, $K^* \cong L_3(2)$, and $\tilde{I}$ is the tensor product of natural modules for the factors of K_0^*.

(iii) $\hat{I}_H = \hat{I}_K \oplus \hat{I}_K^t$ for $t \in T - N_T(K)$, where $\tilde{I}_K = [\tilde{I}_H, K] = C_{\tilde{I}_H}(K^t)$, and one of the following holds:

(a) $\hat{I}_K = \hat{I}$ is the natural module for K^*, or the $2n$-dimensional orthogonal module for $K^* \cong L_2(2^n)$ with n even.

(b) $\tilde{I}_K = \tilde{I} \oplus \tilde{I}^s$ for $s \in N_T(K) - N_T(I)$, and $K^* \cong L_3(2)$ with $m(\tilde{I}) = 3$.

(c) $\tilde{I}_K$ is the sum of four isomorphic natural modules for $K^* \cong L_3(2)$, and

$$O^2(Aut_{C_{H^*}(K_0^*)}(\tilde{I}_H)) \cong \mathbf{Z}_5 \text{ or } E_{25}.$$

(6) If $0 \neq \tilde{W} = [\tilde{W}, K_0]$ is a K_0T-submodule of $\tilde{U}_H$, then one of the following holds:

(a) $\tilde{W} = [\tilde{U}_H, K_0]$.

(b) $\tilde{W}$ and U_H/W are FF-modules for $K_0^*T^*$.

(c) $\tilde{W}$ or U_H/W is a strong FF-module for $K_0^*T^*$.

(7) If $K_0^*T^*$ has no FF-modules, then $\tilde{I}_H = [\tilde{U}_H, K_0]$.

PROOF. Part (1) is straightforward; in particular $O_2(K_0^*T^*) = 1$ as $C_H(\tilde{U}_H) = Q_H$ by F.9.2.3. Thus we may apply suitable results from section D.3. The first statement in (2) is a restatement of F.9.16.3. As $LT = G_{\gamma_0}$, $H = G_{\gamma_1}$, and $B \leq O_2(G_{\gamma_0,\gamma_1})$, $B \leq O_2(LT \cap H)$. Thus as $O_2(H^*) = 1$ by F.9.2.3, and $H = K_0(LT \cap H)$, B^* is faithful on K_0^*, completing the proof of (2).

Adopt the notation of (3); then as $B \leq O_2(G_{\gamma_0,\gamma_1}) \leq T$, B^* acts on $\tilde{I}_X$. As K^* is quasisimple and $K_0 = \langle K^T \rangle$, $C_{K_0^*}(\tilde{I}_X) \leq Z(K_0^*)$. Thus as B^* is faithful on K_0^*, B^* is faithful on $\tilde{I}_X$, so as $B^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, also $B^* \cong Aut_B(\tilde{I}_X) \in \mathcal{Q}(Aut_{K_0T}(\tilde{I}_X), \tilde{I}_X)$, so (3) holds.

Assume $K = K_0$. Then by Theorem D.3.10, either $I = I_T$ or (4iii) holds, so we may assume the former. Then the hypotheses of D.3.9 are satisfied, so as B^* is faithful on K_0^*, D.3.9.1 says (4i) or (4ii) holds. Moreover by (3), K^* is one of the groups listed in the initial statement of (4), using B.4.2, and the values of "q" given in B.4.5. Thus (4) is established.

Finally assume that $K < K_0$. Then by Theorem D.3.21, either $I_T = I_H \trianglelefteq H$ or (5iiic) holds, so we may assume the former. If $I = I_T$ then (5i) or (5ii) holds by D.3.7, so we may asume $I < I_T$. Then conclusion (a) or (b) of (5iii) holds by D.3.6; the possibilities for K^* and the modules in (5iiia) are as stated using A.3.8.3, B.4.2, and B.4.5.

Assume the hypotheses of (6), and assume also that (a) fails. As $K_0 = \langle K^T \rangle$, $C_{K_0^*}(E) \leq Z(K_0^*)$ for $E := \tilde{W}$ and U_H/W; so as B^* is faithful on K_0^*, B^* is faithful on E. Thus as $B^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, (6b) or (6c) holds. Thus (6) is established, and of course (6) implies (7). □

CHAPTER G

Various representation-theoretic lemmas

This chapter contains some technical results (for the most part on $\mathbf{F}_2$-modules) which are useful in various situations. Some are essentially available from other sources, but we include details here for completeness.

G.1. Characterizing direct sums of natural $\mathbf{SL_n(F_{2^e})}$-modules

Assume $q := 2^e$ for some e. We will establish a "local" criterion for an $\mathbf{F}_2 SL_n(\mathbf{F}_q)$-module to be a direct sum of natural modules; this extends a well-known result of G. Higman [**Hig68**] for the case $n = 2$.

So in this section let $n > 1$ be an integer, $F := \mathbf{F}_q$, V an n-dimensional vector space over F, and $L := SL(V)$. We first establish some notation:

DEFINITION G.1.1. Let Λ be the set of F-points (1-dimensional F-subspaces of V) in the projective space $PG(V)$ over F, and θ the set of incident point-hyperplane pairs. For $(A, B) \in \theta$, let $R_{A,B}$ denote the root group of all transvections in L with center A and axis B.

Fix a decomposition $V = V_1 \oplus \cdots \oplus V_n$ with $V_i \in \Lambda$. For $i \neq j$, set $R_{i,j} := R_{V_i,\hat{V}_j}$ and $L_{i,j} := \langle R_{i,j}, R_{j,i} \rangle$, where $\hat{V}_j := \langle V_k : k \neq j \rangle$. Set $V_{i,j} := V_i + V_j$ and $\hat{V}_{i,j} := \hat{V}_i \cap \hat{V}_j$.

Next we record some standard facts:

LEMMA G.1.2. *(1)* $[V, L_{i,j}] = V_{i,j}$, $C_V(L_{i,j}) = \hat{V}_{i,j}$, *and* $L_{i,j} = C_{SL(V)}(\hat{V}_{i,j}) \cap N_{SL(V)}(V_i + V_j)$, *so* $L_{i,j} \cong SL_2(F)$.

(2) $L = \langle L_{i,i+1} : 1 \leq i < n \rangle$.

We now turn our attention to modules U for L over $\mathbf{F}_2$ rather than F. In particular, we adopt the convention that dimensions are over $\mathbf{F}_2$, unless otherwise specified; for example, $\dim(F) = e$ since $F = \mathbf{F}_{2^e}$.

Our main result in the section characterizes direct sums of natural modules as the modules U on which the root groups $R_{A,B}$ act quadratically, with the commutator subspaces $[U, R_{A,B}]$ satisfying suitable regularity conditions:

THEOREM G.1.3. *Let U be an $\mathbf{F}_2 L$-module, and assume $\varphi : \Lambda \to PG(U)$ is an L-equivariant map such that:*

(i) For all $(A, B) \in \theta$, $[U, R_{A,B}] \leq \varphi(A)$; and for all $D \in \Lambda$ with $D \leq B$, $\varphi(D) \leq C_U(R_{A,B})$.

(ii) For each pair of distinct $A_1, A_2 \in \Lambda$, $\varphi(A_1) \cap \varphi(A_2) = 0$.

(iii) $U = \langle \varphi(A) : A \in \Lambda \rangle$.

Let $s := \dim_{\mathbf{F}_2}(\varphi(A))/e$. Then U is the direct sum of s $\mathbf{F}_2 L$-submodules, which are $\mathbf{F}_2 L$-isomorphic to V.

REMARK G.1.4. Recall L acts F-flag-transitively on V, and 2-transitively on Λ. Further in the decomposition of G.1.2, $(V_1, \hat{V}_2) \in \theta$ with $R_{1,2} = R_{V_1,\hat{V}_2}$, and $N_L(V_1) \cap N_L(\hat{V}_2)$ is transitive on $\Lambda \cap \hat{V}_2 - \{V_1\}$. Thus given an $\mathbf{F}_2L$-module U and an L-equivariant map $\varphi : \Lambda \to PG(U)$, if we define $U_i := \varphi(V_i)$, then to verify hypotheses (i)–(iii) of Theorem G.1.3, it suffices to show that $U = U_1 + \cdots + U_n$, $U_1 \cap U_2 = 0$, $[U_2, R_{1,2}] \leq U_1 \leq C_U(R_{1,2})$, and if $n > 2$ also $U_3 \leq C_U(R_{1,2})$.

Before begining the proof of Theorem G.1.3, we prove three results in the special case where $n = 2$.

LEMMA G.1.5. *Assume $n = 2$ and $R \in Syl_2(L)$. Then*

(1) $R = R_{A,A}$ for some $A \in \Lambda$.

(2) If the hypotheses of Theorem G.1.3 are satisfied, then $\varphi(A) = [U, R] = C_U(R)$ and $U = [U, R] \oplus [U, R^g]$ for each $g \in L - N_L(R)$.

(3) If $U = \psi(R) \oplus \psi(R)^g$ for some $g \in L$ and some $N_L(R)$-invariant subspace $\psi(R)$ with $[U, R] \leq \psi(R) \leq C_U(R)$, then the hypotheses of Theorem G.1.3 are satisfied with $\varphi(Ax) := \psi(R)^x$ for $x \in L$.

(4) The following are equivalent:

(a) $U = C_U(R) \oplus C_U(R^g)$ for some $g \in L$.

(b) R is quadratic on U, and $C_U(x) = 0$ for each $x \in L^\#$ of odd order.

(c) R is quadratic on U, and $C_U(x) = 0$ for some $x \in L^\#$ of odd order.

PROOF. Part (1) is well known. Assume the hypotheses of G.1.3 are satisfied. By hypothesis (i) of G.1.3, $[U, R] \leq \varphi(A) \leq C_U(R)$. Let $g \in L - N_L(R)$ and $W := [U, R] + [U, R^g]$. As $L = \langle R, R^g \rangle$, $W = [U, L]$. Thus using hypothesis (iii), $U = [U, L] + \varphi(A) = W + \varphi(A)$, so as $[U, R] \leq \varphi(A)$, $U = [U, R^g] + \varphi(A)$. Then as $[U, R^g] \leq \varphi(Ag)$, we conclude from (ii) that $U = [U, R^g] \oplus \varphi(A)$. Finally suppose that $[U, R] < C_U(R)$. Then $Z := C_U(R) \cap \varphi(Ag) \neq 0$, so as $L = \langle R, R^g \rangle$, $Z \leq C_U(L)$. But then $0 \neq Z = Z^{g^{-1}} \leq \varphi(Ag) \cap \varphi(A) = 0$, a contradiction. This completes the proof of (2).

Assume the hypotheses of (3) and define $\varphi(Ax) := \psi(R)^x$ for $x \in L$. As $\psi(R)$ is $N_L(R)$-invariant, $\varphi(Ax)$ is well-defined, and φ is L-equivariant. By construction and the hypothesis on ψ, $[U, R] \leq \varphi(A) \leq C_U(A)$; thus (i) holds. As L is 2-transitive on Λ and $U = \psi(R) \oplus \psi(R)^g$, (ii) and (iii) hold. So (3) is established.

It remains to prove (4). Suppose first that (a) holds. We claim first that $C_U(R) = C_U(r)$ for each $r \in R^\#$: For if $C_U(R) < C_U(r)$, then as $U = C_U(R) \oplus C_U(R^g)$, $C_{C_U(R^g)}(r) \neq 0$. But then $L = \langle R^g, r \rangle$ centralizes $C_{C_U(R^g)}(r)$, so $0 \neq C_{C_U(R^g)}(r) \leq C_U(R^g) \cap C_U(R) = 0$, and this contradiction establishes the claim. Now $[U, r] \leq C_U(r) = C_U(R)$, so R is quadratic on U. Let $x \in L^\#$ be of odd order. Then x is inverted in L, so as L is 2-transitive on Λ, we may assume that $x = rs^g$ for some $r, s \in R^\#$. Assume that $C_U(x) \neq 0$. Then as r normalizes $\langle x \rangle$,

$$0 \neq C_U(x) \cap C_U(r) \leq C_U(s^g) \cap C_U(r) = 0,$$

using hypothesis (a) and the claim. This contradiction shows that (a) implies (b). Trivially (b) implies (c), so we may assume that (c) holds, and it remains to show that (a) holds. Thus there is $x \in L^\#$ of odd order with $C_U(x) = 0$, and as above we may assume that $x = rs^g$ with $r, s \in R^\#$. As $C_U(x) = 0$, by A.1.44, $U = [U, r] \oplus [U, s^g]$ with $[U, r] = C_U(r)$. Then as R is quadratic on U, $C_U(R) \leq C_U(r) = [U, r] \leq [U, R] \leq C_U(R)$, so $C_U(R) = [U, r] = [U, R]$ and (a) holds. $\square$

We often use G.1.5 in the form of the following corollary:

LEMMA G.1.6. *Let $L := L_2(2^e)$ with $e > 1$, $R \in Syl_2(L)$, and U an $\mathbf{F}_2L$-module. Assume $U = \langle C_U(R)^L \rangle$, and $C_U(R) = C_U(r)$ for each $r \in R^\#$. Then $U/C_U(L)$ is a direct sum of natural modules for L.*

PROOF. Let $U^* := U/C_U(L)$, $g \in L - N_L(R)$, and $S := R^g$. As $U = \langle C_U(R)^L \rangle$, $U = [U, L] + C_U(R)$. Then by Gaschütz's Theorem A.1.39, $U = [U, L] + C_U(L)$, so that $U^* = [U^*, L]$. Next as $L = O^2(L)$ since $e > 1$, by Coprime Action

$$C_{U^*}(L) = C_U(L)^* = 0.$$

Also

$$L = \langle R, S \rangle,$$

so $U^* = [U^*, L] = [U^*, R] + [U^*, S]$. For $r \in R^\#$, $[U, r] \leq C_U(r) = C_U(R)$. Thus $[U^*, R] = [U, R]^* \leq C_U(R)^* \leq C_{U^*}(R)$. Hence

$$[U^*, R] \cap [U^*, S] \leq C_{U^*}(R) \cap C_{U^*}(S) = C_{U^*}(L) = 0,$$

so $U^* = [U^*, R] \oplus [U^*, S]$. We have shown that the hypotheses of G.1.5.3 are satisfied with $[U^*, R]$, U^* in the roles of "$\psi(R)$, U", so by G.1.5.3 and G.1.3, U^* is a direct sum of natural modules for L. □

The next lemma G.1.7, together with G.1.5.2, establishes Theorem G.1.3 in the case where $n = 2$. Notice that by G.1.5.4, the hypotheses of G.1.7 are equivalent to the conditions that R is quadratic on U and $C_U(x) = 0$ for x of order 3 in L. Thus G.1.7 can be regarded as a special case of the well-known result of G.Higman (Theorem 8.2 in G. Higman's U. Michigan lecture notes [**Hig68**]), stating that modules for $SL_2(2^n)$ on which elements of order 3 are fixed point free are direct sums of natural modules. Since Higman's work was never published, we include here a short, independent proof under the extra condition that the Sylow 2-group R of L is quadratic on U. (Higman's treatment also used (without proof) the fact that if N is the natural $FL_2(2^n)$-module and $\sigma \in Aut(F)$ then $Ext^1(N, N^\sigma) = 0$; nowadays a proof of this fact can be found in various places in the literature, such as Corollary 4.5 in [**AJL83**, Cor 4.5].)

LEMMA G.1.7. *Assume that $n = 2$ and $U = C_U(R) \oplus C_U(R^g)$ for some $R \in Syl_2(L)$ and $g \in L$. Then U is the direct sum of $s := \dim_{\mathbf{F}_2}([U, R])/e$ natural $\mathbf{F}_2L$-submodules.*

PROOF. The proof is by induction on $\dim(U) = \dim_{\mathbf{F}_2}(U)$. The lemma holds if $\dim(U) = 0$, so we may assume that $\dim(U) > 0$. The hypotheses fail if $\dim(U) = 1$, so $\dim(U) > 1$. Notice if W is an L-submodule of U, then W and U/W satisfy the hypotheses of the lemma: For by G.1.5.4, the hypothesis that $U = C_U(R) \oplus C_U(R^g)$ is equivalent to the hypothesis that R is quadratic on U and $C_U(x) = 0$ for $x \in L^\#$ of odd order, and the latter hypotheses are inherited by submodules and quotient modules.

Using the Steinberg Tensor Product Theorem (cf. 2.8.5 in [**GLS98**]) and a tensor computation, the natural $\mathbf{F}_qL$-module (regarded as an $\mathbf{F}_2L$-module) is the only irreducible for L of dimension at least 2 on which R is quadratic. Thus we may assume that L is reducible on U. Let W be a maximal L-submodule of U. By the previous paragraph and induction on $\dim(V)$, W is the sum of natural modules and U/W is natural. Thus it remains to show that U splits over W.

Let X be a subgroup of order $q+1$. Then X is inverted by a unique involution from each Sylow 2-group of L. So let $r \in R^{\#}$ invert X; from the proof of G.1.5, $[U,R] = C_U(R) = [U,r] = C_U(r)$. In particular r is free on U, so U splits over W as an $\langle r\rangle X$-module. Let Z be an $\langle r\rangle X$-complement to W in U. If $q = 2$, then $L = \langle r\rangle X$ and we are done. So assume that $q > 2$ for the remainder of the proof.

Next $X = \langle rt\rangle$ for some involution t and $t \in T \in Syl_2(L) - \{R\}$. As $q > 2$, there is $h \in T - \langle t\rangle$, and so $t = t^h$ inverts X^h. On the other hand $\langle t\rangle = N_T(X)$, so $X^h \neq X$. Set $I := Z + Z^h$. As $C_U(X) = 0$, by A.1.44,

$$Z = [Z,r] \oplus [Z,t], \qquad (+)$$

and $[Z,t] \leq C_U(t) = C_U(T)$, so $[Z,t] = [Z^h,t]$. Further there is a unique involution $s \in R$ inverting X^h, and $s \neq r$ since $X^h \neq X = \langle rt\rangle$. As $X\langle r\rangle$ is maximal in L, $L = \langle r,s,t\rangle$. But t acts on Z and Z^h, and hence on I. As $C_U(st) = 0$, $Z^h = [Z^h,s] + [Z^h,t] = [Z^h,s] + [Z,t]$, so we conclude from (+) that

$$I = Z + Z^h = [Z,r] + [Z^h,s] + [Z,t]. \qquad (++)$$

Now $[Z,r] + [Z^h,s] \leq [U,R] \leq C_U(R)$, while $[Z,t,r] \leq Z \leq I$, and we saw $[Z,t] = [Z^h,t]$ so that $[Z,t,s] = [Z^h,t,s] \leq Z^h \leq I$. Thus using (++), we see that r and s also act on I. Hence I is an L-submodule of U, and then so is $I \cap W$. But $U = W \oplus Z = W + I$, while $\dim(Z) = 2\dim([Z,t]) = 2e$ by (+), and $\dim(I) \leq 3\dim([Z,t]) = 3e$ using (++). Thus $\dim(I \cap W) \leq e$. As the minimal dimension of a faithful FL-module is $2e$, we conclude that $I \cap W \leq C_U(L) \leq C_U(X) = 0$. Hence $I = Z$ is an L-complement to W in U, completing the proof. □

We now begin the proof of Theorem G.1.3. The proof involves a series of reductions. Assume the hypotheses of Theorem G.1.3, and choose notation as in G.1.2. Set $U_i := \varphi(V_i)$ and $s := \dim_{\mathbf{F}_2}(\varphi(A))/e$.

Lemma G.1.8. $U = U_1 + \cdots + U_n$.

Proof. By hypothesis (i) of Theorem G.1.3, $[U,R_{i,j}] \leq U_i$, so each $R_{i,j}$ acts on $U' := U_1 + \cdots + U_n$. Hence by G.1.2.2, U' is L-invariant. Then as φ is L-equivariant, $\varphi(A) \leq U'$ for each $A \in \Lambda$, so $U \leq U'$ by hypothesis (iii). □

We define some notation similar to that in Definition G.1.1:

Definition G.1.9. Set $U_{i,j} := U_i + U_j$, $\hat{U}_i := \langle U_k : k \neq i\rangle$, and $\hat{U}_{i,j} := \langle U_k : k \neq i,j\rangle$.

We will proceed by induction on n, using G.1.7 for the base step $n = 2$.

Lemma G.1.10. *$U_{1,2} = U_1 \oplus U_2$ is the sum of s natural modules for $L_{1,2}$. Furthermore $U = U_{1,2} \oplus \hat{U}_{1,2}$, $\hat{U}_{1,2} = C_U(L_{1,2})$, $\hat{U}_2 = U_1 \oplus \hat{U}_{1,2} = C_U(R_{1,2})$, and $U = U_1 \oplus \hat{U}_1$.*

Proof. The argument used to prove G.1.8 shows that $L_{1,2}$ acts on $U_{1,2}$. Observe that the hypotheses of Theorem G.1.3 are satisfied with $L_{1,2}$, $V_{1,2}$, ψ, $U_{1,2}$ in the roles of "L, V, φ, U", where $\psi(A) := \varphi(A)$ for A a point in $V_{1,2}$. Thus by G.1.5.2, we may apply G.1.7 to conclude that $U_{1,2} = U_1 \oplus U_2$, and $U_{1,2}$ is a sum of s natural modules for $L_{1,2}$. By (i), $L_{1,2}$ centralizes $\hat{U}_{1,2}$, and $R_{1,2}$ centralizes U_1. By G.1.8, $U = U_{1,2} + \hat{U}_{1,2}$. As $U_{1,2}$ is a sum of natural modules for $L_{1,2}$, $C_{U_{1,2}}(L_{1,2}) = 0$, so $\hat{U}_{1,2} = C_U(L_{1,2})$ and $U = U_{1,2} \oplus \hat{U}_{1,2}$. Thus $U = U_1 \oplus \hat{U}_1$. As $U_{1,2}$ is a sum of natural modules for $L_{1,2}$, $U_1 = C_{U_{1,2}}(R_{1,2})$, so $\hat{U}_2 = C_U(R_{1,2})$. □

Observe that G.1.10 says that Theorem G.1.3 holds when $n = 2$. So we may assume during the remainder of the proof that $n \geq 3$.

LEMMA G.1.11. $U = U_1 \oplus \cdots \oplus U_n$.

PROOF. By G.1.10 and the L-equivariance of φ, $U_i \cap \hat{U}_i = 0$ for each i. Then the lemma follows from this fact and G.1.8. □

Let T be the Sylow 2-subgroup of L acting on the F-flag $0 < V_1 < V_1 + V_2 < \cdots < V$ from Definition G.1.1.

LEMMA G.1.12. *(1)* $U_1 = C_U(T) = C_U(L_1)$, *where* $L_1 := O^{2'}(N_L(V_1))$.
(2) U_1 *is the direct sum of* s e*-dimensional* $\mathbf{F}_2 N_L(V_1)$*-irreducibles.*

PROOF. As T is the product of root groups $R_{A,B}$ with $V_1 \leq B$, we conclude from (i) that $U_1 \leq C_U(T)$. Similarly as $L_1 = O^{2'}(N_L(V_1))$ is generated by such root groups, $U_1 \leq C_U(L_1)$. Conversely as $R_{1,m} \leq T$ for each $1 < m \leq n$, using G.1.10 and G.1.11 we see

$$C_U(L_1) \leq C_U(T) \leq \bigcap_{1<m\leq n} C_U(R_{1,m}) = \bigcap_{1<m\leq n} \hat{U}_m = U_1,$$

establishing (1). Next $N_L(V_1) = L_1 H$, where H is a Cartan subgroup of $L_{1,2}$, and H acts irreducibly on V_1 of $\mathbf{F}_2$-dimension e in the natural $\mathbf{F}_2 L_{1,2}$-module $V_{1,2}$. Thus (1) and G.1.10 imply (2). □

We now complete the proof of Theorem G.1.3. Let Δ be the set of e-dimensional $\mathbf{F}_2 N_L(V_1)$-irreducibles on U_1 supplied by G.1.12.2. For $W_1 \in \Delta$, let $U(W_1) := \langle W_1^L \rangle$. We will show the modules $U(W_1)$, $W_1 \in \Delta$, are $\mathbf{F}_2 L$-submodules which are $\mathbf{F}_2 L$-isomorphic to V. This will suffice: for by G.1.12,

$$U_1 \leq U' := \langle U(W_1) : W_1 \in \Delta \rangle,$$

so that $U = \langle U_1^L \rangle \leq U'$; hence U will be the direct sum of some subcollection of s of these submodules, and so the proof of Theorem G.1.3 will be complete.

So pick $W_1 \in \Delta$, and for $l \in L$, let $W(V_1^l) := W_1^l$. This is well-defined as $N_L(V_1)$ acts on W_1. Let $W_i := W(V_i)$ for $1 \leq i \leq n$. It will suffice to show that

$$U(W_1) = W_1 + \cdots + W_n. \qquad (*)$$

For then as $W_i \leq U_i$, by G.1.11 $U(W_1) = W_1 \oplus \cdots \oplus W_n$ is of $\mathbf{F}_2$-dimension ne, and by G.1.12, $C_{U(W_1)}(T) = W_1$ of $\mathbf{F}_2$-dimension e is an irreducible for $N_L(V_1)$. However if I is an L-irreducible submodule of $U(W_1)$, then $C_I(T) \neq 0$, so as $N_L(V_1)$ is irreducible on $C_{U(W_1)}(T) = W_1$, $W_1 \leq I$; thus $U(W_1) = \langle W_1^L \rangle \leq I$. That is $U(W_1)$ is an irreducible L-module. Now as the natural module for $SL(V)$ is the unique $\mathbf{F}_2$-irreducible of dimension ne in which $N_L(V_1)$ stabilizes an e-dimensional subspace, it follows that $U(W_1)$ is $\mathbf{F}_2 L$-isomorphic to V.

So it remains to establish (*). By G.1.2.2, it suffices to show that $[W_j, R_{i,j}] \leq W_i$ for each i, j. By G.1.10, $U_{i,j}$ is the sum of s natural modules for $L_{i,j}$, so

$$\frac{q^s - 1}{q - 1} = |Irr(L_{i,j}, U_{i,j})| = |Irr(N_{L_{i,j}}(V_i), U_{i,j})|, \qquad (**)$$

recalling that $N_{L_{i,j}}(V_i)$ acts indecomposably on $V_{i,j}$. By (**):

The map $W \mapsto \langle W^{L_{i,j}} \rangle$ is a bijection between $Irr(N_{L_{i,j}}(V_i), U_i)$ and $Irr(L_{i,j}, U_{i,j})$. (!)

Further using G.1.12 together with the fact that $N_L(V_i) = N_{L_{i,j}}(V_i)L_i$ for $L_i := O^{2'}(N_L(V_i))$, we have $W_i \in Irr(N_{L_{i,j}}(V_i), U_{i,j})$. Hence by (!),

$$\langle W_i^{L_{i,j}} \rangle =: I \in Irr(L_{i,j}, U_{i,j}).$$

So as I is a natural module for $L_{i,j}$, for $l \in L_{i,j}$ with $V_i^l = V_j$, we have $I = W_i + W_i^l = W_i + W_j$, and then $[W_j, R_{i,j}] = W_i$ as desired.

This completes the proof of Theorem G.1.3.

G.2. Almost-special groups

During the proof of the Main Theorem, (cf. Remark F.7.12) we often consider the normal closure U_H of a suitable submodule of the internal module V for $L \in \mathcal{L}_f^*(L,T)$, under the action of $H \in \mathcal{H}_*(T,M)$. The analysis then bifurcates depending on whether U_H is abelian or non-abelian. When U_H is nonabelian, the structure of U_H resembles that of a special 2-group. In this section we assume hypotheses which hold in this situation, and derive a few consequences of the hypotheses. In particular we use the results on $\mathbf{F}_2$-modules in the previous section to obtain information about 2-chief factors for L.

So in this section, we assume:

Hypothesis G.2.1. *G is a finite group, V is an elementary abelian 2-subgroup of G, $1 \neq V_1 \leq V$, and $L = O^2(L)$ is a subgroup of G, with T a 2-subgroup of G such that*

$$V \leq T \leq N_G(V_1) \cap N_G(V) \cap N_G(L)$$

and $T \in Syl_2(LT)$. Set $G_1 := N_G(V_1)$, $\tilde{G}_1 := G_1/V_1$, and $L_1 := O^2(N_L(V_1))$. Assume H is a subgroup of G_1 containing L_1T such that $T \in Syl_2(H)$, $Q_H := O_2(H) = F^*(H)$, and L_1T is irreducible on $\tilde{V}$. Set $(LT)^* := LT/O_2(LT)$, $U := \langle V^H \rangle$, and $V_L := \langle V_1^L \rangle$.

Lemma G.2.2. *(1) $\tilde{V} \leq \tilde{U} \leq \Omega_1(Z(\tilde{Q}_H))$.*
(2) $\Phi(U) \leq V_1$.
(3) $U \trianglelefteq T$.
(4) $\tilde{U} \in \mathcal{R}_2(\tilde{H})$.

Proof. As $H \leq G_1 = N_G(V_1)$, $V \leq C_H(V_1) \trianglelefteq H$. By hypothesis $F^*(H) = O_2(H)$, so using 1.1.3.2, $F^*(C_H(V_1)) = O_2(C_H(V_1)) =: Q_1$, and hence as V_1 is central in $C_H(V_1)$, $F^*(\widetilde{C_H(V_1)}) = O_2(\widetilde{C_H(V_1)}) = \tilde{Q}_1 \leq \tilde{Q}_H$, using A.1.8. Next since $V \trianglelefteq T \in Syl_2(H)$, by B.2.14 applied in $\widetilde{C_H(V_1)}$,

$$1 \neq Z(\tilde{T}) \cap \tilde{V} \leq Z(\tilde{Q}_1) \leq \tilde{Q}_H.$$

Then as $L_1T \leq H$ and L_1T is irreducible on $\tilde{V}$, $\tilde{V} \leq \Omega_1(Z(\tilde{Q}_H))$. Hence (1) holds as $U = \langle V^H \rangle$, and then (1) implies (2), and B.2.13 implies (4). By (1), $U \leq T$, so (3) holds. □

Next we consider the normal closure of U in L under the assumption that U is not abelian and V_L is the natural module for $L^* \cong L_2(2^n)$, $SL_3(2^n)$, or $\Omega_4^\epsilon(2)$. Under these hypotheses, the 2-chief factors of L are almost determined using Theorem G.1.3.

The hypothesis that $U^* \neq 1$ in the next two lemmas is equivalent to the hypothesis that U is nonabelian, which is in turn equivalent to the hypothesis that

$b = 2$, where b is the amalgam parameter defined by the pair LT,H, discussed in Remark F.7.12. When $b = 2$, U resembles a special group by G.2.2; hence the terminology "almost special" .

LEMMA G.2.3. *Assume $L^* \cong L_2(2^n)'$, $V = V_L$ is the natural module for L^*, V_1 is a 1-dimensional $\mathbf{F}_{2^n}$-subspace of V, and $U^* \neq 1$. Let $g \in L - N_L(V_1)$ and set $I := \langle U, U^g\rangle$ and $W := U \cap O_2(I)$. Then*

(1) $L^*U^* \cong L_2(2^n)$, and $U^* \in Syl_2(L^*U^*)$.

(2) $I = LU$.

(3) $S := O_2(I) = WW^g$.

(4) S has an I-series

$$1 =: S_0 \leq S_1 \leq S_2 \leq S_3 := S$$

with $S_1 := V \leq Z(S)$ and $S_2 := U \cap U^g$.

(5) $[S_2, I] \leq S_1$, and S_2 is elementary abelian.

(6) $S/S_2 = W/S_2 \oplus W^g/S_2$ is the sum of s natural modules for L^*U^* for some nonnegative integer s, with $W/S_2 = C_{S/S_2}(U^*)$.

(7) $|U| = 2^{n(s+3)}|S_2 : S_1|$.

PROOF. This is essentially 8.15 in [**Asc94**], or 17.7 in [**Asc86b**]; but we include a proof here for completeness. As $L^* = F^*(L^*T^*)$ and $U^* \neq 1$, U^* is a nontrivial 2-subgroup of $Aut(L^*)$, and $N_{L^*}(U^*)$ is a proper subgroup of L^*. Thus if $n = 1$, then $L^* \cong \mathbf{Z}_3$ and $L^*U^* = Aut(L^*) \cong L_2(2^n)$. In this case set $K := LU$; if $n > 1$, set $K := L$. Thus in any case $K^* \cong L_2(2^n)$ and $K^*U^* \leq Aut(K^*)$.

As V_1 is a 1-dimensional $\mathbf{F}_{2^n}$-subspace of the natural module V for K^*, $N_K(V_1)^*$ is a Borel subgroup of K^*. By Hypothesis G.2.1, $L_1T \leq H$ and hence $N_K(V_1) = L_1(T \cap K) \leq N_K(U)$, with $N_K(U)^* \leq N_{K^*}(U^*)$. Thus $N_{K^*}(U^*) = N_K(V_1)^*$ is a Borel subgroup of K^* by maximality of $N_K(V_1)^*$ in K^*. As $T \leq H \leq N_G(U)$, $1 \neq Z(T^*) \cap U^* \leq T^* \cap K^*$, so as $N_L(V_1)^*$ is irreducible on $(T \cap K)^*$, we conclude that $U^* = (T \cap K)^* \in Syl_2(K^*)$, completing the proof of (1).

Let $g \in L - N_L(V_1)$; as U^* is Sylow in $K^* \cong L_2(2^n)$, $K^* = \langle U^*, U^{g*}\rangle = I^*$, so $LU = O_2(LU)I$. As $O_2(LU) \leq T \leq N_G(U)$, $U^{LU} = U^{O_2(LU)I} = U^I$; in particular, we can take $g \in I$, and we make this choice during the remainder of the proof. Then $I \leq \langle U^{LU}\rangle = \langle U^I\rangle \leq I$, so LU acts on $\langle U^{LU}\rangle = I$. Then as $LU = O_2(LU)I$,

$$LU/I = O_2(LU)I/I \cong O_2(LU)/(O_2(LU) \cap I),$$

so LU/I is a 2-group and hence $L = O^2(L) \leq O^2(LU) \leq I$, so $I = LU$, completing the proof of (2).

We turn to the proofs of (3)–(5). By hypothesis $V = V_L \trianglelefteq LT$, so $S_1 := V$ is normal in I. As L is irreducible on V, $S := O_2(I)$ centralizes V. As $V \leq U$ by construction, $S_1 = V \leq U \cap U^g =: S_2$. Indeed $[U, S_2] \leq V_1 \leq V$ by G.2.2.1, and similarly $[U^g, S_2] \leq V_1^g \leq V$, so $[S_2, I] \leq S_1$. Further $\Phi(S_2) \leq \Phi(U) \cap \Phi(U^g) = V_1 \cap V_1^g = 1$, as the 1-space V_1 is a TI-set under L, completing the proof of (5). In particular, S_2 is normal in I, completing the proof of (4).

Set $W := U \cap S$; as $g \in I \leq N_G(S)$, $W^g = U^g \cap S$, and as S acts on U, S acts on W and W^g. Set $P := WW^g$. As S acts on U, $[U, S] \leq U \cap S = W \leq P$, and similarly $[U^g, S] \leq P$. Therefore P is normal in $I = \langle U, U^g\rangle$ and $[S, I] \leq P$ so $S/P \leq Z(I/P)$.

Set $I^+ := I/P$, so that $S^+ \leq Z(I^+)$. We have seen that

$$L_2(2^n) \cong L^*U^* = I^* \cong I/O_2(I) \cong I^+/S^+.$$

Therefore as $U \cap S = W = U \cap P$, $U^+ \cong U^* \cong E_{2^n}$, and as $S^+ \leq Z(I^+)$, $S^+ = Z(I^+)$.

Assume first that $n = 1$. Then $U^* \cong U^+$ is of order 2, so $I^+ = \langle U^+, U^{g+} \rangle$ is dihedral. Then as $I^+/S^+ \cong I^* \cong S_3$ and $S^+ = Z(I^+)$, we conclude $I^+ \cong S_3$ or D_{12}. The latter case is impossible, as we saw U is conjugate to U^g in I. Therefore $S^+ = 1$ and hence $O_2(I) = S = P$, so (3) holds in this case.

So assume that $n > 1$. As $L \leq I$ and $I^+/Z(I^+) \cong L_2(2^n)$, $L^+/Z(L^+) \cong L_2(2^n)$. From the proof of (1), $U^* = [U^*, L_1^*] \in Syl_2(L^*)$, so as L_1 acts on U, $[U^+, L_1^+] \leq U^+ \cap L^+$ and $[U^+, L_1^+]Z(L^+) \in Syl_2(L^+)$. Thus $|[U^+, L_1^+] : [U^+, L_1^+] \cap Z(L^+)| = 2^n = |U^+|$, so $U^+ = [U^+, L_1^+]$ is a complement to $Z(L^+)$ in a Sylow 2-group of L^+, and hence $Z(L^+) = 1$ using Gaschutz's Theorem A.1.39. Then $I^+ = \langle U^+, U^{g+} \rangle = L^+$ is simple, completing the proof of (3).

Let $\hat{I} := I/S_2$. Then $\hat{S} = \hat{W} \oplus \hat{W}^g$ by construction, and $[\hat{S}, U^*] \leq \hat{W} \leq C_{\hat{S}}(U^*)$. Thus the hypotheses of G.1.5.3 are satisfied with respect to the action of I^* on $\hat{S}$ and $\psi(U^*) = \hat{W}$, so (6) holds by G.1.5.3 and Theorem G.1.3. Finally (1)–(6) imply (7); for example $|V| = 2^{2n}$, $|W : S_2| = 2^{ns}$, and $|U : W| = |U^*| = 2^n$. □

We have an analogue of G.2.3 in the case $L^* \cong A_5$ and V the A_5-module:

LEMMA G.2.4. *Assume $L^* \cong A_5$, V_L is the A_5-module for L^*, V_1 is of order 2, $V = [V_L, L_1]$, and $U^* \neq 1$. Let $g \in L - N_L(V_1)$ and set $I := \langle U, U^g \rangle$ and $W := U \cap O_2(I)$. Then*

(1) $U^* = O_2(L_1^*) \in Syl_2(L^*)$.

(2) $I = LU$.

(3) $S := O_2(I) = WW^g$.

(4) S has an I-series

$$1 =: S_0 \leq S_1 \leq S_2 \leq S_3 := S$$

with $S_1 := V_L \leq Z(S)$ and $S_2 := V_L(U \cap U^g)$.

(5) $[S_2, I] \leq S_1$.

(6) $S/S_2 = W/S_2 \oplus W^g/S_2$ is the sum of s natural $L_2(4)$-modules for L^* for some nonnegative integer s, with $W/S_2 = C_{S/S_2}(U^*)$.

(7) $|U| = 2^{2s+5}|S_2 : S_1|$.

(8) If $[\tilde{U}, L_1] = \tilde{U}$ and $O_2(L_1) \leq Q_H$, then $I = L$ is an A_5-block, $S = V_L$, and $U = O_2(L_1) \cong Q_8^2$.

PROOF. Much of the proof is the same as that of G.2.3; we supply details that are different. As V_L is the natural module for L^* and V_1 is a T-invariant point of V_L, V_1 is a singular point in V_L, and $L_1^* = C_{L^*}(V_1)$ is a Borel subgroup of L^*. As $1 \neq U^* \trianglelefteq L_1^*T^*$, $U^* = O_2(L_1^*)$, so (1) holds. Then the proof of (2) is exactly like that of G.2.3.2.

Next V is a hyperplane of V_L and $[V_L, U^*] = V > V_1$, so $V_L \cap U = V$ by G.2.2.1. Next $V_L = VV^g \leq S$, and $S_1 := V_L$ is normalized by I. Then we argue as in the proof of (3)–(6) of G.2.3, with only a few differences: V_L serves in the role of "V" for most of those arguments; also $V_L \not\leq U$, but by definition $S_1 = V_L \leq V_L(U \cap U^g) = S_2$. Then as $V = V_L \cap U$, (1) and (6) imply (7).

Finally assume the hypotheses of (8), and set $H^+ := H/Q_H$. As $\tilde{U} = [\tilde{U}, L_1]$ and $V_1 = [V, L_1]$, $U = [U, L_1] \leq L_1 \leq L$, so $I = L$ by (2). As $O_2(L_1) \leq Q_H$ by hypothesis, $L_1^+ \cong \mathbf{Z}_3$. Then as $\tilde{U} = [\tilde{U}, L_1]$, V_1 is the unique central chief factor for L_1 on U, and $U \leq O_2(L_1)$. In particular as L_1 centralizes $S_2/S_1 = (U \cap U^g)S_1/S_1$

by (5), $S_1 = S_2$. Then by (6), $S/S_1 = [S/S_1, L_1]$, so $S \leq O_2(L_1)V_L \leq Q_H V_L$. Thus $[S/V_L, U] \leq [Q_H V_L/V_L, U] \leq V_1 V_L/V_L = 1$, so $S = V_L$ by (6). That is L is an A_5-block with $S = V_L$. Therefore $O_2(L_1) \cong Q_8^2$ and $V = V_L \cap O_2(L_1)$, so as $U^* = O_2(L_1^*)$ and $U \leq O_2(L_1)$, it follows that $U = O_2(L_1)$, completing the proof of (8) and the lemma. □

The methods for the 2-dimensional case in G.2.3 can be extended to the 3-dimensional case:

LEMMA G.2.5. *Assume $L^* \cong SL_3(2^n)$, $V = V_L$ is the natural module for L^*, and V_1 is a 1-dimensional $\mathbf{F}_{2^n}$-subspace of V. Further assume $U^* \neq 1$, let $g, h \in L$ with $V = V_1 \oplus V_1^g \oplus V_1^h$, and set $I := \langle U, U^g, U^h \rangle$ and $W := U \cap O_2(I)$. Then*

(1) $U^* = O_2(N_L(V_1)^*) \cong E_{2^{2n}}$ and $V_1 = [V, U^*]$.

(2) $I = LU$.

(3) $S := O_2(I) = WW^gW^h$.

(4) S has an I-series

$$1 =: S_0 \leq S_1 \leq S_2 \leq S_3 \leq S_4 := S,$$

with $S_1 := V \leq Z(S)$, $S_2 := U \cap U^g \cap U^h$, and $S_3 := (W \cap W^g)(W \cap W^h)(W^g \cap W^h)$.

(5) $[S_2, I] \leq S_1$.

(6) $S_3/S_2 = (W \cap W^g)/S_2 \oplus (W \cap W^h)/S_2 \oplus (W^g \cap W^h)/S_2$ is the sum of s copies of the dual of V as an $\mathbf{F}_2L$-module for some nonnegative integer s.

(7) $S/S_3 = W/S_3 \oplus W^g/S_3 \oplus W^h/S_3$ is the sum of r copies of V as an $\mathbf{F}_2L$-module for some nonnegative integer r.

(8) $|U| = 2^{n(2s+r+5)}|S_2 : S_1|$, and L_1 has $r + s + 2$ noncentral 2-chief factors, each of which is a natural $L_2(2^n)$-module.

PROOF. The proof is similar to that of G.2.3, and is essentially contained in 8.16 in [**Asc94**] or 17.7 in [**Asc86b**]; still we sketch a proof here for completeness. Let $F := \mathbf{F}_{2^n}$. The proofs of (1)–(5) are essentially the same as those in G.2.3, so we will simply mention a few differences: This time $N_L(V_1)$ is not a Borel subgroup, but the maximal parabolic stabilizing the point V_1, so U^* is the unipotent radical of that parabolic. Arguments based on the pair U, U^g are now extended to the triple U, U^g, U^h, and lead to an extra term in the series in (4) defined by the intersection of pairs from the triple. The L-invariance of S_3 also emerges in the details provided below.

It remains to establish (6)–(8). Adopt the notation of section G.1 in discussing the action of L^* on V. In particular if (A, B) is an incident F-point-line pair in V, let $R^*_{A,B}$ be the root group of transvections in L^* with center A and axis B. For example U^* is partitioned by the root groups $R^*_{V_1,B}$ as B ranges over the lines of V through V_1. Write $\hat{V}$ for the dual space of V; then $R^*_{A,B} = R^*_{\hat{B},\hat{A}}$ where for $C \leq V$, $\hat{C}$ is the annihilator of C in $\hat{V}$. Let $\hat{\Lambda}$ be the set of F-points of $\hat{V}$.

Let $\hat{S} := S/S_2$, $W_1 := W$, $W_2 := W^g$, $W_3 := W^h$, and $W_{i,j} := W_i \cap W_j$, so that $S_3 = W_{1,2}W_{1,3}W_{2,3}$. Define $\varphi : \hat{\Lambda} \to PG(\hat{S}_3)$ by $\varphi(\hat{D}) := \hat{W}_{1,2}$ for $D := V_1 + V_1^g$, and extend to an L^*-equivariant map. This makes sense as U^* and U^{g*} centralize $\hat{W}_{1,2}$, so that $O^{2'}(N_{L^*}(D)) = \langle U^*, U^{g*} \rangle$ centralizes $\hat{W}_{1,2}$, and then

$$N_L(D^*) = O^{2'}(N_L(D))^*(N_L(V_1)^* \cap N_L(V_1^g)^*)$$

acts on $\hat{W}_{1,2}$. Similarly

$$R^*_{\hat{D},\hat{V}_1} = R^*_{V_1,D} \leq U^* \leq C_L(\hat{W}_{1,2})^* \cap C_L(\hat{W}_{1,3})^*$$

and $R^*_{V_1,D} \leq N_L(V_2)^* \leq N_L(W_2)^*$, so

$$[W_{2,3}, R^*_{V_1,D}] \leq W_2 \cap [U,S] \leq W_2 \cap W_1 = W_{1,2}.$$

Therefore, in the notation of G.1.2, $[\hat{W}_k, R^*_{i,j}] = 0$ for $k \neq j$ and $[\hat{W}_j, R^*_{i,j}] = \hat{W}_i$, so $L^* = \langle R^*_{i,j} : i,j \rangle$ acts on $\hat{S}_3$. Then the restriction $\varphi : \hat{\Lambda} \to PG(\hat{S}_3)$ satisfies the hypotheses of Theorem G.1.3 using Remark G.1.4, so G.1.3 implies (6).

Similarly let $\bar{S} = S/S_3$ and observe Theorem G.1.3 implies (7), when we define $\varphi : \Lambda \to PG(\bar{S})$ by $\varphi(V_1) := \bar{W}$ and extend to an L^*-equivariant map. Finally (1)–(7) imply (8). □

LEMMA G.2.6. *Relax the assumptions in Hypothesis G.2.1 that L_1T is irreducible on $\tilde{V}$, and assume instead that G.2.2 holds. Assume $L^* \cong E_9$, V_L is the $\Omega_4^+(2)$-module for $L^*U^* = \Omega_4^+(2)$, V_1 is of order 2, and $V := V_1^\perp$ is the hyperplane of V_L orthogonal to V_1. Let $g \in L$ with $V_1^g \not\leq V$ and set $I := \langle U, U^g \rangle$ and $W := U \cap O_2(I)$. Then*

(1) $I = LU$.

(2) $S := O_2(I) = WW^g$.

(3) S has an I-series

$$1 =: S_0 \leq S_1 \leq S_2 \leq S_3 := S$$

with $S_1 := V_L \leq Z(S)$ and $S_2 := V_L(U \cap U^g)$.

(4) $[S_2, I] \leq S_1$.

(5) $L^*U^* = M_1^* \times M_2^*$ with $M_i^* \cong S_3$, and $S/S_2 = S^1/S_2 \oplus S^2/S_2$, where $S^i/S_2 = [S, M_i]S_2/S_2 = C_{S/S_2}(M_{3-i})$ is the sum of s natural modules for M_i^* for some nonnegative integer s.

PROOF. As V_L is the natural module for $L^*U^* = \Omega_4^+(2)$, and V_1 is a T-invariant point of V_L, V_1 is a singular point in V_1, and $U^* = C_{L^*U^*}(V_1)$. As V_1^g is not orthogonal to V_1, $I^* = L^*U^*$. Then the proof of (1) is exactly like that of G.2.3.2.

Set $P := WW^g$, $S_1 := V_L$, and $S_2 := V_L(U \cap U^g)$. Arguing as in the proof of G.2.3, with V_L in the role of "V", $P \trianglelefteq I$, $[I,S] \leq P$, and $[I, S_2] \leq S_1$. Set $I^+ := I/P$. Then $S_2^+ \leq Z(I^+)$, $U^+ \cong U^* \cong E_4$ since $U \cap O_2(I) \leq P$, and $L^+ \cong L^* \cong E_9$ as $[S,I] \leq P$ and $L = O^2(L)$. Then as $g \in L$, $I^+ = \langle U^+, U^{g+} \rangle = U^+L^+$, so $S^+ = O_2(I^+) = 1$, and hence $S = P$. This establishes (2)–(4).

Let $\hat{I} := I/S_2$. Then $\hat{S} = \hat{W} \oplus \hat{W}^g$ and $[\hat{S}, U^*] \leq \hat{W} \leq C_{\hat{S}}(U^*)$, so U^* is quadratic on $\hat{S}$. Now $L^*U^* = M_1^* \times M_2^*$ with $M_i^* \cong S_3$. Let $L_i := O^2(M_i)$ and $\langle u_i^* \rangle = U^* \cap M_i^*$. Now $S^1 := [\hat{S}, L_1] = [\hat{S}, L_1, u_1] \oplus [\hat{S}, L_1, u_1^l]$ for some $l^* \in L_1^{*\#}$. Further u_2 centralizes $[\hat{S}, L_1, u_1]$ by the quadratic action, and u_2 centralizes M_1^*, so u_2 centralizes $\hat{S}^1$. Thus $M_2^* = \langle u_2^{*L_2} \rangle$ centralizes $\hat{S}^1$. Therefore as $C_{\hat{S}}(L) = 1$ since $L = O^2(L)$, We conclude that $\hat{S} = \hat{S}_0 \oplus C_{\hat{S}}(L)$, where $\hat{S}_0 := \hat{S}^1 \oplus \hat{S}^2$ and $C_{\hat{S}_0}(U) = [S_0, U]$. Also $\hat{S} = \hat{S}_0\hat{W}$, so as $[\hat{S}, U] \leq \hat{W} \leq C_{\hat{S}}(U)$, $C_{\hat{S}}(U) = \hat{W}$. Then as $\hat{W} \cap \hat{W}^g = 0$, $\hat{S} = \hat{S}_0$ and $\hat{S}^i = C_{\hat{S}}(M_{3-i})$. As $\hat{S}^i = [\hat{S}, L_i]$, $\hat{S}^i$ is the sum of natural modules for M_i^*, completing the proof of (5). □

G.3. Some groups generated by transvections

In this section we prove a result on groups containing a subgroup of rank at least 2 consisting of $\mathbf{F}_2$-transvections with a common center. The result may well be in the literature in some form; at least it can be easily derived from results in the literature. As usual in such cases, for completeness we provide a proof.

LEMMA G.3.1. *Let V be an $\mathbf{F}_2$-space and $G \leq GL(V)$. For X a point in V, let $A(X)$ be the group of $\mathbf{F}_2$-transvections in G with center X. Assume Δ_0 is an orbit of G on points such that $m(A(X)O_2(G)/O_2(G)) > 1$ for $X \in \Delta_0$. Set $V_0 := \langle \Delta_0 \rangle$ and $G_0 := \langle A(X) : X \in \Delta_0 \rangle$. Let $\Delta_1, \ldots, \Delta_s$ denote the orbits of G_0 on Δ_0, $V_i := \langle \Delta_i \rangle$, and $G_i := \langle A(X) : X \in \Delta_i \rangle$. Then*

(1) $V_0 = V_1 \oplus \cdots \oplus V_s$, $G_0 = G_1 \times \cdots \times G_s$, $[G_i, V_j] = 0$ for $1 \leq i, j \leq s$ with $i \neq j$, and G is transitive on $\{G_i : 1 \leq i \leq s\}$ and $\{V_i : 1 \leq i \leq s\}$.

(2) $Aut_{G_1}(V_1) = GL(V_1)$, Δ_1 is the set of points in V_1, $C_{G_1}(V_1)$ is an elementary abelian 2-group, and $m(A(X)) = \dim(V_1) - 1 + m(C_{A(X)}(V_1))$.

(3) $\dim V_1 \geq 3$.

PROOF. For $\Gamma \subseteq \Delta_0$, set $K_\Gamma := \langle A(X) : X \in \Gamma \rangle$ and $W_\Gamma := \langle \Gamma \rangle$. For $X \in \Gamma$, $[V, A(X)] \leq W_\Gamma$, so K_Γ acts on W_Γ and centralizes V/W_Γ. Thus $C_{K_\Gamma}(W_\Gamma)$ is quadratic on V, and hence is elementary abelian.

Define $\mathcal{S}$ to be the set of $\Gamma \subseteq \Delta_0$ such that K_Γ acts on Γ and $Aut_{K_\Gamma}(W_\Gamma) = GL(W_\Gamma)$. Observe that if $\Gamma \in \mathcal{S}$, then the triple Γ, K_Γ, W_Γ satisfies the conclusions of (2) in the roles of "Δ_1, G_1, V_1". This follows from the discussion in the previous paragraph, and the following remark: For $X \in \Gamma$, $A(X)$ is $N_G(X)$-invariant with $Aut_{A(X)}(W_\Gamma) \neq 1$, so as $P := N_{GL(W_\Gamma)}(X)$ is irreducible on $O_2(P) = C_{GL(W_\Gamma)}(W_\Gamma/X)$, it follows that $Aut_{A(X)}(W_\Gamma) = O_2(P)$ is of rank $\dim(W_\Gamma) - 1$.

Let $X, Y \in \Delta_0$. If $X \leq C_V(A(Y))$, then $A(Y)$ acts on $A(X)$, so as $A(X)$ and $A(Y)$ are TI-sets in G and $|A(X)| = |A(Y)|$ as X and Y are conjugate, $[A(X), A(Y)] = 1$ by I.6.2.1. Then $A(X)$ acts on $[V, A(Y)] = Y$, so also $Y \leq C_V(A(X))$. Thus if $X \not\leq C_V(A(Y))$, then by symmetry also $Y \not\leq C_V(A(X))$. Set $K := K_{\{X,Y\}}$ and $W := W_{\{X,Y\}}$. Then $Aut_K(W) = GL(W) \cong L_2(2)$, so $X^K \in \mathcal{S}$ by paragraph two.

We now prove (1). Since G is transitive on Δ_0, it is transitive on the orbits $\{\Delta_1, \ldots, \Delta_s\}$ of G_0 on Δ_0, and hence also transitive on $\{V_i : i\}$ and $\{G_i : i\}$. As $\Delta_0 = \bigcup_i \Delta_i$, $V_0 = \langle V_i : 1 \leq i \leq s \rangle$. If $i \neq j$, then $X \in \Delta_i$ is not G_0-conjugate to $Y \in \Delta_j$, so it follows from the discussion above that G_i centralizes V_j and G_j. In particular $W_1 := V_1 \cap \langle V_i : i > 1 \rangle \leq C_{V_1}(G_1)$. Once we have established (2), we will know that G_1 induces $GL(V_1)$ on V_1, so $C_{V_1}(G_1) = 0$, and hence $V_0 = V_1 \oplus \cdots \oplus V_s$. Similarly $[V, G_1 \cap \langle G_i : i > 2 \rangle] \leq W_1 = 0$, so that $G_0 = G_1 \times \cdots \times G_s$, completing the proof of (1) modulo (2).

Thus it remains to prove (2), so without loss $G_1 = G_0 = G$, and so $\Delta_1 = \Delta_0 =: \Delta$. The proof is by induction on $|\Delta|$. For $X \in \Delta$, $A(X) \not\leq O_2(G)$ by hypothesis, so by earlier discussion there is $Y \in \Delta$ with $[A(X), A(Y)] \neq 1$, $\Sigma := X^{\langle A(X), A(Y) \rangle} \in \mathcal{S}$, and $|\Sigma| = 3$. Pick $\Gamma \in \mathcal{S}$ of maximal order. If $\Gamma = \Delta$, then (2) holds as $\Gamma \in \mathcal{S}$. Thus we may assume that $\Gamma \neq \Delta$.

If $W := W_\Gamma \leq C_V(Y)$ for each $Y \in \Delta - \Gamma$, then the earlier discussion shows that $\langle A(Y) : Y \in \Delta - \Gamma \rangle$ centralizes $K := K_\Gamma$; then $K \trianglelefteq \langle \Delta \rangle = G$, contradicting transitivity of G on Δ. Thus there exist $X \in \Gamma$ and $Y \in \Delta - \Gamma$ with $X \not\leq C_V(Y)$. Set $I := W + Y$, $H := \langle K, A(Y) \rangle$, and $\theta := X^H$. By paragraph one, H acts on

I; set $H^* := H/C_H(I)$ and $N^* := N_{GL(I)}(W)$. As $\Gamma \in \mathcal{S}$, $Aut_K(W) = GL(W)$, so $N^* = N_H(W)^* O_2(N^*)$ and $N_H(W)$ acts irreducibly on the unipotent radical $O_2(N^*)$. Hence either $N^* = N_{H^*}(W)$, or $N_{H^*}(W) = K^*$ is a complement to $O_2(N^*)$ in N^*.

Assume that $N_{H^*}(W) = N^*$. Then $N_{H^*}(W)$ is maximal in $GL(I)$, so as $A(Y)$ does not act on W, we conclude that $H^* = GL(I)$ and hence $\theta \in \mathcal{S}$, contradicting the maximality of $|\Gamma|$. Therefore $N_{H^*}(W) = K^*$. Thus $m(A(X)^*) = m(W) - 1 = m(I) - 2$, so

$$m(C_I(A(X))) = 2.$$

Suppose that $|\Gamma| = 3$, so that $m(W) = 2$. Then $m(A(X)/C_{A(X)}(W)) = 1 = m(I/W)$, so as $m(A(X)O_2(G)/O_2(G)) > 1$ by hypothesis, there is $1 \neq a \in C_{A(X)}(W)$ with $a \notin O_2(G)$. But if $\Delta - \Gamma \subseteq C_V(a)$, then $a \in C_G(W) \cap C_G(V/W) \leq O_2(G)$, contrary to our choice of a. Therefore we can pick Y so that $Y \not\leq C_V(a)$. Thus $a \notin C_H(I)$. But a centralizes W and I/W, so that $1 \neq a^* \in O_2(K^*)$ by Coprime Action, contradicting $K^* \cong GL(W) \cong L_2(2)$.

Therefore $|\Gamma| > 3$, so that $m(W) > 2$. As K^* is irreducible on W and as $A(Y)$ does not act on W, or on $C_V(K)$ if $C_V(K) \neq 0$, H^* is irreducible on I. If $m(W) = 3$, then H^* is an irreducible subgroup of $GL_4(2)$ containing an $L_3(2)$-subgroup K^* generated by transvections, so that $H^* = GL(I)$, contradicting the maximality of $|\Gamma|$.

Thus $m(W) > 3$. Let W_1 be a hyperplane of W containing X, $I_1 := W_1 + Y$, $H_1 := \langle A(Y), A(X_1) : X_1 \leq W_1 \rangle$, and $\Gamma_1 := X^{H_1}$. As $m(W) > 3$,

$$m(A(X)/C_{A(X)}(W_1)) = m(W_1) - 1 \geq 2.$$

Thus by induction on $|\Delta|$, (2) holds for the triple Γ_1, H_1, I_1, so that $Aut_{H_1}(I_1) = GL(I_1)$ and $X = C_{I_1}(A(X))$. However each point of I is contained in some I_1, so we conclude $X = C_I(A(X))$. This contradicts our earlier remark that $m(C_I(A(X))) = 2$, and completes the proof of (2).

Finally (3) follows from our hypothesis that $m(AO_2(G)/O_2(G)) > 1$, because $m_2(L_2(2)) = 1$. This completes the proof of G.3.1. □

G.4. Some subgroups of $\mathbf{Sp_4(2^n)}$

In this section, V is a 4-dimensional symplectic space over $F := \mathbf{F}_{2^n}$, and $\hat{G} := Sp(V)$. We establish a result needed for E.2.7:

THEOREM G.4.1. *Let G_0 be a $\mathcal{K}$-subgroup of $\hat{G} = Sp(V)$, and assume G_0 is irreducible on V and the set D of F-transvections in G_0 is nonempty. Let $G := \langle D \rangle$. Then one of the following holds:*

(1) $G = G_0 = C_{\hat{G}}(\sigma)$ for some field automorphism σ of $\hat{G}$. Thus $G \cong Sp_4(2^k)$ for some divisor k of n.

(2) $G_0 = G$ preserves a quadratic form Q on V, and $G = C_{O(V,Q)}(\sigma)$ for some field automorphism σ of $\hat{G}$. Thus $G \cong O_4^{\epsilon}(2^k)$ for some divisor k of n and $\epsilon := \pm 1$.

(3) G_0 preserves a decomposition $V = V_1 \oplus V_2$ of V, where V_1 is a nondegenerate projective line, $V_2 := V_1^{\perp}$, $G_0 = G\langle t \rangle$ where t is an involution with $V_1^t = V_2$ and $G = G_1 \times G_2$, where $G_i := \langle C_D(V_{3-i}) \rangle$ is isomorphic to $L_2(2^k)$ or D_{2m} for some divisor k of n, or m of $2^n \pm 1$.

REMARK G.4.2. In [**McL67**] and [**McL69**], McLaughlin considers subgroups G of $GL(V)$ generated by root groups of transvections, without the assumption that

$\dim(V) = 4$ or that G preserves a symplectic form. In [**Kan79**], Kantor considers the general case. To keep our treatment self-contained, we sketch a proof in the case of subgroups of $Sp_4(2^n)$.

Notice as G_0 is faithful and irreducible on V and $G \trianglelefteq G_0$, that

LEMMA G.4.3. $O_2(G_0) = 1 = O_2(G)$. *In particular G is not abelian.*

DEFINITION G.4.4. Let $\hat{D}$ denote the set of transvections in $\hat{G}$. For $d \in \hat{D}$, let $\hat{R}_d$ be the root group of d in $\hat{G}$, $R_d := \hat{R}_d \cap G$, and $V_d := [V, d]$ the center of d. Let $D_d := C_D(d) - R_d$ and $A_d := D - C_D(d)$. For $J \subseteq \hat{D}$, let $L_J = \langle J \rangle$ and $V_J := \langle V_j : j \in J \rangle$. If P is a projective point in V, then $P := V_d$ for some $d \in \hat{D}$, and we set $\hat{X}_P := \langle C_{\hat{D}}(d) \rangle$, $\hat{X}_P^+ := \hat{X}_P / O_2(\hat{X}_P)$, and $X_P := \langle D_d \rangle$. Let $\mathcal{A}$ denote the graph on D in which a is adjacent to b iff $b \in A_a$. Let Σ denote the set of field automorphisms of $\hat{G}$. Let $\mathcal{I}$ consist of all involutions in G of the form ab, where $a \in D$ and $b \in D_a$.

Observe that as G is not abelian by G.4.3, $A_d \neq \emptyset$. Also $\hat{X}_P = O^{2'}(\hat{Q})$ for $\hat{Q}$ a maximal parabolic of $\hat{G}$, so $\hat{X}_P^+ \cong L_2(2^n)$, and $O_2(\hat{X}_P)$ is the 3-dimensional orthogonal module (nonsplit when $n > 1$) for $L_2(2^n)$ described in I.2.3.1, with $Z(\hat{X}_P) = \hat{R}_d$.

The following facts are well known and easy:

LEMMA G.4.5. *(1) D is a set of odd transpositions of G; that is, $|ab| = 2$ or $|ab|$ is odd for each $a, b \in D$.*

(2) If $a \in D$ and $b \in A_a$, then $V_{a,b}$ is a nondegenerate line.

(3) If $a \in D$ and $b \in D_a$, then $V_{a,b}$ is a totally singular line, and $C_{\hat{G}}(ab) \in Syl_2(\hat{G})$.

LEMMA G.4.6. *The following are equivalent:*

(1) $\mathcal{A}$ is disconnected.

(2) G is not transitive on D by conjugation.

(3) G_0 is imprimitive on V.

(4) Case (3) of Theorem G.4.1 holds.

PROOF. Let $a \in D$. By G.4.5, D is a set of odd transpositions of G; thus an elementary argument shows that (1) and (2) are equivalent; cf. Exercise 2.2.1 in [**Asc97**].

Assume that $V = V_1 \oplus \cdots \oplus V_r$ is a G_0-invariant decomposition of V with $r > 1$. Then $r = 2$ or 4 and $\dim(V_i) = 4/r$.

Suppose first that $r = 4$. Then a has a cycle (V_1, V_2) on $X := \{V_1, \ldots, V_4\}$, so as a is a transvection, $V_a \leq V_1 + V_2 =: U$ and $V_1 \not\leq V_a^\perp$, so U is nondegenerate. Then a centralizes $V_3 + V_4$, so a induces a transposition on X. As G_0 is irreducible on V, G_0 is transitive on X, so G_0 induces S_4 or D_8 on X. As $O_2(G_0) = 1$ by G.4.3, $F^*(G_0)$ is contained in the kernel K of the action of G_0 on X. As K is abelian of odd order and $[K, a] \neq 1$, there is $b \in A_a \cap aK$. As $\langle a \rangle K$ acts on U and on $V_3 + V_4$, $U = V_{a,b}$ and $V_3 + V_4 = U^\perp$ by G.4.5.2. Thus $\{U, U^\perp\}$ is also a system of imprimitivity for G_0, with two summands, so we may take $r = 2$. But now $\dim(V_1) > \dim(V_a)$, so a acts on V_1 and $V_a \leq V_i$ for $i = 1$ or 2. Then the connected components of $\mathcal{A}$ are $D_i := \{d \in D : V_d \leq V_i\}$, so (3) implies (1).

Visibly (4) implies (2) and (3), so it remains to assume (1) and show that (4) holds. Let D_1 be a connected component of $\mathcal{A}$, $D_2 := D - D_1$, $G_i := \langle D_i \rangle$, and

$a \in D_1$. Recall there exists $b \in A_a$. Let $V_1 := V_{a,b}$ and $V_2 := V_1^{\perp}$; by G.4.5, V_1 is a nondegenerate line and $V_{D_2} \leq V_2$. Let $d \in D_2$; by symmetry there is $e \in A_d$ and $V_{d,e}$ is a line contained in $V_{D_2} \leq V_2$, so $V_2 = V_{d,e}$. Thus $G_2 = \langle C_D(V_1) \rangle$, and by symmetry $G_1 = \langle C_D(V_2) \rangle$. As G_0 is irreducible on V, there exists $t \in G_0$ interchanging G_1 and G_2. From the structure of $Sp(V_i)$, G_i is described in case (3) of Theorem G.4.1. Thus G_i is self-normalizing in $C_{\hat{G}}(V_{3-i}) \cong L_2(2^n)$, so $G_1 \times G_2 = N_{G_0}(V_1)$, completing the verification of case (3) of Theorem G.4.1. □

By lemma G.4.6 we may assume that $\mathcal{A}$ is connected. Thus by G.4.6, G is transitive on D and G_0 is primitive on V.

Lemma G.4.7. $G_0 = G$.

Proof. Suppose U is a nonzero G-submodule of V. As G_0 is irreducible on V and $G \trianglelefteq G_0$, $C_V(G) = 0$ and $V = [V, G]$. Thus $[U, a] \neq 0$ for some $a \in D$, so $V_a \leq U$. Then as G is transitive on D, $V_D \leq U$. Thus $G = \langle D \rangle$ centralizes V/U, so as $V = [V, G]$, $U = V$; that is, G is irreducible on V. Therefore if $G_0 > G$, then by induction on the order of G_0, G is described in (1) or (2) of Theorem G.4.1. But then G is self-normalizing in $\hat{G}$, so $G_0 = G$. □

Lemma G.4.8. *Assume that $a \in D$ and H is an a-invariant subgroup of G irreducible on V. Let $G_1 := H\langle a \rangle$, I an irreducible $\mathbf{F}_2G_1$-submodule of V, $F_0 := End_{\mathbf{F}_2G_1}(I)$, and write V_0 for I regarded as an F_0G_1-module. Then F_0 is a subfield of F, $F_0 = End_{\mathbf{F}_2H}(V_0)$, V_0 is an absolutely irreducible self-dual F_0H-module, $V = V_0 \otimes_{F_0} F$, $G_1 \leq C_{\hat{G}}(\sigma) \cong Sp_4(F_0)$ for some field automorphism σ of $\hat{G}$ determined by a generator of $Gal(F/F_0)$, and, up to scalar multiplication, G_1 preserves a unique symplectic form on V.*

Proof. As H is irreducible on V, V is a homogeneous $\mathbf{F}_2H$-module, and hence I is also an irreducible H-module and there is a G_1-chief series $\mathcal{S}$ on V in which each factor is $\mathbf{F}_2H$-isomorphic to I. Let $E := \mathrm{End}_{\mathbf{F}_2H}(I)$. Then either $E = F_0$, or a induces an automorphism of order 2 on E and $m([I, a]) = m(I)/2$. However in the latter case, the existence of $\mathcal{S}$ says that $m([V, a]) = m(V)/2$, impossible as a induces a transvection on V. Therefore $E = F_0$.

Let $F_1 := F \cap F_0$, and let V_1 be I regarded as an F_1G_1-module. By 26.4 in [**Asc86a**], $V = V_1^F = V_1 \otimes_{F_1} F$, so a induces an F_1-transvection on V_1. Thus as $[V_1, a]$ is a 1-dimensional F_0-subspace of V_1, $F_0 = F_1 \leq F$ and $V_0 = V_1$. Then by Theorem 26.6 in [**Asc86a**], V_0 is an absolutely irreducible F_0G-module and V_0 can be written over no proper subfield of F_0. As $V = V_0^F$, G is contained in $C_{GL(V)}(\sigma)$ for a suitable field automorphism σ of $GL(V)$ determined by a generator $\bar{\sigma}$ of $Gal(F/F_0)$.

As $G \leq Sp(V)$, V is isomorphic to its dual V^* as an FG-module, so $V_0^F \cong (V_0^F)^* \cong (V_0^*)^F$, and hence $V_0^* \cong V_0^{\tau}$ for some $\tau \in Aut(F_0)$ by 26.6 in [**Asc86a**]. Let F_2 be the fixed field of τ and ρ the lift of τ to F with fixed field F_2. Then $V \cong V^* = (V_0^{\tau})^F \cong (V_0^F)^{\rho} = V^{\rho}$, so V_0 can be written over F_2 by 26.3 in [**Asc86a**]. Thus $F \cong F_2$ and hence $V_0 \cong V_0^*$.

As V_0 is self-dual, G preserves a symplectic form f on V_0. Then f extends to an F-form f^K on $V = V_0^F$ preserved by G. As V is absolutely irreducible, f^K is unique up to scalar multiplication, so $\hat{G}$ is the isometry group of $f^K\bar{\sigma}$ and f^K. Thus σ acts on $\hat{G}$, completing the proof. □

LEMMA G.4.9. *Either*

(1) $F^*(G)$ is a nonabelian simple group containing a member of $\mathcal{I}$, and irreducible on V; or

(2) $F^*(G) \cong \Omega_4^+(2^k)$ for some $k > 1$ dividing n, and case (2) of Theorem G.4.1 holds.

PROOF. Let $a \in D$. We first establish a preliminary result: Suppose that X is a normal subgroup of G which is irreducible on V and not in the center of G. We claim that there is $e \in D_a \cap aX$ so that $ae \in \mathcal{I}$: Assume the claim fails. As $G = \langle D \rangle$ is transitive on D, while X is normal but not central in G, $[X, a] \neq 1$. Thus as $O_2(G) = 1$, by the Baer-Suzuki Theorem there exists $b \in A_a \cap a^X$. As X is irreducible on V, there is $c \in a^X$ with $V_c \not\leq V_{a,b}$. Then $V_{a,b,c}$ is of F-dimension 3, so $P := V_{a,b,c}^{\perp}$ is a point of V. Let $L := L_{a,b,c}$ and $d \in \hat{D}$ with $P = V_d$; then $L \leq \hat{X}_P$. Recall the definition of $\hat{X}_P^+$ from G.4.4, and that $\hat{X}_P^+ \cong L_2(2^n)$. As $L_{a,b} \cong D_{2m} \cong L_{a,b}^+$ for some odd $m > 1$, Dickson's Theorem A.1.3 says:

(a) L^+ is isomorphic either to D_{2m_0} for some odd m_0, or $L_2(2^k)$ for some $k > 1$, and

(b) each $L_{a,b}$-invariant 2-subgroup of $\hat{X}_P$ is contained in $O_2(\hat{X}_P)$.

By (b), $O_2(L) \leq O_2(\hat{X}_P)$, so as $\hat{R}_a \cap O_2(\hat{X}_P) = 1$, $[O_2(L), a]^{\#} \subseteq \mathcal{I}$, and therefore $[O_2(L), a] = 1$ by our assumption. Hence $O_2(L) \leq C_{O_2(\hat{X}_P)}(L_{a,b}) = \hat{R}_d \leq C_G(L)$. If $d \in D \cap aX$ then $ad \in \mathcal{I} \cap X$, again contrary to assumption; thus

(c) $O_2(L) \cap aX \subseteq \hat{R}_d \cap a(X \cap L) = \emptyset$.

By (c), $c \notin O_2(L)$, so by (a), c lies in A_a or A_b. Thus a, b, c are fused under $O^2(L)$, so $L = \langle a \rangle O^2(L)$. As $|L : L \cap X| \leq 2$, $O^2(L) \leq X$, so (c) says that $O_2(L) \leq O^2(L)$. As $\dim(V_{a,b,c}) = 3$, L is not dihedral, so L^+ is not dihedral. Thus $L^+ \cong L_2(2^k)$ for some $k > 1$ by (a). If $R_a \cap L \in Syl_2(L)$, then $V_{a,b,c} \leq [V, R_a] + [V, R_b] = V_{a,b}$, a contradiction. Thus there is $e \in D_a \cap L$, and hence $ae \in \mathcal{I}$, establishing the claim.

We now turn to the proof of the lemma. Applying the claim to G in the role of "X", we conclude that there is $e \in D_a$, so that $i := ae \in \mathcal{I}$.

Suppose first that $W := O_p(G) \neq 1$ for some odd prime p. By G.4.5.3, $C_G(i)$ is a 2-group, so i inverts W. Thus if W is cyclic, W is centralized by a or e, and hence centralized by $G = \langle D \rangle$ as G is transitive on D. Therefore W is noncyclic. But the Sylow p-subgroups of G are abelian, so W is noncyclic abelian and hence not homogeneous on V, contradicting our assumption that G is primitive on V.

Therefore $O_p(G) = 1$ for each odd prime p, while $O_2(G) = 1$ by G.4.3. Thus $H := F^*(G)$ is the product of simple components. Let K be a component of G and set $G_1 := \langle K, a \rangle$.

Suppose first that $K^a \neq K$. Then $C_{KK^a}(a) =: J \cong K$, so $K \cong L_2(2^k)$ for some $k > 1$ dividing n, since these are the only nonabelian simple sections of $C_{\hat{G}}(a)$. Further for j an involution in J, $aj \in D_a$, so $j \in \mathcal{I}$. Thus $C_G(j)$ is a 2-group, so $H = KK^a$. As G is primitive on V, V is a homogeneous H-module, so if H is not irreducible on V, then V is the sum of two faithful 2-dimensional irreducible FH-modules, impossible as $GL_2(F)$ contains no subgroup isomorphic to H. Thus H is irreducible on V, so by G.4.8, $V = V_0^F = V_0 \otimes_{F_0} F$ for some subfield F_0 of F and $\mathbf{F}_2 G_1$-submodule V_0 satisfying the various restrictions in G.4.8. As V_0 is a homogeneous F_0K-module, V_0 is the sum of two isomorphic natural $\mathbf{F}_2K$-modules, so $F_0 = \mathrm{End}_{F_2H}(V_0) = \mathbf{F}_{2^k}$. Further as an F_0H-module, V_0 is the tensor product of natural modules for K and K^a, so V_0 is the orthogonal module for $G_1 \cong \Omega_4^+(2^k)$. Thus $G_1 \trianglelefteq N_{GL(V_0)}(H)$, so as $G = \langle D \rangle$, $G = G_1$. Moreover the representation of G on V is determined up to quasiequivalence; and by G.4.8, up to scalar multiplication, G preserves a unique symplectic form on V, so $\hat{G}$ is transitive on its subgroups whose representation is quasiequivalent to that of G_1. Now for Q a quadratic form of Witt index 2 on V, and σ a field automorphism of $\hat{G}$ determined by a generator of $Gal(F/F_0)$, $C_{O(V,Q)}(\sigma)$ is such a subgroup, so case (2) of Theorem G.4.1 holds, and hence also conclusion (2) of our lemma.

Thus we may suppose instead that a acts on each component of G, so as $G = \langle D \rangle$, $K \trianglelefteq G$. If K is not irreducible on V, then V is the sum of two 2-dimensional FK-modules, so $K \cong L_2(2^k)$ and the modules are natural. But then $m(V_a) = m(V)/2$, contrary to $\dim_F(V_a) = 1$. Therefore K is irreducible on V, so by the claim with K in the role of "X", K contains $j \in \mathcal{I}$. Then as $C_G(j)$ is a 2-group by G.4.5.3, $K = F^*(G)$, and hence (1) holds. □

We can now complete the proof of Theorem G.4.1. Let $a \in D$, $H := F^*(G)$, and $G_1 := H\langle a \rangle$. Then G_1 is an SQTK-group with $F^*(G_1) = H$. By G.4.9 we may assume that H is simple and irreducible on V, and there is $i \in \mathcal{I} \cap H$. By G.4.5.3, $C_{G_1}(i)$ is a 2-group. Inspecting the centralizers of involutions in the groups in Theorem C (A.2.3) (cf. 16.1.4 and 16.1.5 where most cases are considered), we conclude that $H \cong L_2(p)$ for p a Fermat or Mersenne prime, $L_2(2^r)$, $Sz(2^r)$, $Sp_4(2^r)$, or $L_3(4)$. By G.4.8, $V = V^F = V \otimes_{F_0} V_0$ for a subfield F_0 and F_0G-module V_0 satisfying the restrictions of that lemma. In particular V_0 is an absolutely irreducible 4-dimensional H-module on which a induces a transvection, so we conclude that either

(i) $G_1 \cong Sp_4(2^k)$, $F_0 = \mathbf{F}_{2^k}$, and V_0 is the natural H-module, or

(ii) $H \cong L_2(2^{2k})$, a induces a field automorphism on H, $F_0 = \mathbf{F}_{2^k}$, and V_0 is the orthogonal module for $G_1 \cong O_4^-(2^k)$.

Then arguing as in the proof of lemma G.4.9, Theorem G.4.1 holds.

G.5. $\mathbf{F_2}$-modules for $\mathbf{A_6}$

The results of this section are known, and can be found in various places in the literature. However for the convenience of the reader and to maintain a self-contained treatment, we provide the easy proofs here.

In this section $G := A_6$, $T \in Syl_2(G)$, and G_1 and G_2 are the two subgroups of G of index 15 containing T. Let $F := \mathbf{F}_2$ be the field of order 2, and $S_6 \cong G_0 \leq Aut(G)$.

LEMMA G.5.1. *(1) There are four irreducible FG-modules, denoted* 1, 4_1, 4_2, *and* 16.

(2) The module 4_i is the 4-dimensional irreducible on which G_i fixes a point and G_{3-i} fixes a line.

(3) The module 16 is the restriction V of the 16-dimensional Steinberg module for G_0 to G, and $\mathbf{F}_4 \otimes_F V = U_1 \oplus U_2$, where U_1 is an 8-dimensional $\mathbf{F}_4G$-module, and $U_2 = U_1^t = U_1^\sigma$ for $t \in G_0 - G$, and $\langle \sigma \rangle = Gal(\mathbf{F}_4/F)$.

PROOF. As in H.6.1, it follows from the list on page 77 of [**GLS98**] that the F-irreducibles for $G_0 \cong Sp_4(2)$ may be denoted by 1, 4_1, 4_2, and the projective Steinberg module $V_0 \cong 4_1 \otimes 4_2$. Further 4_i restricts to an absolutely irreducible

FG-module, so (2) holds. Write V for the restriction of V_0 to G. By Clifford's Theorem, either V is an irreducible FG-module or $V = V_1 \oplus V_2$ is the sum of two irreducible FG-modules interchanged by members of $G_0 - G$. Moreover another application of Clifford's Theorem tells us that each irreducible FG-module not in $\Delta := \{1, 4_1, 4_2\}$ is isomorphic to a summand of V.

Let K be a minimal splitting field for G over F, and for D an FG-module let $D_K := K \otimes_F D$. Since G has 5 conjugacy classes of elements of odd order, the set Γ of irreducible KG-modules is of order 5; and as the members of Δ are absolutely irreducible, $\Delta_K := \{D_K : D \in \Delta\}$ is a subset of Γ of order 3. Thus either

(a) V is irreducible, $K = \mathbf{F}_4$, and $V_K = U_1 \oplus U_2$ with $\dim_K(U_1) = 8$, $U_2 = U_1^\sigma$, and $\Gamma - \Delta_K = \{U_1, U_2\}$; or

(b) V is not irreducible, V_1 is absolutely irreducible, $K = F$, and $\Gamma - \Delta = \{V_1, V_2\}$.

In case (a), as V_0 is an absolutely irreducible FG_0-module, $U_1^t = U_2$ for $t \in G_0 - G$, so the lemma holds. So assume that case (b) holds. As $V_0 = 4_1 \otimes 4_2$, $\dim(C_{V_0}(X)) = 4$ for each X of order 3 in G using (2); so as $V_1^t = V_2$ and t acts on X^G, $\dim(C_{V_i}(X)) = 2$ for $i = 1, 2$. Thus the character χ_i of $U_i := \mathbf{F}_4 \otimes_F V_i$ on each element of odd order distinct from 5 in G is the same on V_1 and V_2, so the characters differ on elements of order 5. This is impossible as $U_i = U_i^\sigma$, and we may choose $t \in G_0 - G$ and y of order 5, such that $y^t = y^2$, so

$$\chi_i(y) = \chi_i^\sigma(y) = \chi_i(y^2) = \chi_i(y^t) = \chi_{3-i}(y).$$

□

LEMMA G.5.2. *If V is an irreducible FG_0-module on which G_1 and G_2 fix lines, then V is the 16-dimensional Steinberg module for G_0.*

PROOF. From G.5.1, G_0 has four irreducibles: 1, 4_1, 4_2, and the Steinberg module 16. Further G_i fixes no line in 4_i. □

In the remainder of the section, let V be the module 4_1.

The goal of the section is to describe the 15-dimensional permutation module U on $\Omega := G/G_1$. As G is transitive on $V^\#$ and G_1 is the stabilizer of a point of V, the permutation representations of G on $V^\#$ and Ω are equivalent, so we may view the two sets as the same. We also view U as the power set of Ω with addition equal to symmetric difference. For $S \subseteq \Omega$, write S' for the complement $\Omega - S$ of S in Ω. The *weight* of S is its order as a subset of Ω. Let U_0 be the *core* of U; that is U_0 is the submodule of vectors of even weight.

Recall that V has the structure of a symplectic space preserved by G. For $X \subseteq \Omega$, write $X^\perp$ for the subset of Ω orthogonal to X in V. Let $\mathcal{L}$ be the set of totally singular lines of V; we abuse notation and regard $l \in \mathcal{L}$ as the 3-subset $l^\#$ of Ω. As G is transitive on $\mathcal{L}$ and G_2 is the stabilizer of a member of $\mathcal{L}$, the permutation representations of G on G/G_2 and $\mathcal{L}$ are equivalent.

Recall that the *radical* $J(M)$ of a module M is the intersection of all maximal submodules of M, so that $M/J(M)$ is the largest semisimple quotient of M. Define $J^1(M) := J(M)$ and $J^i(M) := J(J^{i-1}(M))$ for $i > 1$ recursively. Recall the notion of a covering of a module from Remark I.1.5, and the description of the covering of 4_i from I.2.3.1.

LEMMA G.5.3. *(1) $U = U_0 \oplus F\Omega$.*

(2) U_0 *has* G*-chief series*

$$0 = U_5 < U_4 < \cdots < U_1 < U_0$$

where $U_n := J^n(U_0)$.

(3) $U_2 = \langle l' : l \in \mathcal{L}\rangle$, *and* U_0/U_2 *is the* 5*-dimensional cover of* 4_1.

(4) $U_4 = \{C_x : x \in \Omega\} \cup \{0\}$ *is isomorphic to* 4_1, *where* $C_x := (x^\perp)'$.

(5) U_2/U_4 *is the* 5*-dimensional cover of* 4_2.

(6) If W *is an* FG*-module such that* $W = \langle wG\rangle$ *for some* $w \in W^\#$ *fixed by* G_1, *and* $\langle wG_2\rangle$ *is a line, then* W *is isomorphic to* 4_1 *or its* 5*-dimensional cover.*

PROOF. Recall G_1 and G_2 are the stabilizers of $x \in \Omega$ and $l \in \mathcal{L}$ with $x \in l$.

Assume the hypotheses of (6), and let $W(x) := Fw$, and $W(l) := \langle wG_2\rangle$. Thus $W(l)$ is a line by hypothesis. For $g \in G$, let $W(xg) := W(x)g$ and $W(lg) := W(l)g$. Finally let $S(x) := \langle W(k) : x \in k \in \mathcal{L}\rangle$. As x is on three members of $\mathcal{L}$, it follows that $m(S(x)) \le 4$, and $S(x)$ contains $W(y)$ for each point y in $x^\perp$. Next $l = \{x, x_1, x_2\}$ and each $v \in \Omega$ is orthogonal to some point of l, so $W = \langle wG\rangle = S(x) + S(x_1) + S(x_2)$. Then as $m(S(x)) \le 4$ and $W(l) \le S(x) + S(x_i)$, $m(W) \le 8$.

Let $U_{\mathcal{L}} := \langle \mathcal{L}\rangle \le U$ and $\bar{U} := U/U_{\mathcal{L}}$. Then $\bar{U}$ is the largest FG-module satisfying the hypotheses of W in (6), so $W = \bar{U}/\bar{U}_W$ for some $\bar{U}_W \le \bar{U}$ and $\dim(\bar{U}) \le 8$. Also the 5-dimensional cover of 4_1 satisfies the hypotheses for W, so there is $\bar{U}_+ \le \bar{U}$, such that $\bar{U}/\bar{U}_+$ is that cover. In particular $m(\bar{U}_+) \le 3$, so by G.5.1.1, G centralizes $\bar{U}_+$. On the other hand, as $\bar{x} \in \langle \bar{x}G_2\rangle = [\langle \bar{x}G_2\rangle, G_2]$, $\bar{U} = [\bar{U}, G]$. Thus as $\bar{U}/\bar{U}_+$ has no proper cover since $H^1(G, 4_1)$ is 1-dimensional by I.1.6.1, it follows that $\bar{U} = U/U_{\mathcal{L}}$ is of rank 5. This establishes (6).

Next U has a nondegenerate G-invariant quadratic form q over F, in which Ω is an orthonormal basis; that is, $q(u)$ is the weight of u mod 2, and (u, v) is the weight of $u \cap v$ mod 2. Thus $F\Omega$ is a fixed point of G, and U_0 is the orthogonal complement to $F\Omega$ in U, so (1) holds. Moreover this shows that U_0 is self-dual.

The projection of x on U_0 is x' of weight 14. Similarly the projection of $U_{\mathcal{L}}$ on U_0 is $U_2 := \langle l' : l \in \mathcal{L}\rangle$, $U_0/U_2 \cong U/U_{\mathcal{L}}$ is the 5-dimensional cover of 4_1, and $J^2(U) \le U_2$. Thus (3) holds, modulo showing $U_2 = J^2(U)$. Further $m(U_2) = 14 - 5 = 9$.

Next observe that for distinct $u, v \in \Omega$, $C_u + C_v = C_{u+v}$: Namely $w \in C_u + C_v$ iff w is orthogonal in V to exactly one of u and v iff w is not orthogonal to $u + v$. It follows that

$$U_4 := \{C_y : y \in \Omega\} \cup \{0\}$$

is a subspace of U isomorphic to V under the FG-isomorphism $y \mapsto C_y$.

Let l_i, $1 \le i \le 3$ be the members of $\mathcal{L}$ through x. Observe next that $l_1 + l_2 + l_3 = C_x$, since $u \in l_1 + l_2 + l_3$ iff u is on an even number of the three lines iff u is on none of the lines. Thus $U_4 \le U_2$, with $m(U_2/U_4) = 5$ as $m(U_2) = 9$. So applying the analogue of (6) obtained via an outer automorphism nontrivial on the Dynkin diagram, it follows that U_2/U_4 is isomorphic to the 5-dimensional cover of 4_2.

We have shown that U_0 has five composition factors, two isomorphic to each of 1 and 4_1, and one isomorphic to 4_2. In particular there is a 1-dimensional factor in each of $J(U_0/U_2)$ and $J(U_2/U_4)$, so it follows that all 1-dimensional factors are in $J(U_0)$. Thus U_0 has no 1-dimensional quotient modules, and then as U_0 is self-dual, U_0 has no 1-dimensional submodules. Similarly U_2/U_4 has a 4_2-quotient but no 4_2-submodule, so U_0 has no 4_2-submodule, and hence no 4_2 quotient. Thus $U_0/J(U_0)$ and its dual $Soc(U_0)$ are the sum of $s := 1$ or 2 copies of 4_1. As U_0/U_2 is the cover of 4_1, $Soc(U_0)$ has at most one 4_1 summand, so $s = 1$. Thus $U_2 = J^2(U_0)$ is of codimension 1 in $J(U_0)$. Finally as U_0 has no 1-dimensional submodule, neither does U_2, so as $J(U_2/U_4)$ is of rank 1, the preimage U_3/U_4 of $J(U_2/U_4)$ does not split over U_4, so $U_4 = J^2(U_2) = J^4(U_0)$. This completes the proof of (2)–(5) and hence of the lemma. □

G.6. Modules with $\mathbf{m(G, V) \leq 2}$

In this section we assume:

HYPOTHESIS G.6.1. *G is a finite group with $O_2(G) = 1$, which is a quotient of an SQTK-group, V is a faithful $\mathbf{F}_2G$-module, and a is an involution in G such that $m([V, a]) \leq 2$ and $V := \langle [V, a]^G \rangle$. Set $K := \langle a^G \rangle$.*

LEMMA G.6.2. *(1) $V = [V, K]$.*
(2) $V = [V, O^2(K)]$.
(3) If $H \leq G$ with $K = \langle a^H \rangle$, and U is an H-submodule of V with $[V, a] \leq U$, then $U = V$.

PROOF. First $[V, a] \leq [V, K]$ as $a \in K$ by Hypothesis G.6.1. Further as $K \trianglelefteq G$, $[V, K]$ is G-invariant, so $V = \langle [V, a]^G \rangle \leq [V, K]$, proving (1). Next if $W := [V, O^2(K)] < V$, then as $K/O^2(K)$ is a 2-group,

$$[V, K]/W = [V/W, K/O^2(K)] < V/W,$$

contrary to (1). Thus (2) holds. Similarly under the hypotheses of (3), $[V, K] = \langle [V, a]^H \rangle \leq U$, so (1) implies (3). □

LEMMA G.6.3. *Assume U is a K-submodule of V, and either $K = \langle a^K \rangle$ or U is G-invariant. Then one of the following holds:*
(1) $[U, K] = 0$.
(2) $U = V$.
(3) a induces a transvection on U and V/U.

PROOF. Let $H := G$ if U is G-invariant, and $H := K$ otherwise. Thus by hypothesis, $K = \langle a^H \rangle$ and U is H-invariant. If $[U, a] = 0$, then as $K = \langle a^H \rangle$ and U is H-invariant, (1) holds. Thus we may assume $[U, a] \neq 0$. Similarly if $[V, a] \leq U$ then (2) holds by G.6.2.3. Thus as $m([V, a]) \leq 2$, we may assume $[U, a] = [V, a] \cap U$ is of rank 1, and that $m([V/U, a]) = 1$. Hence (3) holds. □

The following result describes groups generated by a conjugacy class of $\mathbf{F}_2$-transvections. While the result could presumably be extracted from the literature, (cf. also Remark G.4.2) we will deduce it from Theorem B.5.6 on FF-modules.

LEMMA G.6.4. *Assume Hypothesis G.6.1 and that a induces an $\mathbf{F}_2$-transvection on V. Then*
(1) $K = K_1 \times \cdots K_s$ and $V = V_1 + \cdots + V_s$, where $s \leq 2$, $V_i := [V, K_i]$, $[V_j, K_i] = 0$ for $i \neq j$, $a \in K_1$, and G is transitive on $\{K_1, \ldots, K_s\}$.
(2) One of the following holds:
(a) $K_i \cong L_n(2)$, $2 \leq n \leq 5$, and V_i is the natural module for K_i.
(b) $K_i \cong S_n$, $5 \leq n \leq 8$, and V_i is the natural module for K_i.
(c) $K_i \cong S_6$ or S_8, and V_i is the core of the permutation module for K_i.
(3) Either
(i) $s = 1$ and $G = K$, or

(ii) $s = 2$, $K_i \cong L_2(2)$, $L_3(2)$, *or* S_5, *and* $G = K\langle t\rangle$, *where* t *is an involution with* $K_1^t = K_2$.

(4) If a *is in the center of a Sylow* 2*-subgroup of* G, *then* $s = 1$ *and* $G = K \cong L_n(2)$, S_6, *or* S_7.

Proof. Let $A := \langle a\rangle$. As a induces an $\mathbf{F}_2$-transvection on V, $A \in \mathcal{P}(G, V)$. Thus Hypothesis B.5.3 holds with K in the role of "G" by Hypothesis G.6.1. Therefore we can appeal to Theorem B.5.6, to conclude that one of the following holds:

(I) $F^*(K)$ is quasisimple, and its action on V is described in Theorem B.5.1.

(II) $K = K_1 \times K_2$ and $\tilde{V} := V/C_V(K) = \tilde{V}_1 \oplus \tilde{V}_2$, where $V_i := [V, K_i]$, and $\tilde{V}_i$ is the natural module for $K_i \cong L_2(2)$, $L_3(2)$, or S_5.

(III) $K \cong L_2(2)$ and $V = [V, K] \oplus C_V(K)$ with $m([V, K]) = 2$.

Notice that the fact that $m(A) = 1$ excludes conclusions (4) and (5) of B.5.6, as well as cases (b) and (c) of B.5.6.3, and shows that K_i is $L_2(2)$, $L_3(2)$, or S_5 in case (a) of B.5.6.3.

By Hypothesis G.6.1, $K = \langle a^G\rangle$, so in case II we may choose $a \in K_1$, and there is $t \in G$ interchanging K_1 and K_2. Furthermore in case II, $V = [V, K] \oplus C_V(K)$, since the 1-cohomology of $\tilde{V}_i$ is zero when K_i is $L_2(2)$ or S_5, and by an appeal to B.4.8.2 and the fact that $m(A) = 1$ when K_i is $L_3(2)$. Thus in cases II and III, $V = [V, K] \oplus C_V(K)$. But by G.6.2.1, $V = [V, K]$, so $C_V(K) = 0$. Further K_i is self-normalizing in $GL(V_i)$, so we conclude $G = K\langle t\rangle$ when $s = 2$ and $G = K$ when $s = 1$. Thus (1)–(3) hold in cases II and III.

Thus we may assume case I holds. Here we appeal to Theorem B.5.1.1, to conclude that $U := [V, F^*(K)] \in Irr_+(V, F^*(K))$ and $\tilde{U} := U/C_U(F^*(K))$ is described in B.4.2. Cases (ii)–(iv) of B.5.1.1 do not hold as $m(A) = 1$. Then as $m(A) = 1$, B.4.2 says that K and $\tilde{U}$ are described in (a) or (b) of (2). Now $V = U$ by G.6.2.2. From the 1-cohomology of $\tilde{V}$ in I.1.6, or from B.4.8.2 when K is $L_3(2)$, either $C_V(K) = 0$ giving $U = \tilde{U}$ in (a) and (b) of (2), or (2c) holds. Once again K is self-normalizing in $GL(V)$, and hence $K = G$. Thus (1)–(3) are established in case I also.

It remains to prove (4), so we may assume that $a \in Z(T)$ for some $T \in Syl_2(G)$. As $a \in Z(T)$, and we chose $a \in K_1$ when $s = 2$ in (1), T acts on K_1. Therefore as $s \leq 2$ and G is transitive on $\{K_1, \ldots, K_s\}$, we conclude that $s = 1$ and hence $G = K$ by (3). Further if K is S_5 or S_8, then transpositions are not 2-central, so these cases do not occur. Hence (4) holds. This completes the proof of G.6.4. □

G.7. Small-degree representations for some SQTK-groups

In our treatment of $L \in \mathcal{L}_f^*(G, T)$ with $L/O_2(L) \cong L_n(2)$, and particularly in section 12.8, it is useful to have a classification of irreducible $\mathbf{F}_2$-modules for SQTK-groups G which are "small" in the sense that their degree is bounded above roughly by $2(m_2(G)+1)$. In some cases where $F^*(G)$ is quasisimple and irreducible, one could appeal to lists of modular character degrees in the literature, such as in the Modular Atlas [**JLPW95**]; but as those results often appear without explicit proof, and as we usually only require lower bounds on the minimal degree of a nontrivial irreducible, we will avoid extensive appeals to the literature, and instead provide a more self-contained treatment.

This first short section produces results for many of the groups appearing in Theorem C (A.2.3). By contrast, the groups of Lie type and characteristic 2 will be treated in section G.9, using knowledge of the $\mathbf{F}_2$-representations of small degree obtainable from the Lie theoretic literature; we will quote [**GLS98**] as a standard reference. Results on 2-ranks of automorphism groups of simple groups can be found in [**Asc82a**] and 5.2.10 and 5.6.1 in [**GLS98**]; we have also listed many of these in Table 7.2.1 and chapter H. We quote those standard results below without explicit reference.

So in this section, G is a finite group and V is a faithful $\mathbf{F}_2G$-module. Set $m := m_2(G)$ and $d := m(V)$.

LEMMA G.7.1. *Let p be an odd prime, $d(p)$ the order of 2 in the multiplicative group of $\mathbf{F}_p$, and P a p-subgroup of G. Then*

(1) If $|P| = p$, then $d \geq d(p)$.

(2) If $P \cong p^{1+2}$, then $d \geq p \cdot d(p)$.

(3) If P is elementary abelian, and n is the minimal length of an orbit of $N_G(P)$ on hyperplanes of P, then $d \geq n \cdot d(p)$, and in case of equality $m(C_V(H)) = d(p)$ for each hyperplane H of P with $C_V(H) \neq 0$.

PROOF. As $d(p)$ is the degree of each faithful irreducible for $\mathbf{Z}_p$ over $\mathbf{F}_2$, (1) holds. Let E be an elementary abelian subgroup of P, and $0 < U = [U, E] \leq V$. Then by Generation by Centralizers of Hyperplanes A.1.17,

$$U = \bigoplus_{H \in \mathcal{H}} C_U(H),$$

where $\mathcal{H}$ is the set of hyperplanes H of E with $C_U(H) > 0$. By (1), $m(C_U(H)) \geq d(p)$, so $d \geq |\mathcal{H}|d(p)$. In particular if $K \leq N_G(E) \cap N_G(U)$ then $d \geq kd(p)$, where k is the minimal length of an orbit of K on hyperplanes of E. Applying this observation with $E := P$, $U := [V, P]$, and $K := N_G(E)$, we obtain (3). Finally if $P \cong p^{1+2}$ take $U := [V, Z(P)]$, E any subgroup of P of index p, and $K := P$. Then $n = p$, so (2) follows. □

Our next result provides rough lower bounds on d for most cases in Theorem C which are not of Lie type and characteristic 2:

LEMMA G.7.2. *Assume $L := F^*(G)$ is a quasisimple SQTK-group irreducible on V. Then*

(1) If $L/Z(L)$ is sporadic, then $d \geq 10$.

(2) If $L \cong L_2(p^e)$ with p an odd prime and $p^e > 9$, then $d > 6$, and $d > 8$ unless $p^e = 17$.

(3) If $L \cong (S)L_3^\epsilon(p)$ with p an odd prime and $d \leq 8$, then $L \cong U_3(3)$ and $d = 6$.

(4) If $L \cong J_4$, then $d \geq 110$.

(5) If $L \cong J_2$ then $d \geq 12$.

(6) If $L \cong HS$, He, or Ru, then $d \geq 20$.

PROOF. As L is an SQTK-group, $L/Z(L)$ is listed in Theorem C. Note that if 11 divides the order of $L_n(2)$, then $n \geq 10$. Therefore as the order of each sporadic SQTK-group in Theorem C other than J_2, He, and Ru is divisible by 11, (1) follows for those remaining sporadics. Thus (5) and (6) will complete the proof of (1).

Similarly if p is a prime divisor of $|L_8(2)|$, then $p \leq 7$ or $p = 17$, 31, or 127, while p^2 does not divide the order of $|L_8(2)|$ for $p > 7$. Thus if $L \cong L_2(p^e)$ and $d \leq 8$ then $p^e \leq 9$ or $p^e = 25$, 49, 17, 31, or 127. As $L_8(2)$ has no Frobenius subgroup of order $25 \cdot 12$, $49 \cdot 24$, $31 \cdot 15$, or $127 \cdot 63$, and 17 does not divide $|L_6(2)|$, (2) holds.

Assume $L \cong (S)L_3^\epsilon(p)$ and $d \leq 8$. As p^3 does not divide $|L_8(2)|$ for $p > 3$, we conclude $p = 3$. Now $U_3(3) \cong G_2(2)'$; so if L is $U_3(3)$, then from the Lie theoretic literature, L has an irreducible of degree 6, but no other faithful irreducible of degree less than 14. Since 13 divides the order of $L_3(3)$ and $d(13) = 12$, G.7.1.1 implies that $L \not\cong L_3(3)$. So (3) is established.

Assume that $L \cong J_4$, HS, He, or Ru. Then G has a subgroup $P \cong p^{1+2}$, for $p = 11$, 5, 7, or 5, respectively. Thus (4) and (6) follow from G.7.1.2.

Finally assume that $L \cong J_2$. Then a Sylow 5-group P of L is isomorphic to E_{25}, and $N_L(P)$ has two orbits of length 3 on the hyperplanes of P, so (5) follows from G.7.1.3. □

We can compare the lower bounds in G.7.2 with the following upper bound (*), which holds for suitable modules in sections G.10 and 12.8:

LEMMA G.7.3. *Assume $L := F^*(G)$ is a quasisimple SQTK-group irreducible on V, and*

$$d \leq 2(m + 1). \tag{$*$}$$

Then

(1) If $L/Z(L)$ is of Lie type and odd characteristic, then either L is isomorphic to $L_2(4)$, $L_3(2)$, $Sp_4(2)'$, or $\hat{A}_6$; or $L \cong G_2(2)'$ and $d = 6$.

(2) If $L/Z(L)$ is sporadic, then one of the following holds:

(i) $G \cong Aut(M_{12})$ and V is the 10-dimensional core of a 12-dimensional permutation module.

(ii) $L \cong M_{22}$ or M_{24}, and V is either the Todd (cocode) module or its dual the code module.

(iii) $G \cong \mathbf{Z}_2/\hat{M}_{22}$ and $d = 12$.

(3) If $L/Z(L) \cong A_7$ then $L \cong A_7$ and $d = 4$ or 6.

PROOF. Suppose $L/Z(L)$ is of Lie type and odd characteristic. Then as L is an SQTK-group, it follows from Theorem C that $L \cong L_2(p^e)$, $(S)L_3^\epsilon(p)$, or $\hat{A}_6$. In the first case $m \leq 3$, and $m = 2$ if $p^e = 17$. Thus $d \leq 8$ and $d \leq 6$ if $p^e = 17$ by (*). Then by G.7.2.2, $p^e \leq 9$, so $L \cong L_2(4)$, $L_3(2)$, or $Sp_4(2)'$. In the second case, $m \leq 3$, so $d \leq 8$ by (*). Then $L \cong U_3(3) \cong G_2(2)'$ and $d = 6$ by G.7.2.3. Therefore (1) is established.

Suppose next that $L/Z(L)$ is sporadic, and recall the references for 2-ranks indicated before G.7.1. By G.7.2, $d \geq 10$, so $m \geq 4$ by (*). Thus $L/Z(L)$ is not M_{11} or J_1, as $m_2(Aut(L)) \leq 3$ in those groups; and if $L/Z(L) \cong M_{12}$, then $G \cong Aut(M_{12})$ as $m_2(M_{12}) = 3$. Similarly if $L/Z(L) \cong J_4$, then $m_2(Aut(L)) = 11$, so G.7.2.4 supplies a contradiction. If $L/Z(L) \cong M_{12}$, M_{22}, M_{23}, or M_{24}, then $m_2(Aut(L)) = 4$, 5, 4, or 6, respectively, so $d \leq 10$, 12, 10, 14 by (*). Now James [**Jam73**] shows that if L is simple then the minimum dimension of a faithful $\mathbf{F}_2L$-irreducible is 10, 10, 11, 11. Therefore L is not M_{23}, and if $L \cong M_{12}$ then $d = 10$ and $G \cong Aut(M_{12})$ by an earlier remark. Further James shows that the only irreducibles whose degrees satisfy this bound are those in conclusions (i) and (ii) of (2). When $L \cong \hat{M}_{22}$, James shows $d \geq 12$, so (*) forces $m \geq 5$. Thus G is $\hat{M}_{22}$ extended by an involutory outer automorphism, as $m_2(M_{22}) = 4$.

The remaining sporadics in Theorem C are J_2, HS, He, and Ru. For these, $m_2(Aut(L)) = 4$, 5, 6, 6, respectively, so that $d \leq 10$, 12, 14, 14 by (*). Then parts (5) and (6) of G.7.2 complete the proof of (2).

Finally suppose $L/Z(L) = A_7$. Then $m \leq 3$, so $d \leq 8$ by (*). Thus if $L \cong A_7$, then (3) holds by B.4.11. Finally observe that $\hat{A}_7$ has Sylow 3-group $P \cong 3^{1+2}$, whose faithful irreducibles have dimension 6 by G.7.1.2; then as $Z(P) = Z(L)$, $d \equiv 0 \mod 6$. Thus $d = 6$ and $L \leq SL_3(4)$ as $\mathbf{F}_4 = End_{\mathbf{F}_2 Z(L)}(V)$, contrary to $\hat{A}_7 \not\leq SL_3(4)$. This contradiction shows L is not $\hat{A}_7$, completing the proof of (4). □

G.8. An extension of Thompson's dihedral lemma

In order to determine the representations of solvable SQTK-groups satisfying condition (*) of G.7.3, we will prove a slight extension of Thompson's dihedral lemma. Throughout the section we assume:

HYPOTHESIS G.8.1. *A is an elementary abelian 2-subgroup of a 2-group P acting on a nontrivial nilpotent group X of odd order. Form the semidirect product G of X by P, and regard X and P as subgroups of G.*

LEMMA G.8.2. *Assume $B \leq A$ with B faithful on X. Then there exists a complement D to B in A, such that B is faithful on $C_X(D)$; in particular $D = C_A(C_X(D))$.*

PROOF. Choose a counterexample with $n := m(B)$ and $|X|$ minimal; observe that $n > 0$ as we may take D to be A when $B = 1$. Thus B is nontrivial on X, so B is nontrivial on $C_X(E)$ for some hyperplane E of A by Generation by Centralizers of Hyperplanes A.1.17. If $n = 1$, B is faithful on $C_X(E)$, and we may take $D = E$; hence $n > 1$.

Let F be a hyperplane of B. By minimality of n, there is a complement D' to F in A, with $D' = C_A(C_X(D'))$. Notice $B = F \times B'$ with $B' := B \cap D'$ of rank 1. By Coprime Action B' is faithful on $[X, D']$, so by minimality of n, B' is faithful on $C_{[X,D']}(D)$ for some complement D to B' in D'. Thus D is a complement to B in A, and

$$C_B(C_X(D)) \leq B \cap C_A(C_X(D')) \cap C_A(C_{[X,D']}(D)) \leq C_{B'}(C_{[X,D']}(D)) = 1,$$

so that B is faithful on $C_X(D)$, completing the proof. □

LEMMA G.8.3. *If $U \leq X$ then $O_2(C_G(U)) = O_2(G)$.*

PROOF. This follows from 31.14.2 in [**Asc86a**] by induction on $|\pi(U)|$. □

The next lemma is Thompson's $A \times B$-Lemma A.1.18, stated for nilpotent groups rather than p-groups.

LEMMA G.8.4. *If P centralizes $Y \leq X$ and P is faithful on X, then P is faithful on $C_X(Y)$.*

PROOF. As P is faithful on X, $O_2(G) = 1$. Let $U := C_X(Y)Y$, and observe $C_X(U) \leq C_X(Y) \leq U$. If P is not faithful on $C_X(Y)$, then $1 \neq B := C_P(C_X(Y)) \leq O_2(PU) \leq C_G(U) \leq PU$, so $1 \neq O_2(C_G(U))$, contrary to G.8.3, since $O_2(G) = 1$. □

LEMMA G.8.5. *Assume $B \leq A$, B centralizes an A-invariant subgroup Y of X, and B is faithful on X. Then B is faithful on $C_X(DY)$ for some complement D to B in A.*

PROOF. By G.8.4, B is faithful on $X' := C_X(Y)$. Then the lemma follows from G.8.2 applied to X' in the role of "X". □

Now we restate and prove Thompson's Dihedral Lemma A.1.5:

LEMMA G.8.6. *Assume $n := m(A) > 0$ and A is faithful on X. Then there exists $H \leq G$ with $A \leq H = H_1 \times \cdots \times H_n$, and $H_i \cong D_{2p_i}$ for suitable odd primes p_i.*

PROOF. Assume false, and pick a counterexample with n minimal. As A is faithful on X, each involution in A inverts an element of odd prime order in X. Thus the result holds if $n = 1$, so we may assume that $n > 1$. Let B be a hyperplane of A. By G.8.2, there exists $a \in A - B$ such that B is faithful on $X' := C_X(a)$ and $\langle a \rangle = C_A(X')$. By G.8.5 with $\langle a \rangle$, X' in the roles of "B, Y", a is faithful on $C_X(X'B')$ for some complement B' to $\langle a \rangle$ in A. Now the complement B' is also faithful on X', so without loss, $B = B'$. As B is faithful on X', by minimality of n there exists $H_1 \times \cdots \times H_{n-1} \leq X'B$ as in the lemma, and we let $H_n := \langle a, h \rangle$ for $h \in C_X(X'B)$ of prime order inverted by a. This completes the proof. □

LEMMA G.8.7. *Assume A is faithful on X, Y is an A-invariant subgroup of X, and set $B := C_A(Y)$, $n := m(A)$, and $m := m(B)$. Assume that $0 < m < n$. Then there exists a complement D to B in A, with $B \leq H_1 \times \cdots \times H_m \leq BC_X(DY)$, and $D \leq H_{m+1} \times \cdots \times H_n \leq YD$ with $H_i \cong D_{2p_i}$ for suitable odd primes p_i.*

PROOF. By G.8.5, there exists a complement D to B in A with B faithful on $C_X(YD)$. As $B = C_A(Y)$, D is faithful on Y. Now the lemma follows from applications of G.8.5 to the actions of B on $C_X(YD)$ and D on Y. □

Finally to prove G.9.2 in the next section, we will require the following extension of Thompson's dihedral lemma, which gives more information about the factorization in the lemma.

LEMMA G.8.8. *If A is faithful on X and nontrivial on $O_p(X)$, then there exists $H \leq G$ with $A \leq H = H_1 \times \cdots \times H_n$, $n := m(A)$, $H_i \cong D_{2p_i}$ for suitable odd primes p_i, and $p_1 = p$.*

PROOF. Let $B := C_A(O_p(X))$; then $m := m(B) < n := m(A)$ by hypothesis. If $B = 1$, the result is the usual dihedral lemma G.8.6, so we may take $m > 0$. Thus the lemma follows from G.8.7 applied with $O_p(X)$ in the role of "Y". □

We close the section with a well-known consequence of G.8.4, which we use at various places in the proof of the Main Theorem.

LEMMA G.8.9. *Let H be a finite group with $O_2(H) = 1$, for $K \leq H$, write $\Lambda(K)$ for the subgroup generated by all elements of K of prime order, and let $\Theta(H)$ be the set of all subgroups θ of H such that*

$$\theta = E(H) \prod_{d \in \pi(F(H))} \theta_d,$$

for some choice of supercritical subgroups θ_d of $O_d(H)$.

(1) If $\Lambda(C_{F^*(X)}(X)) \leq X \trianglelefteq \trianglelefteq F^*(H)$ and U is a 2-subgroup of $N_H(X)$, then U is faithful on X.

(2) Let $\theta \in \Theta(H)$. Then each 2-subgroup of H is faithful on θ.

PROOF. Assume the hypotheses of (1). As $X \trianglelefteq\trianglelefteq F^*(H)$, $X = F(X)E(X)$, with $E(X)$ the product of components of H and $F(X) \le F(H)$. For each component L of H, $L = \Lambda(L)$, and L centralizes $F(H)$ and all other components in $E(H)$. So as $\Lambda(C_{F^*(H)}(X)) \le X$ by hypothesis, $E(H) = E(X)$. Thus

$$X = F(X)E(H) \text{ with } F(X) \le F(H). \qquad (+)$$

Let $V := C_U(E(H))$; then V is faithful on $F(H)$, and by (+) it suffices to show $C := C_V(F(X)) = 1$. Applying G.8.4 to $F(H)$, $F(X)$, C in the roles of "X, Y, P", we conclude from that C is faithful on $C_{F(H)}(F(X)) = C_{F(H)}(X)$ in view of (+). So by Coprime Action, C is faithful on $\Lambda(C_{F(H)}(X)) \le X \cap F(H) = F(X)$. As C centralizes $F(X)$ by definition, we conclude $C = 1$, so (1) is established.

Assume the hypotheses of (2) and let U be a 2-subgroup of H; then U acts on the characteristic subgroup θ of $F^*(H)$. By construction $E(H) \le \theta$, while $\Lambda(C_{O_d(H)}(\theta_d)) \le \theta_d$ as θ_d is a supercritical subgroup of $O_d(H)$. Thus $\Lambda(C_{F^*(H)}(\theta)) \le \theta$, so (1) completes the proof. □

G.9. Small-degree representations for more general SQTK-groups

Throughout this section we assume:

HYPOTHESIS G.9.1. *G is a finite group, V is a faithful $\mathbf{F}_2G$-module, $d := m(V)$, $E_{2^m} \cong A \le G$, and*

$$d \le 2(m+1). \qquad (*)$$

We first show that if $F^*(G)$ is of odd order, then the upper bound in (*) is also a lower bound—unless A centralizes $O^3(F(G))$, in which case at least we obtain the lower bound (**) in G.9.2 below.

LEMMA G.9.2. *Assume that A acts faithfully on a nilpotent subgroup X of G of odd order. Assume further that:*

$$d \le 2m \text{ if } [O^3(X), A] = 1. \qquad (**)$$

Then

(1) $[O^{3,5}(X), A] = 1$,

and there exists a subgroup $H = G_1 \times \cdots \times G_r$ of XA containing A such that:

(2) $V = U_1 \oplus \cdots \oplus U_r$, where $U_i := [V, G_i]$.

(3) If $[O^3(X), A] \ne 1$, then $G_1 \cong D_{10}$ and $m(U_1) = 4$.

(4) Set $e := 1$ if $[O^3(X), A] \ne 1$ and $e := 0$ otherwise. Then for $e < i \le r$, either $G_i \cong S_3$ and $m(U_i) = 2$, or U_i is the orthogonal module of rank 4 for $G_i = \Omega_4^+(2)$.

(5) If $[O^3(X), A] = 1$ then $d = 2m$.

(6) If $[O^3(X), A] \ne 1$ then $d = 2(m+1)$.

PROOF. We begin with some observations which we apply to proper subspaces of V in an inductive context. First, observe that (5) and (6) say that if an elementary abelian 2-group B of rank b acts on a nilpotent subgroup Y of odd order, and U is a faithful $\mathbf{F}_2YB$-module, then $m(U) \ge 2b$, and $m(U) \ge 2(b+1)$ if $[O^3(Y), B] \ne 1$. Second, if $YB = K_1 \times \cdots K_b$ with $K_i \cong D_{2p_i}$ for suitable primes p_i, then this is the unique such decomposition of YB by the Krull-Schmidt Theorem A.1.15. Third, if the inequalities $m(U) \ge 2b$ or $2(b+1)$ are equalities, then hypothesis (*) or (**) holds, so the lemma says that $YB = I_1 \times \cdots \times I_s$ and $U = U_1 \oplus \cdots \oplus U_s$, where $U_i = [U, I_i]$, $I_1 = K_1 \cong D_{10}$ if $[O^3(Y), B] \ne 1$, the I_i are products of one or two of the K_j, and the action of I_i on U_i is as described in (4). Fourth, when $m(U_i) = 4$, $I_i \cong \Omega_4^+(2)$ since the subgroup $S_3 \times S_3$ of $O_4^+(2)$ generated by transvections decomposes into two factors of rank 2.

Fifth, by G.8.6, A is contained in a subgroup $H = H_1 \times \cdots \times H_m$ of AX such that $H_i \cong D_{2p_i}$; indeed by G.8.8, we can choose $p := p_1$ to be large in some situations: If $[O^3(X), A] = 1$, then of course $p_i = 3$ for all i, and in particular $p = 3$. On the other hand if (1) fails, then $[O_q(X), A] \neq 1$ for some prime $q > 5$, while if (1) holds but $[O^3(X), A] \neq 1$, then $[O_5(X), A] \neq 1$. Then by G.8.8 we may choose $p = q > 5$ or $p = 5$ in the respective cases.

Now we begin the proof of the lemma, starting with the case $m = 1$. Here we take $r := 1$ and $G_1 := H_1 = H$. By (*), $d \leq 2(m + 1) = 4$, so by G.7.1.1, $p \leq 5$, with $d = 4$ in case $p = 5$. Similarly if $[O^3(X), A] = 1$ then $d \leq 2m = 2$ by (**), so that $p = 3$ and $d = 2$ by G.7.1.1. Thus the lemma holds when $m = 1$.

Thus we assume for the rest of the proof that $m \geq 2$. Let $\langle a \rangle := H_m \cap A$ and $\langle x \rangle := O(H_m)$. Then by A.1.44, $V = U \oplus W \oplus W^x$, where $U := C_V(x)$ and $W := C_{[V,x]}(a)$. Let J consist of those indices $j < m$ such that $O(H_j)$ is nontrivial on $[V, x] = W \oplus W^x$; then as $O(H_j)$ has prime order, $O(H_j)$ is faithful on $[V, x]$. Set $n := |J|$, and notice $n < m$ since $m \notin J$. Set $P := \prod_{j \in J} H_j$, $I := \{1, \ldots, m-1\} - J$, $k := |I| = m - 1 - n$, and $Q := \prod_{i \in I} H_i$. Then $\{I, J, \{m\}\}$ is a partition of $\{1, \ldots, m\}$, so $H = P \times Q \times \langle a \rangle \langle x \rangle$. As each nontrivial normal subgroup of P contains $O(H_j)$ for some $j \in J$, and $O(H_j)$ is faithful on $[V, x]$, P is faithful on $[V, x] = W \oplus W^x$, and hence also on W as P centralizes H_m. As $O(Q)$ centralizes $[V, x] = W \oplus W^x$, Q is faithful on U.

Now we obtain some useful information by induction on m:

(J1) $m(W) \geq 2n$, and in case of equality $p_j = 3$ for all $j \in J$, and the action of P on W is as described in (4).

As $n < m$ and P is faithful on W, (J1) follows by induction on m and observations one and three.

(J2) Assume $1 \in J$. Then (1) holds. If in addition $[O^3(X), A] \neq 1$, then $p = 5$, $m(W) \geq 2(m + 1)$, and in case of equality the action of P on W is as described in (4).

As $1 \in J$, induction on m and observation five show that (1) holds, and that $p = 5$ if $[O^3(F(X)), A] \neq 1$. Then the remaining remarks in (J2) follow from our first and third observations.

By similar inductive arguments using $k < m$ we obtain:

(I1) $m(U) \geq 2k$, and if equality holds, then the action of Q on U described in (4), with $p_i = 3$ for all $i \in I$.

(I2) Assume that $1 \in I$. Then (1) holds. If in addition $[O^3(X), A] \neq 1$, then $p = 5$, $m(U) \geq 2(k+1)$, and if equality holds, then the action of Q on U is described in (4).

The remainder of the proof is broken into several cases. As $m \geq 2$, $1 \in I \cup J$, so that at least one of I or J is nonempty, and (1) follows from (J2) and (I2).

We first consider the case where $[O^3(X), A] = 1$. Then (3) and (6) hold vacuously, $p_i = 3$ for all i, and $d \leq 2m$ by (**). Thus it remains to establish (2), (4), and (5).

First suppose that $J \neq \emptyset$. Then $n \geq 1$ and appealing to (I1) and (J1),

$$d = m(U) + 2m(W) \geq 2k + 4n = 2(m-1) + 2n \geq 2m \qquad (!)$$

since $n \geq 1$. Therefore (5) holds and all inequalities in (!) are equalities, so that $n = 1$, $m(W) = 2$, and $m(U) = 2k$. Then by (I1), the action of Q on $U = [V, Q]$ is described in (4). We choose notation so that $J = \{m-1\}$, and let $G_i := H_i$ and $U_i := [V, G_i]$ for $i \leq m-2$. As the action of Q on $[V, Q]$ is described in (4), $Q = G_1 \times \cdots \times G_{m-2}$ and $[V, Q] = U_1 \oplus \cdots \oplus U_{m-2}$. Finally set $G_{m-1} := H_{m-1} \times H_m$, and observe that $[V, G_{m-1}] = W + W^x$ is of rank 4, and then that (2) and (4) hold in this subcase.

So we may suppose that $J = \emptyset$. Then by (I1), $m(U) \geq 2k = 2(m-1)$. Since $m([V, H_m]) \geq 2$, $d \geq 2m$, so that $d = 2m$ and (5) holds. As the inequality is an equality, $m([V, H_m]) = 2$ and $m(U) = 2(m-1) = 2k$. Therefore by (I1), the action of Q on U is given in (4). This time we take $G_i := H_i$ for $i < m$, and take $G_m := H_m \cong S_3$. Now (2) and (4) hold.

We now turn to the case $[O^3(X), A] \neq 1$. Here (5) is vacuous. If $1 \in J$ then $n \geq 1$, and using (I1) and (J2),

$$d \geq 2k + 4(n+1) = 2m + 2(n+1) > 2(m+1),$$

contradicting (*). Therefore $1 \in I$, and using (I2) and (J1),

$$d \geq 2(k+1) + 4n = 2m + 2n = 2(m+1) + 2n - 2. \qquad (!!)$$

Suppose first that $J \neq \emptyset$. Now $n \geq 1$, and $d \geq 2(m+1)$, and hence $d = 2(m+1)$ by (*), so that (6) holds. As all inequalities in (!!) are equalities, $n = 1$, $m(W) = 2$, and $m(U) = 2(k+1)$. By (I2) the action of Q on U is described in (4). As $m(W) = 2$ and $n = 1$, we may choose notation so that $J = \{m-1\}$ and $p_{m-1} = 3$. Once again we take $G_i := H_i$ for $i < m-1$ and $G_{m-1} := H_{m-1} \times H_m = \Omega^+(W + W^x)$. Thus (2), (3), and (4) hold.

Finally suppose that $J = \emptyset$. Here (I2) says $m(U) \geq 2(k+1) = 2m$, so as $m([V, H_m]) \geq 2$, we conclude $d \geq 2(m+1)$. Again (*) implies $d = 2(m+1)$, so that (6) holds. As the inequality is an equality, $m([V, H_m]) = 2$ and hence $p_m = 3$ and $m(U) = 2m = 2(k+1)$. Now (I2) shows that the action of Q on U is described in (4). Take $G_i := H_i$ for $i < m$ and $G_m := H_m$. Then (2), (3), and (4) hold. This finally completes the proof of G.9.2. □

The following two results are the main theorems in this section. They determine the irreducible $\mathbf{F}_2G$-modules satisfying (*) for a larger class of SQTK-groups.

Theorem G.9.3. *Assume G is a quotient of an SQTK-group, L is a component of G with $[L, A] \neq 1$, and $H \trianglelefteq G$ is irreducible on V, where either $H = L$, or $H = LL_2$ with L_2 a component of G such that the representation of L on V is quasiequivalent to the representation of L_2 on V or its dual. Then one of the following holds:*

(0) $m = 4$, $d = 10$, $H \cong M_{12}$, and V is the 10-dimensional core of a 12-dimensional permutation module.

(1) $H \cong M_{22}$ or M_{24}; $d = 10$ or 11; $m \geq 4$ or 5; and V is a Todd module or code module in either case.

(2) $m = 5$, $d = 12$, and $G \cong \mathbf{Z}_2/\hat{M}_{22}$.

(3) $m = 2$, $d = 4$, and $G \cong A_7$.

(4) $m \geq 2$, $d = 6$, and $H \cong A_7$.

(5) $m \geq 2$, $d = 6$, *and* $H \cong \hat{A}_6$.

(6) V *is the natural module for* $H \cong L_2(2^n)$, $d = 2n$, *and* $m \geq n-1$.

(7) $H \cong L_2(2^n)$, n *even*, $d = 2n$, $m \geq n-1$, *and* V *is the* $\Omega_4^-(2^{n/2})$*-module.*

(8) V *is a natural module for* $H \cong SL_3(2^n)$, $d = 3n$, *and* $m \geq (3n-2)/2$.

(9) V *is a natural module for* $H \cong Sp_4(2^n)'$, $d = 4n$, *and* $m \geq 2n-1$.

(10) V *is a natural module for* $H \cong G_2(2^n)'$, $d = 6n$, *and* $m \geq 3n-1$.

(11) V *is a natural module for* $G \cong L_4(2)$, $d = 4$, *and* $m \geq 1$.

(12) V *is the* 6*-dimensional orthogonal module for* $H \cong A_8$, $d = 6$, *and* $m \geq 2$.

(13) $d = 5$ *or* 10, $G \cong L_5(2)$, *and* $m \geq 2$ *or* 4, *respectively.*

(14) $d = 9$, $G \cong L_3(2) \times L_3(2)$, $m = 4$, *and* V *is the tensor product of natural modules for the factors.*

(15) V *is the orthogonal module for* $H \cong \Omega_4^+(2^n)$, $n > 1$, $d = 4n$, *and* $m \geq 2n-1$.

(16) $d = 8$, $m = 3$, $H \cong L_2(8)$, *and* $V \otimes_{\mathbf{F}_2} \mathbf{F}_8 \cong N \otimes N^\sigma \otimes N^{\sigma^2}$, *where* N *is the natural module for* H, *and* $\langle\sigma\rangle = Gal(\mathbf{F}_8/\mathbf{F}_2)$.

(17) $H \cong (S)L_3(2^{2n})$ *for* $n \leq 2$, $d = 9n$, $m = 4n$, *and* $V \otimes_{\mathbf{F}_{2^n}} \mathbf{F}_{2^{2n}} \cong N \otimes N^\sigma$, *where* N *is the natural* $SL_3(2^{2n})$*-module and* σ *is the involutory automorphism of* $\mathbf{F}_{2^{2n}}$.

THEOREM G.9.4. *Assume* G *is a quotient of an SQTK-group, and* G *is irreducible on* V. *Assume further that* A *is faithful on* $F(G)$, *and set* $K := \langle A^G\rangle$. *Then one of the following holds:*

(1) $m = 1$, $d = 2$, *and* $G \cong S_3$.

(2) $m = 1$, $d = 4$, *and* $K \cong D_{10}$.

(3) $m = 1$, $d = 4$, *and either* $G = \Gamma L_2(4)$ *or* $O_3(O_4^+(2)) \leq G \leq O_4^+(2)$.

(4) $m = 2$, $d = 4$, *and* G *is of index at most* 2 *in* $O_4^+(2)$.

(5) $m = 2$, $d = 6$, *and* G *is a subgroup of* $SD_{16}/3^{1+2}$ *containing* $E_4/3^{1+2}$.

By hypothesis, in each Theorem there is an SQTK-group $\hat{G}$ and a normal subgroup $\hat{N}$ of $\hat{G}$, such that $G = \hat{G}/\hat{N}$. Thus by B.5.2 we may assume:

$$\text{The preimage of } F(G) \text{ in } \hat{G} \text{ is nilpotent,} \tag{!}$$

and:

$$\text{If } L \text{ is a component of } G\text{, then } L = \hat{L}\hat{N}/\hat{N} \text{ for some component } \hat{L} \text{ of } \hat{G}. \tag{!!}$$

Observe that as G is faithful and irreducible on V, $O_2(G) = 1$.

We first prove Theorem G.9.4. So assume A is faithful on $F(G)$. Let $B := O_3(G)$.

Suppose that $[F(G), A]$ is not a 3-group. By G.9.2 we can pick $A \leq H = G_1 \times \cdots \times G_r \leq G$ as in G.9.2 with $O(H) \leq F(G)$. In particular $G_1 \cong D_{10}$ by G.9.2.3, so $X := O(G_1) \leq O_5(G)$, and for $i > 1$, $G_i \cong S_3$ or $\Omega_4^+(2)$ by G.9.2.4, so $O(G_i) \leq B$. By G.9.2, $V = U_1 \oplus \cdots \oplus U_r$ with $U_i := [G_i, V]$, $m(U_1) = 4$, and $m(U_i) = 2$ or 4 for $i > 1$.

Now $O_5(G)$ centralizes $B \geq O_3(G_i)$, and hence acts on U_i. Thus $O_5(G)$ centralizes U_i if $m(U_i) = 2$. Similarly if $m(U_i) = 4$ then $O_3(G_i) = X_{i,1} \times X_{i,2}$ with $X_{i,j} \cong \mathbf{Z}_3$ centralized by $O_5(G)$, and $U_i = [V, X_{i,i}] \oplus [V, X_{i,2}]$ with $[V, X_{i,j}]$ of rank 2 acted on by $O_5(G)$, so again $O_5(G)$ centralizes U_i. Therefore $[V, O_5(G)] = U_1$, so as G is faithful and irreducible on V, $V = U_1$ is of rank 4. As a Sylow 5-subgroup of $GL(V) \cong L_4(2)$ has order 5, $O_5(G) = X = O_5(G_1)$. Also G_1 is the subgroup generated by all involutions in $N_{GL(V)}(X)$, so conclusion (2) of Theorem G.9.4 holds.

Therefore we may assume that $[F(G), A]$ is a 3-group. As A is faithful on $F(G)$, A is faithful on B, and $m_3(B) \geq m$ by A.1.5. Let $\hat{B}$ be a Sylow 3-subgroup of the preimage in $\hat{G}$ of B; by (!), $\hat{B} \trianglelefteq \hat{G}$. As A is faithful on B, A is a section of $Aut_{\hat{G}}(\hat{B})$. However as $\hat{G}$ is an SQTK-group, $m_3(\hat{B}) \leq 2$, sections of $Aut_{\hat{G}}(\hat{B})$ are of 2-rank at most 2, and hence $m \leq 2$.

As G is faithful and irreducible on V, $V = [V, B]$, so d is even. Suppose first that $m = 1$; then $d \leq 2(m + 1) = 4$ by (*), so $d = 2$ or 4. In the first case conclusion (1) of Theorem G.9.4 holds. In the second case $G \leq GL(V) \cong L_4(2)$ and $V = [V, B]$, so either

(a) $F \cong \mathbf{Z}_3$ and $N_{GL(V)}(B) = \Gamma L_2(4)$, or

(b) $F \cong E_9$ and $N_{GL(V)}(F) = O_4^+(2)$.

As G is an irreducible on V and A centralizes $O_5(G)$, it follows that conclusion (3) of the Theorem holds.

Thus we may take $m = 2$. Therefore $d \leq 6$ by (*), so as d is even and $m = 2$, $d = 4$ or 6. Also $m_3(B) \geq 2$ as observed earlier. If $d = 4$ then (b) holds, and hence conclusion (4) of the Theorem holds. Thus we may assume that $d = 6$.

Let $B \leq Q \in Syl_3(GL(V))$; then $Q \cong \mathbf{Z}_3$ wr $\mathbf{Z}_3$, so $J(Q) \cong E_{27}$. Thus if $m_3(B) > 2$, then $J(B) = J(Q)$, so $J(B)$ is weakly closed in B and hence $G \leq N_{GL(V)}(J(B)) \cong S_3$ wr S_3. Indeed $B = B_1 \times B_2 \times B_3$ and $V = V_1 \oplus V_2 \oplus V_3$, here $\{B_1, B_2, B_3\}$ are the hyperplanes of B with nontrivial fixed points on V, and $V_i := C_V(B_i)$ is of rank 2. Then as G is irreducible on V, G is transitive on $\{V_1, V_2, V_3\}$, so we may take $Q \leq G$. Now if $Q \not\leq B$ then from the structure of S_3 wr S_3, G is irrreducible on $J(Q)$. As $m = 2$, this is impossible by A.1.31.3. Thus $B = Q$, so as $N_{GL(V)}(Q)/Q \cong E_4$, $G = N_{GL(V)}(Q)$. But then there is an involution $t \in N_G(Q)$ with $C_Q(t)$ noncyclic, contrary to A.1.31.1.

Therefore $m_3(B) = 2$. Let R be a supercritical subgroup of B. As A is faithful on B, A is faithful on R, so as $m = 2$, R is noncyclic by A.1.5. Thus $R \cong E_9$ or 3^{1+2} by A.1.24.

Suppose that $R \cong E_9$. Then as G is irreducible on V, the set Δ of subgroups of R of order 3 with fixed points on V is of order 3, G is transitive on Δ, and

$$V = \bigoplus_{D \in \Delta} C_V(D).$$

Thus up to conjugacy in $GL(V)$, $\Delta = \{V_1, V_2, V_3\}$ for V_i as above, so $N_{GL(V)}(R) \leq N_{GL(V)}(J(Q))$. Then as $m_3(B) = 2 = m$, it follows from the structure of S_3 wr S_3 that $R = C_G(R)$ and $G/R \cong \mathbf{Z}_2 \times S_3$; hence $G \cong E_4/3^{1+2}$, whereas $O_3(G)$ has no characteristic subgroup $R \cong E_9$.

This leaves the case where $R \cong 3^{1+2}$. Then R is a maximal subgroup of Q, so $R = B$ as $m_3(B) = 2$. Then $N_{GL(V)}(R)/R \cong GL_2(3)$. Since $m = 2$, some $a \in A^{\#}$ inverts $R/Z(R)$, so if $Q \leq G$ then $C_Q(a)$ is noncyclic, contrary to A.1.31.1. Therefore R is Sylow in G, so $G \leq SD_{16}/3^{1+2}$, and now (5) holds.

This completes the proof of Theorem G.9.4.

We next prove Theorem G.9.3, so we may assume the hypotheses of that Theorem. By (!!) there is $\hat{L} \in \mathcal{C}(\hat{G})$ such that $\hat{L}\hat{N}/\hat{N} = L$. Similarly if $H = LL_2$, there is $\hat{L}_2 \in \mathcal{C}(\hat{G})$ with $L_2 = \hat{L}_2\hat{N}/\hat{N}$. By hypothesis H is irreducible on V.

Suppose first that $H = LL_2 > L$. Let $I \in Irr_+(L, V)$, $F := End_L(I) = \mathbf{F}_{2^n}$, and $k := \dim_F(I)$. By hypothesis the representation of L on V is quasiequivalent to the representation of L_2 on V or its dual, so $L_2 \cong L$, and $J \in Irr_+(L_2, V)$ is quasiequivalent to I or I^*. Therefore $F = End_{L_2}(J)$, $k = \dim(J)$, and as an FH-module, $V = I \otimes_F J$, so $d = nk^2$.

As $L_2 \cong L$ and G is a quotient of an SQTK-group, A.1.34.3 says that $L \cong L_2(2^r)$, $Sz(2^r)$, $L_2(p)$, or J_1. Set $m_L := m_2(L)$; then $m_2(L) = m_2(Aut(L))$ in each case, so $m_2(G) = 2m_L$, and replacing A by a subgroup of higher rank if necessary, we may take $m = m_2(G)$ and $A \leq H$. If $k = 2$, then $L \cong L_2(2^n)$ and by (*),

$$4n = nk^2 = d \leq 2(m + 1),$$

so $m \geq 2n-1$ and conclusion (15) of Theorem G.9.3 holds. Similarly if $k = 3$, then $L \cong L_2(7) \cong L_3(2)$ and $n = 1$, so that (14) holds. Thus we may assume $k \geq 4$, so $d \geq 16$, and hence $2m_L = m \geq 7$ by (*), so $m_L \geq 4$. Therefore $L \cong L_2(2^r)$ or $Sz(2^r)$ with $r = m_L \geq 4$.

Now $\mathbf{F}_{2^r}$ is a splitting field for L, so $n \leq r$. If $nk > 2(r + 1)$, then

$$d = nk^2 = \frac{(nk)^2}{n} > \frac{4(r + 1)^2}{n} \geq \frac{4(r + 1)^2}{r} > 4r + 8 > 2(2r + 1) = 2(m + 1),$$

contrary to (*). Therefore $m(I) = nk \leq 2(r + 1) = 2(m_2(L) + 1)$, so (*) is satisfied for the action of L on I. Thus by induction on d, $L \cong L_2(2^r)$, as $Sz(2^r)$ does not appear in G.9.3. As $k \geq 4$, the pair L, I does not appear in case (6) of G.9.3, so we are in case (7) with $r = 2n$ and I the orthogonal module for L of dimension $k = 4$ over $\mathbf{F}_{2^n}$. Then $d = nk^2 = 16n = 4m > 2(m + 1)$, contrary to (*).

Thus we may assume that $H = L$. Then either H is simple and described in Theorem C (A.2.3), or H is one of the quasisimple groups described in A.3.6.2. If $L/Z(L)$ is not of Lie type and characteristic 2, then G and its representation on V are described in cases (2) and (3) of G.7.3, which in turn appear as conclusions (0)–(4) of G.9.3. Thus we may assume $L/Z(L)$ is of Lie type and characteristic 2. If $L \cong \hat{A}_6$ then $V = [V, Z(L)]$. As in our argument for $\hat{A}_7$ in G.7.3, all faithful irreducibles for a Sylow 3-subgroup of G are of rank 6, and hence $d \equiv 0 \mod 6$. Then as $m_2(Aut(L)) = 3$, $d = 6$, so conclusion (5) holds. Therefore we may assume L is not $\hat{A}_6$.

In all remaining cases, L is a quotient of the universal group of its type defined by the Steinberg relations (cf. sec 2.9 of [**GLS98**]), so the small dimensional representations of L are known. (Cf. pages 26–27 and 77–78 of [**GLS98**].) If $L/Z(L) \cong L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$, $Sp_4(2^n)'$, $G_2(2^n)'$, ${}^3D_4(2^n)$, ${}^2F_4(2^n)$, $L_4(2)$, or $L_5(2)$, then the 2-rank of $Aut(L)$ is n, n, $n+1$, $2n$, $3n$, $3n$, $5n$, $5n$, 4, or 6, respectively. Then (*) rules out $Sz(2^n)$, $U_3(2^n)$, ${}^3D_4(2^n)$, and ${}^2F_4(2^n)$, since the standard theory of the representations of these groups in their defining characteristic (cf. section 2.8 of [**GLS98**]) shows that $d = 4n$ (with $n \geq 3$), $6n$, $24n$, $26n$, respectively. For $L_3(2^n)$, $Sp_4(2^n)'$, and $G_2(2^n)'$, the natural module satisfies (*), and these appear in conclusions (8), (9), and (10) of G.9.3; but (*) rules out the irreducibles of the next higher dimension, in the respective cases: $8n$ or $9n/2$ if n is even; $16n$ or $16n/2$ if n is even; or $14n$—except for $n \leq 2$ and $H \cong (S)L_3(2^n)$, which appears in conclusion (17) ofG.9.3. Finally up to quasiequivalence, $L_2(2^n)$, $L_4(2)$, and $L_5(2)$ each admit two modules satisfying the bound, and these appear as conclusions (6), (7), (11), (12), (13) of G.9.3. Further (*) rules out the irreducibles of the next higher dimension $8n/3$ if 3 divides n, 14, 24—except for $L_2(8)$, which appears as conclusion (16) of G.9.3.

This completes the proof of Theorem G.9.3.

G.10. Small-degree representations on extraspecial groups

In this section we axiomatize the situation which arises in section 12.8 (see especially 12.8.12.1), where the centralizer of an involution acts on the central quotient of a normal "almost-extraspecial" 2-subgroup U (i.e., U is nonabelian and $|\Phi(U)| = 2$) preserving the symplectic form defined by the commutator map.

So in this section, we assume the following hypothesis:

HYPOTHESIS G.10.1. *(1) V, (,) is a d-dimensional symplectic space over $\mathbf{F}_2$, and G is a group of isometries of V.*

(2) $V_1 = \langle v_1 \rangle$ is a point of V, W is a totally isotropic subspace of V containing V_1, X is an elementary abelian 2-subgroup of $C_G(V_1)$, and $X_0 \leq X$ such that

(a) $V = \langle V_1^G \rangle$ and $|G : C_G(V_1)|$ is odd.

(b) $m(W) + m(X/X_0) = d - 1$.

(c) $[X, V_1^{\perp}] \leq W$.

(d) $C_V(x) \leq V_1^{\perp}$ for each $x \in X - X_0$.

(e) X induces the full group of transvections on W with center V_1.

We observe first that this hypothesis implies the upper bound (*) discussed in the previous sections G.7 and G.9; however we will not need to quote the results of section G.9 until section G.11.

LEMMA G.10.2. *$d \leq 2(m(X/X_0) + 1) \leq 2(m_2(G) + 1)$.*

PROOF. As W is totally isotropic, $m(W) \leq d/2$, so by (b) of Hypothesis G.10.1,

$$m(X/X_0) = d - m(W) - 1 \geq \frac{d}{2} - 1,$$

so the lemma holds. □

LEMMA G.10.3. *If U is a nonzero G-submodule of V then $O_2(G/C_G(U)) = 1$. In particular $O_2(G) = 1$.*

PROOF. Let $\hat{V} := V/U^{\perp}$. Then $\hat{V}$ is dual to U as a G-module, so it suffices to show that $O_2(G/C_G(\hat{V})) = 1$. By Hypothesis G.10.1.2.1, $V = \langle V_1^G \rangle$ and $\hat{V}_1$ is centralized by a Sylow 2-group of G, so $\hat{V} = \langle \hat{V}_1^G \rangle \in \mathcal{R}_2(\hat{G})$ by B.2.13, and the lemma follows. □

LEMMA G.10.4. *Assume $1 \neq H \trianglelefteq G$, U is a nonzero proper irreducible HX-submodule of V, and either*

(1) $H = G$, or

(2) $V_1 \not\leq U$ and $[U, X] \neq 0$.

Then $d = 4$, $m(U) = 2$, and G is solvable.

PROOF. Set $A := V_1^{\perp} \cap U$. In case (1), U is a proper nonzero G-submodule of V. Thus also $U^{\perp} < V$, so as $V = \langle V_1^G \rangle$, V_1 is contained in neither U nor $U^{\perp}$. Therefore A is a hyperplane of U. In case (2), $V_1 \not\leq U$ by hypothesis, so that $V_1 \not\leq U$ in either case; we will see in a moment that A is a hyperplane of U in case (2) also.

By Hypothesis G.10.1.2.e, $C_W(X) = V_1$, so as $V_1 \not\leq U$, $W \cap U = 0$. Thus $[A, X] \leq W \cap U = 0$ by Hypothesis G.10.1.2.c. In case (2), $[U, X] \neq 0$ by hypothesis, so $A < U$ and hence as promised A is a hyperplane of U.

Therefore in either case, X induces a group of transvections on U with axis A. Thus $A = C_U(x)$ for $x \in X - X_0$ by Hypothesis G.10.1.2.d, and hence $C_X(U) \leq X_0$. Then using Hypothesis G.10.1.2.b and recalling $U \cap W = 0$,

$$m(U) - 1 \geq m(X/C_X(U)) \geq m(X/X_0) = d - m(W) - 1 \geq m(U) - 1,$$

and hence all inequalities are equalities. Therefore X induces the full group of transvections on U with axis A, $m(U) = m(X/X_0) + 1$, and W is a complement to U in V. By Hypothesis G.10.1.2, W is totally isotropic, so $m(W) \leq d/2$ and hence $m(U) \geq d/2$. By hypothesis HX is irreducible on U, so A contains no nonzero HX-submodule of U. Therefore as X induces the full group of transvections on U with axis A, it follows from the dual of G.3.1 that $Aut_{HX}(U) = GL(U)$. Further

$$O_2(HX) \leq R_0 := C_{HX}(U)$$

as HX is irreducible on U.

Suppose that $m(U) = 1$. Then $d = 2$, so G is irreducible on V by Hypothesis G.10.1.2.a, so case (1) of our lemma does not hold. Further as $m(U) = 1$, $[X, U] = 0$ and hence neither does case (2). Next suppose $m(U) = 2$. Then $d \leq 4$, so as U is proper and the dimension d of the symplectic space V is even, $d = 4$. As $H \trianglelefteq G \leq Sp(V) \cong S_6$ and H acts on a 2-dimensional subspace of V, we conclude G is solvable, so the conclusion of the lemma holds. Thus we may assume that $m(U) > 2$, and it remains to derive a contradiction.

By hypothesis HX is irreducible on U, so $[U, H] \neq 0$ as $m(U) > 1$. Further $H \trianglelefteq G$, so $1 \neq Aut_H(U) \trianglelefteq Aut_G(U)$. Then as $Aut_{HX}(U) = GL(U)$ is simple, $Aut_{HX}(U) = Aut_H(U)$. Thus $HX = HR_0$, and U is the natural module for $H/C_H(U) = GL(U)$. As $\dim(U) > 2$, U is not a self-dual H-module. Thus U is not nondegenerate, so as H is irreducible on U, U is totally isotropic. Thus $m(U) \leq d/2$, so as we showed earlier that $m(U) \geq d/2$, we conclude $m(U) = d/2$. Therefore $d > 4$ as $m(U) > 2$. Also $U = U^\perp$, so V/U is isomorphic to the dual of U as an HX-module. In particular R_0 also centralizes V/U, so $R_0 \leq O_2(HX)$, and we conclude $R_0 = O_2(HX)$ using an earlier observation. But as $H \trianglelefteq G$, $O_2(H) \leq O_2(G) = 1$ by G.10.3, so that R_0 centralizes H. However R_0 is contained in the unipotent radical R in $Sp(V)$ of the stabilizer P of U, and now $H \cong GL(U)$ is a complement to R in P, so $C_R(H) = Z(P) = 1$ as $d > 4$. Thus $R_0 = 1$, so $HX = H = GL(U)$ is faithful on U, and in particular $X \leq H$.

We show next that V splits over U as an HX-module. By Hypothesis G.10.1.2.a, V_1 is centralized by a Sylow 2-subgroup T of G. Then $T \cap H \in Syl_2(H)$, and $T \cap H$ acts on $V_1^\perp \cap U = A$, so $A_1 := C_A(T \cap H)$ is a point. As X induces the full group of transvections on U with axis A, there is $x \in X$ with $A_1 = [U, x]$; let Y be the subgroup of H inducing transvections on U with center A_1, so in particular $x \in Y$. Now Y is the unipotent radical of the stablizer in $Sp(V)$ of A_1, so that $Y \leq T \cap H$, and Y induces the group of transvections on V/U with axis $A_1^\perp/U$. We saw earlier that W is an X-complement to U in V. Thus $[V, x] = [U, x] \oplus [W, x] = A_1 \oplus V_1$. Now $Y \leq T \leq C_G(V_1)$ and Y centralizes A_1, so $[V, x, Y] = 0$. Therefore as $N_H(Y)$ is 2-transitive on $Y^\#$ and $x \in Y^\#$, Y is quadratic on V. Therefore V splits over U as an H-module by B.4.9. Thus $V = U \oplus U'$ for some H-submodule U' isomorphic to the dual of U.

Observe that we have symmetry between U and U': In case (1) this is clear. Furthermore only case (1) is used in the proof of the next lemma to show when $d > 4$ that G is irreducible on V; thus in case (2) we may assume that G is irreducible on V. As U' is the dual of U, H is irreducible on U' and $[U', X] \neq 0$. By Hypothesis G.10.1.2.a, $C_G(V_1)$ contains a Sylow 2-group T of G. Then as $\{U, U'\} = Irr_+(H, V)$ and G is irreducible on V, there exists $t \in T$ with $U^t = U'$; so as $V_1 \not\leq U$, also $V_1 \not\leq U'$. This establishes the symmetry in case (2).

By symmetry X induces the full group of transvections with axis $V_1^{\perp} \cap U'$ on U'; but this is impossible as $\dim(U) > 2$ and U' is dual to U as an X-module, so that X induces transvections on U' with a common center. □

LEMMA G.10.5. *Either*

(1) G is irreducible on V, or

(2) $d = 4$, and either $G \cong S_3 \times S_3$ is generated by transvections on V, or G is the extension of $O_3(G) \cong E_9$ by an involution inverting $O_3(G)$.

PROOF. Assume $0 < U$ is a proper G-submodule of V, and choose U minimal subject to this constraint, so that G is irreducible on U. Then by G.10.4, $m(U) = 2$ and $d = 4$. Now $O_2(G) = 1$ by G.10.3, so G is a subgroup of $GL(U) \times GL(V/U) \cong L_2(2) \times L_2(2)$. If $O(G) \cong \mathbf{Z}_3$, then $G \cong S_3$, so as $V = \langle V_1^G \rangle$ and $|V_1^G| \leq 3$, $m(V) \leq 3$, contrary to $d = 4$. Thus $O(G) \cong E_9$, so $O(G)$ has two 2-dimensional irreducibles U and $U^{\perp}$, which must be G-invariant. If $m(G) > 1$ then $G \cong S_3 \times S_3$, so (2) holds; otherwise by Hypothesis G.10.1, $m(X) = 1$ and X is nontrivial on a totally isotropic line W, so X is not generated by a transvection, and again (2) holds. □

In the remainder of the section assume:

HYPOTHESIS G.10.6. *G is irreducible on V and L is a component of G with $[L, X] \neq 1$. Set $H := \langle L^G \rangle$.*

LEMMA G.10.7. *$C_V(H) = 0$, $V = [V, H]$, and for each $I \in Irr_+(H, V)$:*

(1) I is an irreducible H-module, I is a TI-set under G, and V is a semisimple H-module.

(2) Either I is totally isotropic or I is nondegenerate.

(3) $V = I_1 \oplus \cdots \oplus I_k$ with $I_1 := I$ and $I_i \in Irr_+(H, V)$ such that either

(a) I is nondegenerate and $I^{\perp} = I_2 \oplus \cdots \oplus I_k$, or

(b) I is totally isotropic, and we may order the summands so that I_2 is dual to I as an H-module, $I + I_2$ is nondegenerate, and $(I + I_2)^{\perp} = I_3 \oplus \cdots \oplus I_k$.

(4) $m(I) > 2$.

PROOF. As H is normal in G and G is faithful and irreducible on V, $V = [V, H]$ and $C_V(H) = 0$. Thus each $I \in Irr_+(H, V)$ is an irreducible H-module. Then by Clifford's Theorem, V is the direct sum of G-conjugates of I, so (1) holds. Further $\mathrm{Rad}(I) := I \cap I^{\perp}$ is either I or 0. In the first case I is totally isotropic, and in the second I is nondegenerate, so (2) holds. If I is nondegenerate, $V = I \oplus I^{\perp}$, and as V is the direct sum of copies of conjugates of I, so is $I^{\perp}$, and hence (3a) holds. If I is totally isotropic, then $I \leq I^{\perp}$ and $V = I^{\perp} \oplus I_2$ for some $I_2 \in Irr_+(H, V)$ with $I_2 \cong V/I^{\perp}$ isomorphic to the dual of I. Then $I + I_2$ is nondegenerate, so $V = (I + I_2) \oplus (I + I_2)^{\perp}$, and (3b) holds. Of course (4) holds as L is nonsolvable. □

During the remainder of the section, we assume:

Hypothesis G.10.8. *Hypotheses G.10.1 and G.10.6 hold, and in addition:*
(3) Either:
(i) $V_1 = C_V(X)$*, or*
(ii) $|L^G| \leq 2$*.*
(4) $X \trianglelefteq T \in Syl_2(C_G(V_1))$*.*

Lemma G.10.9. *(1)* $T \in Syl_2(G)$*.*
(2) $[X, T \cap H] \leq X \cap H$*, so* $\Phi([X, T \cap H]) = 1$*.*

Proof. Part (1) follows from G.10.1.2.a. By Hypothesis G.10.8.4, $X \trianglelefteq T$, so $[X, T \cap H] \leq X \cap H$. Also $\Phi([X, T \cap H]) \leq \Phi(X \cap H) = 1$ using Hypothesis G.10.1.2, so (2) holds. □

Lemma G.10.10. *If* U *is a nonzero* HX*-submodule of* V*, then* $V_1 \leq U$ *and* $[U, X] \neq 0$*.*

Proof. Replacing U by a minimal nonzero HX-submodule of U, we may assume HX is irreducible on U. The conclusion of G.10.4 does not hold as H is not solvable; therefore if $[U, X] \neq 0$, then $V_1 \leq U$, so the lemma holds. Thus we may assume $[U, X] = 0$, and it remains to derive a contradiction. As $C_U(H) = 0$ by G.10.7, $m(U) > 1$, so $C_V(X) > V_1$ as $m(V_1) = 1$. Thus $|L^G| \leq 2$ by Hypothesis G.10.8.3. Now $[H, X] < H$ as $X \leq C_G(U)$ and $C_V(H) = 0$. Therefore as H is the product of at most two components, and $[H, X]$ is a proper normal subgroup of H, $[H, X]$ is a component L of H by 31.4 in [**Asc86a**]. Hence $|L^G| = 2$, and so by G.10.9.1 there is $t \in T - N_T(L)$ with $H = LL^t$. This is impossible, as T normalizes X by Hypothesis G.10.8.4, so T normalizes $[H, X] = L$, contrary to $L \neq L^t$. □

Lemma G.10.11. *For* $I \in Irr_+(H, V)$*,* X *acts on* I *iff* $V_1 \leq I$*.*

Proof. If $V_1 \leq I$ then X acts on I as I is a TI-set by G.10.7.1. Conversely if X acts on I then $V_1 \leq I$ by G.10.10. □

Lemma G.10.12. *If* $I \in Irr_+(H, V)$ *with* $V_1 \not\leq I$ *then*
(1) $m(W \cap I) \leq 1$*, and*
(2) If $I \cap W \neq 0$*, then* $N_X(I)$ *is a hyperplane of* X*.*

Proof. By part (e) of Hypothesis G.10.1.2, $C_X(w)$ is a hyperplane of X for each $w \in W - V_1$. If $0 \neq w \in I \cap W$, this hyperplane normalizes I as I is a TI-set by G.10.7.1, while X does not normalize I by G.10.11, so (2) holds. Further if $w' \in W - (V_1 + w)$, then $C_X(w) \neq C_X(w')$ by Hypothesis G.10.1.2.e. Thus if $w, w' \in I$, then $X = C_X(w)C_X(w')$ acts on I, contrary to G.10.11. Thus (1) holds. □

Lemma G.10.13. *If* $V_1 \leq I \in Irr_+(H, V)$*, then either* I *is totally isotropic, or* H *is irreducible on* $V = I$*.*

Proof. Assume that I is not totally isotropic. Thus by G.10.7.2, I is nondegenerate, so $V = I \oplus I^\perp$. By G.10.11, X acts on I, so X acts on $I^\perp$. Thus if $I < V$, $V_1 \leq I \cap I^\perp = 0$ by G.10.10, a contradiction. □

Lemma G.10.14. *Assume* $V_1 \not\leq I \in Irr_+(H, V)$ *and* $x \in N_X(I)^\#$ *with* $[H, x] \not\leq C_H(I)$*. Then one of the following holds:*
(1) $I \leq V_1^\perp$ *and* x *induces an* $\mathbf{F}_2$*-transvection on* I *with center* $I \cap W$*.*
(2) $I \not\leq V_1^\perp$ *and* x *induces a transvection on* I *with axis* $I \cap V_1^\perp$*.*

(3) $I \not\leq V_1^\perp$, $m([I,x]) \leq 2$, *and* x *induces a transvection on* $I \cap V_1^\perp$ *with center* $I \cap W$.

PROOF. As $[H,x] \not\leq C_H(I)$, $[I,x] \neq 0$. By G.10.12.1, either $I \cap W = 0$ or $I \cap W$ is a point. By Hypothesis G.10.1.2.c, $[I \cap V_1^\perp, x] \leq I \cap W$, so if $I \leq V_1^\perp$ then (1) holds. Thus we may assume $I \not\leq V_1^\perp$. If x centralizes $V_1^\perp \cap I$, then (2) holds, so we may assume $[I \cap V_1^\perp, x] = I \cap W$ is a point. Thus (3) holds. □

THEOREM G.10.15. *(1) One of the following holds:*

(i) V *is a homogeneous* H*-module, and there exists a unique* X*-invariant* $I \in Irr_+(H,V)$*: the one containing* V_1.

(ii) $V = I \oplus I^x$ *for* $I \in Irr_+(H,X)$, *where* $x \in X - N_X(I)$ *and* I *is not* H*-isomorphic to* I^x. *Further* $v_1 = i + i^x$, *where* $I \cap W = \langle i \rangle$, $\langle i \rangle = C_I(N_X(I))$ *if* $V_1 = C_V(X)$, $N_X(I)$ *is faithful on* H *and on* I, *and* $N_X(I)$ *induces the full group of transvections on* $I \cap V_1^\perp$ *with center* $\langle i \rangle$.

(2) One of the following holds:

(I) V *is a homogeneous* H*-module,* I *is self-dual,* H *is faithful on* I, *and either* I *is totally isotropic or* $I = V$.

(II) V *is not a homogeneous* H*-module,* I *is not self-dual,* H *is faithful on* I, I *is totally isotropic, and* I^x *is isomorphic to the dual of* I *as an* H*-module for* $x \in X - N_X(I)$.

(III) V *is not a homogeneous* H*-module,* I *is self-dual and nondegenerate,* H *is not faithful on* I, $d = 8$, $H = LL^x$ *for* $x \in X - N_X(I)$, $X = (X \cap H)\langle x \rangle \cong E_8$, $I = [V,L]$ *is the* A_5*-module for* $L \cong L_2(4)$, *and* $I^x = C_V(L)$.

(3) $H/C_H(I) < GL(I)$.

PROOF. Recall from G.10.7 that each $I \in Irr_+(H,V)$ is an irreducible H-module of rank at least 3. By A.1.42, there exists $I \in Irr_+(H,V,T \cap HX)$, I is an H-homogeneous component of $I_X := \langle I^X \rangle$, and I_X is the direct sum of the X-conjugates of I.

Suppose that $I_X = I$. Then by G.10.11, $V_1 \leq I$ and I is the unique X-invariant member of $Irr_+(H,V)$. Thus X acts on the H-homogeneous component of I in V, and hence on the sum of the remaining H-homogeneous components, so by G.10.10, V is H-homogeneous and conclusion (i) of (1) holds. Thus as H is faithful on V, H is faithful on I. If $I = V$, then I is a self-dual H-module as $H \leq Sp(V)$, so conclusion (I) of (2) holds. If $I < V$, then by G.10.7.3, some $I' \in Irr_+(H,V)$ is isomorphic to the dual I^* of I, so I is a self-dual H-module as V is H-homogeneous. Also I is totally isotropic by G.10.13, so again conclusion (I) of (2) holds. Finally if $H/C_H(I) = GL(I)$ then I is not self-dual under H since $m(I) > 2$, whereas we just saw I is self-dual. Thus (3) holds.

Therefore for the remainder of the proof, we assume $I < I_X$. Let $x \in X$ with $I^x \neq I$; then by definition of $Irr_+(H,V,X)$, $I^x \not\cong I$ as H-module. Also $V_1 \not\leq I$ by G.10.11. Set $W_x := [V_1^\perp \cap I, x]$; then $W_x \leq W$ by Hypothesis G.10.1.2.c. As W_x is isomorphic to $V_1^\perp \cap I$ under the X-equivariant map $\varphi : v \mapsto v + v^x$ of V into $[V,x]$, $m(W_x) = m(V_1^\perp \cap I) \geq m(I) - 1 \geq 2$. Suppose that $V_1 \leq W_x$. Then $V_1 \leq I + I^x$, so since I_X is the direct sum of the X-conjugates of I and X centralizes V_1, X acts on $I + I^x$. Hence X permutes $\{I, I^x\}$, and so $Y := N_X(I)$ is a hyperplane of X. On the other hand if $V_1 \not\leq W_x$, then for $w \in W_x^\#$, $C_X(w)$ is a hyperplane of X and $C_X(w) \neq C_X(w')$ for $w' \in W_x - (V_1 + w)$, so $X = C_X(w)C_X(w')$ acts on $I + I^x$, and hence again X permutes $\{I, I^x\}$ so that $|X : Y| = 2$.

Therefore in any case $I_X = I \oplus I^x$ and $|X : Y| = 2$. Thus $V_1 \leq I_X$ by G.10.10. Define V_I to be the H-homogeneous component of V containing I. Then X acts on $V_I \oplus V_I^x$, and hence on the sum of the remaining H-homogeneous components, so $V = V_I \oplus V_I^x$ by G.10.10. As $C_V(X) = C_{V_I}(Y)\varphi$ and $V_1 \leq I_X$, $v_1 = i\varphi$ for some $i \in C_I(Y)$. In particular $V_1 \leq W_x$, and if $V_1 = C_V(X)$, then $C_I(Y) = \langle i \rangle$.

By Hypothesis G.10.1.2.e, X induces the full group of transvections on W with center V_1, and hence also on any subspace of W containing V_1. Therefore the complement Y to $\langle x \rangle$ in X induces the full group of transvections with center V_1 on W_x. Thus as $\varphi : I \cap V_1^{\perp} \to W_x$ is a Y-isomorphism, Y induces the full group of transvections on $I \cap V_1^{\perp}$ with center $\langle i \rangle$. Therefore $i \in [V_1^{\perp}, X] \leq W$, so by G.10.12.1, $\langle i \rangle = W \cap I$.

To complete the proof that conclusion (ii) of (1) holds, and hence complete the proof of (1), it remains to show that $I_X = V$ and that Y is faithful on I and H. We now show that (3) implies each of these statements: Namely assume that $H/C_H(I) < GL(I)$. As HX is irreducible on I_X, I_X is either totally isotropic or nondegenerate. If I_X is totally isotropic, then as $V_1 \leq I_X$, $I_X \leq V_1^{\perp}$. Then by the previous paragraph, Y induces the full group of transvections on $V_1^{\perp} \cap I = I$ with center $\langle i \rangle$, so G.3.1 shows $H/C_H(I) = GL(I)$, contrary to our assumption that (3) holds. Thus I_X is nondegenerate, so HX acts on $I_X^{\perp}$, and as usual $V = I_X$ by G.10.10. Therefore Y is faithful on I. Since H is irreducible on I, $C_{GL(I)}(Aut_H(I))$ is of odd order, so $C_Y(H) \leq C_Y(I) = 1$. Thus we have established our claim that (3) suffices to complete the proof of (1).

Therefore it remains to establish (2) and (3). Notice that as $V = V_I \oplus V_I^x$ is the direct sum of copies of $I \oplus I^x = I_X$, H is faithful on I_X. Further by G.10.7.3, the dual I^* of I is isomorphic to I or I^x; and in the latter case $C_H(I) = C_H(I^*) = C_H(I^x)$, so H is faithful on I. Put another way, if H is not faithful on I, then I is self-dual.

Assume that $C_H(I) \not\leq Z(H)$. Then some component K of H centralizes I, so as K is faithful on $I_X = I \oplus I^x$ and H is irreducible on I and I^x, $I = C_{I_X}(K)$ and K is faithful on I^x. Therefore as G is transitive on the components of H, $H = H_I H_I^x$, where H_I is the product of those components of H faithful on I, H_I centralizes I^x, and H_I^x is faithful on I^x and centralizes I. Indeed $H = H_I \times H_I^x$ as H is faithful on I_X.

Next by G.10.9.3, $X_H := [x, T \cap H] \leq X \cap H$ and $\Phi(X_H) = 1$. So as $T \cap H \in Syl_2(H)$ and $H = H_I \times H_{I^x}$, $\Phi(T \cap H) \cong \Phi(X_H) = 1$ and $X_H = [x, T \cap H] = C_{T \cap H}(x)$ is the full x-diagonal subgroup of $T \cap H = (T \cap H_I) \times (T \cap H_I^x)$. Then since X is elementary abelian by hypothesis, $X \cap H \leq C_{T \cap H}(x) = X_H$, so $X \cap H = X_H$. Thus $Aut_{X_H}(I) = Aut_{T \cap H_I}(I) \cong T \cap H_I \in Syl_2(H_I)$ is elementary abelian.

Notice that $X_H \leq N_X(I) = Y$, and X_H is faithful on I. If an element of X_H induces an $\mathbf{F}_2$-transvection on I, then as H_I is semisimple, faithful, and irreducible on I, a result of McLaughlin [**McL69**] shows that $H_I = GL(I)$ or $Sp(I)$. (We only need the case where H is an SQTK-group, so that G.6.4 suffices). This is impossible as H_I has abelian Sylow 2-subgroups, so no element of X_H induces a transvection on I. Hence $I \not\leq V_1^{\perp}$ by G.10.14, so $V_1^{\perp} \cap I$ is a hyperplane of I. We saw that Y induces the full group of transvections on $I \cap V_1^{\perp}$ with center $\langle i \rangle$, and as H is not faithful on I, that I is self-dual. Therefore as H is irreducible on I, G.12.12 says that HY preserves a symplectic form f on I for which $V_1^{\perp} \cap I$ is the subspace of I orthogonal to i under f, and $Aut_{HY}(I)$ satisfies one of the conclusions of Theorem G.12.1 on I: $Aut_{H_I Y}(I)$ is $Sp(I)$; $O(I,q)$ or $\Omega(I,q)$ for some quadratic form q on I; or $m(I) = 4$ and $Aut_{H_I Y}(I)$ is $Sp(I)' \cong A_6$ or S_5 with I the $L_2(4)$-module. As H_I has abelian Sylow 2-groups it follows that $m(I) = 4$ and either I is the orthogonal module for $Aut_{H_I Y}(I) \cong O_4^-(2)$ or $\Omega_4^-(2)$, or I is the $L_2(4)$-module for $Aut_{H_I Y}(I) \cong S_5$. The latter case is impossible, as there the Sylow 2-group $Aut_{X_H}(I)$ of $Aut_H(I)$ does not induce a group of transvections on $V_1^\perp \cap I$ with center $\langle i \rangle$. In the former case $X = X_H\langle x \rangle$ as X is abelian, so conclusion (III) of (2) holds, as does (3). Hence by our earlier reduction, the proof is complete in the case where $C_H(I) \not\leq Z(H)$.

We turn to the remaining case where $C_H(I) \leq Z(H)$ and we must show that (3) and conclusion (II) of (2) hold. We will prove (3) first, so assume $H/C_H(I) = GL(I)$. Since $O_2(H) = 1$ by G.10.3, and the Schur multiplier of $GL(I)$ is a 2-group, we conclude $C_H(I) = 1$ and $H \cong GL(I)$. As $I^x \not\cong I$ as an H-module, and $m(I) \geq 3$, x induces an outer automorphism on H. This is impossible as $\Phi([T \cap H, x]) = 1$ by G.10.9.2 and $T \cap H \in Syl_2(H)$.

This contradiction establishes (3), and hence by our earlier remark completes the proof of (1). Thus it remains to show conclusion (II) of (2) holds. Recall (3) also implies that $V = I \oplus I^x$.

Suppose first that I is nondegenerate. Then $Aut_{HY}(I) \leq Sp(I)$, and $I^x = I^\perp$ by G.10.7.3. As before we may apply G.12.12 to conclude $Aut_{HY}(I)$ is $Sp(I)$; or contains $\Omega(I,q)$ for some quadratic form q on I associated to $(\ ,\ )$; or $m(I) = 4$, and $Aut_H(I)$ is $Sp(I)'$, or S_5 with I the $L_2(4)$-module. However as I^x is not isomorphic to I, x induces an outer automorphism on $H/C_H(I) = E(H/C_H(I))$ not stabilizing the equivalence class (under conjugacy in $Inn(H)$) over $\mathbf{F}_2$ of I, which is impossible in each case: Namely the equivalence class is $Aut(E(H/C_H(I))$-invariant unless $E(H/C_H(I)) \cong Sp(V)$ or A_6, and in those cases $\Phi([x, T \cap H]) \neq 1$, contrary to G.10.9.2.

Therefore by G.10.7.3, I is totally isotropic and I^x is dual to I as an H-module, so H is faithful on I by an earlier remark. Thus we have shown that conclusion (II) of (2) holds in this case, completing the proof of Theorem G.10.15 at last. □

LEMMA G.10.16. *Assume $a \in X^\#$ with $m([V,a]) \leq 2$, and let $K := \langle a^G \rangle$. Then*
(1) $[H,a] \neq 1$.
(2) a acts on each member of $Irr_+(H,V)$.
(3) $H \leq K$ and $Irr_+(H,V) = Irr_+(K,V)$.
(4) Either
(i) H is irreducible on V, or
(ii) $V = I_1 \oplus I_2$, where I_1 and I_2 are isomorphic $H\langle a \rangle$-submodules of V, and a induces a transvection on I_i. In addition either $H\langle a \rangle \cong Sp(I_1)$ or $O(I_1,q)$ for some quadratic from q on I_1, or I_1 is the natural module for $H\langle a \rangle \cong S_n$.

PROOF. Let $I \in Irr_+(H,V)$. If $[H,a] = 1$ then H acts on $[V,a]$, impossible as each irreducible for H on V is of rank at least 3. Thus (1) holds. Similarly if $I \neq I^a$, then $m([V,a]) \geq m([I,a]) = m(I) > 2$, contrary to hypothesis. Thus (2) holds. By (1), $1 \neq [H,a] \leq K$, so K contains some component of H. Then as G is transitive on the components of $H = \langle L^G \rangle$, $H \leq K$. By (2), K acts on each member of $Irr_+(H,V)$, so (3) holds.

If H is not faithful on I, then conclusion (III) of Theorem G.10.15.2 holds. But then $X = (X \cap H)\langle x \rangle$ with $V = I \oplus I^x$ and $X \cap H$ diagonally embedded in H. Hence $m([V,y]) > 2$ for each $y \in X^\#$ as no element of H induces a transvection on I, contrary to the hypothesis that $m([V,a]) \leq 2$ for some $a \in X^{\#}$. Therefore H is faithful on I. Thus by (1), $[I,a] \neq 0$. By Clifford's Theorem, V is the sum of k conjugates of I for some $k \geq 1$. By (2), $m([V,a]) \geq k$, and a induces a transvection on each conjugate of I in case of equality. Thus as $m([V,a]) \leq 2$, either $k = 1$ so that (4i) holds, or $k = 2$, $V = I_1 \oplus I_2$ for some $I_i \in Irr_+(H,V)$, and a induces a transvection on I_i. In the latter case by [**McL69**] (again we only need the case where H is an SQTK-group, so that G.6.4 suffices.), I_i is the natural module for $H\langle a\rangle \cong GL(I_i)$, $Sp(I_i)$, $O^{\pm}(I_i)$, or S_n. The first case is out by Theorem G.10.15.3. In the remaining cases, (4.ii) holds. □

G.11. Representations on extraspecial groups for SQTK-groups

In this section we assume Hypothesis G.11.1, which strengthens Hypothesis G.10.1 by adding some extra conditions which are satisfied in section 12.8. See in particular 12.8.12.4.

HYPOTHESIS G.11.1. *Hypothesis G.10.1 holds, and*

(3) Either

(i) $V_1 = C_V(X)$, *or*

(ii) there exists $a \in X$ *with* $m([V,a]) \leq 2$ *and* $V_1 \leq [V,a]$.

(4) $X \trianglelefteq T$ *for some* $T \in Syl_2(C_G(V_1))$.

The next result is an analogue of the Main Theorem of Timmesfeld [**Tim78**], in that we use it to determine the list of possibilities for $\dot{H} := H/C_H(\hat{U})$ and its action on the quotient $U/Z(U)$ of the almost-extraspecial group U in section 12.8. We prove:

THEOREM G.11.2. *Assume Hypothesis G.11.1, and in addition assume G is a quotient of an SQTK-group. Then one of the following holds:*

(1) $d = 2$ *and* $G \cong S_3$.

(2) $d = 4$ *and G is a subgroup of* $Sp(V) \cong S_6$ *of order divisible by* 10 *or* 18.

(3) $d = 6$ *and G is a subgroup of* $SD_{16}/3^{1+2}$ *containing* $E_4/3^{1+2}$.

(4) $d = 6$ *and V is the natural module for* $F^*(G) \cong A_7$.

(5) $d = 6$ *and G is of index at most* 2 *in* $O(V,q) \cong S_8$ *for some quadratic form q on V of Witt index* 3 *associated to* (,).

(6) $d = 6$ *and V is the natural module for* $G \cong G_2(2)$.

(7) $d = 8$, *V is the orthogonal module for* $F^*(G) \cong \Omega_4^+(4)$, *and* $X \not\leq F^*(G)$.

(8) $d = 8$, $G \cong S_7$, *and* $V = I \oplus I^x$, *where I is a totally isotropic $E(G)$-submodule of rank* 4 *and* $x \in X - N_X(I)$.

(9) $d = 8$, $G \cong S_3 \times S_5$ *or* $S_3 \times A_5$, *and V is the tensor product of the natural module for S_3 and the natural module or A_5-module for* $L_2(4)$.

(10) $d = 8$, *G is the extension of* $A_6 \times \mathbf{Z}_3$ *by* $x \in X$ *inverting* $O_3(G)$ *and inducing a transposition on* $E(G)$, *and V is the tensor product of the natural modules for A_6 and* $\mathbf{Z}_3$.

(11) $d = 8$, $F^*(G) \cong L_2(8)$, *and* $V \otimes_{\mathbf{F}_2} \mathbf{F}_8 \cong N \otimes N^{\sigma} \otimes N^{\sigma^2}$, *where N is the natural module for* $F^*(G)$, *and* $\langle\sigma\rangle = Gal(\mathbf{F}_8/\mathbf{F}_2)$.

(12) $d = 8$, $F^*(G) = L \times L^x$, *and* $V = [V,L] \oplus [V,L^x]$, *with* $[V,L]$ *the orthogonal module for* $L \cong L_2(4)$, *and* $X = \langle x\rangle(X \cap LL^x) \cong E_8$.

(13) $d = 12$ *and* $G \cong \mathbf{Z}_2/\hat{M}_{22}$.

REMARK G.11.3. As is to be expected, many of these possibilities do arise in simple QTKE-groups: for example, conclusions (1), (2), (11), and (13) arise as sections of centralizers of involutions with large extraspecial 2-subgroups in M_{12}; $G_2(3)$, $U_4(2)$, $L_4^\epsilon(3)$, J_2 and J_3; ${}^3D_4(2)$; and J_4. Similarly (3), (7), and (9) arise in the non-quasithin shadows $U_5(2)$, HN, and $\Omega_8^-(2)$.

For the remainder of the section assume G, V is a counterexample to Theorem G.11.2.

By Hypothesis G.11.1.4, $X \trianglelefteq T$ for some Sylow 2-subgroup T of $N_G(V_1)$, and by Hypothesis G.10.1.2.a, $T \in Syl_2(G)$. By Hypothesis G.10.1.1, $G \leq Sp(V)$ and by G.10.3, $O_2(G) = 1$.

LEMMA G.11.4. *G is irreducible on V.*

PROOF. If not then by G.10.5, G, V is described in G.10.5.2. But then G is a subgroup of $Sp_4(2)$ of order divisible by 18, so conclusion (2) of Theorem G.11.2 holds, contrary to the choice of G, V as a counterexample. □

As we observed earlier, G.10.2 allows us to use the results of section G.9 in this section. The primary applications of Theorem G.9.4 and Theorem G.9.3 appear in lemmas G.11.5 and G.11.7—although at other times we obtain the hypotheses of section G.9 by other means than G.10.2.

LEMMA G.11.5. *X is not faithful on $F(G)$.*

PROOF. By G.10.2, $d \leq 2(m(X)+1)$, so if X is faithful on $F(G)$, then G, V, X appear in one of the cases of Theorem G.9.4. But then either conclusion (1), (2), or (3) of Theorem G.11.2 holds, or $d = 4$ and G is $\Gamma L_2(4)$. The first case contradicts the choice of G, V as a counterexample to Theorem G.11.2. In the second as X is faithful on $F(G)$, X is of order 2 and not normal in a Sylow 2-group of G, contrary to a remark above. □

By G.11.5 there is a component L of G with $1 \neq [L, C_X(F(G))] \leq [L, X]$, and by G.11.4, G is irreducible on V. Thus Hypothesis G.10.6 is satisfied with $H := \langle L^G \rangle$. As in the proof of Theorem G.9.3, L is simple and described in Theorem C (A.2.3), or L is quasisimple and described in A.3.6.2. Further either $H = L$, or $H = LL^t$ for some $t \in T - N_T(L)$ and L is isomorphic to $L_2(2^n)$, $Sz(2^n)$, $L_2(p)$, or J_1. In particular case (ii) of Hypothesis G.10.8.3 is satisfied, while Hypothesis G.10.8.4 holds by Hypothesis G.11.1.4. Thus Hypothesis G.10.8 is satisfied, so we may appeal to the results in section G.10, and in particular to Theorem G.10.15.

LEMMA G.11.6. *(1) V is a homogeneous H-module.*
(2) There is a unique X-invariant $I \in Irr_+(H, V)$: the one containing V_1.
(3) Either $I = V$ or I is totally isotropic.
(4) I is a self-dual H-module.
(5) H is faithful on I.
(6) $H < GL(I)$.

PROOF. Suppose first that conclusion (III) of Theorem G.10.15.2 holds. Thus $H = LL^x$ and $V = [L, V] \oplus [L^x, V]$ with $[L, V]$ the A_5-module for $L \cong L_2(4)$, and $X = \langle x \rangle X_H$, where $X_H := X \cap H \cong E_4$. In particular $End_L([V, L]) = \mathbf{F}_2$, so $C_G(H) = 1$, and hence $H = F^*(G)$. Thus conclusion (12) of Theorem G.11.2 holds, contrary to choice of G, V as a counterexample.

Assume next that conclusion (II) of Theorem G.10.15.2 holds. Then H is faithful on each $I \in Irr_+(H, V)$, and conclusion (ii) of Theorem G.10.15.1 holds, so $Y := N_X(I)$ is also faithful on I. Observe that the hypotheses of Theorem G.9.3 are now satisfied with $Aut_{HY}(I)$, I in the roles of "G, V": namely by Theorem G.10.15.1, Y induces the full group of transvections on $V_1^{\perp} \cap I$ with center $\langle i \rangle$, so that $m(Y) \geq m(I) - 2$. Thus $m(I) \leq m(Y) + 2 \leq 2(m_2(Aut_{HY}(I)) + 1)$, and hence (*) of section G.9 holds. Further if $H \neq L$ then L^t is the second component of H, and t defines a quasiequivalence of the L-module I with the L^t-module I^t. Finally by G.10.15, $I^t = I$ or I^x and I^x is isomorphic to I^*, so the L-module I is quasiequivalent to the L^t-module I or its dual. Therefore $Aut_{HY}(I)$ and I are described in Theorem G.9.3. Since I is not a self-dual H-module in G.10.15.2.II, one of conclusions (1), (3), (5), (8), (11), (13), (14), or (17) of Theorem G.9.3 holds. By Theorem G.10.15.3, conclusion (11) and also conclusion (13) with $d = 5$ are out, as is conclusion (8) when $n = 1$. As $m_2(Aut(H)) \geq m(Y) \geq m(I) - 2$, conclusions (1), (5), the remaining case of (13) with $d = 10$, (14), and (17) are out. Similarly in conclusion (8) we must have $n \leq 2$, so that $H \cong SL_3(4)$ since we just saw that $n > 1$. This last case is out by G.10.9.2, as then x induces an outer automorphism on H with $\Phi([x, T \cap H]) \neq 1$. Finally in case (3), I is totally isotropic by Theorem G.10.15.2, and $I^x \not\cong I$ so that x induces an outer automorphism on $H \cong A_7$. Also $End_L(I) \cong \mathbf{F}_2$ so that $C_G(H) = 1$. Thus conclusion (8) of Theorem G.11.2 holds, contrary to the choice of G, V as a counterexample.

Therefore conclusion (I) of Theorem G.10.15.2 holds. Thus V is a homogeneous H-module, so conclusion (i) of Theorem G.10.15.1 also holds. Now all the conclusions of G.11.6 follow from Theorem G.10.15. This completes the proof of G.11.6. □

Lemma G.11.7. *(1) X is not faithful on I.*
(2) H is not irreducible on V.

Proof. If (2) fails, then $V = I$, so that X is faithful on I, and hence (1) fails also; thus (2) is a consequence of (1), and it suffices to assume X is faithful on I and derive a contradiction.

By G.10.2, $d \leq 2(m(X/X_0) + 1)$. Also if $L < H$ then by G.11.6.1, the representations of L and L^t on I are quasiequivalent. Thus the hypotheses of Theorem G.9.3 are satisfied with $Aut_{HX}(I)$, I in the roles of "G, V". As I is a self-dual H-module by G.11.6.4, we must consider the complement of the set of cases analyzed in the proof of G.11.6: namely conclusions (0), (2), (4), (6), (7), (9), (10), (12), (15), and (16) of Theorem G.9.3. Conclusion (4) is out, as in that case conclusion (4) of Theorem G.11.2 holds: for example $H = F^*(G)$ since $End_H(I) \cong \mathbf{F}_2$.

Suppose that $I < V$. Then as V is a homogeneous H-module by G.11.6.1, $m(V) = kd_I$, where $d_I := m(I)$ and $k := d/d_I \geq 2$, so

$$d_I \leq \frac{2(m(X) + 1)}{k} \leq m(X) + 1 \leq m_2(Aut_G(I)) + 1.$$

Inspecting the remaining conclusions, we conclude case (9) holds with $n = 1$ and $k = 2$: namely I is the natural module for $Aut_{E(G)}(I) \cong A_6$, and $Aut_G(I) \cong S_6$ since the inequality requires $m_2(Aut_G(I)) = 3$. As V is homogeneous and $k = 2$, $|Irr_+(H, V)| = 3$ and $C_{GL(V)}(H) \cong L_2(2)$. As G is irreducible on V by G.11.4, G is transitive on $Irr_+(G, V)$, so $O_3(C_G(H)) \neq 1$ and hence $C_G(H) \cong \mathbf{Z}_3$ or $L_2(2)$. As S_6 contains $S_3 \times S_3$, we conclude from A.1.31.1 that the former case holds, and further that some $x \in X - H$ inverts $O_3(G)$. But now conclusion (10) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample.

Therefore G is irreducible on V. Then conclusions (2), (12), and (16) of Theorem G.9.3 are out, since they appear as conclusions (13), (5), and (11) of Theorem G.11.2: In cases (12) and (16) of Theorem G.9.3, $End_H(I) \cong \mathbf{F}_2$, so $F^*(G) = H$. In case (2) of Theorem G.9.3, $End_H(I) \cong \mathbf{F}_4$, with $Z(H) = End_H(I)^{\#}$, so $C_G(H) = Z(H)$ and again $F^*(G) = H$.

This leaves conclusions (0), (6), (7), (9), (10), and (15) of Theorem G.9.3.

We begin by excluding some small cases: In conclusion (7), if $n = 2$ then $d = 4$ and conclusion (2) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample. Thus we may assume that $n \geq 4$ when conclusion (7) of Theorem G.9.3 holds. Similarly $n > 2, 1$ when conclusions (6), (9) of Theorem G.9.3 hold, respectively. Suppose that conclusion (10) of Theorem G.9.3 holds with $n = 1$. By the usual argument $F^*(G) = H$, so $G \cong U_3(3)$ or $G_2(2)$. If G is $U_3(3)$, then $m_2(G) = 2$, so by G.10.2, X is a normal 4-subgroup of T and $X_0 = 1$. But then X is uniquely determined in T, and centralizes the unique T-invariant 2-subspace V_2 of V. Thus V_2 must be contained in the T-invariant 3-subspace W, contrary to Hypothesis G.10.1.2.e. This contradiction shows $G \cong G_2(2)$, so conclusion (6) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample. Therefore $n > 1$ when conclusion (10) of Theorem G.9.3 holds.

A similar argument eliminates conclusion (0): As usual $F^*(G) = H \cong M_{12}$, so that $G \cong Aut(M_{12})$ as $m_2(G) = 4$ by G.10.2. But then G.10.2 also forces $X_0 = 1$, so $X \cong E_{16}$. Then for $x \in X - H$, $C_G(x) \cong E_4 \times L_2(4)$, so $X \in Syl_2(C_G(x))$. Also x inverts an element of order 11 in H, so $C_V(x) = [V, x]$ is of rank 5. Now as in the proof of H.11.1, $[V, x, E(C_G(x))]$ is the $L_2(4)$-module for $E(C_G(x))$, so $m(C_{[V,x]}(X)) \geq 2$, and hence case (i) of Hypothesis G.11.1.3 does not hold. But $m([V, a]) > 2$ for each involution $a \in X$ by H.11.1.2, so case (ii) of Hypothesis G.11.1.3 also fails. This shows case (0) cannot hold.

We have reduced to cases: (6) with $n \geq 3$; (7) with $n \geq 4$; (9) and (10) with $n \geq 2$; and (15). Thus in all cases, H preserves an $\mathbf{F}_{2^m}$-structure on V for $m > 1$, and indeed we can take $m = n$ in all cases except (7), where we take $m = n/2$.

We restrict ourselves for the time being to the cases other than (15). First assume that X preserves an $\mathbf{F}_{2^r}$-structure on V for some $r > 1$ dividing n. Then (3.i) of Hypothesis G.11.1 does not hold, so (3.ii) holds; that is $m([V, a]) \leq 2$ for some $a \in X$. Thus $r = 2$ and a induces an $\mathbf{F}_4$-transvection on V. This eliminates conclusion (10) of Theorem G.9.3, and as we are excluding (15) by assumption, conclusion (7) or (9) of Theorem G.9.3 holds in this case, with $d = 8$, $H \cong \Omega_4^-(4)$ or $Sp_4(4)$, and $HX \cong O_4^-(4)$ in the former case. By G.10.2, $m(X/X_0) \geq 3$, so for some $x \in X$, $\dim_{\mathbf{F}_4}(C_V(x)) = 2$. Let V_2 be the $\mathbf{F}_4$-point containing V_1; then $V_2^\perp$ is a 3-dimensional $\mathbf{F}_4$-subspace contained in $V_1^\perp$, so as $\dim_{\mathbf{F}_4}(C_V(x)) = 2$, $[V_2^\perp, x]$ contains an $\mathbf{F}_4$-point U. By Hypothesis G.10.1.2.c, $U \leq W$, contrary to Hypothesis G.10.1.2.e.

Therefore X does not preserve an $\mathbf{F}_{2^r}$-structure on V for any $r > 1$ dividing n, except possibly in case (15) of Theorem G.9.3. However in case (7), HX preserves an $\mathbf{F}_{2^{n/2}}$-structure and $n \geq 4$, so conclusion (7) of Theorem G.9.3 cannot hold.

Thus conclusion (6), (9), or (10) holds, and HX induces inner-field automorphisms on V, since in case (9) of Theorem G.9.3, any automorphism nontrivial on the Dynkin diagram does not preserve V. Hence H preserves an $\mathbf{F}_{2^n}$-structure on V, so X preserves an $\mathbf{F}_{2^{n/2}}$-structure on V. Therefore by the previous paragraph, $n = 2$ and some $x \in X$ induces a field automorphism on L; so case (9) or (10) holds, and $L \cong Sp_4(4)$ or $G_2(4)$, respectively. But then $\Phi([x, T \cap H]) \neq 1$, contrary to G.10.9.2.

This leaves conclusion (15) of Theorem G.9.3. Here there exists $\tau \in O_4^+(2^n)$ interchanging the two components of H and preserving the $\mathbf{F}_{2^n}$-structure. Further all involutions in $N_{Sp(V)}(H)$ are contained in $H\langle \tau, t\rangle$, where t centralizes τ, induces a field automorphism on H, and $m([V,t]) = m([V, t\tau]) = 2n$. Thus $m(C_V(X)) \geq n/2$, and $m([V,x]) \geq n$ for all $x \in X^\#$, so it follows from Hypothesis G.11.1.3 that $n = 2$. Further if $X \leq H$, then $m(C_V(X)) \geq 2$ and $m([V,x]) = 4$ for each $x \in X^\#$, contrary to Hypothesis G.11.1.3. Thus $X \not\leq H$. Also $End_H(V) \cong \mathbf{F}_4$, so $C_G(H)$ is of order at most 3. But $\Omega_4^+(4)$ contains a subgroup isomorphic to $S_3 \times S_3$, so as G is a quotient of an SQTK-group, we conclude $C_G(H) = 1$ from A.1.31.1. Thus conclusion (7) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample. This completes the proof of G.11.7. □

Set $d_I := \dim(I)$ and $k := d/d_I$; thus by G.11.6.1, V is the direct sum of k copies of I as an H-module, and by G.11.7.2, $k > 1$. Let $F := \mathbf{F}_{2^e} = End_H(I)$, $G_0 := N_{GL(V)}(H)$, $C_0 := C_{G_0}(H)$, and $U := Hom_H(I, V)$. As H is homogeneous on V, by 13.4 in [**Asc86a**], there is an F-space structure U_F on the $\mathbf{F}_2$-space U; hence there is a representation $\varphi : G_0 \to P\Gamma L(U_F)$ with $C_0\varphi = PGL(U_F)$, and a bijection $\psi : Irr_+(H,V) \to \Delta$, where Δ is the set of F-points of U_F, such that $\dim_F(U_F) = k$, $\ker(\varphi) = HF^\#$, and $(Jg)\psi = (J\psi)(g\varphi)$ for each $J \in Irr_+(H,V)$ and $g \in G_0$.

Let $C := C_G(H)$ and $m_C := m(C_X(H))$. Observe also that $C_X(H) = C_X(I) < X$, as H is faithful and irreducible on I and as X acts on I. By G.11.7.1, $m_C > 0$, so $C \neq 1$. As $O_2(G) = 1$, also $O_2(C) = 1$. As $C \leq C_0$, $C\varphi \leq PGL_k(F)$. If $F = \mathbf{F}_2$, then $PGL(U_F) \cong GL(U_F)$, so φ is faithful on C_0, and then $C \leq L_k(2)$.

LEMMA G.11.8. *(1)* $k > 2$.
(2) $C_V(X) = V_1$.

PROOF. First assume (2) fails. Then by Hypothesis G.11.1.3, $m([V,a]) \leq 2$ for some $a \in X^\#$. Thus H has at most two chief factors on V by G.10.16.4, so $k \leq 2$, and hence (1) also fails. Thus it will suffice to assume $k = 2$, and derive a contradiction.

We first claim that I is not the natural module for $H \cong A_n$, or a rank-4 module for $H \cong A_7$. For if so, $F = \mathbf{F}_2$, so $C \leq L_k(2) = L_2(2)$. Thus as $m_C > 0$ and $O_2(C) = 1$, $C \cong L_2(2)$. Then, unless $H \cong A_5$, a 2-element of C centralizes a noncyclic 3-subgroup of H, contrary to A.1.31.1 in view of our hypothesis that G is a quotient of an SQTK-group. On the other hand if $H \cong A_5$, then as $C = C_G(H) \cong S_3$, we conclude $F^*(G) = H \times O_3(C)$, so that $G \cong S_3 \times A_5$ or $S_3 \times S_5$, and $d = 8$, with V the tensor product of the natural modules for A_5 and $S_3 \cong L_2(2)$. Thus conclusion (9) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample. This establishes the claim.

Using the claim, we can also complete the proof of (2): For if (2) fails, then case (ii) of Hypothesis G.11.1.3 holds, and so case (ii) of G.10.16.4 holds. Since G is a quotient of an SQTK-group, I is the natural module for $H \cong A_n$ by G.6.4, a case which we just eliminated. So (2) holds.

Recall by construction that $X \not\leq C_G(H)$, so that $X \cap C < X$. Let $x \in X - C$. Suppose $W \leq I$. Then as $[V_1^\perp, x] \leq W$ by G.10.1.2.c, x induces a transvection on V/I with axis $V_1^\perp/I$. So by G.6.4, V/I is the natural module for $HX/C_{HX}(I) \cong$

$L_n(2)$ or S_n. As $I \cong V/I$ as an H-module, the latter case is impossible by the claim, and the former by G.11.6.6. This contradiction shows there is $w \in W - I$.

Let $Y := C_X(H)$ and recall that $Y = C_X(I) < X$. Further $Y\varphi \leq PGL_2(F)$, so $Y\varphi$ is semiregular on $\Delta - \{I\psi\}$, and hence Y is semiregular on $Irr_+(H,V) - \{I\}$. Thus $I = C_V(Y) = C_V(y)$ for $y \in Y^\#$. Then as a hyperplane of X centralizes $w \in W - I$ by G.10.1.2.e, we conclude that $m_C = m(Y) = 1$. Let $\overline{HX} = HX/C_{HX}(I)$. Then $m(\bar{X}) = m(X/Y) = m(X) - 1$, so

$$2d_I = d \leq 2(m(X) + 1) = 2(m(\bar{X}) + 2),$$

so $d_I \leq m(\bar{X}) + 2 < 2(m(\bar{X}) + 1)$, since $Y < X$. Thus $\bar{H}\bar{X}$, I satisfy the hypotheses of Theorem G.9.3, so examining the list of Theorem G.9.3 for those cases where $d_I \leq m(\bar{X}) + 2$, we conclude that I is either the natural module for $H \cong L_2(4)$, $L_n(2)$ or A_n for suitable n, $SL_3(4)$, or $Sp_4(4)$, or I is the 4-dimensional module for $H \cong A_7$. By the claim, I is not the A_n-module or the rank-4 module for A_7. By G.11.6.6, I is not the $L_n(2)$-module. In the remaining cases, $H \cong L_2(4)$, $SL_3(4)$, $Sp_4(4)$, so that $F = \mathbf{F}_4$ and hence $C\varphi \leq L_2(4)$. Then as $O_2(C) = 1$, $O^{2'}(C) \cong L_2(4)$, S_3, or D_{10}. Further by (2), $C_V(X) = V_1$, so some $x \in X$ induces a field automorphism on H; hence $m(\bar{X}) \leq m_2(C_{\bar{H}\bar{X}}(\bar{x})) = 2$, 3, or 4, respectively. Thus as $d_I \leq m(\bar{X}) + 2$, $H \cong L_2(4)$. Moreover X does not act on a D_{10}-subgroup of $C_0 \cong GL_2(4)$. Thus $O^{2'}(C)$ is $L_2(4)$ or S_3. We check as usual that $F^*(G) = H \times O^2(C)$ in the respective cases, and then that conclusion (7) or (9) of Theorem G.11.2 holds, contrary to our choice of G, V as a counterexample. This completes the proof of G.11.8. □

Define

$$\mathcal{J} := \{J \in Irr_+(H,V) : J \leq I^\perp\}.$$

As $I < V$ by G.11.7.2, $I \in \mathcal{J}$ by G.11.6.3.

LEMMA G.11.9. *(1) Each $x \in X - C_X(H)$ fixes some $I_x \in \mathcal{J} - \{I\}$.*
(2) x induces a transvection on I_x with center $W \cap I_x$.
(3) I is the natural module for $H \cong A_n$, $5 \leq n \leq 8$.
(4) $|X : C_X(H)| = 2$.

PROOF. Recall $\psi : Irr_+(H,V) \to \Delta$ is an X-equivariant bijection. Thus if I is the unique fixed point of x on $\mathcal{J}$, then $I\psi$ is the unique F-point of U_F fixed by x in the F-hyperplane U_0 of U_F whose points are those in $\mathcal{J}\psi$. Thus $k = \dim_F(U_F) \leq 3$, so $k = 3$ by G.11.8.1. Thus x fixes some $J\psi$ not in U_0. But now $J \notin \mathcal{J}$ so that $J \not\leq I^\perp$, and hence $I + J$ is nondegenerate. Then as $k = 3$, $(I + J)^\perp$ is in $\mathcal{J} - \{I\}$ and is fixed by x, contrary to our assumption. This establishes (1). Now $V_1 \leq I$ by G.11.6.2, so $I_x \leq I^\perp \leq V_1^\perp$, and hence (2) follows from G.10.14. By (2), and since G is a quotient of an SQTK-group, we may apply G.6.4 to conclude that I is the natural module for $H \cong A_n$ or $L_n(2)$. Then (3) holds by G.11.6.6. In particular x induces an outer automorphism in S_n on H. As this holds for each $x \in X - C_X(H)$, we conclude that (4) holds. This completes the proof of G.11.9. □

Observe that we can now derive a contradiction from G.11.9, establishing Theorem G.11.2: Namely by G.11.9.4, $X = \langle x \rangle C_X(H)$, so as we saw $C_X(H) = C_X(I)$, $0 \neq C_I(x) = C_I(X)$. Thus $V_1 = C_V(X) = C_I(X) = C_I(x)$ by G.11.8.2. This is impossible, as $m(C_I(x)) \geq 3$ by parts (2) and (3) of G.11.9. Thus the proof of Theorem G.11.2 is at last complete.

G.12. Subgroups of Sp(V) containing transvections on hyperplanes

In this section, we assume V is a d-dimensional symplectic space over $\mathbf{F}_2$ with bilinear form $(\ , \ )$, and $G \leq Sp(V)$. For $v \in V^{\#}$, let $Q(v) := C_G(v) \cap C_G(v^{\perp}/v)$; and define $\mathcal{V} = \mathcal{V}(G)$ to consist of those $v \in V^{\#}$ which satisfy the condition arising in G.10.15.1.ii:

$$Q(v) \text{ induces the full group of transvections on } v^{\perp} \text{ with center } \langle v \rangle. \qquad (*)$$

The main result in this section is:

THEOREM G.12.1. *Assume $u, v \in \mathcal{V}$ with $(u, v) \neq 0$ and $d \geq 4$. Then one of the following holds:*

(1) $G = Sp(V)$.

(2) There is a quadratic form q on V with bilinear form $(\ , \ )$ such that $G = O(V, q)$ or $\Omega(V, q)$, and v^G is the set of nonzero q-singular vectors in V.

(3) $d = 4$, and either

(a) $G = Sp(V)' \cong A_6$, or

(b) $G \cong S_5$, and V is the $L_2(4)$-module for $E(G)$.

Throughout this section, we assume the hypotheses of Theorem G.12.1.

Let $S := Sp(V)$, and pick $u, v \in \mathcal{V}$ with $(u, v) \neq 0$. Set $\tilde{V} := V/\langle v \rangle$ and $W := v^{\perp}$. For $H \leq S$ and $z \in V$, let H_z be the stabilizer in H of z.

LEMMA G.12.2. *(1) $C_V(x) \leq W$ for each $x \in Q(v)^{\#}$.*

(2) Either

(i) $Q(v)$ is faithful on W and is of rank $d - 2$, or

(ii) $Q(v)$ is the full unipotent radical of S_v of rank $d-1$, and in particular contains the transvection t_v with center v.

(3) $Q(v)$ induces the full group of transvections on $\tilde{V}$ with axis $\tilde{W}$.

PROOF. As $G \leq S$, the action of $Q(v)$ on $\tilde{V}$ is dual to its action on W, so (3) follows from (*). If $x \in Q(v)$ with $C_V(x) \not\leq W$, then by (3), x centralizes $\tilde{V}$, so $x \in \langle t_v \rangle$. Then as $W = C_V(t_v)$, (1) holds.

Let K be the kernel of the action of $Q(v)$ on W. By (*), $|Q(v) : K| = 2^{d-2}$, and of course $K \leq \langle t_v \rangle$, so either (2i) holds, or $K = \langle t_v \rangle$ and $|Q(v)| = 2^{d-1}$. But $Q(v)$ is contained in the unipotent radical $R(v)$ of S_v, and $|R(v)| = 2^{d-1}$, so in the latter case, (2ii) holds. $\square$

LEMMA G.12.3. *(1) $\langle Q(v), Q(u) \rangle$ is irreducible on V.*

(2) There exists $g \in \langle Q(v), Q(u) \rangle$ with $v^g \in V - W$.

PROOF. Let $H := \langle Q(v), Q(u) \rangle$. By (*) and G.12.2.1, $\langle v \rangle = C_V(Q(v))$. Thus if $0 \neq U$ is an H-submodule of V, then $v \in U$, and similarly $u \in U$ as $u \in \mathcal{V}$. By G.12.2.3, $[\tilde{u}, Q(v)] = \tilde{W}$, so $W \leq U$. Then as $u \notin v^{\perp} = W$ by hypothesis, $V = \langle W, u \rangle \leq U$, so (1) is established. Part (1) implies (2). $\square$

LEMMA G.12.4. *(1) If $V - W \subseteq v^G$ then G is transitive on $V^{\#}$.*

(2) If $W^{\#} \subseteq v^G$ then G is transitive on $V^{\#}$.

PROOF. Assume $V - W \subseteq v^G$. As $d \geq 4$, there is $w \in W - \langle v \rangle$. There is $u \in V - W$ with $(u, w) \neq 0$, so as $u \in v^G$ and $w \notin u^{\perp}$, by assumption $w \in u^G = v^G$. This establishes (1).

Next assume that $W^{\#} \subseteq v^G$. Then again for $z \in V - W$, there is $w \in W^{\#} \cap z^{\perp}$. Then as $w^{\perp} - \{0\} \subseteq w^G$ by assumption, while $w \in v^G$, $z \in v^G$, establishing (2). $\square$

LEMMA G.12.5. *If $Q(v)$ is not faithful on W, then $G = Sp(V)$.*

PROOF. Assume $Q(v)$ is not faithful on W. Then by G.12.2.2, $Q(v)$ is the full unipotent radical of S_v and $t_v \in G$. Thus $|Q(v)| = 2^{d-1}$, and by G.12.2.1, $Q(v)$ is semiregular on $V - W$. Thus as $|V - W| = 2^{d-1} = |Q(v)|$, $Q(v)$ is transitive on $V - W$, so $V - W \subseteq v^G$ by G.12.3.2. Therefore G is transitive on $V^\#$ by G.12.4.1, so as $t_v \in G$, $t_z \in G$ for each $z \in V^\#$. Thus $G = S$, as S is generated by its transvections. (cf. 22.4 in [**Asc86a**]). □

Because of G.12.5, we may assume during the remainder of the section that $Q(v)$ is faithful on W. So by G.12.2.2.i, $|Q(v)| = 2^{d-2}$.

LEMMA G.12.6. *(1) $Q(v)$ has two regular orbits on $V - W$, with representatives u and $u + v$.*

(2) For $w \in W - \langle v\rangle$, $w \in (v + w)^{Q(v)}$, so $w \in v^G$ iff $w + v \in v^G$.

(3) Either G is transitive on $V^\#$, or exactly one of u or $u + v$ is in v^G.

PROOF. Part (2) follows from (*). By G.12.2.3, $Q(v)$ is transitive on $\tilde{V} - \tilde{W}$, so each $z \in V - W$ is conjugate under $Q(v)$ to u or $u + v$. As $Q(v)$ is faithful on W, an argument in the proof of G.12.5 shows that $Q(v)$ has two regular orbits on $V - W$, each of length 2^{d-2}, so (1) holds. If both u and $u + v$ are in v^G, then $V - W \subseteq v^G$ by (1), so G is transitive on $V^\#$ by G.12.4.1. On the other hand by G.12.3.2, either u or $u + v$ is in v^G, so (3) is established. □

By G.12.3.2, $v^G \cap (V - W) \neq \emptyset$, so we may take $u \in v^G$. In the remainder of this section, assume G, V is a counterexample to Theorem G.12.1 of minimal dimension d.

LEMMA G.12.7. *If $d = 4$, then G is not transitive on $V^\#$.*

PROOF. Assume $d = 4$ and G is transitive on $V^\#$. Then

$$|G| = |V^\#| \cdot |G_v| = 15 \cdot |G_v|. \qquad (!)$$

Further as $Q(v)$ is faithful on W, so is G_v, so G_v is a subgroup of the stabilizer $GL(W)_v$ of v in $GL(W)$, containing $Q(v) = O_2(GL(W)_v) \cong E_4$, and hence $G_v \cong E_4$, D_8, A_4, or S_4. Thus by (!), $|G| = 60$, 120, 180, 360 in the respective case. Inspecting the subgroups of $Sp(V) \cong S_6$ transitive on $V^\#$, we conclude that the third case is impossible; in the first case that $G \cong L_2(4)$ with V the $L_2(4)$-module, also impossible as (*) is not satisfied; in the second case that G is the extension of this first group by an involutory outer automorphism; and in the fourth case that $G = Sp(V)' \cong A_6$. The second and fourth cases appear as conclusion (3) of Theorem G.12.1, contrary to the choice of G, V as a counterexample. □

LEMMA G.12.8. *Suppose there exists $w \in W \cap v^G - \{v\}$. Let $U := w^\perp$, $\hat{U} := U/\langle w\rangle$, and $G_w^* := G_w/C_{G_w}(\hat{U})$. Then*

(1) $\hat{U}$ is a symplectic space for the bilinear form $(\hat{x}, \hat{y}) := (x, y)$ for $x, y \in U$. Moreover $G_w^* \leq Sp(\hat{U})$.

(2) There exists $u \in v^G \cap (U - W)$.

(3) $\hat{v}, \hat{u} \in \mathcal{V}(G_w^*)$.

(4) If $d > 4$ there is a quadratic form q_w on $\hat{U}$ with bilinear form $(\ ,\ )$, such that $G_w^* = O(\hat{U}, q_w)$ or $\Omega(\hat{U}, q_w)$, and $\hat{v}^{G_w}$ is the set of q_w-singular vectors in $\hat{U}$.

(5) If $d > 4$ and $z \in U - \langle w\rangle$, then $z \in v^G$ iff $q_w(\hat{z}) = 0$.

PROOF. Part (1) is easy and well-known. As $w \neq v$ and $d \geq 4$, there exists $z \in U - W$. By G.12.6.3, at least one of z or $z + v$ is in v^G, so (2) holds and we may take $u \in U$. For $x \in U^\#$, let $P(x) := C_{Q(x)}(w)$. If $x \in v^G$, then by (*), $|Q(x) : P(x)| = 2$, and $P(x)^*$ induces the full group of transvections on $\hat{x}^\perp$ with center $\langle \hat{x} \rangle$. Thus (3) holds.

We next prove (4). Suppose $d > 4$. Then by (1)–(3), the pair $G_w^*, \hat{U}$ satisfies the hypotheses of Theorem G.12.1, so by minimality of d, the pair satisfies one of the conclusions of G.12.1. If conclusion (2) of Theorem G.12.1 holds, we obtain conclusion (4) of G.12.8. Thus we may assume conclusion (1) or (3) of Theorem G.12.1 holds, and it remains to derive a contradiction. In both cases (1) and (3) of Theorem G.12.1, G_w^* is transitive on $\hat{U}^\#$, so as $w \in v^G$, we conclude from G.12.6.2 that $W^\# \subseteq v^G$, and then conclude from G.12.4.2 that G is transitive on $V^\#$.

By G.12.2.3, $Q(v)$ is regular on $\tilde{V} - \tilde{W}$, so by a Frattini Argument $G_{v,\tilde{u}}$ is a complement to $Q(v)$ in G_v. Observe that the line $l := \langle u, v \rangle$ is nondegenerate and $O^2(G_{v,\tilde{u}})$ centralizes l, so $O^2(G_{v,\tilde{u}})$ acts on $l^\perp$, and $l^\perp$ is $O^2(G_{v,\tilde{u}})$-isomorphic to $\tilde{W}$.

Assume first that $d > 6$. Then the pair appears in case (1) of Theorem G.12.1, so $G_w^* = Sp(\hat{U})$ and $O^2(G_w^*)$ contains a transvection on $\hat{U}$; hence $O^2(G_{u,v})$ contains a transvection on $l^\perp$. Then as G is transitive on $V^\#$, we have a contradiction to our assumption that $Q(v)$ is faithful on W.

Therefore $d = 6$. Then as G is transitive on $V^\#$,

$$|G| = |V^\#||G_v| = (2^6 - 1)|Q(v)||G_w^*| = (2^6 - 1)2^4|G_w^*|,$$

so as G_w^* is S_5, A_6, or S_6 (since (1) or (3) of G.12.1 holds for G_w^*), $|G| = (2^6-1)\cdot 2^7 \cdot 15 \cdot a$, with $a := 1, 3$, or 6 in the respective case. Then as $|Sp_6(2)| = (2^6-1)\cdot 2^9 \cdot 15 \cdot 3$, $|Sp(V) : G| \leq 12$—impossible, as $Sp(V)$ is simple and is not isomorphic to a subgroup of S_{12}. This contradiction completes the proof of (4).

It remains to prove (5). By (4) and G.12.6.2, for $z \in U$, $q_w(\hat{z}) = 0$ iff $z \in v^{G_w}$. Thus it remains to take $y \in U$ with $q_w(\hat{y}) = 1$, assume $y = v^g$ for some $g \in G$, and derive a contradiction. Let $H := G_{w,y}$; then $Q(w) \cap H =: R(w)$ is of index 2 in $Q(w)$, and $H/R(w)$ is the stabilizer of the nonsingular vector $\hat{y}$ in the orthogonal group G_w^* of dimension $d - 2$ by (4), so $H^* \cong H/R(w)$ is isomorphic to $Sp_{d-4}(2)$ or $\mathbf{Z}_2 \times Sp_{d-4}(2)$. For $r \in Q(v)$ let α_r be the element of the dual space D of $\hat{U}$ with kernel $\widehat{C_U(r)}$. Then the map $r \mapsto \alpha_r$ is a G_w-isomorphism of $Q(w)$ with D which restricts to an H-isomorphism φ of $R(w)$ with the subspace D_y of D trivial on $\hat{y}$. Let $r_y \in R(w)$ with $C_U(r_y) = y^\perp \cap U$ and $\alpha := \alpha_{r_y}$. Then H acts on $\hat{y}^\perp/\langle \hat{y} \rangle$ and φ induces an H-isomorphism $r + \langle r_y \rangle \mapsto \alpha_r + \langle \alpha \rangle$ of $R(w)/\langle r_y \rangle$ with $D_y/\langle \alpha \rangle$, which is in turn isomorphic with the dual space of $\hat{y}^\perp/\langle \hat{y} \rangle$ via $\alpha_r + \langle \alpha \rangle \mapsto \beta_r$, where $\beta_r : \hat{u} + \langle \hat{y} \rangle \mapsto \alpha_r(\hat{u})$. Therefore $C_{R(w)}(H) = \langle r_y \rangle$ is of order 2, $R(w) \cap Q(y) \leq C_{R(w)}(H)$, and $H/R(w)$ acts as $Sp_{d-4}(2)$ on $R(w)/C_{R(w)}(H)$.

Let $Y := y^\perp$, $\dot{Y} := Y/\langle y \rangle$, and $G_y^+ :=: G_y/Q(y)$. We have shown that $R(w)^+/C_{R(w)^+}(H^+)$ is the natural module for $H^+/R(w)^+ \cong Sp_{d-4}(2)$. Now by (4), $\hat{v}^{G_w}$ is the set of q_w-singular vectors in $\hat{U}$, so that H^+ contains a Sylow 2-subgroup of G_y^+, and hence lies in a parabolic subgroup of G_y^+. But as G_y^+ is $O_{d-2}^\epsilon(2)$ or $\Omega_{d-2}^\epsilon(2)$, the Levi complements of those parabolics are of structure $L_r(2) \times \Omega_{d-2-2r}^\epsilon(2)$ for $1 \leq r \leq \lfloor (d-2)/2 \rfloor$; so no such parabolic of G_y^+ contains $Sp_{d-4}(2)$ unless $d = 6$ and $G_w^* \cong O_4^-(2)$ ($\epsilon \neq +$ since (1) or (3) of G.12.1 holds for

G_w^*). Then arguing as above, $|Sp(V) : G| = 12$, for the same contradiction. This completes the proof of (5), and hence of G.12.8. □

LEMMA G.12.9. *G is not transitive on $V^\#$.*

PROOF. If G is transitive on $V^\#$, then $d > 4$ by G.12.7; further the hypothesis of G.12.8 holds, and then G.12.8.5 says $U^\# \not\subseteq v^G$, a contradiction. □

Define $q : V \to \mathbf{F}_2$ by $q(z) := 0$ if $z = 0$ or $z \in v^G$, and $q(z) := 1$ otherwise. Observe G preserves q, as q is constant on orbits of G on V. Define $z \in V^\#$ to be *singular* if $q(z) = 0$, and *nonsingular* otherwise. Thus by definition v^G is the set of singular vectors.

LEMMA G.12.10. *Assume*

$$q(x + y) = q(x) + q(y) + (x, y) \text{ for all } x, y \in V. \qquad (!)$$

Then

(1) q is a quadratic form on V with bilinear form (,).

(2) $G = O(V, q)$ or $\Omega(V, q)$.

PROOF. Part (1) is trivial, since by definition a function $q : V \to \mathbf{F}_2$ is a quadratic form with bilinear form (,) precisely when (!) is satisfied. By an earlier remark, G preserves q, so $G \leq O(V, q) =: H$. But the unipotent radical $R(v)$ of H_v is of order $2^{d-2} = |Q(v)|$, and $Q(v) \leq R(v)$, so $Q(v) = R(v)$. Thus $\Omega(V, q) = \langle R(v), R(u)\rangle \leq G$, so (2) holds. □

LEMMA G.12.11. *(1) G is transitive on nonsingular vectors.*

(2) $d > 4$.

(3) For $e := 0, 1$, G is transitive on lines $l_e := \langle x, y\rangle$ where $q(x) = q(y) = 1$ and $(x, y) = e$. Further for each such line, $q(x + y) = e$.

PROOF. By G.12.9, G is not transitive on $V^\#$, so as we took $u \in v^G$, $u + v \notin v^G$ by G.12.6.3. Thus by G.12.6.1, the singular and nonsingular vectors in $V - W$ each form regular orbits under $Q(v)$ of size 2^{d-2}. By G.12.3.1, G is irreducible on V, so each vector in $W^\#$ is fused into $V - W$ under G, so (1) holds as $Q(v)$ is transitive on the nonsingular vectors in $V - W$.

Suppose that $v^G \cap W = \{v\}$. Then $v^G = \{v\} \cup u^{Q(v)}$, so G is 2-transitive on v^G of order $2^{d-2} + 1$. Thus the order of $Sp_d(2)$ is divisible by $2^{d-2} + 1$, so $d = 4$. Now G is 2-transitive on the 5 singular vectors of V, and contains the E_4-subgroup $Q(v)$, so we conclude $G \cong A_5$ or S_5 with V the orthogonal module; then conclusion (2) of Theorem G.12.1 holds, contrary to our choice of G, V as a counterexample. This contradiction shows there is $w \neq v$ in $v^G \cap W$, so that the hypothesis of G.12.8 holds.

Assume that $d = 4$, and define a quadratic form q_w on $\hat{U}$ by $q_w(\hat{z}) := q(z)$ for $z \in U$. By G.12.8.3, $\hat{U}$ contains distinct singular vectors $\hat{u}$ and $\hat{v}$, so as $u + v$ is nonsingular, the Witt index of $\hat{U}$ is 1 and U contains 5 singular vectors. As there are $2^{d-2} = 4$ singular vectors in $V - U$, there are 9 singular vectors in V. Further $|G_v| = 4$ or 8 as $|O_2^+(2)| = 2$, and hence $|G| = 36$ or 72. Then as $G \leq Sp(V)$, G is of index at most 2 in the normalizer of a Sylow 3-subgroup of $Sp(V)$. As these normalizers stabilize quadratic forms, conclusion (2) of Theorem G.12.1 holds, contrary to our choice of G, V as a counterexample. This establishes (2).

Let $l := l_e$ be a line as in (3). Observe that to prove (3), it suffices to show l is conjugate to a line in U: For if $l \leq U$, then $q_w(\hat{z}) = q(z)$ for $z \in l^\#$ by G.12.8.5, so the result holds as it holds for G_w^* on $\hat{U}$. Since $\hat{U}$ contains nonsingular vectors, by (1) we may assume $x \in U$ and $y \notin U$. Then $y^\perp \cap U =: Y$ is a complement to $\langle w \rangle$ in U with $(Y, q) \cong (\hat{U}, q_w)$. Therefore as $d > 4$, $Y \cap x^\perp$ contains a singular vector, so we are done by transitivity of G on singular vectors. □

To complete the proof of Theorem G.12.1, it suffices by G.12.10 to show that q satisfies condition (!) of G.12.10. Suppose first $x \in v^G$; then $q(x) = 0$ and as G preserves q, we may take $x = v$. If $y \in W$ then $v + y \in y^G$ by G.12.6.2, so $q(y) = q(y + v)$, and hence (!) holds as $(v, y) = 0$. Similarly if $y \notin W$, then by G.12.9 and G.12.6.3, exactly one of y, $y + v$ lies in v^G, so $q(y) \neq q(y + v)$, and again (!) is satified. Thus we may assume $q(x) = 1 = q(y)$. Then G.12.11.2 completes the proof.

Therefore Theorem G.12.1 is at last established.

We close the section with an application of Theorem G.12.1.

LEMMA G.12.12. *Assume H is a finite group, U is a faithful $\mathbf{F}_2H$-module of dimension $d > 2$, and:*

(i) $K \trianglelefteq H$ and U is an irreducible self-dual K-module.

(ii) There exist an elementary abelian 2-subgroup Q of H, a hyperplane W of U, and $v \in W^\#$, such that Q induces the full group of transvections on W with center $\langle v \rangle$.

Then

(1) There exists a KQ-invariant symplectic form f on U.

(2) W is the subspace $v^\perp$ of U orthogonal to v with respect to f.

(3) $v^K \subseteq \mathcal{V}(KQ)$ and $v^H \not\subseteq W$.

(4) Either

(a) f is H-invariant, or

(b) $H = \Gamma L_2(U_{\mathbf{F}_4})$ for some $\mathbf{F}_4$-space structure $U_{\mathbf{F}_4}$ on U preserved by H.

(5) KQ and its action on U satisfy one of the conclusions of Theorem G.12.1, as does the action of H on U, if H preserves f.

PROOF. Let $\mathcal{L}$ be the space of K-invariant bilinear forms on U and $F := End_K(U)$. By (i) and Exercise 9.1 in [**Asc86a**], $\mathcal{L}$ is a 1-dimensional F-space, where F-scalar multiplication on $\mathcal{L}$ is defined by

$$(\alpha \cdot f)(x, y) := f(x\alpha, y), \text{ for } f \in \mathcal{L},\ \alpha \in F, \text{ and } x, y \in U,$$

and each nonzero member of $\mathcal{L}$ is nondegenerate. Further the $\mathbf{F}_2$-subspace $\mathcal{S}$ of symmetric forms is nonzero, and H acts on $\mathcal{L}$ via the diagonal action. As $|\mathcal{S}^\#|$ is odd, Q fixes some $f \in \mathcal{S}^\#$, so $KQ \leq O(U, f) = Sp(U)$. Thus (1) holds. In particular d is even by (1), so as $d > 2$, $d \geq 4$.

Suppose (2) fails. Then there exists $w \in W - v^\perp$. Thus $l := \langle v, w \rangle$ is nondegenerate, so $W = l \oplus W'$, where $W' := l^\perp \cap W$. As $[W, Q] \leq \langle v \rangle$, Q acts on l, and hence on W'. Thus $[W', Q'] \leq \langle v \rangle \cap W' = 0$. But by (ii), $C_W(Q) = \langle v \rangle$, so $W' = 0$. Thus $W = l$ is of rank 2, contradicting $d \geq 4$. Hence (2) holds.

By (1), (2), and (ii), $v \in \mathcal{V}(KQ)$, so $v^K \subseteq \mathcal{V}(KQ)$. As K is irreducible on U, $v^K \not\subseteq W$. Thus (3) holds. Further (3) says that KQ satisfies the hypotheses of Theorem G.12.1, and hence also the conclusions of that theorem, as does H if H preserves f. Thus (5) is established. Now by inspection of the groups in

Theorem G.12.1, either $F = \mathbf{F}_2$; or $d = 4$, $KQ \cong S_5$, and U is the $L_2(4)$-module for $K \cong L_2(4)$, and $F = \mathbf{F}_4$. In the first case f is the unique symplectic form preserved by K, so as $K \trianglelefteq H$, (4a) holds. In the latter case (4a) or (4b) holds. $\square$

CHAPTER H

Parameters for some modules

The main purpose of this chapter is to establish the values of various parameters for the $\mathbf{F}_2G$-modules V with $1 < \hat{q}(G, V) \leq 2$ appearing in Tables 7.1.1 and 7.2.1. However we do not require each value in each case, so in those cases where the values are not needed (or where the proofs are particularly straightforward), we will not supply proofs that the values are correct. In some cases, we also establish more specialized facts required elsewhere in the proof of the Main Theorem.

There are also several sections on $\mathbf{F}_2$-permutation modules for $L_3(2)$, again used in various places in the proof of the Main Theorem.

Throughout most of this chapter M is a finite group, V is a faithful $\mathbf{F}_2M$-module, $L := M^\infty$, and $T \in Syl_2(M)$. We will be determining the values of the following parameters, the latter two given in Definitions E.3.1 and E.3.9:

$$m_2 := m_2(M)$$

$$m := m(M, V)$$

$$a := a(M, V)$$

as well as the following parameters discussed before Table 7.2.1, the latter related to the existence of $(F-1)$-offenders:

$$\beta := \min\{m(V/U) : U < V \text{ and } O^2(C_M(U)) \neq 1\}$$

$$\alpha := \min\{m(V/U) : m(V/U) \geq 2 \text{ and } m_2(C_M(U)) \geq m(V/U) - 1\}$$

If $m_2(C_M(U)) < m(V/U) - 1$ for all proper subspaces U of V, set $\alpha := \infty$.

The chapter is divided into sections primarily corresponding to the groups L occurring in Table 7.2.1, with subsections devoted to the various $\mathbf{F}_2L$-modules in that Table.

Sometimes we will need to assume the following hypothesis:

$$M \text{ is a homomorphic image of an SQTK-group.} \qquad (!)$$

In a number of cases we use the following elementary fact:

Lemma H.0.1. *Let $F := \mathbf{F}_{2^n}$, and suppose V admits the structure V_F of an FL-module, with $\dim_F(V_F) = d$. Assume $x \in M$ is of odd prime order p, and x induces a field automorphism on V_F. Then*

$$m(V/C_V(x)) = \frac{(p-1)nd}{p} \geq \frac{2nd}{3}.$$

H.1. $\Omega_4^\epsilon(2^n)$ on an orthogonal module of dimension 4n (n > 1)

Here $\epsilon = +1$ or -1. We assume that $n > 1$, since when $n = 1$, $\epsilon = -1$ and the module is the A_5-module, which is an FF-module and well understood. Recall that $L_2(2^{2n}) \cong \Omega_4^-(2^n)$, $L_2(2^n) \times L_2(2^n) \cong \Omega_4^+(2^n)$, and V can be viewed as a 4-dimensional module for L over $F := \mathbf{F}_{2^n}$.

$m_2 = 2n$: Achieved on the Sylow group $T \cap L$ of L.

$m = n$: Achieved by an orthogonal F-transvection.

$a = n$: We do not need this fact and do not supply a proof.

$\beta \geq 2n$: Again this is not a fact we need.

$\alpha \geq 2$ for $n = 2$, and $\alpha = \infty$ if $n > 2$: We establish this in the following lemma:

Lemma H.1.1. *For $A \in \mathcal{A}^2(M)$,*

(1) $m(A) \leq m(V/C_V(A)) - n + 1$.

(2) Either $m(A) \leq m(V/C_V(A)) - 2$, or $n = 2$ and $A = \langle t \rangle$, where t induces an F-transvection on V.

(3) $m(A) \leq m(V/C_V(A)) - n$ if $A \leq L$.

Proof. Let $t \in M$ be an involution. If $t \in L$, then $\dim_F(C_V(t)) = 2$ and $m_2(C_L(C_V(t))) = n$. So if $A \leq L$, then either $m(A) \leq n$ and $m(V/C_V(A)) \geq 2n$, or $m(V/C_V(A)) = 3n$ and $m(A) \leq m_2(L) = 2n$; thus the lemma holds in this case. So suppose that $t \in A - L$. If A contains no orthogonal transvection then $\epsilon = +1$, $m(V/C_V(t)) = 2n$, $\langle t \rangle = C_M(C_V(t))$, and $m(A) \leq m_2(C_L(t)) + 1 = n + 1$, so the lemma holds in this case. Thus we may assume that t is a transvection, so $m(V/C_V(t)) = n$, $\langle t \rangle = C_M(C_V(t))$, and $C_V(t)$ is the 3-dimensional orthogonal module for $C_L(t) \cong \Omega_3(2^n)$. Now it is easy to check that the lemma holds, recalling our assumption that $n \geq 2$. □

H.2. $SU_3(2^n)$ on a natural 6n-dimensional module

The module V can be regarded as a 3-dimensional module for L over $F := \mathbf{F}_{2^{2n}}$.

$m_2 = n + 1$: Achieved by $\Omega_1(T \cap L)$ extended by a graph automorphism.

$m = 2n$: achieved by unitary F-transvections.

$a \leq n$: A graph automorphism induces a field automorphism on V, and hence is contained in no member of $\mathcal{A}_2(M, V)$.

$\beta \geq 4n$ assuming Hypothesis (!): If not, there is $x \in M$ of odd prime order with $m(V/C_V(x)) < 4n$. By H.0.1, x does not induce a field automorphism on L, so x induces an inner-diagonal automorphism, and hence preserves the F-structure. Therefore $\dim_F(C_V(x)) = 2$, so $V = C_V(x) \perp [V, x]$, and $[V, x]$ is a 1-dimensional F-eigenspace for x with eigenvalue λ of order p dividing $2^n + 1$. But now there is $E = \langle e_1, e_2, e_3 \rangle \leq L\langle x \rangle$, with $V = V_1 \perp V_2 \perp V_3$ the sum of 1-dimensional subspaces V_i, and e_i of order p with $V_i := [V, e_i]$. Finally there is a transvection $s \in L$ interchanging e_1 and e_2 and centralizing V_3, so $C_E(s)$ is noncyclic, contradicting (!) and A.1.31.1.

$\alpha = \infty$: V is not an $(F-1)$-module as $m_2 + 1 < 2n = m$.

H.3. $Sz(2^n)$ on a natural 4n-dimensional module

The module V can be regarded as a 4-dimensional symplectic space for L over $F := \mathbf{F}_{2^n}$. Since n is odd, T contains no field automorphisms, so $T \leq L$.

We recall Definition E.2.6 of the Suzuki type of an involution in the overgroup $Sp_4(2^n)$ of L in $GL(V)$.

$m_2 = n$: Achieved by $\Omega_1(T)$.

$m = 2n$: Involutions are of Suzuki type c_2 in $Sp_4(2^n)$ and act freely on V.

$a \leq n$: As $m_2 = n$, nothing is being asserted.

$\beta \geq 8n/3$: If $x \in L$ is of odd prime order, then $|x|$ divides $2^n - 1$ or $2^{2n} + 1$; in the latter case x is fixed-point-free on V, while in the former the eigenvalues of x are of multiplicity 1 (cf. p.133 in [**Suz62**]). Thus we may assume that $x \notin L$. Next if x induces an inner automorphism on L, then as $End_L(V) \cong F$ and $x \notin L$, the projection y of x on L has order dividing $2^n - 1$; thus $\mathrm{codim}_F(C_V(x)) \geq 3$ as x acts as a scalar if $y = 1$, and the eigenvalues of y are of multiplicity 1 otherwise. If x induces a field automorphism, then $m(V/C_V(x)) \geq 8n/3$ by H.0.1.

$\alpha = \infty$: V is not $(F-1)$-module as $m = 2m_2$ and $m_2 > 1$.

H.4. (S)$L_3(2^n)$ on modules of dimension 6 and 9

$m_2 = 2n$: Achieved on the unipotent radical of either maximal parabolic.

H.4.1. $SL_3(2^n)$ on a natural module 3n plus its dual $(3n)^*$.
Recall that we typically denote the dual of module X by X^*.

Let $F := \mathbf{F}_{2^n}$. Then $V = V_1 \oplus V_2$, where we can view V has a 3-dimensional F-module for L and V_2 as the dual V_1^* of V_1.

$m = 2n$: realized by a transvection on V_1 (and hence also on V_2).

$a = n$: Realized by a root group, so $a \geq n$. To prove $a \leq n$, suppose $A \in \mathcal{A}_{n+1}(M, V)$. If $a \in A^\#$ induces an outer automorphism on L, then $\langle a \rangle = C_M(C_V(a))$, a contradiction. Thus $A \leq L$ and A is contained in the unipotent radical Q of some maximal parabolic. Let R be a root subgroup of Q. Then $m(A/A \cap R) \leq m(Q/R) = n$, so $A \cap R \neq 1$, and hence $A \leq C_M(C_V(A \cap R)) = R$, contradicting $m(A) > n$.

$\beta \geq 4n$ assuming Hypothesis (!): Arguing as in the treatment of $SU_3(2^n)$, if x is of odd prime order with $m(V/C_V(x)) < 4n$, then by H.0.1, x induces an inner-diagonal automorphism on L. Thus x preserves the F-structure on V, and $\dim_F(C_{V_j}(x)) = 2$ for $j = 1$ or 2. Then again arguing as in the case $SU_3(2^n)$, $x \in L$ by (!) and A.1.31.1, so the determinant of x on V_j is 1, contradicting $\dim_F(C_{V_j}(x)) = 2$.

$\alpha = \infty$ unless $n = 1$, where $\alpha = 2$: This follows as $m = 2n = m_2$, and $m(V/C_V(A)) \geq 3n$ unless A is contained in a root subgroup of L.

We need a few more facts in the case where $n = 1$. Pick a standard basis $\{1, 2, 3\}$ for V_1, and (enlarging M if necessary) pick $t \in M - L$ acting as the transpose-inverse automorphism on L with $\{1^t, 2^t, 3^t\}$ the dual basis for the dual space V_2 of V_1. Thus if A is the matrix of $g \in L$ on V_1 with respect to our standard basis, then the matrix of g on V_2 with respect to $\{1^t, 2^t, 3^t\}$ is given by the transpose-inverse of A. The proofs of the following two lemmas are straightforward.

LEMMA H.4.1. *LT preserves the orthogonal form on V with associated symplectic form $(v_1, v_2) = v_2(v_1)$ for $v_i \in V_i$ (regarding v_2 as a member of V_1^*), which makes V_1 and V_2 totally singular. In particular*

$$(i, j^t) = \delta_{i,j}.$$

LEMMA H.4.2. *There are three orbits of nonzero vectors on V under LT, with representatives 1 which is singular and not 2-central, $1 + 2^t$ which is singular and 2-central, and $1 + 1^t$ which is nonsingular.*

LEMMA H.4.3. *Let $E_4 \cong A \leq M$ and $W := \langle C_V(a) : a \in A^\# \rangle$. Then*

(1) If $A \not\leq L$, then $m(C_V(A)) = 2$ and $m(W) = 5$.

(2) If $A \leq L$, then up to conjugacy, $C_V(A) = \langle 1 \rangle \oplus \langle 2^t, 3^t \rangle$ is of rank 3, $W = V_1 + C_V(A)$ is of rank 5, and $[W, A] = \langle 1 \rangle$.

PROOF. If $A \leq L$, then conjugating in $L\langle t \rangle$, we may take A to be the group of transvections on V_1 with center $\langle 1 \rangle$, and then (2) is easy. If $A \not\leq L$, conjugating in L we may take $A = \langle t, i \rangle$, where i is the involution in L with center $\langle 1 \rangle$ and axis $\langle 1, 2 \rangle$. Then $C_V(A) = \langle 1 + 1^t, 2 + 2^t \rangle$ and $W = C_V(t) + C_V(i)$ is of rank

$$\dim(C_V(t)) + \dim(C_V(i)) - \dim(C_V(A)) = 3 + 4 - 2 = 5,$$

establishing (1). □

H.4.2. $\mathbf{L_3(2^{2n}).2}$ on the tensor module 9n.

Let $\hat{L} := SL_3(2^{2n})$, $F := \mathbf{F}_{2^n}$, $E := \mathbf{F}_{2^{2n}}$, N the natural $E\hat{L}$-module, and σ the involutory automorphism of E. Let $B := \{x_1, x_2, x_3\}$ be a basis for N, and N^σ the Galois conjugate of N as an $E\hat{L}$-module. That is, if $g \in \hat{L}$ has matrix $(g_{i,j})$ on N with respect to B, then N^σ is the $E\hat{L}$-module with basis $B^\sigma = \{x_1^\sigma, x_2^\sigma, x_3^\sigma\}$ such that g has matrix $(g_{i,j}^\sigma)$ on N^σ with respect to B^σ.

Let $U := N \otimes N^\sigma$ regarded as an $E\hat{L}$-module. Let γ be the semilinear map on U such that $(eu)\gamma = e^\sigma(u\gamma)$ for each $u \in U$ and $e \in E$, and such that $(x_i \otimes x_j^\sigma)\gamma = x_j \otimes x_i^\sigma$ for all i, j. Then by construction, γ commutes with $\hat{L}$ on U. Therefore (cf. 25.7 in [**Asc86a**]) we can regard V as the FL-space $C_U(\gamma)$ of fixed points of γ on U, where $L := \hat{L}/C_{\hat{L}}(U)$. Observe V is the F-span of

$$\{x_j \otimes x_j^\sigma, e(x_j \otimes x_k^\sigma) + e^\sigma(x_k \otimes x_j^\sigma) : j \neq k,\ e \in E\}.$$

Further $N_{GL(V)}(L)$ is L extended by the scalar maps in $F^\#$ and a field automorphism; so in particular each involution in M preserves the F-structure. We also write σ for the field automorphism induced on $\hat{L}$ by σ; then σ acts as the semilinear map on N and U fixing the bases B and $B \otimes B^\sigma$, with $(ew)\sigma = e^\sigma(w\sigma)$ for $e \in E$ and $w \in N, U$.

Let $0 < Z < W < N$ be the maximal $(T \cap L)$-invariant E-flag in N, and for $X := Z, W$ set $P_X := N_L(X)$, and let Q_X be the unipotent radical of P_X. We may choose notation so that $Z = Ex_1$ and $W = Ex_1 + Ex_2$. For Y a σ-invariant subgroup of L, let Y_σ denote the fixed points of σ on Y.

LEMMA H.4.4. *(1) $C_U(Q_W) = W \otimes W^\sigma$ is the 4−dimensional orthogonal module for P_W/Q_W.*

(2) For $r \in L_\sigma$ an involution, $m(C_V(r)) = 5n$ and $C_M(C_V(r)) = R_\sigma$, where R is the root subgroup of L containing r.

(3) Assume $A \in \mathcal{A}^2(M)$ and either $m(A) \geq 3n$, or $A \leq L$ with $m(A) > n$. Then $m(V/C_V(A)) \geq 5n$.

(4) If $A \in \mathcal{A}^2(L)$ with $m(A) = 4n$, then there is no x of order $2^{2n}-1$ in $N_M(A)$ with $m(C_V(A\langle x\rangle)) = 3n$.

(5) Assume $A \in \mathcal{A}^2(M)$ with $m(A) = 4n$ and $m(V/C_V(A)) = 5n$. Then there does not exist $1 \neq x$ of order $2^n - 1$ centralizing $C_V(A)$.

(6) For i an involution in $M - L$, $m(C_V(i)) = 6n$, $\langle i\rangle = C_M(C_V(i))$, and $C_L(i) \cong SL_3(2^n)$ acts faithfully on 3-dimensional modules $[V,i]$, $C_V(i)/[V,i]$, and $V/C_V(i)$.

(7) $L \cong L_3(2^{2n})$ if n is odd, and $SL_3(2^{2n})$ if n is even.

PROOF. For $\lambda \in E$, define $r(\lambda)$ to be the element of L fixing x_1 and x_2 with $r(\lambda) : x_3 \mapsto x_3 + \lambda x_1$. Then setting $r := r(1)$, an easy tensor calculation shows that $C_U(r) = (W \otimes W^\sigma) + Eu$, where $u := x_1 \otimes x_3^\sigma + x_3 \otimes x_1^\sigma$, and $C_M(C_U(r)) = R_\sigma$. This establishes (1) and (2). Similarly $C_{\hat{L}}(U)$ is the set of scalar maps on N with respect to a scalar λ with $\lambda^{2^n+1} = \lambda\lambda^\sigma = 1$, so (7) holds.

Each involution in $M - L$ is conjugate to the field automorphism σ. Further $C_U(\sigma)$ is the F-span of $B \otimes B^\sigma$, which is isomorphic to the F-tensor product $C_N(\sigma) \otimes C_N(\sigma)$ as an F-module for $C_L(\sigma) \cong SL_3(2^n)$. Then from our earlier description of V as $C_U(\gamma)$, (6) holds.

Let $A \in \mathcal{A}^2(M)$. Suppose first that either $m(A) \geq 3n$, or $m(A) > n$ with $A \leq L$. Then either $A \leq L$, or $n = 1$ and there is $i \in A - L$. In the latter case, $m(A) = 3 = m_2(C_M(i))$, and (6) says $m(V/C_V(i)) = 3$, and $m(C_V(i)/C_V(A)) \geq 2$, so (3) holds in this case. Thus we may take $A \leq L$, so as $m(A) > n$, $A \not\leq R_\sigma$ for any root group R, so (3) follows from (2).

Thus it remains to prove (4) and (5), so we may assume $m(A) = 4n$, and x is a counterexample to (4) or (5). Hence $A = Q_X$ for $X = Z$ or W—and as $m(C_V(Q_Z)) = n$, $A = Q_W$. But then we calculate using (1) that $C_M(C_V(A)) = AD$, where D consists of those elements inducing scalar action via λ on W with $\lambda^{2^n+1} = 1$. This establishes (5), since $(2^n + 1, 2^n - 1) = 1$. Similarly (4) follows from (1). □

Recall the notation $\check{\Gamma}_{k,A}(V)$ from Definition E.3.30.

LEMMA H.4.5. *Assume $A \in \mathcal{A}_{2n+1}(T,V)$. Then*

(1) $A \leq Q_W$.

(2) If $m(A) > 3n$, then $H := \check{\Gamma}_{3n,A}(V)$ is an F-hyperplane of V, so A is not quadratic on H.

(3) $a = 3n$.

PROOF. If $a \in A - L$ then $\langle a\rangle = C_M(C_V(a))$ by H.4.4.6, contradicting $A \in \mathcal{A}_2(M,V)$. Thus $A \leq L$, and as $A \leq T$, $A \leq P_W$. Then as $m(P_W/Q_W) = 2n$ A centralizes $C_V(A \cap Q_W)$ as $A \in \mathcal{A}_{2n+1}(T,V)$, so (1) holds. Assume $m(A) > 3n$. Then for each root group R of Q_W and each $r \in R^\#$, there is a $N_{\hat{L}}(R)$-conjugate τ of σ centralizing r. Then $B := A \cap R_\tau$ is of corank at most $3n$ in A, so $C_V(R_\tau) = C_V(B) \leq H$ using H.4.4.2. Indeed from the description of $C_U(R_\sigma)$ in paragraph one of the proof of H.4.4.2, $E \otimes_F H$ is the E-hyperplane of U spanned by $W \otimes W^\sigma$, $x_j \otimes x_3^\sigma$, $x_3 \otimes x_j^\sigma$, $j = 1, 2$. Thus (2) holds. Finally if $A \in \mathcal{A}_{3n+1}(M,V)$, then A centralizes H, so (2) implies (3). □

$m = 3n$: This follows from (2) and (6) of H.4.4.

$a = 3n$: This is H.4.5.3.

$\beta \geq 4n$: Suppose $x \in M$ is of odd prime order p, with $m(V/C_V(x)) \leq 4n$. Then by H.0.1, x induces inner-diagonal automorphisms on L. Extending E if necessary, we may assume x is diagonalizable on N with eigenvalues α_i on basis vectors x_i for N, $1 \leq i \leq 3$. Then x has eigenvalue $\alpha_i \alpha_j^q$ on $x_i \otimes x_j^\sigma$, where $q := 2^n$. Thus by hypothesis there are at least 5 pairs (i,j) with $\alpha_i = \alpha_j^{-q}$. We conclude that, up to a permutation of $\{1,2,3\}$, $\alpha_1 = \alpha_2 = \lambda$ and $\alpha_3 = \mu$ with $\lambda^{q+1} = \mu^{q+1} = 1$, and so $\dim_F(C_V(x)) = 5$ exactly.

$\alpha = \infty$—unless $n = 1$, where $\alpha = 5$: If we assume that we are given $A \in \mathcal{A}^2(M)$ with $m(V/C_V(A)) \leq m(A)+1$, then as $m = 3n$, $m(A) \geq 3n-1$. If $a \in A - L$, then by H.4.4.6, $m(A) \leq m(C_M(a)) = 2n+1$ and $m(V/C_V(A)) > 3n$, contradicting our assumption. Thus $A \leq L$, so $m(A) \leq 4n = m_2$, and by H.4.4.3, $m(V/C_V(A)) \geq 5n$, with equality only when A is the radical Q_W. Thus the assertion holds.

Again we require extra information in the case $n = 1$.

Lemma H.4.6. *Assume $n = 1$. Then*

(1) If $y \in M$ is of odd order with $C_V(y)$ of rank 5, then $\langle y \rangle = C_M(C_V(y))$ is of order 3.

(2) $L_W := P_W^\infty$ acts as A_5 on $C_V(Q_W)$, and as $L_2(4)$ on $[V, L_W]/C_V(Q_W)$.

(3) $C_M(C_V(Q_W)) = Q_W$ or $Q_W D$, where $D \cong \mathbf{Z}_3$ and $LD \cong PGL_3(4)$.

(4) Let $L_Z := P_Z^\infty$. Then L_Z has chief series $0 < V_1 < V_2 < V$ on V, where: $V_1 := C_V(Q_Z)$ is of rank 1; V_2/V_1 is the $L_2(4)$-module for L_Z/Q_Z and Q_Z induces the full group of F-transvections on V_2 with center V_1; and V/V_2 is the A_5-module for L_Z/Q_Z.

(5) $V_2 \leq [V, L_W]$.

(6) $V_2/V_1 = C_{V/V_1}(Q_Z)$.

Proof. The first part of (2) follows from H.4.4.1, and an easy calculation completes the proof of (2). Then (4) follows from (2) and the fact that the action of P_Z on V is dual to that of P_W. From the discussion of β above, the element y in (1) is of order 3 with $C_N(y) = W = \langle x_1, x_2 \rangle$ and $C_U(y) = W \otimes W^\sigma + F(x_3 \otimes x_3^\sigma)$. Thus (1) holds. Similarly the proof of H.4.4.5 establishes (3).

Let $\hat{V} := V/V_1$. As L_Z is irreducible on $\hat{V}_2$ and V/V_2, $\hat{V}_2 = C_{\hat{V}}(Q_Z)$, establishing (6). As $V_2 = [V_2, L_W \cap L_Z]$, (5) holds. $\square$

H.4.3. Modules for $\mathbf{L_3(4)}$.

Lemma H.4.7. *Let $L_3(4) \cong L := F^*(M)$ be irreducible on V and t an involution in M. Then $m([V,t]) \geq 3$, with equality iff V is the tensor module 9 or its dual and t is a field automorphism.*

Proof. Let σ denote a field automorphism of L, and B, N, N^*, and A be the basic irreducibles (in the usual Lie-theoretic language; cf. H.6.1 and the discussion before it) for $\hat{L} = SL_3(4)$ of dimensions 1, 3, 3, and 8 over $\mathbf{F}_4$. Then the nontrivial $\mathbf{F}_4\hat{L}$-irreducibles are of the form $U = M_1 \otimes M_2^\sigma$, with M_1 and M_2 basic and not both B. A generator z for $Z(\hat{L})$ has eigenvalue $\lambda := \omega_1 \omega_2^2$ on U, where ω_i is the eigenvalue for z on M_i. Thus U is an L-module iff $\lambda = 1$ iff one of: $M_1 = M_2 = N$, or $M_1 = M_2 = N^*$, or M_1 and M_2 are both in $\{B, A\}$. Let E be the field of definition of V; then $\mathbf{F}_4 \otimes_E V$ is one of these irreducibles. If $M_1 = M_2$ is N or N^*, then V is the tensor module 9 or its dual, and the lemma follows from H.4.4. Thus we may assume that $M_1, M_2 \in \{A, B\}$, but not both are B. Now each involution in M inverts an element y of order 3 in L, and we compute directly that $m([V, y]) \geq 12$, so $m([V, t]) \geq 6$, establishing the lemma. $\square$

H.4.4. $\mathbf{L_3(2)} \wr \mathbf{Z_2}$ on its 9-dimensional tensor product module.

As this module is perhaps less familiar than those in the other sections, more details are provided. For the parameter values of Table 7.2.1:

$m_2 = 4$: Obvious.
$m = 3$: See H.4.10 below.
$a = 2$: See H.4.11.1.
$\beta \geq 6$ and $\alpha = 3$: See the end of the subsection.

In this subsection for $i = 1, 2$, V_i is a 3-dimensional vector space over $\mathbf{F}_2$ with basis $\{x^i_j : 1 \leq j \leq 3\}$, and $V := V_1 \otimes V_2$ is the tensor product space. Define

$$x_{i,j} := x^1_i \otimes x^2_j \text{ and } y_{i,j} := x_{i,j} + x_{j,i}$$

Further set $L_i := GL(V_i)$, and let $t : V_1 \to V_2$ be the isomorphism defined by $t : x^1_j \mapsto x^2_j$. Represent $L_0 := L_1 \times L_2$ on V via the tensor product representation, and extend t to V by defining

$$t : x_{i,j} \mapsto x_{j,i}.$$

The representations of L_0 and the involution t on V are faithful, so we can view L_0 and $\langle t \rangle$ as subgroups of $GL(V)$, and let $G := L_0\langle t \rangle \leq GL(V)$. Define

$$V_{i,m} := \langle x^i_j : 1 \leq j \leq m \rangle,$$

and for $m = 1, 2$, define

$$G_m := (N_{L_1}(V_{1,m}) \times N_{L_2}(V_{2,m}))\langle t \rangle.$$

Define $R_i := C_{L_i}(V_{i,2})$ and $R := R_1R_2 = O_2(G_2)$. Let T_i be the Sylow 2-group of L_i stabilizing the flag $0 < V_{i,1} < V_{i,2} < V_i$, and $T := T_1T_2\langle t \rangle$. We have immediately:

LEMMA H.4.8. *(1) $L_1^t = L_2$; and the isomorphism $l \mapsto l^t$ of L_1 with L_2 is induced by the isomorphism $t : V_1 \to V_2$.*
(2) $G = L_0\langle t \rangle$ is the wreath product of $L_3(2)$ by $\mathbf{Z}_2$.
(3) G_m $(m = 1, 2)$ are maximal parabolic subgroups of G.
(4) $T \in Syl_2(G)$.
(5) $[V, t] = \langle y_{1,2}, y_{1,3}, y_{2,3} \rangle$ is of dimension 3, and

$$C_V(t) = [V, t] \oplus \langle x_{1,1}, x_{2,2}, x_{3,3} \rangle.$$

LEMMA H.4.9. *(1) L_0 has 3 orbits on $V^\#$ with representatives $x_{1,1}$, $y_{1,2}$, and $d := x_{1,1} + x_{2,2} + x_{3,3}$.*
(2) Let $V_5 := \langle x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{3,1} \rangle$. Then $C_G(x_{1,1}) = G_1 \cong S_4$ wr $\mathbf{Z}_2$ and G_1 is irreducible on $V_5/\langle x_{1,1} \rangle$ and V/V_5.
(3) $C_G(y_{1,2})$ is of index 6 in G_2 and isomorphic to $(S_3 \times \mathbf{Z}_2)/E_{16}$.
(4) $C_G(d) \cong Aut(L_3(2))$ and $V = [V, C_G(d)] \oplus \langle d \rangle$ with $[V, C_G(d)]$ the Steinberg module for $C_G(d)$.

PROOF. Visibly $G_1 \leq C_G(x_{1,1})$, so as G_1 is maximal in G, $G_1 = C_G(x_{1,1})$. The remainder of (2) is an easy calculation. Similarly $C_{G_2}(y_{1,2}) \cong (S_3 \times \mathbf{Z}_2)/E_{16}$, and G_2 is the unique maximal subgroup of G containing $C_{G_2}(y_{1,2})$, so $C_G(y_{1,2}) = C_{G_2}(y_{1,2})$, establishing (3). Finally as the Steinberg module is a summand of the tensor product of the natural module and its dual, there is a diagonal copy L of $L_3(2)$ in L_0 such that $V = C_V(L) \oplus [V, L]$ with $[V, L]$ the Steinberg module; by a Frattini Argument, we may take t to act on L. As L is maximal in L_0, $L = C_{L_0}(v)$ for $v \in C_V(L)^{\#}$. By a counting argument, $x_{1,1}$, $y_{1,2}$, and v are representatives for the orbits of L_0 on $V^{\#}$, so $d \in v^{L_0}$ and the lemma is established. □

LEMMA H.4.10. *G has three conjugacy classes of involutions:*

(1) The involutions in $L_1 \cup L_2$, with representative $s \in L_1$, such that $V_{1,2} = C_{V_1}(s)$ and $V_{1,1} = [V_1, s]$. Further $[V, s] = \langle x_{1,1}, x_{1,2}, x_{1,3}\rangle$.

(2) The diagonal involutions in L_0 with representative $r := ss^t$. Further

$$[V, r] = \langle x_{1,1}, x_{1,2}, x_{2,1}, y_{1,3}\rangle$$

and $C_V(r) = \langle [V, r], x_{2,2}\rangle$.

(3) The involutions in $G - L_0$ with representative t. Further $m([V, t]) = 3$, $C_V(t)$ contains d, and t is contained in no $(F-1)$-offender.

PROOF. This is straightforward, except possibly for the last sentence of (3), which we now prove: By H.4.8, $C_{L_0}(t) \cong L_3(2)$ acts faithfully on $[V, t]$ and on $C_V(t)/[V, t]$, so $m(V/C_V(A)) \geq 5$ for each noncyclic $A \in \mathcal{A}^2(C_G(t))$ containing t, and such A are of rank at most 3. □

LEMMA H.4.11. *(1) $a(G, V) = 2$.*

(2) $\langle \mathcal{A}_2(T, V)\rangle = R$.

PROOF. Suppose $A \in \mathcal{A}_2(T, V)$. Then $C_V(A) = C_V(B)$ for each hyperplane B of A, so $m(A) > 1$. If furthermore $A \in \mathcal{A}_3(T, V)$, then $B \in \mathcal{A}_2(T, V)$ so $m(A) > 2$.

Suppose first that $t \in A$. Let $G_t := C_G(t)$, $T_t := C_T(t)$, and $\bar{G}_t := G_t/\langle t\rangle$. Then by H.4.8, $\bar{G}_t \cong L_3(2)$ acts faithfully on $[V, t]$ and on $C_V(t)/[V, t]$, with $\bar{A} \in \mathcal{A}_2(\bar{T}_t, C_V(t))$. Hence as the maps

$$\alpha : L_1 \to C_{L_0}(t) \text{ defined by } l \mapsto ll^t$$

and

$$\beta : V_1^* \to [V, t] \text{ defined by } x_i^{1*} \mapsto y_{j,k}$$

for $\{1, 2, 3\} = \{i, j, k\}$ define a quasiequivalence of the actions of $\bar{G}_t$ on the dual V_1^* of V_1 with $[V, t]$, we conclude that $\bar{A} = \langle \bar{r}, \bar{b}\rangle$: where r is as in H.4.10.2, and $b := ff^t$ where $f \in T_1$ is the transvection on V_1 with axis $\langle x_1^1, x_3^1\rangle$ and center $\langle x_1^1\rangle$. This is impossible as $x_{2,2} \in C_V(\langle t, r\rangle) - C_V(b)$. Thus in view of H.4.10, we conclude that $\mathcal{A}_2(T, V) \subseteq L_0$.

If $Aut_A(V_1) = 1$, then $A \leq T_2$, so $A \leq R_2$ and hence $A = R_2 \leq R$. Otherwise $Aut_A(V_1) \neq 1$, so $Aut_A(V_1) \in \mathcal{A}_2(Aut_T(V_1), V_1)$, so $Aut_A(V_1)$ is the group R_1 of transvections on V_1 with axis $V_{1,2}$. Similarly either $A = R_1$ or $Aut_A(V_2) = R_2$. So $A \leq R_1R_2 = R$. Conversely $R_i \in \mathcal{A}_2(T, V)$ for $i = 1, 2$, so (2) is established. Further if $A \in \mathcal{A}_3(T, V)$, we saw in the first paragraph that $m(A) > 2$, so $A_i := A \cap R_i \neq 1$ for $i = 1, 2$. Therefore $A \leq C_R(C_V(A_i)) = R_i$, a contradiction. Hence (1) holds. □

LEMMA H.4.12. *(1) Each subgroup of order 3 in G has exactly three noncentral chief factors on V.*

(2) If P is a diagonal subgroup of order 3 in G_1, then P has two noncentral chief factors on $O_2(G_1)$.

(3) $C_G(U)$ is a 2-group for each $U \leq V$ with $m(V/U) < 6$.

PROOF. An easy calculation. □

By H.4.12.1, $m(V/C_V(x)) = 6$ for each element x of order 3 in M, and an easy calculation gives $m(V/C_V(y)) \geq 6$ for y of order 7. Thus $\beta \geq 6$.

Finally we show $\alpha = 3$: By definition, R_1 centralizes the 6-space $V_{1,2} \otimes V_2$, so R_1 is an $(F-1)$-offender with $m(V/C_V(R_1)) = 3$. On the other hand as $m = 3$, $\alpha > 2$, so we are done.

H.5. 7-dimensional permutation modules for $\mathbf{L_3(2)}$

In this section V is a 3-dimensional vector space over $\mathbf{F}_2$, and $L := GL(V) \cong L_3(2)$. Let $T \in Syl_2(L)$, $p \in V^\#$ the vector (projective point) fixed by T, and l the projective line in V fixed by T. For $x := p, l$, let L_x be the stabilizer in L of x; thus L_p and L_l are the maximal parabolics of L over T.

Let $\mathcal{L}$ be the set of lines in V, $\Omega := V^\#$ the set of points, and U_0 the permutation module for L on Ω over $\mathbf{F}_2$. As usual we regard U_0 as the power set of Ω, with addition given by symmetric difference. Let U be the *core* (the subspace of subsets of even order) of U_0.

The following results on U are all well known; for completeness we supply the easy proofs. First as $|\Omega|$ is odd:

LEMMA H.5.1. *$U_0 = U \oplus \mathbf{F}_2\Omega$ as an $\mathbf{F}_2L$-module.*

For $S \subseteq \Omega$, write u_S for the complement $S + \Omega$ of S in Ω. Define:

$$\mathcal{O}_1 := \{u_q : q \in \Omega\},\ \mathcal{O}_2 := \{u_r : r \in \mathcal{L}\},\ \mathcal{O}_4 := \{u_B : B \text{ is a basis of } V\},$$

and let $\mathcal{O}_3$ be the set of vectors in U_0 of weight 2.

LEMMA H.5.2. *(1) For $r, s \in \mathcal{L}$, $u_r + u_s = u_t$, where t is the third line through the common point $r \cap s$.*

(2) $C_U(T) = \langle u_p, u_l \rangle$ is of dimension 2.

(3) $\dim(C_U(O_2(L_p))) = \dim(C_U(O_2(L_l))) = 3$.

(4) $\dim([U, O_2(L_p)]) = \dim([U, O_2(L_l)]) = 3$.

(5) $C_U(L_p) = \langle u_p \rangle$ and $C_U(L_l) = \langle u_l \rangle$.

(6) U is a self-dual $\mathbf{F}_2L$-module.

(7) For $Y \in Syl_3(L_l)$, $C_U(Y) = \langle u_l, u_q \rangle$, where $\langle q \rangle = C_V(Y)$.

PROOF. Part (1) is an easy calculation. For $X \leq L$, $\dim(C_{U_0}(X))$ is the number $o(X)$ of orbits of X on Ω, so $\dim(C_U(X)) = o(X) - 1$ since X centralizes Ω. Then it is easy to calculate that (3) holds, $\dim(C_U(T)) = 2 = \dim(C_U(Y))$, and $\dim(C_U(L_p)) = \dim(C_U(L_l)) = 1$. As T fixes p and l, it centralizes u_p and u_l, so as $\dim(C_U(T)) = 2$, (2) holds. Similarly (5) and (7) follow.

As U is nondegenerate with respect to the quadratic form on U_0 in which the basis Ω is orthonormal, (6) holds. By (6), $\dim(C_U(X)) = \dim(U/[U, X])$, so (3) implies (4). □

LEMMA H.5.3. *Let $W := \mathcal{O}_2 \cup \{0\}$. Then*

(1) W is an $\mathbf{F}_2L$-submodule of U isomorphic to the dual of V.

(2) U/W is $\mathbf{F}_2L$-isomorphic to V.

(3) $\mathcal{O}_i$, $1 \leq i \leq 4$, are the orbits of L on $U^\#$.

(4) $|\mathcal{O}_1| = |\mathcal{O}_2| = 7$, $|\mathcal{O}_3| = 21$, and $|\mathcal{O}_4| = 28$.

(5) U is indecomposable, and hence W is the socle of U.

PROOF. By H.5.2.1, W is a subspace of U. As $|W^{\#}| = |\mathcal{L}| = 7$, $\dim(W) = 3$. As L is faithful on $\mathcal{L}$, L is faithful on W. Therefore $L = GL(W)$, and then as L_l stabilizes $u_l \in W$, U is isomorphic to the dual of V. Thus (1) is established.

By (1), $\dim(U/W) = 3$. As $U = \langle u_p^L \rangle$, $U/W = \langle (W + u_p)^L \rangle$, so L is nontrivial on U/W. Thus L is faithful on U/W as L is simple, so $L = GL(U/W)$; then as L_p stabilizes $W + u_p$, (2) holds.

Visibly L is transitive on $\mathcal{O}_i$ for $1 \leq i \leq 4$ and (4) holds. Then as $\sum_i |\mathcal{O}_i| = 63 = |U^{\#}|$, (3) holds. By (4), $\mathcal{O}_1$ is the only orbit of length 7 on $U - W$, while for distinct $p, q \in \Omega$, $u_p + u_q = \{p, q\} \in \mathcal{O}_3$ is of weight 2, so $\mathcal{O}_1 \cup \{0\}$ is not a subspace of U. On the other hand L is transitive on $(U/W)^{\#}$ by (2). Therefore U does not split over W, and hence (5) holds. □

LEMMA H.5.4. *Let $Z := C_U(T)$. Then*

(1) $C_U(L_p) \not\leq [Z, L_l]$.

(2) $C_U(L_l) \leq [Z, L_p]$.

(3) If a, b, c are distinct elements of $\mathcal{O}_1$, then $a + b + c \neq 0$.

PROOF. By H.5.3.1, $C_W(L_l) \leq [C_W(T), L_p]$, so (2) follows from H.5.2.5.

Let S be a subset of Ω of order 3. Then $\sum_{s \in S} u_s = u_S$ is of weight 4, so (3) holds. By parts (2) and (5) of H.5.2, $\langle Z^{L_l} \rangle = [Z, L_l] \oplus \langle u_l \rangle$, with $\dim([Z, L_l]) = 2$. Thus $u_p \notin [Z, L_l]$ by (3), so (1) follows from H.5.2.5. □

LEMMA H.5.5. *Assume M is an $\mathbf{F}_2L$-module such that*

(1) $M = \langle x^M \rangle$ for some $1 \neq x \in C_M(L_p)$, and

(2) $\dim(\langle x^{L_l} \rangle) = 2$.

Then M is isomorphic to V as an $\mathbf{F}_2L$-module.

PROOF. By (1), there is a surjection $\varphi : U_0 \to M$ with $\varphi(p) = x$; let $K := \ker(\varphi)$. By (2), $u_l + \Omega \in K$; hence the sum Ω of the L-conjugates of $u_l + \Omega$ lies in K, so that $W + \mathbf{F}_2\Omega \leq K$. Then $K = W + \mathbf{F}_2\Omega$ and $M \cong U_0/K \cong V$ by H.5.3.2. □

H.6. The 21-dimensional permutation module for $\mathbf{L_3(2)}$

In this section, V is a 3-dimensional vector space over $\mathbf{F}_2$, and $L := GL(V) \cong L_3(2)$. Let $T \in Syl_2(L)$, with $p \in V^{\#}$ the vector (projective point) fixed by T and l the projective line in V fixed by T. For $x = p, l$, let L_x be the stabilizer in L of x; thus L_p and L_l are the maximal parabolics of L over T.

In this section, and at a few other points in our work, we describe irreducible modules via the highest-weight representation theory of the Lie type groups in their natural characterstic, using section 2.8 of [**GLS98**] as our standard reference. In particular we recall that the Steinberg Tensor Product Theorem (2.8.5 in [**GLS98**]) expresses those irreducibles as algebraic conjugates of *basic* irreducible modules $M(\lambda_i)$ for fundamental weights λ_i.

So now for $L \cong L_3(2)$, we assume the familiar list of irreducible $\mathbf{F}_2L$-modules (e.g., page 77 of [**GLS98**]) indexed by sums of the fundamental weights λ_1 and λ_2:

LEMMA H.6.1. *L has four irreducible $\mathbf{F}_2L$-modules up to isomorphism:*

(1) The 1-dimensional $\mathbf{F}_2L$-module $M(0)$.

(2) $M(\lambda_1) \cong V$.

(3) $M(\lambda_2)$ isomorphic to the dual V^* of V.

(4) The 8-dimensional adjoint (also, Steinberg) module $M(\lambda_1 + \lambda_2)$.

Let U be the permutation module for L on $\Omega := L/T$ over $\mathbf{F}_2$. Thus U is the 21-dimensional transitive permutation module for L over $\mathbf{F}_2$. The following results on U are well known; for completeness we supply the easy proofs.

As usual let $J(M)$ denote the radical of a module M, namely the intersection of all maximal submodules.

LEMMA H.6.2. *(1) There exist $\mathbf{F}_2L$-submodules U_i of U, $1 \leq i \leq 4$, such that*

$$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4,$$

$U_1 \cong M(0)$ is the trivial module, U_2 and U_3 are the 6-dimensional cores of the 7-dimensional permutation modules on L/L_p and L/L_l with quotients $M(\lambda_1)$ and $M(\lambda_2)$, respectively, and U_4 is the Steinberg module $M(\lambda_1 + \lambda_2)$.

(2) $J(U) = J(U_2) \oplus J(U_3) \cong M(\lambda_1) \oplus M(\lambda_2)$ is of dimension 6.

PROOF. Define $W := W_1 \oplus W_2 \oplus W_3 \oplus W_4$ with $W_1 \cong M(0)$, W_2 and W_3 the cores of the permutation modules on L/L_p and L/L_l, respectively, and W_4 the Steinberg module. Thus

$$J(W) = \bigoplus_{i=1}^{4} J(W_i),$$

and W_0 and W_4 are irreducible, so $J(W) = J(W_2) \oplus J(W_3)$. By H.5.3, for $i = 2, 3$, $W_i/J(W_i) \cong M(\lambda_{i-1})$ and $J(W_i) \cong M(\lambda_{5-i})$. Thus (1) implies (2).

Let $0 \neq w_i$ be a fixed point of T on W_i. For $i = 1$ and 4, W_i is irreducible, so $W_i = \langle w_i^L \rangle$. For $i = 2, 3$, W_i is the core of the permutation module for L on L/L_{x_i}, where $x_2 := p$ and $x_3 := l$, so we can choose w_i to be a fixed point of L_{x_i} such that $W_i = \langle w_i^{L_i} \rangle$. Let $w := \sum_i w_i$; then w is a fixed point of T, and we will show that $W = \langle w^L \rangle$. Thus W is a homomorphic image of U, so as $\dim(U) = 21 = \dim(W)$, the lemma will follow.

Let $M := \langle w^L \rangle$; it remains to show that $W = M$. Set $\hat{W} := W/J(W)$. As the projection $\hat{w}_i$ of $\hat{w}$ on $\hat{W}_i$ is nontrivial and $\hat{W}_i$ is irreducible, $\hat{W}_i$ is a homomorphic image of $\hat{M}$. Thus as $\hat{W}_i$ is a homogeneous component of $\hat{W}$, it follows that $\hat{M} = \hat{W}$. That is, $W = M + J(W)$, so indeed $M = W$. □

During the remainder of the section, define U_i as in H.6.2.

LEMMA H.6.3. *Let M be the Steinberg module for L. Then*

(1) M is projective.

(2) M is self-dual and invariant under $Aut(L)$.

(3) L_p and L_l each have three noncentral chief factors on M.

(4) T stabilizes a unique point m of M, and $T = C_M(m)$.

(5) $M = [M, L_l] = [M, L_p]$.

PROOF. Part (1) is well known. By H.6.1, M is the unique irreducible of its dimension, so (2) holds.

Let X be of order 3 in L. Then X has 7 orbits of length 3 on Ω, so $\dim(C_U(X)) = 7$. Further $U_1 \leq C_U(X)$, and $\dim(C_{U_i}(X)) = 2$ for $i = 2, 3$ by H.5.2.7. Hence $\dim(C_{U_4}(X)) = 2$, so X has three noncentral chief factors on U_4. Thus (3) holds.

Let $Z(T) =: \langle z \rangle$. Then we compute that T has 1,2,2,1 orbits of length 1,2,4,8 on z^L, where $|zz'| = 1, 2, 4, 3$ for z' in the respective class of orbits. Thus T has 6 orbits on Ω, so $\dim(C_U(T)) = 6$. But by H.5.2.2, $\dim(C_{U_i}(T)) = 2$ for $i = 2, 3$, so T has a unique fixed point m on U_4. If $T < C_L(m)$, then $C_L(m)$ is a maximal parabolic L_x for $x := p$ or l. This is impossible, as M is not a quotient of the permutation module on L/L_x by H.5.3. Thus $T = C_L(m)$, completing the proof of (4), and further $C_M(L_x) = 0$. This last fact, together with (2), implies (5). □

We need the following general technical lemma:

LEMMA H.6.4. *Assume $M = A + B$ is a module with a finite composition series, such that $A/J(A) \cong S$ is simple, and B has no quotient isomorphic to S. Let $K \leq M$ be a submodule. Then either $A \leq K$, or M/K has a quotient isomorphic to S.*

PROOF. Choose a counterexample M with a minimal number n of composition factors. If $n = 1$, then $M = A \cong S$ and the lemma is trivial, so we may assume that $n > 1$. Also we may assume that M/K has no quotient isomorphic to S, and it remains to show that $A \leq K$. Let C be a simple submodule of K, and set $\hat{M} := M/C$. Then $\hat{M} = \hat{A} + \hat{B}$, and neither $\hat{B}$ nor $\hat{M}/\hat{K} \cong M/K$ has a quotient S. Next $\hat{A} = (A + C)/C \cong A/(A \cap C) \cong A/C$ or A, for $C \leq A$ or $C \not\leq A$, respectively. Assume either that the latter case holds, or that the former case holds with $C \leq J(A)$. Then $S \cong A/J(A) \cong \hat{A}/J(\hat{A})$, so $\hat{A} \leq \hat{K}$ by minimality of n. Then $A \leq K$ under either of our present assumptions, so the lemma holds. Finally if $C \leq A$ but $C \not\leq J(A)$, then $A = C + J(A)$, so $A = C \leq K$, completing the proof. □

LEMMA H.6.5. *Assume M is a faithful $\mathbf{F}_2L$-module such that for some $m \in C_M(T)$, $M = \langle m^L \rangle$ and $\dim(\langle m^{L_l} \rangle) = 2$. Then M is a quotient of $U_2 \oplus U_4$. In particular, M is isomorphic to one of: the core U_2 of the permutation module on L/L_p, the Steinberg module S, V, $V \oplus S$, or $U_2 \oplus S$.*

PROOF. As T fixes m and $M = \langle m^L \rangle$, there is a surjective homomorphism $\pi : U \to M$ with $u\pi = m$, where $u \in U$ is the fixed point of T of weight 1 when regarded as a subset of Ω. Let $K := \ker(\pi)$.

We first observe that M has no quotient module M^+ isomorphic to the dual V^* of V or to the trivial module U_1: Namely V^* contains a unique nonzero vector v fixed by T, and v is also fixed by L_l. However as v is unique, the image m^+ of m in M^+ is v; so as L_l is nontrivial and irreducible on $W := \langle m^{L_l} \rangle$ by hypothesis, $W \cong \langle m^{+L_l} \rangle$, whereas L_l fixes $v = m^+$.

Now we may apply H.6.4 with U, U_1 or U_3, U_1 or V^* in the roles of "M, A, S", and the complement in H.6.2.1 to U_1 or U_3 in the role of "B". In view of the previous paragraph, we conclude from H.6.4 that K contains U_1 and U_3, so $M \cong U/K$ is a quotient of $U/(U_1 + U_3) \cong U_2 \oplus U_4$. Thus the lemma holds. □

H.7. $\mathbf{Sp_4(2^n)}$ on natural 4n plus the conjugate $\mathbf{4n^t}$.

Here $V = V_1 \oplus V_2$ is a module for L over $F := \mathbf{F}_{2^n}$; with V_1 natural, and $V_2 := V_1^t$ conjugate to V_1 via an outer automorphism t nontrivial on the Dynkin diagram of L.

$m_2 = 3n$: Achieved by unipotent radicals of maximal parabolics.

$m = 3n$: Achieved by root involutions. Involutions of Suzuki type c_2 (cf. Definition E.2.6), and involutions in $M - L$, are free on V.

$a < 2n$: Let W be the 2-subspace of V_1 stabilized by $N_T(V_1)$. We first claim that $a(N_M(V_1), V_1) = 2n$, with $B \in \mathcal{A}_{2n}(N_T(V_1), V_1)$ precisely when $B \leq C_T(W)$

and $m(B/(B \cap R)) = 2n$ for each root subgroup R of $N_T(V_1)$ acting as transvections on V_1: For each such subgroup B is indeed in $\mathcal{A}_{2n}(N_T(V_1), V_1)$. Conversely, assume $B \in \mathcal{A}_{2n}(N_M(V_1), V_1)$. If B centralizes W, then $m(B/B \cap R) \geq 2n$ as $C_{V_1}(B \cap R) > W$, so $m(B/(B \cap R)) = 2n$ since $m(C_T(W)/R) = 2n$. In particular, $B \notin \mathcal{A}_{2n+1}(N_M(V_1), V_1)$. If B does not centralize W, then $Aut_B(W) \in \mathcal{A}_{2n}(Aut_T(W), W)$, which is impossible as $m(T/C_T(W)) = n$, completing the proof of the claim.

Next if $A \in \mathcal{A}_{2n}(T, V)$, then $1 \neq N_A(V_1)$ has corank at most 1 in A and $1 < 2n$, so $0 \neq C_{V_1}(N_A(V_1)) \leq C_{V_1}(A)$; hence $A \leq N_T(V_1)$ and then $A \in \mathcal{A}_{2n}(N_M(V_1), V_1)$, so $A \leq C_T(W)$ by the claim. By symmetry, $A \leq C_T(U)$, where U is the 2-subspace of V_2 stabilized by $N_T(V_1)$. Thus $A \leq C_T(W) \cap C_T(U) = Z(T_L)$, where $T_L := T \cap L$ if $n > 1$, or $T_L := T \cap L_0$ if $n = 1$ where $L_0 \cong S_6$. Then as $m(A) \geq 2n = m(Z(T_L))$, $A = Z(T_L)$; thus $A \cap R \neq 1$, where R is the root subgroup of $Z(T_L)$ acting as transvections on V_1, giving the contradiction $C_V(A \cap R) > C_V(A)$.

$\beta \geq 4n$, assuming Hypothesis (!): Suppose x is of odd prime order p and $m(V/C_V(x)) < 4n$. Then x does not induce a field automorphism on L by H.0.1, so arguing as in the treatment of $SL_3(2^n)$ on the sum of a natural module and its dual, (!) and A.1.31.1 say $x \in L$. Therefore $m(V_i/C_{V_i}(x)) = 2n$ or $4n$ as V_i is a symplectic space, contradicting $m(V/C_V(x)) < 4n$. In case $n = 1$, each element of order 3 is fixed-point-free on one of the two modules, so that $\beta \geq 6$.

V is not an $(F-1)$-module, so $\alpha = \infty$: Suppose $m(A) \geq m(V/C_V(A)) - 1$. As $m = 3n$, $m(A) \geq 3n - 1$. If $a \in A - N_A(V_1)$, then $m(V/C_V(a)) = 4n$, while $m_2(C_M(a)) \leq n + 1$, a contradiction. Thus A acts on V_1, and as $m(A) \geq 3n - 1$, $A \leq T_L$. Then either A is not contained in a conjugate of $Z(T_L)$, so $m(V/C_V(A)) \geq 5n > m(A) + 1$, or we may take $A \leq Z(T_L)$ of rank $2n$, so that $A = Z(T_L)$, $n = 1$, and $m(V/C_V(A)) = 4 > m(A) + 1$.

H.8. $\mathbf{A_7}$ on $\mathbf{4 \oplus \bar{4}}$

We can take $M \cong S_7$ and $V = V_1 \oplus V_2$, with $V_1^t =: V_2$ for t an involution in $M - L$.

$m_2 = 3$: Achieved on the two classes of E_8-subgroups of M.

$m = 4$: Achieved by all involutions.

$a = 2$: $\mathcal{A}_2(T, V) = \{A\}$, where A is the 4-subgroup of L with $m(C_{V_i}(A)) = 2$.

$\beta \geq 4$: There is x is of order 3 in L with $m(C_{V_i}(x)) = 2$.

V is not an $(F-1)$-module, so $\alpha = \infty$: As $m = 4$, if $m(V/C_V(A)) \leq m(A) + 1$, then $m(A) = m_2 = 3$ and $C_V(a) = C_V(A)$ for each $a \in A^\#$. But there is $a \in A - L$ with $C_{V_1}(a) = 0 \neq C_{V_1}(A \cap L)$.

H.9. $\mathbf{Aut(L_n(2))}$ on the natural n plus the dual $\mathbf{n^*}$

The following lemmas are needed in section 12.1.

In this section assume $M \cong Aut(L_n(2))$, with $n = 4$ or 5, $T \in Syl_2(M)$, $L := E(M)$, V is an $\mathbf{F}_2M$-module, and $V = V_1 \oplus V_2$, with V_1 the natural module for L and $V_2 = V_1^t$ for $t \in T - L$. Thus V_2 is the dual of V_1 as an $\mathbf{F}_2L$-module. Let $T_1 := N_T(V_1)$, and for $v \in V$, let $M_v := C_M(v)$ and $L_v := O^2(C_L(v))$.

LEMMA H.9.1. *(1) There is a unique nondegenerate quadratic form q on V preserved by M in which V_1 is totally singular.*

(2) M has three orbits $\mathcal{O}_m$, $1 \le m \le 3$ on $V^\#$: Namely $\mathcal{O}_1 := V_1^\# \cup V_2^\#$ consists of singular vectors; $\mathcal{O}_2$ consists of the singular diagonal vectors, which are also the nonzero vectors centralized by a Sylow 2-subgroup of M; and $\mathcal{O}_3$ is the set of nonsingular diagonal vectors.

(3) Let v_1 generate $C_{V_1}(T_1)$. Then $M_{v_1} = L_{v_1} \cong L_{n-1}(2)/E_{2^{n-1}}$ with $V_1 = [V_1, L_{v_1}]$ and $U_2 := [V_2, L_{v_1}] = V_2 \cap v_1^\perp$ a hyperplane of V_2 and a natural module for $L_v/O_2(L_v)$. Further $T_1 = C_T(v_1) \in Syl_2(M_{v_1})$.

(4) Let $v_2 \in V_2 - U_2$ and $v := v_1 + v_2$. Then $v \in \mathcal{O}_3$, $M_v = \langle t\rangle L_v \cong Aut(L_{n-1}(2))$ with $L_v = O^2(M_{v_1,v_2})$ and $t \in T_v - T_1$, where $T_v := C_T(v) \in Syl_2(M_v)$. Further $V_i = \langle v_i\rangle \oplus W_i$, where $W_i := V_i \cap \langle v_1, v_2\rangle^\perp$, L_v acts naturally on W_1, and W_2 is dual to W_1.

PROOF. Part (1) follows as V_2 is the dual of V_1. As L is transitive on $V_i^\#$, $\mathcal{O}_1$ is an orbit, and as L_{v_1} is transitive on $V_2^\# \cap v_1^\perp$ and $V_2 - v_1^\perp$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are the remaining orbits. Thus (2) holds. Part (3) is straightforward. By construction, the vector v of (4) is in $\mathcal{O}_3$, and its stabilizer is $G_{v_1,v_2}\langle t\rangle$, where $t \in T - T_1$ interchanges v_1 and v_2. Then (4) also follows in a straightforward way. □

Again we recall the notation of Definition E.3.30.

LEMMA H.9.2. *Assume that $n = 5$. Let A denote an elementary abelian 2-subgroup of M. Then*

(1) M is transitive on involutions in $M - L$. If t is such an involution, then $C_L(t) \cong S_6$ and the map $\theta : V_1 \to [V, t] = C_V(t)$ given by $\theta : w \mapsto w + w^t$ is a $C_L(t)$-isomorphism. In particular $C_V(t) = \langle v\rangle \oplus [C_V(t), C_L(t)]$ with $[C_V(t), C_L(t)]$ a natural module for $C_L(t)$, and v nonsingular centralizing $C_L(t)$.

(2) If $m(A) \ge 5$ then $A \le L$.

(3) Let U be the T_1-invariant 3-subspace of V_1, and A_0 the unipotent radical of $N_L(U)$. Then A_0^M is the set of E_{64}-subgroups of M, and if $m(A) = 5$, then A is conjugate to a hyperplane of A_0.

(4) If A is a hyperplane of A_0, then

$$C_V(A) = [V, A] = C_V(A_0) = [V, A_0] = U \oplus (V_2 \cap U^\perp)$$

is of rank 5.

(5) If $m(A) \ge 5$, then $V = \check{\Gamma}_{3,A}(V)$.

PROOF. The first two statements in part (1) are well known; cf. [**AS76a**] for example. As $V_1^t = V_2$, the map θ is a $C_L(t)$-isomorphism. By Lemma H.9.1.4, each $v \in \mathcal{O}_3$ is centralized by an involution $t \in M - L$ such that $C_{L_v}(t) \cong S_6$. Thus $C_L(t) = C_{L_v}(t)$, and hence by H.9.1.4, $V_1 = \langle v\rangle \oplus W_1$ as an L_v-module, so we conclude that the rest of (1) holds. Part (1) implies (2).

Let U and A_0 be defined as in (3). Assume $m(A) \ge 5$. Then by (2), $A \le L$. Thus we may take $A \le T_1$, so A acts on A_0 as $A_0 \trianglelefteq T_1$. Now $N_L(A_0) = K$ where $K/A_0 = K_1/A_0 \times K_2/A_0$, $K_1/A_0 \cong L_3(2)$, $K_2/A_0 \cong L_2(2)$, and A_0 is the tensor product of the natural modules for the factors. Further $\mathcal{A}(T) = \{A_1, A_0\}$, where $A_1 = A_0^t$ for $t \in T - T_1$. Also $A_1 \le K_1$, $A_0 \cap A_1 \cong E_{16}$, and A_0 and A_1 are the maximal elementary abelian subgroups of $J(T) = A_0A_1$. Thus to prove (3), it suffices to show that each A of rank at least 5 is conjugate to a subgroup of $J(T)$.

Suppose first that $A \le K_1$. Then as $m_2(L_3(2)) = 2$ and $m(A) \ge 5$, $m(A \cap A_0) \ge 3$. Thus as A_0 is a sum of isomorphic natural modules E for K_1/A_0, AA_0/A_0 is conjugate to a subgroup of the group $J(T)/A_0$ of transvections on E with a common axis, so A is conjugate to a subgroup of $J(T)$ as required. So we may suppose that $A \not\leq K_1$. Then there is $a \in A - K_1$, and for each such a, $A \cap A_0 \leq A_a := C_{A_0}(a)$ of rank 3; and choosing E to be a K_1-irreducible in A_0 which is not a-invariant, $e \mapsto [a, e]$ is an $(A \cap K_1)$-isomorphism of E with A_a. Thus $A \cap K_1 \not\leq A_0$ and $m(C_{A_a}(A)) = m(C_E(A \cap K_1)) \leq 2$. Therefore

$$5 \leq m(A) = m(AA_0/A_0) + m(A \cap A_0) \leq 3 + m(C_{A_a}(A)) \leq 5,$$

so we conclude that A is of rank 5, $m(AA_0/A_0) = 3$, and $A \cap A_0 = C_{A_a}(A)$ is of rank 2. It follows that $AA_0 = J(T)\langle a\rangle$, where a is an involution in K_2, so $A \cap J(T) \not\leq A_0$ and $|A : A \cap J(T)| = 2$. As A_0 and A_1 are the maximal elementary abelian subgroups of $J(T)$, $A \cap J(T) \leq A_1$, so $|A : A \cap A_1| = 2$. Thus $|AA_1 : A_1| = 2$, so by symmetry between A_0 and A_1, our argument shows that A is conjugate in $N_M(A_1)$ to a subgroup of A_0. This completes the proof of (3).

Take A to be a hyperplane of A_0. If A centralizes some $W \leq V_1$ properly containing U, then $A \leq B$, where B is the group of transvections on V_1 with axis W. As $m(B) = 4$, this contradicts $m(A) = 5$. Thus $U = C_{V_1}(A)$. Similarly if A centralizes some $W \leq V_2$ with $U^\perp \cap V_2 =: U_2 < W$, then $A \leq C_{A_0}(W)$ of rank at most 4, again contrary to $m(A) = 5$. Thus $C_V(A) = U \oplus U_2$, and by duality, $[V, A] = [V, A_0] = U \oplus U_2$. Hence (4) is established.

Let $U_i := C_{V_i}(A_0)$ and $U_i < W_i < V_i$ with $m(W_i/U_i) = 1$. Since $Aut_{A_0}(W_i)$ induces a group of transvections on W_i with axis U_i, $m(A_0/C_{A_0}(W_i)) \leq 3, 2$, for $i = 1, 2$, respectively. Thus $1 \neq A \cap C_{A_0}(W_i)$ is of corank at most 3 in A, so $W_i \leq \check{\Gamma}_{3,A}(V)$, and hence (5) follows. □

Lemma H.9.3. *Assume that $n = 4$, and regard M as S_8. Then*

(1) M has two orbits on involutions in $M - L$: the transpositions t with $C_L(t) \cong S_6$, and u of cycle type $2^3, 1^2$ with $C_L(u) \cong \mathbf{Z}_2 \times S_4$.

(2) If t is an involution in $M - L$, then $\theta : V_1 \to C_V(t) = [V, t]$ defined by $\theta : v \mapsto v + v^t$ is a $C_{\bar{L}}(t)$-isomorphism.

(3) M has three orbits on the E_{16}-subgroups of M, with representatives A_i, $0 \leq i \leq 2$, where $A_0 := J(T_1)$ has orbit structure 4^2, and A_1 and A_2 have orbit structure 2^4 and $2^2, 4$, respectively, and are not contained in L.

(4) $C_V(A_0)$ is of rank 4, and $V = \check{\Gamma}_{2,A_0}(V)$.

(5) $C_V(A_1)$ is of rank 1, and $\check{\Gamma}_{2,A_1}(V) = C_V(r)$ is of rank 6, where r is the element in A_1 inducing a transvection on V_1.

(6) $C_V(A_2) = \{v + v^a : v \in C_{V_1}(B_2)\}$ is of rank 2, where $B_2 := A_2 \cap L$, and a is any element in $A_2 - B_2$. Further $\check{\Gamma}_{2,A_2}(V) = C_V(B_2) + C_V(a)$ is of rank 6.

Proof. Parts (1) and (3) follow from the representation of M as S_8. Part (2) follows as $V_1^t = V_2$. The group A_0 is the unipotent radical of the parabolic of L stabilizing a 2-subspace U_1 of V_1. As in the proof of H.9.2.5, for each hyperplane $U_1 < W_1 < V_1$, $m(A_0/C_{A_0}(W_1)) = 2$, so that $W_1 \leq \check{\Gamma}_{2,A_0}(V)$, and (4) holds.

Let r be the element of $B_1 := A_1 \cap L$ inducing a transvection of V_1; that is, r is the element of cycle type 2^4. Then B_1 is a hyperplane of A_1 and $C_{V_i}(r) =: U_i$ is a hyperplane of V_i, with the kernel K_i of the action of $C_L(r)$ on U_i given by the E_8-subgroup of transvections on V_i with axis U_i. Now $B_1K_i = O_2(C_L(r))$ acts on U_i as the group of transvections with center $\langle v_i\rangle := [V_i, r]$. Then $C_V(A_1) = \langle v_1 + v_2\rangle$ and $\check{\Gamma}_{1,B_1}(U_i) = \check{\Gamma}_{1,B_1/\langle r\rangle}(U_i) = U_i$. It follows that $U_1 \oplus U_2 = C_V(r) \leq \check{\Gamma}_{2,A_1}(V)$. But any 4-subgroup of A_1 contains some $1 \neq b \in B_1$, and $C_{V_i}(b) \leq U_i$, so (5) holds.

Let $B_2 := A_2 \cap L$. By construction, the set $\mathcal{R}$ of elements in B_2 acting as transvections on V_1 is of order 3, while $C_{V_i}(b) = C_{V_i}(B_2) =: W_i$ is of rank 2 for each $b \in B_2^\# - \mathcal{R}$. Thus $C_V(A_2) = \{v + v^a : a \in W_1\}$ is of rank 2, where $a \in A_2 - B_2$. Moreover as the three transvections in B_2 have distinct axes on V_1, $\check{\Gamma}_{2,B_2}(V_1) = V_1$. Then $C_V(a) = \check{\Gamma}_{2,A_2/\langle a\rangle}(C_V(a)) = \check{\Gamma}_{2,A_2}(C_V(a))$ by (2). Further $C_V(a) + C_V(B_2) = [V, A_2] = C_V(a') + C_V(B_2)$ for each $a' \in A_2 - B_2$, so (6) follows. $\square$

H.10. A foreword on Mathieu groups

In the remainder of the chapter, we consider small modules for the Mathieu groups, where most of the parameters were originally computed in Table 11.2 of [**Asc82a**]. In keeping with our philosophy of minimizing appeals to outside references, we reprove those results here. In the case of the Mathieu groups, a bound on the parameter $a(M, V)$ was established in section E.4.

The parameter β (measuring the codimension of the fixed points of elements of odd order) can be retrieved for the Mathieu groups M_{22}, M_{23}, and M_{24} on their code and cocode modules by elementary means from section 21 in [**Asc94**].

DEFINITION H.10.1. For $A \in \mathcal{A}^2(M)$, define

$$\xi_V(A) := \langle C_V(a) : a \in A^\# \rangle.$$

Notice that $\xi_V(A) = \check{\Gamma}_{m(A)-1,A}(V)$ using the notation of Definition E.3.30.

H.11. $\mathbf{M_{12}}$ on its 10-dimensional module

$m_2 = 4$: See Lemma H.11.1.1 below.

$m = 4$: Lemma H.11.1.2.

$a \leq 2$: Lemma E.4.3.

$\beta \geq 6$: As V is the 10-dimensional noncentral chief factor in the 12-dimensional permutation module, $m(C_V(x)) = o(x) - 2$ for $x \in L$ of odd order, where $o(x)$ is the number of orbits of x on the 12 points. Hence the value is achieved by an element x of cycle type $1^3, 3^3$.

Not an $(F-1)$-module, so $\alpha = \infty$: Lemma H.11.1.3.

LEMMA H.11.1. *(1)* $m_2 = 4$.

(2) $m = 4$.

(3) V is not an $(F-1)$-module.

(4) If $E_{16} \cong A \leq M$, then $M \cong Aut(M_{12})$, $m(C_V(A)) \leq 3$, and $m(\xi_V(A)) \geq 8$.

(5) If A is a noncyclic quadratic subgroup of M, then either each member of $A^\#$ is 2-central, or $m(A) = 2$ and $m(V/C_V(A)) > 4$.

PROOF. From the list of centralizers of involutions in $Aut(M_{12})$ (cf. Table 5.3b in [**GLS98**]), L has two classes z^L and i^L of involutions, there is one class j^L in $Aut(L) - L$, $m_2(L) = 3$, and $m_2(Aut(L)) = 4$. Thus (1) holds. As V is the 10-dimensional noncentral chief factor in the 12-dimensional permutation module V_0 for L, while z, i fix 4, 0 points from the set Ω of 12 points permuted by L, it follows that $m(C_V(z)) = 6 = m(C_V(i))$. Also $C_L(i) \cong \langle i\rangle \times S_5$, and we may choose j so that $C_L(j) = \langle i\rangle \times E$, where $E := E(C_L(i))$. Moreover j inverts an element of order 11 which is fixed-point-free on V, so $C_V(j) = [V, j]$ is of rank 5. Thus (2) holds.

Let $D := \langle i, j\rangle$ and $d \in D^\#$. By the Thompson $A \times B$-Lemma, E is faithful on $C_V(D)$; and then as $m(C_V(d)/[V,d]) \leq 2$, E is faithful on $[V,d]$. Thus as $D = C_M(d) \cap C_M(E)$, if $d \in A$ with A noncyclic and quadratic on V, then $A = D$, establishing (5).

Assume that $E_{16} \cong A \leq M$. Then from the first paragraph, $M \cong Aut(M_{12})$ and we may assume $j \in A$, and $A = D \times (A \cap E) \in Syl_2(C_M(j))$. Then as $E \cong A_5$ is faithful on $C_V(j)$ of rank 5, $C_V(A) = C_{C_V(j)}(A)$ is of rank at most 3. Finally as $A \in Syl_2(C_M(j))$, there is a 2-central involution z in A with $m(C_{C_V(t)}(z)) = 3$, so as $m(C_V(z)) = 6$, $m(\xi_V(A)) \geq m(C_V(t) + C_V(z)) = 8$, establishing (4).

Suppose B is an $(F-1)$-offender on V. Then $m(B) < 4$ by (1) and (4); therefore by (2), $m(B) = 3$ and $C_V(B) = C_V(b)$ is of rank 6 for each $b \in B^\#$. Then B is quadratic on V, so we may assume that $z \in B$ by (5). Thus $C_V(B) = C_V(z)$, and hence B is in the pointwise stabilizer of the 4 points of Ω fixed by z, a contradiction since that stabilizer is isomorphic to Q_8. This completes the proof of (3). $\square$

H.12. $\mathbf{3M_{22}}$ on its 12-dimensional modules

$m_2 \leq 5$: See Lemma H.12.1.13 below.
$m = 4$: Lemma H.12.1.10.
$a \leq 3$: E.4.3.
$\beta = 8$: H.12.1.11.
V is not an $(F-1)$-module, so that $\alpha = \infty$: Lemma H.12.1.12.

LEMMA H.12.1. *Let $L = \hat{M}_{22}$ and G a finite group with $F^*(G) = L$.*

(1) There is a unique 12-dimensional faithful irreducible $\mathbf{F}_2L$-module V.

(2) There is an $\mathbf{F}_4$-space structure $V_{\mathbf{F}_4}$ on V and a unitary $\mathbf{F}_4$-form f on V with $L \leq O(V_{\mathbf{F}_4}, f)$. Thus $L \leq O(V,q)$, where q is the $\mathbf{F}_2$-quadratic form on V defined by $q(u) := f(u,u)$ for $u \in V$.

(3) If U is a faithful irreducible $\mathbf{F}_2G$-module with $m([V,t]) \leq 4$ for some involution t of G, then U is $\mathbf{F}_2L$-isomorphic to V.

(4) Let $T \in Syl_2(L)$. There are exactly two maximal subgroups $M \cong \hat{A}_6/E_{16}$ and $N \cong (S_5/E_{16}) \times \mathbf{Z}_3$ of L containing T.

(5) T stabilizes a unique isotropic $\mathbf{F}_4$-point V_1 in (V,f), and $N_L(V_1) = N$.

(6) Let $Q_M := O_2(M)$ and $V_M := C_V(Q_M)$. Then $V_M = [V, Q_M]$ is totally isotropic of $\mathbf{F}_4$-dimension 3, and $Q_M = C_L(V_M)$.

(7) M has two orbits on isotropic $\mathbf{F}_4$-points in V_M with representatives V_1 and V_2. Further $N_M(V_2) = N_L(V_2) \cong (A_5/E_{16}) \times \mathbf{Z}_3$, and V_1 and V_2 are representatives for the orbits of L on isotropic $\mathbf{F}_4$-points.

(8) We can choose notation so that $[V,t] = V_1 + V_2$ for some involution $t \in Q_M$.

(9) Let $T \leq T_G \in Syl_2(G)$. Then $C_V(T_G) = C_{V_1}(T_G)$, and $m(C_V(T_G)) = 1$ if $L < G$.

(10) $m(G,V) = 4$.

(11) $\beta = 8$.

(12) V is not an $(F-1)$-module.

(13) $m_2(G) \leq 5$.

PROOF. By lemma 8.3 in James [**Jam73**], (1) holds, and $m(W) \geq 12$ for each faithful irreducible $\mathbf{F}_2L$-module W. Further as $Z := Z(L)$ is of order 3, L preserves an $\mathbf{F}_4$-space structure $V_{\mathbf{F}_4}$ on V. By uniqueness of V, $V_{\mathbf{F}_4} \cong V_{\mathbf{F}_4}^*$ or $V_{\mathbf{F}_4}^{*\theta}$, where

$\langle\theta\rangle = Gal(\mathbf{F}_4/\mathbf{F}_2)$ and $V^*_{\mathbf{F}_4}$ is the dual of $V_{\mathbf{F}_4}$ as an $\mathbf{F}_4L$-module. As 11 does not divide the order of $Sp_6(4)$, it must be the latter, so (2) holds.

Let U be a faithful irreducible $\mathbf{F}_2G$-module and $P \in Syl_3(L)$. Then $P \cong 3^{1+2}$ with $Z = Z(P)$. As U is faithful and irreducible, $C_U(Z) = 0$, so each irreducible $\mathbf{F}_2P$-submodule I of V is of rank 6, and $m([I,X]) = 4$ for each X of order 3 in P distinct from Z. Thus $m(U) = 6k$ and $m([U,X]) = 4k$, where $k := m(U)/6 \geq 2$ since $m(U) \geq 12$ by the previous paragraph. In particular $m([V,X]) = 8$, so to complete the proof of (11), it remains to show $m([V,y]) \geq 8$ for y of prime order $p > 3$ in L.

Let t be an involution in G; then t inverts some Y of order 3 in G, so $m([U,t]) \geq m([U,Y])/2 \geq 4k/2 = 2k$. Thus if $m([U,t]) \leq 4$ then $k \leq 2$, so that $k = 2$ by the previous paragraph.; thus (3) follows from (1). Moreover $k = 2$ for V, so $m(G,V) \geq 2k = 4$. Part (4) is well-known (cf. p. 9 in [**Asc86b**]).

Let V_1 be an isotropic $\mathbf{F}_4$-point in (V,f) stabilized by $T \in Syl_2(L)$. There are 693 isotropic points in (V,f), so $|V_1^L| \leq 693$. But by (4) the only overgroups H of T with $|L : H| \leq 693$ are M and N, so $N_L(V_1)$ is one of these groups. As $Z \leq M = M^\infty$, while Z is faithful on V_1, it follows that $N_L(V_1) = N$ so that (5) holds. Similarly we may take $P \leq M$, so from paragraph two, $V_M \in Irr_+(P,V)$ is of $\mathbf{F}_4$-dimension 3. Then V_M is not M-isomorphic to $V_M^{*\theta}$, so V_M is totally isotropic. As M is maximal in L, $M = N_L(V_M)$, so (6) holds. The first statement in (7) follows from the structure of the 3-dimensional $\mathbf{F}_4(M/Q_M)$-modules. As M and $K \cong SL_3(4)$ are the only maximal subgroups of L containing $N_M(V_2)$, and K fixes no $\mathbf{F}_4$-point since $Z \leq K^\infty$, $N_L(V_2) = N_M(V_2)$. Then $693 = |V_1^L| + |V_2^L|$, completing the proof of (7).

Let $Q_N := O_2(N)$. We've seen that $m([V,x]) = 8$ for x of order 3 in N^∞, so N^∞ has two chief factors on $V_1^\perp/V_1$, and these factors are distinct Galois conjugate $L_2(4)$-modules for N^∞/Q_N. So $s \in T - N^\infty$ must interchange two distinct $\mathbf{F}_4$-irreducibles, and hence N is irreducible on $V_1^\perp/V_1$. Thus $[V_1^\perp, Q_N] = V_1$, so $\dim_{\mathbf{F_4}}([V,t]) = 2$ for $t \in Q_N \cap Q_M^\#$. Furthermore any involution u in $G - L$ inverts Z, so that $m([V,u]) = 6$, completing the proof of (10). Now (8) follows from the structure of V_M.

From the structure of the 3-dimensional $\mathbf{F}_4$-module $V_M = C_V(Q_M)$ for $M/Q_M \leq \Gamma L_3(4)$, $C_{V_M}(T)$ is the $\mathbf{F}_4$-point V_1. If $L < G$, then elements of $T_G - T$ invert Z, so $m(C_{V_1}(T_G)) = 1$. Thus (9) holds as $C_V(T) \leq C_V(Q_M)$.

As the minimal dimension of a faithful $\mathbf{F}_4\mathbf{Z}_{11}$-module is 5, $m([V,y]) \geq 10$ for y of order 11 in L. Then as L has a Frobenius subgroup of order 55, $m([V,y']) \geq 8$ for y' of order 5, completing the proof of (11).

Recall $K \cong SL_3(4)$ is the stabilizer of one of the 22 points permuted by L. As V is self-dual, V is an extension of a natural module for K by its dual. Thus if A is an elementary abelian 2-subgroup of K, then $m(V/C_V(A)) \geq 4$, and $m(V/C_V(A)) \geq 6$ if $m(A) > 2$, so V is not an $(F-1)$-module for K. Choose $T_K := T \cap K \in Syl_2(K)$. By H.14.3.1, $T_K = J(T \cap L) = Q_M Q_N$ and $m_2(Aut(M_{22})) = 5$. Thus (13) holds.

Suppose A is an $(F-1)$-offender in T; then $3 \leq m(A) \leq 5$ by (10) and (13). But if $a \in A - L$, then a induces a field automorphism on $C_V(b)$ of rank 8 for $b \in A \cap L^\#$, so $m(V/C_V(A)) \geq 8$, contradicting A an $(F-1)$-offender. Thus $A \leq L$, so $m(A) = 3$ or 4, and in the latter case $A \in \mathcal{A}(T \cap L)$, so that $A \leq T_K$ by the previous paragraph. In the former case, we may take $a \in A \cap Q_M^\#$ and $C_V(A) = C_V(a)$. But then $A \leq Q_M \leq T_K$ by (6). Thus in any case, $A \leq K$, whereas by the previous paragraph, V is not an $(F-1)$-module for K; this completes the proof of (12). □

H.13. Preliminaries on the binary code and cocode modules

In this section, let $M^0 := M_{24}$ be the largest Mathieu group and $(X, \mathcal{C})$ the Steiner system for M^0, as described in chapter 6 of [**Asc94**].

DEFINITION H.13.1. As in section 19 of [**Asc94**], let V_0 denote the binary permutation module for M^0 on $X := \{1, \ldots, 24\}$, and identify V_0 with the power set of X by identifying $v \in V_0$ with its support. Under this identification, addition is symmetric difference. Let $V_{\mathcal{C}}$ be the *12-dimensional (Golay) code submodule* of V_0 generated by $\mathcal{C}$, and form $\tilde{V}_0 := V_0/V_{\mathcal{C}}$ as in section 19 of [**Asc94**]. Thus $\tilde{V}_0$ is the *12-dimensional Todd module* or *12-dimensional cocode module* for M^0.

The *weight* of a vector $v \in V_0$ is just its size as a set. We also speak of the *weight* of $\tilde{v} \in \tilde{V}_0$, which is the minimum of the weights of the vectors in the coset $\tilde{v}$. Similarly we set $\bar{V}_0 := V_0/\langle X \rangle$ and define the *weight* of $\bar{v}$ to be the weight of either v or $v + X$.

Let V_{core} be the *core* of V_0, consisting of those vectors in V_0 of even weight. Then $\tilde{V}_{core}$ is an 11-dimensional irreducible $\mathbf{F}_2 M^0$-module called the *11-dimensional Todd module* or *11-dimensional cocode module* for M^0.

For $Y \subseteq X$, write e_Y for the subset Y regarded as a vector of V_0. Then e_1 and $e_{1,2}$ are of weight 1 and 2, respectively. The corresponding stabilizers $M^0_1 := M^0_{e_1} = M^0_{\tilde{e}_1}$ and $M^0_{1,2} := M^0_{e_1,e_2} = M^0_{\tilde{e}_1,\tilde{e}_2}$ are M_{23} and M_{22}, respectively, and $M^0(\{1,2\}) := M^0_{e_{1,2}} = M^0_{\tilde{e}_{1,2}}$ is $Aut(M_{22})$, an extension of M_{22} by $\mathbf{Z}_2$. Therefore $\tilde{V}_{core}/\langle \tilde{e}_{1,2} \rangle$ is a 10-dimensional module for M_{22} called the *10-dimensional Todd module* or *10-dimensional cocode module*. The restriction of a Todd module for M_{24} to M_{23} is the corresponding Todd module for M_{23}.

There is a nondegenerate symmetric bilinear form $(\, , \,)$ over $\mathbf{F}_2$ on V_0, defined by $(u, v) := |u \cap v|$, and of course this form is preserved by M^0. Notice $V_{core} = X^{\perp}$, and $V_{\mathcal{C}}$ is a totally singular subspace of V_0, so $\tilde{V}_0$ is dual to V_0 as an M^0-module via $(\, , \,)$. We also have the *11-dimensional code module* $\bar{V}_{\mathcal{C}}$ for $M^0 \cong M_{24}$ and M_{23}, and the cocode module $\tilde{V}_{core}$ is dual to the code module $\bar{V}_{\mathcal{C}}$ via $(\, , \,)$. Finally the *10-dimensional code module* for M_{22} is the submodule of $\bar{V}_{\mathcal{C}}$ orthogonal to $\bar{e}_{1,2}$, and is dual to the 10-dimensional cocode module.

It is well known that:

PROPOSITION H.13.2. *The code and cocode modules of dimensions* 10, 11*, and* 11*, for* $M \cong M_{22}$*,* M_{23}*, and* M_{24}*, respectively, are absolutely irreducible* $\mathbf{F}_2 M$*-modules.*

PROOF. See 22.5 in [**Asc97**] for the cocode modules. Then the result holds for the code modules by duality. □

In the remainder of this chapter, let L be M_{22}, M_{23}, or M_{24}; let M be a subgroup of $Aut(L)$ containing L; and let V be a code or cocode module for M of dimension 10, 11, or 11, respectively. Observe that if L is M_{23} or M_{24} then $L = Aut(L)$, so $M = L$, while if L is M_{22} then $|M : L| \leq 2$.

We conclude this section by recording the values of our parameters for the code and cocode modules. Note that as the code module is dual to the cocode module, the values of m and β on the two modules are the same, and of course m_2 is independent of the module.

The value of β can be retrieved by elementary means from section 21 of [**Asc94**], so we do not supply a proof here; the value turns out to be always achieved only by the class $3A$ in the Modular Atlas [**JLPW95**]. For the determination of α, see H.15.2 and H.16.5 below.

Parameters for $\mathbf{M_{24}}$ on the code and cocode modules.
$m_2 = 6$: See H.14.1.1 below.
$m = 4$: H.14.4.3.
$a \leq 3$: E.4.3.
$\beta = 6$.
$\alpha \geq 5$ on the cocode module, and $\alpha \geq 7$ on the code module.

Parameters for $\mathbf{M_{23}}$ on the code and cocode modules.
$m_2 = 4$: H.14.2.1.
$m = 4$: H.14.4.3.
$a \leq 3$: E.4.3.
$\beta = 6$.
$\alpha \geq 5$ on the cocode module, and the code module is not an $(F-1)$-module.

Parameters for $\mathbf{M_{22}}$ on the code and cocode modules.
$m_2 \leq 5$: H.14.3.1.
$m \geq 3$: H.14.4.4.
$a \leq 3$: E.4.3.
$\beta = 6$.
$\alpha \geq 5$ on the cocode module, and $\alpha \geq 6$ on the code module.

In the remaining sections of the chapter, we will verify the information in our tables above, and establish a few supporting results.

H.14. Some stabilizers in Mathieu groups

We refer the reader to chapter 6 of [**Asc94**] for a discussion of the Steiner system $(X, \mathcal{C})$ and the definition of the terminology used in this section. Much of this language is originally due to Todd in [**Tod66**] and some to Conway [**Con71**].

Pick an octad $\mathbf{O}$, trio $\mathbf{T}$, and sextet $\mathbf{S}$ in the Steiner system, stabilized by the same Sylow 2-subgroup T_0 of M^0. Thus $\mathbf{O}$ is one of the three octads in $\mathbf{T}$, each of which is the union of a pair of the tetrads in $\mathbf{S}$. Choose notation so that 1 and 2 are in the same tetrad $\mathbf{B}$ of $\mathbf{S}$ and in $\mathbf{O}$, and define $T := T_0 \cap M$.

If $M = M_{23}$, let $\mathbf{Q} := \mathbf{S} - \mathbf{B}$ be a quintet and $\mathbf{H} := \mathbf{O} - \{\mathbf{1}\}$ be a heptad. If $L = M_{22}$, let $\mathbf{Q} := \mathbf{S} - \mathbf{B}$ be a quintet and $\mathbf{H} := \mathbf{O} - \{\mathbf{1}, \mathbf{2}\}$ be a hexad. For $Y \in \{\mathbf{O}, \mathbf{T}, \mathbf{S}, \mathbf{Q}, \mathbf{H}\}$, let M_Y be the *global stabilizer* in M of Y, and $K_Y := O_2(M_Y)$ be the "kernel" of the global stabilizer.

LEMMA H.14.1. *Let $L = M_{24}$. Then*

(1) $K_{\mathbf{S}}$ and $K_{\mathbf{T}}$ are the unique E_{64}-subgroups of T.

(2) $K_{\mathbf{S}}$ is the natural module for $M_{\mathbf{S}}/K_{\mathbf{S}} \cong \hat{S}_6$, so $K_{\mathbf{S}}$ has an $\mathbf{F}_4$-structure preserved by $M_{\mathbf{S}}^{\infty}$.

(3) $K_{\mathbf{T}}$ is the tensor product of the natural modules for the factors of $M_{\mathbf{T}}/K_{\mathbf{T}} \cong L_3(2) \times S_3$.

(4) $K_{\mathbf{O}}$ *is the natural module for* $M_{\mathbf{O}}/K_{\mathbf{O}} \cong L_4(2)$.
(5) Each $A \in \mathcal{A}^2(M)$ *with* $m(A) \geq 5$ *is fused into* $K_{\mathbf{S}}$ *or* $K_{\mathbf{T}}$.
(6) $K_{\mathbf{O}} \cap K_{\mathbf{S}} \cong E_4$ *is a* 1*-dimensional* $\mathbf{F}_4$*-subspace of* $K_{\mathbf{S}}$ *with* 15 $M_{\mathbf{S}}$*-conjugates.*

PROOF. Parts (2)–(4) are well known; see e.g. 19.9, 20.2, and 19.8 in [**Asc94**] for (2), (3), and (4), respectively. In particular, $K_{\mathbf{S}}$ and $K_{\mathbf{T}}$ are E_{64}-subgroups of T. Conversely let $M_{\mathbf{O}}^* := M_{\mathbf{O}}/K_{\mathbf{O}} \cong L_4(2)$, and $A \in \mathcal{A}^2(T)$ with $m(A) \geq 5$ so that $A^* \neq 1$. Then $K_{\mathbf{S}}^*$ is the unipotent radical of the parabolic stabilizing the line $E := K_{\mathbf{S}} \cap K_{\mathbf{O}}$ of $K_{\mathbf{O}}$, $K_{\mathbf{T}}^*$ is the unipotent radical of the parabolic stabilizing the plane $K_{\mathbf{T}} \cap K_{\mathbf{O}}$, and setting $B := A \cap K_{\mathbf{O}}$,

$$m(A^*) \geq 5 - m(B) = m(K_{\mathbf{O}}/B) + 1 > m(K_{\mathbf{O}}/C_{K_{\mathbf{O}}}(A^*)). \qquad (*)$$

Thus $A^* \leq K_{\mathbf{S}}^*$ or $K_{\mathbf{T}}^*$, since the latter groups are the maximal strong FF*-offenders in T^* of $M_{\mathbf{O}}^* \cong L_4(2)$ on its natural module $K_{\mathbf{O}}$. Also from the action of $K_{\mathbf{T}}$ on $K_{\mathbf{O}}$, $K_{\mathbf{T}}$ and $K_{\mathbf{O}}$ are the maximal elementary abelian 2-subgroups of $K_{\mathbf{T}}K_{\mathbf{O}}$, so $A \leq K_{\mathbf{T}}$ if $A^* \leq K_{\mathbf{T}}^*$. Thus we may assume that $A^* \leq K_{\mathbf{S}}^*$ but A^* does not lie in any $M_{\mathbf{O}}$-conjugate of $K_{\mathbf{T}}^*$; then as A^* is a strong FF*-offender in $K_{\mathbf{T}}^*$, $m(A^*) > 2$, so there is $a \in A$ with $C_{K_{\mathbf{O}}}(a) = E$, and hence $A \leq C_{K_{\mathbf{S}}K_{\mathbf{O}}}(a) = K_{\mathbf{S}}$. Thus (1) and (5) are established. Further $E \cong E_4$ and $N_{M_{\mathbf{O}}}(E) \leq N_M(K_{\mathbf{S}}) \cap N_M(E) = N_{M_{\mathbf{S}}}(E)$, which implies (6). □

From H.14.1 we can deduce analogous statements for the smaller Mathieu groups:

LEMMA H.14.2. *Let* $L = M_{23}$*. Then*

(1) $K_{\mathbf{Q}}$ *and* $K_{\mathbf{H}}$ *are the unique* E_{16}*-subgroups of* T.
(2) $K_{\mathbf{Q}}$ *is the natural module for* $M_{\mathbf{Q}}/K_{\mathbf{Q}} \cong \Gamma L_2(4)$.
(3) $K_{\mathbf{H}}$ *is a* 4*-dimensional irreducible for* $M_{\mathbf{H}}/K_{\mathbf{H}} \cong A_7$.

LEMMA H.14.3. *Let* $L = M_{22}$*. Then*

(1) $K_{\mathbf{Q}} \cap L$ *and* $K_{\mathbf{H}}$ *are the unique* E_{16}*-subgroups of* $T \cap L$*; and if* $M > L$*, then* $K_{\mathbf{Q}}$ *is the unique* E_{32}*-subgroup of* T.

(2) $K_{\mathbf{Q}} \cap L$ *is the* $L_2(4)$*-module; and if* $M > L$*, then* $K_{\mathbf{Q}}$ *is a* 5*-dimensional indecomposable for* $M_{\mathbf{Q}}/K_{\mathbf{Q}} \cong S_5$.

(3) $K_{\mathbf{H}}$ *is a natural module for* $M_{\mathbf{H}}/K_{\mathbf{H}} \cong A_6, S_6$*—for* $M = L$, $M > L$*, respectively.*

We sketch a proof of the two lemmas. As $M_{\mathbf{O}}^0$ acts as A_8 on the octad $\mathbf{O}$ with kernel $K_{\mathbf{O}}$, part (3) of each lemma follows from H.14.1.4. Similarly $M_{\mathbf{S}}^0$ acts as S_6 on the 6 tetrads of $\mathbf{S}$ with kernel $O_{2,3}(M_{\mathbf{S}}^0)$ acting as A_4 on each tetrad, so part (2) of each lemma follows from H.14.1.2. The proof of part (1) is analogous to that of H.14.1.1.

LEMMA H.14.4. *(1)* M_{24} *has two classes of involutions: the* 2*-central involutions* z *with* $Fix_X(z)$ *an octad, and the non-2-central involutions* t *which are fixed-point-free on* X.

(2) M_{23} *and* M_{22} *each have one class of involutions, and involutions fix pointwise a heptad or hexad.*

(3) If z *is a* 2*-central involution in* L*, then* $m(V/C_V(z)) = 4$*, while if* t *is a non-2-central involution in* M_{24}*, then* $m(V/C_V(t)) = 5$.

(4) If $M \cong Aut(M_{22})$ there are two classes of involutions in $M - L$, one fixed-point-free on X and one fixing pointwise an octad not containing $\{1, 2\}$. Further $m(V/C_V(i)) = 5$, 3, for i in the respective cases.

PROOF. Part (1) is well known; cf. 21.1 in [**Asc94**]. Further if z is 2-central, then $C_{M^0}(z)$ is transitive on $Fix_X(z)$, so (cf. 5.21 in [**Asc86a**]) $L = M_{23}$ or M_{22} is transitive on $z^{M^0} \cap L$, and (2) holds. Next t inverts a subgroup Y of order 11 in M^0, and $m(C_V(Y)) = 1$, so $C_V(t) = C_V(Y) \oplus [V, Y, t]$ is of rank 6. Similarly if $M \cong Aut(M_{22})$, $m(V/C_V(i)) = 5$ if i is fixed-point-free on X.

It is convenient to postpone proofs of the remainder of (3) and (4) until the discussion of the cocode module in the next section; note that codimensions of involution centralizers on the code module are the same as those on the cocode module by duality. □

H.15. The cocode modules for the Mathieu groups

In this section we assume V is the cocode module for M. We recall (cf. 19.10 in [**Asc94**]) that M^0 has two orbits $\mathcal{O}_2$ and $\mathcal{O}_4$ on its cocode module $V := \tilde{V}_{core}$, where $\mathcal{O}_k$ consists of the vectors of weight k. Further each coset $\tilde{v}$ of weight 2 contains a unique vector of weight 2, so $M^0_{\tilde{v}} \cong Aut(M_{22})$, while the vectors of weight 4 in a coset $\tilde{u}$ of weight 4 form a sextet $\mathbf{S}$ and $M^0_{\tilde{u}} = M^0_{\mathbf{S}}$. Similarly the orbits of M_{23} and M_{22} on their cocode modules V are described in 22.3 and 22.4 in [**Asc97**]; in particular there is an orbit $\mathcal{O}_2$ of vectors of weight 2 intersecting the fixed set of L in exactly one point, so that part (1) of the following lemma holds:

LEMMA H.15.1. *(1) M has an orbit $\mathcal{O}_2$ on $V^\#$ such that $M_v \cong Aut(M_{22})$, M_{22}, $L_3(4)$, for $L \cong M_{24}$, M_{23}, M_{22} respectively.*

(2) $V = [V, M_v]$.

PROOF. To prove the second part of the lemma, observe that $V/\langle v\rangle$ is the cocode module for M_v, so in particular M_v is irreducible on $V/\langle v\rangle$, and the extension does not split since M_v stabilizes no vector in the code module, and hence stabilizes no hyperplane in its dual V. □

LEMMA H.15.2. *(1) K_Y is the unique $(F-1)$-offender in T, where $Y := \mathbf{O}$, $\mathbf{H}$, $\mathbf{H}$ is an octad, heptad, or hexad in the respective cases.*

(2) $U := C_V(K_Y)$ is an orthogonal module of rank $d := 6$, 6, 5, for $M_Y/K_Y \cong \Omega_6^+(2)$, A_7, $O_5(2)$, respectively.

(3) If $I \leq U$ with $\mathcal{O}_2 \cap I = \emptyset$, then $m(I) \leq 3$, 4, 4, respectively.

(4) $W := \xi_V(K_Y)$ is a hyperplane of V.

(5) $\langle z\rangle = C_M(C_V(z))$ for each 2-central involution z, and $U = C_V(A)$ for each noncyclic subgroup A of K_Y.

(6) K_Y induces the full group of transvections on V/U with axis W/U, so W/U is isomorphic to K_Y as an M_Y/K_Y-module.

PROOF. By 19.1 in [**Asc94**], $M^0_{\mathbf{O}}$ is the split extension of $K_{\mathbf{O}} \cong E_{16}$ by $L_4(2)$ acting naturally on $K_{\mathbf{O}}$, and $M^0_{\mathbf{O}}$ acts as A_8 on $\mathbf{O}$ with kernel $K_{\mathbf{O}}$. Let U_0 be the subspace of $\tilde{V}_{core}$ consisting of those $\tilde{v}$ with $v \in \mathbf{O}$. It follows that $U_0 = C_V(K_{\mathbf{O}})$ and $|U_0| = 2^6$ with $\mathcal{O}_2 \cap U_0$ of order 28, and then that (2) holds for $M^0 = M_{24}$, with $\mathcal{O}_2 \cap U_0$ the set of nonsingular vectors in the orthogonal space U_0. In particular if $I \leq U_0$ with $I \cap \mathcal{O}_2 = \emptyset$, then I is totally singular, so $m(I) \leq 3$, and (3) holds in this case.

Further this shows that (2) holds when L is M_{23} or M_{22}, as $K_{\mathbf{H}} = K_{\mathbf{O}}$ and $M_{\mathbf{H}}$ is the stabilizer in $M_{\mathbf{O}}$ of 1 or $\{1,2\}$, respectively. If $L = M_{23}$, then $\mathcal{O}_2 \cap U$ consists of the seven vectors $\tilde{v}$ with $1 \in v$. As $M_{\mathbf{H}}$ is irreducible on U, each hyperplane of U contains such a vector by A.1.43, so (3) holds. Similarly (3) holds if $L = M_{22}$ since U has rank 5 while $\mathcal{O}_2 \cap U$ consists of the images of the six vectors v with $v \cap \{1,2\} = 1$, and in particular is nonempty.

Next we complete the proof of parts (3) and (4) of H.14.4, postponed from the previous section: we first complete the proof of part (3) of that lemma, establish (4)–(6) of the present lemma, and then complete the proof of part (4) of H.14.4 at the end of the paragraph. For $z \in K_{\mathbf{O}}^{\#}$, z is 2-central, and z inverts a subgroup D of order 5 with $m([V,D]) = 8$, so $m(V/C_V(z)) \geq 4$. On the other hand z centralizes U_0, and if $v \in V$ is of weight 2 with $|v \cap \mathbf{O}| = 1$, then $[\tilde{v}, z] \in V - U_0$, so $C_V(z) \not\leq U_0$; hence $m(C_V(z)) \geq 7$, so $m(V/C_V(z)) = 4$ and $C_V(z) = U_0 + \langle [\tilde{v}, z] \rangle$. Further (4)–(6) hold in this case, with $W := \xi_V(K_Y)$ the image of $\mathbf{O}^{\perp} \cap V_{core}$ in $V = \tilde{V}_{core}$. The cocode module for M_{23} is the restriction to M_{23} of the cocode module for M_{24} and $K_{\mathbf{O}} = K_{\mathbf{H}}$, so $m(V/C_V(z)) = 4$ for M_{23} also. This also shows that (4)–(6) hold for M_{23}, as they hold for M_{24}, and that if (1) holds for M_{24}, it also holds for M_{23}. Finally if $L = M_{22}$, then $V = \tilde{V}_{core}/\langle \tilde{e}_{1,2} \rangle$, so $m(V/C_V(z)) \leq m(\tilde{V}_{core}/C_{\tilde{V}_{core}}(z)) = 4$, and z still inverts a subgroup of order 5 in M_{22}; therefore the inequality is an equality, and $C_V(z) = C_{\tilde{V}_{core}}(z)/\langle \tilde{e}_{1,2} \rangle$. This last fact says that (4)–(6) hold for M_{22}, since they hold for M_{24}. Furthermore when $M > L$, there is $i \in K_{\mathbf{Q}} - L$ fixing 4 points of $\mathbf{O}$, with $\tilde{e}_{1,2} \in [U_0, i]$, so $m(V/C_V(i)) < m(\tilde{V}_{core}/C_{\tilde{V}_{core}}(i)) = 4$; hence $m(V/C_V(i)) = 3$, since i is 2-central in M_{24} so that $m(C_V(i)) \leq 7$. This completes the postponed parts of the proof of H.14.4, as well as of (4)–(6) of the present lemma.

Thus it remains to establish (1) for M_{24} and M_{22}, since we saw (1) holds for M_{23} if it holds for M_{24}. As $U = C_V(K_Y)$ and $m(V/U) = m(K_Y) + 1$, K_Y is an $(F-1)$-offender on V. Conversely, suppose $A \leq T$ is an $(F-1)$-offender distinct from K_Y. By (5), $U = C_V(B)$ for each noncyclic subgroup B of K_Y, so as $m \geq 3$ by H.14.4, no proper subgroup of K_Y is an $(F-1)$-offender, and hence $A \not\leq K_Y$.

Next for $a \in A^{\#}$,

$$m(A) \geq m(V/C_V(A)) - 1 \geq m(V/C_V(a)) - 1 \geq m - 1 \geq 2, \qquad (*)$$

as $m \geq 3$. In particular $A \cap L \neq 1$, so we can choose $a \in A \cap L$; therefore $m(V/C_V(a)) = 4$ and hence $m(A) \geq 3$ by (*). In case of equality, $C_V(A) = C_V(a)$ has corank 4 in V, so that a is 2-central for each $a \in A \cap L$, and $A \cap L$ is of rank at least 2, contrary to (5). Hence $m(A) > 3$, so as $m_2(T/K_Y) = 4$ or 3 for $L \cong M_{24}$ or M_{22}, respectively, either $A \cap K_Y \neq 1$, or $A \cap K_Y = 1$, $M = M_{24}$, and $AK_Y/K_Y = J(T/K_Y)$ is of rank 4. But in the latter case $m(A) = 4$, $m(U/C_U(A)) = 5$, and $[W/U, A] \neq 0$ by (6), contradicting A an $(F-1)$-offender.

Hence $B := A \cap K_Y \neq 1$, so $U \leq C_V(B)$. Also $m(V/C_V(B)) \geq m(V/C_V(b)) = 4$ for $b \in B^{\#}$ as $K_Y \leq L$, and as $B \neq 1$, $m(U/C_U(A)) > 1$ by (2). Thus

$$m(A) \geq m(V/C_V(A)) - 1 \geq m(V/C_V(B)) + m(U/C_U(A)) - 1 > m(V/C_V(B)) \geq 4. \qquad (**)$$

Now arguing as before on $m_2(T/K_Y)$, we conclude that B is noncyclic, so $U = C_V(B)$ by (5). Thus $m(V/C_V(A)) > m(V/C_V(B)) = 5$ by (**), so L is M_{24} and $A = K_{\mathbf{T}}$ or $K_{\mathbf{S}}$ by H.14.1.1. However we will see in later H.15.3.1 that $m(C_V(K_{\mathbf{S}})) = 1$, and in H.15.6 that $m(C_V(K_{\mathbf{T}})) = 3$, so (as the proofs of these later lemmas are independent of this lemma) neither $K_{\mathbf{S}}$ nor $K_{\mathbf{T}}$ is an $(F-1)$-offender, completing the proof of the lemma. □

LEMMA H.15.3. *Let $M := M_{24}$. Then*

(1) $M_{\mathbf{S}}$ has chief series $0 < V_1 < V_7 < V$, where $V_1 := C_V(K_{\mathbf{S}}) = \langle \tilde{e}_B \rangle$ for B a tetrad of $\mathbf{S}$, V_7/V_1 is of rank 6 and isomorphic to the dual of $K_{\mathbf{S}}$ as a $M_{\mathbf{S}}$-module, and V/V_7 is a natural module for $M_{\mathbf{S}}/O_{2,3}(M_{\mathbf{S}}) \cong S_6$.

(2) $K_{\mathbf{S}}$ is the full group of transvections on V_7 with center V_1.

(3) $V = \xi_V(K_{\mathbf{S}})$.

(4) $V_7/V_1 = C_{V/V_1}(K_{\mathbf{S}})$.

PROOF. The dual of 20.4 in [**Asc94**] and its proof says that $M_{\mathbf{S}}$ has a chief series whose terms have the desired rank and structure as $M_{\mathbf{S}}$-modules. Thus $V_1 = C_V(M_{\mathbf{S}})$, so as $M_{\mathbf{S}}$ fixes $\tilde{e}_B$, $V_1 = \langle \tilde{e}_B \rangle$. Now for $a \in K_{\mathbf{S}}^{\#}$, $[V_7, a] \leq V_1$. Further we can choose a non-2-central, so that $m(C_V(a)) = 6$ by H.14.4.3; thus a induces a transvection on V_7 with center V_1, and $C_V(a) \leq V_7$. Then as $M_{\mathbf{S}}$ is irreducible on $K_{\mathbf{S}}$, it follows that $K_{\mathbf{S}}$ acts faithfully as the full group of transvections on V_7 with center V_1, establishing (2), and showing that $V_1 = C_V(K_{\mathbf{S}})$, so that the proof of (1) is complete. But we can also choose a 2-central, so that $m(C_V(a)) = 7$ by H.14.4.3; thus $C_V(a) \not\leq V_7$ as $K_{\mathbf{S}}$ is faithful on V_7, and hence (3) holds as $M_{\mathbf{S}}$ is irreducible on V/V_7. Finally (4) holds as M_S is irreducible on V_7/V_1 and V/V_7. □

LEMMA H.15.4. *Let $M := Aut(M_{22})$. Then*

(1) $M_{\mathbf{Q}}$ has chief series $0 < V_1 < V_2 < V_6 < V$, where $V_1 := C_V(K_{\mathbf{Q}})$ is of rank 1, $V_2 := C_V(K_{\mathbf{Q}} \cap L)$ is of rank 2, V_6/V_1 is isomorphic to the dual of $K_{\mathbf{Q}}$ as a $M_{\mathbf{Q}}$-module, and V/V_6 is the S_5-module for $M_{\mathbf{Q}}/K_{\mathbf{Q}}$.

(2) $K_{\mathbf{Q}}$ is the full group of transvections on V_6 with center V_1.

(3) $V = \xi_V(A)$, and $m(C_V(A)) = 2$ for each hyperplane A of $K_{\mathbf{Q}}$.

(4) $V = \langle C_V(B) : E_4 \cong B \leq K_{\mathbf{Q}} \rangle$.

PROOF. Write $0 < U_1 < U_7 < U := \tilde{V}_{core}$ for the series of H.15.3 under $M^0_{\mathbf{S}}$. Then $V = U/\langle \tilde{e}_{1,2} \rangle$ and $\tilde{e}_{1,2} \in U_7$; let V_1 be the image of U_1 and V_6 the image of U_7 in V, so that $0 < V_1 < V_6 < V$ is an $M_{\mathbf{Q}}$-series, since $M_{\mathbf{Q}}$ is the stabilizer of $\tilde{e}_{1,2}$ in $M^0_{\mathbf{S}}$. Then (2) follows from H.15.3.2, $V_1 = C_V(K_Q)$, and V_6/V_1 is the dual of $K_{\mathbf{Q}}$ as $M_{\mathbf{Q}}$-module. In particular as $A_0 := K_{\mathbf{Q}} \cap L$ is of index 2 in $K_{\mathbf{Q}}$, $V_0 := C_{V_6}(A_0)$ is an M_Q-invariant subspace of dimension 2. Next $m(C_V(x)) = 4$ for x of order 3 in $M_{\mathbf{Q}}$, so $m(C_{V/V_6}(x)) = 2$. Therefore as U/U_7 is a natural module for $M_{\mathbf{S}}/O_{2,3}(M_{\mathbf{S}}) \cong S_6$, and $M_{\mathbf{Q}}/K_{\mathbf{Q}} \cong S_5$, V/V_6 is the S_5-module for $M_{\mathbf{Q}}/K_{\mathbf{Q}}$. Thus $M_{\mathbf{Q}}$ is irreducible on V/V_6, so $V_2 = V_0$, completing the proof of (1). Further if $A \neq A_0$ is a hyperplane of $K_{\mathbf{Q}}$, then A contains an involution i with $m(C_V(i)) = 5$, so $C_V(i) = C_{V_6}(i)$, and thus $C_V(A) = C_{V_6}(A)$ is of rank 2 by (2).

Next from the structure of $K_{\mathbf{O}}$, $E := K_{\mathbf{O}} \cap K_{\mathbf{S}} \cong E_4$, and by H.15.3.2, $m(C_{U_7}(E)) = 5$. So as $C_U(E) = C_U(K_{\mathbf{O}})$ is of rank 6 by H.15.2.5, $C_U(E) \not\leq U_7$. As $K_{\mathbf{O}} = K_{\mathbf{H}}$, $E \leq K_{\mathbf{H}} \cap K_{\mathbf{Q}}$, so that $C_V(E) \not\leq V_6$. Thus (4) holds as $M_{\mathbf{Q}}$ is irreducible on V/V_6. Finally if D is a hyperplane of $K_{\mathbf{Q}}$ and B a 4-subgroup of $K_{\mathbf{Q}}$, then there is $b \in B \cap D^{\#}$, so as $C_V(B) \leq C_V(b) \leq \xi_V(D)$, (4) says $V = \xi_V(D)$, completing the proof of (3). □

LEMMA H.15.5. *Let $M := Aut(M_{22})$, $E_{16} \cong A \leq T$, and $W := \xi_V(A)$. Then one of the following holds:*

(1) $A = K_{\mathbf{H}}$*,* $m(C_V(A)) = 5$*, and* $m(V/W) = 1$*.*

(2) A *is fused into* $K_{\mathbf{Q}}$ *under* M*,* $m(C_V(A)) = 2$*, and* $W = V$*.*

(3) $A = O_2(C_M(i))$ *for some* $i \in A - L$ *fixing an octad not containing* $\{1, 2\}$*, so that* $C_M(i)/A \cong L_3(2)$*,* $m(C_V(A)) = 3$*, and* $W = V$*.*

(4) $m(A \cap K_{\mathbf{H}}) = 3$*,* $m(C_V(A)) = 3$*, and* $m(V/W) \leq 1$*.*

PROOF. Suppose $A \leq L$. Then A is as described in (1) or (2) by H.14.3.1, as are $C_V(A)$ and W by H.15.2 and H.15.4.3.

Thus we may assume that $A \not\leq L$. Set $M^*_{\mathbf{H}} := M_{\mathbf{H}}/K_{\mathbf{H}}$ and $U := C_V(K_{\mathbf{H}})$, so that $m(U) = 5$ and $U = C_V(B)$ for each noncyclic subgroup B of $K_{\mathbf{H}}$ by (2) and (5) of H.15.2. As $m(A) = m(K_{\mathbf{H}})$ and $K_{\mathbf{H}} = C_{M_{\mathbf{H}}}(K_{\mathbf{H}})$,

$$m(A^*) = m(A) - m(A \cap K_{\mathbf{H}}) \geq m(K_{\mathbf{H}}/C_{K_{\mathbf{H}}}(A)),$$

so A^* is an FF*-offender in $M^*_{\mathbf{H}} \cong S_6$ on $K_{\mathbf{H}}$. Now $K^*_{\mathbf{Q}}$ is the E_8-subgroup of $M^*_{\mathbf{H}}$ generated by three transpositions, so by B.3.2.5, either $A^* \leq K^*_{\mathbf{Q}}$ or A^* is the remaining E_8-subgroup "A_0" of T^*.

First assume that the latter case holds, and let $P_Z := O_2(O^2(C_{M_{\mathbf{H}}}(Z))$, where $Z := Z(T)$. Now $C_M(Z) \leq M_{\mathbf{H}}$ and $P_Z \cong Q_8^2$. Further there is $i \in A$ centralizing the hyperplane $P_Z \cap K_{\mathbf{H}}$ of $K_{\mathbf{H}}$ and the supplement $B := A \cap P_Z$ to $P_Z \cap K_{\mathbf{H}}$ in P_Z; thus $i \in C_M(P_Z) \cong E_4$, and hence $Z\langle i \rangle = C_M(P_Z)$. Therefore $Fix_X(i)$ is an octad not containing $\{1, 2\}$, and $C_L(i) \cong L_3(2)/E_8$. Now up to conjugacy under $C_L(Z)$, there are three E_8-subgroups of P_Z: $P_Z \cap K_{\mathbf{H}}$, $O_2(C_L(i))$, and $K_{\mathbf{Q}} \cap P_Z$. Thus as B is contained in neither $K_{\mathbf{H}}$ nor any $C_L(Z)$-conjugate of $K_{\mathbf{Q}}$, $B = O_2(C_L(i))$, so $A = O_2(C_M(i))$. Therefore by H.14.4.4, $C_V(A) = [V, i]$ is of rank 3 and $C_V(Z) \not\leq C_V(i)$, so as $C_L(i)$ is irreducible on $V/C_V(i)$, (3) holds in this case.

Next assume A^* is of order 2 in $K^*_{\mathbf{Q}}$. Then A is conjugate to $\langle i \rangle(P_Z \cap K_{\mathbf{H}})$. As $P_Z \cap K_{\mathbf{H}}$ is noncyclic, $C_V(P_Z \cap K_{\mathbf{H}}) = U$, and then $C_V(A) = C_U(i)$ is of rank 3. Hence W contains $U + C_V(i)$ of rank 9, and (4) holds.

Finally we may take $A^* \leq K^*_{\mathbf{Q}}$ with $m(A^*) > 1$. Thus there is $a \in A \cap L - K_{\mathbf{H}}$, so $a \in K_{\mathbf{H}}(K_{\mathbf{Q}} \cap L)$. Then as $K_{\mathbf{Q}} \cap L$ and $K_{\mathbf{H}}$ are the maximal elementary abelian subgroups of $K_{\mathbf{H}}(K_{\mathbf{Q}} \cap L)$, $a \in K_{\mathbf{Q}} \cap L$ and $A \leq C_{AK_{\mathbf{H}}}(a) \leq C_{K_{\mathbf{Q}}K_{\mathbf{H}}}(a) = K_{\mathbf{Q}}$. As $m(C_V(K_{\mathbf{Q}})) = 1 = m(K_{\mathbf{Q}}/A)$, $m(C_V(A)) \leq 2$. Hence (2) holds by H.15.4.3. □

LEMMA H.15.6. *Let* $M := M_{24}$*. Then*

(1) $M_{\mathbf{T}}$ *has chief series* $0 < V_3 < V_9 < V$*, where* $V_3 := C_V(K_{\mathbf{T}})$ *is of rank* 3*, and* $V_9 := [M^\infty_{\mathbf{T}}, V] = [K_{\mathbf{T}}, V]$ *is of corank* 2 *in* V*.*

(2) $M_{\mathbf{T}}$ *induces* $GL(V_3)$ *on* V_3*, and* $GL(V/V_9)$ *on* V/V_9*.*

(3) V_9/V *is the tensor product of the natural modules for the factors of* $M_{\mathbf{T}}/K_{\mathbf{T}}$*.*

PROOF. This is essentially the dual of 20.7 in [**Asc94**]. □

LEMMA H.15.7. *Let* $M := M_{23}$ *or* M_{24}*, with* D *of odd prime order* p *in* M*, and assume* $W := C_V(D)$ *is of rank at least* 4*. Then*

(1) $p = 3$*,* $m(W) = 5$*, and up to conjugacy in* M^0*,* $N_{M^0}(D)$ *is a complement to* $K_{\mathbf{S}}$ *in* $M_{\mathbf{S}}$*.*

(2) W *is isomorphic to the core of a* 6*-dimensional permutation module for* $N_{M^0}(D)/D \cong S_6$ *on* $Fix_X(D)$ *with* $C_V(K_{\mathbf{S}}) = C_V(N_{M^0}(D))$*.*

(3) If $I \leq W$ *with* $I \cap \mathcal{O}_2 = \emptyset$*, then* $m(I) \leq 2, 4$ *for* $M = M_{24}, M_{23}$*, respectively.*

PROOF. Part (1) is contained in section 21 of [**Asc94**], while (2) is contained in Exercise 7.4 in [**Asc97**]. In particular if $M = M_{24}$, then $\mathcal{W} := \mathcal{O}_2 \cap W$ is the set of vectors $\tilde{e}_v$ with $v \in Fix_X(D)$ of weight 2; thus if $I \cap \mathcal{O}_2 = \emptyset$, I is a totally singular subspace of W, so (3) holds in this case. If $M = M_{23}$, then $1 \in Fix_X(D)$, and $\mathcal{W}$ consists of those $\tilde{e}_v$ with $1 \in v$, so $\mathcal{W} \neq \emptyset$ and hence $I < W$ for each such I, so again (3) holds. □

Lemma H.15.8. *Let $M := M_{24}$, and define V_7 and V_9 as in H.15.3 and H.15.6. Then $V_7 \leq V_9$, but $V_7 \not\leq V_3$.*

Proof. Let $Y \in Syl_3(O_{2,3}(M_{\mathbf{S}}))$. Then from H.15.3, $V_7 = V_1 + [V,Y]$. Next $Y \leq M_{\mathbf{T}}^{\infty}$, so from H.15.6.1, $[V,Y] \leq V_9$ and $V_1 = C_V(T) \leq V_9$, so $V_7 = V_1 + [V,Y] \leq V_9$. □

H.16. The code modules for the Mathieu groups

In this section, we assume that V is the code module for M.

Lemma H.16.1. *Let $M := M_{24}$. Then*

(1) $M_{\mathbf{S}}$ has chief series $0 < V_4 < V_{10} < V$, where $V_4 := C_V(K_{\mathbf{S}})$ is of rank 4, and $V_{10} := [V, M_{\mathbf{S}}^{\infty}]$ is a hyperplane of V.

(2) V_4 is a natural module for $M_{\mathbf{S}}/O_{2,3}(M_{\mathbf{S}}) \cong Sp_4(2)$.

(3) $K_{\mathbf{S}}$ induces the full group of transvections on V/V_4 with axis V_{10}/V_4, so V_{10}/V_4 is isomorphic to $K_{\mathbf{S}}$ as an $M_{\mathbf{S}}$-module.

Proof. Each part of the lemma follows by dualizing the corresponding statement in H.15.3. □

Lemma H.16.2. *Let $M := M_{24}$, $V_4 := C_V(K_{\mathbf{S}})$, $\langle z \rangle = Z(T)$, and $P_z := O_2(C_M(z))$. Then*

(1) $M_{\mathbf{O}}$ has chief series $0 < V_1 < V_5 < V$, where $V_1 := C_V(K_{\mathbf{O}})$ and $m(V_k) = k$.

(2) $K_{\mathbf{O}}$ induces the full group of transvections on V_5 with center V_1, so V_5/V_1 is isomorphic to the dual of $K_{\mathbf{O}}$ as an $M_{\mathbf{O}}$-module.

(3) V/V_5 is the 6-dimensional orthogonal module for $M_{\mathbf{O}}/K_{\mathbf{O}}$, with $(V_4+V_5)/V_5$ a singular point and $(C_V(z) + V_5)/V_5$ a totally singular 3-subspace.

(4) $V_4 = C_V(K_{\mathbf{O}} \cap K_{\mathbf{S}})$.

(5) $\langle z \rangle = C_M(C_V(z))$.

(6) $P_z = K_{\mathbf{O}}D$, where $V_z := C_{V_5}(z)$ is of rank 4, and $D := C_M(V_z) \cong E_{16}$.

(7) $m(C_V(B)) = 8 - m(B)$ for each B with $z \in B \leq D$.

(8) All elements of $C_M(z)$ inducing transvections on $C_V(z)$ are in P_z.

Proof. Parts (1) and (2) follow by dualizing corresponding statements in H.15.2. Part (3) is 20.6.2 in [**Asc94**]. By H.14.1.6, $E := K_{\mathbf{O}} \cap K_{\mathbf{S}} \cong E_4$. By (3),

$$(C_V(z) + V_5) \cap (C_V(e) + V_5) = V_4 + V_5$$

for $e \in E - \langle z \rangle$, and $m(C_{V_5}(E)) = 3$ by (2), so $m(C_V(E)) = 4$ and hence (4) holds.

Next by (1) and (2), $M_z := C_M(z)$ induces the full stabilizer in $GL(V_z)$ of V_1 on $V_z = C_{V_5}(z)$, so $C_{M_z}(V_z) \leq P_z$ and $|D| = |M_z|/|N_{GL(V_z)}(V_1)| = 16$. Thus D is the unique M_z-supplement to $K_{\mathbf{O}}$ in P_z of order 16, so defining U_1 and U_2 as in 39.1 of [**Asc94**], $K_{\mathbf{O}} = U_1$ and $D = U_2 \cong E_{16}$. In particular (6) holds, and M_z is irreducible on $D/\langle z \rangle$, so $C_M(C_V(z)) = \langle z \rangle$ or D. But in the latter case, $N_M(D) = \langle C_M(d) : d \in D^{\#} \rangle$ induces $L_4(2)$ on D, impossible as $M_{\mathbf{O}}$ is the unique proper overgroup of M_z in M. Thus (5) holds. We saw $C_{M_z}(V_z) \leq P_z$, while by

(3), $C_{M_z}(C_V(z)/V_z) \leq P_z$, so (8) holds. By 39.1.3 in [**Asc94**], $K_{\mathbf{S}} \cap D \cong E_8$. Now $K_{\mathbf{S}} \cap D$ centralizes $V_4 + V_z$ of rank 5, and then as M_z induces $GL(C_V(z)/V_z)$ on $C_V(z)/V_z$, and induces the stabilizer in $GL(D)$ of z on D, (7) holds. □

LEMMA H.16.3. *Let $M := Aut(M_{22})$. Then*

(1) $M_{\mathbf{Q}}$ has chief series $0 < V_4 < V_8 < V_9 < V$, where $V_4 := C_V(K_{\mathbf{Q}})$ and V_k is of rank k for $k = 4, 8, 9$.

(2) V_4 is the S_5-module for $M_{\mathbf{Q}}/K_{\mathbf{Q}}$.

(3) $K_{\mathbf{Q}}$ is the full group of transvections with axis V_9/V_4 on V/V_4, so V_9/V_4 is isomorphic to $K_{\mathbf{Q}}$ as an $M_{\mathbf{Q}}$-module.

(4) $C_M(V_4) = K_{\mathbf{Q}}$ is a 2-group.

PROOF. Each of the first three parts is the dual of the corresponding statement in H.15.4. As $M_{\mathbf{Q}}$ is maximal in M, $N_M(V_4) = M_{\mathbf{Q}}$, so $C_M(V_4) = C_{M_{\mathbf{Q}}}(V_4) = K_{\mathbf{Q}}$. □

LEMMA H.16.4. *Let $M := Aut(M_{22})$. Then*

(1) $M_{\mathbf{H}}$ has chief series $0 < V_1 < V_5 < V_6 < V$, where $m(V_k) = k$, $V_1 := C_V(K_{\mathbf{H}})$, and $V_5 := [V, K_{\mathbf{H}}]$.

(2) $K_{\mathbf{H}}$ is the full group of transvections on V_5 with center V_1, so V_5/V_1 is isomorphic to the dual of $K_{\mathbf{H}}$ as an $M_{\mathbf{H}}$-module.

(3) V/V_5 is the core of a 6-dimensional permutation module for $M_{\mathbf{H}}/K_{\mathbf{H}} \cong S_6$.

(4) $V_5 \leq V_9$ but $V_5 \not\leq V_4$.

(5) $V_5/V_1 = C_{V/V_1}(K_{\mathbf{H}})$.

PROOF. Parts (1)–(3) are the duals of corresponding statements in H.15.2. Indeed froom 20.6 in [**Asc94**], V_5 is the union of the 15 trios through **O**, while from 20.4 in [**Asc94**], V_4 consists of the 15 octads through **S**. In particular the 7 octads in the 3 trios through **O** and **S** form the rank-3 subspace $V_4 \cap V_5$ invariant under $M_{\mathbf{H}} \cap M_{\mathbf{Q}}$, so $M_{\mathbf{H}} \cap M_{\mathbf{Q}}$ is the parabolic of $M_{\mathbf{H}}$ irreducible on $(V_4 \cap V_5)/V_1$ and $V_5/(V_4 \cap V_5)$. Thus as the hyperplane V_9 is $(M_{\mathbf{H}} \cap M_{\mathbf{Q}})$-invariant, $V_5 \leq V_9$, establishing (4).

Let $\hat{V} := V/V_1$. As $M_{\mathbf{H}}$ is irreducible on $\hat{V}_5$, $\hat{V}_5 \leq C_{\hat{V}}(K_{\mathbf{H}})$. Then as $V_5 = [V, K_{\mathbf{H}}]$ and $M_{\mathbf{H}}$ is irreducible on V/V_6, $C_{\hat{V}}(K_{\mathbf{H}}) = \hat{V}_5$ or $\hat{V}_6$. Let Y be of order 3 in $C_{M_{\mathbf{H}}}(z)$, for $z \in K_{\mathbf{H}}^{\#}$. As $[Y, z] = 1$, $m([V_5, Y]) = 2$ by (2); therefore as $m([V, Y]) = 6$, we conclude $[V/V_6, Y] = V/V_6$. However $m(C_V(z)) = 6$ and $m(C_{V_5}(z)) = 4$, so $m((V_5 + C_V(z))/V_5) = 2$, and hence $V_5 + C_V(z) = V_5 + [C_V(z), Y]$. Therefore $C_{V_6}(z) = C_{V_5}(z)$, so $[\hat{V}_6, z] \neq 1$, completing the proof of (5). □

LEMMA H.16.5. *(1) Up to conjugation, $K_{\mathbf{S}}$ is the unique $(F-1)$-offender in M_{24}.*

(2) M_{23} has no $(F-1)$-offenders.

(3) Up to conjugation, $K_{\mathbf{Q}}$ is the unique $(F-1)$-offender in $Aut(M_{22})$.

(4) If $M \cong Aut(M_{22})$, then $C_M(C_V(z)) = C_M(O^2(C_M(z))) \cong E_4$ for each 2-central involution z.

PROOF. Let z generate $Z(T)$; we first prove (4). Now V is a hyperplane of $U := \bar{V}_{\mathcal{C}}$, and $C_V(z)$ is a hyperplane of $C_U(z)$. By H.16.2.8, all elements inducing transvections on $C_U(z)$ are in P_z, so $C_M(C_V(z)) \leq P_z \cap M = K_O\langle d\rangle$, where d is an element of the subgroup D defined in that result, and d has cycle $(1, 2)$. Now

$\langle z, d\rangle = C_M(O^2(C_M(z)))$, and using H.16.2.6, $C_U(\langle z, d\rangle)$ is the $C_M(z)$-invariant hyperplane of $C_U(z)$ and $\langle z, d\rangle = C_{M^0}(C_U(\langle z, d\rangle))$, establishing (4).

Next by H.16.1.1 and H.16.3.1, $K_{\mathbf{S}}$ and $K_{\mathbf{Q}}$ are $(F-1)$-offenders in M_{24} and $Aut(M_{22})$, respectively. Further (1) implies (2), so we may take M to be M_{24} or $Aut(M_{22})$, and assume that $A \leq T$ is an $(F-1)$-offender, but A is not $K_{\mathbf{S}}$ or $K_{\mathbf{Q}}$, and it remains to derive a contradiction. Proceeding as in the proof of H.15.2,

$$m(A) \geq m(V/C_V(a)) - 1 \geq 2, \qquad (*)$$

for $a \in A^{\#}$, so we can pick $a \in L$ and hence $m(V/C_V(a)) \geq 4$. Thus by (*), $m(A) \geq 3$, with equality only if a is 2-central and $C_V(a) = C_V(A)$, so that H.16.2.5 and (4) supply a contradiction. Thus $m(A) \geq 4$.

Assume first that $M = Aut(M_{22})$. As A is not $K_{\mathbf{Q}}$, $m(A) = 4$ by H.14.3.1, so A is described in H.15.5. If $A = K_{\mathbf{H}}$, then $m(C_V(A)) = 1$ by H.16.4.1, so A is not an $(F-1)$-offender. If $A = O_2(C_M(i))$ for $i \in M - L$ fixing an octad not containing $\{1, 2\}$, we saw $C_M(i)/A \cong L_3(2)$, which has three natural or dual constituents on the cocode module, and hence also on the code module V. Since $m(C_V(A)) < 6$ in view of (4), we see $m(C_V(A)) \leq 4$, so again A is not an $(F-1)$-offender. In the fourth case of H.15.5, we saw $K_{\mathbf{H}} \cap A$ is a hyperplane of $K_{\mathbf{H}}$, and we must have $m(C_V(K_{\mathbf{H}} \cap A)) \geq m(C_V(A)) \geq 5$, so $m(C_V(K_{\mathbf{H}})) > 1$, contrary to H.16.4.1. This leaves the case where A is a hyperplane of $K_{\mathbf{Q}}$. Here $K_{\mathbf{Q}} \cap L =: D$ is partitioned by the five $M_{\mathbf{Q}}$-conjugates of $E := K_{\mathbf{H}} \cap K_{\mathbf{Q}}$ of rank 2, and the hyperplane $A \cap D$ must contain one of these conjugates, say E. But using H.16.2.4, we see $C_V(E) = V_4 = C_V(K_{\mathbf{Q}})$, so $m(C_V(A)) \leq 4$, contrary to $m(C_V(A)) \geq 5$.

Therefore we may take M to be M_{24}. Assume $m(A) = 4$. Then $m(C_V(A)) \geq 6$. If A consists entirely of non-2-central involutions, then $C_V(A) = C_V(a)$ for all $a \in A^{\#}$. But we may take $a \in K_{\mathbf{S}}$, so $A \leq K_{\mathbf{S}}$ by H.16.1.1, whereas E_{16}-subgroups of $K_{\mathbf{S}}$ contain 2-central involutions. Thus there is a 2-central involution z in A, so A centralizes a hyperplane of $C_V(z)$, and hence $A \leq P_z$ by H.16.2.8. Since A centralizes a hyperplane of V_z, $|A : A \cap D| \leq 2$ by H.16.2.2, and in case of equality $A \cap D$ centralizes $C_V(A) + V_z = C_V(z)$, contrary to H.16.2.5. Therefore $A = D$, so that $m(C_V(A)) = 4$ by H.16.2.7, contrary to A an $(F-1)$-offender.

If $m(A) = 6$, then as $A \neq K_{\mathbf{S}}$, $A = K_{\mathbf{T}}$ by H.14.1.1. This is impossible, as on the dual V^* of V, $[V^*, K_{\mathbf{T}}])$ is of codimension 2 by H.15.6.1, so $m(C_V(K_{\mathbf{T}})) = 2$. Therefore $m(A) = 5$, so by H.14.1.5 we may take $A \leq K_{\mathbf{T}}$ or $K_{\mathbf{S}}$. In the first case, as $m(K_{\mathbf{O}} \cap K_{\mathbf{T}}) = 3$, $K_{\mathbf{O}} \cap A$ is noncyclic. But then $m(C_V(K_{\mathbf{O}} \cap A)) \geq m(C_V(A)) \geq 5$, so $m(C_V(B)) \geq 3$ for B a hyperplane of $K_{\mathbf{O}}$ containing $K_{\mathbf{O}} \cap A$, and then $m(C_V(K_{\mathbf{O}})) > 1$, contrary to H.16.2.1. Hence $A \leq K_{\mathbf{S}}$, and using the previous argument, $A \cap (K_{\mathbf{O}} \cap K_{\mathbf{S}})^g$ is cyclic for each of the 15 conjugates of $K_{\mathbf{O}} \cap K_{\mathbf{S}}$ under $M_{\mathbf{S}}$. Thus A contains at most one vector from each of those $\mathbf{F_4}$-points, while from H.14.1.6, there are only six other $\mathbf{F_4}$-points, so A contains at least 16 vectors living in those six points, and hence contains at least four of those six points. However as $M_{\mathbf{S}}$ is 2-transitive on the six points, any three generate $K_{\mathbf{S}}$, so $A = K_{\mathbf{S}}$, contradicting our assumption. □

LEMMA H.16.6. *Let* $M := M_{24}$, *and* $\mathbf{Z}_3 \cong Y \leq M$ *with* $m(V/C_V(Y)) = 6$. *Then* $Y = C_M(C_V(Y))$.

PROOF. Let $W := C_V(Y)$. By the dual of H.15.7, up to conjugacy in M, $N_M(Y) =: M_Y$ is a complement to $K_{\mathbf{S}}$ in $M_{\mathbf{S}}$, $C_V(K_{\mathbf{S}}) \leq W$, and W is a quotient of the permutation module for $M_Y/Y \cong S_6$. As M_Y is irreducible on $K_{\mathbf{S}}$, while

$m(C_V(K_{\mathbf{S}})) = 4$, $C_{K_{\mathbf{S}}}(W) = 1$. As $M_{\mathbf{S}}$ is maximal in M, $M_{\mathbf{S}} = N_M(C_V(K_{\mathbf{S}}))$, so as $K_{\mathbf{S}} = C_{M_{\mathbf{S}}}(C_V(K_{\mathbf{S}}))$ and $Y = C_{M_Y}(W)$, $C_M(W) = C_{M_{\mathbf{S}}}(W) = C_{K_{\mathbf{S}}M_Y}(W) = YC_{K_{\mathbf{S}}}(W) = Y$. □

LEMMA H.16.7. *Assume* $L \cong M_{22}$ *or* M_{24}, $s := 2$ *or* 3, *and* $E_{2^{2s}} \cong A$ *and* $\mathbf{Z}_3 \cong Y$ *are subgroups of* M *with* $A = [A, Y]$. *Then* $m(V/C_V(AY)) \neq 2(s+1)$.

PROOF. Assume otherwise. As $A = [A, Y]$, $A \leq L$.

Assume first that $L \cong M_{22}$. Then by H.14.3.1, $s = 2$ and we may take $A = K_{\mathbf{H}}$ or $K_{\mathbf{Q}} \cap L$. Now $m(C_V(K_{\mathbf{H}})) = 1$ by H.16.4.1, so $m(V/C_V(K_{\mathbf{H}}Y)) \geq 9 > 2(s+1)$, and hence $A = K_{\mathbf{Q}} \cap L$. From the series in H.16.3.1, $V_4 := C_V(K_{\mathbf{Q}}) = C_V(K_{\mathbf{Q}} \cap L)$ is of rank 4. Now $Y \leq N_M(K_{\mathbf{Q}} \cap L) = M_{\mathbf{Q}}$ since this group is maximal in M. Also H.16.3 shows V_4 is the S_5-module, so $m(V/C_V(AY)) = 8$ rather than 6.

Therefore $M = M_{24}$. Suppose first $s = 3$. Then $A = K_{\mathbf{S}}$ or $K_{\mathbf{T}}$ by H.14.1.1. However $m(C_V(K_{\mathbf{T}})) = 2$ by the dual of H.15.6, so that $A = K_{\mathbf{S}}$. Hence by H.16.1, $V_4 := C_V(A)$ is of rank 4, and no element of order 3 in $M_{\mathbf{S}}$ centralizes exactly a hyperplane of V_4. But again we have $Y \leq N_M(A) = M_{\mathbf{S}}$ since $M_{\mathbf{S}}$ is maximal in M, so that $m(V/C_V(AY)) \neq 8$.

This leaves the case $s = 2$, where $m(A) = 4$ and $m(C_V(AY)) = 5$. But by the dual of H.15.7.1, $m(C_V(Y)) = 5$, so $C_V(A) = C_V(Y)$. Now H.16.6 supplies a contradiction. This completes the proof of the lemma. □

LEMMA H.16.8. *If* $M := M_{23}$ *and* $E_{16} \cong A \leq T$, *then* $m(V/\xi_V(A)) \leq 1$.

PROOF. By H.14.2.1, $A = K_{\mathbf{H}}$ or $K_{\mathbf{Q}}$. Assume first that $A = K_{\mathbf{H}}$, and define V_5 as in H.16.2.1. Now V is the restriction of the code module for M^0 to M, and $K_{\mathbf{O}} = K_{\mathbf{H}}$, while $m((C_V(a) + V_5)/V_5) \geq 2$ for $a \in K_{\mathbf{H}}^{\#}$, since $m(C_V(a)) = 7$. Hence $\xi_V(A) = V$ using H.16.2.3. Thus we may take $A = K_{\mathbf{Q}}$. Define V_{10} as in H.16.1.1; we will prove that $V_{10} = \xi_V(A)$. Now the subgroup $K_{\mathbf{O}} \cap K_{\mathbf{S}}$ of M^0 lies in $K_{\mathbf{Q}}$, and $K_{\mathbf{O}} \cap K_{\mathbf{S}} = \langle a, b \rangle$ with $C_V(a) \cap C_V(b) = C_V(K_{\mathbf{O}} \cap K_{\mathbf{S}}) = V_4$ of rank 4 by H.16.2.4. Thus

$$m(C_V(a) + C_V(b)) = m(C_V(a)) + m(C_V(b)) - 4 = 10,$$

so as $C_V(c) \leq V_{10}$ for all $c \in K_S^{\#}$ by H.16.1.3, $\xi_V(K_{\mathbf{Q}}) = V_{10}$, completing the proof. □

CHAPTER I

Statements of some quoted results

In this chapter we continue to follow the conventions of section A.1 with regard to quoted results: typically we provide statements and at least a reference for the proof, with fuller proofs given where no good reference seems available.

I.1. Elementary results on cohomology

In this section we record various standard facts on cohomology involving the simple SQTK-groups listed in Theorem C (A.2.3), and the quasithin $\mathcal{K}$-groups listed in Theorem B (A.2.2).

REMARK I.1.1. We recall some well known concepts. Let G be a perfect (i.e., $G = [G, G]$) finite group. A *covering (perfect central extension)* of G is a surjective group homomorphism $\varphi : \hat{G} \to G$ such that $\hat{G}$ is perfect and $\ker(\varphi) \leq Z(\hat{G})$. The group $\hat{G}$ is called a *covering group* of G. There is a *universal covering* $\psi : \tilde{G} \to G$ with the property that for each covering φ of G there exists a unique group homorphism $\theta : \tilde{G} \to \hat{G}$ with $\theta\varphi = \psi$. The group $\tilde{G}$ is the *universal covering group* and $\ker(\psi)$ is the *Schur multiplier* of G. See for example section 33 in [**Asc86a**] for proofs of these facts and further discussion.

LEMMA I.1.2. *If G is perfect and $m_r(G) \leq 1$ for r an odd prime, then the Schur multiplier of G is an r'-group.*

PROOF. Standard; e.g., 33.14 in [**Asc86a**]. □

One reference for the next two results is [**GLS98**, Sec 6.1]; see in particular Definition 6.1.1 and Tables 6.1.2 and 6.1.3. A briefer discussion appears in the Atlas [**C$^+$85**], pages xv–xvi and the tables.

LEMMA I.1.3. *The simple SQTK-groups G with a nontrivial Schur multiplier are listed below. Except where otherwise indicated, the multiplier is cyclic of order d, where d is given by:*

(1) $d = 2$ when G is $L_4(2)$, $G_2(4)$, M_{12}, J_2, HS, Ru, or $L_2(q)$, $q \neq 9$ is an odd prime power.

(2) $d = 3$ when G is $L_3^\epsilon(q)$, $q \equiv \epsilon \mod 3$, and G is not $L_3(4)$.

(3) $d = 6$ when G is A_6 or A_7.

(4) M_{22} has multiplier $\mathbf{Z}_{12}$.

(5) $Sz(8)$ has multiplier E_4.

(6) The multiplier of $L_3(4)$ is $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_3$.

LEMMA I.1.4. *The simple quasithin $\mathcal{K}$-groups G which are not strongly quasithin and have a nontrivial Schur multiplier are listed below. Except where otherwise indicated, the multiplier is cyclic of order d, where d is given by:*

(1) $d = 2$ *when* G *is* A_9*,* $L_2(q)$*,* $q > 9$ *an odd prime power,* $U_4(2)$*, or* $L_4(3)$*.*

(2) $d = 3$ *when* G *is* J_3*,* McL*,* $G_2(3)$*, or* $L_3^\epsilon(q)$*,* $q \equiv \epsilon \mod 3$ *an odd prime power.*

(3) $d = 4$ *if* G *is* $L_4(r)$*,* $r > 3$ *a Fermat prime, or* $U_4(s)$*,* $s > 3$ *a Mersenne prime.*

(4) The multiplier of $U_5(4)$ *is* $\mathbf{Z}_5$*.*

(5) The multiplier of $U_4(3)$ *is* $E_9 \times \mathbf{Z}_4$*.*

REMARK I.1.5. We recall more well known concepts, this time concerning covers of modules. Let G be a finite group and V a faithful $\mathbf{F}_2G$-module. Assume first that $C_V(G) = 0$, and define a *dual covering* of V to be an $\mathbf{F}_2G$-module W such that $[W, G] \leq V \leq W$ and $C_W(G) = 0$. There exists a *universal dual covering* $U := U(G, V)$; that is U is a dual covering, and each dual covering of V is a submodule of $U(G, V)$. The *1-cohomology group* $H^1(G, V)$ is the group $U(G, V)/V$. (This group is often denoted by $Ext^1_G(\mathbf{F}_2, V)$). In particular if V is *projective*, then $H^1(G, V) = 0$.

Now assume instead that $V = [V, G]$. A *covering* of V is an $\mathbf{F}_2G$-module $W = [W, G]$ such that $W/W_0 \cong V$ for some $W_0 \leq C_W(G)$. Again there is a *universal covering* U; that is, U is a covering such that each covering of V is a quotient of U. Indeed $U = U(G, V^*)^*$, where W^* denotes the dual of W, so $m(U_0) = m(H^1(G, V^*))$.

LEMMA I.1.6. *Let* G *be a group,* V *a* $\mathbf{F}_2G$*-module, and* $d := \dim_{\mathbf{F}_2}(H^1(G, V))$*. Then*

(1) If V *is the natural module for* $G = A_n$*, then* $d = 0$ *when* $n = 5, 7$*, and* $d = 1$ *when* $n = 6, 8$*. The* A_5*-module is projective.*

(2) If V *is the natural module for* $G = L_2(2^n)$*,* $n > 1$*, then* $d = n$*.*

(3) If V *is the* 4*-dimensional orthogonal module for* $G = L_2(2^{2n}) \cong \Omega_4^-(2^n)$*, then* $d = 0$*. The orthogonal module for* $L_2(4)$ *is the* A_5*-module, and hence is projective.*

(4) If V *is the natural module for* $G = SL_3(2^n)$*, then* $d = 0$ *when* $n > 1$*, and* $d = 1$ *when* $n = 1$*.*

(5) If V *is the natural module for* $G = Sp_4(2^n)'$ *or* $G_2(2^n)'$*, then* $d = n$*.*

(6) If V *is the natural module for* $G = L_4(2)$ *or* $L_5(2)$*, then* $d = 0$*.*

(7) If V *is a* 10*-dimensional irreducible for* $G = L_5(2)$*, then* $d = 0$*.*

(8) $d = 0$ *for each* 4*-dimensional irreducible for* $G = A_7$*.*

(9) If V *is a faithful irreducible for* $G = \hat{A}_6$ *of dimension* 6*, then* $d = 0$*.*

PROOF. A standard reference for the Chevalley groups is Jones-Parshall [**JP76**]. For the alternating groups, see e.g. Exercise 6.3 in [**Asc86a**]. □

REMARK I.1.7. (a) By Remark I.1.5, the results about $H^1(G, V)$ in I.1.6 give us information about the coverings and dual coverings of the modules V discussed in that lemma. For example, we recorded I.1.6.1 earlier as B.3.3.1, and used it to establish B.3.3.3.

(b) We can also use Remark I.1.5 and I.1.6 to study FF-modules: For example let $L \cong G_2(2^n)'$ and suppose that W is an FF-module for a group G with $L = F^*(G)$, and W is a dual covering of the natural module V. We claim that $W = V$: Part (13) of B.4.6 shows that up to conjugacy, there is a unique FF*-offender A_1 on V, and that $m(V/C_V(A_1)) = m(A_1)$. Thus $W = VC_W(A_1)$, so W splits over V

as an A_1-module. Next V is self-dual, so from Remark I.1.5, W^* is a covering of V. As W splits over V as an A_1-module, W^* splits over $C_{W^*}(L)$ as an A_1-module. But by B.4.6.3, $C_{W^*}(L) \leq [W^*, A_1]$, so $W^* = V$ and hence $W = V$.

I.2. Results on structure of nonsplit extensions

Sometimes we require more detailed information on the structure of the nonsplit extensions in I.1.3 and I.1.6.

The following result is needed only to show that certain coverings groups of groups in Theorems B (A.2.2) and C (A.2.3) are not strongly quasithin groups:

LEMMA I.2.1. *Let G be a covering group of $G^* \cong U_4(3)$, $G_2(3)$, $\Omega_7(3)$, $O'N$, J_3, or McL, with $Z(G)$ a 3-group. Then $m_3(G) > 2$.*

PROOF. From the list of Schur multipliers in I.1.4, $Z(G)$ is of order 3, except possibly when $G^* \cong U_4(3)$, in which case $Z(G)$ is a subgroup of E_9.

We apply A.1.31.1 to a suitable subgroup X of G in the role of "G"; thus it will suffice to produce a subgroup X with $Y := F^*(X) = O_3(X)$, a normal subgroup H of X, and an involution t in X with $C_{\bar{Y}}(\bar{t})$ noncyclic, where $\bar{X} := X/H$.

Take X to be the preimage in G of a subgroup X^* of G^* isomorphic to A_6/E_{3^4}, $\Omega_5(3)/E_{3^5}$, $\mathbf{Z}_2/E_{3^4}$, $\mathbf{Z}_2/E_{3^3}$, A_6/E_{3^4}, for G^* isomorphic to $U_4(3)$, $\Omega_7(3)$, $O'N$, J_3, McL, respectively. When G^* is $O'N$ or J_3, choose X^* inside a subgroup isomorphic to $E_9 \times A_6$ or $\mathbf{Z}_3 \times A_6$, respectively. When G^* is $G_2(3)$, take X to be a Sylow 3-group of G extended by an involution. Let $H := Z(G)$ unless G^* is J_3, where $H := 1$.

By construction, X satisfies the necessary conditions, except possibly in the case where G^* is J_3, where we must show that $C_Y(t)$ is noncyclic. This holds as $C_{Y^*}(t)$ is of order 3 and its preimage splits over $Z(G)$: the splitting occurs as $C_G(t)^* \cong A_5/Q_8D_8$, and $Z(G) \not\leq [C_G(t), C_G(t)]$ by I.1.2. □

Our next result collects facts about coverings of groups in Theorem C which are needed at various points during the proof of the Main Theorem.

LEMMA I.2.2. *Let K be quasisimple with $Z := Z(K) \neq 1$, let $S \in Syl_2(K)$, and set $K^* := K/Z$.*

(1) If K^* is A_5 or A_6 and Z is a 2-group, then $K \cong SL_2(5)$ or $SL_2(9)$, respectively.

(2) If K^* is A_6 or A_7 then $m_3(K) = 2$.

(3) Assume $K^* \cong L_3(4)$ and Z is a 2-group, and let P be the preimage of a maximal parabolic of K^* and $R := O_2(P)$. Then

(a) If $\Phi(Z) \neq 1$ then $Z(S) = Z$ and R is nonabelian.

(b) If $\Phi(Z) = 1$ then involutions in K^* lift to involutions in K, $\Phi(R) = 1$, $R = [R, P]$, and Z is centralized by a graph-field automorphism of K.

(4) If $K^* \cong Sz(8)$ then involutions in K^* lift to involutions in K, and $Z = Z(S)$.

(5) Assume K^* is $G_2(4)$, J_2, M_{12}, or HS. Then

(a) 2-central involutions of K^* lift to involutions of K.

(b) Non-2-central involutions of K^* lift to elements of order 4, and each involutory outer automorphism of K inverts some such element of order 4.

(c) Assume K^* is not $G_2(4)$, let $u \in K - Z$ with u an involution, and set $X := O^2(C_K(u))$. Then there exists a unique $v \in uZ$ such that there exists $x \in O_2(X)$ with $x^2 = v$.

(6) Assume $K^* \cong M_{22}$ and Z is a 2-group. Let J be the preimage in K of a subgroup of K^* isomorphic to S_5/E_{16}. Then

(a) $Z \leq O^2(J)$, $\Phi(O_2(J)) = 1$ if $|Z| = 2$, and $O_2(J) \cong \mathbf{Z}_4 * Q_8^2$ if $|Z| = 4$.

(b) Assume $|Z| = 2$, $u \in K - Z$ is an involution, and set $X := O^2(C_K(u))$. Then there exists a unique $v \in uZ$ such that there exists $x \in O_2(X)$ with $x^2 = v$.

(7) Assume $K^* \cong Ru$, let i^* be a non-2-central involution in K^*, and I the preimage in K of $E(C_{K^*}(i^*))$. Then

(a) I splits over Z.

(b) i^* lifts to an element of order 4, and 2-central involutions of K^* lift to involutions, so that $m_2(K) \geq 6$.

(c) Let $u \in K - Z$ be an involution and set $X := O^2(C_K(u))$ and $Y := C_{O_2(X)}(\Phi(O_2(X)))$. Then $[Y, Y]$ is of order 2 and $[Y, Y]^* = \langle u^* \rangle$.

PROOF. Many of these facts can be retrieved from the literature; but in most cases no good direct reference exists, so we sketch proofs.

We will begin with some standard observations which are often useful. Using the well known result of Cartan-Eilenberg that the restriction of mod-p cohomology to a Sylow p-subgroup is injective (Generalized Thompson Transfer A.1.37 suffices in this case), we have:

(i) If Z is a p-group and H is the preimage in K of a subgroup H^* of K^* such that $(|H^* : K^*|, p) = 1$, then $Z \leq [H, H]$, and if $H^* = O^p(H^*)$ then $H = O^p(H)$.

Also:

(ii) Assume $|Z| = 2$, $H^* \leq K^*$, and the Sylow 2-subgroups of H^* are not cyclic or dihedral. Then some involution in H^* lifts to an involution of H, and if all involutions in H^* are fused in K^*, then each involution in H^* lifts to an involution of H.

For by the restriction in (ii) on the Sylow 2-groups of H^*, a Sylow 2-group of H is not cyclic or quaternion, so there are involutions in $H - Z$ (cf. Exercise 8.4 in [**Asc86a**]), and hence (ii) follows.

Part (1) of the lemma is well known; it follows from I.1.3.1 and the existence of $SL_2(5)$ and $SL_2(9)$.

Assume K^* is A_6 or A_7, Z is a 3-group, and let $P \in Syl_3(K)$. By I.1.3.3, $|Z| = 3$, so as $P^* \cong E_9$ is inverted in K^*, it follows that P is of exponent 3, and P is nonabelian by (i), so $P \cong 3^{1+2}$. Thus (2) holds.

Assume the hypothesis and notation of (3), and let L be the preimage of a Levi complement of P^*. By (i), P is perfect. Suppose $Z \cong E_4$; then $K = \tilde{K}/W$ for some characteristic subgroup of the universal covering group $\tilde{K}$ of K^*, so $Aut(K^*) = Aut(K)$. A diagonal outer automorphism of order 3 is irreducible on Z and centralizes a graph-field automorphism β, so β centralizes Z, establishing the last statement in (3b).

Suppose that Z is of order 2. Then as K has one class of involutions, involutions in K^* lift to involutions by (ii). Thus $\Phi(R) = 1$, and L splits over Z by (1), so $R = [R, L]$ as P is perfect. Therefore (b) holds when $|Z| = 2$, and hence also when $Z \cong E_4$, considering the quotients K/U for the three subgroups U of order 2 in Z.

It remains to prove (a), where by I.1.3.4 we may assume that $Z \cong \mathbf{Z}_4$ by passing to a suitable quotient of K. Suppose R is abelian and let $X \in Syl_5(P)$. As $C_{R^*}(X) = 1$, $R = [R, X] \times Z$ by Coprime Action, so $V := [R, X]\Phi(Z) = \Omega_1(R) \trianglelefteq$

P. Hence $R/\Phi(Z)$ splits over $Z/\Phi(Z)$ as a P-module, contrary to the previous paragraph. Therefore R is nonabelian, so as P is irreducible on R^*, $Z = Z(R)$, and hence $Z(S) = Z$, completing the proof of (3).

The first statement of (4) follows from (ii) and the fact that K^* has one class of involutions. Next $N_{K^*}(S^*)$ is irreducible on $Z(S^*)$ and on $S^*/Z(S^*)$, each of rank 3. Thus either S is of class 3 with $Z = Z(S)$, or S is of class 2 with $Z(S)^* = Z(S^*)$; since it remains to prove the second part of (4), we may assume the latter. Then $m([S,S]) \leq \binom{m(S/Z(S))}{2} = 3 = m([S,S]^*)$, so $[S,S]$ is a complement to Z in $Z(S)$, contrary to (i). This completes the proof of (4).

Assume that K^* is one of the four groups in (5), and let i^* be a non-2-central involution in K^*. We claim that to establish (a) and (b), it suffices to prove:

(iii) i^* lifts to an element of order 4, and a conjugate t of each involutory outer automorphism of K^* centralizes i^* with $i^*t \in t^{K^*}$.

For as K^* has two classes of involutions, (a) follows from (ii) and (iii). Further as $i^*t \in t^{K^*}$ and t is an involution, so is it; so t inverts i, establishing (b).

We next establish the second statement in (iii): Unless K^* is HS, K^* is transitive on the set Δ of involutory outer automorphisms, and from the structure of $C_{Aut(K^*)}(i^*)$, i^* centralizes a member of Δ. So assume instead that K^* is HS. Then there are two orbits of K^* on Δ with representatives t_1 and t_2, where $K_1^* := C_{K^*}(t_1) \cong S_8$, $K_2^* := C_{K^*}(t_2) \cong O_4^-(2)/E_{16}$, and $K_i^* := C_{K^*}(i^*) \cong \mathbf{Z}_2 \times Aut(A_6)$. The involutions in K_1^* with cycle structure 2^2 or 2^4 are 2-central, while those with cycle structure 2 or 2^3 are fused to i^*, with $t_1 z$ fused to t_1 if z is of cycle type 2^4 as $\langle t_1, z\rangle$ is the center of a Sylow 2-subgroup of $K_1^*\langle t_1\rangle$. Further we may take t_1 to centralize the subgroup $\mathbf{Z}_2 \times S_6$ of index 2 in K_i^*; in particular we may take $[t_1, t_2] = 1$. Thus K_1^* has two orbits on $i^{*K^*} \cap K_1^*$, so K_i^* has two orbits on $K_i^*\langle t_1\rangle \cap t_1^{K^*}$. Hence these orbits have representatives t_1 and $t_1 y$, where y induces an outer automorphism in S_6 on $E(K_i^*)$. Further $t_1 i^* \in t_1^{K_i^*}$; and for e an involution in $E(K_i^*)$, $t_1 e \in t_2^{K^*}$ as $t_1 e$ is not fused to t_1, while $t_1 e i^* \in (t_1 e)^{K_i^*}$. This establishes the second statement in (iii) when $K^* \cong HS$. Therefore to complete the proof of (iii), it remains to show that i^* lifts to an element of order 4.

We first treat the case $K^* \cong G_2(4)$. Let H be $\cdot 0$ and $\bar{H} := H/Z(H)$, so that H is the universal covering group of $\bar{H} \cong Co_1$. We quote some results from [**Asc94**]: The "root 4-involutions" $\bar{i}$ of $\bar{H}$ are the $\bar{H}$-conjugates of root 4-involutions of Suz-subgroups of $\bar{H}$, which are in turn defined on page 269 of [**Asc94**]. By Lemma 46.9, these involutions lift to elements of order 4 in H. Further by Lemma 49.5, $\bar{J} := E(C_{\bar{H}}(\bar{i})) \cong G_2(4)$ with the non-2-central involutions in $\bar{J}$ fused to $\bar{i}$. Thus the preimage J of $\bar{J}$ is the universal covering group of $G_2(4)$. This completes the proof of (5) when $K^* \cong G_2(4)$.

Thus we may assume that K^* is J_2, M_{12}, or HS. Let u^* be a 2-central involution in K^*, and set $X := O^2(C_K(u))$ and $R := O_2(X)$. In each case $R^* = [R^*, X^*]$ is of symplectic type with $\Phi(R^*) = \langle u^*\rangle$, so by Coprime Action $Z(R)^* = Z(R^*)$, X centralizes $Z(R)$, and R is of class 2 with $[R,R]Z = \langle u\rangle Z$. Therefore $[R,R]$, and hence also $\langle u\rangle Z$, is of exponent 2, so u^* lifts to an involution u. In particular, this establishes (a), and shows that $[R,R]Z = \Phi(R)Z = \langle u\rangle \times Z \cong E_4$. Let $\mathcal{R}$ be the set of $r \in R - Z(R)$ with $|r^*| = 4$.

Suppose that K^* is J_2. Then $X^* = C_{K^*}(u^*) \cong A_5/Q_8D_8$, so X is perfect by (i). As X is transitive on $\mathcal{R}^*$, $r^2 =: v$ is independent of the choice of $r \in \mathcal{R}$, and $R/\langle v\rangle$ is elementary abelian, establishing (c). Let $\bar{X} := X/\langle v\rangle$. Then as $R^*/\langle u^*\rangle$ is a projective X/R-module, $\bar{R}$ splits over $\bar{Z}$, so as X is perfect, $X/[R,X] \cong SL_2(5)$. Thus as $R = [R,X]Z$, there are no involutions in $X - R$, so that i^* lifts to an element of order 4. This completes the proof of (5) when K^* is J_2.

Assume that K^* is M_{12}. Then $X^* \cong \mathbf{Z}_3/Q_8^2$ has five orbits $\mathcal{O}_j^*$ of length 3 on $(R^*/\langle u^*\rangle)^\#$, with $\mathcal{O}_1 \cup \mathcal{O}_2$ containing the images of non-2-central involutions, $\mathcal{O}_3$ the images of 2-central involutions, and $\mathcal{O}_4 \cup \mathcal{O}_5 = \mathcal{R}^*$. For $j = 4, 5$, let $v_j := r_j^2$ for $r_j \in \mathcal{O}_j$, $V_j := \langle v_j \rangle$, and $\bar{X} := X/V_4$. As X has two irreducibles of rank 2 on $\bar{R}/\bar{Z}$, $\bar{R} \cong E_{32}$, Q_8^2, or $E_4 \times Q_8$, so the number c of cosets of $\bar{Z}$ in $\bar{R}$ containing elements of order 4 is 0, 6, or 12, respectively. If i^* lifts to an involution then $c = 0$ or 3, and the latter is impossible, so $V_4 = \Phi(R)$. But then $\bar{R} = [\bar{R}, X] \times \bar{Z}$, and as there is a 2-central involution w^* in $C_{K^*}(u^*) - X^*$, $C_K(u) = X\langle w \rangle \times Z$, contrary to (i). Therefore i^* lifts to i of order 4, establishing (b). Furthermore $c = 6$ or 9, and the latter is impossible, so $v_4 = v_5$, establishing (c), and completing the proof of (5) when $K^* \cong M_{12}$.

Finally assume that K^* is HS. As $X^* \cong A_5/(\mathbf{Z}_4 * Q_8^2)$ is transitive on $\mathcal{R}^*$, we argue as in the case of J_2 that (c) holds. Let M be the preimage of $M^* \cong L_3(2)/\mathbf{Z}_4^3$; by (i), M is perfect. Let $Q := O_2(M)$ and $V := Z(R)$. Then $Q \leq X \leq C_K(V)$ and $Q = \langle V^M \rangle$, so Q is abelian. Thus $Q = [Q,Y] \times Z$ by Coprime Action, where $Y \in Syl_7(M)$, and $\Phi(Q) =: E \cong E_8$. Set $\bar{M} := M/E$. If $\bar{Q}$ splits over $\bar{Z}$ then as M is perfect, $M/[Q,M] \cong SL_2(7)$, impossible as $M^* - Q^*$ contains 2-central involutions. Each involution in M/Q is fused in M to jQ, where $j \in R$ is an involution. As $\bar{Q}$ is indecomposable, $C_{\bar{Q}}(j) = [\bar{Q}, j]$ by B.4.8.2, so all involutions in $\bar{j}\bar{Q}$ are conjugate to $\bar{j}$. Finally $jE \subseteq R$ and all involutions in R^* are 2-central, so i^* lifts to an element of order 4, completing the proof of (5).

Assume the hypothesis of (6). Then K^* has one class of involutions, so involutions lift to involutions by (ii). Let I be the preimage in K of an A_6/E_{16}-subgroup of K^* and R the preimage of $O_2(I^*)$. By (i), I is perfect. As there is a complement L^* to R^* in I^* and involutions lift, there is a complement $L \cong A_6$ to R. Thus as I is perfect, $R = [R, L]$ is the core of the permutation module for L if $|Z| = 2$, and $R \cong \mathbf{Z}_4 * Q_8^2$ if $|Z| = 4$. In particular (6b) is now an easy calculation in the split extension I, as I^* contains $C_{K^*}(u^*)$ for a suitable involution $u^* \in R^*$. Further we calculate in this extension that $Z \leq O^2(I \cap J)$, so (6a) holds.

Assume the hypotheses and notation of (7). Then $I^* \cong Sz(8)$, and there is an element of order 3 in K^* faithful on I^*. As this automorphism is irreducible on the Schur multiplier of I^* of order 4 whereas Z is of order 2, it follows that (7a) holds. Then as the involutions in I^* are 2-central in K^*, such involutions lift to involutions of K. Pick u as in (7c) to be such an involution, and set $Q^* := O_2(C_{K^*}(u^*))$ and $U^* := \Phi(Q^*)$. The structure of X and Y is described in section J.2. Let $B \in Syl_5(X)$. By J.2.2, J.2.4, and J.2.5, $U^* \cong E_{32}$ and $Y^* = C_{Y^*}(B) \times [U^*, B]$ with $C_{Y^*}(B) \cong Q_8$. By J.2.6.3, all involutions in U^* are 2-central, so the preimage U of U^* is elementary abelian as such involutions lift to involutions. Thus $YZ = C_{YZ}(B) \times [U, B]$ and X centralizes u by Coprime Action so $YZ \leq XZ \leq C_K(u)$. Therefore $[Y, Y] = \langle [y, x] \rangle$ is of order 2, where $C_{Y^*}(B) = \langle y^*, x^* \rangle$. That is, (7c) holds. Further $[y, x]$ is centralized by $C_K(u^*)$, so u is in the center of a Sylow 2-subgroup S of K.

Let M be the preimage in K^* of a $L_3(2)/2^{3+8}$-subgroup of K^* with $S \leq M$; by (i), M is perfect. Let $P^* := O_2(M^*)$ and $V^* := Z(P^*)$; thus P^* is special, V^* is the natural module for M^*/P^*, and P^*/V^* is the Steinberg module for M^*/P^*.

Then $V \cong E_{16}$ and as $u \in Z(S)$, V splits over Z as an M-module by B.4.8.2. Let $\bar{M} := M/[V,M]$. If $Z \not\leq [P,M]$, then as M is perfect, $M/[P,M] \cong SL_2(7)$, a contradiction as involutions in $M^* - P^*$ are 2-central by J.2.6, and hence lift to involutions in K. Therefore $\bar{P}$ is extraspecial of width 4. However we can choose notation so that a Sylow 2-subgroup R of the preimage of $N_{K^*}(I^*)$ is contained in P with $R \cap V = [V,M]$. If R splits over Z, then $\bar{R} \cong E_{64}$, whereas $m_2(\bar{P}) \leq 5$ as $\bar{P}$ is extraspecial of width 4. Thus R does not split over Z, establishing (7b) and completing the proof of the lemma. □

We also sometimes require additional information on the structure of coverings of certain $\mathbf{F}_2G$-modules:

LEMMA I.2.3. *Let $G = LT$ with $L = F^*(G)$ isomorphic to $L_2(2^n)$ ($n > 1$), $Sp_4(2^n)'$, or $G_2(2^n)'$. Let $T \in Syl_2(G)$, $F := \mathbf{F}_{2^n}$, and V a covering of the natural $\mathbf{F}_2G$-module $\bar{V}$. For $i = 1, 2$, let $\bar{V}_i$ be the T-invariant F-subspace of $\bar{V}$ with $\dim_F(\bar{V}_i) = i$, where $\bar{V}$ is regarded as an FL-module, and V_i the preimage in V of $\bar{V}_i$. If L is $L_2(2^n)$ or $L = Sp_4(2^n)'$, let R denote the root group of L inducing transvections on $\bar{V}$ with center $\bar{V}_1$. Let S denote the long root group in $Z(T \cap L)$ if $L = G_2(2^n)'$. Then*

(1) If V is the universal covering of $\bar{V}$, then as an FL-module, V is the orthogonal module for G of dimension 3, 5, or 7 over F, respectively, and $V_L := C_V(L)$ is the radical of V of F-dimension 1. Further:

(a) $V_1 = C_V(T \cap L) = P \oplus V_L$, for a unique singular projective point P of V.

(b) For $r \in R^\#$, $[V,r]$ is a nonsingular point in V_1 distinct from V_L. Further if $n > 1$ and $s \in R - \langle r \rangle$, then $[V,r] \cap [V,s] = 0$, so $V_1 = [V,R] \leq C_V(R)$.

(c) $[V,S]$ is a totally singular F-subspace of dimension 2, which is a complement to V_L in V_2.

(2) $C_{V_1}(T) \not\leq C_V(L)$ and $V_i \leq Z(O_2(N_G(V_i)))$.

(3) If $n > 1$, then $C_V(L) \leq V_1 = [V,R] \leq C_V(R)$.

PROOF. We recall that $L_2(2^n) \cong \Omega_3(2^n)$ and $Sp_4(2^n) \cong \Omega_5(2^n)$, where the notation $\Omega_{2m-1}(q)$ denotes the stabilizer in $\Omega_{2m}(q)$ of a nonsingular point W_L in a $2m$-dimensional orthogonal space U over $\mathbf{F}_q$. Similarly $G_2(2^n)$ is contained in the stabilizer $Sp_6(q)$ of a nonsingular point W_L of an 8-dimensional orthogonal space U; see e.g. B.4.6, based on [**Asc87**] and its references. Moreover in each case L is indecomposable on the subspace $W := W_L^\perp$ of U orthogonal to W_L, and $\bar{V} \cong W/W_L$. Hence as $n = \dim(H^1(L,\bar{V}))$ by I.1.6, the first statement of (1) is established. Then we compute directly in the orthogonal space W that (a)–(c) hold.

Next by Remark I.1.5, V is a quotient of W. Now parts (a) and (b) of (1) imply (2) and (3); for example in proving (2), observe that $N_G(\bar{V}_i)$ is irreducible on $\bar{V}_i$ and acts on the unique totally singular complement to W_L in V_i. □

LEMMA I.2.4. *Let G be finite group, p a prime, $P \in Syl_p(G)$, and $\mathcal{D}$ the Alperin-Goldschmidt conjugation family for P in G. Let $Q \leq P$. Then*

(1) If $Q \trianglelefteq N_G(D)$ for each $D \in \mathcal{D}$ such that $Q \leq D$, then Q is weakly closed in P with respect to G.

(2) Assume that $N_G(E) \leq N_G(D)$ whenever $D, E \in \mathcal{D}$ with $D \leq E$. Then each member of $\mathcal{D}$ is weakly closed in P with respect to G.

PROOF. Visibly (1) implies (2). Assume the hypotheses of (1) and let $R \in Q^G$ with $R \leq P$. Then by the Alperin-Goldschmidt Fusion Theorem (cf. Theorem 16.1 in [**GLS96**]), there exist subgroups $Q =: Q_1, \ldots, Q_n := R$ of P, $D_i \in \mathcal{D}$, and $x_i \in N_G(D_i)$ such that $\langle Q_i, Q_{i+1} \rangle \leq D_i$ and $Q_i^{x_i} = Q_{i+1}$ for $1 \leq i < n$. We show that $Q = Q_i$ for each i to complete the proof. If the claim fails, pick i minimal subject to $Q \neq Q_{i+1}$. Then $Q = Q_i \leq D_i$, so by the hypotheses of (1), $Q \trianglelefteq N_G(D_i)$, and hence $Q_{i+1} = Q^{x_i} = Q$, contrary to the choice of i. □

The next result seems to have been well-known to experts since the description of the Bruhat decomposition; the only explicit statement and proof we know of in the literature is 4.1 of Grodal [**Gro02**]. Here is an easy proof using the approach of the previous lemma:

LEMMA I.2.5. *If G if a finite group of Lie type in characteristic p, then each unipotent radical is weakly closed (with respect to G) in each unipotent overgroup.*

PROOF. Let $P \in Syl_p(G)$ and $\mathcal{D}$ the set of unipotent radicals of the proper parabolics of G over P. Then (see e.g. 3.1.6 in [**GLS98**], using the Borel-Tits Theorem 3.1.3 in [**GLS98**]) $\mathcal{D}$ is the Alperin-Goldschmidt conjugation family for P in G. Further (see e.g. 47.4.1 in [**Asc86a**] and the definition of the parabolic subgroups and their radicals just before that result) $N_G(E) \leq N_G(D)$ whenever $D, E \in \mathcal{D}$ with $D \leq E$. Finally the unipotent subgroups of G are its p-subgroups. Therefore the lemma follows from I.2.4. □

I.3. Balance and 2-components

Next we record some standard results on "L-balance", taken from the first two sections of [**GLS96**]. We first recall two notions from Definition 4.5 on page 18 in [**GLS96**], expressed in the language of section A.3: A *2-component* of a finite group H is some $L \in \mathcal{C}(H)$ such that $L/O(L)$ is quasisimple. The *2-layer* $L_{2'}(H)$ of H is the product of the 2-components of H. Notice $O_{2',E}(H) = O(H)L_{2'}(H)$.

LEMMA I.3.1. *Let H be a finite $\mathcal{K}$-group, P a 2-subgroup of H, and L a 2-component of $C_H(P)$. Then*

(1) $\langle L^{O_{2',E}(H)} \rangle = \langle K^P \rangle$ for some 2-component K of H.

(2) If L centralizes $O(H)$, then L and K are components of $C_H(P)$ and H, respectively.

(3) If H is an SQTK-group and $m_3(L) = 2$, then $L \leq K \trianglelefteq H$ and L and K are quasisimple.

PROOF. Part (1) is essentially parts (i) and (ii) of 5.22 in [**GLS96**], with H, 2, L in the roles of "X, p, I". As H is a $\mathcal{K}$-group and each simple group S in $\mathcal{K}$ satisfies the Schreier Property (i.e., $Out(S)$ is solvable), H satisfies property (S_2) on page 5 of [**GLS96**]: for $T \in Syl_2(S)$, $C_{Aut(S)}(T)$ is 2-solvable. Let $X := \langle L^{O_{2',E}(H)} \rangle$ and $\bar{H} := H/O(H)$. Lemma 5.22 in [**GLS96**] also says that the projection $\bar{L}_K$ of $\bar{L}$ on $\bar{K}$ is a 2-component of $C_{\bar{K}}(N_P(K))$; indeed $\bar{L}_K$ is quasisimple as $\mathcal{K}$-groups satisfy the B_2-property (cf. Def. 2.4 on page 5 of [**GLS96**]), a fact we need in order to establish (2) in the next paragraph.

Assume that L centralizes $O(H)$; then so does X, so K is a component of H. Then as $\bar{L}_K$ is a component of $N_{\bar{K}}(N_P(K))$, L is a component of $N_H(P)$, so (2) holds.

Finally assume the hypotheses of (3). Then $m_3(\bar{K}) \geq m_3(\bar{L}_K)$, so (2) of Theorem A (A.2.1) says that $m_3(K) = 2$ as $m_3(L) = 2$. Thus A.3.8.3 says that $K \trianglelefteq H$, and K and L are quasismple as $K/O(K)$ and $L/O(L)$ are quasisimple. □

Usually we use the following special case of lemma I.3.1:

LEMMA I.3.2. *Let H be a finite $\mathcal{K}$-group, t an involution in H, and L a 2-component of $C_H(t)$. Set $K := \langle L^{O_{2',E}(H)} \rangle$. Then either*

(1) K is a t-invariant 2-component of H, or

(2) $K = JJ^t$ for some 2-component J of H such that $J \neq J^t$ and $J/O_{2',2}(J) \cong L/O_{2',2}(L)$.

PROOF. By I.3.1 with $\langle t \rangle$ in the role of "P", either (1) holds, or $K = KK^t$ for some 2-component J of H with $J \neq J^t$, and we may assume the latter. Setting $\bar{K} := K/O_{2',2}(K)$, $\bar{L}$ is a 2-component of $C_{\bar{K}}(t)$. Further $\bar{K} = \bar{J} \times \bar{J}^t$, so $C_{\bar{K}}(t) \cong \bar{J}$ is a full diagonal subgroup of $\bar{K}$. Therefore as $\bar{L}$ is a 2-component of $C_{\bar{K}}(t)$ and $\bar{J}$ is simple, $\bar{L} = C_{\bar{K}}(t) \cong \bar{J}$, so (2) holds. □

I.4. Recognition Theorems

In this section we list and discuss certain recognition theorems used to identify groups, usually by 2-local hypotheses. In some cases we prove our own recognition theorems; but when there is an outside appeal, we include some discussion of the appeal and an indication of how difficult the proof of that external result is. In particular we indicate how more modern proofs of such results (sometimes just restricted to our special case) are relatively easy, so that we are not relying on major classification theorems from the earlier literature. This requires some discussion of history, since the modern proof is usually a second or third proof of the original result.

LEMMA I.4.1. *Let G be a finite group with one conjugacy class of involutions, such that $C_G(z)$ is the dihedral group D_{2^n} for an involution z in G and some integer $n \geq 2$. Then*

(1) If $n = 2$, then $G \cong A_4$ or A_5.

(2) If $n = 3$, then $G \cong L_3(2)$ or A_6.

(3) If $n > 3$, then $G \cong L_2(p)$ for some Fermat or Mersenne prime p.

PROOF. All of these results follow from the classification of groups with dihedral Sylow 2-subgroups by Gorenstein and Walter in [**GW64**]. However with modern techniques and the strong condition on $C_G(z)$, there are much easier proofs: For example, all parts of the lemma can be retrieved from Bender's elementary proof of the Brauer-Suzuki-Wall Theorem in [**Ben74a**] and Gorenstein's discussion of Zassenhaus groups in [**Gor80**]. However, since Bender treats the cases $n = 2$ and $n = 3$ differently from the case $n > 3$, and only supplies a sketch of a proof in those first two cases, we supply a few more details below:

First, part (1) follows from Exercise 16.6 in [**Asc86a**]. Second, part (2) is a special case of Theorem 3 in [**AMS01**], so assume $n > 3$. Then by the Brauer-Suzuki-Wall Theorem (cf. [**Ben74a**]), G is a Zassenhaus group satisfying the hypotheses of Theorem 13.3.5 in chapter 13 of [**Gor80**], with the parameter "n" of that Theorem odd. Hence by that result, $G \cong L_2(q)$ for some odd prime power q. Then as $C_G(z)$ is a 2-group, the restrictions on q in conclusion (3) of the lemma follow. □

We recall also that simple groups with semidihedral Sylow 2-subgroups were classified by Alperin-Brauer-Gorenstein in [**ABG70**]. However to deduce our result I.4.3 below, it will suffice to quote an earlier special case of the general result, namely:

LEMMA I.4.2. *Let G be a finite group with one conjugacy class of involutions, such that $C_G(z) \cong GL_2(3)$ for an involution z in G. Then $G \cong L_3(3)$ or M_{11}.*

PROOF. This is a special case of a result announced by Brauer [**Bra57**]; a proof of this special case also appears in Theorem 6 of Wong [**Won64a**]. A more modern elementary proof using no character theory appears in Theorem 1.2 of R. Solomon and S. K. Wong [**SW91**]. □

Now adding our hypothesis (E) of even characteristic, we can use the results above to deduce a more general result. Recall that a group G is of even characteristic if each maximal 2-local M of G of odd index satisfies $F^*(M) = O_2(M)$. By the standard result 31.16 in [**Asc86a**], this is equivalent to the hypothesis that $F^*(H) = O_2(H)$ for each 2-local H of odd index in G.

LEMMA I.4.3. *Let G be a finite group with no subgroup of index 2 such that G is of even characteristic. Let $T \in Syl_2(G)$.*

(1) If T is dihedral then $G \cong A_6$ or $L_2(p)$ for some Fermat or Mersenne prime p.

(2) If T is semidihedral then $G \cong L_3(3)$ or M_{11}.

PROOF. Since $G = O^2(G)$ and T is dihedral or semidihedral, the following easy and standard transfer argument shows that G has one class of involutions and one class of elements of order 4: Let S denote the cyclic subgroup of index 2 in T. Then Thompson Transfer A.1.36 shows that each involution in $T - S$ is conjugate to the involution z of S. Hence G has one class of involutions. Now assume that T has elements of order 4, so that $|T| \geq 8$. Then T is transitive on the set $\{j, j^{-1}\}$ of elements of order 4 in S. We claim that G has one class of elements of order 4. If T is dihedral, then $\{j, j^{-1}\}$ is the set of elements of order 4 in T; so we may assume that T is semidihedral. Let D be the dihedral subgroup of T of index 2. Then Generalized Thompson Transfer A.1.37 shows that each element of order 4 in $T - D$ is conjugate to j, so again G has a unique class of elements of order 4.

Let z be the involution in $Z(T)$, and set $H := C_G(z)$. Assume T is dihedral. Then involutions a in $H - \langle z \rangle$ are not H-conjugate to z as $z \in Z(H)$, so Thompson Transfer shows $a \notin O^2(H)$. Then Cyclic Sylow 2-Subgroups A.1.38 shows that $H = TO(H)$. But as G is of even characteristic, $O(H) = 1$, so $H = T$. Thus we have the hypotheses of I.4.1, so (1) follows since $A_4 \cong L_2(3)$, $A_5 \cong L_2(5)$, and $L_3(2) \cong L_2(7)$.

So assume T is semidihedral of order 2^{n+1}. Set $\tilde{H} := H/\langle z \rangle$. Now $\tilde{T} \cong D_{2^n}$ with $n \geq 3$, so $Z(\tilde{T}) = \langle \tilde{j} \rangle$, where $j^2 = z$. As G is of even characteristic, $F^*(H) = O_2(H)$, and hence $F^*(\tilde{H}) = O_2(\tilde{H})$ by A.1.8. Therefore by B.2.14, $\tilde{U} := \langle \tilde{j}^H \rangle$ is an elementary abelian subgroup of $Z(O_2(\tilde{H}))$. As G is transitive on its elements v of order 4, and as $z = v^2$ for each v of order 4 in H, $C_G(z) = H$ is also transitive on its elements of order 4. Let U be the preimage of $\tilde{U}$ in H; then $U = \langle v \in T : v^2 = z \rangle$, so $U \cong Q_{2^n}$ as T is semidihedral. Therefore as $\tilde{U}$ is abelian, $n = 3$. As $\tilde{U} \leq Z(O_2(\tilde{H}))$ with $\tilde{U}$ of index 2 in $\tilde{T}$ but not central in $\tilde{T}$, $U = O_2(H)$. As $F^*(H) = O_2(H)$, $C_H(U) = Z(U) = \langle z \rangle$, so $\tilde{H} \leq Aut(U) \cong S_4$. Then as $|T : U| = 2$ and H

is transitive on the 6 elements of order 4 in U, $\tilde{H} = Aut(U)$. Hence H is the semidirect product of U with S_3, so that $H \cong GL_2(3)$. Then (2) holds by I.4.2. □

Define a finite group G to be of $U_3(3)$-*type* if G has an involution z such that $F^*(C_G(z)) \cong \mathbf{Z}_4 * Q_8$, $C_G(z)/F^*(C_G(z)) \cong S_3$, and z is not weakly closed in $O_2(C_G(z))$ with respect to G.

Lemma I.4.4. *Each group of $U_3(3)$-type is isomorphic to $U_3(3)$.*

Proof. The original proof of this result is due to P. Fong in [**Fon67**]. Fong considers a slightly more general class of groups. He shows the group is 2-transitive on the normalizers of Sylow 3-subgroups, and appeals to a result of Suzuki. A more modern elementary proof in [**Asc02a**] avoids character theory and the appeal to Suzuki. After calculating the group order, $U_3(3)$ is identified using Corollary F.4.24 of this work. □

Define a finite group G to be of $G_2(3)$-*type* if G has subgroups H and M such that

(G1) H has normal subgroups H_1 and H_2 with $H_1 \cong H_2 \cong SL_2(3)$, $|H : H_1H_2| = 2$, $\mathbf{Z}_2 \cong H_1 \cap H_2$, and $H = C_G(H_1 \cap H_2)$; and

(G2) $F^*(M) \cong E_8$ and $M/F^*(M) \cong L_3(2)$.

Lemma I.4.5. *Each group of $G_2(3)$-type is isomorphic to $G_2(3)$.*

Proof. The group $G_2(3)$ was first characterized by Janko via the centralizer of an involution in [**Jan69**]. In [**FW69**] and [**Fon70**], Fong and Wong characterized groups with related but more general centralizers, although in the special case of $G_2(3)$, they appeal to Janko's paper to handle the case where H is strongly 3-embedded in G. On the other hand, Janko appeals to Thompson's N-group paper [**Tho68**] to handle the case where H is *not* stongly 3-embedded in G. Both Fong-Wong and Thompson identify G as $G_2(3)$ essentially by constructing a BN-pair for G. A more modern elementary proof in [**Asc02b**] avoids character theory, and identifies $G_2(3)$ using the results of Delgado-Stellmacher in the Green Book [**DGS85**], and Corollary F.4.26 of this work. □

We need a means for identifying M_{12} and A_9; the following result suffices:

Lemma I.4.6. *Let G be a finite group, z an involution in G, $H := C_G(z)$, $Q := F^*(H)$, and $X \in Syl_3(H)$. Assume*

(a) Q is extraspecial of order 32,

(b) $H/Q \cong S_3$ and $C_Q(X) = \langle z \rangle$, and

(c) z is not weakly closed in Q with respect to G.

Then one of the following holds:

(1) There is a normal E_8-subgroup V of G with $G/V \cong L_3(2)$.

(2) $G \cong A_8$ or A_9, and the two Q_8-subgroups of Q are not normal in H.

(3) $G \cong M_{12}$, and the two Q_8-subgroups of Q are normal in H.

Proof. In [**Won64b**], W. Wong characterizes M_{12} via the centralizer of a 2-central involution and the condition that G has at most two classes of involutions; of course there is also a nonsimple example arising in (1). Wong's result can also be retrieved from a theorem of Brauer and Fong with less natural hypotheses. Both proofs are highly character-theoretic. For a short, modern, character-free proof of I.4.6, see [**Asc03b**]. □

Define a finite group G to be of *type* $\mathcal{H}(2,\Omega_4^-(2))$ if G has an involution z such that for $H := C_G(z)$, $Q := F^*(H) \cong Q_8D_8$, $H/Q \cong A_5$, and z is not weakly closed in Q with respect to G. We say G is of *type* J_2 or J_3 if G is of type $\mathcal{H}(2,\Omega_4^-(2))$, and G has 2 or 1 classes of involutions, respectively.

LEMMA I.4.7. *(1) Each group of type J_2 is isomorphic to J_2.*
(2) Each group of type J_3 is isomorphic to J_3.

PROOF. The general structure of groups of type J_2 and J_3 was determined by Janko in [**Jan68**]. The uniqueness of J_2 as a rank 3 group of permutations on the cosets of $U_3(3)$ was proved by Hall and Wales in [**HW68**]. A modern treatment of the uniqueness of groups of type J_2 appears in section 47 of [**Asc94**].

In unpublished work, Thompson showed J_3 has a $\mathbf{Z}_2/L_2(16)$-subgroup. In [**HM69**], G. Higman and J. McKay proved the uniqueness of groups of type J_3 via a coset enumeration on such a subgroup. There is a less computational uniqueness proof for J_3 in [**Fro83**], which at least for the time being we quote. B. Baumeister is working on a geometric characterization of J_3 that we hope might be used for this work. □

Define a finite group G to be of *type HS* if there exists an involution z in G and $E_8 \cong V \leq G$, such that $H := C_G(z)$ and $M := N_G(V)$ satisfy:
(HS1) $F^*(H) =: Q \cong \mathbf{Z}_4 * Q_8^2$ and $H/Q \cong S_5$.
(HS2) $V \leq Q$, $F^*(M) \cong \mathbf{Z}_4^3$, and $M/F^*(M) \cong L_3(2)$.

LEMMA I.4.8. *Each group of type HS is isomorphic to HS.*

PROOF. The original 2-local characterization of HS is due to D. Parrott and S. K. Wong in [**PW70**], and Z. Janko and S. K. Wong in [**JW69**]. Parrott and Wong prove there is a unique simple group of order $|HS|$ by using character theory to show that such a group has a rank 3 permutation representation of degree 100 on the cosets of a subgroup isomorphic to M_{22}, and then appealing to a theorem of Wales on rank 3 groups. Janko and Wong characterize HS by the general structure of the centralizer of a 2-central involution; they use the Thompson Order Formula 45.6 in [**Asc86a**] to calculate the group order, and then appeal to Parrott-Wong.

There is a short, modern, character-free proof in [**Asc03a**]. which uses Corollary F.4.26 of this work to produce an $L_3(4)$-subgroup, after which the construction of an M_{22}-subgroup is easy. □

We say a simple group G is of *type* J_1 if G contains a 2-central involution z such that $C_G(z) \cong L_2(4) \times \mathbf{Z}_2$.

LEMMA I.4.9. *Each simple group of type J_1 is isomorphic to J_1.*

PROOF. The original uniqueness proof is due to Janko in [**Jan66**]. We mention also that in [**Ben74b**], Bender uses a counting argument to calculate the order of a group of type J_1; this can be used to simplify Janko's proof. □

I.5. Characterizations of $L_4(2)$ and $Sp_6(2)$

In this section we prove a recognition theorem for $L_4(2)$ and $Sp_6(2)$.

Let G be a group and $\mathcal{F} := (G_i : i \in I)$ a family of subgroups of G. As in section 4 of [**Asc94**], for $J \subseteq I$ define

$$G_J := \bigcap_{j\in J} G_j.$$

Further set $Q_J := O_2(G_J)$ and $L_J := O^2(G_J)$.

Let D be a Coxeter diagram of type A_3 or C_3, and write $D = A_3$ or $D = C_3$, in the respective cases. Define a *D-system* to be a pair $(G, \mathcal{F})$, where $\mathcal{F} := (G_0, G_1, G_2)$ is a family of subgroups of G such that

(D1) $G_0/Q_0 \cong L_3(2)$ if $D = A_3$, and $G_0/O_2(G_0) \cong A_6$ or S_6 if $D = C_3$.

(D2) $G_1/Q_1 \cong L_2(2) \times L_2(2)$ and $G_2/Q_2 \cong L_3(2)$.

(D3) For each $i \in I := \{0, 1, 2\}$, $T := G_I \in Syl_2(G_i)$, and $G_{i,j}$, $j \in I - \{i\}$, are the rank-1 parabolics of G_i over T.

(D4) $G = \langle \mathcal{F} \rangle$ and $\ker_T(G) = 1$.

(D5) If $D = C_3$, then either $Z(G_0) \neq 1$ or $|Q_0| > 2^5$.

The main theorem of this section is:

THEOREM I.5.1. *Let $(G, \mathcal{F})$ be a D-system. Then*
(1) If $D = A_3$, then $G \cong L_4(2)$.
(2) If $D = C_3$, then $G \cong Sp_6(2)$.

REMARK I.5.2. Theorem I.5.1 follows as a special case from more general results: Theorem 3 in [**Asc84**] plus Tits' classification of spherical buildings [**Tit74**]. However as is our usual practice, we give here an elementary proof of this more specialized result, which we use to identify $Sp_6(2)$ in the proof of the Main Theorem and to eliminate certain configurations of 2-local subgroups during that proof.

Hypothesis (D5) in the definition of D-system is not actually necessary, and could be eliminated at the expense of a longer proof. However in the situations in which we apply the Theorem, this hypothesis is obviously satisfied, so we include it here in order to simplify the proof.

In the remainder of this section, assume $(G, \mathcal{F})$ is a D-system. Form the coset geometry $\Gamma := \Gamma(G, \mathcal{F})$ as in Example 4 in section 4 of [**Asc94**]. For $i \in I$, let Γ_i be the coset space G/G_i; we call the members of Γ_i *points*, *lines*, and *planes* for $i = 0$, 1, 2, respectively. These are the *vertices* of Γ regarded as a simplicial complex. Adopt the notation of section 4 of [**Asc94**], and let $\mathcal{F}_i := \{ G_{i,j} : j \in I - \{i\} \}$.

Set $x := G_0$, $l := G_1$, and $\pi := G_2$, regarding these subgroups as cosets and hence members of Γ. Thus G_0 is the stabilizer G_x in G of the point x, and similarly $G_1 = G_l$ and $G_2 = G_\pi$. For F a simplex of Γ, write Q_F for $O_2(G_F)$ and G^{*F} for G_F/Q_F. Set $L_F := O^2(G_F)$. Let R_x be the subgroup of Q_x fixing each point collinear with x. (That is, points y such that x and y are incident with a common line of the geometry).

For a vertex v, let $\Gamma(v)$ denote the *residue* of V: the subcomplex of all simplices $F - \{v\}$ such that F is a simplex of Γ containing v. Γ is *residually connected* if Γ and each residue $\Gamma(v)$ is connected.

LEMMA I.5.3. *(1) Γ is residually connected.*
(2) G is flag-transitive on Γ.
(3) $\Gamma(G_i) \cong \Gamma(G_i, \mathcal{F}_i)$.
(4) $\Gamma(l)$ is a generalized digon with three points and three planes.
(5) Γ has a string diagram with string ordering $0, 1, 2$.

PROOF. By (D4), $G = \langle \mathcal{F} \rangle$, and by (D3), $G_i = \langle \mathcal{F}_i \rangle$ for each $i \in I$. Then (1) follows from 4.5 in [**Asc94**]. By (D2) and (D3), $G_1 = G_{0,1}G_{1,2}$ with $|G_1 : G_{1,i}| = 3$

for $i = 0, 2$, so (4) follows from 4.2 in [**Asc94**]. Part (4) implies (5); while (1), (5), and 4.11 in [**Asc94**] imply (2) and (3). □

The *polar space* for $Sp_4(2)$ is the generalized quadrangle determined by points and isotropic lines in the natural module.

LEMMA I.5.4. *(1)* $\Gamma(x)$ *is the projective plane of order* 2 *if* $D = A_3$, *and the polar space of* $Sp_4(2)$ *if* $D = C_3$.
(2) $\Gamma(\pi)$ *is the projective plane of order* 2.
(3) Q_J *is the kernel of the action of* G_J *on* $\Gamma(G_J)$.
(4) G *is faithful on* Γ.

PROOF. By construction, $\ker_{G_I}(G)$ is the kernel of the action of G on Γ, so (4) follows from (D4). Similarly $\ker_{G_I}(G_i)$ is the kernel of the action of G_i on $\Gamma(G_i)$, so as $G_I = T \in Syl_2(G_i)$ by (D3), (3) holds. Parts (1) and (2) follow from I.5.3.3, the description of G_i/Q_i in (D1) and (D2), and the description of $G_{i,j}$ in (D3). □

REMARK I.5.5. By lemmas I.5.3.3 and I.5.4, Hypothesis ($\Gamma 0$) of section 38 of [**Asc94**] is satisfied. Thus we can appeal to results in that section. As in that section, write $\Gamma_i(a)$ for the vertices of type i in the residue $\Gamma(a)$.

LEMMA I.5.6. $\Gamma_0(l) \subseteq \Gamma_0(\pi)$, *and* $\Gamma_2(l) \subseteq \Gamma_2(x)$.

PROOF. This is 38.1 in [**Asc94**]. □

LEMMA I.5.7. *(1)* $Q_{0,1} = Q_0Q_1$ *and* $Q_{1,2} = Q_1Q_2$.
(2) For $i = 0, 2$, Q_i *is transitive on* $\Gamma_i(l) - \{G_i\}$.
(3) $G_l^{*l} = L_{0,1}^{*l}Q_2^{*l} \times L_{1,2}^{*l}Q_0^{*l}$ *acts faithfully as* $L_2(2) \times L_2(2)$ *on* $\Gamma_2(l) \times \Gamma_0(l)$.
(4) $G_{0,1} = G_{l,x,y}Q_0$ *for* $y \in \Gamma_0(l)$.
(5) $R_x \leq Q_2$.
(6) $Q_0 \cap Q_1 \neq Q_0 \cap Q_1^g$ *for* $g \in G_0 - G_{0,1}$.

PROOF. By I.5.4.2, $\Gamma(\pi)$ is the plane of order 2. In this plane, each point is collinear with x, so R_x is contained in the kernel Q_2 of the action of G_2 on $\Gamma_0(\pi)$. Thus (5) holds.

We next prove (1). By (D2), $G_1^{*l} \cong L_2(2) \times L_2(2)$, so $|Q_{1,i} : Q_1| = 2$ for $i = 0, 2$. Thus as $|Q_{1,i}^{*G_i}| > 2$ for $i = 0, 2$, $Q_1^{*G_i} \neq 1$. But $Q_j \trianglelefteq G_{j,k}$, so that $Q_iQ_1 \leq O_2(G_{1,i}) = Q_{1,i}$; and unless $i = 0$ and $G_0/Q_0 \cong S_6$, $G_{1,i}$ is irreducible on $Q_{1,i}^{*G_i}$, so that $Q_1^{*G_i} = Q_{1,i}^{*G_i}$ and hence $Q_{i,1} = Q_1Q_i$. Thus to complete the proof of (1), it remains to consider the case where $G_0/Q_0 \cong S_6$ and $Q_{0,1} > Q_0Q_1$, and to derive a contradiction. As $|Q_{1,0} : Q_1| = 2$, it follows that $Q_0 \leq Q_1$. Thus Q_0 fixes $\Gamma_0(l)$ pointwise, so as $Q_0 \trianglelefteq G_0$, $Q_0 = R_x$. Thus $Q_0 \leq Q_2$ by (5). As $G_0/Q_0 \cong S_6$, $|T : Q_0| = 16$. As we have shown that $Q_{1,2} = Q_1Q_2$, $|Q_1^{*\pi}| = |Q_{1,2}^{*\pi}| = 4$ and $|Q_2 : Q_1 \cap Q_2| = |Q_{1,2} : Q_1| = 2$. Thus $|T : Q_1 \cap Q_2| = 2|T^{*\pi}| = 16$, so as $Q_0 \leq Q_1 \cap Q_2$, $Q_0 = Q_1 \cap Q_2$. Thus $Q_0 \trianglelefteq G_{1,2}$, so Q_0 is normal in $\langle G_0, G_{1,2} \rangle = \langle \mathcal{F} \rangle = G$ by (D4), so $Q_0 \leq \ker_T(G) = 1$ again using (D4). Then $G_0 \cong A_6$ or S_6 has trivial center, contrary to (D5), so the proof of (1) is complete.

Next by I.5.6, $\Gamma_0(l) \subseteq \Gamma_0(\pi)$, so Q_2 is trivial on $\Gamma_0(l)$. As $|\Gamma_0(l)| = 3$ and $L_{0,1}$ fixes x, $L_{0,1} = O^2(G_{0,1})$ is also trivial on $\Gamma_0(l)$. Similarly $L_{1,2}Q_0$ is trivial on $\Gamma_2(l)$. Then by I.5.3.3.2, $Q_2L_{0,1}$ acts transitively on $\Gamma_2(l)$ as S_3, and the corresponding statement for $Q_0L_{1,2}$ also holds. That is (3) is established. In particular (2) holds. As $G_{0,1}$ fixes x and acts on $\Gamma_0(l)$, (2) implies (4).

Suppose $Q_0 \cap Q_1 = Q_0 \cap Q_1^g$ for some $g \in G_0 - G_{0,1}$. Then $Q_0 \cap Q_1 \trianglelefteq \langle G_{0,1}, g\rangle = G_0$, as $G_{0,1}$ is maximal in G_0 by (D3). Then as $Q_0 \cap Q_1$ is the subgroup of Q_0 fixing $\Gamma_0(l)$ pointwise, $Q_0 \cap Q_1 = R_x$, so $Q_0 \cap Q_1 \leq Q_2$ by (5). As $|Q_1 : Q_0 \cap Q_1| = 2$, it follows that $|Q_1^{*\pi}| \leq 2$, whereas we saw earlier that $|Q_1^{*\pi}| = 4$. This contradiction establishes (6). □

LEMMA I.5.8. *(1) Each pair x, y of distinct collinear points is incident with a unique line $x \odot y$ of Γ.*

(2) G is faithful on Γ_0.

PROOF. Part (1) follows from 38.8 in [**Asc94**]: Namely from I.5.4.1 and (D1), G_x is primitive on points and planes of $\Gamma(x)$. This observation together with I.5.7 verifies the hypotheses of 38.8 in [**Asc94**].

Let K be the kernel of the action of G on Γ_0; then $K \leq G_0$. As G_0/Q_0 is faithful on $\Gamma_1(x)$, (1) says $K \leq Q_0$. Thus $K \leq \ker_T(G) = 1$ by (D4). □

Let Λ be the bipartite graph $\Gamma_0 \cup \Gamma_1$ and Δ the collinearity graph on Γ; that is, the points are the vertices of Δ, and points are adjacent if they are collinear. Let $x^\perp$ be the set of points collinear with x, and $\Delta(x) := x^\perp - \{x\}$. Write $\Delta^2(x)$ for the set of points at distance 2 from x in Δ. For $y \in \Delta(x)$, define $x \odot y$ as in I.5.8.1.

LEMMA I.5.9. *(1) G_x is transitive on $\Delta(x)$, and $|\Delta(x)| = 14$ or 30 for $D = A_3$ or C_3, respectively.*

(2) If $D = A_3$, then Δ is of diameter 1 and G is 2-transitive on Δ of order 15.

(3) If $D = C_3$, then

(i) G_x is transitive on geodesics in Λ of length 4 with origin x, and there are $2^5 \cdot 15$ such geodesics.

(ii) For each $z \in \Delta^2(x)$ and each $r \in \Gamma_1(x)$, z is collinear with a unique point on r.

(iii) $|\Delta^2(x)| = 32$.

(iv) Δ is of diameter 2 and order 63.

(v) For each $l \in \Gamma_1$ and $p \in \Delta$, p is collinear with one or all points on l.

(vi) For $a, b \in \Delta(x) - \{x\}$, $a \in \Delta(b)$ iff $x \odot a$ and $x \odot b$ are coplanar in $\Gamma(x)$.

PROOF. By construction, G_x is transitive on $\Gamma_1(x)$ of order k, where $k := 7$ or 15 for $D = A_3$ or C_3, respectively. Next by I.5.8.1, each $y \in \Delta(x)$ is on a unique line l through x; while by I.5.3.4, l has three points, and by I.5.7.2, $G_{x,l}$ is transitive on $\Gamma_0(l) - \{x\}$. Thus (1) holds.

Assume that $D = A_3$. We prove (2); notice by (1) and connectivity of Δ in I.5.3.3.1, it suffices to show that if $y \in \Gamma_0(l) - \{x\}$, then $y^\perp = x^\perp$. Let $z \in \Delta(y)$; then as $\Gamma(y)$ is a projective plane, there exists a plane π incident with l and $y \odot z$. Then by I.5.6, x, z are incident with π, so as $\Gamma(\pi)$ is a projective plane, x and z are incident with a common line $x \odot z$ in $\Gamma(\pi)$. This completes the proof of (2).

Finally assume that $D = C_3$. From the first paragraph, G_x is transitive on the 30 geodesics xly in Λ of length 2. Let $z \in \Delta(y)$ and $k := y \odot z$. If π is a plane in $\Gamma(y)$ incident with l and k, then as in the previous paragraph, $z \in \Delta(x)$. Thus z is collinear with all points on each line through y coplanar with k, and if $d_\Lambda(x, z) = 4$, then k is not coplanar with l in $\Gamma(y)$. By I.5.4.1, $\Gamma(y)$ is the polar space for $Sp_4(2)$, so the set S of lines k in $\Gamma(y)$ not coplanar with l has size 8; now $G_{l,y}$ is transitive on S and hence so is $G_{x,y,l}$ snce $G_{l,y} = G_{x,y,l}Q_y$ by I.5.7.4.

Next $Q_l \cap Q_y =: Q_{x,y}$ is of index 2 in Q_y, and by I.5.7.6, $Q_l \cap Q_y \neq Q_k \cap Q_y$. Thus $Q_{x,y}$ is transitive on $\Gamma_0(k) - \{y\}$. Therefore if there exists $z \in \Delta(y) - \Delta(x)$, then (3i) holds. So suppose instead that $\Delta(y) \subseteq \Delta(x)$. Then $x^\perp = y^\perp$, so Δ is of diameter 1, and hence $|\Delta| = 31$ by (1). But then

$$|G| = |\Delta| \cdot |G_x| = 31 \cdot 9 \cdot 5 \cdot |T|,$$

so $|G|$ is not divisible by 7, whereas $|G_2|$ *is* divisible by 7. This contradiction establishes (3i), so there exists $z \in \Delta(y)$ with $d_\Lambda(x, z) = 4$. As $Q_{x,y}$ is transitive on $\Gamma_0(k) - \{y\}$, $\{y\} = \Delta(z) \cap \Gamma_0(l)$; thus for each $r \in \Gamma_1(x)$, z is collinear with at most one point on r. This completes the proof of (3vi) and establishes (3v) modulo (3iv).

Now in the polar space $\Gamma(y)$, there are three lines l_i, $1 \leq i \leq 3$, which are coplanar with both l and k. Let π_i be the plane through l and l_i in $\Gamma(y)$. Then there are three lines $l_{i,j}$ (including l) through x in π_i, and each contains a unique point $y_{i,j}$ on l_i. As k and l_i are coplanar, $y_{i,j} \in \Delta(z)$, so $y_{i,j}$ is the unique point on $l_{i,j}$ collinear with z.

Next the 7 lines $l_{i,j}$ are those coplanar with l in the polar space $\Gamma(x)$. Then by symmetry between l and $l_{i,j}$, each line in $\Gamma(x)$ coplanar with $l_{i,j}$ contains a unique point in $\Delta(z)$, so (3ii) holds. Now by (3ii) and I.5.8.1, there are exactly 15 geodesics from x to z in Λ, so (3iii) follows from (3i). Further by (3ii), Δ is of diameter 2: for if $a \in \Delta(x)$, then z is collinear with a point on $x \odot a \in \Gamma_1(x)$, so that $d(a, z) \leq 2$. Now (3iv) follows from (1) and (3iii). □

Let U be the $\mathbf{F}_2G$-permutation module on Δ. By I.5.9, U is of dimension 15 or 63 for $D = A_3$ or C_3, respectively. As usual view U as the power set of Δ. Let K be the submodule of U generated by the vectors $u_l = \Gamma_0(l)$, $l \in \Gamma_1$. Set $\bar{U} := U/K$.

Lemma I.5.10. *If* $D = A_3$, *then* $\dim(\bar{U}) = 4$ *and* G *acts faithfully on* $\bar{U}$ *as* $GL(\bar{U}) \cong L_4(2)$.

Proof. For $a \in \Gamma_0 \cup \Gamma_2$, define $K(a) := \langle u_l : l \in \Gamma_1(a)\rangle$. Then $\dim(K(x)) \leq |\Gamma_1(x)| = 7$. Let $K^* := K/K(x)$.

Next $\Gamma_0(\pi)$ is a basis for the permutation module $U(\pi) := \langle \Gamma_0(\pi)\rangle$ on $G_2/G_{0,2}$, and $K(\pi)$ is the 4-dimensional subspace $\mathbf{F}_2\Gamma_0(\pi) \oplus W$ where W is described in H.5.3. Further $K(x) \cap K(\pi)$ contains $\langle u_l : l \in \Gamma_1(x, \pi)\rangle$ of dimension 3, so $\dim(K(\pi)^*) \leq 1$. However there are 7 planes through x, and each line r is on at least one of those planes, so $K^* = \langle K(\pi)^* : \pi \in \Gamma_2(x)\rangle$ is of dimension at most 7. Thus $\dim(K) \leq 14$, so $\bar{U} \neq 0$.

Suppose that $x, y \in \Delta$ with $\bar{x} = \bar{y}$. Then $x + y \in K$. But $\overline{x + y} = \bar{z}$, where z is the third point on $x \odot y$. Thus $z \in K$, so $\Delta = z^G \subseteq K$, contrary to $\bar{U} \neq 0$. Hence the map $\varphi : x \mapsto \bar{x}$ is an injection of Δ into $\bar{U}$. On the other hand for all $x \neq y \in \Delta$, $\overline{x + y} \in \bar{\Delta}$, so $\bar{\Delta} \cup \{0\}$ is a subspace of $\bar{U}$, and hence $\bar{U}^\# = \bar{\Delta}$. Thus $|\bar{U}^\#| = 15$, so $\dim(\bar{U}) = 4$. Further by I.5.8.2, G is faithful on Δ, so as φ is an injection, G is faithful on $\bar{U}$. Thus $G \leq H := GL(\bar{U}) \cong L_4(2)$. Further G_0 is the stabilizer in G of $\bar{x} \in \bar{U}^\#$, so $G_0 \leq H_{\bar{x}}$. As $G_0/Q_0 \cong L_3(2)$, G_0 is irreducible on $O_2(H_{\bar{x}})$ and $Q_0 = G_0 \cap O_2(H_{\bar{x}})$, so either $G_0 = H_{\bar{x}}$ or $Q_0 = 1$. The latter is impossible by I.5.7.2, and in the former $G_0 = H$ as G_0 is transitive on $\bar{U}^\#$. Thus the lemma is established. □

Because of I.5.10, we may assume during the remainder of the section that $D = C_3$.

Let Σ be the set of pairs (a, b) with $a \in \Delta$ and $b \in \Delta^2(a)$. For $(x, y) \in \Sigma$, let $\Delta(x, y) := \Delta(x) \cap \Delta(y)$ and

$$\mathcal{L}(x, y) := \{a \odot b : a, b \in \Delta(x, y) \text{ with } a \text{ and } b \text{ collinear}\}.$$

By I.5.9.3:

LEMMA I.5.11. *$(\Delta(x, y), \mathcal{L}(x, y))$ is the geometry of points and lines for $Sp_4(2)$.*

For $(x, y) \in \Sigma$, define

$$x\Box y := \bigcap_{z\in\Delta(x,y)} z^{\perp}.$$

That is, $x\Box y$ is the *nonsingular line* through x and y in Γ. Further for $(a, b) \in \Sigma$, set $\Sigma(a, b) := \Sigma \cap (\Delta(a, b) \times \Delta(a, b))$; observe that if $(x, y) \in \Sigma(a, b)$, then also $(a, b) \in \Sigma(x, y)$. For $(x, y) \in \Sigma(a, b)$, define

$$n_{a,b}(x, y) := \bigcap_{d\in\Delta(x,y)\cap\Delta(a,b)} d^{\perp} \cap \Delta(a, b).$$

Thus $n_{a,b}(x, y)$ is the *nonsingular line* through x and y in $\Delta(a, b)$.

LEMMA I.5.12. *For $(x, y) \in \Sigma$, either*
(1) $x\Box y = \{x, y\}$, or
(2) $x\Box y = n_{a,b}(x, y)$ for all $(a, b) \in \Sigma(x, y)$, so $x\Box y$ is of order 3.

PROOF. By definition, $x, y \in x\Box y \subseteq n_{a,b}(x, y)$. By I.5.11, $n_{a,b}(x, y)$ is of order 3. Now if $n_{a,b}(x, y)$ is independent of (a, b), then (2) holds; and otherwise (1) holds. □

LEMMA I.5.13. *For each $(x, y) \in \Sigma$, $x\Box y$ is of order 3.*

PROOF. Assume the lemma is false. Then by I.5.12 and I.5.9.3i, $x\Box y = \{x, y\}$ for all $(x, y) \in \Sigma$. Let $\Xi := \Delta(x, y)$ and R the subgroup of Q_0 acting on Ξ. Thus $R_x \leq R$. Further for each $l \in \Gamma_0(x)$, R fixes l and hence also fixes the unique point of Ξ on l, so R fixes l pointwise. Thus $R = R_x$.

As R_x fixes each point of Ξ, R_x acts on $x\Box y$; so as $x\Box y = \{x, y\}$, R_x fixes y. As this holds for each $y \in \Delta^2(x)$, R_x is in the kernel of the action of G on Δ, so $R_x = 1$ by I.5.8.2.

Next $G_{x,y} \cap Q_0 \leq R = R_x = 1$, so by I.5.9, $32 = |G_0 : G_{x,y}| \geq |Q_0|$. Thus $Z := Z(G_x) \neq 1$ by (D5). As $Z(G_x/Q_x) = 1$, $Z \leq Q_x$, and in particular Z fixes $\Gamma_2(l)$ pointwise. Similarly as $Z(G^{*\pi}_{0,2}) = 1$, $Z \leq Q_2$, so Z fixes $\Gamma_0(l)$ pointwise, and hence $Z \leq Q_x \cap Q_l$. But then as $Z \trianglelefteq G_x$, $Z \leq R_x$, contradicting $R_x = 1$. □

LEMMA I.5.14. *For each $(x, y) \in \Sigma$ and each $d \in \Delta$, d is collinear with one or all members of $x\Box y$.*

PROOF. In view of I.5.13, $x\Box y = \{x, y, z\}$ for some point z. By I.5.12.2, $x\Box y = n_{a,b}(x, y)$ for all $(a, b) \in \Sigma(x, y)$; but we saw that $(x, y) \in \Sigma(a, b)$, so from the polar-space structure on $\Delta(a, b)$ in I.5.11, distinct members of $x\Box y$ are not collinear. Then each distinct pair (a, b) from $x\Box y$ lies in $\Sigma(x, y)$, so by transitivity of G on Σ in I.5.9.3i, $\Sigma(x, y) = \Sigma(a, b)$ for each such pair (a, b). For $a \in x\Box y$, let $\theta(a) := \Delta(a) - \Sigma(x, y)$. Thus $|\theta(a)| = 15$, so as $|\Delta^2(x)| = 32$ by I.5.9.3iii, it follows that

$$\{x\Box y, \Sigma(x, y), \theta(x), \theta(y), \theta(z)\}$$

is a partition of Δ. Finally if $p \in \Sigma(x, y)$, p is collinear with all members of $x\Box y$, while if p is in any other member of the partition, then p is collinear with exactly one member of $x\Box y$. This completes the proof. $\Box$

Let J be the submodule of U generated by K and $x\Box y$, $(x, y) \in \Sigma$. Define $\tilde{U} := U/J$.

Define a symmetric bilinear form f on U by $f(x, y) = 0$ if x and y are collinear, and $f(x, y) = 1$ otherwise. For $z \in \Delta$, z is collinear with one or all of the points on $x \odot y$ by I.5.9.3ii, so $f(z, u_{x\odot y}) = 0$ and hence K is in the radical of f. Similarly by I.5.14, $x\Box y$ is in the radical of f for each $(x, y) \in \Sigma$. Thus J is a proper subspace of U, and there is an induced form f on $\tilde{U}$ defined by $f(\tilde{u}, \tilde{v}) := f(u, v)$.

Lemma I.5.15. *If $D = C_3$, then $(\tilde{U}, f)$ is a 6-dimensional symplectic space over $\mathbf{F}_2$, and G acts faithfully on $\tilde{U}$ as $Sp(\tilde{U}, f) \cong Sp_6(2)$.*

Proof. The proof is much like that of I.5.10. First as we just observed, $J < U$, so $\tilde{U} \neq 0$. Next as $x \odot y \in J$ for $y \in \Delta(x)$, and $x\Box z \in J$ for $z \in \Delta^2(x)$, $\tilde{x} + \tilde{y} \in \tilde{\Delta}$ for each pair of distinct $x, y \in \Delta$. Then an argument in the last paragraph of the proof of I.5.10 shows that $\varphi : \Delta \to \tilde{U}$ defined by $\varphi : x \mapsto \tilde{x}$ is a bijection of Δ with $\tilde{U}^\#$. In particular by I.5.8.2, G is faithful on $\tilde{U}$. As $|\Delta| = 63$, $\dim(\tilde{U}) = 6$. As $f(\tilde{x}, \tilde{y}) = 1$ for $(x, y) \in \Sigma$, f is a nondegenerate form on $\tilde{U}$, so $(\tilde{U}, f)$ is a 6-dimensional symplectic space. Thus $G \leq H := Sp(\tilde{U}, f)$ and $G_0 \leq H_{\tilde{x}}$. By I.5.7.6, G_0^∞ is nontrivial on Q_0, and $Z(G_x) \neq 1$ by (D5); so $|Q_0| \geq 32$, and hence $Q_0 = O_2(H_{\tilde{x}})$ and $|H_{\tilde{x}} : G_0| \leq 2$. Therefore $|H : G| \leq 2$, so as H is simple, $H = G$. $\Box$

Notice I.5.10 and I.5.15 complete the proof of Theorem I.5.1.

I.6. Some results on TI-sets

Lemma I.6.1. *Let X be a 2-subgroup of a group G and $M = N_G(X)$. Then each of the following implies that X is a TI-set in G:*

(1) For each $x \in X^\#$, $C_G(x) \leq M$ and M controls fusion of elements of X.
(2) $M = !\mathcal{M}(L)$ for some $L \leq C_G(X)$.

Proof. First A.1.7.2 implies (1). Next assume the hypotheses of (2). As $M = !\mathcal{M}(L)$ and L centralizes X, $C_G(x) \leq M$ for each $x \in X^\#$. If $x^g \in X$ then

$$M = !\mathcal{M}(L) = !\mathcal{M}(C_G(x^g)) = !\mathcal{M}((L^g) = M^g,$$

so $g \in M$ as $M \in \mathcal{M}$. Thus M controls fusion in X, so X is a TI-set by (1). $\Box$

A *natural module* for a dihedral group G of order $2m$, $m > 1$ odd, is a faithful irreducible $\mathbf{F}_2G$-module.

Lemma I.6.2. *Let V, A be elementary abelian 2-subgroups of the same rank which are TI-subgroups of a finite group G, such that $A^g \cap V = 1$ for all $g \in G$ with $V \neq A^g$. Assume $U := N_V(A) \neq 1$ and set $X := \langle V, A\rangle$ and $Y := UN_A(V)$.*

(1) If $U = V$ then $X = V \times A$. In particular, this holds if $C_A(V) \neq 1$.
(2) If $U < V$, then

(a) $Y = U \times N_A(V) \trianglelefteq X$ and $U = C_Y(v) = [Y, v]$ for each $v \in V - U$.
(b) $\bar{X} := X/Y \cong D_{2m}$ (m odd), $L_2(2^n)$, or $Sz(2^n)$, and $m(\bar{V}) = m_2(\bar{X})$.
(c) The $\bar{X}$-composition factors of Y are natural modules for $\bar{X}$, and afford a direct sum if $\bar{Y} \cong L_2(2^n)$ or D_{2m}.

Proof. This is essentially lemma 2.14 in Timmesfeld [**Tim75**]; here is a proof:

Set $B := N_A(V)$. As $V \cap A = 1$, $Y = B \times U$; then as V is a TI-subgroup of G, $B = C_A(u) \neq 1$ for each $u \in U^\#$. As u is an involution, $[A, u] \leq C_A(u) = B$, and hence A acts on Y; by symmetry so does V, and thus $Y \trianglelefteq X$. Notice also for $v \in V - U$ that $C_Y(v) = UC_B(v) = U$, since $U = C_V(b)$ for $b \in B^\#$ again by symmetry. Thus conclusion (a) of (2) holds and $m(U) = m(C_Y(v)) \geq m(B \times U)/2$, so $m(U) \geq m(B)$. By symmetry $m(B) = m(U) = m(BU)/2$.

If $U = V$ then $B = A$ since $m(V) = m(A)$ by hypothesis, establishing (1). Thus we may assume that $U < V$, so V and A are noncyclic.

We claim that $\bar{V}$ is a TI-set in $\bar{X}$ and $\bar{V}$ is the strong closure of $\bar{V} \cup \bar{A}$ in $N_{\bar{X}}(\bar{V})$: For let $C \in \{A, V\}$ and suppose that $\bar{c}^x \in N_{\bar{X}}(\bar{V})$ for some $x \in X$ and $\bar{c} \in \bar{C}^\#$. Then $(C \cap Y)^x = C_Y(c^x)$, while c^x acts on $C_Y(\bar{V}) = U$, so $C_U(c^x) \neq 1$. Thus $1 \neq V \cap C^x$, and hence $C \in V^G$ by hypothesis, and $x \in N_X(V)$ as V is a TI-set in G, establishing the claim.

By the claim, $\bar{A} \in \bar{V}^{\bar{X}}$ and $\bar{V}$ satisfies the hypotheses of Shult's Fusion Theorem I.8.3 in $\bar{X}$. Let $n := m(\bar{V})$. If $n = 1$, then $\bar{X} = \langle \bar{V}, \bar{A} \rangle \cong D_{2m}$, and $m > 1$ is odd since $\bar{A} \in \bar{V}^{\bar{X}}$; thus (2) holds in this case. Otherwise I.8.3 shows that $\bar{X}$ is $L_2(2^n)$, $Sz(2^n)$, or $(S)U_3(2^n)$. Then since $\bar{X} = \langle \bar{V}, \bar{A} \rangle$, $\bar{X}$ is $L_2(2^n)$ or $Sz(2^n)$. Thus (b) holds. Finally (c) follows as $C_Y(v) = U$ for $v \in V - U$; for the direct sum, cf. G.1.3 when $\bar{X}$ is $L_2(2^n)$, and observe that a Sylow 2-subgroup of order 2 acts freely when $\bar{X}$ is D_{2m}. □

I.7. Tightly embedded subgroups

In this section, we state and prove some elementary lemmas on tightly embedded subgroups from [**Asc75**] and [**Asc76**]. The proofs are sometimes simpler than the originals, but in keeping with the style of exposition in this work, we supply more details.

Throughout this section we assume

G is a finite group and K is a tightly embedded subgroup of G.

Recall this means that K has even order, while $K \cap K^g$ is of odd order for each $g \in G - N_G(K)$. We work in the following setup:

$H \leq G$, $1 \neq Q \in Syl_2(H \cap K)$, and S is a 2-subgroup of H containing Q.

For example this setup is satisfied when $H = G$, $Q \in Syl_2(K)$, and S is any 2-subgroup of G containing Q. We also often use the following observation:

Remark I.7.1. If $g \in H$, then Q^g is Sylow in $H \cap K^g$, so Q^g, K^g satisfy the hypotheses of our setup in the roles of "Q, K".

Lemma I.7.2. *(1) $Q = S \cap K$*

(2) If $g \in H$ with $Q \neq Q^g \leq S$, then $Q \cap Q^g = 1$.

(3) Q is a TI-subgroup of S under $N_G(S)$.

(4) If T is a nontrivial TI-subgroup of S then T is tightly embedded in S.

(5) For each $1 \neq X \leq Q$, $N_G(X) \leq N_G(K)$ and $N_S(X) \leq N_S(Q)$.

Proof. As Q is Sylow in $H \cap K$, and S is a 2-subgroup of H containing Q, (1) holds. Assume the hypotheses of (2). If $Q \cap Q^g \neq 1$, then as K is tightly embedded in G, $K = K^g$, so by (1) and I.7.1, $Q = S \cap K = S \cap K^g = Q^g$, contrary to assumption. Thus (2) holds. A similar proof establishes (3), and (4) is immediate from the definitions. Assume $1 \neq X \leq Q$. For $g \in N_G(X)$, $1 \neq X \leq K \cap K^g$, so

$g \in N_G(K)$ as K is tightly embedded. Then if $g \in S$, g normalizes $S \cap K = Q$, completing the proof of (5). □

LEMMA I.7.3. *If* $|S : Q| \leq |Q|$, *then* $Q \trianglelefteq S$.

PROOF. This is essentially 4.4.1 in [**Asc75**]. If Q is weakly closed in $N_S(Q)$ with respect to S, then $N_S(N_S(Q)) \leq N_S(Q)$, so that $S = N_S(Q)$, and the lemma holds. Thus we may assume there is $s \in S$ with $Q \neq Q^s \leq N_S(Q)$. Thus $Q \cap Q^s = 1$ by I.7.2.2, so

$$|S| \geq |N_S(Q)| \geq |QQ^s| = |Q|^2 \geq |S : Q||Q| = |S|,$$

so that all inequalities are equalities. In particular $S = N_S(Q)$, completing the proof. □

LEMMA I.7.4. *Assume* $Q \neq Q^g \leq N_S(Q)$ *for some* $g \in H$. *Then* $\langle Q, Q^g \rangle = Q \times Q^g$.

PROOF. This is essentially 4.4.2 in [**Asc75**]. As $Q^g \leq N_S(Q)$, $\langle Q, Q^g \rangle = QQ^g$. As $|QQ^g| \leq |Q|^2$, the triple Q^g, QQ^g, K^g satisfies the hypotheses of "Q, S, K" in I.7.3, so $Q^g \trianglelefteq QQ^g$ by that lemma. Then as $Q \cap Q^g = 1$ by I.7.2.2, the lemma is established. □

LEMMA I.7.5. *Assume* $\Phi(Q) \neq 1$. *Then for distinct* $Q^h, Q^k \in Q^H \cap S$, $\langle Q^h Q^k \rangle = Q^h \times Q^k$.

PROOF. As $Q^h \cap Q^k = 1$ by I.7.2.2, so we must show that distinct members of $Q^H \cap S$ commute. This is essentially 2.5 in [**Asc76**]. Let $\Delta := Q^H \cap S$, Γ a maximal set of pairwise commuting members of Δ, and $W := \langle \Gamma \rangle$. Conjugating in H if necessary, we may assume that $Q \in \Gamma$. As distinct members of Γ commute, $\Gamma \subseteq \Delta \cap W \subseteq N_\Delta(\Gamma)$. Suppose $\Gamma = N_\Delta(\Gamma)$. Then $\Gamma = \Delta \cap W$, so that $N_S(W) = N_S(\Gamma)$, and hence $\Gamma = \Delta \cap N_S(W)$, so that $N_S(N_S(W)) \leq N_S(\Gamma) = N_S(W)$. Therefore $S = N_S(W)$ and $\Gamma = \Delta$, so the lemma holds in this case.

Thus we may assume there is $T \in N_\Delta(\Gamma) - \Gamma$, and it remains to derive a contradiction. We first show:

$$\text{If } N_T(Q) \neq 1, \text{ then } [T, Q] = 1. \qquad (*)$$

For if $N_T(Q) \neq 1$, there is an involution x in $N_T(Q)$. Let z be an involution in $C_{Z(Q)}(x)$; as distinct members of Γ commute, $z \in Z(W)$. Then as z centralizes x, $z \in N_S(T)$ by I.7.2.5, and hence $[z, T] \leq Z(W) \cap T$. Thus if $[z, T] \neq 1$ then $Q \leq C_S([z, T])) \leq N_S(T)$ by I.7.2.5, so Q centralizes T by I.7.4. On the other hand, if $[z, T] = 1$, then $T \leq C_S(z) \leq N_S(Q)$ by I.7.2.5, so that T centralizes Q by I.7.4. Thus (*) is established.

By maximality of Γ, T does not centralize some member of Γ, say Q. Thus $N_T(Q) = 1$ by (*), so T acts regularly on Q^T. Then as $Q \trianglelefteq W$, $T \cap W = 1$. Now $\Phi(Q) \neq 1$ by hypothesis, so there is x of order 4 in T; then $t := x^2$ is an involution in T. As $N_T(Q) = 1$, $Q \neq Q^t$, so as $Q \trianglelefteq W$, $\langle Q, Q^t \rangle = Q \times Q^t$ by I.7.4. Therefore $Y := \{qq^t : q \in Q\} = C_{QQ^t}(t) \leq C_W(t)$. By I.7.2.5, $C_W(t)$ acts on T, so $[T, C_W(t)] \leq T \cap W = 1$. Thus x centralizes $C_W(t)$ and in particular centralizes Y.

Let u be an involution in Q, and set $A := \langle u^{\langle x \rangle} \rangle$. As T is regular on Q^T, A is elementary abelian, and $u \neq u^t$, so that $\langle x \rangle$ is faithful on A. Also $uu^t \in Y \leq C_A(x)$, so $uu^t = u^x u^{tx}$, and hence $m(A) \leq 3$. Then as t is an involution, $B := C_A(t)$ satisfies $m(B) \geq m(A)/2 \geq m(A) - 1$. Hence x centralizes A/B, and x centralizes

B as $B \le C_W(t)$. Thus $\langle x\rangle$ is faithful and quadratic on A, impossible as x is of order 4. □

LEMMA I.7.6. *If* $g \in H$ *with* $Q \ne Q^g \le S$ *and* $N_{Q^g}(Q) \ne 1$, *then* $QQ^g = Q \times Q^g$.

PROOF. If $\Phi(Q) \ne 1$, the result follows from I.7.5, while if $\Phi(Q) = 1$ it follows from I.6.2.1 as $\langle Q, Q^g\rangle$ is a 2-group. □

LEMMA I.7.7. *Assume* $g \in G - N_G(K)$ *and* $1 \ne R \in Syl_2(N_K(K^g))$. *Let* $R \le S \in Syl_2(K^gR)$, *and set* $S_0 := S \cap K^g$ *and* $S_1 := N_{S_0}(R)$. *Then*

(1) $N_S(R) = S_1 \times R$ *with* $S_1 = C_{S_0}(R)$.

(2) $S_1 \cong R$.

(3) $S_1 \in Syl_2(N_{K^g}(K))$.

(4) If $S_1 < S_0$, *then* R *is abelian.*

(5) Assume $S_1 < S_0$ *and let* $W := \langle R^S\rangle$. *Then one of the following holds:*

(a) $|R| = 2$ *and* S *is dihedral or semidihedral.*

(b) $W = RS_1 = R \times S_1 \cong R \times R$.

(c) $R \cong E_{2^n}$ *for some* $n \ge 2$, W *is special of order* 2^{3n} *with center* S_1, *and* $|S : W| = 2$.

(6) $S_1 \trianglelefteq S_0$.

PROOF. This is essentially Theorem 3 in [**Asc75**], together with part (1) of Theorem 2 in [**Asc75**]. Observe that the setup we've been working in is satisfied with K^gR, R, S in the roles of "H, Q, S", so we can appeal to the earlier lemmas in this section.

As K is tightly embedded in G and $K \ne K^g$, $R \cap K^g = 1$, so $S_0 \cap R = 1$. Thus $[S_1, R] \le S_0 \cap R = 1$, so $S_1R = S_1 \times R$ and $S_1 = C_{S_0}(R)$. Further using the Dedekind Modular Law, $S = RS_0$ and $N_S(R) = RN_{S_0}(R) = RS_1$, completing the proof of (1).

Suppose that $S_1 = S_0$. Then parts (4) and (5) of the lemma are vacuous, and (6) is immediate. As $S_0 \in Syl_2(K^g)$, (3) holds. Further S_0 normalizes some $R_1 \in Syl_2(K)$ containing R, and applying I.7.3 to S_0, S_0R_1 in the roles of "Q, S", we conclude that $R_1 \le N_K(S_0)$, so $R_1 \le N_K(K^g)$ by I.7.2.5. Then $R = R_1$ by definition of R, so that (2) holds.

Thus we may assume that $S_1 < S_0$, so there is $x \in N_S(RS_1) - RS_1$ and $R \ne R^x \le RS_1$. By I.7.4, $\langle R, R^x\rangle = R \times R^x$, so $|S_1| = |N_S(R) : R| \ge |R^x| = |R|$. Let $S_1 \le S_2 \in Syl_2(N_{K^g}(K))$; then

$$|N_{K^g}(K)|_2 = |S_2| \ge |S_1| \ge |R| = |N_K(K^g)|_2. \qquad (*)$$

Indeed (*) also holds when $S_1 = S_0$ by the previous paragraph.

By (*), $|N_{K^g}(K)|_2 > 1$, so we have symmetry between K and K^g; then by that symmetry and (*), $|N_K(K^g)|_2 \ge |N_{K^g}(K)|_2$. Hence all inequalities in (*) are equalities, so that $|S_1| = |R|$ and $S_1 = S_2 \in Syl_2(N_{K^g}(K))$, establishing (3). As $|S_1| = |R|$, $S_1 \times R = R^x \times R$, so $R^x \cong S_1R/R \cong S_1$, proving (2). Also $S_1R = S_1R^x$ and S_1 and R^x centralize R, so $R \le Z(S_1R)$, proving (4).

If $|R| = 2$, then $C_S(R) = RS_1 \cong E_4$, so S is dihedral or semidihedral by a lemma of Suzuki. (cf. Exercise 8.6 in [**Asc86a**]). Thus conclusion (a) of (5) holds in this case, and then $S_1 = Z(S) \cap S_0 \trianglelefteq S$, so (6) holds. Thus we may assume that $|R| > 2$.

Recall $W = \langle R^S \rangle$. If $W = RS_1$, then conclusion (b) of (5) holds, and $S_1 = S \cap W \trianglelefteq S$, so (6) holds. Thus we may assume that $RS_1 < W$. However if $\Phi(R) \neq 1$, then $W \leq N_S(R) = RS_1$ by I.7.5 and (1), contrary to assumption; thus $\Phi(R) = 1$, so $R \cong E_{2^n}$ for some $n \geq 2$. Also there is $y \in N_S(RS_1) - RS_1$ with $R^y \leq N_S(RS_1)$, but $R^y \not\leq RS_1 = N_S(R)$. Then by I.7.6, $N_{R^y}(R) = 1$, so $|R^{R^y}| = 2^n$. Then as $R \cap R^a = 1$ for $a \in R^{y\#}$,

$$|(\bigcup_{b \in R^y} R^{b\#}) \cup S_1^{\#}| = (2^n - 1)(2^n + 1) = 2^{2n} - 1 = |(RS_1)^{\#}|,$$

so each element in $RS_1 - S_1$ is in a unique R^b with $b \in R^y$. Therefore $R^S \cap RS_1 = R^{R^y}$ for any such y, and $C_{S_1R}(a) \leq S_1$ for $a \in R^{y\#}$, so as $m(C_{S_1R}(a)) \geq m(S_1R)/2 = n$, we conclude $S_1 = C_{S_1R}(a) = [S_1R, a]$. It follows that $U := S_1RR^y$ is special of order 2^{3n}, and $S_1 = Z(U)$. As $RS_1 = N_S(R)$ and R^y is transitive on $R^S \cap RS_1$, $N_S(RS_1) = U$. Then as RS_1 and R^yS_1 are the maximal elementary abelian subgroups of U, $|N_S(U) : U| = 2$. Thus if $U = W$, then (5c) and (6) hold, so we may assume that $U < W$, and it remains to derive a contradiction.

As $U < W$, there is $s \in S$ with $R^s \leq N_S(U)$ but $R^s \not\leq U$. As $|N_S(U) : U| = 2 < |R^s|$, there is an involution $i \in R^s \cap U$. Then as RS_1 and R^yS_1 are the maximal elementary abelian subgroups of U, i is contained in one of these subgroups, say RS_1. But then $1 \neq i \in N_{R^s}(R)$, so $R^s \leq N_S(R) = RS_1 \leq U$ by I.7.6 and (1), contrary to the choice of R^s. □

I.8. Discussion of certain results from the Bibliography

In this section we discuss several results from our bibliography, and provide proofs of some of the easier results, rather than quoting them as Background References.

I.8.1. Some results related to strongly embedded subgroups. In this subsection, for the convenience of the reader we provide a brief discussion of several results related to the notion of a "strongly embedded subgroup". This notion is originally due to Thompson.

Recall (see Definition 17.1 in [**GLS96**]) that a proper subgroup M of a finite group G is *strongly embedded* in G if M has even order, and for some $T \in Syl_2(M)$, $N_G(Q) \leq M$ for each $1 \neq Q \leq T$. There are various equivalent formulations of this definition, such as:

LEMMA I.8.1. *Let G be a finite group, M a proper subgroup of G of even order, and $T \in Syl_2(M)$. Then the following are equivalent:*

(1) M is strongly embedded in G.

(2) $|M \cap M^g|$ is odd for each $g \in G - M$.

(3) $N_G(T) \leq M$ and $C_G(i) \leq M$ for each involution $i \in T$.

(4) Each involution in G fixes a unique point in the permutation representation of G on G/M.

PROOF. See 17.11 in [**GLS96**] or 46.1 in [**Asc86a**]. □

For the most part we will need only elementary results on strongly embedded subgroups due to Feit and Thompson. In particular, we will usually only need Exercise 16.5 in [**Asc86a**] or Lemma 7.6 in [**Asc94**]. Occasionally however we will need deeper results, like the classification of groups with a strongly embedded subgroup due to Bender [**Ben71**] and Suzuki [**Suz62**]; this is stated as Theorem SE on page 20 of [**GLS99**] in our Background References.

Sometimes we will need one of a number of theorems related to the Bender-Suzuki Theorem. Historically the first such theorem was due to Shult in [**Shu**]; a special case appears as Theorem I.8.3 below. Shult proved his theorem before Bender completed the classification of groups with a strongly embedded subgroup. However Shult's Fusion Theorem can be derived as a corollary to Theorem I.8.2 below, which is essentially part (2) of Theorem 2 in [**Asc75**]. Theorem I.8.2 is in turn a corollary of any one of a number of results originally proved by Aschbacher, such as the main result Theorem 1 of [**Asc73**]. We will derive Theorem I.8.2 from Theorem ZD in [**GLS96**]; a somewhat weaker version of Theorem ZD was proved by Aschbacher as Theorem 3.3 in [**Asc75**].

Recall a subgroup K of a finite group G is *tightly embedded* in G if K is of even order but $K \cap K^g$ is of odd order for each $g \in G - N_G(K)$. Also recall that a *Bender group* is a simple group of Lie type in characteristic 2 of Lie rank 1: i.e., $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$.

THEOREM I.8.2. *Let G be a finite group, K a tightly embedded subgroup of G, $\mathcal{Z} := I(K)^G$ where $I(K)$ is the set of involutions in K, and $L := \langle \mathcal{Z} \rangle$. Assume K is not normal in G, and $N_J(K)$ is of odd order for each $J \in K^G - \{K\}$. Then either*

(1) L is of 2-rank 1 and $L = O(L)\langle z \rangle$, where $z \in I(K)$, or

(2) $L/O(L)$ is a Bender group, $O(L) \leq M := N_G(K)$, and $(M \cap L)/O(L)$ is a Borel subgroup of $L/O(L)$.

PROOF. We first claim that $\mathcal{Z}$ is *product-disconnected* in G with respect to M, in the sense of Definition ZD on page 20 of [**GLS99**]: As K is of even order, $\mathcal{Z} \cap M \supseteq I(K) \neq \emptyset$, giving condition (a) of that definition. Let $z \in \mathcal{Z} \cap M$; then $z \in K^g$ for some $g \in G$. But by hypothesis, $|N_J(K)|$ is odd for $J \in K^G - \{K\}$, so $K^g = K$ and hence $g \in N_G(K) = M$. This gives condition (b) of the definition and shows that

$$\mathcal{Z} \cap M = I(K). \tag{$*$}$$

Finally if $v \in C_{\mathcal{Z}}(z) - \{z\}$, then as K is tightly embedded by hypothesis, $v \in M$ by I.7.2.5. Hence $v \in \mathcal{Z} \cap M = I(K)$ by (*), so that also $vz \in I(K) \subseteq \mathcal{Z}$. Since any member of $\mathcal{Z}$ is conjugate to a member of $I(K)$, we conclude $\mathcal{Z}$ is closed under products of commuting pairs. In particular if $1 \neq xy \in M$ for some commuting pair x, y from $\mathcal{Z}$, then $xy \in I(K)$ by (*), so that $C_G(xy) \leq M$. This establishes condition (c) of the definition, completing the proof of the claim.

Set $\bar{G} := G/O(L)$, $X := [L, L]$, $M_L := M \cap L$, and $M_X := M \cap X$. By the claim, $\mathcal{Z}$ is product-disconnected in G with respect to M, so $\bar{L}$ and $\bar{M}_L$ are described in Theorem ZD in [**GLS99**]. If L is of 2-rank 1, then by the Brauer-Suzuki Theorem (XII.7.1 in [**Fei82**]), $\bar{z} \in Z(\bar{L})$, so $\bar{L} = \langle \bar{\mathcal{Z}} \rangle = \langle \bar{z} \rangle$, and hence (1) holds. Thus we may assume that $m_2(L) =: n > 1$. Then by Theorem ZD in [**GLS99**], $\bar{X}$ is a Bender group, $O(L) \leq M_X$, and $\bar{M}_X$ is a Borel subgroup of $\bar{X}$. Further there is $u \in M$ with $u^2 = 1$ such that $\bar{L} = \langle \bar{u} \rangle \times \bar{X}$ with $\bar{\mathcal{Z}} \subseteq \bar{u}\bar{X} - \{\bar{u}\}$. In particular if $u = 1$, then (2) holds, so we may assume that u is an involution, and it remains to derive a contradiction.

Let $z \in I(K)$, $z, u \in S \in Syl_2(M_L)$, and $V := \Omega_1(S \cap X)$. Since $\bar{\mathcal{Z}} \subseteq \bar{u}\bar{X} - \{\bar{u}\}$, $z = ui$ for some involution i in V. As $\bar{M}_X$ is a Borel subgroup of the Bender group

$\bar{X}$, $\bar{V} = [\bar{M}_X, \bar{i}] = [\bar{M}_X, \bar{z}]$, so as $z \in K \trianglelefteq M$, $V \leq K$. Therefore $u \in \langle z \rangle V \leq K$; thus $u \in \mathcal{Z}$, contrary to $\bar{\mathcal{Z}} \subseteq \bar{u}\bar{X} - \{\bar{u}\}$. This contradiction completes the proof of Theorem I.8.2. $\square$

THEOREM I.8.3 (Shult's Fusion Theorem). *Assume G is a finite group, $T \in Syl_2(G)$, and V is an elementary abelian TI-subgroup of G strongly closed in T with respect to G. Assume further that $m_2(V) =: n > 0$, and V is not normal in G. Set $L := \langle V^G \rangle$. Then either*

(1) $n = 1$ and $L = O(L)V$.

(2) $n > 1$, L is quasisimple, $L \cong L_2(2^n)$, $Sz(2^n)$, or $(S)U_3(2^n)$, and $V = \Omega_1(T \cap L)$.

PROOF. Set $\bar{L} := L/O(L)$ and $M_L := N_L(V)$. As V is a TI-subgroup of G, V is tightly embedded in G. As V is strongly closed in T with respect to G, $N_J(V) = 1$ for $J \in V^G - \{V\}$. Thus L is described in Theorem I.8.2. If $m := m_2(L) = 1$, then (1) holds, so we may assume that $m > 1$. Thus by I.8.2, $\bar{L}$ is a Bender group, and $\bar{M}_L$ a Borel subgroup of $\bar{L}$ with $O(L) \leq M_L$. Then $V_T := \Omega_1(T \cap L)$ is abelian of rank m, and $\bar{M}_L$ is transitive on $\bar{V}_T^\#$, so as $M_L = N_L(V)$ it follows that $V = V_T$ and hence $n = m$. Finally as $O(L) \leq M_L$, $[O(L), V] \leq O(L) \cap V = 1$, so $L = \langle V^G \rangle$ centralizes $O(L)$, and hence L is quasisimple. Then inspecting the Schur multipliers of Bender groups in I.1.3, we conclude that L is either simple or $SU_3(2^n)$, so (2) holds. $\square$

REMARK I.8.4. Assume that V is a strongly closed abelian subgroup of T. Then $N_G(V)$ controls fusion in V by Burnside's Fusion Lemma A.1.35. Thus by I.6.1.1, V is a TI-subgroup of G iff $C_G(v) \leq N_G(V)$ for each $v \in V^\#$. This gives an equivalent set of hypotheses for applying Theorem I.8.3 which is sometimes more convenient.

LEMMA I.8.5. *Let G be a finite group, M a proper subgroup of G, and V a nontrivial normal elementary abelian 2-subgroup of M such that*

$$\text{Each } v \in V^\# \text{ fixes a unique point in } G/M \text{ by right multiplication.} \qquad (*)$$

Set $L := \langle V^G \rangle$; then:

(1) Either

(i) $|V| = 2$ and $L = O(L)V$, or

(ii) L is a quasisimple Bender group, $V = \Omega_1(T_L)$ for some $T_L \in Syl_2(L)$, and $M \cap L = N_L(V)$ is a Borel subgroup of L.

(2) $M/C_M(V)$ is solvable.

PROOF. By (*) and 7.3 in [**Asc94**], V is strongly closed in M with respect to G and $C_G(v) \leq M$ for each $v \in V^\#$. Hence by Remark I.8.4, we can appeal to Theorem I.8.3 to conclude that (1) holds. Thus it remains to prove (2). If $|V| = 2$, then $M = C_G(V)$, so we may assume that L is a Bender group. Thus $Aut_{Aut(L)}(V)$ is solvable, so (2) also holds in this case. $\square$

CHAPTER J

A characterization of the Rudvalis group

In this chapter we obtain a 2-local characterization of the Rudvalis sporadic simple group Ru; that is we, prove that Ru is the unique group satisfying certain 2-local hypotheses, and use this result to recognize Ru in the proof of Theorem 14.7.75.

J.1. Groups of type Ru

Define a finite group G to be of *type Ru* if

(Ru1) There is a subgroup L of G such that $S := O_2(L)$ is special with center V, V is the natural module for $L/S \cong L_3(2)$, S/V is the Steinberg module, and $L = N_G(V)$.

(Ru2) Let $T \in Syl_2(G)$, $Z := Z(T)$, $H := C_G(Z)$, $Q := O_2(H)$, $U := \langle V^H \rangle$, $\tilde{H} := H/Z$, and $H^* := H/Q$. Assume $H^* \cong S_5$, $\tilde{U}$ is the $L_2(4)$-module for H^*, and Q/U is a 6-dimensional indecomposable for H^*.

The main theorem of this chapter is:

THEOREM J.1.1. *Each group of type Ru is isomorphic to Ru.*

Rudvalis [**Rud84**] discovered the Rudvalis group Ru: that is in studying rank 3 permutation groups, he discovered that the properties defining "Rudvalis rank 3 groups" (in the sense of Definition J.5.1 in section 5) appeared to be consistent, and he generated considerable information about such groups. In [**CW73**], Conway and Wales proved the existence of Rudvalis rank 3 groups; uniqueness follows from a result of Wales in [**Wal69**] For completeness, we have included a proof of the uniqueness of Rudvalis rank 3 groups in the final section of this chapter.

In [**Par76**], Parrott characterizes Ru via the centralizer of a 2-central involution; these hypotheses are weaker than our hypotheses of "type Ru". In this chapter we give a shorter, less computational proof of our weak version of Parrott's theorem. To produce a subgroup of G isomorphic to ${}^2F_4(2)$, Parrott generates a portion of the character table of G, and then uses the Brauer trick to show that a suitable subgroup F of G is proper. Then he uses his involution-centralizer characterization [**Par72**] of the Tits group ${}^2F_4(2)'$ to identify F as ${}^2F_4(2)$.

We instead identify F using Theorem F.4.8. In particular, we avoid the character theory and its associated computations, and we avoid reliance on Parrott's work in [**Par72**] on the Tits group.

For $X \leq G$, let $I(X)$ denote the set of involutions in X.

J.2. Basic properties of groups of type Ru

In this section, we assume G is a group of type Ru.

We choose T so that $N_T(V) \in Syl_2(L)$. Then:

LEMMA J.2.1. *(1) $T \leq L$.*
(2) $Z = C_V(T)$ is of order 2.

PROOF. We chose $N_T(V) \in Syl_2(L)$, $Z = Z(T)$ by (Ru2), and from the structure of L described in (Ru1), $|C_V(T)| = 2$. Thus (2) holds. Then using (Ru2), $|H|_2 = 2^{14} = |L|_2$, so (1) holds. □

Set $K := O^2(H)$, $L_1 := O^2(C_L(Z))$, $R_1 := O_2(L_1)$, $H_C := C_H(U)$, and choose $B \in Syl_3(L_1)$. Let z be a generator of Z.

LEMMA J.2.2. *(1) $H \cap L = L_1T$ and $QS = O_2(L_1T) = R_1$. In particular, $Q \leq K$.*
(2) $U \cong E_{32}$.
(3) $H_C/U = C_{Q/U}(K)$ and Q/H_C is the $L_2(4)$-module for K^*.
(4) Q induces the group of transvections on U with center Z.
(5) For each L-conjugate U_α of U distinct from U, U_α^* is of order 2 and $V = U_\alpha \cap H_C$.

PROOF. We use (Ru1) and (Ru2) without further reference to deduce various properties: As H is transitive on $\tilde{U}^\#$, and $V \leq U$ with $\Phi(V) = 1$, $\Phi(U) = 1$. Then since $\tilde{U} \cong E_{16}$, and Z is of order 2 by J.2.1.2, (2) holds.

As V is the natural module for L/S, $H = C_G(Z)$, and $T \leq L$ by J.2.1.1, $H \cap L = L_1T$ is the minimal parabolic of L centralizing Z. Further $V = [V, L_1]$, and as S is the Steinberg module for L/S, also $S/V = [S/V, L_1]$. Hence $S = [S, L_1]$, so that $S \leq O_2(L_1) = R_1$. Then as $O_2(L_1T) = R_1S$, $R_1 = O_2(L_1T)$. If $S = Q$ then $V = Z(S) = Z(Q) \trianglelefteq H$, contradicting $U = \langle V^H \rangle$ of rank 5. Thus as $|S| = 2^{11} = |Q|$, $S \not\leq Q$, so as L_1 is irreducible on R_1/Q, $R_1 = QS$, completing the proof of (1).

Next as $V = Z(S)$, $[V, Q] = [V, QS] = [V, R_1] = Z$, so that $[U, Q] = Z$ since $U = \langle V^H \rangle$. Then as H is irreducible on $\tilde{U}$, $C_U(Q) = Z$, establishing (4).

Observe that Hypothesis F.9.1 is satisfied, with Z, H, H, V, V in the roles of "V_1, G_z, H, V_+, V". Then by F.9.7, U/H_C is dual to $U/C_U(Q) = U/Z = \tilde{U}$ as an H-module, so since Q/U is a 6-dimensional indecomposable for H^*, $H_C/U = C_{Q/U}(K)$, and hence (3) holds.

Let $U_\alpha := U^l \in U^L - \{U\}$, $Z_\alpha := Z^l$, and $Q^+ := Q/H_C$. Then $U_\alpha \trianglelefteq C_G(Z_\alpha)$, and as $V \trianglelefteq L$, $V \leq U \cap U_\alpha$ and $m(U_\alpha/V) = m(U/V) = 2$. As $H \cap L = L_1T$ and $L/S \cong L_3(2)$, L is 2-transitive on $L/(H \cap L)$, and hence also on U^L. Next $U \leq C_T(V) = S$, so as L is irreducible on S/V and $V < U$, $S = \langle U^L \rangle$. Thus $U_\alpha \not\leq Q$, since $QS \not\leq Q$ and L is 2-transitive on U^L. So $1 \neq U_\alpha^* \leq S^* = R_1^*$. On the other hand S/V is abelian, so $[U, U_\alpha] \leq V$.

Suppose $U_\alpha^* = R_1^*$. Then $m(U_\alpha^*) = 2 = m(U_\alpha/V)$, so $V = U_\alpha \cap Q$. On the other hand as $Z_\alpha \leq U$, $C_Q(Z_\alpha)$ is of index 2 in Q by (4). Then $[C_Q(Z_\alpha), U_\alpha] \leq U_\alpha \cap Q = V$, so U_α centralizes the hyperplane $C_Q(Z_\alpha)^+$ of Q^+, a contradiction since H^* contains no $\mathbf{F}_2$-transvections on the natural $L_2(4)$-module Q^+.

Thus U_α^* is of order 2, and the argument of the previous paragraph also shows that $U_\alpha \cap Q \not\leq H_C$. Then $U_\alpha > U_\alpha \cap Q > U_\alpha \cap H_C \geq V$ while $m(U_\alpha/V) = 2$, so $U_\alpha \cap H_C = V$, completing the proof of (5). $\square$

Recall $B \in Syl_3(L_1)$. Let B_7 denote a B-invariant subgroup of L of order 7, set $S_0 := C_S(B_7)$, and let s denote an element of $S_0^\#$.

Recall from Definition 2.4.13 that a *Suzuki 2-group* is a 2-group admitting a cyclic group of automorphisms transitive on its involutions.

LEMMA J.2.3. *(1)* $S_0 = [S_0, B] \cong E_4$.

(2) $z^G \cap S_0 = \emptyset$.

(3) $C_L(s) = S_0[C_S(s), B_7]B_7$, *with* $C_S(s) = S_0 \times [C_S(s), B_7] \in Syl_2(C_L(s))$ *of order* 2^8 *and exponent* 4.

(4) B *normalizes* $C_S(s)$, S_0, *and* $[C_S(s), B_7]$.

(5) $[C_S(s), B_7]$ *is a nonabelian Suzuki* 2*-group.*

(6) For each $t \in S - V$, $[t, S] = V$ *and* $m(C_S(t)/V) = 5$.

(7) $C_Q(B) \cong Q_8$.

(8) $C_{S/V}(R_1) = U/V$.

(9) $U \cap U^l = V$ *for each* $l \in L - H$.

PROOF. We first observe:

$$\text{For each } t \in S,\ m(S/C_S(t)) = m([S, t]) \leq 3. \tag{$*$}$$

This follows as S is special with center V; cf. 8.5.4 in [**Asc86a**].

Let $S^+ := S/V$. As S^+ is the Steinberg module, T fixes a unique point $\langle u^+\rangle$ of S^+, which must lie in U^+ since $U \trianglelefteq T$. Then since L_1 is irreducible on U^+, $C_{S^+}(R_1) = \langle C_{S^+}(T)^{L_1}\rangle = U^+$, establishing (8). Also $C_L(u^+) = T \leq N_L(U^+)$, and $N_L(U^+)$ is transitive on $U^{+\#}$, so U^+ is the unique member of U^L containing u^+ by A.1.7.2, and (9) holds.

We next claim for any preimage u of u^+:

$$V = [u, S]. \tag{!}$$

For by J.2.2.1, $R_1 = QS$, so $[\tilde{u}, S] = [\tilde{u}, R_1] = \tilde{V}$ as $\tilde{U}$ is the natural module for K^*. Further as T centralizes u^+, T acts on $[\langle u\rangle V, S] = [u, S]$, so $Z = C_V(T) \leq [u, S]$, completing the verification of (!).

Assume $t \in S - V$ with $W := [S, t] < V$, let P be the parabolic of L stabilizing W, and $S_P := \langle t^P\rangle$. As $[t, S] = W \trianglelefteq P$, $[S_P, S] = W$. But P contains a Sylow 2-subgroup T_0 of L, and T_0 acts on S_P, so $\langle w^+\rangle := C_{S^+}(T_0) \leq S_P^+$. Thus $[w, S] \leq W < V$, contrary to (!). Thus $[t, S] = V$, and (*) completes the proof of (6).

As S^+ is the Steinberg module for L/S and V is the natural module,

$$S_0 \cong S_0^+ = C_{S^+}(B_7) \cong E_4,$$

and $S_0 = [S_0, B]$, so that (1) holds. As H is a $7'$-group, (2) holds. By (6), $|C_S(s)| = 2^8$. As S is special, $C_S(s)$ is of exponent at most 4. As $m(C_S(s)^+) = 5$, as B_7 acts on $C_S(s)^+$, and as B_7 has two noncentral chief factors of rank 3 on S^+, it follows that $C_S(s) = S_0[C_S(s), B_7]$. Now the only maximal subgroup of $\bar{L} := L/S$ containing $\bar{B}_7$ is $\bar{B}_7\bar{B}$, so $C_{\bar{L}}(s) = \bar{B}_7$ using (1).

Let $\dot{L} := L/S$; as $Aut(\dot{L}) \cong Aut(L_3(2))$ acts on the Steinberg module S^+, $[S^+, B_7] = S_1^+ \oplus S_2^+$ is the sum of two nonisomorphic irreducibles of rank 3 for B_7. By the previous paragraph, we may take $S_1 = [C_S(s), B_7]$. Then B normalizes

$C_S(B_7) = S_0$ and S_1 and S_2, so (4) holds. As B is transitive on $S_0^{+\#}$, and acts on S_0 and on S_1, $S_3 := S_0S_1 = S_0 \times S_1$.

As B_7 is transitive on $S_1^{+\#}$ and on $V^\#$, either (5) holds or S_1 is abelian. Suppose the latter case holds. Then S_3 is abelian, so by (6), $S_3 = C_S(t)$ for each $t \in S_3 - V$. But for $l \in L - N_L(S_3)$, there exists $t \in S_3 \cap S_3^l - V$ as $m(S_3^+) = 5 > m(S^+)/2$, and then $S_3 \neq S_3^l \leq C_S(t)$, contradicting $S_3 = C_S(t)$. This establishes (5) and shows $C_S(s)$ has exponent 4, so the proof of (3) is also complete.

Finally $S_B := C_Q(B) \leq H_C \leq C_T(V) = S$, and from J.2.2.3, $|S_B| = 8$ with $Z = S_B \cap V$. Thus $S_B^+ := S_{B,1}^+ \oplus S_{B,2}^+$, where $S_{B,i} := S_B \cap S_i$. Let $S_{B_i}^+ =: \langle t_i^+ \rangle$; an involution j in $Aut(\dot{L})$ acting on $\dot{B}_7$ and centralizing $\dot{B}$ interchanges S_1^+ and S_2^+, and hence $t_1^+t_2^+ = C_{S_B^+}(j)$. Therefore an involution in $N_{\dot{L}}(\dot{B}) \cap C_{\dot{L}}(j)$ centralizes $t_1^+t_2^+$ and hence interchanges t_1^+ and t_2^+. Therefore by (5), $S_{B_i} \cong \mathbf{Z}_4$. Then as $|S_B| = 8$, either (7) holds or S_B is abelian, and we may assume the latter. By (6), $m(S/C_S(t_1)) = 3$. But B acts on $S/C_S(t_1)$, and as S_B is abelian, $S_B \leq C_S(t_1)$; so all B-chief factors on $S/C_S(t_1)$ are of rank 2, whereas $m(S/C_S(t_1))$ is odd. This completes the proof of (7), and hence of the lemma. □

LEMMA J.2.4. *For $h \in H_C - U$:*
(1) $[\tilde{h}, Q] = \tilde{U}$.
(2) $[h, Q] = U$.
(3) $C_Q(h) = \langle h \rangle U$.

PROOF. Set $H_B := C_Q(B)$. It follows from J.2.2.3 that $H_C = H_BU$.

We first prove (1). Since Q centralizes $\tilde{U}$, if $h_1 \in hU$ with $[\tilde{h}_1, Q] = \tilde{U}$, then also $[\tilde{h}, Q] = \tilde{U}$, so we may assume that $h \in H_B$. As $h \notin U$ and B is irreducible on R_1/S, $C_{R_1}(h) \leq S$ by J.2.3.8. By J.2.2.1, $R_1 = QS$ and $4 = |QS : S| = |Q : Q \cap S|$, so as $\tilde{Q} = Q/Z$ with $|Z| = 2$, $[\tilde{h}, Q] \neq 1$. Then as Q centralizes $\tilde{U}$, $1 \neq [\tilde{h}, Q] = [\langle \tilde{h} \rangle \tilde{U}, Q]$. Therefore as $[h, K] \leq U$ by J.2.2.3, and K is irreducible on $\tilde{U}$, (1) follows.

By J.2.3.7, $H_B \cong Q_8$, so $Z = [h, H_B]$. Thus (2) follows from (1). By (1), $C_Q(\tilde{h}) = H_C$, so $C_Q(h) \leq H_C$. Thus as $U \leq Z(H_C)$ with $H_C = H_BU$ and $H_B \cong Q_8$, (3) follows. □

LEMMA J.2.5. $C_Q(B) \cong Q_8$ *and* $C_H(B)/B \cong SD_{16}$.

PROOF. By J.2.3.7, $C_Q(B) \cong Q_8$. We will see during the proof of the next lemma (before the only appeal to the present lemma in the final sentence of that proof) that there is an involution $j \in H - K$ centralizing B. As Q/U is a 6-dimensional indecomposable for H^* with $H_C/U = C_{Q/U}(K)$ by J.2.2.3, $[H_C/U, j] \neq 1$, so $[C_Q(B), j] \not\leq Z$, and hence the lemma holds. □

LEMMA J.2.6. *(1) H is transitive on the involutions in $H - K$, there are* 640 *such involutions, and all are in z^G.*
(2) All involutions in $L - S$ are in z^G.
(3) All involutions in Q are in z^G.
(4) $U = \Omega_1(H_C)$, $Z = \Phi(H_C)$, and $U^\# \subseteq z^G$.

PROOF. As $H_C = C_Q(B)U$ with $C_Q(B) \cong Q_8$, and $\Phi(C_Q(B)) = Z$ while $U = Z(H_C)$ is of exponent 2, we conclude that $U = \Omega_1(H_C)$ and $Z = \Phi(H_C)$. Then as all elements of $U^\#$ are fused to z under $H \cup L$, (4) holds. In particular all involutions in H_C are in z^G. Further defining U_α as in part (5) of J.2.2, that result shows $U_\alpha \cap Q \not\subseteq H_C$, so $z^G \cap Q \not\subseteq H_C$. By (3) and (1) of J.2.2, H is transitive on $(Q/H_C)^\#$ and $Q \not\leq S$, so each involution in $Q - H_C$ is fused to an involution in $L - S$. Next each involution in $L - S$ is fused to an involution in T inverting B, and all such involutions are in $H - K$. Thus there are involutions in $z^G \cap (H - K)$, (2) implies (3), and (2) will follow when we establish transitivity of H on involutions in $H - K$. So it suffices to prove (1).

Now H^* is transitive on the 10 involutions in $H^* - K^*$, and if i is any involution in $H - K$, then i is free on Q/U and $\tilde{U}$. So all involutions in $H - K$ are H-conjugate to i or iz, there are $10 \cdot 8 \cdot 4 \cdot 2 = 640$ such involutions, and $O^2(C_{H^*}(i^*)) = O^2(C_H(i))^*$. Thus we may take $i \in C_H(B)$. Then by J.2.5, iz is fused to i in $C_Q(B)$, completing the proof of (1), and hence of J.2.6. □

Set $L_B := C_G(B)^\infty$.

LEMMA J.2.7. *(1)* $L_B \cong \hat{A}_6$.
(2) $C_G(B)/B \cong M_{10}$.
(3) $N_G(B)/B \cong Aut(A_6)$.
(4) G has two classes of involutions, with representatives s and z.
(5) For $B_5 \in Syl_5(H)$, $C_G(B_5) = O_5(C_G(B_5))C_Q(B_5)$ with $C_Q(B_5) \cong Q_8$ and $|O_5(C_G(B_5))| = 5^3$.
(6) B_5 is not conjugate to a Sylow 5-subgroup of L_B.
(7) Involutions in L_B and also involutions inducing a transposition on L_B are in z^G, while involutions inducing diagonal outer automorphisms on L_B are in s^G.
(8) K has two orbits on involutions in $K - Q$, one each in s^G and z^G. Also $s^G \cap H = s^H \subseteq K - Q$, and $|z^G \cap (K - Q)| = 480$.
(9) All involutions in $C_G(B)$ lie in z^G.
(10) $T_s := C_S(B_7)[C_S(s), B_7] = C_H(s) \in Syl_2(C_G(s))$.
(11) G has one conjugacy class of elements of order 3.

PROOF. Let $T_B \in Syl_2(C_H(B))$ and set $G_B := C_G(B)$. By J.2.5, T_B is semidihedral of order 16, so $Z = Z(T_B)$ and hence $T_B \in Syl_2(G_B)$. Further $T_B = C_Q(B)\langle j \rangle$ where $j \in H - K$ is an involution, so by J.2.6.1, $j \in z^G$. As B is Sylow in $H = C_G(z)$, $B^G \cap H = B^H$, so $z^G \cap G_B = z^{G_B}$ using A.1.7.1. Thus j is fused to z in G_B.

Set $G_B^+ := N_G(B)$ and $\hat{G}_B^+ := G_B^+/B$; then $\hat{T}_B = C_{\hat{G}_B}(\hat{z})$ is isomorphic to $T_B \cong SD_{16}$. Thus all involutions in T_B are contained in the dihedral subgroup $S_B \cong D_8$ of T_B. Take $f \in T_B - S_B$ of order 4; then $f^2 = z$, so if f is G_B-fused into S_B, the fusion occurs in $C_H(B) = T_B B$, which is not the case. Hence by Generalized Thompson Transfer A.1.37, $f \notin O^2(G_B)$. We saw $j \in z^{G_B}$, so $|G_B : O^2(G_B)| = 2$, and $O^2(G_B)$ has Sylow group $S_B \cong D_8$ and one conjugacy class of involutions. Thus by I.4.1, $\widehat{O^2(G_B)} = \hat{L}_B$ is isomorphic to $L_3(2)$ or A_6. Then as $T_B \cong SD_{16}$, $\hat{L}_B$ is not $L_3(2)$ and $\hat{G}_B \cong M_{10}$, proving (2). Also (9) is established.

We next consider a Sylow 5-subgroup B_5 of H. First $C_H(B_5) = B_5 T_5$, where $T_5 := C_Q(B_5) = C_{H_C}(B_5)$, and $Aut_H(B_5) \cong \mathbf{Z}_4$. As $H_C = C_Q(B)U$ with $C_Q(B)$ centralizing U, and $C_Q(B) \cong Q_8$, also $T_5 \cong Q_8$, with $T_5 \cap U = Z$. Set $G_{B_5} := C_G(B_5)$. As $Z = Z(T_5)$ and $T_5 \in Syl_2(C_H(B_5))$, $T_5 \in Syl_2(G_{B_5})$. By the Brauer-Suzuki Theorem (cf. Theorem 15.2 in [**GLS96**]), $Z \leq Z^*(G_{B_5})$, so setting $P := O(G_{B_5})$, $G_{B_5} = PC_H(B_5) = PT_5$.

Next from the structure of K^* and the Baer-Suzuki Theorem, each involution in $K - Q$ is fused in H to an involution inverting B_5. Now for $l \in L - H$, U^l contains such an involution by J.2.2.5, so we may choose notation so that there is $i \in z^G \cap H \cap U^l$ inverting B_5. We claim $[i, T_5] \neq 1$; for assume otherwise: First $T_5 \leq C_G(V) \leq C_G(Z^l) = H^l$, so as T_5 centralizes the hyperplane $V\langle i \rangle$ of U^l, and H^* contains no $\mathbf{F}_2$-transvections on the natural $L_2(4)$-module $\tilde{U}$, $T_5 \leq Q^l$. By J.2.2.5, there is $w \in U^l \cap Q - H_C$. By J.2.4.1, $C_{\tilde{Q}}(\tilde{t}) = \tilde{H}_C$ for $t \in T_5 - Z$, so $[\tilde{T}_5, w] \cong E_4$ and hence $[w, T_5] \neq Z^l$, contradicting $T_5 \leq Q^l$. Thus the claim is established.

By the claim i is fused to iz under T_5. Thus as $C_P(z) = B_5$ since $C_H(B_5) = B_5T_5$, and B_5 is inverted by i, using Generation by Centralizers of Hyperplanes A.1.17, we conclude

$$P = B_5 C_P(i) C_P(iz),$$

with $C_P(i) \cong C_P(iz)$ and $C_P(i) \cap C_P(iz) = C_P(\langle i, z \rangle) = 1$, so $|P| = 5|C_P(i)|^2$. As the subgroups of H of odd order are of order 1, 3, or 5, $|C_P(i)| = 1$, 3 or 5. If $|C_P(i)| = 3$, then P is a subgroup of order 45 in G_B, contrary to (2). Thus P is a 5-group, so $C_G(B_5)$ is a $\{2,5\}$-group, and hence (6) holds. By (6), $B_5 \notin Syl_5(G)$, so $B_5 \notin Syl_5(G_{B_5})$, and hence $B_5 < P$, so $|P| = |B||C_P(i)|^2 = 5^3$, completing the proof of (5).

Set $T_s := C_S(s)$. Then $s \notin z^G$ by J.2.3.2, so $s \notin Q$ by J.2.6.3, and hence $1 \neq s^* \in R_1^* \leq K^*$ by J.2.2.1. Then $C_{H^*}(s^*) \leq T^*$, so that $C_H(s) = T_s$. Next there is $h \in H$ such that $s_0 := s^h$ inverts B, so $T_+ := \langle s_0 \rangle T_B \in Syl_2(G_+)$, and then $\hat{G}_+ = \langle \hat{s}_0 \rangle \hat{G}_B \cong \mathbf{Z}_2 \times M_{10}$ or $Aut(A_6)$. Similarly each involution in $K - Q$ inverts some conjugate of B, so some conjugate of z inverts B by J.2.6.1.

Assume that $\hat{G}_+ \cong \mathbf{Z}_2 \times M_{10}$. Then G_B has two orbits on involutions inverting B, with representatives $c \in Z(L_BT_+)$ and cz. From the previous paragraph, one of these G_B-orbits lies in $s_0^G = s^G \neq z^G$, and one lies in z^G. Further c centralizes $C_Q(B)$, so since $H_C = C_Q(B)U$, $c \in C_H(H_C/U) = K$. Let r be an involution in $T_B - Z$; by J.2.5, all such involutions are in $H - K$, so $rc \in H - K$ and hence $rc \in z^G$ by J.2.6.1. As $r \in T_B$, $r \in z^{G_B}$ by (9), so $rc \in (cz)^{G_B}$, and hence $cz \in z^G$ so that $c \in s^G$. Thus $\{c\} = s^G \cap G_+$, so $c = s_0 = s^h$. Now $s^G \cap H \subseteq K - Q$ in view of parts (1) and (3) of J.2.6, and by the previous paragraph, each involution in $K - Q$ is conjugate to an involution inverting B, so $s^G \cap H = s^H$. Hence by A.1.7.1, $z^G \cap C_G(s) = z^{C_G(s)}$. As z is 2-central in G, each Sylow 2-subgroup of $C_G(s)$ contains an element of z^G in is center, so that z is in the center of some Sylow group of $C_G(s)$. Then as $T_s \in Syl_2(C_H(s))$, $T_s \in Syl_2(C_G(s))$. This is impossible, as T_s is of exponent 4 by J.2.3.3, while $T_+ \leq C_G(c)$ contains $T_B \cong SD_{16}$ of exponent 8.

Thus $\hat{G}_B \cong Aut(A_6)$, so (3) is established. This time G_B has two orbits on involutions inverting B with representatives f and d, where f induces a transposition on $\hat{L}_B \cong A_6$, and d a diagonal outer automorphism in $PGL_2(9)$. Arguing as in the previous paragraph, one of f or d is in s^G, and the other in z^G. By J.2.6, involutions in $H - K$ and Q lie in z^G, and as each involution in $K - Q$ inverts a K-conjugate of B, all involutions in $K - Q$ lie in s^G or z^G. Hence (4) holds. As d centralizes an element of L_B of order 5, $d \notin z^G$ by (6), so $d \in s^G$ by (4), and hence (7) holds.

Next T_+ is transitive on the set $d^{G_B} \cap T_+$ of involutions in $T_+ \cap dL_B$, so $s^G \cap H = s^H$ as before. Also $\{f, fz\} = f^{T_+}$ is the set of involutions in $T_+ - Z$

centralizing $C_{\tilde{Q}}(B)$, so as earlier, $f \in K$. On the other hand if $\mathcal{R}$ is the set of involutions in $T_B - Z$, then $\{fr : r \in \mathcal{R}\} \subseteq H - K$. Thus T_+ is transitive on members of $z^G \cap T_+ \cap K$ inverting B, so we conclude that H is transitive on $z^G \cap K - Q$. Finally K^* has 15 involutions, and each involution $i \in K - H$ is free on Q/H_C and $\tilde{U}$, and inverts a conjugate of B, so

$$|z^G \cap (K - Q)| = 15 \cdot 16 \cdot \beta = 240\beta,$$

where $\beta := |z^G \cap iC_Q(B)|$. We just showed that $\{f, fz\} = z^G \cap fC_Q(B)$, so $\beta = 2$, and hence the proof of (8) is complete.

Since $s^G \cap H = s^H$ and $T_s = C_H(s)$, an argument above shows that $T_s \in Syl_2(C_G(s))$, so the remainder of (10) follows from J.2.3.3.

Let R be a Sylow 3-subgroup of G_+; we may choose notation so that $\langle f, z\rangle \leq T_R := N_{T_+}(R) \in Syl_2(N_{G_+}(R))$. From the structure of $Aut(A_6)$, $R/B \cong E_9$, $B = C_R(z)$, $N_{G_+}(R) = T_R R$, $T_R \cong SD_{16}$, and $R = C_R(f)C_R(fz)B$, with $R_f := C_R(f) \cong C_R(fz) \cong \mathbf{Z}_3$. As $f \in z^G$ and $B \in Syl_3(H)$, we conclude $R_f \in B^G$.

Suppose that $R \cong 3^{1+2}$. Then (1) holds. Further as $B = Z(R)$ char R, $R \in Syl_3(G)$. Therefore as T_R is transitive on $\hat{R}^\#$, and $R_f \in B^G$, (11) is established, and hence also the lemma.

Thus we may assume instead that $R \cong E_{27}$, and it remains to derive a contradiction. Set $G_R := N_G(R)$. Then $C_G(R) \leq C_{G_B}(R) = R$, so that $G_R/R \leq GL_3(3)$. As R is an abelian Sylow subgroup of $N_G(B)$, G_R is transitive on $B^G \cap R$ by Sylow's Theorem and A.1.7.1; in particular, R_f is conjugate to B in G_R. On the other hand, $G_R \cap H$ acts on $C_R(z) = B$, so $G_R \cap H = T_R B \leq N_{G_R}(B)$. Therefore as $Z = Z(T_R)$ char T_R, $T_R \in Syl_2(G_R)$, and hence $|B^{G_R}| = |G_R : RT_R|$ is odd. Now T_R has orbits of length 1, 4, and 8 on the points of R, and $N_{T_R}(R_f) = \langle z, f\rangle$ is of index 4 in T_R, so R_f lies in the T_R-orbit of length 4. Since the order of $GL_3(3)$ is not divisible by 5, it follows that $|G_R : RT_R| = 13$. But this is also impossible, as $GL_3(3)$ has no subgroup of order $2^4 \cdot 13$. This contradiction finally completes the proof of J.2.7. □

LEMMA J.2.8. *(1) H is transitive on the involutions in $Q - U$, and there are 240 such involutions, all in z^G.*

(2) All involutions in $S \cap Q$ are fused into U under L.

(3) For $u \in U - V$, all involutions in $C_S(u)$ are in z^G.

PROOF. Define $U_\alpha := U^l$ as in J.2.2.5, and set $D_\alpha := U_\alpha \cap Q$ and $Z_\alpha := Z^l$; by J.2.2.5, $m(U_\alpha/D_\alpha) = m(U_\alpha^*) = 1$, while $U_\alpha \cap H_C = V$ with $m(U_\alpha/V) = 2$, so that $D_\alpha \not\leq H_C$. As H is transitive on $(Q/H_C)^\#$ of order 15, all involutions in $Q - U$ are conjugate to a member of wH_C, for $w \in D_\alpha - H_C$. Thus there are $15 \cdot |I(wH_C)|$ involutions in $Q - U$.

By J.2.2.4, $D := C_U(w)$ is of order 16, $[w, U] = Z$, and $[D_\alpha, H_C] \leq [S, S] = V$. Notice D contains V since $V \leq D_\alpha$. By J.2.4.3, $[\tilde{H}_C, w] \cong E_4$, so as $Z = [U, w] \leq [H_C, w] \leq V$, we conclude that $[H_C, w] = V$. Also for $h \in H_C - U$, w does not invert h as $[\tilde{h}, w] \neq 1$. Thus $I(wH_C) = wC_U(w) = wD$ is of order 16, and hence there are 240 involutions in $Q - U$.

Next for $d \in D - V$, $H_C D_\alpha$ is a subgroup of $C_S(d)$ of order 2^8, so $C_S(d) = H_C D_\alpha$ by J.2.3.6. We just saw that $I(wH_C) = wD$, so $UD_\alpha = \Omega_1(C_S(d)) \cong E_8 \times D_8$. Let $y \in Q - C_G(Z_\alpha)$; then $D_\alpha^y \cap D_\alpha = V$ by J.2.3.9, while y acts on wH_C, so that $w^y \in I(wH_C) = wD$. Thus $w^y \in wD - D_\alpha = wdV$, so $[w, Q] = D$

as $[w, H_C] = V$. Therefore Q is transitive on wD, completing the proof of (1). By (1), all involutions in $(S \cap Q) - U$ are fused into U_α under L_1, so (2) holds. Finally as $C_S(d) = H_C D_\alpha \leq Q$, and each involution in $U - V$ is fused to d in L, J.2.6.3 implies (3). □

LEMMA J.2.9. *(1) $C_G(s) = S_0 \times L_s$, where $L_s \cong Sz(8)$.*
(2) $N_G(S_0) = C_G(s)B$, and B induces outer automorphisms on L_s.
(3) $z^G \cap C_G(s) = I(L_s)$.

PROOF. Set $G_s := C_G(s)$. By J.2.7.10, $T_s := C_S(s) = S_0 S_1 \in Syl_2(G_s)$, and $T_s = C_H(s)$, where $S_1 := [C_S(s), B_7]$. By parts (1) and (5) of J.2.3, $T_s = S_0 \times S_1$ with $S_0 \cong E_4$, and S_1 is a Suzuki 2-group. Thus $V = \Omega_1(S_1)$.

Set $L_s := O^2(G_s)$. By Generalized Thompson Transfer A.1.37.2, $S_0 \cap L_s = 1$, so $S_1 \in Syl_2(L_s)$. Further as $S_1 \leq C_{L_s}(z) \leq C_H(s) = S_0 S_1$, $C_{L_s}(z) = S_1$ since $S_0 \cap L_s = 1$. Therefore as B_7 is transitive on $V^\#$, $C_{L_s}(v) = S_1$ for all $v \in V^\#$, and V is a TI-subgroup of L_s by I.6.1.1.

By J.2.7.8, $s^G \cap H = s^H$, so $z^G \cap G_s = z^{G_s}$ by A.1.7.1. By J.2.7.7, there is $t \in s^G \cap N_G(B)$ with $X := N_G(B) \cap C_G(t) \cong \mathbf{Z}_2 \times Sz(2)$. In particular, the members of $z^G \cap X$ are not contained in $O_2(C_G(t))$, so $z \notin O_2(G_s)$ as $z^G \cap G_s = z^{G_s}$. Thus V is not normal in L_s, so $N_{L_s}(V)$ is a strongly embedded subgroup of L_s as V is a TI-subgroup of L_s and $V = \Omega_1(S_1)$. Therefore as L_s is a SQTK-group, $L_s \cong Sz(8)$. As S_0 centralizes the Borel subgroup $B_7 S_1$ of L_s, $[L_s, S_0] = 1$, establishing (1). As $S_0^\#$ is fused in G, it follows that $L_s = L_{s_0}$ for each $s_0 \in S_0^\#$. Hence B acts on L_s, and as B is faithful on S_1, B is faithful on L_s, so (2) follows. As $[s_0, S] = V$ for all $s_0 \in S_0^\#$ by J.2.3.6, each element of $S_0 V - V$ is fused into S_0 under S, so (3) holds. □

LEMMA J.2.10. *Let $p := 7$ or 13, and $Y \in Syl_p(O^2(C_G(s)))$. Then*
(1) $Y \in Syl_p(G)$ is of order p.
(2) $N_G(Y) \leq N_G(S_0)$.
(3) $YS_0 = C_G(Y)$, and $Aut_G(Y)$ is cyclic of order $p - 1$.

PROOF. Notice (2) and J.2.9 imply (1) and (3), so it suffices to establish (2). By J.2.9, $S_0 \times Y = C_{G_s}(Y)$ and $S_0 = C_G(Y) \cap N_G(S_0)$. Thus $S_0 \in Syl_2(C_G(Y))$, and then by Burnside's Normal p-Complement Theorem 39.1 in [**Asc86a**], $C_G(Y) = O(C_G(Y))S_0$. Using Generation by Centralizers of Hyperplanes A.1.17 and $C_{G_s}(Y) = S_0 \times Y$, we conclude that $Y = O(C_G(Y))$. Finally since $Aut_{N_G(S_0)}(Y) = Aut(Y)$, (2) holds. □

J.3. The order of a group of type Ru

In this section, we continue the hypotheses and notation of section J.2. We calculate the order of our group G of type Ru using the order estimate in A.6.5.

THEOREM J.3.1. *For each group G of type Ru, $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$.*

We first establish some preliminary numerical results. By J.2.7.4, G has two classes of involutions with representatives z and s. Moreover by J.2.9.3:

LEMMA J.3.2. *If $u, v \in z^G$ with $uv = vu \neq 1$, then $uv \in z^G$.*

Observe that J.3.2 gives hypothesis (*) of A.6.5, with z, s in the roles of "z, t". Also J.2.9.3 shows:

LEMMA J.3.3. $|z^G \cap C_G(s)| = |I(L_s)| = 5 \cdot 7 \cdot 13 = 455$.

LEMMA J.3.4. $|z^G \cap H| = 1391$.

PROOF. From J.2.6.1, J.2.7.8, J.2.8.1, and J.2.6.3:

$$z^G \cap H = j^H \cup i^H \cup u^H \cup U^\#,$$

where $j \in H - K$, $i \in K - Q$, and $u \in Q - U$; further j^H, i^H, u^H, and $U^\#$ are of orders 640, 480, 240, and 31, respectively. Hence the lemma holds. □

We conclude from J.3.3, J.3.4, and A.6.5 that $|G : H| \leq 455(455 + 1391) = 455 \cdot 1846$. So as $1846 < 1890 = 42 \cdot 45$, we obtain:

LEMMA J.3.5. $|G : H| \leq 5 \cdot 7 \cdot 13 \cdot 1846 < m_H \cdot 42$, *where* $m_H := 3^2 \cdot 5^2 \cdot 7 \cdot 13$.

We are now in a position to establish Theorem J.3.1. Observe by J.2.7 and J.2.10 that $|G : H| = m_H \cdot k_H$ for some odd integer k_H, and by J.3.5, $k_H < 42$. Thus it remains to show that $k_H = 29$. Indeed it suffices to show $k_H \equiv 29 \equiv 3 \mod 13$: for by parts (3) and (11) of J.2.7, $|G|_3 = 27$, so $k_H \neq 3$. Thus if $k_H \neq 29$, then as k_H is odd, $k_H \geq 29 + 13 = 42$, contrary to J.3.5.

Let $Y \in Syl_{13}(L_s)$. By J.2.10, $Y \in Syl_{13}(G)$ and

$$|N_G(Y) : Y| = 48 \equiv -4 \mod 13.$$

Thus by Sylow's Theorem, $|G : Y| \equiv -4 \mod 13$. But

$$|H| = 2^{14} \cdot 3 \cdot 5 \equiv -5 \mod 13,$$

and $m_H/13 \equiv 2 \mod 13$. So as $|G : Y| \cdot 13 = |G| = |H| m_H k_H$,

$$k_H \equiv \frac{-4}{2 \cdot (-5)} \equiv \frac{2}{5} \equiv 3 \mod 13$$

completing the proof of Theorem J.3.1.

We close this section by determining the normalizer of a Sylow 5-subgroup of G:

LEMMA J.3.6. $N_G(B_5) = PR$, *where* $P := F^*(N_G(B_5)) \cong 5^{1+2}$ *is a Sylow 5-subgroup of* G, *and* $R \cong \mathbf{Z}_4$ *wr* $\mathbf{Z}_2$ *with* $z \in Z(R)$.

PROOF. By J.2.7, $C_G(B_5) = PC_Q(B_5)$, with $P := O(C_G(B_5))$ of order 5^3 and $C_Q(B_5) \cong Q_8$. Then as $Aut(B_5) = Aut_H(B_5) \cong \mathbf{Z}_4$, $N_G(B_5) = PR$ with R a subgroup of H of order 32. Now R acts faithfully and irreducibly on P/B_5, so $P/B_5 \cong E_{25}$; hence $P \cong 5^{1+2}$ or E_{5^3}, and R is a Sylow 2-subgroup of $GL_2(5)$, so $R \cong \mathbf{Z}_4$ wr $\mathbf{Z}_2$.

By Theorem J.3.1, $P \in Syl_5(G)$. Further if $P \cong 5^{1+2}$ then $N_G(P) = N_G(B_5) = PR$, so the lemma holds. Thus we may assume that $P \cong E_{5^3}$, and it remains to derive a contradiction.

Let $M_P := N_G(P)$. Then $H \cap M_P$ acts on $C_P(z) = B_5$, so $PR = H \cap M_P$. Then as $Z = \Omega_1(Z(R))$, $R \in Syl_2(M_P)$. Also $C_G(P) \leq C_G(B_5) = B_5 C_Q(B_5)$, so $C_G(P) = P$ and $M_P/P \leq GL_3(5)$. As R is Sylow in M_P and $|M_P/P|$ is coprime to 5, we conclude that either $M_P = RP$ or $|M_P/P| = 3 \cdot 32$. But by Sylow's Theorem $|G : N_G(P)| \equiv 1 \mod 5$, so the latter case is impossible.

Therefore $M_P = RP = N_G(B_5)$. But from the proof of J.2.7, R contains $z^g \neq z$, so $C_P(z^g) \in B_5^G$; then by Burnside's Fusion Lemma A.1.35 applied to P, $C_P(z^g) \in B_5^{M_P}$, contradicting $B_5 \trianglelefteq M_P$. □

J.4. A ${}^2\mathbf{F}_4(\mathbf{2})$-subgroup

In this section, we continue the hypotheses and notation of section J.2. Our object in this section is to construct a subgroup of G isomorphic to ${}^2F_4(2)$.

Let V_2 denote the T-invariant 4-subgroup of V. Set $\hat{H} := H/U$, $L_2 := O^2(N_L(V_2))$, and $R_2 := O_2(L_2T)$. Pick $g \in L_2 - H$, write U_α for U^g, Z_α for Z^g, and let $D_\alpha := U^g \cap Q$. By J.2.2.5, there is $v := v_\alpha \in U_\alpha - D_\alpha$.

LEMMA J.4.1. *(1)* $\langle v^* \rangle = U_\alpha^* = Z(T^*)$ *is of order* 2.

(2) v *inverts some* $B_5 \in Syl_5(H)$, *and* v^* *inverts exactly two members of* $Syl_5(H^*)$.

(3) Let $Q_5 := [Q, B_5]$. *Then* $U \leq Q_5$, $\hat{Q}_5 \cong E_{16}$, $C_Q(B_5) \cong Q_8$, *and* $\hat{Q} = \widehat{C_Q(B_5)} \oplus \hat{Q}_5$.

(4) $Z(\hat{H})$ *is of order* 2.

(5) $V = U_\alpha \cap H_C$.

(6) $\hat{D}_\alpha = [\widehat{C_Q(V_2)}, U_\alpha] \leq \hat{Q}_5$ *is of order* 2.

(7) $[\hat{Q}, U_\alpha] \leq \hat{Q}_5$.

PROOF. By J.2.2.5, U_α^* is of order 2. Further as $g \in L_2 - H$, $V_2 = ZZ_\alpha$, so as V_2 is T-invariant, so is $\tilde{Z}_\alpha$. Then as $[Z_\alpha, Q] = Z$ by J.2.2.4, $T^* = C_{H^*}(\tilde{Z}_\alpha) = C_H(Z_\alpha)^*$, so as $U_\alpha \trianglelefteq H^g$, $U_\alpha^* = Z(T^*)$, establishing (1). The final remarks in (2) follow from (1) and the structure of S_5, and then the first remark follows from the Baer-Suzuki Theorem.

As $Q_5 \trianglelefteq Q$, $Z = Z(Q) \leq Q_5$, so $U = [U, B_5]Z \leq Q_5$. By J.2.7.5, $C_Q(B_5) \cong Q_8$. The remainder of (3) follows from J.2.2.3, as does (4). Part (5) follows from J.2.2.5. Then by (1) and (5), $\hat{D}_\alpha$ is of order 2. From the action of K on $\hat{Q}$, $m(\hat{Q}/C_{\hat{Q}}(U_\alpha)) = 2$, so as $m(Q/C_Q(V_2)) = 1$ by J.2.2.4, and $[C_Q(V_2), U_\alpha] \leq Q \cap U_\alpha = D_\alpha$, the equality in (6) holds. As $\hat{Q} = C_{\hat{Q}}(K)\hat{Q}_5$, and $D_\alpha \leq Q \leq K$ by J.2.2.1, it follows that $[\hat{Q}, U_\alpha] \leq \hat{Q}_5$, so (7) holds, as does (6). □

Let Q_0 be the preimage of $Z(\hat{H})$ in H, $W_0 := \langle U^{L_2} \rangle$, and $W := W_0 Q_0$. By J.4.1, we may choose $B_5 \in Syl_5(H)$ inverted by $v =: v_\alpha$; set $Q_5 := [Q, B_5]$.

LEMMA J.4.2. *(1)* $T = R_2 Q$.

(2) $\hat{W} \cong E_{16}$ *and* $|W| = 2^9$.

(3) $Q_5/(Q_5 \cap W) \cong E_4$.

PROOF. First $T = R_1 R_2$ from (Ru1), and $R_1 = SQ$ by J.2.2.1, so (1) holds.

Next $L_1 T = N_L(U)$, with $R_2 = L_2 S \cap L_1 T$ of index 3 in $L_2 S$, so $L_2 T$ induces S_3 on $U^{L_2} = \{U, U_\alpha, U_\alpha^t\}$ for $t \in T - R_2$, and hence $W_0 = UU_\alpha U_\alpha^t$. Thus as $T = R_2 Q$, $W_0 = UU_\alpha [U_\alpha, Q]$. Finally by J.4.1.1 and the structure of the K-module $\hat{Q}$ (cf. Remark 14.7.64), $[\hat{Q}, U_\alpha] \cong E_4$, with $[\hat{Q}, U_\alpha] \cap \hat{H}_C = 0$. Thus as $\hat{D}_\alpha \leq [\hat{Q}, U_\alpha]$ with $|U_\alpha : D_\alpha| = 2$, $|W| = 2^9$. Then as U_α^* centralizes $[\hat{Q}, U_\alpha]\hat{Q}_0 \cong E_8$, (2) holds. Indeed $[\hat{Q}, U_\alpha] \leq \hat{Q}_5$ by J.4.1.7, so $\hat{W}_0 \cap \hat{Q}_5 \geq [\hat{Q}, U_\alpha] \cong E_4$. Then as $\hat{Q}_0 \cap \hat{Q}_5 = 0$, (3) holds. □

Set $L^+ := L/V$, $S_2^+ := [S^+, R_2]$, $G_2 := L_2 T$, and $G_2^- := G_2/S_2$, where S_2 is the preimage of S_2^+ in S.

LEMMA J.4.3. *(1)* $m(S_2^+) = 6$.

(2) $S_2 = W$.

PROOF. First the Steinberg module S^+ for $L/S \cong L_3(2)$ is a free T/S-module, so as $m(R_i/S) = 2$ for $i = 1, 2$, $m(C_{S^+}(R_i)) = 2 = m(S^+/[S^+, R_i])$. In particular, (1) holds. Further as $m(S_2^+) = 6$ and R_1/S is a 4-group, $m(C_{S_2^+}(R_1)) \geq 2$, so $C_{S_2^+}(R_1) = C_{S^+}(R_1)$. Thus $U \leq S_2$ by J.2.3.8, so $W_0 = \langle U^{G_2} \rangle \leq S_2$ as G_2 acts on S_2. Finally by J.4.2.1, $T = R_2 Q$, so $\hat{Q}_0 = [\hat{H}_C, T] = [\hat{H}_C, R_2]$, and hence $Q_0^+ \leq [S^+, R_2] = S_2^+$ as $H_C \leq C_T(V) = S$ and $U \leq S_2$. Thus $W = Q_0 W_0 \leq S_2$, while by J.4.2.2, $|W| = 2^9 = |S_2|$, so (2) holds. □

LEMMA J.4.4. *Let $X \in Syl_3(G_2)$.*

(1) $Q_5^- \cong E_4$, and $Q_5^- \cap R_2^- = \langle a^- \rangle \leq Z(T^-)$, with a^- of order 2.

(2) $A^- := \langle (a^-)^{G_2} \rangle \cong E_4$ is a complement to S^- in R_2^-, and X is irreducible on R_2/A.

(3) Let A denote the preimage of A^- in G_2. Then A acts on $B_5 Q_5$ and Q_5, and $AB_5Q_5/O_2(ABQ_5) \cong Sz(2)$.

(4) $AQ_5S = T$.

(5) Q_5 acts on XA for some $X \in Syl_3(G_2)$, and $Q_5XA/A \cong L_2(2)$.

PROOF. As $Q = Q_5 H_C$, $R_1 = QS = Q_5 S$ from J.2.2.1. Also

$$Q_5^- = Q_5 S_2/S_2 \cong Q_5/(Q_5 \cap S_2) = Q_5/(Q_5 \cap W) \cong E_4$$

by J.4.3.2 and J.4.2.3. Therefore as $|R_1 : S| = 4$, Q_5^- is a complement to S^- in R_1^-. As $m((R_2 \cap R_1)/S) = 1$, S^- is a hyperplane of $R_2^- \cap R_1^-$, so $Q_5^- \cap R_2^-$ is of order 2. Let a^- denote the involution generating $Q_5^- \cap R_2^-$.

As X is transitive on $(S^-)^\#$ and $(R_2/S)^\#$, and a^- is an involution in $R_2^- - S^-$, it follows that $\Phi(R_2^-) = 1$, and (2) holds if $a^- \in Z(T^-)$. Then as $T = R_1 R_2 = Q_5 S R_2 = Q_5 R_2$, and Q_5^- and R_2^- are elementary abelian, $R_2^- \cap Q_5^- \leq Z(T^-)$. Thus (1) and (2) are established. Further $[A^-, Q_5] = C_{A^-}(Q_5^-) = \langle a^- \rangle$, so $[A, Q_5] \leq Q_5 S_2 = Q_5 W$ by J.4.3.2. Thus A acts on $Q_5 W$, so A acts on $Q_5 W \cap Q = Q_0 Q_5$, and hence also on $O^2(N_H(Q_0 Q_5)) = B_5 Q_5$ and on $O_2(B_5 Q_5) = Q_5$.

We saw in paragraph one that $R_1 = Q_5 S$, and by (2), $R_2 = AS$, so $T = R_1 R_2 = AQ_5 S$, establishing (4). By (4), $A^* S^* = T^*$, with $S^* < T^*$ since S^* is abelian, so that $A^* \not\leq S^*$; so as $1 \neq W^* = S_2^* \leq S^*$ and A^* act on B_5^*, $|N_{T^*}(B_5^*)| > 2$. Thus as $N_{H^*}(B_5^*) \cong Sz(2)$, $A^* W^* B_5^* = N_{H^*}(B_5^*)$, completing the proof of (3). Part (5) follows from the Baer-Suzuki Theorem, since $|Q_5 A : A| = 2$ and $|G_2| = 2^{14} \cdot 3$. □

We next define certain subgroups of G that provide an amalgam, which we will show generates a subgroup of G isomorphic to $^2F_4(2)$.

Set $T_F := AQ_5$ and $F_1 := T_F B_5$; observe T_F and F_1 are subgroups of G by J.4.4.3.

By J.4.4.5, Q_5 acts on XA for some $X \in Syl_3(G_2)$. Set $F_2 := T_F X$; then F_2 is a subgroup of G.

By J.4.4.3, $F_1/O_2(F_1) \cong Sz(2)$, and by J.4.4.5, $F_2/O_2(F_2) \cong L_2(2)$. In particular for $i = 1, 2$, T_F is self-normalizing in F_i, and $F^*(F_i) = O_2(F_i)$. Set $F := \langle F_1, F_2 \rangle$. To establish Hypothesis F.1.1 with F, F_1, F_2, T_F in the roles of "G_0, L_1, L_2, S", it remains only to verify condition (e) of that hypothesis, which we do next:

LEMMA J.4.5. *(1) $O_2(F) = 1$.*

(2) $\alpha := (F_1, T_F, F_2)$ is the amalgam of $^2F_4(2)$, and the inclusion map from α into F is a faithful completion.

(3) $O_2(F_1) = Q_5 Q_0 = F_1 \cap Q$.

(4) $G_2 = F_2S$, $O_2(F_2) = A$, $F_2 \cap S = W = S_2$, *and* F_2 *is irreducible on* $R_2/O_2(F_2)$.

PROOF. Set $K_F := O_2(F)$. If (1) fails then $C_{K_F}(T_F) \neq 1$. But using J.4.1.3, $U \leq Q_5Q_0 \leq T_F$, with $C_G(U) = H_C$ and $C_{H_C}(Q_0Q_5) = Z$ by J.2.4 and J.2.2.4. So as Z is of order 2, we conclude that $Z \leq K_F$. Then $V_2 = \langle Z^{F_2} \rangle \leq K_F$, so that $U = \langle V_2^{F_1} \rangle \leq K_F$. Therefore $W_0 = \langle U^{F_2} \rangle \leq K_F$. This is impossible, as $U^g \not\leq O_2(F_1)$ for $g \in L_2 - L_1$ by J.2.2.5. So (1) is established.

In particular we have established part (e) of Hypothesis F.1.1. Thus by F.1.9, $\alpha := (F_1, T_F, F_2)$ is a weak BN-pair of rank 2, and as T_F is self-normalizing in F_i, α appears in the list of F.1.12.

Now $G_2 = XT$, so by J.4.4.4, $G_2 = XAQ_5S = F_2S$. Therefore $F_2/(S \cap F_2) \cong G_2/S$ is isomorphic to the symmetric group S_4. By construction, $A \trianglelefteq F_2$ and $F_2/A \cong S_3$. By J.4.4.2, F_2 is irreducible on $R_2/O_2(F_2)$. Again by construction of A in J.4.4.2, $A \cap S = S_2 = W$ by J.4.3.2; thus as $S \cap F_2 \leq O_2(F_2) = A$, $S \cap F_2 = W$, establishing (4). Then using J.4.2.2,

$$|T_F| = |T_F : A||A : W||W| = 2 \cdot 4 \cdot 2^9 = 2^{12}.$$

Therefore α is the amalgam of ${}^2F_4(2)$, since $F_1/O_2(F_1) \cong Sz(2)$, so (2) follows.

As $F_1^* \cong Sz(2)$ has a Sylow 2-subgroup isomorphic to $\mathbf{Z}_4$, and $|T_F| = 2^{12}$, $O_2(F_1) = T_F \cap Q$ has order 2^{10}. Then as $O_2(F_1)$ contains Q_5Q_0 of order 2^{10}, (3) holds. □

DEFINITION J.4.6. Form the coset geometry Γ of the completion F of the amalgam α in J.4.5 as in Definition F.3.1, and adopt the notational conventions of section F.4 such as Definition F.4.1 for the Lie amalgam α. In particular, write x and l for F_1 and F_2 regarded as points and lines in the geometry Γ, and for $y \in \Gamma$, write F_y for the stabilizer in F of y. Thus $F_x = F_1$ and $F_l = F_2$. Recall that Γ_1 and Γ_2 are the F-orbits of x and l, respectively. For $y \in \Gamma_1$, $y = xg$ for some $g \in F$, and we set $z(y) := z^g$; thus $z(y)$ is a generator for $Z(F_y)$, since we saw in the proof of J.4.5.1 that $Z = Z(T_F)$.

The main result in this section is:

THEOREM J.4.7. $F \cong {}^2F_4(2)$.

During the remainder of the section, we work toward a proof of this Theorem using Theorem F.4.8.

Let $\bar{\alpha} := (\bar{F}_1, \bar{F}_{1,2}, \bar{F}_2)$ be the amalgam of parabolics in $\bar{F} := {}^2F_4(2)$, and $\beta : \bar{\alpha} \to \alpha$ an isomorphism. For $E \subseteq F_i$ let $\bar{E} := \beta^{-1}(E)$.

Let $x_4 \in \Gamma^4(x)$; recall this means x_4 is at distance 4 from x in the graph Γ. Set $u := z(x_4)$; thus $u \in z^F$. Computing in the generalized octagon $\bar{\Gamma}$ for $\bar{F}$, $\bar{u}$ inverts a Sylow 5-subgroup of $\bar{F}_1$, so applying β and conjugating in F_1, we may take $u \in T_F$ to invert B_5, and $N_{F_1}(B_5) = B_5I$, where $u, z \in I \cong \mathbf{Z}_4^2$. By J.3.2, $uz \in z^G$. Next by J.3.6, $N_G(B_5) = PR$, where $P := F^*(N_G(B_5)) \cong 5^{1+2}$, and $R \in Syl_2(N_G(B_5))$ is isomorphic to $\mathbf{Z}_4$ wr $\mathbf{Z}_2$ with $z \in Z(R)$. Let $I_z := Q_0 \cap I$, and observe that I_z is the cyclic subgroup of order 4 in I inverted by elements in $R - I$. Moreover I is the unique abelian subgroup of R of index 2, so I is weakly closed in R with respect to G.

Next, as we saw in the proof of J.2.7.5, but now using u in the role of "i", $P = B_5B_uB_{uz}$, where $B_v := C_P(v) \cong \mathbf{Z}_5$ for $v = u$, uz. Observe $I = C_R(u)$

acts on P and hence on B_u. Further $(u, B_u) \in (z, B_5)^G$, and so by the previous paragraph, I is weakly closed in a suitable $R_u \in Syl_2(C_G(u) \cap N_G(B_u))$ with respect to $C_G(u) \cap N_G(B_u)$, and $I_u = I \cap Q_0^f \cong \mathbf{Z}_4$, where $z^f = u$. Then R and R_u induce transvections on $\langle u, z\rangle$ with centers z, u, respectively, so $\langle R, R_u\rangle$ induces $L_2(2)$ on $\langle u, z\rangle$.

For $g \in F$, let $\mathcal{F}(z^g) := (F_1^g)^{H^g}$, and for $v \in \langle u, z\rangle^\#$, let

$$\mathcal{E}(v) := \{E \in \mathcal{F}(v) : \langle u, z\rangle \leq E\}.$$

Now $F_1 = IB_5Q_5$ with $Q_5 = [Q, B_5]$, and I is weakly closed in a Sylow 2-group R of $N_H(B_5)$, so as $H = C_G(z)$, $F_1 =: \theta(z, B_5)$ is canonically determined by the pair (z, B_5). Further by J.4.1.2, u^* acts on exactly two subgroups of H^* of order 5, so $|\mathcal{E}(z)| = 2$. Thus

$$\mathcal{E} := \bigcup_{v \in \langle u,z\rangle^\#} \mathcal{E}(v)$$

is of order 6 since $\langle z\rangle = Z(F_1) \cap \langle u, z\rangle$. Further $IB_v[B_v, O_2(C_G(v))] = \theta(v, B_v) \in \mathcal{E}(v)$.

Next by symmetry between $B_z := B_5$ and B_u, $O_5(N_G(B_u)) = B_zB_uB'_{uz}$, where $B'_{uz} := C_{O_5(N_G(B_u))}(uz)$. Further $[B_{uz}, B_uB_z] = B_z$ and $[B'_{uz}, B_uB_z] = B_u$, so $K_{uz} := \langle B_{uz}, B'_{uz}\rangle$ induces $SL_2(5)$ on B_uB_z and centralizes uz; thus $K_{uz} \cong SL_2(5)$ as B_uB_z is self-centralizing in G by J.3.6. In particular $B_{uz} \neq B'_{uz}$, so $\theta(uz, B_{uz}) \neq \theta(uz, B'_{uz})$, and hence

$$\mathcal{E}(uz) = \{\theta(uz, B_{uz}), \theta(uz, B'_{uz})\}.$$

Also B_{uz} and B'_{uz} centralize I_{uz}, so K_{uz} centralizes I_{uz}.

By symmetry: $\mathcal{E}(u) = \{\theta(u, B_u), \theta(u, B'_u)\}$, where $B'_u := C_G(u) \cap O_5(N_G(B_{uz}))$;

$$B_5 \neq B_{5,v}, \quad \text{where} \quad B_{5,v} := C_{O_5(N_G(B'_v))}(z) \quad \text{for} \quad v \in \{u, uz\};$$

and $K_{z,v} := \langle B_5, B_{5,v}\rangle \cong SL_2(5)$ centralizes I_z. Now for $r \in R - I$, $(B_uB'_{uz})^r = B_{uz}B'_u$, so $K_{z,u}^r = K_{z,uz}$. On the other hand, using J.2.4.3, $C_G(I_z) = IUK_{z,u} = IUK_{z,uz}$, and as $C_U(B_5) = Z$, B_5 is contained in a unique conjugate of $K_{z,u}$ under $C_G(I_z)$. Hence $K_{z,u} = K_{z,uz}$ is r-invariant, and therefore $B'_z := B_{5,u} = B_{5,uz}$ is the I-invariant subgroup of $K_z := K_{u,z}$ of order 5, other than B_5 (cf. J.4.1.2). Therefore B'_z is r-invariant. In particular:

LEMMA J.4.8. *$\mathcal{B} := \{B_v, B'_v : v \in \langle u, z\rangle^\#\}$ is of order* 6.

LEMMA J.4.9. *(1) The commuting graph on $\mathcal{B}$ makes $\mathcal{B}$ into a* 6*-gon with $\langle B_1, B_2\rangle$ isomorphic to 5^{1+2} or $SL_2(5)$ for B_2 at distance* 2 *or* 3 *from B_1, respectively.*

(2) Let

$$W(\mathcal{B}, I) := \langle N_G(I) \cap N_G(B'), N_G(I) \cap N_G(B'B'') : B', B'' \in \mathcal{B} \ \text{ and } \ [B', B''] = 1\rangle.$$

Then $W(\mathcal{B}, I)$ acts transitively as D_{12} on the 6*-gon $\mathcal{B}$ with kernel I.*

PROOF. Part (1) is immediate from the discussion above. Similarly from that discussion, $R := N_G(B_5) \cap N_G(I)$ acts on $\mathcal{B}$ as the reflection fixing B_5 and B'_5, with kernel I. Further $K_{uz}I$ acts as $GL_2(5)$ on B_5B_u, so $R' := N_{K_{uz}I}(I)$ is a Sylow 2-group of $K_{uz}I$, interchanging the following pairs: the two I-invariant $\mathbf{F}_5$-points B_5 and B_u of B_5B_u; the I-invariant points B'_{uz} and B_{uz} in the I-invariant Sylow 5-groups of $K_{uz}I$; and the I-invariant points B'_z and B'_u in $O_5(C_G(B'_{uz})$ and

$O_5(C_G(B_{uz})$. Thus R' induces the reflection on $\mathcal{B}$ fixing the edges B_uB_z and $B'_uB'_z$, so (2) holds. □

LEMMA J.4.10. *$\mathcal{E}$ is a 6-gon, where E_1 and E_2 in $\mathcal{E}$ are adjacent if there exist $\langle u,z\rangle$-invariant subgroups B_i of order 5 in E_i with $[B_1,B_2]=1$.*

PROOF. Let $v \in \{u, uz\}$. Observe first that as $\langle u,v\rangle = \Omega_1(I)$, and $I \in Syl_2(N_{F_1}(B_5))$, by Sylow's Theorem and A.1.7.1, $N_{F_1}(\langle u,z\rangle)$ is transitive on the $\langle u,z\rangle$-invariant subgroups of F_1 of order 5.

We claim that if B_5 centralizes some subgroup B of order 5 in $E \in \mathcal{E}(v)$, then B is the unique subgroup of order 5 in E such that $\langle B_5, B\rangle$ is a 5-group. We first show that this claim establishes the lemma: For suppose $[B_1,B_2]=1$ for some $\langle u,z\rangle$-invariant subgroups B_i of order 5 in $E_i \in \mathcal{E}$. By J.4.9 and paragraph one, we may take $B_1 = B_5$ and $E_1 = F_1$. Then as $\langle B_5, B'_v\rangle \cong 5^{1+2}$ for $v \in \{u, uz\}$, by the uniqueness of B_2 in the claim, $E_2 \neq \theta(v, B'_v)$. Further by J.4.5.3, $O^2(\theta(z,B'_z)^*) = (B'_z)^*$ and $O^2(F_1^*) = B_5^*$, so B_5 commutes with no subgroup of order 5 in $\theta(z, B'_z)$, and hence E_2 is $\theta(v, B_v)$ where v is u or uz. Thus the map $\theta(w,B') \mapsto B'$ is an isomorphism of the graphs $\mathcal{E}$ and $\mathcal{B}$, and hence the lemma holds. This completes the proof that the claim is sufficient.

Thus it remains to prove the claim. Suppose B_1 and B_2 are distinct subgroups of E of order 5, with $[B_5,B_1]=1$ and $\langle B_5,B_2\rangle$ a 5-group. Then from the structure of E, $1 \neq O_2(\langle B_1,B_2\rangle) \not\leq Z(\langle B_1,B_2\rangle)$, while $P := O^{5'}(C_G(B_5))$ is a 5-group by J.2.7.5, so $[B_5,B_2] =: D_2 \neq 1$. Therefore as a Sylow 5-group of G is isomorphic to 5^{1+2} by J.3.6, $P_2 := \langle B_5,B_2\rangle \cong 5^{1+2}$ and $D_2 = Z(P_2)$ is of order 5. Then B_5D_2 is normal in P_2 containing B_2, and in P containing B_1, so B_1 and B_2 act on B_5D_2. This is impossible, as $N_G(B_5D_2)/B_5D_2 \leq GL_2(5)$, with $1 \neq O_2(\langle B_1,B_2\rangle) \not\leq Z(\langle B_1,B_2\rangle)$. This contradiction completes the proof of the claim, and hence of the lemma. □

We are now in a position to establish Theorem J.4.7.

Let $p := y_0 \cdots y_6$ be a geodesic in Γ with $y_3 := l$, and let $z_i := z(y_{2i})$. Then $z_1, z_2 \in V_2$. By F.4.6.2, p corresponds to a geodesic $\bar{p}$ in the building $\bar{\Gamma}$ of $\bar{F}$. By F.4.27.4.i, $\bar{z}_i$ is a long-root involution, so from the structure of $\bar{F}_1$, $\bar{z}_{i+2}$ fixes $\bar{y}_{2i}$, and since $d(\bar{y}_0,\bar{y}_6)=6$, $\bar{z}_0,\bar{z}_3 \in O_2(\bar{F}_2)$, $|\bar{z}_0\bar{z}_3|=4$, and $\bar{t}=[\bar{z}_0,\bar{z}_3]$ is a short-root involution in $\bar{V}$, with $C_{\bar{F}}(\bar{t}) \leq \bar{F}_2$ by F.4.27.4.ii, so $C_{\bar{F}_2}(\bar{t}) = O_2(\bar{F}_2)\bar{F}_2$. We apply β to obtain the corresponding statements in F. In particular z_3 fixes y_2, so $y_0 \cdot y_1 \cdot y_2 \cdot y_1z_3 \cdot y_0z_3$ is a path in Γ of length 4 from y_0 to y_0z_3, and hence $d(y_0,y_0z_3)=4$. Thus by F.4.6.1, there is $f \in F$ with $(y_0,y_0z_3)f=(x,x_4)$, so $(z_0,z_0^{z_3})^f=(z,u)$; hence as $[z_0,z_3]=t$, $t^f=(z_0z_0^{z_3})^f=zu$. Let $j := z_3^f$, so that $xj = x_4$. For $y \in z^G$, let $G_y := C_G(y)$. Recall from the remarks at the beginning of the proof of Theorem J.4.7 that $uz \in z^G$. As $z \notin O_2(G_{uz})$, $z_0 \notin O_2(G_t)$.

Let $G_t^* := G_t/O_2(G_t)$. By construction, $z_0 \in O_2(C_{F_2}(t)) - O_2(G_t)$, and we saw $F_2 = O_2(F_2)C_{F_2}(t)$, so $C_{F_2}(t)^* \cong S_4$ and $1 \neq z_0^* \in O_2(C_{F_2}(t))^*$. Next by uniqueness of geodesics of length less than 8 in F.4.6.6, $y_2 = \Gamma(l) \cap \Gamma^2(y_0)$ and $y_4 = \Gamma(l) \cap \Gamma^2(y_6)$. Choose $f_i \in F$ so that $z^{f_i} = z_i$ and set $U(z_i) := U^{f_i}$. Then $z_0 \in U(z_1)$, $z_3 \in U(z_2)$, and by J.4.1.1, $O_2(C_{F_2}(t))^* = U(z_1)^* \times U(z_2)^* = \langle z_0^*, z_3^*\rangle$. Thus z_3^* interchanges the two subgroups of G_t^* of order 5 inverted by z_0^* supplied by J.4.1.2, so j interchanges the two subgroups of $G_{uz}/O_2(G_{uz})$ of order 5 inverted by z, and hence j interchanges the two members of $\mathcal{E}(uz)$, which are opposite in the 6-gon $\mathcal{E}$. Therefore j induces the reflection on $\mathcal{E}$ through the axis perpendicular to the axis through the members of $\mathcal{E}(uz)$. Thus j interchanges F_1 with a member of $\mathcal{E}(u)$ which is adjacent to F_1 in the 6-gon $\mathcal{E}$, and of course as $z^j = u$, that member is $F_1^j = F_{xj} = F_{x_4}$. Therefore by J.4.10, $[B_5, B_u] = 1$ for a suitable $B_u \leq F_{x_4}$. Now for $b \in B_u^{\#}$, $xb \in \Gamma^8(x)$, and as B acts on B_5 and $B_5 \leq F_x$, $B_5 = B_5^b \leq F_x^b = F_{xb}$, so that B_5 is transitive on $\Gamma(xb)$. Hence there is more than one geodesic in Γ from x to xb. That is, Γ is small in the sense of section F.4, so by part (1) of Theorem F.4.8, $F \cong {}^2F_4(2)$.

This completes the proof of Theorem J.4.7.

J.5. Identifying G as Ru

Let $L_2(25)^+$ denote the unique extension of $L_2(25)$ by an outer automorphism of order 2 with semidihedral Sylow 2-subgroups.

Definition J.5.1. Define a group G of permutations on a set Ω to be a *Rudvalis rank 3 group* if

(Rua) G is a rank 3 permutation group on Ω, and for $\omega \in \Omega$, $F := G_\omega \cong {}^2F_4(2)$.

(Rub) There is $\delta \in \Omega$ with stabilizer F_δ a parabolic subgroup of F of order $2^{12} \cdot 5$.

(Ruc) There is $\gamma \in \Omega$ with $F_\gamma \cong L_2(25)^+$.

Our first task is to show that each group G of type Ru is a Rudvalis rank 3 group on G/F. Then we prove that Rudvalis rank 3 groups are determined up to isomorphism; hence as Ru is such a group, each is isomorphic to Ru. This will establish Theorem J.1.1.

Initially in this section, we continue the hypotheses and notation of section J.2. Let $\Omega := G/F$, and write ω for F regarded as a point in Ω. Thus $F = G_\omega$. Theorem J.4.7 showed that $F \cong {}^2F_4(2)$, so $|F| = |{}^2F_4(2)| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 13$, and hence we conclude from Theorem J.3.1 that

Lemma J.5.2. $|\Omega| = 2^2 \cdot 5 \cdot 7 \cdot 29 = 4060$.

For $\alpha, \beta \in \Omega$, we let $G(\{\alpha, \beta\})$ denote the setwise stabilizer of $\{\alpha, \beta\}$.

Lemma J.5.3. *F has an orbit δF on Ω of length 1755 with $F_\delta = F_1$ and $G(\{\omega, \delta\}) = F_1 Q$.*

Proof. Observe F_1 that is of index 2 and in particular normal in $F_1 Q$, so $F_1 \leq F \cap F^h$ for $h \in Q - F_1$. As F_1 is maximal in F, it follows that $F_1 = F_\delta$, where $\delta := \omega h$. Hence $F_1 = G_{\omega,\delta}$, so the lemma follows. □

In the remainder of the section, define δ as in J.5.3.

Lemma J.5.4. *(1) $\Omega = \{\omega\} \cup \delta F \cup \gamma F$ with γF of length 2304, and $F_\gamma \cong L_2(25)^+$.*

(2) $G(\{\omega, \delta\}) \cong Aut(L_2(25))$.

(3) G is a Rudvalis rank 3 group on Ω.

Proof. Let $P \in Syl_5(F)$. Then $N_G(P) \not\leq F$ by J.3.6, so $P \leq F \cap F^g$ for some $g \in N_G(P) - F$. Let $\gamma := \omega g$. Then $\gamma F \subseteq \Sigma := \Omega - (\{\omega\} \cup \delta F)$, and $|\Sigma| = 2304 = 2^8 \cdot 3^2$ using J.5.2 and J.5.3. Thus

$$|F_\gamma| \geq \frac{|F|}{2304} = 2^4 \cdot 3 \cdot 5^2 \cdot 13 =: N.$$

But up to conjugacy, there are two maximal subgroups $N_F(P)$ and F_+ of F containing P, and $F_+ \cong L_2(25)^+$. As $|N_F(P)| < N$ and $|F_+| = N$, we conclude that F_γ is conjugate to F_+ and $\Sigma = \gamma F$, so that (1) and (2) hold. Then (1) and J.5.3 imply (3). □

In the remainder of this section, assume G is a Rudvalis rank 3 group on a set Ω, pick $\omega \in \Omega$, and let $F := G_\omega$, and δ and γ satisfy (Rub) and (Ruc). We appeal to the basic theory of rank 3 permutation groups (cf. section 16 of [**Asc86a**]), and adopt much of the usual notation used in that theory. For $g \in G$ let $\Delta(\omega g) := \delta F g$ and $\Sigma(\omega g) := \gamma F g$. From the theory of rank 3 groups, the relation * defined by $\omega * \delta$ if $\delta \in \Delta(\omega)$ is symmetric. Thus we can regard Ω as the (undirected) graph such that for $\rho \in \Omega$, $\Delta(\rho)$ is the set of vertices adjacent to ρ in Ω. Further $\Sigma(\rho)$ is the set of vertices at distance two from ρ in Ω.

We often abbreviate $\Delta := \Delta(\omega)$, and we set $\Delta(\omega, \delta) := \Delta \cap \Delta(\delta)$.

Lemma J.5.5. *(1) G is a rank 3 permutation group on Ω, with parameters $k := 1755$, $l := 2304$, $\lambda := 730$, and $\mu := 2^2 \cdot 3 \cdot 5 \cdot 13 = 780$.*

(2) We can identify Δ with the set Γ_1 of points in the building Γ of F so that $\Delta(\omega, \delta) = \{\rho \in \Delta : 0 < d_\Gamma(\delta, \rho) < 8\}$.

(3) F_δ has three orbits on $\Delta(\omega, \delta)$, of lengths 10, 80, and 640.

(4) F_γ is transitive on $\Delta(\omega, \gamma)$ of order 780, and on $\Delta \cap \Sigma(\gamma)$ of order 975, with $F_{\gamma,\sigma} \cong Sz(2)$ for $\sigma \in \Delta(\omega, \gamma)$, and $F_{\gamma,\tau} \in Syl_2(F_\gamma)$ for $\tau \in \Delta(\omega) \cap \Sigma(\gamma)$.

(5) F_γ is transitive on $\Delta(\gamma) \cap \Sigma(\omega)$ of order 975, and this is the unique orbit of F_γ on $\Sigma(\omega)$ of order 975.

Proof. As G is a Rudvalis rank 3 group on Ω, F has orbits $\Delta = \delta F$ and $\Sigma = \gamma F$ of lengths $k := |F : F_\delta| = 1755$ and $l := |F : F_\gamma| = 2304$, respectively. As $\Delta \cong F/F_1$, we can identify Δ with the set of points of the coset complex Γ of Definition J.4.6. Thus from F.4.6, F_δ has five orbits on Δ, of lengths 1, 10, 80, 640, and 2^{10}. From 16.3.2 in [**Asc86a**],

$$3^2 \cdot 2^8 = l = \frac{k(k-\lambda-1)}{\mu} = \frac{3^3 \cdot 5 \cdot 13 \cdot (k-\lambda-1)}{\mu},$$

and so

$$|\Sigma(\delta) \cap \Delta| = k - \lambda - 1 = \frac{2^8 \cdot \mu}{3 \cdot 5 \cdot 13}.$$

Thus $\Sigma(\delta) \cap \Delta$ is a union of orbits of F_δ on Δ, with the order of this union divisible by 2^8. From the lengths of the F_δ-orbits given above, it follows that $\Sigma(\delta) \cap \Delta$ is the orbit of length 2^{10}, so (1)–(3) follow. In particular, G is transitive on paths $\omega_0\omega_1\omega_2$ with $d_\Omega(\omega_0, \omega_2) = 2$, and we can choose as a representative the path $\delta\omega\rho$, where $\rho \in \Sigma(\delta) \cap \Delta$. Then by (2), ρ is *opposite* to δ in the building Δ, so $G_{\delta,\omega,\rho} \cong Sz(2)$. This transitivity implies that F_γ is transitive on $\Delta(\omega, \gamma)$ and $F_{\gamma,\sigma} \cong Sz(2)$ for $\sigma \in \Delta(\omega, \gamma)$, so $|\Delta(\omega, \gamma)| = |F_\gamma|/20 = 780$. This leaves $k - 780 = 975$ points in $\Delta(\omega) \cap \Sigma(\gamma)$. Let $\mathcal{O}_i$, $1 \leq i \leq r$, denote the orbits of F_γ on these points. If $u_i \in \mathcal{O}_i$, then $|F_{\gamma,u_i}|$ divides $|F_1| = 2^{12} \cdot 5$, so either F_{γ,u_i} is a 2-group, or $|F_{\gamma,u_i}| = 5 \cdot 2^a$ for some a. In the first case, $|F_{\gamma,u_i}| \leq |F_\gamma|_2 = 16$ using J.5.4.1, so $|\mathcal{O}_i| \geq |F_\gamma|_{2'} = 975$, with equality iff $F_{\gamma,u_i} \in Syl_2(F_\gamma)$. In the second case, from the subgroup structure of F_γ, $|F_{\gamma,u_i}| = 5$, 10, or 20, so $|\mathcal{O}_i| = 3120$, 1560, or 780. Thus the first case occurs, and the remainder of (4) holds.

By (4), G is transitive on triples $\eta := (\omega_1, \omega_2, \omega_3)$ with $\omega_2 \in \Delta(\omega_1)$ and $\omega_3 \in \Sigma(\omega_1, \omega_2) := \Sigma(\omega_1) \cap \Sigma(\omega_2)$; further $|G_\eta| = 16$. This implies the first remark in

(5), and leaves $l - 976 = 1328$ points in $\Sigma(\omega, \gamma)$. So as $1328 \equiv 2 \mod 13$, a Sylow 13-subgroup P_{13} of F_γ fixes a point $u \in \Sigma(\omega, \gamma)$. As $N_{F_\gamma}(P_{13})$ is of order $4 \cdot 13$, and is the unique maximal subgroup of F_γ containing P_{13}, $|uF_\gamma| = 300$, 600, or 1200, and is congruent to 1, 2, or 4 modulo 13, respectively. It follows that F_γ has either one orbit of length 600, or two of length 300, leaving 728 points. Thus there is no orbit of length 975 on $\Sigma(\omega, \gamma)$, completing the proof of (5). □

Write $Aut(\Omega)$ for the automorphism group of the graph Ω. The following theorem is our main tool in proving the uniqueness of Rudvalis rank 3 groups:

LEMMA J.5.6. *$G = Aut(\Omega)$.*

PROOF. Let $I := Aut(\Omega)$. Then $G \leq I$ and G is transitive on Ω, so it suffices to show that $F = G_\omega = I_\omega$. Further I_ω induces a group of automorphisms on the subgraph $\Delta = \Delta(\omega)$ of Ω, so it suffices to show that I_ω is faithful on Δ and $Aut(\Delta) = F$.

As F_γ is a maximal subgroup of F, F is primitive on $\Sigma := \Sigma(\omega)$, so

$$\{\gamma\} = \{u \in \Sigma : \Delta(\omega, u) = \Delta(\omega, \gamma)\}.$$

Hence the kernel $I_{\omega,\Delta}$ of the action of I_ω on Δ fixes each $\gamma \in \Sigma$, so $I_{\omega,\Delta} = 1$; that is I_ω is faithful on Δ.

Thus to complete the proof of J.5.6, it remains to establish that $F = Aut(\Delta) =: A$. Observe that $A = Aut(\Delta^c)$, where Δ^c is the complementary graph of Δ. By J.5.4, we can identify Δ with the set of points Γ_1 in the building Γ of F via $\delta g \mapsto F_1 g$ for $g \in F$. By J.5.5.2, $\Delta^c(\delta)$ is the set of opposites of δ in Γ.

It is presumably well-known that the opposite relation determines the building Γ; moreover this fact implies $A = Aut(\Gamma) = F$, completing the proof of J.5.6 (and hence of Theorem J.1.1, as we see at the end of the section). However for completeness, we also provide below a fairly detailed sketch of a proof that $F = A$. □

By the previous remarks, we must retrieve the lines of the building Γ from the opposite relation; this is more or less equivalent to identifying the set $\Gamma^2(u)$ of points at distance 2 in Γ from a point u. Thus we are led to the following definition: For $u \in \Delta$, let $\theta(u)$ denote the orbit of F_u on $\Delta(\omega, u)$ of length 10 discussed in the proof of J.5.5; subject to the identification in J.5.5.2, $\theta(u) = \Gamma^2(u)$.

The next lemma shows how the sets $\theta(u)$ are determined from the opposite relation defined by Σ:

LEMMA J.5.7. *(1) $\theta(\delta) = \{u \in \Delta(\omega, \delta) : |\Delta(u, \omega) \cap \Sigma(\delta)| = |\Sigma(\delta)|/2 = 2^9\}$.*

(2) Let $u \in \theta(\delta)$, $\{l\} = \Gamma(\delta) \cap \Gamma(u)$, and $\{u'\} = \Gamma(l) - \{\delta, u\}$. Then u' is the unique $w \in \theta(\delta)$ such that $\Delta(w, \omega) \cap \Sigma(\delta)$ is a complement to $\Delta(u, \omega) \cap \Sigma(\delta)$ in $\Delta \cap \Sigma(\delta)$.

We first show that J.5.7 implies that $F = A$: For by J.5.7.1, A_δ acts on $\theta(\delta)$. Then as F_2 is 2-transitive on the set $\Gamma(l)$ of three points on the line $l \in \Gamma_2$, it follows from J.5.7.2 that A permutes the collection $\{\Gamma(k) : k \in \Gamma_2\}$. Thus A acts on the geometry Γ' with point set Δ, line set $\{\Gamma(k) : k \in \Gamma_2\}$, and incidence equal to inclusion. Of course $\Gamma' \cong \Gamma$, so $F \leq A \leq Aut(\Gamma) = F$, completing the proof of the claim.

So to complete the proof of J.5.6, it remains to establish J.5.7. We will require some lemmas which allow us to count geodesics. We first establish some notation:

Let u denote a member of $\Delta(\omega,\delta)$, and set $d := d(\delta,u) = d_\Gamma(\delta,u)$, so that $d = 2$ or 4 or 6. Set $\Lambda(u) := \Delta(\omega,u) \cap \Sigma(\delta)$. Thus $\Lambda(u)$ consists of the points v which are opposite to δ in the building Γ, but not opposite to u, so that $d(u,v) = 2$ or 4 or 6.

Recall as Γ is a generalized octagon that there is a unique geodesic from u to any point v of Γ with $d(u,v) \leq 6$; write $p(v)$ for this geodesic and let $\mathcal{G}(u)$ denote the set of geodesics with origin u and end in $\Lambda(u)$. Since $p(v)$ is unique:

LEMMA J.5.8. *The map $v \mapsto p(v)$ is a bijection of $\Lambda(u)$ with $\mathcal{G}(u)$.*

Because of the uniqueness of such geodesics, we have the following convexity property:

LEMMA J.5.9. *If $y \in \Gamma$ with $e := d(\delta,y) < 8$, then there is a unique member of $\Gamma(y) \cap \Gamma^{e-1}(\delta)$.*

Given a path $q := w_0 \cdots w_s$ in Γ and $0 \leq i \leq i' \leq s$, we say that q *increases on* $[i,i')$ if $d(\delta,w_j) = d(\delta,w_i) + j$ for each $0 \leq j \leq i' - i$.

Let $p := y_0 \cdots y_r \in \mathcal{G}(u)$ and set $d_i := d(\delta,y_i)$ for $i \leq r$. Then $y_0 = u$, $r \leq 6$, $d_0 = d$, and $d_i \leq 8 = d_r$ since the path p ends in y_r opposite to δ.

LEMMA J.5.10. *(1) If $d_i < d_{i+1}$ then p increases on $[i, \min\{i+8-d_i, r\})$.*

(2) If $d_i = 8-(r-i)$ then p increases on $[i,r)$.

(3) If $d = 4$ then one of the following holds:

(a) $r = 4$ and p increases on $[0,4)$.

(b) $r = 6$, p increases on $[0,4)$, and $d_5 = 7$.

(c) $r = 6$, $d_1 = 3$, and p increases on $[1,6)$.

(4) If $d = 6$ then one of the following holds:

(i) $d_i \geq 7$ for all $i > 0$.

(ii) $d_4 = 6$ and $d_i \geq 7$ for all $i \neq 0,4$.

(iii) $d_1 = 5$ and $d_i \geq 6$ for all $i \neq 1$.

(iv) $d_2 = 4$ and p increases on $[2,6)$.

PROOF. Assume that $d_i < d_{i+1}$. In particular, $d_i \leq 7$. By assumption p increases on $[i,i+1)$, so (1) holds if $d_i = 7$; hence we may assume $d_i \leq 6$. Then $d_{i+1} \leq 7$, so by J.5.9, $\{y_i\} = \Gamma(y_{i+1}) \cap \Gamma^{d_i}(\delta)$. Further $y_{i+2} \neq y_i$ since p is a geodesic, so $d_{i+2} = d_{i+1} + 1 > d_{i+1}$. Then (1) follows by induction on $8 - d_i$.

Assume that $d_i = 8-(r-i)$. Then

$$8 = d_r \leq d_{i+1} + d(y_{i+1},y_r) \leq (d_i+1)+(r-(i+1)) = (8-(r-i)+1)+(r-(i+1)) = 8,$$

so all inequalities are equalities. In particular $d_{i+1} = d_i + 1$, so that (2) follows from (1).

Assume that $d = 4$. If $r = 4$, then (a) holds by (2), so we may assume that $r = 6$. If $d_1 = 5$, then p increases on $[0,4)$ by (1), and in particular $d_4 = 8$, so that $d_5 = 7$; that is, (b) holds. Finally if $d_1 = 3$, then (c) holds by (2).

Assume that $d = 6$. Suppose that $d_1 = 7$. Then $d_2 = 8$ by (1); in particular (i) holds if $r = 2$, so we may assume that $r = 4$ or 6. Then $d_3 = 7$, so that $d_4 = 6$ or 8. In the second case, (i) holds, and in the first case, (ii) holds by (2). Suppose instead that $d_1 = 5$, so that $d_2 = 4$ or 6. In the second case, (iii) holds by (1). In the first case , (iv) holds by (2). $\square$

LEMMA J.5.11. *Let $y \in \Gamma$ with $e := d(\delta, y) < 8$, and N_e the number of geodesics beginning at y and increasing on $[0, 8-e)$. For $y' \in \Gamma(y) \cap \Gamma^{e+1}(\delta)$, let N'_e be the number of such geodesics which do not contain y'. Then*

(1) If e is even, then $N_e = 8^{(8-e)/2}$ and $N'_e = 6 \cdot 8^{(6-e)/2}$.
(2) If e is odd, then $N_e = 2 \cdot 8^{(7-e)/2}$ and $N'_e = 8^{(7-e)/2}$.

PROOF. Set $N_8 := 1$. If e is even then y is a point so as there are 5 lines incident with each point, we conclude from J.5.9 that $|\Gamma(y) \cap \Gamma^{e+1}(\delta)| = 4$. Thus $N_e = 4N_{e+1}$ and $N'_e = 3N_{e+1}$. Similarly if e is odd then y is a line, so as there are 3 points incident with each line, from J.5.9 we obtain $N_e = 2N_{e+1}$ and $N'_e = N_{e+1}$. Then the lemma follows by induction on $8 - e$. □

Notice the next lemma implies J.5.7.1. During the proof of the lemma, we will also establish J.5.7.2.

LEMMA J.5.12. *For $d := 2$, 4, or 6, $|\Lambda(u)| \leq 2^9$ for each $u \in \Gamma^d(\delta)$, with equality iff $d = 2$.*

PROOF. By J.5.8, it suffices to establish the analogous result for $|\mathcal{G}(u)|$.

Suppose first that $d = 2$; then by J.5.9, there is a unique line k through δ and u in Γ and p is a geodesic from u to $v \in \Gamma^8(\delta)$ iff δkp is a geodesic from δ to v. Further there is a unique geodesic from δ to v through k, so there are exactly $2^{10} = |\Delta \cap \Sigma(\delta)|$ geodesics from δ to $\Gamma^8(\delta)$ through k. Therefore as $F_{\delta,k}$ is transitive on $\{u, u'\} := \Gamma(k) - \{\delta\}$, $|\mathcal{G}(u)| = 2^9$. Thus $|\Lambda(u)| = 2^9$ by J.5.8, and we see that $\Lambda(u')$ is a complement to $\Lambda(u)$ in $\Delta \cap \Sigma(\delta)$. Conversely suppose $\Lambda(u') = \Lambda(w)$ for some $w \in \theta(\delta) - \{u'\}$. Then $O_2(F_1) = F_{\delta,u'}F_{\delta,w}$ acts on $\Lambda(u')$, which is impossible, as $O_2(F_1)$ is regular on the opposites to δ. This proves J.5.7.2.

Next suppose that $d = 6$; we count the number of geodesics arising in each of the subcases of part (4) of J.5.10, using J.5.11.

In subcase (i), we claim that there are $8 \cdot 4^{(r-2)/2}$ geodesics of length r: First $d_j = 7$ if j is odd and $d_j = 8$ if j is even. Therefore by J.5.11.1, there are $N_6 = 8$ choices for $y_0y_1y_2$, so the claim holds when $r = 2$. Then if $r > 2$, y_3 is one of the 4 members of $\Gamma(y_2)$ distinct from y_1; thus $d_3 = 7$, and y_4 is determined uniquely as $N'_7 = 1$ by J.5.11.2, so the claim holds when $r = 4$. Similarly if $r = 6$, there are 4 choice for y_5y_6, completing the proof of the claim. By the claim, there are $2^3 + 2^5 + 2^7 = 21 \cdot 2^3$ geodesics in subcase (i).

Next we claim that there are $3 \cdot 2^6 = 2^3 \cdot 24$ geodesics in subcase (ii): From the previous paragraph, there are 32 choices for $y_0 \cdots y_3$; then $\{y_4\} = \Gamma(y_3) \cap \Gamma^6(\delta)$ by J.5.9, and by J.5.11.1 there are $N'_6 = 6$ choices for y_5y_6, establishing the claim.

In the remaining two subcases, $\{y_1\} = \Gamma(u) \cap \Gamma^5(\delta)$. In subcase (iii), p increases on $[1, 4)$, so by J.5.11.2 there are $N'_5 = 8$ choices for $y_0 \cdots y_4$; then as earlier, if $r = 6$, there are 4 choices for y_5y_6, so there are $8 + 32 = 40 = 2^3 \cdot 5$ geodesics in subcase (iii). Finally in subcase (iv), $\{y_2\} = \Gamma(y_1) \cap \Gamma^4(\delta)$, and then by J.5.11.1, there are $N'_4 = 48$ geodesics in this subcase.

Summing, we get a total of $2^3(21+24+5+6) = 2^6 \cdot 7 < 2^9$ geodesics, completing the proof when $d = 6$.

Finally suppose that $d = 4$. We consider the subcases of part (3) of J.5.10. By J.5.11.1, there are $N_4 = 2^6$ geodesics in subcase (a), 2^8 choices in subcase (b), and 2^6 in subcase (c). This gives a total of $2^7 \cdot 3 < 2^9$ geodesics, and completes the proof of J.5.12. □

As mentioned previously, J.5.12 completes the proof of J.5.7, and hence of J.5.6.

To complete our uniqueness proof for Rudvalis rank 3 groups, we recall next that there is a canonical construction of the graph Ω, depending only on F. This result is essentially Theorem 1 in [**Wal69**], but we provide a proof: Let

$$\Omega' := \{\infty\} \cup \Gamma_1 \cup F/E,$$

where $E := F_\gamma$. Define adjacency in Ω' as follows: The vertex ∞ is adjacent to the members of Γ_1. Vertices $u, v \in \Gamma_1$ are adjacent iff $d_\Gamma(u, v) \leq 6$. Vertices u and Ef are adjacent iff u is in the orbit of E^f on Γ_1 of length 780. Finally Ef and Eg are adjacent iff Eg is in the orbit of E^f of length 975.

By J.5.5:

LEMMA J.5.13. *The graphs Ω and Ω' are isomorphic.*

LEMMA J.5.14. *Each Rudvalis rank 3 group is isomorphic to Ru.*

PROOF. If $\dot{G}$ is a Rudvalis rank 3 group on $\dot{\Omega}$, then by J.5.13 and symmetry between G and $\dot{G}$, $\dot{\Omega} \cong \Omega' \cong \Omega$ as graphs. Then by J.5.6, $\dot{G} = Aut(\dot{\Omega}) \cong Aut(\Omega) = G$. $\square$

We are now in a position to complete the proof of Theorem J.1.1, by showing that if G is of type Ru, then $G \cong Ru$. Namely by J.5.4.3, G is a Rudvalis rank 3 group, so the Theorem follows from J.5.14.

CHAPTER K

Modules for SQTK-groups with $\hat{q}(G,V) \le 2$.

In this chapter, we supply a proof of Theorems B.4.2 and B.4.5, results which are stated in chapter A.

That is, we consider pairs G, V such that G is an SQTK-group, $L := F^*(G)$ is quasisimple, V is a faithful $\mathbf{F}_2G$-module, L is irreducible on V, and $\hat{q}(G,V) \le 2$. The quasisimple groups L and modules V which arise are determined by Guralnick and Malle in [**GM02**] and [**GM04**], without the assumption that G is strongly quasithin. We appeal to their papers to obtain a list of pairs (L,V); thus Guralnick and Malle do most of the work for us. There is a bit more to be done however: we obtain information about the possible offending subgroups and the parameters $q(G,V)$ and $\hat{q}(G,V)$.

This version of the proof of Theorems B.4.2 and B.4.5 replaces an earlier draft of ours, which was direct but very ad-hoc. That draft was based in part on unpublished work such as Cooperstein-Mason [**CM**]. Those unpublished results are now subsumed by the more general results of Guralnick-Malle. We are grateful to Robert Guralnick and Gunter Malle for providing us with prepublication copies of their work. Indeed the group theory community is in their debt for filling this long-standing gap in the literature on representations.

Notation and overview of the approach

Let T denote a Sylow 2-subgroup of G, and $T_L := T \cap L$. As in Theorem B.4.5, we set $q := q(G,V)$ and $\hat{q} := \hat{q}(G,V)$; thus we are assuming in this section that $\hat{q} \le 2$.

We recall the parameters $q(G,V)$ and $\hat{q}(G,V)$ from Definitions B.1.1 and B.4.1, and the sets $\mathcal{Q}(G,V)$ and $\hat{\mathcal{Q}}(G,V)$ from Definition D.2.1. In particular q and $\hat{q}$ denote the minimal value of the ratio $r_{A,V} = m(V/C_V(A))/m(A)$, as A varies over elementary abelian 2-subgroups of T such that A is quadratic ($[V,A,A] = 0$) or cubic ($[V,A,A,A] = 0$) on V, respectively. Thus $\hat{q} \le q$.

Our proofs of Theorems B.4.2 and B.4.5 are organized as follows: We successively examine each of the families of simple groups arising in Theorem C (A.2.3). For each group, we appeal to Guralnick-Malle to determine the possible modules V. Since we are assuming that L is faithful and irreducible on an $\mathbf{F}_2$-module, $O_2(L) = 1$, so $Z(L)$ is of odd order. Then from I.1.3, $Z(L)$ is of order 1 or 3.

To complete the treatment of a pair (L,V), we determine or bound the values of $\hat{q}$ and q, and in some instances we determine the offending subgroups. In treating a pair (L,V), we often begin by exhibiting an elementary abelian 2-subgroup X of G such that $q_0 := r_{X,V}$ is a candidate for q or $\hat{q}$. To show that q_0 is indeed minimal, we need to show $r_{A,V} \ge q_0$ for each elementary abelian 2-group A which is quadratic or cubic, respectively. Sometimes we show that A is uniquely determined.

Recall the definition in E.3.1 of the parameter $m := m(G, V)$; namely m is the minimum value of $m(V/C_V(a))$ for a an involution in G. Note for example that

$$m(V/C_V(A)) = r_{A,V} m(A); \text{ so if } m > r_{A,V}, \text{ then } m(A) > 1. \qquad (*)$$

Recall from Definition B.1.1 that V is an FF-module if $r_{A,V} \leq 1$ for some nontrivial elementary abelian 2-subgroup A of G. In that event, the Thompson Replacement Theorem B.1.4.3 says that there exists a quadratic offender A with $r_{A,V} \leq 1$, so $q \leq 1$. In particular if $\hat{q} \leq 1$, then $q \leq 1$. Moreover in this case we are only concerned with the parameter q, so we do not bother to calulate $\hat{q}$. The pairs with $q \leq 1$ appear in Theorem B.4.2. In that Theorem, we must determine the group $\langle \mathcal{P}(G, V) \rangle$, as defined in Definition B.1.2. In some cases we show that A is uniquely determined in T. Sometimes we appeal to results in sections B.3 and B.4.

K.1. Alternating groups

Assume that case (1) of Theorem C holds. Then $L/Z(L) \cong A_n$ for $5 \leq n \leq 8$.

Because of the isomorphisms $A_5 \cong L_2(4)$ and $A_8 \cong L_4(2)$, we postpone the discussion of these groups until section K.3 on groups of Lie type in characteristic 2. Thus in the remainder of this section, we assume that $n = 6$ or 7. Then by Theorem 6.2 and Table 6.3 of [**GM02**], L is A_6, $\hat{A}_6$, or A_7. The natural A_n-module V for L is discussed in section B.3. The natural module is an FF-module for $N_{GL(V)}(L) \cong S_n$, and the value of q and the FF-offenders are listed in B.3.2 and B.3.4, as required either for cases (3), (5), and (6) of Theorem B.4.2, or to verify the row of Table B.4.5 if $G = A_7$. Thus we need only concern ourselves with the remaining modules.

K.1.1. A_6**.** When $L \cong A_6$, Table 6.3 of [**GM02**] shows that $\dim(V) = 4$. Up to equivalence, there are exactly two choices for V, but these two modules are quasiequivalent (conjugate under $Aut(L)$). The A_6-module is a representative for the quasiequivalence class, so our treatment of this case is complete by the discussion above.

K.1.2. A_7**.** Next assume that $L \cong A_7$. Then Table 6.3 of [**GM02**] shows that $\dim(V) = 4$ or 6; the case in that table with $\dim(V) = 8$ for $G = S_7$ is ruled out here by our assumption that L is irreducible on V, although it appears in case (2) of Theorem D.3.10, where that assumption is relaxed.

When $m(V) = 6$, V is the A_7-module discussed above, so we may assume that $m(V) = 4$. Here there are two choices for V, but the modules are quasiequivalent, with L the stabilizer of a quasiequivalence class, so that $G = L \cong A_7$. The module is obtained by restriction from $GL(V) \cong A_8$ to L.

Let $R_1 := \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$ and $X := \langle (5, 6, 7) \rangle$. Transvections in $L_4(2)$ are of cycle type 2^4 in A_8 and centralize elements of order 3 of cycle type 3^2, so this is the class of elements of order 3 with nonzero fixed points on V. Thus $V = [V, X]$, so as X centralizes R_1 and is inverted by $(1, 2)(5, 6)$, $C_V(R_1) = C_V(r) = [V, r]$ is of rank 2 for each $r \in R_1^{\#}$, while $m(C_V(R_2)) = 1$ where $R_2 := \langle (1, 2)(3, 4), (1, 2)(5, 6) \rangle$. Thus R_1 acts quadratically on V with $r_{R_1,V} = 1$, and R_1 is the unique FF-offender on V in T. Hence case (7) of B.4.2 holds.

K.1.3. $\hat{A}_6$**.** Finally assume that $L \cong \hat{A}_6$. Again from Table 6.3 of [**GM02**] there are two modules V of rank 6, but the modules are quasiequivalent, with $\hat{S}_6$ the stabilizer of an equivalence class. Thus G is $\hat{A}_6$ or $\hat{S}_6$. Further V can be viewed as a 3-dimensional module V^F over $F := \mathbf{F}_4$ for L, with an involution in $\hat{S}_6 - L$ inducing a field automorphism on V^F. Thus $L \leq SL(V^F) =: J$ and $T_L \leq S \in Syl_2(J)$ with $Z(T_L) \leq Z(S)$. Let T_1 and T_2 be the 4-subgroups of T_L. Then $T_i \leq S_i$, where S_i is the unipotent radical of a maximal parabolic J_i of J over S, since the radicals are the maximal elementary abelian 2-subgroups of S. Since T_L is nonabelian, $S_1 \neq S_2$. We may take J_i to stabilize an i-dimensional subspace of V^F. Thus R_2 is quadratic on V with $r_{R_2,V} = 1$, but $r_{R_1,V} = 2$. If $G = \hat{S}_6$, then each involution $t \in T - T_L$ induces a field automorphism on V^F, so t lies in no FF*-offender. That is R_2 is the unique offender in T. Hence case (8) of Theorem B.4.2 holds.

K.2. Groups of Lie type and odd characteristic

Assume that case (2) of Theorem C holds. Then $L/Z(L) \cong L_2(p)$, $L_2(p^2)$, or $L_3^\epsilon(p)$ for an odd prime p. Moreover we may assume in this section that $L/Z(L)$ is not $A_6 \cong L_2(9)$, as that case was treated in the previous section. Finally as $L_2(5) \cong L_2(4)$, $L_2(7) \cong L_3(2)$, and $U_3(3) \cong G_2(2)'$, we may assume $L/Z(L)$ is none of those groups, and instead we treat them in section K.3 as groups of Lie type in characteristic 2. It follows from Guralnick-Malle [**GM02**] that $\hat{q} > 2$ for all remaining choices of V.

K.3. Groups of Lie type and characteristic 2

Now assume that case (3) or (4) of Theorem C holds, so that $L/Z(L)$ is of Lie type and characteristic 2. Choose r so that $L/Z(L)$ is defined over $F := \mathbf{F}_{2^r}$ for some r. Let $\hat{L}$ be the universal group of Lie type such that $L/Z(L)$ is a quotient of $\hat{L}$.

We recall from the Steinberg Tensor Product Theorem (see e.g. 2.8.5 in [**GLS98**]) that the irreducible $F\hat{L}$-modules are of the form $M_1^{\sigma_1} \otimes \cdots \otimes M_k^{\sigma_k}$, where $M_1, \ldots, M_k$ are *basic* $F\hat{L}$-modules and $\sigma_1, \ldots, \sigma_k$ are suitable members of $Aut(F)$. Further setting $K := End_{\mathbf{F}_2\hat{L}}(V)$, we can regard V as a $K\hat{L}$-module V^K, and $V^F := V^K \otimes_K F$ is an irreducible $F\hat{L}$-module, and hence has a decomposition

$$M_1^{\sigma_1} \otimes \cdots \otimes M_k^{\sigma_k}. \tag{!}$$

Finally

$$m(V) = \prod_{i=1}^{k} \dim_F(M_i) \cdot |K : \mathbf{F}_2|. \tag{!!}$$

We recall also (e.g. 2.8.7 in [**GLS98**]) that the *Steinberg module* for $\hat{L}$ is the tensor product of the algebraic conjugates of a basic Steinberg module; this has dimension equal to the order of a Sylow 2-subgroup of $\bar{L}$, and is projective.

We first consider the cases where $L/Z(L)$ is a classical group. From Theorem C, $L/Z(L)$ is $L_2(2^r)$, $U_3(2^{r/2})$, $L_3(2^r)$, $Sp_4(2^r)'$, $L_4(2)$, or $L_5(2)$.

K.3.1. $\mathbf{L_2(2^r)}$**.** Assume that $L/Z(L) \cong L_2(2^r)$. As $Z(L)$ has order 1 or 3, and the Schur multiplier of $L_2(2^r)$ is a 2-group by I.1.3, $L \cong L_2(2^r)$ is simple. Further $L = \hat{L}$. Then by Proposition 4.1 and Table 7 of Guralnick-Malle [**GM04**], V is of rank $2r$ over $\mathbf{F}_2$; and either $K = F$ and V^F is the natural module $N = M(\lambda_1)$, or

$r = 2s$ is even, $|F : K| = 2$, and $V^F = N \otimes N^\sigma$ where $\langle\sigma\rangle = Gal(F/K)$. Finally in the latter case, we can regard V^K as a 4-dimensional orthogonal space over K; that is, V is the orthogonal module for $L \cong \Omega_4^-(2^s)$.

K.3.1.1. *The natural module.* Assume that $V^F = N$. Let $X := T_L$; then $C_{V^F}(X) = [V^F, X]$ is an F-point of V^F, so that $r_{R,V} = 1$ and X is an FF-offender on V. As involutions in $T - T_L$ induce field automorphisms on V^F and L, X is the unique offender in T and $q = 1$. Hence case (1) of B.4.2 holds.

K.3.1.2. *The orthogonal module.* Next assume $r = 2s$ is even and V is the orthogonal module for $L = \Omega_4^-(2^s)$. Recall $|F : K| = 2$ and V^K is an orthogonal space of Witt index 1 for L. Let t be an involution in $N_{O(V^K)}(T_L)$ inducing a field automorphism on L; then t induces a K-transvection on the orthogonal space V^K. Hence $V_t := [V^K, t]$ is a nonsingular K-point in V^K.

If $s = 1$, then $G \cong A_5$ or S_5, and V is the A_5-module for L. Thus when $s = 1$, from the information in B.3.2.4 we conclude when $G \cong S_5$ that conclusion (5) of Theorem B.4.2 holds, and when $G \cong A_5$ we verify Table B.4.5 for this case. So we assume from now on that $s > 1$. Hence by H.1.1.1, V is not an FF-module for G.

Let $L_t := C_L(t)$, $R_t := C_{T_L}(t)$, and $X := R_t\langle t\rangle$; then L_t is the stabilizer in L of V_t, so $L_t \cong L_2(2^s)$ is a 3-dimensional orthogonal group on $V_t^\perp$, and hence X acts quadratically on V^K with $\dim_K(C_{V^K}(X)) = 2$. In particular $r_{X,V} = 2s/(s+1)$ and $r_{R_t,V} = 2$.

Next T_L is partitioned by the $N_L(T_L)$-conjugates of R_t, and for $g \in T_L - R_t$, $C_{V^K}(\langle R_t, g\rangle)$ is a K-point of V^K. Therefore for each $A \leq T_L$ which is not contained in a conjugate of R_t, $r_{A,V} \geq 3/2$ with equality iff $A = T_L$, and A is cubic but not quadratic on V. Thus $q = 2$ and $\hat{q} = 3/2$ when $G = L$, so B.4.5 holds in that case.

Therefore we may assume that $t \in G$. Then from the previous two paragraphs, $q = 2s/(s+1)$, $\hat{q} \leq 3/2$, and each member of $\mathcal{Q}(G,V)$ is conjugate to X, completing the verification of Table B.4.5 in this case.

K.3.2. $\mathbf{L_n(2^r)}$ for $\mathbf{n \geq 3}$. Assume that $L/Z(L) \cong L_n(2^r)$ for $n \geq 3$. Then by Theorem C, either $n = 3$, or $n = 4, 5$ with $r = 1$. By I.1.3, either $L = \hat{L}$ is simple, or $L/Z(L) \cong L_3(2^r)$ with r even, $\hat{L} \cong SL_3(2^r)$, and $L = \hat{L}$ or $\hat{L}/Z(\hat{L})$. By Proposition 4.6 and Table 7 of [**GM04**], up to quasiequivalence one of the following holds:

(a) $K = F$ and V^F is the natural module $N = M(\lambda_1)$ for $L \cong SL_n(2^r)$.

(b) L is $L_4(2)$ or $L_5(2)$, $F = K = \mathbf{F}_2$, and $\dim(V) = 6$ or 10, respectively.

(c) $r = 2s$ is even, $n = 3$, $|F : K| = 2$, and $V^F = N \otimes N^\sigma$, where $\langle\sigma\rangle = Gal(F/K)$.

Observe also that if $L < O^{2'}(G)$, then $r = 2s$ is even, so $n = 3$ and the stabilizer of the equivalence class of V is the subgroup of $Aut(L)$ trivial on the Dynkin diagram of L, so G is contained in that group.

K.3.2.1. *The natural module.* Assume that V^F is the natural module of dimension n over F. As V is a G-module, G is trivial on the Dynkin diagram of L, so if $t \in G - L$ is an involution, then t induces a field automorphisms on L and V^F. Let X be the centralizer in L of a hyperplane of V; then $m(X) = r(n-1)$ and $m(V/C_V(X)) = r$, so that $r_{X,V} = 1/(n-1)$. Thus $q \leq 1$, and our initial candidate X for a subgroup exhibiting the minimal value q satisfies $q_0 := 1/(n-1)$. Let $A \in \mathcal{P}(T,V)$.

Assume that $a \in A - L$. Then by an earlier remark, $n = 3$, $r = 2s$ is even, and t induces a field automorphism on L and V^F. As $m(V/C_V(s)) = 3s > 1$, $A > \langle a \rangle$. Thus $m(A) \leq m(C_G(a)) = 2s+1$ and $m(C_V(A)) \leq 2s$, so that $r_{A,V} \geq 4s/(2s+1) > 1$, contrary to $A \in \mathcal{P}(G,V)$.

Therefore $A \leq L$. For $n = 3,4,5$, $m(A) \leq j := m_2(SL_n(2^r)) = 2r, 4, 6$. Thus if $r_{A,V} < q_0$, then $m(V/C_V(A)) < q_0 j = r$, $4/3$, $3/2$. This forces A to centralize an F-hyperplane of V^F, so A is contained in a conjugate of X and $r_{A,V} = r/m(A) \geq q_0$. This completes the proof that $q = 1/(n-1)$, and that conclusion (3) or (9) of B.4.2 holds in this case.

K.3.2.2. *The 6-dimensional module for $L_4(2)$.* When $L \cong L_4(2)$ and $\dim(V) = 6$, V is the orthogonal module for L regarded as $\Omega_6^+(2)$. This module is also the A_8-module for L regarded as A_8, and hence the information contained in B.3.2 shows that one of the possibilities in case (10) of Theorem B.4.2 holds.

K.3.2.3. *The 10-dimensional module for $L_5(2)$.* We turn to the remaining subcase of (b), where $L \cong L_5(2)$ with $\dim(V) = 10$. Again V is determined up to quasiequivalence with L the stabilizer of the equivalence class of V, so $G = L$. Further $F = K = \mathbf{F}_2$ and V is the exterior square of a natural module Γ for L. In section 12.5 there is a fairly complete description of V, and a list of various subspaces of V and their stabilizers in L; we adopt the notation established there, and appeal to lemmas 12.5.2 and 12.5.5.

Let X be the centralizer of the hyperplane Γ_4 of Γ; thus X is denoted by $\bar{R}_6$ in section 12.5, and from the discussion in 12.5.2 and using the description of V as the exterior square of Γ, $m(X) = 4$ and $V_6 = C_V(X)$ is of rank 6, so that $r_{X,V} = 1$, and hence $q \leq 1$. We will show that X is the unique member of $\mathcal{P}(T,V)$, so that case (11) of B.4.2 holds.

Assume $A \in \mathcal{P}(T,V) - \{X\}$. First by 12.5.5.1, the set of all $C_V(x)/V_6$ for $x \in X^\#$ is the set of points of V/V_6, so $V_6 = C_V(Y)$ for each noncyclic subgroup Y of X. Hence $X \in \mathcal{P}^*(G,V)$. Thus as $X = C_T(X)$, each $B \in \mathcal{P}^*(G,T)$ contained in A is not contained in X, so we may take $A \in \mathcal{P}^*(G,T)$. Now by B.7.1, $Aut_A(V_6) \in \mathcal{P}^*(Aut_T(V_6), V_6)$, and if $r_{Aut_A(V_6),V_6} = 1$ then $V = V_6 + C_V(A)$. But from B.3.2.6, $q(Aut_G(V_6), V_6) = 1$ since $Aut_G(V_6) \cong A_8$, so indeed $V = V_6 + C_V(A)$. This is contrary to $r_{A,V} \leq 1$ since L_6/V_6 is faithful on V/V_6.

K.3.2.4. *The tensor product module for $L_3(2^{2s})$.* Finally we assume that case (c) holds. Then V is of rank $9s$ over $\mathbf{F}_2$. Recall that this module is discussed in detail in chapter H, especially in H.4.4; we adopt the notation of the subsection containing that result. In particular the Galois automorphism σ is also regarded as a field automorphism of L. The stabilizer of the quasiequivalence class of V is the subgroup of $Aut(L)$ trivial on the Dynkin diagram of L, so G is contained in that group.

Let z be an involution in T_L and R the root group of z; without loss z centralizes σ. From the discussion in the proof of H.4.4.2, $[V,z]$ is of rank $5s$, and $C_G([V,z]) =: R_\sigma$ is the group of fixed points of σ on R, and by that lemma, $m(R_\sigma) = s$ while $m(V/C_V(R_\sigma)) = 4s$. By H.4.4.6, $m(V/C_V(\sigma)) = 3s$ and $C_L(\sigma)$ acts faithfully as $SL_3(2^s)$ on $[V,\sigma]$ and $C_V(\sigma)/[V,\sigma]$. Thus if A is quadratic on V with $r_{A,V} \leq 2$, then $m(A) > 1$ by (*), so that $A \leq L$ as $C_L(\sigma)$ is faithful on $[V,\sigma]$. Then $m(V/C_V(A)) \geq 4s$, so $m(A) \geq 2s$ by (*), contrary to $m(C_G([V,z])) = s$. We conclude that $q > 2$.

Let X be the centralizer in L of an F-line W in the natural module N; thus X is the group denoted by Q_W in the discussion preceding H.4.4. As X acts quadratically on N, it acts cubically on V. Further by H.4.4.1, $m(X) = 4s = m(C_V(X))$ so that $r_{X,V} = 5/4$. Hence $\hat{q} \leq 5/4$.

Conversely assume that A is an elementary abelian subgroup of T with $r_{A,V} \leq 5/4$. From the discussion above of $C_G(\sigma)$ and its action on V, $m(V/C_V(B)) \geq 5s$ for $\langle\sigma\rangle < B \leq C_G(\sigma)$ and $m_2(C_G(\sigma)) = 2s + 1$, so $A \leq L$; thus $A \leq Y$, where $Y = X$ or Y is the unipotent radical Q_Z of the stabilizer of an F-point Z in N. We saw $m(C_V(z)) = 5s$, so $m(A) \geq 16s/5$. Hence $A \cap R \neq 1$ for each root group R in Y, which allows us to compute that $C_V(A) = C_V(Y)$ is of rank $4s$ or s for $Y = X$ or Q_Z, respectively. Therefore as $r_{A,V} \leq 5/4$, $A = X$, which completes the verification of Table B.4.5 in this case.

K.3.3. $U_3(2^s)$**.** Assume $L/Z(L) \cong U_3(2^s)$. Then $s \geq 2$ as L is quasisimple. Here $r = 2s$, and when s is even, $L = \hat{L}$ is simple, while when s is odd, $\hat{L} \cong SU_3(2^s)$ with $Z(\hat{L})$ of order 3. By Theorem 4.10 and Table 7 of [**GM04**], V^F is the natural module for $L = \hat{L}$ of dimension 3 over $F = K$ and of rank $6s$ over $\mathbf{F}_2$.

Now $X := \Omega_1(T_L)$ is of rank s, and X induces a group of transvections on V^F with fixed axis and center. Thus $C_V(X)$ is of rank $4s$ and $r_{X,V} = 2$. Next if t is an involution in $T - L$, then t induces a field automorphism on V^F, so X is the unique offender in T, and $q = \hat{q} = 2$, completing the verification of Table B.4.5 in this case.

K.3.4. $Sp_4(2^r)$**.** Assume that $L/Z(L)$ is $Sp_4(2^r)'$. As we have already treated $Sp_4(2)' = A_6$, we may assume that $r > 1$. Hence $L/Z(L)$ is $Sp_4(2^r)$, with Schur multiplier a 2-group by I.1.3, so that $L \cong Sp_4(2^r)$. Also $\hat{L} = L$.

By Proposition 6.2 and Table 9 of [**GM04**], up to quasiequivalence either V^F is a natural module N for $L = \hat{L}$ of dimension 4 over $F = K$, or $r = 2s$ is even, $|F : K| = 2$, and $V^F = N \otimes N^\sigma$, where $\langle\sigma\rangle = Gal(F/K)$.

K.3.4.1. *The natural module.* Assume first that V^F is the natural module. The stabilizer of the equivalence class of V is the subgroup of $Aut(L)$ trivial on the Dynkin diagram, so G is contained in that subgroup.

Let X be the centralizer in L of a totally isotropic 2-dimensional F-subspace of V^F; then $m(X) = 3r$ and $m(C_V(X)) = 2r$, so that $r_{X,V} = 2/3$. In particular $q \leq 2/3$.

Involutions t in $G - L$ induce field automorphisms on L and V, so $m_2(C_G(t)) = (3r + 2)/2$ and $m(C_V(t)) = 2r$. Further if $1 \neq A \leq C_L(t)$, then $m(V/C_V(A\langle t\rangle)) \geq 5r/2$. Thus t is contained in no FF-offender on V.

Finally let A be an elementary abelian subgroup of T_L. Then $m(A) \leq 3r$ and either $A \leq X$ or A is contained in the unipotent radical of the stabilizer of the F-point of V^F centralized by T_L. In particular if $A \not\leq X$ then $m(V/C_V(A)) = 3r$, so $r_{A,V} \geq 1$. Thus $q = 2/3$, so that case (3) of B.4.2 holds.

K.3.4.2. *The tensor product module.* Assume $r = 2s$ is even and $V^F = N \otimes N^\sigma$. Again G is contained in the subgroup of $Aut(L)$ trivial on the Dynkin diagram of L. Also $|F : K| = 2$ and V is of rank $16s$ over $\mathbf{F}_2$. Define X as in the previous case; then X centralizes an F-line U in N and $C_{V^F}(X) = U \otimes U^\sigma$, so $m(C_V(X)) = 4s$ and $r_{X,V} = 2$. As X acts quadratically on N, it acts cubically on V, so $\hat{q} \leq 2$.

Our argument is similar to that used to deal with the tensor product module for $(S)L_3(2^{2s})$; but we do not have the elementary calculations already exhibited in chapter H as we did in that case. We will show that $q > \hat{q} = 2$; so assume A is an elementary abelian subgroup of T with $r_{A,V} \leq 2$.

Arguing as in the proof of H.4.4.6, $m(C_V(\sigma)) = 10s$ and $C_L(\sigma)$ acts faithfully on $[V,\sigma]$ and $C_V(\sigma)/[V,\sigma]$. Thus $m_2(C_L(\sigma)) = 3s$ and $m(V/C_V(B)) \geq 8s$ for $\langle\sigma\rangle < B \leq C_G(\sigma)$ (and $m(V/C_V(B)) > 8s$ if $m(B) = 3s+1$), so $A \leq L$.

Next L has three classes b_1, a_2, and c_2 of involutions in the notation of Definition E.2.6. A standard tensor calculation shows that $m(C_V(i)) = 10s$ for $i \in b_1$, and involutions in a_2 and c_2 are free on V. Further if we pick $j \in L$ to be an involution centralizing σ, then $C_T([V,j]) =: R_{j,\sigma}$ of rank s is the subgroup of fixed points of σ on R_j, where R_j is the root group of j if $j \in b_1$ or a_2, and $R_j = \alpha(j)$ is the group defined in section 11 of [**AS76a**] if $j \in c_2$. Thus A is not quadratic on V, so $q > 2$. Further $m(A) \geq 3s$, and $m(A) \geq 4s$ unless $A^\# \subseteq b_1$. However for $i, i' \in b_1$, $ii' \in b_1$ only if $R_i = R_{i'}$, so $m(A) \geq 4s$. Similarly for each $j \notin b_1$, $C_T(C_V(j)) = R_{j,\sigma}$ is of rank s, so $m(A) > 4s$.

Let X' be the unipotent radical of the stabilizer of the F-point Z of N stabilized by T_L. Thus X and X' are the maximal elementary abelian subgroups of T_L, each is of rank $6s$, and $X \cap X' = R_iR_j$ is of rank $4s$, for some $i \in b_1$ and $j \in a_2$. Thus A is contained in $Y \in \{X, X'\}$. From the previous paragraph, $m(A) > 4s = |Y : R_k|$ for each root group $R_k \leq Y$, so $A \cap R_k \neq 1$. However we calculate that given any set Δ of nontrivial elements from the various root groups contained in Y, $C_{V^F}(\langle\Delta\rangle))$ is either $U \otimes U^\sigma$ or $Z \otimes Z^\sigma$ for $Y = X$ or X', respectively. It follows that $A = X$ and $\hat{q} = 2$, so that case (iii) of B.4.5 holds in this case.

We have completed the treatment of the cases where $L/Z(L)$ is a classical group. The groups in (3) or (4) of Theorem C which are not classical are $G_2(2^r)'$, $Sz(2^r)$, ${}^2F_4(2^r)'$, and ${}^3D_4(2^r)$. As $Z(L)$ is of order 1 or 3, and the Schur multiplier of each of these groups is a 2-group by I.1.3, L is simple. By Theorem 8.1 and Table 17 of [**GM04**], L is $G_2(2^r)'$ or $Sz(2^r)$ and V is the natural module.

K.3.5. $G_2(2^r)'$**.** Assume V is the natural module of rank $6m$ for $G_2(2^r)'$. Then V is described in B.4.6. In particular, that lemma shows that case (4) of B.4.2 holds unless $r = 1$ and $G = L = G_2(2)'$, where it shows that the row in Table B.4.5 for $G_2(2)'$ is correct.

K.3.6. $Sz(2^r)$**.** Assume finally that V is the natural module of rank $4r$ for $Sz(2^r)$, where $r \geq 3$ is odd. Then $|Out(L)|$ is odd, so $L = O^{2'}(G)$ and $\Omega_1(T) =: X$ is of rank r. Now involutions of X are free on V, and $m(C_V(X)) = 2r$ so that $r_{R,V} = 2$. As X is quadratic on V, we conclude that $\hat{q} = q = 2$, and X is the unique member of $\mathcal{Q}(T,V)$. This verifies the row for $Sz(2^r)$ in Table B.4.5.

K.4. Sporadic groups

It remains to consider the case where $L/Z(L)$ is a sporadic group appearing in conclusion (5) of Theorem C. Then by Theorem 6.6 and Table 6.7 of [**GM02**], L is M_{12}, $\hat{M}_{22}$, M_{22}, M_{23}, M_{24}, or J_2, and V is of dimension 10,12,10,11,11,12 over $\mathbf{F}_2$, respectively. In the last three cases, the module of dimension 10 or 11 can be either the code module or the cocode module, as described in chapter H.

Observe that V is not an FF-module, so $\hat{q} > 1$; this follows from H.11.1.3, H.12.1, H.15.2, and H.16.5 for the Mathieu groups; for J_2, see the discussion in the final subsection of this section. Thus the sporadic groups do not appear in Theorem B.4.2, and to establish Theorem B.4.5, it remains to show: If L is M_{24} or M_{23} on either of its 11-dimensional modules, or M_{22} on its cocode module, or M_{12} on its

10-dimensional module, then $q > 2$; if V is the code module for M_{22}, then $q \geq 2$; and if L is J_2 then $\hat{q} = 2 < q$. Thus we may assume that (L, V) is one of these pairs, and in addition we may assume that $A \leq T$ with $r_{A,V} \leq 2$, and that A is quadratic unless possibly L is J_2.

Next observe that $m > 2$ when L is not J_2 by H.14.4. The module for J_2 is the natural module for the overgroup $G_2(4)$, so that $m = 4$ in that case using B.4.6.4. Thus by (*) we have:

$$m(V/C_V(A)) \leq 2\ m(A) \text{ and } m(A) > 1. \qquad (**)$$

Given a group M and a faithful $\mathbf{F}_2M$-module U, write $\mathcal{D}(M, U)$ for the set of noncyclic elementary abelian 2-subgroups of M which are quadratic on U.

K.4.1. Preliminaries for $\mathbf{M_{23}}$ and $\mathbf{M_{24}}$. In this subsection, assume that $M \cong M_{24}$ and V is the code or cocode module for M. Since the restriction of V to M_{23} is the code or cocode module for M_{23}, the results we obtain apply to L given by either M_{23} and M_{24}.

By H.14.4, M has two classes of involutions with representatives z and t, where z is 2-central, t is not 2-central, $m([V, z)] = 4$, and $m([V, t]) = 5$. By H.15.2.5 and H.16.2.5, $C_M(C_V(z)) = \langle z \rangle$. Hence $m(C_V(B)) \leq 6$ for each 4-subgroup B of M. Then since $m(A) > 1$ by (**), we conclude:

LEMMA K.4.1. *If L is M_{23} or M_{24}, then $m(A) > 2$.*

LEMMA K.4.2. *If $D \in \mathcal{D}(M, V)$ such that $D^\# \subseteq t^M$, then $m(D) \leq 2$.*

PROOF. Assume D is such a subgroup. By B.4.7.3, D is quadratic on V iff D is quadratic on the dual of V; so we may assume that V is the code module. By 21.1.4 in [**Asc94**], we may take $t \in K_{\mathbf{S}}$ (as defined before H.14.1). By H.16.1, $V_4 = C_V(K_{\mathbf{S}})$ is of rank 4 and $m([V/V_4, t]) = 1$. Therefore $V_4 \leq [V, t]$ as $m([V, t]) = 5$. Thus as D is quadratic on V, $D \leq C_{M_{\mathbf{S}}}(V_4) = K_{\mathbf{S}}$. But by H.14.1.2, $K_{\mathbf{S}}$ is a natural module for $M_{\mathbf{S}}/K_{\mathbf{S}} \cong \hat{A}_6$; so from the structure of that module, $K_{\mathbf{S}}$ is the kernel of the action of $C_L(t)$ on V_4, and the maximal rank of a subgroup E of $K_{\mathbf{S}}$ with $E^\# \subseteq t^G$ is 2, achieved by a 1-dimensional $\mathbf{F}_4$-subspace. □

K.4.2. The code and cocode modules for the Mathieu groups.

In this subsection M is M_{24}, $(X, \mathcal{C})$ the Steiner system for M, and U is the code module for M. See section 5 of chapter H for a discussion of this setup, and as in that section let $\mathbf{O} \in \mathcal{C}$ be an octad and let $K_{\mathbf{O}}$, $M_{\mathbf{O}}$ be the pointwise, global stabilizer of $\mathbf{O}$ in M, respectively. From H.14.1.4, $K_{\mathbf{O}}$ is the natural module for $\bar{M}_{\mathbf{O}} := M_{\mathbf{O}}/K_{\mathbf{O}} \cong L_4(2)$. Adopt the notation in H.16.2.1, and let $z \in K_{\mathbf{O}}^\#$; by that lemma $Fix_X(z) = \mathbf{O}$ (so $C_M(z) \leq M_{\mathbf{O}}$), $V_1 = C_U(K_{\mathbf{O}})$ is of rank 1, and V_5 is $M_{\mathbf{O}}$-invariant of rank 5, with $K_{\mathbf{O}}$ inducing the group of transvections with center V_1 on V_5. Set $V_z := C_{V_5}(z)$ and $D_z := C_M(V_z)$.

LEMMA K.4.3. *Let $d \in D_z - \langle z \rangle$ and $E := \langle d, z \rangle$. Then*

(1) $O_2(C_M(z)) = K_{\mathbf{O}}D_z$, $D_z \cap K_{\mathbf{O}} = \langle z \rangle$, $m(V_z) = 4$, and $D_z \cong E_{16}$.
(2) $[U, z] = V_z$ is of rank 4.
(3) $E = D_z \cap D_d$.
(4) If $z \in D$ for $D \in \mathcal{D}(M, U) \cup \mathcal{D}(M, U^*)$, then $D \in E^{M_{\mathbf{O}}}$.
(5) $C_U(E) = [U, E] = V_z + V_d$ is of rank 6.
(6) Let U^* be the dual of U; then $m(U^*/C_{U^*}(E)) = 6$.

PROOF. Part (1) follows from H.16.2.6. By H.16.2.1, $K_{\mathbf{O}}$ centralizes U/V_5, so $[U,z] \leq C_{V_5}(z) = V_z$. From the previous subsection, $m([U,z]) = 4$, so (1) implies (2).

Let $M_E := C_M(E)$. As $O_2(C_M(z))$ is extraspecial with $d \in O_2(C_M(z))$, there is (cf. 8.7.3 in [**Asc94**]) $g \in M$ interchanging z and d via conjugation. Thus $E \leq D_z \cap D_d$. As $\overline{C_M(z)}$ is the stabilizer in $\bar{M}_{\mathbf{O}} = GL(K_{\mathbf{O}})$ of z, we conclude from (1) that $C_M(z)$ induces the stabilizer in $GL(D_z)$ of z on D_z, with kernel D_z. Thus M_E is irreducible on D_z/E; so if (3) fails, then $D_z = D_d$. Then by symmetry between z and d, $\langle C_M(z), C_M(d)\rangle$ induces $GL(D_z)$ on D_z, which is not the case as $M_{\mathbf{O}}$ is the unique overgroup in M of the form $L_4(2)/E_{16}$. This establishes (3). Then (2) and (3) imply (4) in the case where $D \in \mathcal{D}(M,U)$, and say that E centralizes $[U,E] = V_z + V_d$. As U/V_5 is the orthogonal module for $\bar{M}_{\mathbf{O}}$, $(V_d+V_5)/V_5 = [U/V_5, d]$ is of rank 2, so $m([U,E]) = 6$. Then as we saw that $C_M(C_U(z)) = \langle z\rangle$, (5) follows. Finally (5) and B.4.7 imply (6); while as (4) holds when $D \in \mathcal{D}(M,U)$, B.4.7 says it also holds when $D \in \mathcal{D}(M,U^*)$. □

LEMMA K.4.4. *Assume L is M_{22} and V is the code module for G. Then*

(1) If $D \in \mathcal{D}(G,V)$, then $m(D) = 2$ and $m(V/C_V(D)) \geq 4$.

(2) Let V^* be the dual space of V. If $D \in \mathcal{D}(G,V^*)$, then $m(D) = 2$ and also $m(V^*/C_{V^*}(D)) \geq 5$.

PROOF. First V is a hyperplane of U containing V_5, and by H.14.4.2, z^L is the unique class of involutions in L.

Suppose $D \in \mathcal{D}(G,V)$. As $|G:L| \leq 2$ and $m(D) > 1$ by definition, $D \cap L \neq 1$. Thus we may take $z \in D$. By H.14.4.3, $m([V,z]) = 4$, so $V_z = [U,z] = [V,z]$ by K.4.3.2. Hence D lies in the subgroup D_z of M. Since $m(D) > 1$, D is a 4-subgroup of D_z by K.4.3.3. By K.4.3.4, $m(U/C_U(D)) = 5$, so as V is a hyperplane of U, we conclude that $m(V/C_V(D)) \geq 4$, establishing (1).

Next suppose $D \in \mathcal{D}(G,V^*)$. Then by B.4.7.3, also $D \in \mathcal{D}(G,V)$; so by the previous paragraph, we may take $z \in D$, $[V,z] = V_z$ is of rank 4, and D is a 4-subgroup of D_z. Therefore $[U,D] = V_z + V_d$ is of rank 6 for $d \in D - \langle z\rangle$ by K.4.3.5. Thus $m(V_z \cap V_d) = 2$, and as V is a hyperplane of U, $[V,d]$ is of corank at most 1 in $[U,d] = V_d$. Therefore $[V,D] = V_z + [V,d]$ is of rank at least 5, so $m(V^*/C_{V^*}(D)) = m([V,D]) \geq 5$ by B.4.7. This establishes (2). □

K.4.3. The parameter q on the code and cocode modules. We now turn to the proof of B.4.5 when V is the code or cocode module for $L = M_{22}$, M_{23}, or M_{24}. By (**), $m(A) > 1$, so we may that assume $A \in \mathcal{D}(G,V)$.

If L is M_{24} or M_{23}, then V is the code module U or its dual, the cocode module U^*. Then $m(A) > 2$ by K.4.1. Further if L is M_{24}, then we may take $z \in A$ by K.4.2; while if L is M_{23}, all involutions in L are in z^M, so again we may take $z \in A$. But now $m(A) = 2$ by K.4.3.4, a contradiction completing the proof in these cases.

Thus we may assume that $L \cong M_{22}$. By K.4.4, $m(A) = 2$ and V is not the cocode module; and if V is the code module, then $m(V/C_V(A)) \geq 4 = 2\, m(A)$, so $q \geq 2$. Thus the proof is complete.

K.4.4. $\mathbf{M_{12}}$. Recall that $Aut(M_{12})$ is the subgroup of M_{24} stabilizing a vector $\bar{e}_Y$ in the code module U, where Y is a dodecad in the Steiner system X for $M = M_{24}$. Hence the 10-dimensional irreducible V for M_{12} is the quotient $U/\langle \bar{e}_Y\rangle$ of U, and a hyperplane of the dual U^*. Thus we may take $V \leq U^*$.

We observed during the proof of H.11.1 that an involution z in the 2-central class fixes points of Y, and $m([V,z]) = 4$. Then z fixes points of X, so z is 2-central in M. Also $m([V,z]) = m([U^*,z])$, so as $V \leq U^*$, $[V,z] = [U^*,z]$.

Assume that $A \in \mathcal{D}(G,V)$ satisfies (**). Then by H.11.1.5, each member of $A^\#$ is 2-central, so we may take $z \in A$, and from the previous paragraph, $[V,a] = [U^*,a]$ for each $a \in A$. Thus as $A \in \mathcal{D}(G,V)$, also $A \in \mathcal{D}(M,U^*)$. Then we conclude from parts (4) and (6) of K.4.3 that $m(A) = 2$ and $m(U^*/C_{U^*}(A)) = 6$. Therefore as V is a hyperplane of U^*, $m(V/C_V(A)) \geq 5 > 2\ m(A)$, contradicting (**).

K.4.5. $\mathbf{J_2}$. In our final case, $L \cong J_2$. Therefore as $|Out(J_2)| = 2$, $|G:L| \leq 2$. Further V is obtained by restriction from the 12-dimensional module for $M = G_2(4)$ to L, and information about this module can be derived from B.4.6. In addition if $L < G$, then an involution $i \in G - L$ induces a field automorphism on M, and hence also of L and V, so $m([V,i]) = 6 = m(C_V(i))$. Then as $C_L(i) \cong PGL_2(7)$, $C_L(i)$ acts faithfully on $[V,i]$; thus if $i \in A$, then $m(A) \leq m(C_G(i)) = 3$ while $m(C_V(A)) \leq m(C_{[V,i]}(A)) \leq 4$, so $m(V/C_V(A)) \geq 8$, contrary to (**). Therefore $A \leq L$.

Observe that G contains no FF*-offenders, since the offenders in M are of rank 6 by B.4.6, whereas $m_2(Aut(J_2)) = 4$. Thus $\hat{q} > 1$.

Let z be a 2-central involution of G, and t a non-2-central involution of L. From parts (4) and (6) of B.4.6, $m([V,z]) = 4$ and $m([V,t]) = 6$.

By (**), $m(A) \geq 2$, and $m(V/C_V(A)) \leq 4$ in case $m(A) = 2$. Thus if $m(A) = 2$ then $A^\# \subseteq z^G$, we may take $z \in A$, and $C_V(A) = C_V(z)$ is of rank 8. But B.4.6.4 shows that $C_M(C_V(z))$ is the root group R_z of M containing z—whereas G contains no such root group, since $\langle z \rangle = Z(C_L(z))$ and $R_z \leq Z(C_M(z))$.

Therefore $m(A) \geq 3$. Hence as $z^G \cap C_G(z) \subseteq O_2(C_G(z))$ (e.g. see Note 2 on page 268 of [**GLS98**]), while $m_2(O_2(C_G(z))) = 2$, A contains a conjugate of t. Thus we may assume that $t \in A$.

Next $C_L(t) = C \times E$ where $C \cong E_4$ and $E \cong A_5$. Also from B.4.6.6, $W := [V,C] = C_V(C) = C_V(t) = [V,t]$ is of rank 6, and the E-module W is the extension of $C_W(E)$ of rank 2 by the $L_2(4)$-module. As $m(A) \geq 3$, $A \not\leq C$, so $C_V(A) = C_W(A)$ is of rank 4. Then by (**), $m(A) = 4 = m_2(G)$, so $A \in Syl_2(C_G(t))$ and $\hat{q} = 2$. As $W = [V,t]$ is not centralized by A, A is not quadratic on V, so $q > 2$. This completes the proof of Theorem B.4.5 when L is J_2.

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Index

To Pam and Judy

Contents of Volumes I and II

Volume II: Main Theorems; the classification of simple QTKE-groups

In Volume II we establish our Main Theorem classifying the simple QTKE-groups. The proof uses machinery from Volume I. Also in chapter 16 we establish the Even Type Theorem, which uses our Main Theorem to provide a classification of the quasithin group satisfying the "even type" hypothesis of the Gorenstein-Lyons-Solomon project [**GLS94**].

Introduction to Volume II

The treatment of the "quasithin groups of even characteristic" is one of the major steps in the Classification of the Finite Simple Groups (for short, the Classification). Geoff Mason announced a classification of a subclass of the quasithin groups in about 1980, but he never published his work, and the preprint he distributed [**Mas**] is incomplete in various ways. In two lengthy volumes, we treat the quasithin groups of even characteristic; in particular we close that gap in the proof of the Classification.

Each volume contains an Introduction discussing its contents. For further background, the reader may also wish to consult the Introduction to Volume I; that volume records and develops the machinery needed to prove our Main Theorem, which classifies the simple quasithin $\mathcal{K}$-groups of even characteristic. Volume II implements the proof of the Main Theorem.

Section 0.1 of this Introduction to Volume II gives the statement of the two main results of the paper, together with a few definitions necessary to state those results. Section 0.2 discusses the role of quasithin groups in the larger context of the Classification; we also compare the hypotheses of the original quasithin problem with those of more recent alternatives, and give some history of the problem. In sections 0.3 and 0.4, we introduce further fundamental concepts and notation, and give an outline of the proofs of our two main theorems.

The Introduction to Volume I describes the references we appeal to during the course of the proof; see section 0.12 on recognition theorems and section 0.13 on Background References.

0.1. Statement of Main Results

We begin by defining the class of groups considered in our Main Theorem, and since the definitions are somewhat technical, we also supply some motivation. For definitions of more basic group-theoretic notation and terminology, the reader is directed to Aschbacher's text [**Asc86a**].

The quasithin groups are the "small" groups in that part of the Classification where the actual examples are primarily the groups of Lie type defined over a field of characteristic 2. We first translate the notion of the "characteristic" of a linear group into the setting of abstract groups: Let G be a finite group, $T \in Syl_2(G)$, and let $\mathcal{M}$ denote the set of maximal 2-local subgroups of G. [1] We define G to be of *even characteristic* if

$$C_M(O_2(M)) \leq O_2(M) \text{ for all } M \in \mathcal{M}(T),$$

[1] A *2-local subgroup* is the normalizer of a nonidentity 2-subgroup.

where $\mathcal{M}(T)$ denotes those members of $\mathcal{M}$ containing T. The class of simple groups of even characteristic contains some families in addition to the groups of Lie type in characteristic 2. In particular it is larger than the class of simple groups of characteristic 2-type (discussed in the next section), which played the analogous role in the original proof of the Classification.

The Classification proceeds by induction on the group order. Thus if G is a minimal counterexample to the Classification, then each proper subgroup H of G is a $\mathcal{K}$-*group*; that is, all composition factors of each subgroup of H lie in the set $\mathcal{K}$ of known finite simple groups.

Finally quasithin groups are "small" by a measure of size introduced by Thompson in the N-group paper [**Tho68**]. Define

$$e(G) := \max\{m_p(M) : M \in \mathcal{M} \text{ and } p \text{ is an odd prime}\}$$

where $m_p(M)$ is the *p-rank* of M (namely the maximum rank of an elementary abelian p-subgroup of M). When G is of Lie type in characteristic 2, $e(G)$ is a good abstract approximation of the Lie rank of G. Janko called the groups with $e(G) \leq 1$ "thin groups", leading Gorenstein to define G to be *quasithin* if $e(G) \leq 2$. The groups of Lie type of characteristic 2 and Lie rank 1 or 2 are the "generic" simple quasithin groups of even characteristic.

Define a finite group H to be *strongly quasithin* if $m_p(H) \leq 2$ for all odd primes p. Thus the 2-locals of quasithin groups are strongly quasithin.

We combine the three principal conditions into a single hypothesis:

Main Hypothesis. Define G to be a *QTKE-group* if

(QT) G is quasithin,
(K) all proper subgroups of G are $\mathcal{K}$-groups, and
(E) G is of even characteristic.

We prove:

THEOREM 0.1.1 (Main Theorem). *The finite simple QTKE-groups consist of:*

(1) (Generic case) Groups of Lie type of characteristic 2 *and Lie rank at most* 2*, but* $U_5(q)$ *only for* $q = 4$.

(2) (Certain groups of rank 3 *or* 4*)* $L_4(2)$, $L_5(2)$, $Sp_6(2)$.

(3) (Alternating groups:) A_5, A_6, A_8, A_9.

(4) (Lie type of odd characteristic) $L_2(p)$, *p a Mersenne or Fermat prime;* $L_3^\epsilon(3)$, $L_4^\epsilon(3)$, $G_2(3)$.

(5) (sporadic) M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, HS, He, Ru.

We recall that there is an "original" or "first generation" proof of the Classification, made up by and large of work done before 1980; and a "second generation" program in progress, whose aim is to produce a somewhat different and simpler proof of the Classification. The two programs take the same general approach, but often differ in detail. Our work is a part of both efforts.

In particular Gorenstein, Lyons, and Solomon (GLS) are in the midst of a major program to revise and simplify the proof of part of the Classification. We also prove a corollary to our Main Theorem, which supplies a bridge between that result and the GLS program. We now discuss that corollary:

There is yet another way to approach the characterization of the groups of Lie type of characteristic 2. The GLS program requires a classification of quasithin groups which again satisfy (QT) and (K), but instead of condition (E) they impose a more technical condition (see p. 55 of [**GLS94**], and 16.1.1 in this work):

(E') G is of *even type*.

The condition (E') allows certain components [2] in the centralizers of involutions t (including involutions in $Z(T)$, which are not allowed under our hypothesis of even characteristic); but these components can only come from a restricted list. To be precise, a quasithin group G is of even type if:

($E'1$) $O(C_G(t)) = 1$ for each involution $t \in G$, and

($E'2$) If L is a component of $C_G(t)$ for some involution $t \in G$, then one of the following holds:

(i) $L/O_2(L)$ is of Lie type and in characteristic 2, but L is not $SL_2(q)$, $q = 5, 7, 9$ or $A_8/\mathbf{Z}_2$, and if $L/O_2(L) \cong L_3(4)$ then $O_2(L)$ is elementary abelian.

(ii) $L \cong L_3(3)$ or $L_2(p)$, p a Fermat or Mersenne prime.

(iii) $L/O_2(L)$ is a Mathieu group, J_2, J_4, HS, or Ru.

In order to supply a bridge between our Main Theorem and the GLS program, we also establish (as Theorem 16.5.14):

THEOREM 0.1.2 (Even Type Theorem). *The Janko group J_1 is the only simple group of even type satisfying (QT) and (K) but which is not of even characteristic.*

Since the groups appearing as conclusions to our Main Theorem are in fact of even type, the quasithin simple groups of even type consist of J_1 together with that list of groups.

0.2. Context and History

In this section we discuss the role of quasithin groups in the Classification, focusing on motivation for our basic hypotheses. We also recall some of the history of the quasithin problem. Occasionally we abbreviate 'Classification of the Finite Simple Groups" by CFSG.

0.2.1. Case division according to notions of even or odd "characteristic". The Classification of the Finite Simple Groups proceeds by analyzing the p-local subgroups of an abstract finite simple group G for various primes p. Further for various reasons, which we touch upon later, the 2-local subgroups are preferred.

On the other hand the generic example of a simple group is a group G of Lie type over a field of some prime characteristic p, which is the *characteristic* of that group of Lie type. Such a group G can be realized as a linear group acting on some space V over a finite field of characteristic p, and the local structure of G is visible from this representation. For example if $g \in G$ is a p'-element (i.e., $(|g|, p) = 1$) then g is semisimple (i.e., diagonalizable over some extension field), so its centralizer $C_G(g)$ is well-behaved in that it is essentially the direct product of quasisimple groups of Lie type in characteristic p corresponding to the eigenspaces of g. There are standard methods for exploiting the structure of these *components*. On the other hand, if g is a p-element, then g is unipotent (i.e., all its eigenvalues are 1), so $C_G(g)$ has no components; instead its structure is dominated by the unipotent subgroup

$$F^*(C_G(g)) = O_p(C_G(g))$$

[2]See section 31 of [**Asc86a**] for the definition of a *component* of a finite group (namely quasisimple subnormal subgroup), and corresponding properties.

and in particular is more complex, so that this centralizer is more difficult to deal with.

We seek to translate these properties of linear groups, and in particular the notion of "characteristic", into analogous notions for abstract groups. If G is a finite group and p is a prime, G is defined to be of *characteristic p-type* if each p-local subgroup H of G satisfies

$$F^*(H) = O_p(H),$$

or equivalently $C_H(O_p(H)) \leq O_p(H)$. Every group of Lie type in characteristic p is of characteristic p-type; indeed for large p, they are the only examples of p-rank at least 2—though for small primes, there are groups of characteristic p-type which are not of Lie type in characteristic p.

If a simple group G of p-rank at least 3 is "connected" at the prime p (as discussed in the next section) but is not of characteristic p-type, then the centralizer of some element of order p will behave like the centralizer of a semisimple element in a group of Lie type—that is, it will have components, making it easier to analyze. Thus the aim is to find a prime p such that G has a reasonably rich p-local structure, but G is not of characteristic p-type. Recall also that one chooses p to be 2 whenever possible. The original proof of the Classification partitioned the simple groups into two classes: those of characteristic 2-type, and those not of characteristic 2-type; furthermore different techniques were used to analyze the two classes.

In the remainder of this subsection, we'll try to give some insight into how more recent work (done since the original proof of the Classification) has suggested that it is useful and natural to change the boundary of this even/odd partition. We mentioned earlier that in the GLS program, the notion of even type replaces the notion of characteristic 2-type. However for the purpose of dealing with quasithin groups, our notion of even characteristic seems to be more natural than that of even type. Notice that a group of characteristic 2-type *is* of even characteristic, since the former hypothesis requires all 2-locals to be of characteristic 2, while the latter imposes this constraint only on locals containing the Sylow group T. Thus the class of groups of even characteristic is larger than the class of groups of characteristic 2-type, since the 2-locals in the former class are more varied.

In a moment, we will discuss two classes of groups where this extra flexibility is useful. But before doing so, we'll say a word about the influence of these groups and others on our work. In December 1996, Helmut Bender gave a talk at the conference in honor of Bernd Fischer's 60th birthday, in which he suggested approaching classification problems like ours with a list of groups in mind, to serve as a guide to where difficulties are likely to occur. However, that list should include not only the "examples"—the groups which appear in the conclusion of the theorem; it should also include "shadows"—groups not in the conclusion, but whose local structure is very close to that of actual examples, since these configurations of local subgroups will also arise in the analysis, and typically they can be eliminated only with real effort. Thus in our exposition, we try to emphasize not only how the examples arise, but also where the shadows are finally eliminated. Our Index lists occurrences in the proof of examples and shadows.

In particular we must deal with shadows of the following two classes which are QTKE-groups but not simple—since it is hard to recognize *locally* that the groups are not simple.

Two non-simple configurations. Let L be a simple group of Lie type in characteristic 2, and assume either

(a) $G = L\langle t\rangle$ is L extended by an involutory outer automorphism t, or

(b) $G = (L \times L^t)\langle t\rangle$, for some involution t; i.e., G is the wreath product of L by $\mathbf{Z}_2$.

Then G is in fact of even characteristic, but rarely of characteristic 2-type, since $C_G(t)$ usually has a component. However the components of $C_G(t)$ are of Lie type in characteristic 2, so G is also usually of even type. During the proof of the CFSG, groups with the 2-local structure of those in (a) and (b) often arise. Under the original approach, lengthy and difficult computations were required, to reduce to a situation where transfer could be applied to show the group was not simple. In the opinion of GLS (and we agree), the proof should be restructured to avoid these difficulties.

This is achieved in GLS by replacing the old partition into characteristic 2-type/not characteristic 2-type by the partition into even type/odd type, while we achieve it for quasithin groups with the partition into even characteristic/not even characteristic. Locals like those in the two classes of nonsimple groups above are allowed under both the even characteristic hypothesis and the even type hypothesis, but were not allowed under the older characteristic 2-type hypothesis. Thus under the old approach, such groups would be treated in the "odd" case by focusing on the "semisimple" element t—rather artificially, as its order is *not* coprime to the characteristic of its components—and usually at great expense in effort. Under the new approach, such groups arise in the "even" case, where the focus is not on $C_G(t)$.

In the generic situation when G is "large" (see the next subsection for a discussion of size), GLS are able to avoid considering such centralizers by passing to centralizers of elements of odd prime order, which can therefore be naturally regarded as semisimple. However, quasithin groups G are "small", and in particular the p-rank of G is too small to pass to p-locals for odd p; so we avoid difficulties when G is of even characteristic by using unipotent methods applied to overgroups of T, rather than semisimple methods applied to $C_G(t)$. The case where G is of even type but not of even characteristic is discussed later in section 0.4 of this Introduction. There we will again encounter local subgroups resembling those in our two classes, when they appear as shadows in the proof of the Even Type Theorem.

0.2.2. Case division according to size. After the case division into characteristic 2-type/not characteristic 2-type or even type/odd type described above, both generations of the CFSG proceed by also partitioning the simple groups according to notions of size. Here the underlying idea is that above some critical size, there should be standard "generic" (i.e., size-independent) methods of analysis; but that "small" groups will probably have to be treated separately.

In the even/odd division of the previous subsection, we indicated that the generic examples for the even part of the partition should be the groups of Lie type in characteristic 2. For these groups the appropriate measure of size is the Lie rank of the group, and as we mentioned in section 0.1, $e(G)$ is a good approximation of the Lie rank for G of Lie type and characteristic 2. From this point of view, the quasithin groups are the small groups of even characteristic, so our critical value defining the partition into large and small groups occurs at $e(G) = 2$.

This leaves the question of *why* the boundary of the partition according to size occurs when $e(G) = 2$, rather than $e(G) = 1$ or 3 or something else. The answer is that when one passes to p-locals for odd primes p, $e(G) \geq 3$ is needed in order to use signalizer functors. (See e.g. chapter 15 of [**Asc86a**]). Namely such methods can only be applied to subgroups E which are elementary abelian p-groups of rank at least 3, and E needs to be in a 2-local because of connectedness theorems for the prime 2 (which will be discussed briefly in the next section). Using both signalizer functors and connectedness theorems for the prime 2, one can show that the centralizer of some element of E looks like the centralizer of a semisimple element in a group of Lie type and characteristic 2. Then this information is used to recognize G as a group of Lie type. [3]

Thus, in both programs, the two partitions of the simple groups indicated above, into groups of "even" and "odd" characteristic, and into large and small groups, give rise to a partition of the proof of the Classification into four parts. Since groups of even characteristic include those of characteristic 2-type, our Main Theorem determines the groups in one of the four parts—the small even part—in the first generation program.

To integrate our result into the GLS second-generation proof, we need to reconcile our notion of "even characteristic" with the GLS notion of "even type". The former notion is more natural in the context of the unipotent methods of this work, but the latter fits better with the GLS semisimple methods. Our Even Type Theorem provides the transition between the two notions, and is relatively easy to prove. We will say a little more about that result in section 0.4 of this introduction. The Main Theorem, together with the Even Type Theorem, determine the groups in the small even part of the second generation program.

0.2.3. Some history of the quasithin problem. We close this section with a few historical remarks about quasithin groups, and more generally small groups of even characteristic.

The methods used in attacking the problem go back to Thompson in the N-group paper [**Tho68**]; in an N-group, all local subgroups are assumed to be solvable. In particular, Thompson introduced the parameter $e(G)$, and used weak closure arguments, uniqueness theorems, and work of Tutte [**Tut47**] and Sims [**Sim67**]. We discuss some of these techniques in the next section; a more extended discussion appears in the Introduction to Volume I.

Groups G of characteristic 2-type with $e(G)$ small were subsequently studied by various authors. Note that $e(G) = 0$ means that all 2-locals are 2-groups, which is impossible in a nonabelian simple group of even order by an elementary argument going back to Frobenius; cf. the Frobenius Normal p-Complement Theorem 39.4 in [**Asc86a**]. In [**Jan72**], Janko defined G to be *thin* if $e(G) = 1$, and used Thompson's methods to determine all thin groups of characteristic 2-type in which all 2-locals are solvable. His student Fred Smith extended that classification from thin to quasithin groups in [**Smi75**]. The general thin group problem was solved by Aschbacher in [**Asc78b**]. Mason went a long way toward a complete treatment of the general quasithin case in [**Mas**], which unfortunately has never been published. See however his discussion of that work in [**Mas80**].

[3]In both the original proof of CFSG and in the GLS project, the case $e(G) = 3$ requires special treatment.

There have since been new treatments of portions of the N-group problem due to Stellmacher [**Ste97**] and to Gomi and his collaborators [**GT85**], using an extension of the Tutte-Sims theory which has come to be known as the *amalgam method*. The Thin Group Paper [**Asc78b**] used some early versions of such extensions due to Glauberman, which eventually were incorporated in the proof of the Glauberman-Niles/Campbell Theorem [**GN83**]. Goldschmidt initiated the "modern" amalgam method in [**Gol80**], and this was extended and the amalgam method modified in [**DGS85**] by Goldschmidt, Delgado, and Stellmacher, and in [**Ste92**] by Stellmacher. Those techniques and more recent developments are used in places in this work; our approach is a bit different from the standard approach, and is described briefly in section 0.10 of the Introduction to Volume I.

0.3. An Outline of the Proof of the Main Theorem

In this section we introduce some fundamental concepts and notation, and give a rough outline of the proof of the Main Theorem. Throughout the section, assume G is a simple QTKE-group and $T \in Syl_2(G)$. Recall that $\mathcal{M}$ is the set of maximal 2-local subgroups of G, and $\mathcal{M}(T)$ is the collection of maximal 2-locals containing T.

0.3.1. Setting up the Thompson amalgam strategy. An overall strategy for studying groups of even characteristic originated in Thompson's N-group paper [**Tho68**]; generically it involves exploiting the interaction of distinct maximal 2-locals $M, N \in \mathcal{M}(T)$. (We sometimes refer to this as the "Thompson amalgam strategy").

Of course prior to this generic case, we must first deal with the "disconnected" case where T lies in a unique maximal 2-local. To indicate that $|\mathcal{M}(T)| = 1$, we will usually write $\exists!\mathcal{M}(T)$, to emphasize the existence of the unique maximal 2-local overgroup of T. Recall that in the generic conclusion of the Main Theorem, where G is of Lie type of Lie rank at least 2, there are distinct maximal parabolics above T. So for us, the disconnected case will have as its generic conclusion the groups of Lie type of characteristic 2 and Lie rank 1. We handle this in Theorem 2.1.1, which says:

Theorem 2.1.1 If G is a simple QTKE-group such that $\exists!\mathcal{M}(T)$, then G is a rank 1 group of Lie type and characteristic 2, $L_2(p)$ with $p > 7$ a Mersenne or Fermat prime, $L_3(3)$, or M_{11}.

A finite group G is *disconnected* at the prime 2 if the *commuting graph* on vertices given by the set of nonidentity 2-elements of G (whose edges are pairs of vertices which commute as subgroups) is disconnected. The groups of Lie type and characteristic 2 of Lie rank 1 are the simple groups of 2-rank at least 2 which are disconnected at the prime 2. The classification of these groups is due to Bender [**Ben71**] and Suzuki [**Suz64**]; indeed the groups (namely $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$) are often referred to as *Bender groups.* However when working with groups of even characteristic, a weaker notion of disconnected group is also important: namely a group G of even characteristic should be regarded as disconnected if $\exists!\mathcal{M}(T)$ for $T \in Syl_2(G)$.

In view of Theorem 2.1.1, henceforth we will assume that $|\mathcal{M}(T)| \geq 2$. Thompson's strategy now fixes a particular maximal 2-local $M \in \mathcal{M}(T)$. Then instead of working with another maximal 2-local, it will be more advantageous (for reasons which will emerge below) to work with a subgroup H which is *minimal* subject to $T \leq H$, $H \not\leq M$, and $O_2(H) \neq 1$. For example if G is a group of Lie type and characteristic 2, then M is a maximal parabolic over T, and $H = O^{2'}(P)$, where P is the unique parabolic of Lie rank 1 over T not contained in M. Similar remarks hold for other simple groups G with diagram geometries.

We introduce some further definitions to formalize this approach in our abstract setting. We will need to work not only with 2-local subgroups, but also with various subgroups of 2-locals, so we define

$$\mathcal{H} = \mathcal{H}_G := \{H \leq G : O_2(H) \neq 1\},$$

and for $X \subseteq G$, define $\mathcal{H}(X) = \mathcal{H}_G(X) := \{H \in \mathcal{H} : X \subseteq H\}$. Note that any $H \in \mathcal{H}$ lies in the 2-local $N_G(O_2(H))$, and hence is contained in some member of $\mathcal{M}$. Thus as G is quasithin, each $H \in \mathcal{H}$ is in fact *strongly quasithin*; that is H satisfies:

(SQT) $m_p(H) \leq 2$ for each odd prime p.

In addition each $H \in \mathcal{H}$ must also be a $\mathcal{K}$-group by our hypothesis (K), so H in fact satisfies

(SQTK) H is a $\mathcal{K}$-group satisfying (SQT).

The possible simple composition factors for SQTK-groups are determined in Theorem C (A.2.3) in Volume I. The proof of the Main Theorem depends on general properties of $\mathcal{K}$-groups, but also on numerous special properties of the groups in Theorem C, so we refer to the list of groups in that Theorem frequently throughout our proof. We must also occasionally deal with proper subgroups which are not contained in 2-locals. Such groups are quasithin $\mathcal{K}$-groups but not necessarily SQTK-groups; thus we also require Theorem B (A.2.2), which determines all simple composition factors of such groups.

In view of Theorem 2.1.1, the set

$$\mathcal{H}(T, M) := \{H \in \mathcal{H}(T) : H \not\leq M\}$$

is nonempty. Write $\mathcal{H}_*(T, M)$ for the minimal members of $\mathcal{H}(T, M)$, partially ordered by inclusion. Note that for $H \in \mathcal{H}_*(T, M)$, $H \cap M$ is the unique maximal subgroup of H containing T by the minimality of H. Further if $N_G(T) \leq M$ (and we will show in Theorem 3.3.1 that this is usually the case), then T is not normal in H. These conditions give the definition of an abstract *minimal parabolic*, originating in work of McBride; see our definition B.6.1. The condition strongly restricts the structure of H. In particular, the possibilities for H are described in sections B.6 and E.2. In the most interesting case, $O^2(H/O_2(H))$ is a Bender group, so H does resemble a *minimal parabolic* in the Lie theoretic sense for a group of Lie type: namely $O^{2'}(P)$ where P is a parabolic of Lie rank 1.

Thus for each $M \in \mathcal{M}(T)$, we can choose some $H \in \mathcal{H}_*(T, M)$. By the maximality of M, $\langle M, H\rangle$ is not contained in a 2-local subgroup, so that $O_2(\langle M, H\rangle) = 1$. Thompson's weak closure methods and the later amalgam method depend on the latter condition, rather than on the maximality of M, so often we will be able to replace M by a smaller subgroup. We say U is a *uniqueness subgroup* of G if $\exists!\mathcal{M}(U)$. Furthermore we usually write $M = !\mathcal{M}(U)$ to indicate that M is the unique overgroup of U in $\mathcal{M}$. Notice that if $M = !\mathcal{M}(U)$, then from the definition of uniqueness subgroup, $O_2(\langle U, H\rangle) = 1$, so again we can apply weak closure arguments or the amalgam method to the pair U, H.

In the next subsection 0.3.2, we describe how to obtain a uniqueness subgroup U with useful properties, while subsection 0.3.3 discusses how to determine a list of possiblities for U. Here is a brief summary: No nontrivial subgroup T_0 of T can be normal in both U and H; in particular, $Z := \Omega_1(Z(T))$ is not in the center of Y for some $Y \in \{U, H\}$. This places strong restrictions on the $\mathbf{F}_2$-module $\langle Z^Y \rangle$, and on the action of Y on this module. Our approach concentrates on the situation where Y is the uniqueness group U. Roughly speaking, we can classify the possibilities for U and $\langle Z^U \rangle$, resulting in a list of cases to be analyzed when $Y = U$. The bulk of the proof of the Main Theorem then involves the treatment of these cases, a process which is outlined in the final subsection 0.3.4.

0.3.2. Finding a uniqueness subgroup. We put aside for a while the groups M and H from the previous subsection, to see how the hypothesis that G is a QTKE-group gives strong restrictions on the structure of 2-local subgroups of G.

We begin with the definition of objects similar to components: For $H \leq G$, let $\mathcal{C}(H)$ be the set of subgroups L of H minimal subject to

$$1 \neq L = L^\infty \trianglelefteq \trianglelefteq H.$$

We call the members of $\mathcal{C}(H)$ the *$\mathcal{C}$-components* of H. To illustrate and motivate this definition, consider the following

Example. Suppose G is a group of Lie type over a field $\mathbf{F}_{2^n}$ with $n > 1$, and H is a maximal parabolic. If H corresponds to an end node of the Dynkin diagram Δ of G, then H^∞ will be the unique member of $\mathcal{C}(H)$. But suppose instead that G is of Lie rank at least 3 and H corresponds to an interior node δ of Δ. Then the minimality of a $\mathcal{C}$-component L of H says that L covers only that part of the Levi complement corresponding to just one connected component of $\Delta - \{\delta\}$. Furthermore H^∞ is then the product of the $\mathcal{C}$-components of H, and distinct $\mathcal{C}$-components commute modulo $O_2(H)$.

We list some facts about $\mathcal{C}$-components and indicate where these facts can be found; see also section 0.5 of the Introduction to Volume I. In section A.3 we develop a theory of $\mathcal{C}$-components in SQTK-groups. Then in 1.2.1 we use this theory to show that two of the properties in the Example in fact hold for each $H \in \mathcal{H}$ in a QTKE-group G: namely $\langle \mathcal{C}(H) \rangle = H^\infty$, and for distinct $L_1, L_2 \in \mathcal{C}(H)$, $[L_1, L_2] \leq O_2(L_1) \cap O_2(L_2) \leq O_2(H)$. The quasithin hypothesis further restricts the number of factors and the structure of the factors in such commuting products: If $L \in \mathcal{C}(H)$, then either $L \trianglelefteq H$, or $|L^H| = 2$ and $L/O_2(L) \cong L_2(2^n)$, $Sz(2^n)$, $L_2(p)$ with p an odd prime, or J_1. In particular for $S \in Syl_2(H)$, $\langle L^S \rangle \trianglelefteq H$, and $\langle L^S \rangle$ is L or LL^s for some $s \in S$. Moreover 1.2.1.4 shows that almost always $L/O_2(L)$ is quasisimple. Since the cases where $L/O_2(L)$ is not quasisimple cause little difficulty, it is probably best for the expository purposes of this Introduction to ignore the non-quasisimple cases.

To get some control over how 2-locals intersect, and in particular to produce uniqueness subgroups, we also wish to see how $\mathcal{C}$-components of $H \in \mathcal{H}$ embed in other members of $\mathcal{H}$. To do so, we keep appropriate 2-subgroups S of H in the picture, and define $\mathcal{L}(H, S)$ to be the set of subgroups L of H with

$$L \in \mathcal{C}(\langle L, S \rangle),\ S \in Syl_2(\langle L, S \rangle), \text{ and } O_2(\langle L, S \rangle) \neq 1.$$

Again to movitate this definition, consider the case where G is the shadow obtained by extending $G_0 := L_4(2^n)$ for $n > 1$ by an involutory graph automorphism of G_0, with P the middle node maximal parabolic over $T \cap G_0$, and $H := PT$. Then $H \geq \langle L, T\rangle$ for an $L \in \mathcal{L}(G,T)$ with $|L^T| = 2$.

We partially order $\mathcal{L}(G,T)$ by inclusion and let $\mathcal{L}^*(G,T)$ denote the maximal members of this poset. In our earlier example where H is a parabolic of a group of Lie type, notice that any $L \in \mathcal{C}(H)$ is contained in a maximal parabolic determined by some end node. Thus the $\mathcal{C}$-components of such parabolics are the members of $\mathcal{L}^*(G,T)$.

In an abstract QTKE-group G, the members of $\mathcal{L}^*(G,T)$ can be used to produce uniqueness subgroups: For by 1.2.4, when $S \in Syl_2(H)$, any $L \in \mathcal{L}(H,S)$ is contained in some $K \in \mathcal{C}(H)$. Then a short argument in 1.2.7 shows that whenever $L \in \mathcal{L}^*(G,T)$,

$$N_G(\langle L^T\rangle) = !\mathcal{M}(\langle L, T\rangle).$$

Thus $\langle L, T\rangle$ is a uniqueness subgroup in our language, achieving the goal of this subsection.

But it could also happen (for example in a group of Lie type over the field $\mathbf{F}_2$) that the visible 2-locals are solvable, so that $\mathcal{L}(G,T)$ is empty. To deal with such situations, and with the case where $L/O_2(L)$ is not quasisimple for some $L \in \mathcal{L}^*(G,T)$, we also show that certain solvable minimal T-invariant subgroups are uniqueness subgroups. The quasithin hypothesis allows us to focus on p-groups of small rank: Define $\Xi(G,T)$ to consist of those T-invariant subgroups $X = O^2(X)$ of G such that

$XT \in \mathcal{H}$, $X/O_2(X) \cong E_{p^2}$ or p^{1+2} for some odd prime p, and T is irreducible on the Frattini quotient of $X/O_2(X)$.

For example, in the extension of $L_4(2^n)$ discussed above, if we take $n = 1$ instead of $n > 1$, then $H = PT \in \Xi(G,T)$ for $p = 3$.

If X is not contained in certain nonsolvable subgroups, then XT will be a uniqueness subgroup. Thus we are led to define $\Xi^*(G,T)$ to consist of those $X \in \Xi(G,T)$ such that XT is not contained in $\langle L, T\rangle$ for any $L \in \mathcal{L}(G,T)$ with $L/O_2(L)$ quasisimple. We find in 1.3.7 that if $X \in \Xi^*(G,T)$, then

$$N_G(X) = !\mathcal{M}(XT),$$

so that XT is a uniqueness subgroup.

0.3.3. Classifying the uniqueness groups and modules. We now return to our pair M, H with $M \in \mathcal{M}(T)$ and $H \in \mathcal{H}_*(T,M)$ from subsection 0.3.1. The structure of H is restricted since H is a minimal parabolic, but *a priori* M could be a fairly arbitrary quasithin group, subject to the constraint $F^*(M) = O_2(M)$; in particular, the composition factors of M could include arbitrary simple SQTK-groups acting on arbitrary "internal modules" (elementary abelian M-sections) involved in $O_2(M)$

To obtain a more tractable set of possibilities, we exploit a uniqueness subgroup U produced by one of the two methods in the previous subsection 0.3.2; that is, we take U of the form $\langle L, T\rangle$ with $L \in \mathcal{L}^*(G,T)$, or XT with $X \in \Xi^*(G,T)$, and take $M := N_G(O^2(U)) = !\mathcal{M}(U)$. Recall that $Z := \Omega_1(Z(T))$ cannot be central in both U and H. The case where $Z \leq Z(U)$ for all choices of U is essentially a "small" case, treated in Part 6, so most of the analysis deals with the case $[Z, U] \neq 1$.

We introduce notation to cover both the situations discussed in subsection 0.3.2: Define $\mathcal{X}$ to consist of those subgroups $X = O^2(X)$ of G such that $F^*(X) = O_2(X)$. For example $\mathcal{L}(G,T)$ and $\Xi(G,T)$ are contained in $\mathcal{X}$. To describe the members with a "faithful action", write $\mathcal{X}_f$ for those $X \in \mathcal{X}$ such that $[\Omega_1(Z(O_2(X))), X] \neq 1$, with a similar use of the subscript to define subsets $\mathcal{L}_f(G,T)$ and $\Xi_f(G,T)$. Our analysis focuses on the faithful uniqueness groups U in $\mathcal{L}^*_f(G,T)$ and $\Xi^*_f(G,T)$.

If $Y \in \mathcal{H}(T)$, so that $F^*(Y) = O_2(Y)$ by 1.1.4.6, then by a standard lemma B.2.14, $V := \langle Z^Y \rangle$ is elementary abelian and *2-reduced*: that is, $O_2(Y/C_Y(V)) = 1$. Following Thompson, define $\mathcal{R}_2(Y)$ to be the set of 2-reduced elementary abelian normal 2-subgroups of Y. By B.2.12 (26.2 in [**GLS96**]), the product of members of $\mathcal{R}_2(Y)$ is again in $\mathcal{R}_2(Y)$, so $\mathcal{R}_2(Y)$ has a unique maximal member $R_2(Y)$. We regard $R_2(Y)$ as an $\mathbf{F}_2Y$-module.

Observe that if $L \in \mathcal{L}^*_f(G,T)$ with $L/O_2(L)$ quasisimple, or $X \in \Xi^*_f(G,T)$, then $C_U(R_2(U)) \leq O_{2,\Phi}(U)$. [4] Then the representation of $U/C_U(R_2(U))$ on $R_2(U)$ (or indeed on any $V \in \mathcal{R}_2(U)$ with $V \not\leq Z(U)$) is particularly effective, since for any weakly closed subgroup W of $C_T(V)$, W is normal in the uniqueness subgroup U, so that $N_G(W) \leq M$. That is $M = !\mathcal{M}(U)$ contains the normalizers of various weakly closed subgroups W of T.

For $M := N_G(O^2(U))$ and U a uniqueness subgroup of the form $\langle L, T\rangle$ with $L \in \mathcal{L}^*(G,T)$, or XT with $X \in \Xi^*(G,T)$, we prove in Theorem 3.3.1 that $N_G(T) \leq M$. It follows that T is *not* normal in H in those cases, so that H is a minimal parabolic in the sense of Definition B.6.1, and hence we can use the explicit description of $H/O_2(H)$ from section E.2 mentioned earlier.

We next turn to Theorem 3.1.1, which is used in a variety of ways; it says:

Theorem 3.1.1 If $M_0, H \in \mathcal{H}(T)$, such that T is in a unique maximal subgroup of H, and $R \leq T$ with $R \in Syl_2(O^2(H)R)$ and $R \trianglelefteq M_0$, then $O_2(\langle M_0, H\rangle) \neq 1$.

For example in our standard setup we can take M_0 to be the uniqueness group U and $R := C_T(V)$—and conclude that $R \notin Syl_2(O^2(H)R)$, since $H \not\leq M = !\mathcal{M}(U)$; hence $O_2(\langle U, H\rangle) = 1$. In particular we use Theorem 3.1.1 to rule out the first case which occurs in Stellmacher's *qrc*-lemma D.1.5 (see below), and in the remaining cases the *qrc*-lemma gives us strong information on a module V for the action of U. That information is given in terms of small values of certain parameters, which we now introduce. For X a finite group, let $\mathcal{A}^2(X)$ denote the set of nontrivial elementary abelian 2-subgroups of X. Given a faithful $\mathbf{F}_2X$-module V, define

$$q(X,V) := \min\{\frac{m(V/C_V(A))}{m(A)} : 1 \neq A \in \mathcal{A}^2(X) \text{ such that } 0 = [V,A,A]\}$$

and the analogous parameter correponding to cubic rather than quadratic action:

$$\hat{q}(X,V) := \min\{\frac{m(V/C_V(A))}{m(A)} : 1 \neq A \in \mathcal{A}^2(X) \text{ such that } 0 = [V,A,A,A]\}.$$

For example V is a *failure of factorization module* (FF-module—see section B.1) for X precisely when $q(X,V) \leq 1$.

Using Theorem 3.1.1 and Stellmacher's *qrc*-Lemma (see Theorem D.1.5), we obtain:

Theorem 3.1.6 Let $T \leq M_0 \leq M \in \mathcal{M}(T)$ and $H \in \mathcal{H}_*(T,M)$ Assume $V \in \mathcal{R}_2(M_0)$ with $C_T(V) = O_2(M_0)$, and $H \cap M$ normalizes $O^2(M_0)$ or V. Then one of the following holds:

[4] Here $O_{2,\Phi}(U)$ denotes the preimage of the Frattini subgroup $\Phi(U/O_2(U))$; elsewhere we use similar notation such as $O_{2,F}(U)$, $O_{2,E}(U)$, etc.

(1)$O_2(\langle M_0, H\rangle) \neq 1$, so M_0 is not a uniqueness subgroup of G.
(2) $V \not\leq O_2(H)$ and $\hat{q}(M_0/C_{M_0}(V), V) \leq 2$.
(3) $q(M_0/C_{M_0}(V), V) \leq 2$.

When we apply this result with M_0 our uniqueness subgroup U from subsection 0.3.1, case (1) does not arise, so the module V satisfies $\hat{q} \leq 2$.

In section D.3, we determine the groups and modules satisfying this strong restriction (and a suitable minimality assumption) under the SQTK-hypothesis. Since the most general SQTK-group H of characteristic 2 could have arbitrary internal modules as sections of $O_2(H)$, Theorem 3.1.6 leads to a solution in section 3.2 of the First Main Problem for QTKE-groups:

First Main Problem. Show that a simple QTKE-group G does not have the local structure of the general nonsimple strongly quasithin $\mathcal{K}$-group Q with $F^*(Q) = O_2(Q)$, but instead has a more restrictive structure resembling that of the examples in the conclusion of the Main Theorem, or the shadows of groups with similar local structure.

A solution of the First Main Problem amounts to showing that there are relatively few choices for $L/O_2(L)$ and its action on V, where $L \in \mathcal{L}_f^*(G,T)$, $V \in \mathcal{R}_2(\langle L, T\rangle)$, and $[V, L] \neq 1$. Indeed in most cases, $L/O_2(L)$ is a group of Lie type in characteristic 2 and V is a "natural module" for $L/O_2(L)$. This leads us in section 3.2 to define the *Fundamental Setup* FSU (3.2.1), and to the possibilities for $L/O_2(L)$ and V listed in 3.2.5–3.2.9. The proof can be roughly summarized as follows: Apply Theorem 3.1.6 to $M_0 := U = \langle L, T\rangle$. As M_0 is a uniqueness subgroup, conclusion (1) of 3.1.6 cannot hold. Then from section D.3, the restrictions on q and $\hat{q}$ in conclusions (2) and (3) of 3.1.6 allow us to determine a short list of possibilities for $M_0/C_{M_0}(V)$ and its action on V.

0.3.4. Handling the resulting list of cases. We continue to restrict attention to the most important case where $L \in \mathcal{L}^*(G,T)$ with $L/O_2(L)$ quasisimple, and let $L_0 := \langle L^T\rangle$ and $M := N_G(L_0)$. Then in the FSU, there is $1 \neq V = [V, L_0] \in \mathcal{R}_2(L_0T)$ with $V/C_V(L_0)$ an irreducible L_0T-module. Set $V_M := \langle V^M\rangle$ and $\bar{M} := M/C_M(V_M)$. By 3.2.2, $V_M \in \mathcal{R}_2(M)$, and by Theorems 3.2.5 and 3.2.6, we may choose V so that one of the following holds:

(1) $V = V_M \trianglelefteq M$.
(2) $C_V(L) = 1$, $V \trianglelefteq T$, and V is a TI-set under M. [5]
(3) $\bar{L} \cong L_3(2)$, $L < L_0$, and subcase 3.c.iii of Theorem 3.2.6 holds.

Further the choices for L and V are highly restricted, and are listed in Theorems 3.2.5 and 3.2.6, with further information given in 3.2.8 and 3.2.9.

The bulk of the proof of our Main Theorem consists of a treatment of the resulting list of possibilities for L and V. The analysis falls into several broad categories: The cases with $|L^T| = 2$ are handled comparatively easily in chapter 10; so from now on assume that $L \trianglelefteq M$. The Generic Case where $\bar{L} \cong L_2(2^n)$ (leading to the generic conclusion in our Main Theorem of a group of Lie type and characteristic 2 of Lie rank 2) is handled in Part 2. Most cases where V is not an FF-module for $LT/O_2(LT)$ are eliminated in Part 3. The remaining cases where V is an FF-module are handled in Parts 4 and 5.

[5] Recall a TI-set is a set intersecting trivially with its distinct conjugates.

In order to discuss these cases in more detail, we need more concepts and notation.

First, another consequence of Theorem 3.1.1 (established as part (3) of Theorem 3.1.8) is that either

(i) $L = [L, J(T)]$, or
(ii) $\mathcal{H}_*(T, M) \subseteq C_G(Z)$, where $Z = \Omega_1(Z(T))$.

Here $J(T)$ is the Thompson subgroup of T (cf. section B.2). In case (i), V is an FF-module; so when V is not an FF-module, we know $[Z, H] = 1$ for all $H \in \mathcal{H}_*(T, M)$. In particular $C_V(L) = 1$ since H is not contained in the uniqueness group M for LT, whereas if $C_V(L)$ were nontrivial then $C_Z(L)$ would be nontrivial and centralized by H as well as LT.

Second, in section E.1, we introduce a parameter $n(H)$ for $H \in \mathcal{H}$. The parameter involves the generation of H by minimal parabolics, but the definition of $n(H)$ is somewhat more complicated; for expository purposes one can oversimplify somewhat to say that roughly $n(H) = 1$ unless H has a composition factor which is of Lie type over $\mathbf{F}_{2^n}$—in which case $n(H)$ is the maximum of such n. Thus for example in a twisted group H of Lie type, $n(H)$ is usually the exponent n of the larger of the orders of the fields of definitions of the Levi factors of the parabolics of Lie rank 1 of H. In particular if $H \in \mathcal{H}_*(T, M)$, then either $n(H) = 1$, or (using section B.6) $O^2(H/O_2(H))$ is a group of Lie type over $\mathbf{F}_{2^n}$ of Lie rank at most 2, $O^2(H) \cap M$ is a Borel subgroup of $O^2(H)$, and $n(H) = n$. In that event, we call the Hall $2'$-subgroups of $H \cap M$ *Cartan subgroups* of H. Our object is to show that $n(H)$ is roughly bounded above by $n(L)$, and to play off Cartan subgroups of H against those of L when $L/O_2(L)$ is of Lie type. It is easy to see that if $n(H) > 1$ and B is a Cartan subgroup of $H \cap M$, then $H = \langle H \cap M, N_H(B) \rangle$, so that $N_G(B) \not\leq M$. On the other hand, if $n(H)$ is small relative to $n(L)$ (e.g. if $n(H) = 1$), then weak closure arguments can be effective.

Third, except in certain cases where V is a small FF-module, we obtain the following important result, which produces still more uniqueness subgroups:

Theorem 4.2.13 With small exceptions, if $I \leq LT$ with $L \leq O_2(LT)I$ and $I \in \mathcal{H}$, then I is also a uniqueness subgroup.

Theorem 4.2.13 has a variety of consequences, but perhaps its most important application is in Theorem 4.4.3, to show that (except when V is a small FF-module) if $1 \neq B$ is of odd order in $C_M(V)$, then $N_G(B) \leq M$. In particular from the previous paragraph, if $H \in \mathcal{H}_*(T, M)$ with $n(H) > 1$ and B is a Cartan subgroup of $H \cap M$, then $[V, B] \neq 1$. If $[Z, H] = 1$, this forces B to be faithful on L, so that it is possible to compare $n(H)$ to $n(L)$ and show that $n(H)$ is not large relative to $n(L)$.

0.3.4.1. *Weak Closure methods.* Thompson introduced weak closure methods in the N-group paper [**Tho68**]. When $n(H)$ is small relative to $n(L)$ and (roughly speaking) $q(LT/O_2(LT), V)$ is not too small, weak closure arguments become effective. We will not discuss weak closure in any detail here, but instead direct the reader to the discussion in section 0.9 of the Introduction to Volume I, and to section E.3 of Volume I, particularly the exposition introducing that section and the introductions to subsections E.3.1 and E.3.3. However we will at least say here that weak closure, together with the constellation of concepts and techniques introduced earlier in this subsection, plays the largest role in analyzing those cases in the

FSU where V is not an FF-module. The only quasithin example which arises from those cases is J_4, but shadows of groups like the Fischer groups and Conway groups complicate the analysis, and are only eliminated rather indirectly because they are not quasithin. When V is not an FF-module, the pair $L/O_2(L)$, V is usually sufficiently far from pairs in examples or shadows, that the pair can be eliminated by comparing various paramters from the theory of weak closure.

0.3.4.2. *The Generic Case.* In the Generic Case, $\bar{L} \cong L_2(2^n)$ and $n(H) > 1$ for some $H \in \mathcal{H}_*(T, M)$. We prove in Theorem 5.2.3 that the Generic Case leads to the bulk of the groups of Lie type and characteristic 2 in the conclusion of our Main Theorem; to be precise, one of the following holds:

(1) V is the A_5-module for $L/O_2(L) \cong L_2(4)$.

(2) $G \cong M_{23}$.

(3) G is Lie type of Lie rank 2 and characteristic 2.

To prove Theorem 5.2.3, we proceed by showing that if neither (1) nor (2) holds, and D is a Cartan subgroup of L, then the amalgam

$$\alpha := (LTB, BDT, HD)$$

is a *weak BN-pair* of rank 2 in the sense of the "Green Book" [**DGS85**]; then by Theorem A of the Green Book and results of Goldschmidt [**Gol80**] and Fan [**Fan86**], the amalgam α is determined up to isomorphism. At this point there is still work to be done, as this determines G only up to "local isomorphism". Fortunately there is a reasonably elegant argument to complete the final identification of G as a group of Lie type and characteristic 2; this argument is discussed in the Introduction to Volume I, in section 0.12 on recognition theorems. It also requires the extension 4.3.2 of Theorem 4.2.13 to show that $G = \langle L, H \rangle$.

After dealing with the Generic Case, we still have to consider the situation where $L/O_2(L) \cong L_2(2^n)$ and $n(H) = 1$ for all $H \in \mathcal{H}_*(T, M)$; in Theorem 6.2.20, we show that then either V is the A_5-module for $L/O_2(L)$, or $G \cong M_{22}$. Thus from now on, if $L/O_2(L) \cong L_2(2^n)$, we may assume $n = 2$ and V is the A_5-module.

0.3.4.3. *Other FF-modules.* Next in Theorem 11.0.1, we eliminate the cases where $\bar{L}$ is $SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ for $n > 1$. From the list in section 3.2, this leaves the cases where $\bar{L}$ is essentially a group of Lie type defined over $\mathbf{F}_2$; that is, $\bar{L}$ is $L_n(2)$, $n = 3, 4, 5$; A_n, $n = 5, 6$; or $U_3(3) = G_2(2)'$—and V is an FF-module. Roughly speaking, these cases, together with certain cases where $\mathcal{L}_f(G, T)$ is empty, are the cases left untreated in Mason's unpublished preprint. They are also the most difficult cases to eliminate.

We first show either that there is $z \in Z \cap V^\#$ with $G_z := C_G(z) \not\leq M$, or G is A_8, A_9, M_{22}, M_{23}, M_{24}, or $L_5(2)$. In the latter case the groups appear as conclusions in our Main Theorem, so we may now assume the former.

Let $\tilde{G}_z := G_z/\langle z \rangle$, $L_z := O^2(C_L(z))$, and $V_z := \langle V_2^{L_z} \rangle$, where V_2 is the preimage of $C_{\tilde{V}}(T)$, and $U := \langle V_z^{G_z} \rangle$. Then $\tilde{U} \leq Z(O_2(\tilde{G}_z))$ by B.2.14, and our next task is to reduce to the case where U is elementary abelian. If not, then $U = Z(U)Q_U$, where Q_U is an extraspecial 2-group, and then to analyze $\tilde{G}_z$, we can use some of the ideas from the the theory of groups with a large extraspecial 2-group (cf. [**Smi80**]) in the original CFSG: We first show that if $Z(U) \neq \langle z \rangle$, then $G \cong Sp_6(2)$ or HS. Hence we may assume in the remainder of this case that U is extraspecial. Then we repeat some of the elementary steps in Timmesfeld's analysis in [**Tim78**], followed by appeals to results on $\mathbf{F}_2$-modules in section G.11, to pin down the structure of

G_z. At this point our recognition theorems show that G is $G_2(3)$, $L_4^\epsilon(3)$, or $U_4(2)$. The shadow of the Harada-Norton group F_5 also arises to cause complications.

We have reduced to the case where U is abelian. In this difficult case, we show that only $G \cong Ru$ arises. Our approach is to use a modified version of the amalgam method on a pair of groups (LT, H), where $H \in \mathcal{H}(L_z T)$ with $H \not\leq M$. Using the fact that U is abelian, we can show that $\langle V^{G_z} \rangle$ is abelian, and hence conclude that $[V, V^g] = 1$ if $V \cap V^g \neq 1$. In the context of the amalgam method, this shows that the graph parameter b is odd and greater than 1. Then we show that $q(H/C_H(\tilde{U}), \tilde{U}) \leq 2$, which eventually leads to the elimination of all choices for $L/O_2(L)$, V, $H/C_H(\tilde{U})$, and U other than the 4-tuple leading to the Ru example.

We have completed the outline of our treatment of quasithin groups in the main case, when there is $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple. The case where $L/O_2(L)$ is not quasisimple is handled fairly easily in section 13.1. That leaves:

0.3.5. The case $\mathcal{L}_f(G,T)$ empty. In Part 6 we handle the case $\mathcal{L}_f(G,T) = \emptyset$. Part of the analysis here has some similarities to the $\mathbf{F}_2$-case just discussed, and leads to the groups J_2, J_3, ${}^3D_4(2)$, the Tits group ${}^2F_4(2)'$, $U_3(3)$, M_{12}, $L_3(2)$, and A_6.

To replace the uniqueness subgroup $\langle L, T \rangle$, we introduce the partial order $\lesssim$ on $\mathcal{M}(T)$ defined in section A.5, choose $M \in \mathcal{M}(T)$ maximal with respect to $\lesssim$, and set $Z := \Omega_1(Z(T))$ and $V := \langle Z^M \rangle$. Then by A.5.7, for each overgroup X of T in M with $M = C_M(V)X$, we obtain $M = !\mathcal{M}(X)$. The case where $C_G(Z)$ is not a uniqueness subgroup is relatively easy, and handled in the last section of Part 6; in this case $G \cong L_3(2)$ or A_6. The case where $C_G(Z)$ is a uniqueness subgroup is harder; the subcase where $m(V) = 2$ and $Aut_M(V) \cong L_2(2)$ presents the greatest difficulties, and is handled in Part 5—in tandem with the cases where V is the natural module for $L/O_2(L) \cong L_n(2)$ for $n = 4$ and 5. The elimination of these cases completes the proof of our Main Theorem.

0.4. An Outline of the Proof of the Even Type Theorem

Assume in this section that G is a simple QTK-group of even type, but G is not of even characteristic. We outline our approach for showing G is isomorphic to the smallest Janko group J_1.

As G is of even type, there is an involution $z \in Z(T)$ and a component L of $C_G(z)$. As G is quasithin of even type, the possibilities for L are few. Our object is to show that L is a *standard* subgroup of G: That is, we must show that L commutes with none of its conjugates, $N_G(L) = N_G(C_G(L))$, and $C_G(L)$ is *tightly embedded* in G. This last means that $C_G(L)$ is of even order, but if $g \in G - N_G(L)$ then $C_G(L) \cap C_G(L^g)$ is of odd order. Once this is achieved, the facts that $z \in Z(T)$ and that L is highly restricted will eventually eliminate all configurations except $L \cong L_2(4)$ and $C_G(z) = \langle z \rangle \times L$, where $G \cong J_1$ via a suitable recognition theorem.

Here are some details of the proof. We first observe that if i is an involution in $C_T(L)$ and $|S : C_T(i)| \leq 2$ for some $S \in Syl_2(C_G(i))$, then L is a component of $C_G(i)$: For L is a component at least of $C_{C_G(i)}(z)$, and hence contained in KK^z for some component K of $C_G(i)$ by "L-balance" (see I.3.1). Now the hypothesis that $|S : C_T(i)| \leq 2$, together with the restricted choices for K, leads to $L = K$ as desired.

This fundamental lemma can be used to show first that $L \trianglelefteq C_G(z)$—which is very close to showing that L commutes with none of its conjugates. Then the fundamental lemma also shows that $L \trianglelefteq C_G(i)$ for each involution $i \in C_G(L)$, after which it is a short step to showing that $C_G(L)$ is tightly embedded in G, and L is standard in G.

At this point, we could quote some of the theory of standard subgroups and tightly embedded subgroups (developed in [**Asc75**] and [**Asc76**]) to simplify the remainder of the proof. But since GLS do not use this machinery, we content ourselves with using only elementary lemmas from that theory which are easy to prove; the lemmas and their proofs are reproduced in sections I.7 and I.8. In particular, we use I.8.2 to see that our hypothesis that G is not of even characteristic shows that for some L^g distinct from L, an involution of $C_G(L^g)$ normalizes L; this provides the starting point for our analysis. Then, making heavy use of the fact that z is 2-central, and that the component L is highly restricted by the even type hypothesis, we eliminate all configurations except $N_G(L) \cong \mathbf{Z}_2 \times L_2(4)$. Then we identify $G = J_1$ via the structure of $C_G(z)$ as noted above. Along the way, we encounter various shadows coming from groups which are not perfect, like the groups in the examples in subsection 0.2.1. In most such cases it is possible to apply transfer to contradict $G = O^2(G)$, given the fact that the Sylow 2-group T of G normalizes L.

This shows the advantages of introducing the notion of a group of "even characteristic", and hence of the the partition of the quasithin groups of even type into those of even characteristic, and those of even type which are not of even characteristic: The first subclass we studied via unipotent methods, and the latter by semisimple methods at the prime 2. If instead we had used unipotent methods to treat only the more restricted subclass of groups of characteristic 2-type, then our semisimple analysis at the prime 2 would have had to deal with the shadows of the nonsimple configurations in subsection 0.2.1, in which involution centralizers $C_G(z)$ with components do not contain a Sylow 2-group T of G. When z is not 2-central the road to obtaining T, so that one can show G is not simple via transfer, is much longer and very bumpy.

As a final remark, we recall that for the generic groups of even type, GLS are able switch to semisimple analysis of elements of *odd* prime order, and so are able to avoid dealing with shadows of the nonsimple examples of subsection 0.2.1. Thus they do not need the concept of groups of "even characteristic" in their generic analysis.

Part 1

Structure of QTKE-Groups and the Main Case Division

See the Introductions to Volumes I and II for terminology used in this overview.

In this first Part, we obtain a solution to the First Main Problem: that is, we show that a simple QTKE-group G (with Sylow 2-subgroup T) does not have the local structure of the arbitrary nonsolvable SQTK-group Q with $F^*(Q) = O_2(Q)$, but instead has more restricted 2-locals resembling those in examples and shadows. More precisely, we establish the existence of a "large" member of $\mathcal{H}(T)$ (i.e., a uniqueness subgroup of G) resembling a maximal 2-local in an example or shadow. Then the cases corresponding to the possible uniqueness subgroups will be treated in subsequent Parts of this Volume.

Here is an outline of Part 1:

In chapter 1 we use the results in sections A.2 and A.3 of Volume I to establish tools for working in 2-local subgroups H of G, using the fact that our 2-locals are strongly quasithin. In particular we obtain a good description of the last term H^∞ of the derived series for H, primarily in terms of the $\mathcal{C}$-components of H, and some information about $F(H/O_2(H))$. We then go on to show that certain subgroups of G are "uniqueness subgroups" contained in a unique maximal 2-local M. In particular, we show that members of the sets $\mathcal{L}^*(G,T)$ and $\Xi^*(G,T)$ are uniqueness subgroups.

The "disconnected" case where T itself is a uniqueness subgroup and so contained in a unique maximal 2-local, is treated in chapter 2, which characterizes certain small groups via this property. Consequently after Theorem 2.1.1 is proved, we are able to assume during the remainder of the proof of the Main Theorem that T is contained in at least two maximal 2-locals of G. Hence there exist 2-locals H with $T \leq H \not\leq M$.

Next in chapter 3, we begin by proving two important preliminary results: Theorem 3.3.1 which says that $N_G(T) \leq M$ when $M = !\mathcal{M}(L)$ with L in $\mathcal{L}^*(G,T)$ or $\Xi^*(G,T)$; and Theorem 3.1.1, which among other things is needed to apply Stellmacher's *qrc*-lemma D.1.5 to the amalgam defined by M and H. The *qrc*-lemma gives strong restrictions on certain internal modules U for M via the bound $\hat{q}(Aut_M(U), U) \leq 2$. Section 3.2 then uses those restrictions to determine the list of possibilities for $L/O_2(L)$ with $L \in \mathcal{L}^*_f(G,T)$, and for the internal modules $V \in \mathcal{R}_2(\langle L, T\rangle)$. This provides the Main Case Division for the proof of the Main Theorem. One consequence of Theorem 3.3.1 is that members of $\mathcal{H}_*(T,M)$ are minimal parabolics, in the sense of the Introduction to Volume II.

The first Part concludes with chapter 4, which uses the methods of pushing up from chapter C of Volume I to establish some important technical results: In particular, we show in Theorem 4.2.13 that unless V is an FF-module and L is "small", then for each $I \leq L$ with $O_2(I) \neq 1$ and $L = O_2(L)I$, we have $M = !\mathcal{M}(I)$. This large family of uniqueness subgroups then allows us (in Theorems 4.4.3 and 4.4.14) to control the normalizers of nontrivial subgroups of odd order centralizing V. This control is in turn important later, particularly in Part 2 and in chapter 11, when we deal with cases where $L/O_2(L)$ (or $H/O_2(H)$ for $H \in \mathcal{H}_*(T,M)$) is of Lie type over $\mathbf{F}_{2^n}$ for some $n > 1$, allowing us to exploit the existence of nontrivial Cartan subgroups.

CHAPTER 1

Structure and intersection properties of 2-locals

In this chapter we show how the structure theory for SQTK-groups from section A.3 of Volume I translates into a description of the 2-local subgroups of a QTKE-group G. We then use this description to establish the existence of certain uniqueness subgroups, which are crucial to our analysis. We will concentrate on $\mathcal{C}$-components of 2-locals, and the two families $\mathcal{L}(G,T)$ and $\Xi(G,T)$ of subgroups of G discussed in the Introduction to Volume II.

In this chapter, and indeed unless otherwise specified throughout the proof of the Main Theorem, we adopt the following convention:

NOTATION 1.0.1 (Standard Notation). G is a simple QTKE-group, and $T \in Syl_2(G)$.

Recall from the Introduction to Volume I that a finite group G is a *QTKE-group* if

(QT) G is quasithin,
(K) every proper subgroup of G is a $\mathcal{K}$-group, and
(E) $F^*(M) = O_2(M)$ for each maximal 2-local subgroup M of G of odd index.

Also as in the Introductions to Volumes I and II, let $\mathcal{M}$ denote the set of maximal 2-local subgroups of G, for $X \subseteq G$ define

$$\mathcal{M}(X) := \{N \in \mathcal{M} : X \subseteq N\},$$

and recall that a subgroup $U \leq M \in \mathcal{M}$ is a *uniqueness subgroup* if $M = !\mathcal{M}(U)$. (Which means $\mathcal{M}(U) = \{M\}$ in the notation more common in the earlier literature). The members of $\mathcal{M}$ are of course uniqueness subgroups, but for our purposes it is preferable to work with smaller uniqueness subgroups, which have better properties in various arguments involving amalgams, pushing up, etc. We summarize some useful properties of uniqueness subgroups in the final section of the chapter.

1.1. The collection $\mathcal{H}^e$

DEFINITION 1.1.1. Define $\mathcal{H}^e = \mathcal{H}^e_G$ to be the set of subgroups H of G such that $F^*(H) = O_2(H)$; equivalently $C_H(O_2(H)) \leq O_2(H)$ or $O^2(F^*(H)) = 1$.

Using this notation, Hypothesis (E)—namely that G is of *even characteristic*—just says

$$\mathcal{M}(T) \subseteq \mathcal{H}^e.$$

The property that $H \in \mathcal{H}^e$ has many important consequences which we can exploit later—notably the existence of 2-reduced internal modules for H, such as in lemma B.2.14. Thus we want $\mathcal{H}^e$ to be as large as possible, so in this section we establish several sufficient conditions to ensure that a subgroup is in $\mathcal{H}^e$.

We begin by defining some notation.

DEFINITION 1.1.2. Set

$$\mathcal{H} = \mathcal{H}_G := \{H \leq G : O_2(H) \neq 1\};$$

and for $X \subseteq G$, set

$$\mathcal{H}(X) = \mathcal{H}_G(X) := \{H \in \mathcal{H} : X \subseteq H\}.$$

For $X \subseteq Y \subseteq G$, set

$$\mathcal{H}(X,Y) = \mathcal{H}_G(X,Y) := \{H \in \mathcal{H}(X) : H \not\subseteq Y\}.$$

Define $\mathcal{H}^e(X)$ (resp. $\mathcal{H}^e(X,Y)$) as the intersection of $\mathcal{H}^e$ with $\mathcal{H}(X)$ (resp. $\mathcal{H}(X,Y)$).

The subgroups in $\mathcal{H}$ are the primary focus of our proof, so we record here the following elementary (but important) observations: Notice that by (QT), H is an SQT-group. As G is simple and $O_2(H) \neq 1$, certainly H is proper in G; hence by (K), simple sections of subgroups of H are in $\mathcal{K}$, so that H is an SQTK-group. Then by (2) of Theorem A (A.2.1), all simple sections of H are also SQTK-groups.

We are interested in conditions on members H of $\mathcal{H}$ which will ensure that $H \in \mathcal{H}^e$. For example, in 1.1.4.6 below, we show that each member of the collection $\mathcal{H}(T)$ is in $\mathcal{H}^e$. We begin with some well known results in that spirit, which we use frequently:

LEMMA 1.1.3. *Let $M \in \mathcal{H}^e$. Then*

(1) If $1 \neq N \trianglelefteq \trianglelefteq M$, then $N \in \mathcal{H}^e$.

(2) If X is a 2-subgroup of M, and $XC_M(X) \leq H \leq N_M(X)$, then $H \in \mathcal{H}^e$ and $C_M(X) \in \mathcal{H}^e$.

(3) If $H \leq M$ and $B_1, \ldots, B_n$ are 2-subgroups of H with $B_j \leq N_H(B_i)$ for all $i \leq j$ and $H = \bigcap_{i=1}^n N_M(B_i)$, then $H \in \mathcal{H}^e$.

PROOF. As $N \trianglelefteq \trianglelefteq M$, $O^2(F^*(N)) \leq O^2(F^*(M)) = 1$. Thus (1) holds. If X is a 2-subgroup of M, then $N_M(X) \in \mathcal{H}^e$ by 31.16 in [**Asc86a**], so $C_M(X) \in \mathcal{H}^e$ by (1). If $XC_M(X) \leq H \leq N_M(X)$, then $X \leq O_2(H)$, so $O^2(F^*(H))$ centralizes X, and hence $O^2(F^*(H)) \leq O^2(F^*(C_M(X))) = 1$, so that $H \in \mathcal{H}^e$. Thus (2) holds, and (3) follows from (2) by induction on n. □

For $X \leq G$ let $\mathcal{S}_2(X)$ be the set of nontrivial 2-subgroups of X, and let $\mathcal{S}_2^e(G)$ consist of those $S \in \mathcal{S}_2(G)$ such that $N_G(S) \in \mathcal{H}^e$. Here is a collection of conditions sufficient to ensure that various overgroups and subgroups are in $\mathcal{H}^e$:

LEMMA 1.1.4. *(1) If $U \in \mathcal{S}_2^e(G)$ and $U \leq V \in \mathcal{S}_2(G)$, then $V \in \mathcal{S}_2^e(G)$.*

(2) If $1 \neq U \trianglelefteq T$, then $U \in \mathcal{S}_2^e(G)$. In particular 2-locals containing T are in $\mathcal{H}^e$.

(3) If $U \in \mathcal{S}_2(G)$ and $1 \neq Z(T) \cap U$, then $U \in \mathcal{S}_2^e(G)$.

(4) If $1 \neq N \leq M \leq G$ with $M \in \mathcal{H}^e$ and $C_{O_2(M)}(O_2(N)) \leq N$, then $N \in \mathcal{H}^e$.

(5) If $1 \neq N \leq M \in \mathcal{M}(T)$ with $C_{O_2(M)}(O_2(N)) \leq N$, then $N \in \mathcal{H}^e$.

(6) $\mathcal{H}(T) \subseteq \mathcal{H}^e$.

(7) If $M \in \mathcal{H}^e$, $S \in Syl_2(M)$, and $1 \neq M_1 \leq M$ with $|S : S \cap M_1| \leq 2$, then $M_1 \in \mathcal{H}^e$.

PROOF. Assume the hypotheses of (1) and set $N := N_G(U)$. Then by hypothesis $N \in \mathcal{H}^e$. Now if $U \trianglelefteq V$ then $V \leq N$, so $N_N(V) \in \mathcal{H}^e$ by 1.1.3.2. But

$$O^2(F^*(N_G(V))) \leq C_G(V) \leq C_G(U) \leq N,$$

so $O^2(F^*(N_G(V))) \leq O^2(F^*(N_N(V))) = 1$ as $N_N(V) \in \mathcal{H}^e$. Therefore $N_G(V) \in \mathcal{H}^e$ as desired. This shows that (1) holds when $U \trianglelefteq V$. Then as $U \trianglelefteq \trianglelefteq V$, (1) holds by induction on $|V : U|$.

Under the hypotheses of (2), $N_G(U)$ is contained in some $X \in \mathcal{M}(T)$, and, as we remarked earlier, $X \in \mathcal{H}^e$ by Hypothesis (E). Then as $N_G(U) = N_X(U)$, $N_G(U) \in \mathcal{H}^e$ by 1.1.3.2, proving (2).

For (3), observe $Z(T) \cap U \in \mathcal{S}_2^e(G)$ by (2), and then $U \in \mathcal{S}_2^e(G)$ by (1).

Now assume the hypotheses of (4) and set $R := C_{O_2(M)}(O_2(N))$. As $R \leq N \leq M$ by hypothesis, we conclude $R \leq O_2(N)$; and then $O_2(N)$ and R are centralized by $O^2(F^*(N)) =: L$. Then as $L = O^2(L)$, the Thompson $A \times B$-lemma A.1.18 says L centralizes $O_2(M)$. But $O_2(M) = F^*(M)$ as $M \in \mathcal{H}^e$, so that $L \leq Z(O_2(M))$, and then $L = O^2(L)$ forces $L = 1$. Thus (4) is established.

As G is of even characteristic, $\mathcal{M}(T) \subseteq \mathcal{H}^e$, so (4) implies (5).

If $N \in \mathcal{H}(T)$, then $O_2(N) \neq 1$, so there is M such that

$$T \leq N \leq N_G(O_2(N)) \leq M \in \mathcal{M}(N_G(O_2(N))).$$

Then as $T \in Syl_2(M)$, $M \in \mathcal{H}^e$ by (E), and also $O_2(M) \leq N$ by A.1.6. Therefore $N \in \mathcal{H}^e$ by (5), proving (6).

Finally assume the hypotheses of (7) and set $M_2 := M_1 O_2(M)$. By (4), $M_2 \in \mathcal{H}^e$. But as $|S : S \cap M_1| \leq 2$, $|M_2 : M_1| \leq 2$, and so $M_1 \trianglelefteq M_2$. Then $M_1 \in \mathcal{H}^e$ by 1.1.3.1 establishing (7).

This completes the proof of 1.1.4. □

We also need to control members of $\mathcal{H}$ which are not in $\mathcal{H}^e$. The following result gives some control in an important special case. For example, the subsequent result 1.1.6 shows that the hypotheses are achieved in any sufficiently large subgroup of a 2-local subgroup.

Recall our convention in Notation A.3.5 that $\hat{A}_6$, $\hat{A}_7$, and $\hat{M}_{22}$ denote the nonsplit 3-fold covers of A_6, A_7, and M_{22}.

LEMMA 1.1.5. *Let $H \in \mathcal{H}$, $S \in Syl_2(H)$, and $M \in \mathcal{H}^e(S)$. Assume that*

$$C_{O_2(M)}(O_2(H \cap M)) \leq H,$$

and $M \in \mathcal{H}(C_G(z))$ for some $1 \neq z \in \Omega_1(Z(S))$. Then:

(1) $F^*(H \cap M) = O_2(H \cap M)$.

(2) z inverts $O(H)$.

(3) If L is a component of H, then $L = [L, z] \not\leq M$ and one of the following holds:

(a) L is simple of Lie type and characteristic 2, described in conclusion (3) or (4) of Theorem C (A.2.3), and z induces an inner automorphism on L.

(b) $1 \neq Z(L) = O_2(L)$ and $L/O_2(L)$ is $L_3(4)$ or $G_2(4)$, with z inducing an inner automorphism on L.

(c) $L \cong A_6$ or $\hat{A}_6$, and z induces a transposition on L.

(d) $L \cong A_7$ or $\hat{A}_7$, and z acts on L with cycle structure 2^3.

(e) $L \cong L_3(3)$ or $L_2(p)$, p a Fermat or Mersenne prime, and z induces an inner automorphism on L.

(f) $L/O_2(L)$ is a Mathieu group, J_2, J_4, HS, He, or Ru; and z induces a 2-central inner automorphism on L.

PROOF. Part (1) follows from 1.1.4.4 applied with $H \cap M$ in the role of "N", in view of our hypothesis.

Next $C_G(z) \leq M$ by hypothesis, so

$$C_{O(H)}(z) \leq O(H) \cap M \leq O(H \cap M) = 1$$

by (1), giving (2).

Now assume L is a component of H. If $L \leq M$ then $L \leq E(H \cap M)$, contrary to (1). Thus $L \not\leq M$ so in particular $L \not\leq C_G(z)$.

As $z \in Z(S)$ and $S \in Syl_2(H)$, z normalizes each component of H; so as $L \not\leq C_G(z)$, $L = [L, z]$.

Set $R := N_S(L)$ and $(RL)^* := RL/O_2(RL)$. Then $R \in Syl_2(RL)$ so $R^* \in Syl_2(R^*L^*)$ and z^* is an involution in the center of R^*. By hypothesis, $C_G(z) \leq M$, so $C_H(z) = C_{H \cap M}(z)$. Now $H \cap M \in \mathcal{H}^e$ by (1), so by 1.1.3.2, $C_{H \cap M}(z) \in \mathcal{H}^e$. Since $L \trianglelefteq \trianglelefteq H$ we have

$$C_L(z) \trianglelefteq \trianglelefteq C_H(z) = C_{H \cap M}(z),$$

so $C_L(z) \in \mathcal{H}^e$ by 1.1.3.1. Also $O^2(C_{L^*}(z^*)) = O^2(C_L(z))^*$ by Coprime Action, while $O_2(RL) \cap L \leq O_2(L) \leq Z(L)$, so we conclude $F^*(C_{L^*}(z^*)) = O_2(C_{L^*}(z^*))$ from A.1.8.

If z induces an inner automorphism on L then z centralizes $Z(L)$, so $O(L) = 1$ by (2), and hence $Z(L) = O_2(L)$. Put another way (recalling L is quasisimple), if $O(L) \neq 1$ then z induces an outer automorphism on L.

As H is an SQTK-group, we may examine the possibilities for $L/Z(L)$ appearing on the list of Theorem C.

Suppose first that $L/Z(L)$ is of Lie type and characteristic 2; then L^* appears in conclusion (3) or (4) of Theorem C. Now $z^* \in Z(R^*)$, so from the structure of $Aut(L^*)$, either $z^* \in L^*$, or L^* is A_6 or $\hat{A}_6$ with z^* inducing a transposition on L^*. However in the latter case as $O_2(L) = 1$, or else $L/O(L) \cong SL_2(9)$ by I.2.2.1, so that the transposition z does not centralize a Sylow 2-subgroup of L, contrary to $z \in Z(R)$; hence (c) holds. Thus we may assume $z^* \in L^*$, so by an earlier remark, $O(L) = 1$. Thus either $Z(L) = 1$, so L is simple and (a) holds; or from the list of Schur multipliers in I.1.3, L^* is $L_2(4)$, A_6, $Sz(8)$, $L_3(4)$, $G_2(4)$, or $L_4(2)$. Then as z centralizes a Sylow 2-group of L, when $L^* \cong L_2(4) \cong A_5$, A_6, or $Sz(8)$, we obtain a contradiction from the structure of the covering group L in (1) or (4) of I.2.2, or in 33.15 of [**Asc86a**]. This leaves covers of $L_3(4)$ and $G_2(4)$, which are allowed in (b).

We have shown that the lemma holds if $L/Z(L)$ is of Lie type and characteristic 2. But $A_5 \cong \Omega_4^-(2)$, $A_6 \cong Sp_4(2)'$, and $A_8 \cong L_4(2)$, so if L^* is an alternating group, then from conclusion (1) of Theorem C and I.1.3, $L^* \cong A_7$ or $\hat{A}_7$. As $F^*(C_{L^*}(z^*)) = O_2(C_{L^*}(z^*))$, $z^* \notin L^*$ and z^* is not a transvection, so we conclude z^* has cycle structure 2^3. As z centralizes a Sylow 2-group of L, we conclude that $O_2(L) = 1$, from the structure of the double cover of A_7 in 33.15 of [**Asc86a**]. So (d) holds.

Next assume $L/Z(L)$ is of Lie type and odd characteristic; then L^* appears in conclusion (2) of Theorem C. If $L^* \cong L_2(p^2)$ then as $L_2(9) \cong Sp_4(2)'$, we may assume $p > 3$. Therefore as $z^* \in Z(R^*)$, either $z^* \in L^*$ and $O(C_{L^*}(z^*)) \neq 1$ or z^* induces a field automorphism on L^* so that $C_{L^*}(z^*)$ has a component; in either case, this is contrary to $F^*(C_{L^*}(z^*)) = O_2(C_{L^*}(z^*))$. The same argument eliminates $L_2(p)$ unless p is a Fermat or Mersenne prime, which is allowed in (e); as before, the fact that $z \in Z(R)$ rules out the double covers $SL_2(p)$, the only possibilities with $Z(L) \neq 1$ by I.1.3. Similarly if $L^* \cong L_3^\epsilon(p)$ then as $z^* \in Z(R^*)$, $z^* \in L^*$; then unless $p = 3$, $C_{L^*}(z^*)$ has an $SL_2(p)$-component, for our usual contradiction. Finally $U_3(3) \cong G_2(2)'$ was covered earlier, while if $L^* \cong L_3(3)$ then $Z(L) = 1$ by I.1.3, so conclusion (e) holds.

This leaves the case L^* sporadic, so L^* appears in conclusion (5) of Theorem C. First J_1 is ruled out by the existence of a component in $C_{L^*}(z^*)$. Then as usual $z^* \in L^*$ since $z \in Z(R)$, so that (f) holds.

This completes the proof of 1.1.5. □

LEMMA 1.1.6. *Let B be a nontrivial 2-subgroup of G, $H \leq G$ with $BC_G(B) \leq H \leq N_G(B)$, $S \in Syl_2(H)$, T a Sylow 2-subgroup of G containing S, z an involution in $Z(T)$, and $M \in \mathcal{M}(C_G(z))$. Then the hypotheses of 1.1.5 are satisfied.*

PROOF. As $z \in Z(T)$, $M \in \mathcal{M}(T)$, so $M \in \mathcal{H}^e$ since G is of even characteristic. Thus as $S \leq T$, $M \in \mathcal{H}^e(S)$. Next $B \leq O_2(H) \leq S \leq M$ so that $B \leq O_2(H \cap M)$, and hence

$$C_{O_2(M)}(O_2(H \cap M)) \leq C_G(B) \leq H.$$

Also $z \in C_T(B) \leq T \cap H = S$ as $S \in Syl_2(H)$, so $z \in Z(S)$; hence the hypotheses of 1.1.5 are satisfied. The proof is complete. □

1.2. The set $\mathcal{L}^*(G,T)$ of nonsolvable uniqueness subgroups

In this section we use our results on the structure of SQTK-groups in section A.3 to establish tools for working in 2-local subgroups of G; such appeals are possible since our 2-locals are strongly quasithin. In particular we obtain a description of H^∞ for $H \in \mathcal{H}$, and also properties of the poset of perfect members of $\mathcal{H}$, partially ordered by inclusion. Such results then lead to the existence of uniqueness subgroups of G.

We begin by recalling Definition A.3.1 which defines $\mathcal{C}$-components: For $H \leq G$, let $\mathcal{C}(H)$ be the set of subgroups $L \leq H$ minimal subject to

$$1 \neq L = L^\infty \trianglelefteq\trianglelefteq H.$$

The members of $\mathcal{C}(H)$ are the *$\mathcal{C}$-components* of H. As we will see, usually we can expect there will be $H \in \mathcal{H}$ with $\mathcal{C}(H)$ nonempty.

We recall also that the elementary results in A.3.3 hold for arbitrary finite groups. By contrast, the later results in section A.3 requiring Hypothesis A.3.4 apply only to an SQTK-group X with $O_2(X) = 1$. We apply those results to $H/O_2(H)$ for $H \in \mathcal{H}$, and then pull them back to obtain results about H.

Recall that $\pi(X)$ denotes the set of primes dividing the order of a group X.

PROPOSITION 1.2.1. *Let $H \in \mathcal{H}$. Then*

(1) $\langle \mathcal{C}(H) \rangle = H^\infty$.

(2) If L_1, L_2 are distinct members of $\mathcal{C}(H)$, then $[L_1, L_2] \leq O_2(L_1) \cap O_2(L_2) \leq O_2(H)$.

(3) If $L \in \mathcal{C}(H)$, then either $L \trianglelefteq H$; or $|L^H| = 2$ and $L/O_2(L) \cong L_2(2^n)$, $Sz(2^n)$, $L_2(p)$, p an odd prime, or J_1.

(4) Let $L \in \mathcal{C}(H)$ and $\bar{H} = H/O_2(H)$. Then one of the following holds:

(a) $\bar{L}$ is a simple component of $\bar{H}$ on the list of Theorem C (A.2.3).

(b) $\bar{L}$ is a quasisimple component of $\bar{H}$, $Z(\bar{L}) \cong \mathbf{Z}_3$, and $\bar{L}$ is $SL_3^\epsilon(q)$, $q = 2^e$ or q an odd prime, $\hat{A}_6$, $\hat{A}_7$, or $\hat{M}_{22}$.

(c) $F^*(\bar{L}) \cong E_{p^2}$ for some prime $p > 3$, and $F^*(\bar{L})$ affords the natural module for $\bar{L}/F^*(\bar{L}) \cong SL_2(p)$.

(d) $F^*(\bar{L})$ is nilpotent with $Z(\bar{L}) = \Phi(F^*(\bar{L}))$, $\bar{L}/F^*(\bar{L}) \cong SL_2(5)$, and for each $p \in \pi(F^*(\bar{L}))$:

(i) either $p^2 \equiv 1 \mod 5$ or $p = 5$; and

(ii) either $O_p(\bar{L}) \cong p^{1+2}$, or $O_p(\bar{L})$ is homocyclic of rank 2.

(5) If $L \in \mathcal{C}(H)$ satisfies $O_2(L) \leq Z(L)$ and $m_2(L) > 1$, then L is quasisimple.

PROOF. As we observed at the start of the section, since $H \in \mathcal{H}$, H is an SQTK-group, and hence so is $\bar{H} := H/O_2(H)$. Certainly $O_2(\bar{H}) = 1$—so we may apply the results of section A.3. to $\bar{H}$. Further by A.3.3.4:

(*) The map $L \mapsto \bar{L}$ is an H-equivariant bijection of $\mathcal{C}(H)$ with $\mathcal{C}(\bar{H})$—with inverse $\bar{K} \mapsto K^\infty$, where K is the full preimage of $\bar{K}$ in H.

Thus for $L \in \mathcal{C}(H)$, we have $\bar{L} \in \mathcal{C}(\bar{H})$ and the possibilities in (4) are just those from A.3.6. Similarly the existence of the equivariant bijection in (*), together with A.3.7, A.3.9, and (1) and (3) of A.3.8, implies (2), (1), and (3), respectively.

Assume the hypotheses of (5). If $L/O_2(L)$ is quasisimple, then as $O_2(L) \leq Z(L)$ and L is perfect, L is quasisimple. Thus we may assume that case (4c) or (4d) holds. Then as $O_2(L) \leq Z(L)$, $O_{2,F}(L) = O_2(L) \times O(L)$. Thus $L/O(L)$ is the central extension of the 2-group $O_{2,F}(L)/O(L)$ by $L/O_{2,F}(L) \cong SL_2(p)$. But the multiplier of $SL_2(p)$ is trivial (I.1.3), so we conclude $O_2(L) = 1$. Now $m_2(L) = m_2(L/O(L))$ and $L/O(L) \cong SL_2(p)$ has 2-rank 1, contrary to the hypothesis that $m_2(L) > 1$. This establishes (5), and completes the proof of 1.2.1. □

As we mentioned in the Introduction to Volume II, in the bulk of the proof, there will be $H \in \mathcal{H}$ with H nonsolvable; and in that case by 1.2.1.1, $\mathcal{C}(H)$ is nonempty.

LEMMA 1.2.2. *Let $H \in \mathcal{H}$, $\bar{H} := H/O_2(H)$, $L \in \mathcal{C}(H)$, and p an odd prime.*

(a) If $|L^H| = 2$ and $p \in \pi(\bar{L})$, then $O^{p'}(H) = \langle L^H \rangle$.

(b) If $m_p(L) = 2$ then $L \trianglelefteq H$.

PROOF. Part (b) follows as $m_p(\bar{L}) = 1$ for each of the groups $\bar{L}$ listed in 1.2.1.3.

Assume the hypotheses of (a), and set $L_0 := \langle L^H \rangle$. Recall L_0 is normal in H by 1.2.1.3. Then $m_p(\bar{L}_0) = 2$, so $C_{\bar{H}}(\bar{L}_0)$ is a p'-group as $m_p(H) \leq 2$. As $|L^H| = 2$, $O^{p'}(H)$ normalizes L. Recall from the Introduction to Volume I that we refer to [**GLS98**] for the structure of the outer automorphism groups of the groups listed in Theorem C. For those $\bar{L}$ listed in 1.2.1.3, $O^2(Out(\bar{L}))$ is a group of field automorphisms (or trivial), and $O^2(Aut(\bar{L}))$ splits over $Inn(\bar{L}) \cong \bar{L}$. Therefore if $O^{p'}(H) \not\leq L_0$, there is x of order p in $N_H(L) - L_0$. Then x centralizes nontrivial elements of order p in each factor of $P \in Syl_p(L_0)$, contradicting $m_p(H) \leq 2$. This contradiction gives $O^{p'}(H) \leq L_0$, while $L_0 = O^{p'}(L_0)$ as $\bar{L}$ is simple and $p \in \pi(\bar{L})$. This proves (a). □

Next we extend the notation of $\mathcal{L}(X,Y)$ in Definition A.3.10 to our QTKE-group G. This will help us keep track of the possible embeddings of $\mathcal{C}$-components of a subgroup $H_1 \in \mathcal{H}$ in some other $H_2 \in \mathcal{H}$, as long as H_1 and H_2 share a common Sylow 2-subgroup.

DEFINITION 1.2.3. For H a finite group, and S a 2-subgroup of H, let $\mathcal{L}(H,S)$ be the set of subgroups L of H such that

(1) $L \in \mathcal{C}(\langle L,S\rangle)$,
(2) $S \in Syl_2(\langle L,S\rangle)$, and
(3) $O_2(\langle L,S\rangle) \neq 1$; that is, $\langle L,S\rangle \in \mathcal{H}_H$.

Assume for the moment that $H \in \mathcal{H}$, $S \in Syl_2(H)$, $\bar{H} := H/O_2(H)$, and $L \in \mathcal{L}(H,S)$. Then by Hypotheses (QT) and (K), $\bar{H}$ satisfies Hypothesis A.3.4, with $\bar{S} \in Syl_2(\bar{H})$; so from condition (1) of the definition of $\mathcal{L}(H,S)$ and A.3.3, $\bar{L} \in \mathcal{L}(\bar{H},\bar{S})$, defined only for $\bar{H}$ in section A.3. Also applying 1.2.1.3 to $\langle L,S\rangle$, either $L^S = L$ and $\langle L,S\rangle = LS$, or $L^S = \{L, L^s\}$ and $\langle L,S\rangle = LL^sS$. Further as in A.3.11, $\mathcal{C}(H) \subseteq \mathcal{L}(H,S)$, so when $\mathcal{C}(H)$ is nonempty, $\mathcal{L}(H,S)$ is nonempty.

Now just as in section A.3, we wish to see how members of $\mathcal{L}(H,S)$ embed in H.

LEMMA 1.2.4. *Let $H \in \mathcal{H}$, with $S \in Syl_2(H)$; set $\bar{H} := H/O_2(H)$, and assume $B \in \mathcal{L}(H,S)$. Then $B \leq L$ for a unique $L \in \mathcal{C}(H)$, and the pair $(\bar{B},\bar{L})$ is on the list of lemma A.3.12. In particular*

(+) If S normalizes B, then $L \trianglelefteq H$.

PROOF. We apply A.3.12 to conclude $\bar{B}$ is contained in a unique $\bar{L} \in \mathcal{C}(\bar{H})$, with the pair $(\bar{B},\bar{L})$ on the list of A.3.12. Then using the one-to-one correspondence from A.3.3.4, $\bar{L}$ is the image of a unique $L \in \mathcal{C}(H)$; and as $B \leq O_2(H)L$ we see $B = B^\infty \leq (O_2(H)L)^\infty = L$. This completes the proof, as (+) follows from the uniqueness of L. □

LEMMA 1.2.5. *Let $H \in \mathcal{H}$, $S \in Syl_2(H)$, $R \leq S$ with $|S:R| = 2$, and suppose $L \in \mathcal{L}(H,R)$. Then there exists a unique $K \in \mathcal{C}(H)$ with $L \leq K$.*

PROOF. The proof is much like that of A.3.12. Let $H^* := H/O_\infty(H)$. By 1.2.1.1, $H^\infty = K_1 \cdots K_r$ where $K_i \in \mathcal{C}(H)$, and by 1.2.1.2, $H^{\infty *} = K_1^* \times \cdots \times K_r^*$. Now $L = L^\infty \leq H^\infty$, so for some i (which we now fix), the projection P^* of L^* on $K^* := K_i^*$ is nontrivial. As P^* is a homomorphic image of $L^* \in \mathcal{C}(L^*)$, $P^* \in \mathcal{C}(P^*)$ by A.3.3.4.

As $S \in Syl_2(H)$ and K is subnormal in H, $S \cap K \in Syl_2(K)$, and similarly $R \cap L \in Syl_2(L)$ using our hypothesis that $L \in \mathcal{L}(H,R)$. Then as $R \leq S$, $S \cap L = R \cap L \in Syl_2(L)$, so $S \cap P \in Syl_2(P)$, for P the preimage of P^*. Then $|S \cap P : R \cap P| \leq |S:R| \leq 2$; so $(R \cap P)^* \not\leq O_\infty(P^*)$, as otherwise $P^*/O_\infty(P^*)$ has Sylow 2-groups of order at most 2, and so is solvable using Cyclic Sylow 2-Subgroups A.1.38, contrary to $P^* \in \mathcal{C}(P^*)$ nonsolvable. Hence $[L, R \cap P] \not\leq O_\infty(L)$. However as $(R \cap P)^*$ acts on P^* and permutes the $\mathcal{C}$-components L^R of $\langle L,R\rangle$, $R \cap P$ acts on L; so by A.3.3.7, $L = [L, R \cap P] \leq [L,K] \leq K$. Finally K is unique since $K_i \cap K_j \leq O_\infty(H)$ for any $j \neq i$. This completes the proof of 1.2.5. □

Lemma 1.2.4 gives information about $\mathcal{L}(H,S)$ considered as a set partially ordered by inclusion. This leads us to define $\mathcal{L}^*(H,S)$ to be the maximal members of this poset.

We will focus primarily on the case where the role of S is played by $T \in Syl_2(G)$. In this case when $H \in \mathcal{H}(T)$, then T is also Sylow in H, so an earlier remark now specializes to:

LEMMA 1.2.6. *$\mathcal{C}(H) \subseteq \mathcal{L}(H,T) \subseteq \mathcal{L}(G,T)$ for each $H \in \mathcal{H}(T)$.*

THEOREM 1.2.7 (Nonsolvable Uniqueness Groups). *If $L \in \mathcal{L}^*(G,T)$ then*

(1) $L \in \mathcal{C}(H)$ for each $H \in \mathcal{H}(\langle L,T\rangle)$.

(2) $F^*(L) = O_2(L)$.

(3) $N_G(\langle L^T\rangle) = !\mathcal{M}(\langle L,T\rangle)$.

(4) Set $L_0 := \langle L^T\rangle$ and $Z := \Omega_1(Z(T))$ Then $C_Z(L_0) \cap C_Z(L_0)^g = 1$ for $g \in G - N_G(\langle L^T\rangle)$.

PROOF. Let $H \in \mathcal{H}(\langle L,T\rangle)$. As $T \in Syl_2(G)$, $T \in Syl_2(H)$, so also $L \in \mathcal{L}(H,T)$. Then by 1.2.4, $L \leq K \in \mathcal{C}(H)$ for some K. But by 1.2.6, $\mathcal{C}(H) \subseteq \mathcal{L}(G,T)$; so $L = K$ from the maximal choice of L. Hence (1) holds.

Next by 1.1.4.6, $F^*(H) = O_2(H)$; so as L is subnormal in H, (2) holds by 1.1.3.1.

Set $L_0 := \langle L^T\rangle$. As $L \in \mathcal{C}(H)$, $L_0 \trianglelefteq H$ by 1.2.1.3. Hence $H \leq M := N_G(L_0)$, and as $O_2(L) \neq 1$ by (2), $O_2(M) \neq 1$. In particular if $H \in \mathcal{M}(T)$, we conclude $H = M$. Thus (3) holds.

To prove (4), assume $Z_0 := C_Z(L_0) \cap C_Z(L_0)^g \neq 1$. Then $L_0T, L_0^gT^g \leq C_G(Z_0)$, so using (3), $M = !\mathcal{M}(C_G(Z_0)) = M^g$; but then $g \in N_G(M) = M$ as $M \in \mathcal{M}$, contrary to $g \notin M$. $\square$

Part (3) of 1.2.7 says that if $L \in \mathcal{L}^*(G,T)$ then $\langle L,T\rangle$ is a uniqueness subgroup of G. This fact plays a crucial role through most of our work.

Next we obtain some further restrictions on chains in the poset $\mathcal{L}(G,T)$. For example we see in part (4) of 1.2.8 that for many choices of $L/O_2(L)$, $L \in \mathcal{L}(G,T)$ is already maximal. In parts (2) and (3) of 1.2.8 we see that if L is not T-invariant, then usually L is maximal.

LEMMA 1.2.8. *Let S be a 2-subgroup of G, and $L, K \in \mathcal{L}(G,S)$ with $L \leq K$. Then*

(1) $N_S(L) = N_S(K)$. So if $L \neq L^s$ then $LL^s \leq KK^s$ for $K \neq K^s$.

(2) If $L < \langle L^S\rangle$, then either

(a) $L = K$, or

(b) $L/O_2(L) \cong A_5$, and $K/O_2(K)$ is either J_1 or $L_2(p)$ for some prime p with $p^2 \equiv 1 \mod 5$.

(3) If $L < \langle L^S\rangle$, then either $L \in \mathcal{L}^*(G,S)$, or $L/O_2(L) \cong A_5$.

(4) We have $L \in \mathcal{L}^*(G,S)$ if $L/O_2(L)$ is any of the following: $\hat{A}_7$; $L_2(r^2)$, $r > 3$ an odd prime; $(S)L_3^\epsilon(p)$, p an odd prime; M_{11}, M_{12}, M_{23}, J_1, J_2, J_4, HS, He, Ru, $L_5(2)$, or $(S)U_3(2^n)$; a group of Lie type of characteristic 2 and Lie rank 2, other than $L_3(2)$ or $L_3(4)$.

PROOF. Let $H := \langle K,S\rangle$, and recall $\mathcal{C}(H) = \{K\}$ or $\{K,K^s\}$. By 1.2.4, K is the *unique* $\mathcal{C}$-component of H containing L, so that $N_S(L) \leq N_S(K)$. The opposite inclusion follows from A.3.12, as we check that in each of the embeddings listed there, K does not contain a product of two copies of L, so that L is $N_S(K)$-invariant. Hence (1) holds.

Assume as in (2) that $L \neq L^s$, and that $L < K$; then $K^s \neq K$ by (1). Then by 1.2.1.3, $K/O_2(K)$ is $L_2(2^n)$, $Sz(2^n)$, $L_2(p)$, or J_1, and $L/O_2(L)$ is also in this list. Consulting A.3.12, we see the only possible proper embeddings of L in K are those given in (2). This establishes (2) and (3).

Finally (4) is established similarly: from the list of groups in Theorem C, we extract the sublist *not* occurring as an initial possibility in A.3.12. □

We next wish to study the action of members of $\mathcal{L}(G,T)$ on their internal modules. To do so, we use some of the results from section A.4 of Volume I. Recall from Definition A.4.5 that $\mathcal{X}$ consists of the nontrivial subgroups Y of G satisfying $Y = O^2(Y)$ and $F^*(Y) = O_2(Y)$. Notice the second condition says that $\mathcal{X} \subseteq \mathcal{H}^e$.

Now for $L \in \mathcal{L}(G,T)$, $L = L^\infty$ by the definition of $\mathcal{C}$-component, while $L \in \mathcal{H}^e$ by 1.2.7.2, so that $\mathcal{L}(G,T) \subseteq \mathcal{X}$. Next recall that for $Y \in \mathcal{X}$ and $R \in \mathcal{U}_{N_G(Y)}(Y,2)$, from Definition A.4.6

$$V(Y,R) := [\Omega_1(Z(R)),Y] \text{ and } V(Y) := V(Y,O_2(Y)).$$

There we also defined $\mathcal{X}_f$ to consist of those $Y \in \mathcal{X}$ with $V(Y) \neq 1$. The subscript "f" stands for "faithful"; for example, if $X \in \mathcal{X}_f$ with $X/O_2(X)$ simple, then $X/O_2(X)$ is faithful on the module $V(X)$. Define

$$\mathcal{L}_f(G,T) := \mathcal{L}(G,T) \cap \mathcal{X}_f,$$

and also define

$$\mathcal{L}_f^*(G,T) := \mathcal{L}^*(G,T) \cap \mathcal{X}_f,$$

which of course coincides with $\mathcal{L}_f(G,T) \cap \mathcal{L}^*(G,T)$. Now by definition, elements of $\mathcal{L}_f^*(G,T)$ are maximal in the subposet $\mathcal{L}_f(G,T)$; in the next lemma we see that the converse holds.

LEMMA 1.2.9. *Let $L \in \mathcal{L}_f(G,T)$. Then*

(1) If $L \leq K \in \mathcal{L}(G,T)$, then $V(L,O_2(N_T(L)L)) \leq V(K,O_2(N_T(K)K))$, and so $K \in \mathcal{L}_f(G,T)$.

(2) If L is maximal in $\mathcal{L}_f(G,T)$ with respect to inclusion, then $L \in \mathcal{L}^*(G,T)$, and hence $L \in \mathcal{L}_f^*(G,T)$.

PROOF. Let $L \leq K \in \mathcal{L}(G,T)$ and $R := N_T(L)$. By 1.2.8.1, $R = N_T(K)$. Thus $R \in Syl_2(N_{KR}(L))$, so $O_2(KR) \leq R$ by A.1.6, and $O_2(RL) = C_R(L/O_2(L))$. Hence we may apply parts (2) and (3) of A.4.10 to obtain (1). Then (1) implies (2). □

LEMMA 1.2.10. *Let $T \in Syl_2(G)$, $H \in \mathcal{H}(T)$, and $L \in \mathcal{C}(H)$. Then the following are equivalent:*

(1) $L \in \mathcal{L}_f(G,T)$.

(2) There is $V \in \mathcal{R}_2(H)$ with $[V,L] \neq 1$.

(3) $[R_2(H),L] \neq 1$.

In particular the result applies to $L \in \mathcal{L}^*(G,T)$ and $H \in \mathcal{H}(\langle L,T\rangle)$.

PROOF. We have $F^*(H) = O_2(H)$ by 1.1.4.6, and from 1.2.1.4 we see that all non-central 2-chief factors of L lie in $O_2(L)$. These are the hypotheses for A.4.11, whose conclusions are exactly the assertions of 1.2.10. □

LEMMA 1.2.11. *Let $H \in \mathcal{H}$ with $T \cap H =: T_H \in Syl_2(H)$, and $K \in \mathcal{C}(H)$. Assume $z \in Z := \Omega_1(Z(T))$ lies in $K \cap T_H$ and:*

(a) $z \in [U, O^2(N_H(U))]$ for some elementary abelian 2-subgroup U of H;
(b) $C_T(O_2(H)) \leq T_H$.

Then either K is quasisimple or $H \in \mathcal{H}^e$.

PROOF. Assume K is not quasisimple; we must show $H \in \mathcal{H}^e$. By A.3.3.1, $E(K) = 1$, so $F^*(K) = F(K)$, and hence $C_K(F(K)) \leq F(K)$. We claim first that z centralizes $O(K)$: As H is an SQTK-group, $m_r(O(H)) \leq 2$ for all primes r. Then hypothesis (a) allows us to apply A.1.26.2 to $O^2(N_H(U))$, $[U, O^2(N_H(U))]$ in the roles of "X, V", to conclude $z \in [U, O^2(N_H(U))] \leq C_H(O(H)) \leq C_H(O(K))$. Next our assumption that $z \in K \cap T_H$ gives $z \in C_K(O(K)O_2(K)) \leq C_K(F(K)) \leq F(K)$, and hence $z \in O_2(K) \leq O_2(H)$.

Let $G_z := C_G(z)$ and $H_z := C_H(z)$. As $z \in O_2(H)$, $O^2(F^*(H)) \leq O^2(F^*(H_z))$, so it suffices to show $H_z \in \mathcal{H}^e$. As T_H is Sylow in H and $T_H \leq H_z$, $O_2(H) \leq O_2(H_z)$ by A.1.6. Therefore using (b),

$$C_{O_2(G_z)}(O_2(H_z)) \leq C_T(O_2(H)) \leq T_H \cap G_z \leq H_z.$$

Then as $G_z \in \mathcal{H}^e$ by 1.1.4.3, we get $H_z \in \mathcal{H}^e$ by 1.1.4.4. □

1.3. The set $\Xi^*(G,T)$ of solvable uniqueness subgroups of G

As noted in the Introduction to Volume II, it might happen that there are no nonsolvable locals $H \in \mathcal{H}(T)$, so that $\mathcal{L}(G,T)$ is empty; in this case we will need to produce some solvable uniqueness groups. Notice also that any L occurring in cases (c) or (d) of 1.2.1.4 involves interesting (and potentially tractable) solvable subgroups in $O_{2,F}(L)$.

Motivated particularly by the latter example:

DEFINITION 1.3.1. Define $\Xi(G,T)$ to consist of the subgroups $X \leq G$ such that:

(1) $X = O^2(X)$ is T-invariant with $XT \in \mathcal{H}$,
(2) $X/O_2(X) \cong E_{p^2}$ or p^{1+2} for some odd prime p, and
(3) T is irreducible on the Frattini quotient of $X/O_2(X)$.

Notice that each $X \in \Xi(G,T)$ is in $\mathcal{H}^e$ by 1.1.4.6 and 1.1.3.1, so as $X = O^2(X)$ we see $\Xi(G,T) \subseteq \mathcal{X}$.

Subsets $\Xi_-(G,T)$ and $\Xi_+(G,T)$ of $\Xi(G,T)$ appear in Definition 3.2.12.

We first collect some useful elementary properties of the members of $\Xi(G,T)$:

LEMMA 1.3.2. *Let $X \in \Xi(G,T)$. Then*

(1) X is a $\{2,p\}$-group for some odd prime p and $X = O_2(X)P$ for some $P \in Syl_p(X)$.
(2) $X = \langle P^X \rangle = \langle P^{O_2(X)} \rangle$ and $O_2(X) = [O_2(X), P]$.
(3) $T = O_2(X)N_T(P)$ and $N_T(P)$ is irreducible on $P/\Phi(P)$.
(4) $P = [P, \Phi(N_T(P))]$.
(5) If $H \in \mathcal{H}(XT)$, then $X = O^2(O_2(H)X)$.

PROOF. Part (1) is immediate from condition (2) in the definition of $\Xi(G,T)$ and Sylow's Theorem. As $X = O^2(X)$ in condition (1) of the definition of $\Xi(G,T)$, conclusion (1) now implies

$$X = \langle Syl_p(X) \rangle = \langle P^X \rangle = \langle P^{O_2(X)} \rangle$$

and $O_2(X) = [O_2(X), P]$, giving conclusion (2). Notice $XT = PT$, so the Dedekind Modular Law gives $N_{XT}(P) = PN_T(P)$; then a Frattini Argument on $X = O_2(X)P$ gives $T = O_2(X)N_T(P)$. Now $N_T(P)$ is irreducible on $P/\Phi(P)$ by condition (3) of the definition of $\Xi(G,T)$, so conclusion (3) is proved.

Let $S := N_T(P)$ and $S^* := S/C_T(P)$. Now S^* is irreducible on $P/\Phi(P)$ by (3), so each involution $i^* \in Z(S^*)$ inverts $P/\Phi(P)$. Thus for each $I \leq S$ with $i^* \in I^*$, $P = [P, I]$. In particular if $\Phi(S^*) \neq 1$, we can choose $I = \Phi(S)$, so that (4) holds in this case. Otherwise $\Phi(S^*) = 1$, and then S^* is reducible on $P/\Phi(P)$ by A.1.5. This contradiction completes the proof of (4).

Under the hypotheses of (5), $O_2(H) \leq T$, while by condition (1) of the definition, $T \leq N_G(X)$ and $X = O^2(X)$, so $X = O^2(O_2(H)X)$, as required. □

Assume for the moment that $L \in \mathcal{L}(G,T)$ with $L/O_2(L)$ not quasisimple, as in cases (c) and (d) of 1.2.1.4. Then L is T-invariant by 1.2.1.3. Given an odd prime p, define

$$\Xi_p(L) := O^2(X_p), \text{ where } X_p/O_2(L) := \Omega_1(O_p(L/O_2(L)));$$

then define $\Xi_{rad}(G,T)$ to be the collection of subgroups $\Xi_p(L)$, for $L \in \mathcal{L}(G,T)$ with $L/O_2(L)$ not quasisimple, and $p \in \pi(F(L/O_2(L)))$.

We observe that $X \in \Xi_{rad}(G,T)$ satisfies conditions (2) and (3) in the definition of $\Xi(G,T)$, using the action of $L/O_{2,F}(L) \cong SL_2(r)$ ($r = p$ or 5) in cases (c) and (d) of 1.2.1.4. By construction, $X = O^2(X)$, while X is T-invariant as X char $L \trianglelefteq LT$. Finally $LT \in \mathcal{H}^e$ by 1.1.4.6, so that $1 \neq O_2(LT) \leq O_2(XT)$ by A.1.6, the last requirement of condition (1) of the definition. So we see:

LEMMA 1.3.3. *$\Xi_{rad}(G,T) \subseteq \Xi(G,T)$.*

Define $\Xi^*(G,T)$ to consist of those $X \in \Xi(G,T)$ such that XT is not contained in $\langle L, T\rangle$ for any $L \in \mathcal{L}(G,T)$ with $L/O_2(L)$ quasisimple. So for Ξ (in contrast to $\mathcal{L}$), the superscript * will not denote maximality under inclusion in the poset $\Xi(G,T)$. However the following result will be used in 1.3.7 (which is the analogue of 1.2.7.1) to prove that XT is a uniqueness subgroup for each member X of $\Xi^*(G,T)$. Furthermore the list of possible embeddings of members of $\Xi(G,T)$ in nonsolvable groups appearing in the lemma will also be very useful.

PROPOSITION 1.3.4. *Let $X \in \Xi(G,T)$, $P \in Syl_p(X)$ a complement to $O_2(X)$ in X, and $H \in \mathcal{H}(XT)$. Then either $X \trianglelefteq H$, or $X \leq \langle L^T\rangle$ for some $L \in \mathcal{C}(H)$ with $L/O_2(L)$ quasisimple, and in the latter case one of the following holds:*

(1) L is not T-invariant and $P = (P \cap L) \times (P \cap L)^t \cong E_{p^2}$ for $t \in N_T(P) - N_T(L)$. Either $L/O_2(L) \cong L_2(2^n)$ with n even and $2^n \equiv 1 \mod p$, or $L/O_2(L) \cong L_2(q)$ for some odd prime q.

In the remaining cases, L is T-invariant and satisfies one of:

(2) $P \cong E_{p^2}$ and $L/O_2(L) \cong (S)L_3(p)$.

(3) $P \cong E_{p^2}$, $L/O_2(L) \cong Sp_4(2^n)$ with n even and $2^n \equiv 1 \mod p$, and $Aut_T(P)$ is cyclic.

(4) $p = 3$, $P \cong E_9$, and $L/O_2(L) \cong M_{11}$, $L_4(2)$, or $L_5(2)$.

PROOF. Set $\bar{H} := H/O_2(H)$. We first consider $F(\bar{H})$. So let r be an odd prime, and $\bar{R}$ a supercritical subgroup of $O_r(\bar{H})$. (Cf. A.1.21). As usual $m_r(\bar{R}) \leq 2$ since $m_r(H) \leq 2$. Therefore by A.1.32, $[\bar{R}, \bar{P}] = 1$ if $p \neq r$; while if $p = r$, then either

$\bar{R} = \bar{P}$—or $\bar{R} \cong \mathbf{Z}_p$, $\bar{P} \cong p^{1+2}$, and $\bar{R} = Z(\bar{P})$. In particular by A.1.21, $\bar{P}$ centralizes $O^p(F(\bar{H}))$.

Suppose for the moment that $O_p(\bar{H}) \neq 1$, and choose $r = p$. If $m_p(\bar{R}) = 2$ then $\bar{P} = \bar{R} \trianglelefteq \bar{H}$, so $X = O^2(R) \trianglelefteq H$ by 1.3.2.5, and the lemma holds. Therefore we may assume $m_p(\bar{R}) = 1$. Then as $\bar{R}$ is supercritical, it contains all elements of order r in $C_{O_r(\bar{H})}(\bar{R})$, so $O_p(\bar{H})$ is cyclic.

Thus in any case, we may assume that $O_p(\bar{H})$ is cyclic. In particular $\bar{P} \not\leq F(\bar{H})$ as $\bar{P}$ is noncyclic. Hence as $Aut(O_p(\bar{H}))$ is cyclic and $P = [P, N_T(P)]$, $\bar{P}$ centralizes $O_p(\bar{H})$; therefore as $\bar{P}$ centralizes $O^p(F(\bar{H}))$, $\bar{P}$ centralizes $F(\bar{H})$.

By 1.3.2.3, $N_T(P)$ is irreducible on $\bar{P}/\Phi(\bar{P})$, so as $O_p(\bar{H})$ is cyclic, $\bar{P} \cap O_p(\bar{H}) \leq \Phi(\bar{P})$; therefore as $\bar{P}$ centralizes $F(\bar{H})$, we conclude $C_{\bar{P}}(E(\bar{H})) \leq \Phi(\bar{P})$. Thus there is a component $\bar{L}_1$ of $\bar{H}$ with $[\bar{L}_1, \bar{P}] \neq 1$. By A.3.3.4, there is $L \in \mathcal{C}(H)$ with $\bar{L} = \bar{L}_1$. Set $K := \langle L^T \rangle$, so that $K \trianglelefteq H$ by 1.2.1.3. As $1 \neq [\bar{L}, \bar{P}]$, $[L, P] \not\leq O_2(L)$, so $L \leq [L, P] \leq [K, P]$ by A.3.3.7. Then as T acts on P, $K = \langle L^T \rangle = [K, P]$.

We claim $P \leq K$. Suppose first that $L < K = LL^t$. Then $\Phi(N_T(P)) \leq N_T(L)$ as $|L^T| = 2$. Notice that the groups listed in 1.2.1.3 have $Out(\bar{L})$ abelian. But by 1.3.2.4,

$$P = [P, \Phi(N_T(P))] = [P, N_T(P) \cap N_T(L)],$$

so P induces inner automorphisms on L and then also on K. Then by 1.2.2.a, $P \leq O^{p'}(H) = K$, establishing the claim in this case.

Next suppose that $L = K$. This time we examine $Out(\bar{L})$ for the groups $\bar{L}$ appearing in Theorem C, to see in each case there are no noncyclic p-subgroups U whose normalizer is irreducible on $U/\Phi(U)$—as would be the case for the image of P in $Out(\bar{L})$, if P did not induce inner automorphisms on $\bar{L}$. Thus $\bar{P} \leq \bar{L}C_{\bar{H}}(\bar{L})$. Then as $N_T(P)$ is irreducible on $P/\Phi(P)$, either $P \leq L = K$ as claimed, or $P \cap L \leq \Phi(P)$. However as $C_{\bar{P}}(\bar{L}) \leq \Phi(\bar{P})$, $m_p(\bar{L}) = 2$; so in the case where $P \cap L \leq \Phi(P)$, there exists x of order p in $C_{PL}(\bar{L}) - L$, and hence $m_p(L\langle x \rangle) > 2$, contradicting H an SQTK-group. This completes the proof of the claim.

Thus $P \leq K$ by the claim. Then by 1.3.2.2, $X = \langle P^{O_2(X)} \rangle \leq K$.

We next establish the lemma in the case $L < K = LL^t$. Here $m_p(L) = 1$ by 1.2.1.3, so

$$P = (P \cap L) \times (P \cap L)^t \cong E_{p^2},$$

and by 1.2.1.3, $\bar{L}$ is $L_2(2^n)$, $Sz(2^n)$, $L_2(q)$ for some odd prime q, or J_1. If $\bar{L}$ is $L_2(q)$ for some odd prime q, then conclusion (1) of the lemma holds, so we may assume we are in one of the other cases. Then as $PT = TP$, $\bar{P}$ lies in the Borel subgroup $N_{\bar{K}}(\bar{T} \cap \bar{K})$ of $\bar{K}$ if $\bar{L}$ is a Bender group, and similarly $P \leq N_K(T \cap K)$ when $\bar{L}$ is J_1. In the first case $\bar{P}$ lies in a Cartan subgroup, so $2^n \equiv 1 \mod p$, and in the second, $p = 3$ or 7. Further as P acts on $T \cap K$, $[N_{T \cap K}(P), P] \leq (T \cap K) \cap P = 1$. Therefore $Aut_T(P)$ is isomorphic to a subgroup of $Out(K)$, so as $N_{Aut_T(K)}(Aut_P(K))$ is irreducible on $Aut_P(K)/\Phi(Aut_P(K))$ by 1.3.2.4, $|Out(K)|_2 > 2$. Then as $|Out(\bar{K})| = 2\ |Out(\bar{L})|^2$, $Out(\bar{L})$ is of even order, which reduces us to $\bar{L} \cong L_2(2^n)$, n even—so that conclusion (1) of the lemma holds.

It remains to treat the case $K = L \trianglelefteq H$, where we must show that one of conclusions (2)–(4) holds. Thus $P \leq L$ by the claim.

Suppose first that $p > 3$. Then the possibilities for $\bar{L}$ and $\bar{P}$ with $PT = TP$ are determined in A.3.15. Suppose case A.3.15.3 holds. Then p plays the role of "r" in that result, and it follows that the signs δ and ϵ there conincide. Thus $\bar{L} \cong (S)L_3^{\delta}(q)$ with $q \equiv \delta \mod 4$; further $C_{\bar{L}}(Z(\bar{T}))^{\infty} \cong SL_2(p)$ plays the role of

"K" in that result, so that $\bar{P} \cap C_{\bar{L}}(Z(\bar{T}))^{\infty}$ is cyclic of order p dividing $q-\delta$. This contradicts the irreducible action of $N_T(P)$ on $P/\Phi(P)$. Suppose case A.3.15.2 holds. Then $\bar{L} \cong (S)L_3(p)$ and $\bar{P} \cong E_{p^2}$; here the parabolic $N_{\bar{L}}(\bar{P})$ induces $SL_2(p)$ on $\bar{P}$, and in particular the action of $\bar{T}$ on $\bar{P}$ is irreducible. This case appears as our conclusion (2)—using I.1.3 to see that the only cover of $L_3(p)$ is $SL_3(p)$. In cases (1), (4), (6), and (7) of A.3.15 $\bar{P}$ is cyclic, whereas P is noncyclic, so those cases do not arise here. Thus it remains to consider the cases in A.3.15.5, with $\bar{L}$ of Lie type over $\mathbf{F}_{2^n}$ with $n > 1$, As $P \leq L$, $\bar{P}$ lies in a Cartan subgroup of $\bar{L}$ by that result, so $\bar{L}$ is of Lie rank at least 2, and hence $\bar{L}$ is of one of the following Lie types: A_2, B_2, G_2, ${}^3D_4$, or ${}^2F_4$. As in an earlier case, $Aut_T(P)$ is isomorphic to a subgroup of $Out(\bar{L})$. In the last three types, $Out_T(\bar{L})$ consists only of field automorphisms; so as P is a p-group, $N_T(P)$ normalizes each subgroup of P, contradicting the irreducible action of $N_T(P)$ on $P/\Phi(P)$. If $\bar{L}$ is $Sp_4(2^n)$ then $Out(\bar{L})$ is cyclic as $n > 1$; cf. 16.1.4 and its underlying reference. So as $N_T(P)$ is irreducible, n is even and hence conclusion (3) holds. Finally if $\bar{L}$ is $(S)L_3(2^n)$ then $Out_T(L)$ is the product of groups generated by a field automorphism and a graph automorphism of order 2. However the field automorphism acts on each subgroup of P as above, and any automorphism of P of order 2 is not irreducible on P, so $N_T(P)$ is not irreducible on P. This eliminates $(S)L_3(2^n)$, completing the proof for $p > 3$.

We have reduced to the case $p = 3$. Here a priori $\bar{L}/Z(\bar{L})$ can be any group appearing in the conclusion of Theorem C. To eliminate the various possible cases, ordinarily we first apply the restriction $m_3(L) = 2$ (as P is noncyclic), and then the restriction $PT = TP$; a final sieve is provided by the irreducibility of $N_T(P)$ on $P/\Phi(P)$.

Thus from the cases in conclusion (2) of Theorem C: We do not have $\bar{L} \cong L_2(q^e)$ for $q > 3$, as $m_3(L) = 2$, and $\bar{L}$ is not $L_2(3^2) \cong A_6$ as $PT = TP$. The latter argument eliminates $U_3(3)$; while $\bar{L} \cong L_3(3)$ appears in conclusion (2). The groups $L_3^{\delta}(q)$ for $q > 3$ are eliminated when $q \equiv -\delta \mod 3$ since $m_3(L) = 2$; and when $q \equiv \delta \mod 3$, since $PT = TP$ and $N_T(P)$ is irreducible on $P/\Phi(P)$.

We next turn to conclusion (1) of Theorem C: A_5 is eliminated as $m_3(L) = 2$, and A_6 is impossible since $PT = TP$ as just noted. In A_7 there is indeed a subgroup $PT = TP \cong \mathbf{Z}_2/(A_4 \times A_3)$; but even in $Aut(A_7) = S_7$, we see that $S_4 \times S_3$ fails the requirement $N_T(P)$ irreducible on $P/\Phi(P)$. Finally $A_8 \cong L_4(2)$ appears in conclusion (4) of our proposition, as do the groups $L_4(2)$ and $L_5(2)$ arising in conclusion (4) of Theorem C.

In conclusion (3) of Theorem C, $\bar{L}/Z(\bar{L})$ is of Lie type in characteristic 2. Then as $P \leq L$ and $PT = TP$, $\bar{P}$ is contained in a proper parabolic of $\bar{L}$, and unless possibly $\bar{L}$ is defined over $\mathbf{F}_2$, we have $\bar{P}$ in the Borel subgroup $N_{\bar{L}}(\bar{T} \cap \bar{L})$. The case where $\bar{P}$ is contained in a Borel subgroup was treated above among the embeddings in A.3.15. In the case where $\bar{L}$ is defined over $\mathbf{F}_2$, proper parabolics have 3-rank at most 1, contradicting P noncyclic.

This leaves only conclusion (5) of Theorem C, where $\bar{L}/Z(\bar{L})$ is sporadic. Notice the case $\bar{L} \cong M_{11}$ appears in conclusion (4) of our lemma, while $\bar{L}$ is not J_1 as $m_3(L) = 2$. In the other cases, we use [**Asc86b**] to see that $PT = TP$ rules out all but M_{23}, M_{24}, J_2, J_4—which contain 2-groups extended by $GL_2(4)$, $S_3 \times L_3(2)$, $\mathbf{Z}_2/(S_3 \times \mathbf{Z}_3)$, $S_5 \times L_3(2)$, respectively. In these cases (even in $Aut(J_2)$) $N_T(P)$ is not irreducible on $P/\Phi(P)$.

This completes the proof of 1.3.4. □

We have the following corollaries to Proposition 1.3.4:

PROPOSITION 1.3.5. *If* $X \in \Xi^*(G,T)$ *and* $H \in \mathcal{H}(XT)$, *then* $X \trianglelefteq H$.

PROOF. Notice that the proposition follows from 1.3.4, since by 1.2.6, $\mathcal{C}(H) \subseteq \mathcal{L}(G,T)$ for $H \in \mathcal{H}(T)$. □

LEMMA 1.3.6. *If* $X \in \Xi(G,T)$ *with* $X/O_2(X) \cong p^{1+2}$, *then* $X \in \Xi^*(G,T)$.

PROOF. This is immediate from 1.3.4, which says that if $X \leq \langle L^T \rangle$ with $L/O_2(L)$ quasisimple, then $P \cong E_{p^2}$. □

Now, as promised, we see that if $X \in \Xi^*(G,T)$, then XT is a uniqueness subgroup of G:

THEOREM 1.3.7 (Solvable Uniqueness Groups). *If* $X \in \Xi^*(G,T)$ *then* $N_G(X) = !\mathcal{M}(XT)$.

PROOF. Let $M \in \mathcal{M}(XT)$. By 1.3.5, $X \trianglelefteq M$, so maximality of M gives $M = N_G(X)$. □

Recall that $\Xi_{rad}(G,T)$ consists of the subgroups $\Xi_p(L)$, for $L \in \mathcal{L}(G,T)$ such that $L/O_2(L)$ is not quasisimple; and by 1.3.3, $\Xi_{rad}(G,T) \subseteq \Xi(G,T)$. Define $\Xi^*_{rad}(G,T)$ to consist of those $X \in \Xi_{rad}(G,T)$ such that $X \trianglelefteq L \in \mathcal{L}^*(G,T)$. We see next that XT is a uniqueness subgroup for each $X \in \Xi^*_{rad}(G,T)$. This fact will allow us to avoid most of the difficulties caused by those $L \in \mathcal{L}^*(G,T)$ for which $L/O_2(L)$ is not quasisimple, by replacing the uniqueness group LT with the smaller uniqueness subgroup $\Xi_p(L)T$.

PROPOSITION 1.3.8. $\Xi^*_{rad}(G,T) \subseteq \Xi^*(G,T)$.

PROOF. Let $X \in \Xi^*_{rad}(G,T)$. Then $X \trianglelefteq L \in \mathcal{L}^*(G,T)$ by definition. By 1.3.3, $X \in \Xi(G,T)$, so there is an odd prime p such that $X = O_2(X)P$ for $P \in Syl_p(G)$. Indeed by 1.2.1.4, $p > 3$ and either $L/X \cong SL_2(p)$ or $L/O_{2,F}(L) \cong SL_2(5)$. Thus in any case there is a prime r with $L/O_{2,F}(L) \cong SL_2(r)$; r has this meaning throughout the remainder of the proof of the proposition.

By 1.2.1.3, T normalizes L. Then by 1.2.7.3, $M := N_G(L) = !\mathcal{M}(LT)$. We will see shortly how this uniqueness property can be exploited. As X is characteristic in L, $X \trianglelefteq M$, so we also get $M = N_G(X)$ using the maximality of $M \in \mathcal{M}$.

We will next establish a condition used to apply the methods of pushing up. Set $R := O_2(XT)$. Recall the definition of $C(G,R)$ from Definition C.1.5. We claim that

$$C(G,R) \leq M. \qquad (*)$$

The proof of the claim will require a number of reductions.

We begin by introducing a useful subgroup Y of $N_G(R)$: Recall $X \trianglelefteq LT$, and R is Sylow in $C_{LT}(X/O_2(X))$ by A.4.2.7; so by a Frattini Argument,

$$LT = C_{LT}(X/O_2(X))N_{LT}(R). \qquad (**)$$

Thus if we set $Y := N_L(R)^\infty$, then Y contains X, and also by the factorization (**), $N_Y(P)$ has a section $SL_2(r)$, where $r = p$ or 5. So our construction gives $1 \neq Y \in \mathcal{C}(N_{LT}(R))$, such that $Y/O_2(Y)$ is not quasisimple. Further T normalizes R, so in fact using 1.2.6, $Y \in \mathcal{C}(N_{LT}(R)) \subseteq \mathcal{L}(G,T)$.

Next we obtain some restrictions on L. If $R \trianglelefteq LT$ then any $1 \neq C$ char R is normal in LT, so as $M = !\mathcal{M}(LT)$ by 1.2.7.3, we conclude $N_G(C) \leq M$, establishing our claim (*). Thus we may assume R is not normal in LT.

Suppose next that p is the only odd prime in $\pi(O_{2,F}(L))$; in particular this holds in case (c) of 1.2.1.4. Then $X = O^2(O_{2,F}(L))$, so as R centralizes $X/O_2(X)$,

$$[R, L] \leq C_L(X/O_2(X)) \leq O_{2,F}(L) \leq XR$$

and hence $RX \trianglelefteq RL$. But then $R = O_2(RX) \trianglelefteq LT$, contrary to our assumption. Thus we may assume L is in case (d) of 1.2.1.4, and hence $r = 5$ with either $p = 5$ or $p \equiv \pm 1 \mod 5$. Further we have shown there is an odd prime $q \in \pi(O_{2,F}(L))$ with $p \neq q$. Notice that $q \geq 5$.

For $1 \neq C$ char R, let $L_C := N_L(C)^\infty$, and set $X_q := \Xi_q(L)$. Notice $Y \leq L_C$ and $O_2(X_q) \trianglelefteq XT$, so $O_2(X_q) \leq R$. Therefore $R \in Syl_2(X_qR)$. Then as $q \geq 5$, by Solvable Thompson Factorization B.2.16,

$$X_qR = N_{X_qR}(J(R))C_{X_qR}(\Omega_1(Z(R))).$$

So for $C_0 := J(R)$ or $\Omega_1(Z(R))$, $N_{X_qR}(C_0) \not\leq O_{2,\Phi}(X_q)$. Therefore as Y is irreducible on $X_q/O_{2,\Phi}(X_q)$, we conclude $X_q \leq N_G(C_0)$, so $X_q = [X_q, Y] \leq L_{C_0}$. Hence $\pi(O_{2,F}(L_{C_0}))$ contains at least two odd primes p and q.

We are now in a position to complete the proof of the claim. Assume (*) fails. Then there is $1 \neq C$ char R with $N := N_G(C) \not\leq M$. As $YT \leq N_G(R)$, $N \in \mathcal{H}(YT)$; in particular $Y \in \mathcal{L}(N, T)$ as we saw $Y \in \mathcal{L}(G, T)$. So we may apply 1.2.4 to embed $Y \leq Y_C \in \mathcal{C}(N)$, with the inclusion described in A.3.12. Notice in particular that $Y_C \trianglelefteq N$, by 1.2.2.b, since Y contains X of p-rank 2. Also $Y \leq L_C$ by the previous paragraph, so $L_C = [L_C, Y] \leq Y_C$ as $L_C \in \mathcal{L}(G, T)$.

Now if $X \trianglelefteq Y_C$, then X char Y_C using 1.2.1.4, and hence $N \leq N_G(X) = M$, contrary to our choice of $N \not\leq M$. Thus we may assume X is not normal in Y_C. As $X \trianglelefteq Y$, it follows that $Y < Y_C$. In addition the fact that X is not normal in Y_C means $X \not\leq O_\infty(Y_C)$, which rules out cases (21) and (22) of A.3.12, leaving only case (10) of A.3.12 with $Y_C/O_2(Y_C) \cong L_3(p)$. In particular, $p = r = 5$, $Y = L_C$, and $\pi(O_{2,F}(L_C)) = \{2, p\}$. But we saw earlier that $\pi(O_{2,F}(L_{C_0}))$ contains at least two odd primes, so we conclude that our counterexample C cannot be the subgroup C_0 constructed earlier. That is, $N_G(R) \leq N_G(C_0) \leq M$.

Let $(Y_CR)^* := Y_CR/O_2(Y_CR)$. Then P^* is the unipotent radical of a maximal parabolic of $Y_C^* \cong L_3(5)$, so $\text{И}_{Y_C^*R^*}(P^*, 2) = 1$, giving $R^* = 1$ and hence $R \leq O_2(Y_CR)$. On the other hand $O_2(Y_CR) \leq O_2(XT) = R$, so $R = O_2(Y_CR)$. But then $Y_C \leq N_G(R) \leq M$, impossible as X is normal in M, but not in Y_C. This establishes (*); namely $C(G, R) \leq M$.

We now use (*) and results on pushing up from section C.2 to complete the proof of 1.3.8: Assume $X \notin \Xi^*(G, T)$. Then $XT < \langle K, T\rangle =: H$ for some $K \in \mathcal{L}(G, T)$, with $K/O_2(K)$ quasisimple, and H is described in 1.3.4. As $p > 3$, H does not satisfy 1.3.4.4. Now $Aut_{T\cap L}(P)$ is quaternion in cases (c) and (d) of 1.2.1.4, so $Aut_T(P)$ is not cyclic and hence 1.3.4.3 does not hold. Thus we have reduced to cases (1) and (2) of 1.3.4. As X is not normal in H but $X \trianglelefteq M$, $K \not\leq M$. Further from 1.3.4, R acts on K in each case.

We observe next that the property $R \in \mathcal{B}_2(H)$ from Definition C.1.1 and Hypothesis C.2.3 of Volume I hold for $M_H := H \cap M$: Namely $C(H, R) \leq M_H$ using (*); and then by A.4.2.7, $R = O_2(N_H(R))$ and $R \in Syl_2(\langle R^{M_H}\rangle)$.

Furthermore $H \in \mathcal{H}^e$ by 1.1.4.6. So as $K \not\leq M$ and R acts on K, we have the hypotheses of C.2.7, and we conclude K appears on the list of C.2.7.3. In 1.3.4.2, $K/O_2(K) \cong (S)L_3(p)$, whereas no such K appears in C.2.7.3. So we have reduced to 1.3.4.1 where $K/O_2(K) \cong L_2(2^n)$ or $L_2(q)$ for q an odd prime. Then by C.2.7.3, either K is an $L_2(2^n)$-block or an A_5-block, or else $K/O_2(K) \cong L_2(7) \cong L_3(2)$. But the latter two cases are eliminated as $PT = TP$ with $p > 3$. Therefore K is an $L_2(2^n)$-block and 1.3.4.1 holds, so $H = KK^tT$, where $t \in N_T(P) - N_T(K)$. In particular $[K, K^t] = 1$ as distinct blocks commute by C.1.9, so $X = WW^t$ with $W := O^2(X \cap K)$ and $[W, W^t] = 1$. Thus

$$X/Z(X) = WZ(X)/Z(X) \times W^tZ(X)/Z(X)$$

and then $N_G(X)$ permutes $\{WZ(X), W^tZ(X)\}$ by the Krull-Schmidt Theorem A.1.15. In particular, $L = O^2(L)$ acts on $WZ(X)$. This is impossible, as $N_L(P)$ is irreducible on $P/\Phi(P)$ from the structure of L in cases (c) and (d) of 1.2.1.4.

The proof of 1.3.8 is complete. □

As in the previous section, we want to study the action of members of our new class of solvable uniqueness subgroups on their internal modules. So let $\Xi_f(G,T)$ consist of those $X \in \Xi(G,T)$ with $X \in \mathcal{X}_f$, and let $\Xi^*_f(G,T) := \Xi_f(G,T) \cap \Xi^*(G,T)$.

LEMMA 1.3.9. *Let $X \in \Xi(G,T)$, $L \in \mathcal{L}(G,T)$, and $X \leq K := \langle L^T \rangle$. Then*

(1) If $L/O_2(L)$ is quasisimple, then $L \in \mathcal{L}^*(G,T)$.

(2) If $X \in \Xi_f(G,T)$, then $V(X, C_{T \cap K}(X/O_2(X))) \leq V(K)$ and $L \in \mathcal{L}_f(G,T)$.

PROOF. Part (2) follows from A.4.10, just as in the proof of 1.2.9. Thus it remains to establish (1).

Assume $L/O_2(L)$ is quasisimple. Then L is described in 1.3.4. In cases (2)–(4) of 1.3.4, $L \in \mathcal{L}^*(G,T)$ by 1.2.8.4—unless possibly $L/O_2(L) \cong L_4(2)$. But in the latter case, if (1) fails, then $L < Y \in \mathcal{L}(G,T)$, and from 1.2.4 and A.3.12, $Y/O_2(Y) \cong L_5(2)$, M_{24}, or J_4. Now $X = O_2(X)P$ with $P \cong E_9$ and $N_T(P)$ is irreducible on P, so T acts nontrivially on the Dynkin diagram of $L/O_2(L) \cong L_4(2)$. This is impossible, as no such outer automorphism is induced in $Aut(Y/O_2(Y))$.

Therefore 1.3.4.1 must hold. Then by 1.2.8.2, $P \cong E_9$, $L/O_2(L) \cong A_5$, and $Y/O_2(Y) \cong J_1$ or $L_2(p)$. But again as $N_T(P)$ is irreducible on P, some element of $N_T(L)$ induces an outer automorphism on $L/O_2(L) \cong A_5$, whereas no such automorphism is induced in $N_T(Y)$.

Thus 1.3.9 is established. □

1.4. Properties of some uniqueness subgroups

In this section we summarize some basic properties of the families $\mathcal{L}^*(G,T)$ and $\Xi^*(G,T)$ of uniqueness subgroups, which will be used heavily later.

So we consider some L contained either in $\mathcal{L}^*(G,T)$ or in $\Xi^*(G,T)$. Note that the assertion in 1.4.1.1 below is the starting point (as we just saw in the proof of 1.3.8) for arguments using pushing up (sections C.2 etc.).

LEMMA 1.4.1. *Let $L \in \mathcal{L}^*(G,T) \cup \Xi^*(G,T)$ and set $L_0 := \langle L^T \rangle$ and $Q := O_2(L_0T)$. Then $M := N_G(L_0) =! \mathcal{M}(L_0T)$, so L_0T and $N_G(Q)$ are both uniqueness subgroups, and*

(1) $C(G,Q) \leq M$.

(2) $Q \in \mathcal{U}^*_G(L_0, 2) = Syl_2(C_M(L_0/O_2(L_0)))$.

(3) $C_G(Q) \leq O_2(M) \leq Q$.

(4) If $L \in \mathcal{X}_f$*, then there is* $V \in \mathcal{R}_2(L_0T)$ *with* $[V, L_0] \neq 1$.

(5) If $L \in \mathcal{L}^*_f(G, T)$*, assume that* $L/O_2(L)$ *is quasisimple. Let* $V \in \mathcal{R}_2(L_0, T)$ *with* $[V, L_0] \neq 1$*. Then* $C_{L_0T}(V) \leq O_{2,\Phi}(L_0T)$*,* $C_T(V) = Q$*, and* $\Omega_1(Z(Q)) = R_2(L_0T)$.

Proof. First $M := N_G(L_0) = !\mathcal{M}(L_0T)$ by 1.2.7.3 or 1.3.7. Then since $L_0T \leq N_G(Q)$ by definition of Q, also $M = !\mathcal{M}(N_G(Q))$.

Next if $1 \neq R$ char Q, then embedding $N_G(R) \leq N \in \mathcal{M}$, we have $N_G(Q) \leq N_G(R) \leq N$, forcing $N = M$ as $M = !\mathcal{M}(L_0T)$. So (1) holds.

Now (2) follows from A.4.2.7. By (1), $C_G(Q) \leq M$, and by (2), $O_2(M) \leq Q$. Also $M \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ by 1.1.4.6, so

$$C_G(Q) \leq C_M(Q) \leq C_M(O_2(M)) \leq O_2(M),$$

giving (3).

Next when $L \in \mathcal{L}_f(G, T)$, there exists $V \in \mathcal{R}_2(L_0T)$ with $[V, L] \neq 1$ by 1.2.10.3, while this follows from A.4.11 when $L \in \Xi_f(G, T)$, since there *all* 2-chief factors lie in $O_2(L)$. Thus (4) holds. Finally assume that either $L \in \Xi^*_f(G, T)$ or $L \in \mathcal{L}^*_f(G, T)$ with $L/O_2(L)$ quasisimple. Therefore $L_0Q/O_{2,\Phi}(L_0T) = F^*(L_0T/O_{2,\Phi}(L_0T))$ is a chief factor for L_0T, so as $[V, L_0] \neq 1$, $C_{L_0T}(V) \leq O_{2,\Phi}(L_0T)$. But Q is Sylow in $O_{2,\Phi}(L_0T)$ so $C_T(V) \leq Q$, while as V is 2-reduced, $Q = O_2(L_0T) \leq C_T(V)$. This completes the proof of (5). □

CHAPTER 2

Classifying the groups with $|\mathcal{M}(T)| = 1$

Recall from the outline in the Introduction to Volume II that the bulk of the proof of the Main Theorem proceeds under the Thompson amalgam strategy, which is based on the interaction of a pair of distinct maximal 2-local subgroups containing a Sylow 2-subgroup T of G. Clearly before we can implement that strategy, we must treat the case where T is contained in a unique maximal 2-local subgroup.

In Theorem 2.1.1 of this chapter, we determine the simple QTKE-groups G in which a Sylow 2-subgroup T is contained in a unique maximal 2-local subgroup.

This condition is similar to the hypothesis defining an abstract minimal parabolic B.6.1, where T lies in a unique maximal subgroup of G, so we can expect many of the examples arising in E.2.2 to appear as conclusions in Theorem 2.1.1.

The generic examples of simple QTKE-groups with $|\mathcal{M}(T)| = 1$ are the Bender groups. Recall a *Bender group* is a simple group of Lie type and characteristic 2 of Lie rank 1; namely $L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$. The Bender groups also appear in case (a) of E.2.2.2. In addition, some groups from cases (c) and (d) of E.2.2.2 also satisfy the hypotheses of Theorem 2.1.1, as does M_{11} which is not a minimal parabolic.

However, shadows of various groups which are not simple also intrude, and eliminating them is fairly difficult. We mention in particular the shadows of certain groups of Lie type and Lie rank 2 of characteristic 2, extended by an outer automorphism nontrivial on the Dynkin diagram: namely as in cases (1a) and (2b) of E.2.2, extensions of the groups $L_2(2^n) \times L_2(2^n)$, $Sz(2^n) \times Sz(2^n)$, $L_3(2^n)$, and $Sp_4(2^n)$. These groups are not simple, but they *are* QTKE-groups with the property that the normalizer of a Borel subgroup is the unique maximal 2-local containing a Sylow 2-subgroup. We will eliminate the first two families of shadows in 2.2.5 by first using the Alperin-Goldschmidt Fusion Theorem to produce a strongly closed abelian subgroup, and then arguing that G is a Bender group to derive a contradiction. However it is difficult to see that the shadows of the latter two families are not simple, until we have reconstructed in Theorem 2.4.7 most of their local structure, and are then able to transfer off the graph automorphisms and so obtain a contradiction.

Also certain groups of Lie type and odd characteristic are troublesome: The groups $L_2(p) \times L_2(p)$, p a Fermat or Mersenne prime, extended by a 2-group interchanging the components (a subcase of case (b) of E.2.2.1); and the group $L_4(3) \cong P\Omega_6^+(3)$ extended by a group of automorphisms not contained in $PO_6^+(2)$. These groups are also minimal parabolics but not strongly quasithin. Shadows related to the last group appear in many places in the proof.

2.1. Statement of main result

Our main theorem in this chapter is:

THEOREM 2.1.1. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $M = !\mathcal{M}(T)$. Then G is a Bender group, $L_2(p)$ for $p > 7$ a Fermat or Mersenne prime, $L_3(3)$, or M_{11}.*

Of course the groups appearing in the conclusion of Theorem 2.1.1 also appear in the conclusion of our Main Theorem. Thus after Theorem 2.1.1 is proved, we will be able to assume that $|\mathcal{M}(T)| \geq 2$ in the remainder of our work.

Throughout chapter 2, we assume that G, M, T satisfy the hypotheses of Theorem 2.1.1. Thus $M = !\mathcal{M}(T)$, and hence by Sylow's Theorem, also $M = !\mathcal{M}(T')$ for each Sylow 2-subgroup T' of M, so we are free to let T vary over $Syl_2(M)$.

2.2. Bender groups

As we mentioned, the generic examples in Theorem 2.1.1 are Bender groups. These groups were originally characterized by Bender as the simple groups G with the property that the Sylow 2-normalizer M is *strongly embedded* in G; that is (cf. I.8.1), $N_G(D) \leq M$ for all nontrivial 2-subgroups D of M.

If we assume that G is not a Bender group, then there is $1 < D \leq T$ with $N_G(D) \not\leq M$, so that $N_G(D) \in \mathcal{H}(D, M)$ in our notation. If we pick D so that $U := N_T(D)$ is of maximal order subject to this constraint, then since $M = !\mathcal{M}(T)$ by hypothesis, U is a proper subgroup of T Sylow in $N_G(D)$ with $N_G(U) \leq M$. Our proof will focus on pairs (U, H_U) such that $U \leq T$, $U \in Syl_2(H_U)$, and $H_U \in \mathcal{H}^e(U, M)$. While the pair $(U, N_G(D))$ satisfies the first two conditions, and $N_G(D) \in \mathcal{H}(U, M)$, it may not be the case that $N_G(D) \in \mathcal{H}^e$. Thus to ensure that such pairs exist, we use an approach due to GLS (cf. p. 97 in [**GLS94**]) to produce a nontrivial strongly closed abelian 2-subgroup in the absence of such pairs. Then we argue as in the GLS proof [1] of Goldschmidt's Fusion Theorem, to show that G is a Bender group. Our extra hypotheses makes the proof here much easier. We identify G using a special case of Shult's Fusion Theorem, which appears in Volume I as Theorem I.8.3, and is deduced in Volume I from Theorem ZD in [**GLS99**].

We now begin to implement the GLS approach. Instead of considering arbitrary subgroups D of T, we focus on the members of the Alperin-Goldschmidt conjugation family: Using the language of Theorem 16.1 in [**GLS96**] (a form of the Alperin-Goldschmidt Fusion Theorem):

DEFINITION 2.2.1. Given a finite group G and $T \in Syl_2(G)$, define $\mathcal{D}$ to be the set of all nontrivial subgroups D of T such that

(a) $N_T(D) \in Syl_2(N_G(D))$,
(b) $C_G(D) \leq O_{2',2}(N_G(D))$, and
(c) $O_{2',2}(N_G(D)) = O(N_G(D)) \times D$.

The set $\mathcal{D}$ is called the *Alperin-Goldschmidt conjugation family* for T in G.

Next recall that a subgroup X of T is *strongly closed* in T with respect to G if for each $g \in G$, $X^g \cap T \subseteq X$.

[1] See the proof of Theorem SA in section 24 of [**GLS99**]—but recall that we will not make use of their hypothesis of even type.

PROPOSITION 2.2.2. *Assume for each D in the Alperin-Goldschmidt conjugation family that $N_G(D) \leq M$ for T in G. Then*

(1) Each normal 2-subgroup of M is strongly closed in T with respect to G.

(2) G is a Bender group.

PROOF. We first prove (1). Let U be a normal 2-subgroup of M, $u \in U$ and $g \in G$ with $u^g \in T$. We must show $u^g \in U$, so assume otherwise. By the Alperin-Goldschmidt Fusion Theorem (the elementary result 16.1 in [**GLS96**], proved as X.4.8 and X.4.12 in [**HB85**]), there exist $u =: u_1, \ldots, u_n := u^g$ in T, $D_i \in \mathcal{D}$, and $x_i \in N_G(D_i)$, $1 \leq i < n$, such that $u^g = u^{x_1 \cdots x_{n-1}}$, $\langle u_i, u_{i+1} \rangle \leq D_i$, and $u_i^{x_i} = u_{i+1}$. As $u = u_1 \in U$ but $u_n = u^g \notin U$, there exists a least i such that $u_{i+1} \notin U$. Thus $u_i \in U$, and by hypothesis $x_i \in N_G(D_i) \leq M$; therefore as $U \trianglelefteq M$, also $u_{i+1} = u_i^{x_i} \in U$, contrary to the choice of i. Thus (1) holds.

We could now appeal to Goldschmidt's Fusion Theorem [**Gol74**] to establish (2). However the version of this theorem in our list of Background References (cf. Theorem SA in [**GLS99**]) assumes that G is of even type, whereas in the Main Theorem we assume G is of even characteristic. Fortunately the even type hypothesis is unnecessary, and we now extract an easier version of the proof from section 24 of [**GLS99**] under our own hypotheses:

Let U be a minimal normal 2-subgroup of M. Then U is elementary abelian, M is irreducible on U, $M = N_G(U)$, and U is strongly closed in G by (1). Thus for $u \in U^\#$, $u^G \cap M \subseteq U$ and M controls fusion in U by Burnside's Fusion Lemma, so $u^G \cap M = u^M$. Set $G_u := C_G(u)$.

As $U \trianglelefteq T$, we may choose $z \in Z(T) \cap U^\#$. Hence $G_z \leq M$ as $M = !\mathcal{M}(T)$, so as $z^G \cap M = z^M$, M is the unique point fixed by z in the representation of G by right multiplication on the coset space G/M (cf. 46.1 in [**Asc86a**]). We use this fact to show:

(*) For each 2-subgroup S of G containing z, $N_G(S) \leq M$.

For $C_S(z)$ fixes the unique fixed point M of z on G/M, and hence M is the unique fixed point of $C_S(z)$ on G/M. Then as each subgroup of S is subnormal in S, we conclude by induction on $|S|$ that M is the unique fixed point of S of G/M. Hence $N_G(S) \leq M$.

First assume $G_u \leq M$ for every $u \in U^\#$. Then as U is not normal in G, Remark I.8.4 and Theorem I.8.3 tell us that G is a Bender group.

Thus we may assume that $\mathcal{J} := \{u \in U^\# : G_u \not\leq M\}$ is nonempty, and it remains to derive a contradiction. In particular $U > \langle z \rangle$, so the elementary abelian group U is noncyclic. Let $u \in \mathcal{J}$, set $H := G_u$, $M_H := M \cap H$, and let $U \leq S \in Syl_2(H)$. By (*), $S \leq M$, so conjugating in M we may assume that $S \leq T$. By 1.1.6 applied to the 2-local $G_u = H$, the hypotheses of 1.1.5 are satisfied, so $M_H \in \mathcal{H}^e$ by 1.1.5.1.

Suppose $H \in \mathcal{H}^e$. Then as $S \leq T$ and $S \in Syl_2(H)$, $z \in Z(S) \leq O_2(H)$, so $H \leq N_G(O_2(H)) \leq M$ by (*), contradicting $H \not\leq M$. Thus $H \notin \mathcal{H}^e$.

Let W be any hyperplane of U. Then $|z^M \cap U| > 1$ as U is noncyclic, so $z^M \cap W \neq \emptyset$ by A.1.43. Now as $G_z \leq M$, $C_G(W) \leq C_G(z^M \cap W) \leq M$. Hence using Generation by Centralizers of Hyperplanes A.1.17, $O(H) = \langle C_{O(H)}(W) : m(U/W) = 1 \rangle \leq M$, so $O(H) = 1$ since $M_H \in \mathcal{H}^e$.

Thus as $H \notin \mathcal{H}^e$, there is a component L of H, and by 1.1.5, $L = [L, z] \not\leq M$ and L is described in 1.1.5.3. Set $L_0 := \langle L^H \rangle$. As U is strongly closed in S with respect to H, $Aut_U(L_0)$ is strongly closed in $Aut_H(L_0)$, so by inspection of the groups in 1.1.5.3, L is a Bender group with $Aut_U(L) = \Omega_1(S \cap L)$. In particular, U acts on each component of H.

Let U_L and U_C be the projections of U on L and $C_H(L)$, respectively. As $z \in U \leq U_L U_C$, $N_G(U_L U_C) \leq M$ by (*). As $L = [L, z]$, the projection of z on L is nontrivial, while as L is a Bender group, $N_L(U_L)$ is irreducible on $\Omega_1(S \cap L)$. Therefore $U_L = [z, N_L(U_L)] \leq U$ and hence $U = U_C \times U_L$. In particular $m(U) = m(U_L) + m(U_C)$.

Now pick $u \in \mathcal{J}$ so that L is maximal among components of G_j for $j \in \mathcal{J}$. Let $v \in U_C^{\#}$. Since G_v contains $L \not\leq M$, $v \in \mathcal{J}$, so by earlier remarks, U acts on each component of G_v and $O(G_v) = 1$. Then as $u \in U$, u acts on each component of G_v, so L is contained in a component L_v of G_v by I.3.2. Hence $L = L_v$ by maximality of L.

Suppose $g \in M$ with $U_C^g \cap U_C \neq 1$; we claim that $L = L^g$, so that $U_C = U_C^g$ as $M = N_G(U)$. Assume the claim fails and let $1 \neq v \in U_C \cap U_C^g =: V$. By the previous paragraph, L and L^g are components of G_v, and we may assume $L \neq L^g$, so that $[L, L^g] = 1$. It will suffice to show that M acts on $\{L, L^g\}$, since then M permutes $\{U_C, U_C^g\}$, and hence M acts on $1 \neq V = U_C \cap U_C^g \leq U_C$, contradicting the irreducible action of M on U. Now

$$m(U_L) + m(U_C) = m(U) = 2m(U_L) + m(V),$$

so $m(U_C) = m(U_L) + m(V) > m(U)/2$ since $m(V) > 0$. Then for each $x \in M$, $1 \neq U_C \cap U_C^x$. Hence if $L^x \notin \{L, L^g\}$, by symmetry between x and g, also $[L, L^x] = 1$. Then $U_L \leq U_C^g \cap U_C^x$, so also $[L^x, L^g] = 1$. But now for p an odd prime divisor of $|N_L(U_L)|$, $m_{2,p}(L L^g L^x) > 2$, contradicting G quasithin. This completes the proof of the claim.

The claim shows that U_C is a TI-set under M. Further $Aut_L(U)$ is cyclic and regular on $U_L^{\#}$, and is invariant under $N_{Aut_M(U)}(U_C)$. Hence $(Aut_M(U), U)$ is a Goldschmidt-O'Nan pair in the sense of Definition 14.1 of [**GLS96**]. So by O'Nan's lemma, Proposition 14.2 in [**GLS96**], one of the four conclusions of that result holds. Neither conclusion (i) nor (iii) holds, as M is irreducible on U. As $G_z \leq M$ but $\mathcal{J} \neq \emptyset$, M is not transitive on $U^{\#}$, so conclusion (iv) does not hold. Thus conclusion (ii) holds, so that $N_G(U_C)$ is of index 2 in M. However as M is the unique point fixed by z in G/M, by 7.4 in [**Asc94**], M controls G-fusion of 2-elements of M. Therefore by Generalized Thompson Transfer A.1.37.2, $O^2(G) \cap M \leq N_M(U_C)$, contrary to the simplicity of G. This contradiction completes the proof of Proposition 2.2.2. □

Recall that $\mathcal{S}_2(G)$ is the set of nonidentity 2-subgroups of G, and (cf. chapter 1) that $\mathcal{S}_2^e(G)$ consists of those $S \in \mathcal{S}_2(G)$ such that $N_G(S) \in \mathcal{H}^e$. We next verify:

LEMMA 2.2.3. *The Alperin-Goldschmidt conjugation family lies in* $\mathcal{S}_2^e(G)$.

PROOF. By (b) and (c) of the definition of the Alperin-Goldschmidt conjugation family $\mathcal{D}$ for T in G, $O^{2'}(C_G(D)) \leq D$ for each $D \in \mathcal{D}$. Thus as $D \leq T$, $Z(T) \leq D$. Therefore $D \in \mathcal{S}_2^e(G)$ using 1.1.4.3. □

NOTATION 2.2.4. Define $\delta = \delta_M$ to consist of those $D \in \mathcal{S}_2^e(G)$ such that $D \leq M$, but $N_G(D) \not\leq M$. Let $\delta^* = \delta_M^*$ denote the maximal members of δ under inclusion.

THEOREM 2.2.5. *If* $\delta = \emptyset$, *then* G *is a Bender group.*

PROOF. Let $\mathcal{D}$ be the Alperin-Goldschmidt conjugation family for T in G. By 2.2.3, $\mathcal{D} \subseteq \mathcal{S}_2^e(G)$. Therefore if $\delta = \emptyset$, then $N_G(D) \leq M$ for each $D \in \mathcal{D}$. Hence by Proposition 2.2.2.2, G is a Bender group. □

REMARK 2.2.6. The idea of using the Alperin-Goldschmidt Fusion Theorem and Goldschmidt's Fusion Theorem in this way is due to GLS. This approach allows us to avoid considering the case where the centralizer of some involution i has a component which is a Bender group: For if i is such an involution then $\mathcal{U}(C_G(i)) = \emptyset$ (in the language of Notation 2.3.4 established later), whereas Theorem 2.2.5 allows us to assume $\delta \neq \emptyset$, which supplies us with 2-locals H such that $\mathcal{U}(H) \neq \emptyset$. It is these 2-locals which we will exploit during the remainder of this chapter. In particular, as mentioned in the introduction to the chapter, this allows us to avoid difficulties with the shadows of Bender groups extended by involutory outer automorphisms, and also with the shadows of the wreathed products $L_2(2^n)$ wr $\mathbf{Z}_2$ and $Sz(2^n)$ wr $\mathbf{Z}_2$.

2.3. Preliminary analysis of the set Γ_0

Since the Bender groups appear in the conclusion of Theorem 2.1.1, by Theorem 2.2.5, we may assume for the remainder of this chapter that

$$\delta \neq \emptyset, \text{ so that also } \delta^* \neq \emptyset.$$

Recall from the second paragraph of the previous section that there exist pairs (U, H_U) such that $U \in Syl_2(H_U)$, $N_G(U) \leq M$, and $H_U \in \mathcal{H}(U, M)$. Using the fact that δ is nonempty, we will produce such pairs with H_U in $\mathcal{H}^e(U, M)$. Moreover we will see that we can choose U to have a number of useful properties which we list in the next definition:

NOTATION 2.3.1. Let $\beta = \beta_M$ consist of those $U \in \mathcal{S}_2(G)$ such that

(β_0) $U \leq M$, so in fact $U \in \mathcal{S}_2(M)$;
(β_1) For all $U \leq V \in \mathcal{S}_2(M)$, $N_G(V) \leq M$; and
(β_2) $C_{O_2(M)}(U) \leq U$.

Notice that (β_0)–(β_2) are inherited by any overgroup of U in $\mathcal{S}_2(M)$, so all such overgroups are also in β. Some other elementary consequences of this definition include:

LEMMA 2.3.2. *Assume $U \in \beta$, and $U \leq H \leq G$. Then*

(1) If $U \leq V \in \mathcal{S}_2(G)$, then $V \in \beta$. In particular all 2-overgroups of U in G lie in M.
(2) $|H|_2 = |H \cap M|_2$.
(3) If $H \in \mathcal{H}^e$, then $O_2(H) \in \mathcal{S}_2^e(G)$. In particular $\beta \subseteq \mathcal{S}_2^e(G)$.

PROOF. To prove (1), assume $U \leq V \in \mathcal{S}_2(G)$. Recall that each 2-overgroup V of U in M is in β, so it only remains to show that $V \leq M$. If $U \trianglelefteq V$, then $V \leq N_G(U) \leq M$ by (β_1). So as $U \trianglelefteq\trianglelefteq V$, $V \leq M$ by induction on $|V : U|$, completing the proof of (1).

Next let $U \leq S \in Syl_2(H)$. Then $S \in \beta$ by (1), so $S \leq M$ by (β_0), giving (2).

Finally set $Q := O_2(H)$, so that $Q \leq S$ since $S \in Syl_2(H)$. As $S \leq M$, we may assume that $S \leq T$. Then $O_2(M) \leq T$ as $T \in Syl_2(M)$, so $Z(T) \leq C_{O_2(M)}(S) \leq Z(S)$ by (β_2). Under the hypothesis of (3), $Q = F^*(H)$, so $Z(T) \leq Z(S) \leq$

$C_H(Q) \leq Q$, so $Q \in \mathcal{S}_2^e(G)$ by 1.1.4.3. In particular applying this observation to U in the role of "H", $U = O_2(U) \in \mathcal{S}_2^e(G)$, completing the proof of (3). □

We now use our assumption that $\delta^* \neq \emptyset$ to verify that $\beta \neq \emptyset$:

LEMMA 2.3.3. *Let $D \in \delta^*$ and $S \in Syl_2(N_M(D))$. Then:*
(1) $U \in \beta$ for each U in $\mathcal{S}_2(M)$ with $D < U$.
(2) $D < S$, $|S| < |M|_2$, $S \in \beta$, and $S \in Syl_2(N_G(D))$.

PROOF. To prove (1), assume $D < U \in \mathcal{S}_2(M)$. Then U satisfies (β_0) in Notation 2.3.1. By definition of $D \in \delta$ in Notation 2.2.4, $D \in \mathcal{S}_2^e(G)$, so also $U \in \mathcal{S}_2^e(G)$ by 1.1.4.1; hence by maximality of D, $U \notin \delta$, so that $N_G(U) \leq M$. Applying this observation to any $W \in \mathcal{S}_2(M)$ containing U, we obtain (β_1) for U. Next set $E := O_2(N_G(D))$. If $D < E$, then $N_G(D) \leq N_G(E) \leq M$ by the observation, contradicting $D \in \delta$. Thus $D = O_2(N_G(D))$. We saw $D \in \mathcal{S}_2^e(G)$, so that $N_G(D) \in \mathcal{H}^e$, and hence $C_{O_2(M)}(D) \leq C_{N_G(D)}(D) \leq D$. Thus (β_2) holds for D, and hence also for the 2-overgroup U. This completes the proof that $U \in \beta$, giving (1).

Next let $S \in Syl_2(N_M(D))$; we may assume $S \leq T$, and hence $S = N_T(D)$. As $D \in \delta$, $S \leq N_G(D) \not\leq M = !\mathcal{M}(T)$, so $S < T$. In particular $D < T$, so $D < N_T(D) = S$. Then $S \in \beta$ by (1), and hence $S \in Syl_2(N_G(D))$ by 2.3.2.2, completing the proof of (2). □

We now introduce further notation suggested by the GLS proof of the Global C(G,T)-Theorem, in as yet unpublished notes slated to appear in the GLS series; an outline of their proof appears in Sec 2.10 of [**GLS94**].

NOTATION 2.3.4. Let $\mathcal{U}(G) = \mathcal{U}_M(G)$ denote the set of pairs (U, H_U) such that $U \in \beta$ and $H_U \in \mathcal{H}^e(U, M)$. Write $\mathcal{U} = \mathcal{U}_M$ for the set of $U \in \beta$ such that $\mathcal{H}^e(U, M) \neq \emptyset$. For $H \in \mathcal{H}$, let $\mathcal{U}(H) = \mathcal{U}_M(H)$ consist of those $(U, H_U) \in \mathcal{U}(G)$ such that $H_U \leq H$.

Recall that there exists $D \in \delta^*$. By 2.3.3.2, a Sylow 2-group S of $N_M(D)$ is in β, so $N_G(D) \in \mathcal{H}^e(S, M)$ by the definition of δ in Notation 2.2.4. Thus $(S, N_G(D)) \in \mathcal{U}(G)$ and $S \in \mathcal{U}$, so that

$$\mathcal{U}(G) \text{ and } \mathcal{U} \text{ are nonempty,}$$

and by 2.3.3, $S \in Syl_2(N_G(D))$ and $N_G(S) \leq M$. Observe that if $H, H_1 \in \mathcal{H}$ with $H \leq H_1$ then $\mathcal{U}(H) \subseteq \mathcal{U}(H_1)$.

NOTATION 2.3.5. Let $\Gamma = \Gamma_M$ be the set of all $H \in \mathcal{H}$ such that $\mathcal{U}(H) \neq \emptyset$. Let $\Gamma^* = \Gamma^*_M$ consist of those $H \in \Gamma$ such that $\mathcal{U}(H)$ contains some member (U, H_U) with U of maximal order among members of $\mathcal{U}$, and subject to that constraint, with $|H|_2$ maximal. Let $\Gamma_* = \Gamma_{*,M}$ consist of those $H \in \Gamma$ such that $|H|_2$ is maximal among members of Γ. Finally let $\Gamma_0 = \Gamma_{0,M} := \Gamma^* \cup \Gamma_*$.

If $D \in \delta^*$ and $S \in Syl_2(N_M(D)$, then we saw a moment ago that $(S, N_G(D)) \in \mathcal{U}(N_G(D))$, so that $N_G(D) \in \Gamma$ and hence $\Gamma \neq \emptyset$. As Γ is nonempty, also Γ^* and Γ_* are nonempty.

Observe that by that by 2.3.2.2, $|H|_2 = |H \cap M|_2$ for each $H \in \Gamma$, so the constraints on the maximality of $|H|_2$ amount to constraints on $|H \cap M|_2$.

LEMMA 2.3.6. *If $H \in \Gamma_0$, then $|H|_2 \geq |V|$ for any $V \in \mathcal{U}$.*

PROOF. Let $U \in \mathcal{U}$ be of maximal order and $H_U \in \mathcal{H}^e(U, M)$. Then $|V| \leq |U| \leq |H|_2$ for $H \in \Gamma_0$. □

The remainder of the proof of Theorem 2.1.1 focuses on the members of Γ_0. We need to consider members of Γ maximal in the two different senses of Notation 2.3.5 because: On one hand, at a number of points in the proof we produce members of Γ_* (for example in 2.3.7.1), so we need results on the structure of such subgroups. On the other hand, near the end of the proof, particularly in 2.5.10, we need to work with those $H \in \Gamma$ such that $\mathcal{U}(H)$ contains a member (U, H_U) with $|U|$ maximal in $\mathcal{U}$. Thus at that point we choose $H \in \Gamma^*$.

We often use the following observations to produce members of Γ_0:

LEMMA 2.3.7. *Assume $H \in \Gamma$, and let $(U, H_U) \in \mathcal{U}(H)$ and $U \leq S \in Syl_2(H)$.*

(1) Assume $|T : S| = 2$. Then $H \in \Gamma_*$. If $H_1 \in \Gamma$ with $|H_1|_2 \geq |S|$, then $|H_1|_2 = |S|$, and $H_1 \in \Gamma_*$.

(2) Assume $H \in \Gamma_0$ and $H_1 \in \mathcal{H}(H)$. Then $H_1 \in \Gamma_0$, and S is Sylow in H_1 and $H_1 \cap M$.

(3) Assume $H \in \Gamma_0$ and $S \leq H_1 \in \Gamma$; when $H \in \Gamma^*$, assume in addition that $|U|$ is maximal among members of $\mathcal{U}$ and that $\mathcal{H}^e(U, M) \cap H_1 \neq \emptyset$. Then $H_1 \in \Gamma_0$, and S is Sylow in H_1 and $H_1 \cap M$.

(4) Under the hypotheses of (2) and (3), if $H \in \Gamma^*$, Γ_*, then $H_1 \in \Gamma^*$, Γ_*, respectively.

PROOF. Since $U \leq S$ by hypothesis, $S \leq M$ by 2.3.2.1.

Assume $|T : S| = 2$ and $H_1 \in \Gamma$. As $M = !\mathcal{M}(T)$, $|H_1|_2 \leq |T|/2 = |S| = |H|_2$, so $H \in \Gamma_*$, and if $|H_1|_2 \geq |S|$, then $H_1 \in \Gamma_*$, establishing (1).

Now assume the hypotheses of (2); then $H_1 \in \mathcal{H}(H) \subseteq \Gamma$. When $H \in \Gamma_*$, maximality of $|S|$ forces $H_1 \in \Gamma_*$, with S Sylow in H_1, and hence in $H_1 \cap M$. Thus (2) and the corresponding part of (4) hold in this case. When $H \in \Gamma^*$ there is some $(U, H_U) \in \mathcal{U}(H) \subseteq \mathcal{U}(H_1)$, with U of maximal order in $\mathcal{U}$, so by the maximality of $|S|$ subject to this constraint, $H_1 \in \Gamma^*$ and S is Sylow in H_1 and in $H_1 \cap M$. This completes the proof of (2), along with the corresponding part of (4).

Assume the hypotheses of (3); the proof is very similar to that of (2): Again if $H \in \Gamma_*$, then as $S \leq H_1$, $H_1 \in \Gamma_*$ by maximality of $|S|$. Thus we may assume that $H \in \Gamma^*$. Then by hypothesis $|U|$ is maximal in $\mathcal{U}$ and there is $H_2 \in \mathcal{H}^e(U, M) \cap H_1$. Thus $(U, H_2) \in \mathcal{H}(H_1)$, and hence $H_1 \in \Gamma$. Then by maximality of $|U|$ and maximality of $|S|$ subject to that constraint, $H_1 \in \Gamma^*$. □

The next result 2.3.8 lists various properties of members of Γ. In particular part (4) of that lemma is the basis for our analysis of the case where Γ_0 contains a member of $\mathcal{H}^e$ in the next section.

LEMMA 2.3.8. *Let $H \in \Gamma$, $(U, H_U) \in \mathcal{U}(H)$, and $U \leq S \in Syl_2(H)$. Then*

(1) $|S| < |T|$ and $S \in \beta$. In particular, $S \leq M$, so $S \in Syl_2(H \cap M)$.

(2) $O_2(H_U) \in \mathcal{S}^e_2(G)$.

(3) $(U, H_U) \in \mathcal{U}(N_G(O_2(H)))$, and $N_G(O_2(H)) \in \Gamma$.

(4) If $H \in \Gamma_0 \cap \mathcal{H}^e$, then $C(H, S) \leq H \cap M$, so $H = (H \cap M)L_1 \cdots L_s$ with $s \leq 2$ and L_i an $L_2(2^n)$-block, A_3-block, or A_5-block such that $L_i \not\leq M$ and $L_i = [L_i, J(S)]$.

(5) Assume $H \in \Gamma_0$. Then

(a) $N_G(J(S)) \leq M$.

(b) If $J(S) \le R \le S$ with $|S : R| = 2$ and $C_{O_2(M)}(R) \le R$, then $R \in \beta$.

(c) If $H \in \mathcal{H}^e$ then $C_{O_2(M)}(R_0) \le R_0$ for each overgroup R_0 of $O_2(H)$ in S.

(6) If $H \in \Gamma_0$, then the hypotheses of 1.1.5 are satisfied for each involution $z \in Z(S)$ which is 2-central in M.

PROOF. As $U \in \beta$, $S \in \beta$ by 2.3.2.1, so $S \in Syl_2(H \cap M)$. As $S \le M$, we may assume $S \le T$. As $M = !\mathcal{M}(T)$, $|S| < |T|$ completing the proof of (1). Part (2) follows from 2.3.2.3. Next $N_G(O_2(H)) \in \mathcal{H}(H) \subseteq \Gamma$ and $(U, H_U) \in \mathcal{U}(H) \subseteq \mathcal{U}(N_G(O_2(H)))$, so (3) holds.

Assume $H \in \mathcal{H}^e \cap \Gamma_0$. Then $S \in \beta$ by (1), and $H \in \mathcal{H}^e(S, M)$ so that $(S, H) \in \mathcal{U}(H)$ and $S \in \mathcal{U}$. Assume that $C(H, S) \not\le M$. Then there is a nontrivial characteristic subgroup R of S such that $N_H(R) \not\le M$. Now $N_H(R) \in \mathcal{H}^e$ using 1.1.3.2, so $(S, N_H(R)) \in \mathcal{U}(N_G(R))$ and thus $N_G(R) \in \Gamma$. Then we may apply 2.3.7.3 with $N_G(R)$ in the role of "H_1" to conclude that $S \in Syl_2(N_G(R))$. But $S < T$ by (1), so $S < N_T(S) \le N_T(R)$, contradicting $S \in Syl_2(N_G(R))$. This contradiction shows that $C(H, S) \le H \cap M$. Then as $S \in Syl_2(H)$, we may apply the Local $C(G, T)$-Theorem C.1.29 to complete the proof of (4).

We next turn to (5), so we assume $H \in \Gamma_0$, and set $J := J(S)$. By (1), $S \in \beta$, so $Z(T) \le C_{O_2(M)}(S) \le S$ using (β_2) from the definition in 2.3.1. Then $\Omega_1(Z(T)) \le \Omega_1(Z(S)) \le J$ using B.2.3.7, so that $J \in \mathcal{S}_2^e(G)$ by 1.1.4.3. Suppose $N_G(J) \not\le M$. Then $(S, N_G(J)) \in \mathcal{U}(N_G(J))$ so $N_G(J) \in \Gamma$ and $S \in Syl_2(N_G(J))$ by 2.3.7.3. This is impossible as $S < N_T(S) \le N_T(J)$. Therefore $N_G(J) \le M$, proving part (a) of (5).

Next assume that $H \in \mathcal{H}^e$, and consider any R_0 with $Q := O_2(H) \le R_0 \le S$. By 2.3.7.2, $N_G(Q) \in \mathcal{H}(H) \subseteq \Gamma_0$, with S Sylow in $N_G(Q)$ and $N_M(Q)$. Therefore

$$E := C_{O_2(M)}(Q) \le O_2(N_M(Q)) \le S \le H.$$

Also $F^*(H) = O_2(H) = Q$ as $H \in \mathcal{H}^e$, so $E \le C_H(Q) \le Q$. Then as $Q \le R_0$,

$$C_{O_2(M)}(R_0) \le C_{O_2(M)}(Q) = E \le Q \le R_0,$$

establishing part (c) of (5).

So to complete the proof of (5), it remains to establish part (b). Thus we assume that $J \le R \le S$ with $|S : R| = 2$, and $C_{O_2(M)}(R) \le R$. We must show that $R \in \beta$; as R satisfies (β_0) since $S \le M$, and R satisfies (β_2) by hypothesis, we may assume that (β_1) fails for R, and it remains to derive a contradiction. Then for some $R \le V \in \mathcal{S}_2(M)$, $N_G(V) \not\le M$, and we may choose V maximal subject to this constraint. As usual, we may assume that $V \le T$. By hypothesis $J(S) \le R$, so $J(S) = J(R)$ by B.2.3.3, and hence $N_G(R) \le N_G(J(S)) \le M$ by part (a) of (5). Therefore $R < V$. Further $N_G(V) \not\le M = !\mathcal{M}(T)$, so that $V < T$ and hence $V < N_T(V) := W$. Then W satisfies (β_0), and also (β_2), since this condition is inherited by overgroups of R. Further by maximality of V, $N_G(X) \le M$ for each X satisfying $W \le X \in \mathcal{S}_2(M)$, establishing (β_1) for W. Hence $W \in \beta$. We saw earlier that $J(S) = J \in \mathcal{S}_2^e(G)$, and by hypothesis $J \le R \le V$, so $V \in \mathcal{S}_2^e(G)$ by 1.1.4.1, and hence $N_G(V) \in \mathcal{H}^e$. Then $(W, N_G(V)) \in \mathcal{U}(N_G(V))$ so $N_G(V) \in \Gamma$. However by hypothesis $|S : R| = 2$, while $R < V < W$, so that $|W| > |S|$. This contradicts the maximality of $|H|_2$ in Notation 2.3.5 when $H \in \Gamma_*$, and the maximality of $|U|$ when $H \in \Gamma^*$. This contradiction completes the proof of (5).

It remains to prove (6). By (1), $S \leq M$ and $S \in Syl_2(H \cap M)$. Assume that $H \in \Gamma_0$. Then by 2.3.7.2, $N_G(O_2(H)) \in \Gamma$, and S is Sylow in $N_G(O_2(H))$ and $N_M(O_2(H))$. Thus $C_{O_2(M)}(O_2(H)) \leq O_2(N_M(O_2(H))) \leq S$. Now $O_2(H) \leq O_2(H \cap M)$ by A.1.6, so

$$C_{O_2(M)}(O_2(H \cap M)) \leq C_{O_2(M)}(O_2(H)) \leq S \leq H,$$

establishing one of the hypotheses of 1.1.5. Finally if z is an involution central in $T' \in Syl_2(M)$, then $C_G(z) \leq M = !\mathcal{M}(T')$, establishing the remaining hypothesis for that result. This establishes (6), and so completes the proof of 2.3.8. □

The final section of this chapter will focus on components of a member of Γ_0. Using part (6) of 2.3.8, the next result describes these components.

LEMMA 2.3.9. *Let $H \in \Gamma_0$, $Q := O_2(H)$, $(U, H_U) \in \mathcal{U}(H)$, and $U \leq S \in Syl_2(H)$. Then*

(1) S is Sylow in $N_G(Q)$ and $N_M(Q)$, and $N_G(Q) \in \Gamma_0$. If $H \in \Gamma^*$, then $N_G(Q) \in \Gamma^*$.

(2) $C_{O_2(M)}(Q) \leq S$.

(3) $Z(T) \leq S < T$ for some $T \in Syl_2(M)$ depending on H. In particular, $Z(T) \leq Z(S)$.

(4) $F^*(H \cap M) = O_2(H \cap M)$.

(5) Let z be an involution in $Z(T)$ for T as in (3). Then $C_G(z) \leq M$ and z inverts $O(H)$.

(6) If L is a component of H, then $L = [L, z] \not\leq M$, and L is contained in a component L_Q of $N_G(Q)$.

(7) If L is a component of H then z induces an inner automorphism on L unless possibly $L/Z(L) \cong A_6$ or A_7. Moreover one of the following holds:

(a) L is a Bender group.

(b) $L \cong Sp_4(2^n)'$ or $L_3(2^n)$, or $L/O_2(L) \cong L_3(4)$ or $L \cong \hat{A}_6$.

(c) $L \cong A_7$ or $\hat{A}_7$, and $L \cap M$ is the stabilizer in L of a partition of type $2^3, 1$.

(d) $L \cong L_3(3)$ or M_{11}, and $L \cap M = C_L(z_L)$ where z_L is the projection of z on L.

(e) $L \cong L_2(p)$, p a Fermat or Mersenne prime, and $L \cap M = S \cap L$.

(f) $L \cong M_{22}$ or M_{23}, and $L \cap M \cong A_6/E_{16}$ or A_7/E_{16}, respectively.

(g) $L \cong L_4(2)$, S is nontrivial on the Dynkin diagram of L, and $L \cap M = C_L(z_L)$, where z_L is the projection of z on L.

(8) Assume $|S : R| = 2$, with R containing $J(S)$, $O_2(H)$, and $C_S(R)$. Then $R \in \beta$.

PROOF. By 2.3.8.3, $N_G(Q) \in \Gamma$; then (1) follows from parts (2) and (4) of 2.3.7.

By (1), S is Sylow in $N_M(Q)$, so $C_{O_2(M)}(Q) \leq O_2(N_M(Q)) \leq S$, proving (2). By 2.3.8.1, $S \in \beta$, so in particular $S \leq M$ and $S < T$ for some $T \in Syl_2(M)$, As $F^*(M) = O_2(M)$, $Z(T) \leq O_2(M)$, so as $Q \leq S \leq T$, $Z(T) \leq S$ by (2), completing the proof of (3).

By (3), $Z(T) \leq Z(S)$, so by 2.3.8.6 the hypotheses of 1.1.5 are satisfied for each involution $z \in Z(T)$, and in particular $C_G(z) \leq M$. Therefore 1.1.5.1 implies (4), while 1.1.5.2 says z inverts $O(H)$, completing the proof of (5).

Similarly if L is a component of H, then by 1.1.5.3, $L = [L, z] \not\leq M$, and the possibilities for L are listed in 1.1.5.3. Notice that L is a component of $\langle L, S\rangle$ and S is Sylow in $N := N_G(Q)$ by (1), so by 1.2.4, $L \leq L_Q \in \mathcal{C}(N)$. Since $L = [L, z]$, also $L_Q = [L_Q, z]$. As $N \in \Gamma_0$ by (1), z inverts $O(N)$ by (5); so as $L_Q = [L_Q, z]$, L_Q centralizes $O(N)$. Similarly z centralizes $O_2(N)$ as $z \in Z(S)$ and $S \in Syl_2(N)$, so $L_Q = [L_Q, z]$ centralizes $O_2(N)$. Thus L_Q centralizes $F(L_Q)$, so L_Q is quasisimple by A.3.3.1, and hence L_Q is a component of N. This completes the proof of (6).

To prove (7), we must refine the possibilities listed in 1.1.5.3. If $L/Z(L)$ is a Bender group, then $Z(L) = 1$ by 1.1.5.3, so conclusion (a) of (7) holds in this case. Hence we may assume $L/Z(L)$ is not a Bender group.

In this paragraph, we make a slight digression, to construct some machinery to deal with groups of Lie rank at least 2. Assume $L \leq \langle H_1, H_2\rangle$ with $H_i \in \mathcal{H}^e(S)$. Suppose $H_i \not\leq M$ for some i. Then from the definitions in Notation 2.3.4, $(S, H_i) \in \mathcal{U}(H_i)$, so $H_i \in \Gamma_0$ by 2.3.7.3. Consequently H_i is described in 2.3.8.4.

Now suppose $L/Z(L)$ appears in one of cases (a)–(c) of 1.1.5.3; then as $L/Z(L)$ is not a Bender group, $L/Z(L)$ is a group of Lie type and characteristic 2 of rank at least 2 in Theorem C (A.2.3). If there do not exist two distinct maximal $N_S(L)$-invariant parabolics K_1 and K_2, then (cf. E.2.2.2) $L/Z(L) \cong L_3(2^n)$ or $Sp_4(2^n)'$ with S nontrivial on the Dynkin diagram of $L/Z(L)$, and then conclusion (b) of (7) holds. Thus we may assume K_1 and K_2 exist, take $H_i := \langle K_i, S\rangle$, and apply the observations in the previous paragraph. By (6), $L \not\leq M$, and hence $H_i \not\leq M$ for some i, so K_i is a block described in 2.3.8.4. Then we check that the only groups in (a)–(c) of 1.1.5.3 with such a block are those in conclusions (b) and (g) of (7), keeping in mind that $Z(L) = O_2(L)$ in case (b) of 1.1.5.3. Similar arguments, using generation by a pair of members of $\mathcal{H}^e(S)$ in LS, eliminate those cases where $L/Z(L)$ is M_{12}, M_{24}, J_2, J_4, HS, He, or Ru; thus in case (f) of 1.1.5, $L/Z(L)$ is M_{11}, M_{22} or M_{23}.

If case (d) of 1.1.5.3 holds, then z has cycle structure 2^3 and as $C_G(z) \leq M$, $L\cap M$ contains the stabilizer K in $L\langle z\rangle$ of a partition of type $2^3, 1$ determined by z. So as K is a maximal subgroup of $L\langle z\rangle$ and $L \not\leq M$, $K = M \cap L\langle z\rangle$; thus conclusion (c) of (7) holds.

In the cases $L_3(3)$, $L_2(p)$, M_{11}, M_{22}, M_{23} remaining from (e) and (f) of 1.1.5.3, the description of z determines the maximal subgroup of $L\langle z\rangle$ described in conclusions (d), (e), (d), (f), and (f) of (7), respectively. Finally by 1.1.5.3, z induces an inner automorphism on L, except possibly when $L/Z(L)$ is A_6 or A_7, completing the proof of (7).

Assume the hypotheses of (8). Because we are assuming that $Q \leq R \geq C_S(R)$, $C_{O_2(M)}(R) \leq C_S(R) \leq R$ by (2). Then since $|S : R| = 2$ and $J(S) \leq R$ by hypothesis, we have the hypotheses of 2.3.8.5b, and that lemma completes the proof of (8), and hence of 2.3.9. □

Lemma 2.3.10. *If S is of index 2 in T and $\mathcal{H}(S) \not\subseteq M$, then $S \in \beta$.*

Proof. As $S \leq T \leq M$, condition (β_0) from the definition in Notation 2.3.1 holds. As $|T : S| = 2$, $N_G(S) \leq M = !\mathcal{M}(T)$, and then the only proper 2-overgroups of S are Sylow groups T' of M, so (β_1) holds as $M = !\mathcal{M}(T')$. Finally by hypothesis, there is $H \in \mathcal{H}(S)$ with $H \not\leq M$; enlarging H if necessary, we may assume $H = N_G(O_2(H))$. As $M = !\mathcal{M}(T')$ for $T' \in Syl_2(M)$, $S \in Syl_2(H \cap M)$.

Thus $O_2(H \cap M) \leq S$ and $C_G(O_2(H)) \leq H$, so

$$C_{O_2(M)}(S) \leq C_{O_2(M)}(O_2(H)) \leq O_2(M) \cap H \leq O_2(H \cap M) \leq S,$$

establishing (β_2). □

NOTATION 2.3.11. Set $\Gamma^e = \Gamma^e_M := \Gamma \cap \mathcal{H}^e$ and $\Gamma^e_0 = \Gamma^e_{0,M} := \Gamma_0 \cap \mathcal{H}^e$

The proof of Theorem 2.1.1 now divides into two cases: Either Γ^e_0 is nonempty or Γ^e_0 is empty. In the first case we focus on a member of Γ^e_0; the structure of such groups is described in 2.3.8.4. In the second case 2.3.9 gives us information about the members of Γ_0, particularly about their components. The two cases are treated in the remaining two sections of this chapter.

2.4. The case where Γ^e_0 is nonempty

In this section, we treat the case where Γ^e_0 is nonempty. Here by 2.3.8.4, H has a very restricted structure dominated by χ_0-blocks. We will use this fact to identify the groups in the conclusion of Theorem 2.1.1 which are not Bender groups, and eliminate some difficult shadows. The main result of this section is:

THEOREM 2.4.1. *If there exists $H \in \Gamma_0$ with $F^*(H) = O_2(H)$, then G is $L_2(p)$, $p > 7$ a Mersenne or Fermat prime, $L_3(3)$, or M_{11}.*

In the remainder of this section, we assume that

$H \in \Gamma^e_0$, and the pair G, H afford a counterexample to Theorem 2.4.1.

The groups appearing in the conclusion of Theorem 2.4.1 will emerge during the proof of 2.4.26.

Choose $S \in Syl_2(H)$ as in 2.3.8, and choose $T \in Syl_2(M)$ as in 2.3.9.3; then

S is Sylow in H and in $H \cap M$, $T \in Syl_2(M)$, and $Z(T) \leq S < T$.

LEMMA 2.4.2. *If $S_1 \in Syl_2(H_1 \cap M)$ for $H_1 \in \Gamma^e$, then $S_1 \in Syl_2(H_1)$, $(S_1, H_1) \in \mathcal{U}(H_1)$, and $S_1 \in \mathcal{U}$.*

PROOF. From the definition of Γ in Notation 2.3.5, $\mathcal{U}(H_1)$ contains a member (U, H_U) with $U \leq S_1$. Then $S_1 \in Syl_2(H_1)$ and $S_1 \in \beta$ by 2.3.8.1, so as $H_1 \in \mathcal{H}^e$ it follows that $(S_1, H_1) \in \mathcal{U}(H_1)$ and $S_1 \in \mathcal{U}$. □

In particular

$$S \in \mathcal{U} \text{ and } (S, H) \in \mathcal{U}(H)$$

by 2.4.2. For the remainder of this section, we set

$$Q := O_2(H) \text{ and } G_Q := N_G(Q).$$

LEMMA 2.4.3. *(1) $S \in Syl_2(G_Q)$ and $G_Q \in \Gamma^e_0$. In particular, $F^*(G_Q) = O_2(G_Q) = Q$.*

(2) Assume $H_1 \in \Gamma^e$ and $|H_1|_2 \geq |H|_2$. Then $|H_1|_2 = |H|_2$ and $H_1 \in \Gamma^e_0$.

PROOF. First $S \in Syl_2(G_Q)$ and $G_Q \in \Gamma_0$ by 2.3.9.1. Then using A.1.6, $Q \leq O_2(G_Q) \leq O_2(H) = Q$, so $Q = O_2(G_Q)$. As $(S, H) \in \mathcal{U}(H)$, $Q = O_2(H) \in \mathcal{S}^e_2(G)$ by 2.3.8.2, so $G_Q \in \Gamma^e_0$, completing the proof of (1).

Next assume the hypotheses of (2). Recall $\Gamma_0 = \Gamma^* \cup \Gamma_*$ from the definitions in Notation 2.3.5. If $H \in \Gamma_*$, then $|H|_2 \geq |H_1|_2$ by maximality of $|H|_2$, so as $|H_1|_2 \geq |H|_2$ by hypothesis, $H_1 \in \Gamma_*$, so that $H_1 \in \Gamma^e_0$ in this case. Thus we may assume $H \in \Gamma^*$, so S is a member of $\mathcal{U}$ of maximal order. Choose $S_1 \in Syl_2(H_1 \cap M)$; by

2.4.2, $S_1 \in Syl_2(H_1)$ and $S_1 \in \mathcal{U}$. Then as $|H_1|_2 \geq |H|_2 = |S|$ by hypothesis, we conclude from maximality of $|S|$ over $\mathcal{U}$ that $|S_1| = |S|$. Then by maximality of $|H|_2$ over members of Γ containing a pair with a member of $\mathcal{U}$ of maximal order, we conclude $H_1 \in \Gamma^*$, so that $H_1 \in \Gamma_0^e$ in this case as well, completing the proof. □

Since $H \in \Gamma_0^e$, 2.3.8.4 says $H = (H \cap M)L_1 \cdots L_s$, where L_i is an $L_2(2^n)$-block with $n > 1$, an A_3-block, or an A_5-block; further $L_i \not\leq M$, and $s \leq 2$. Since S, H play the roles of "U, H_U" in the previous section, in the remainder of this section U will instead denote the module $U(L_1) = [O_2(L_1), L_1]$ in the notation of Definition C.1.7. Furthermore we set:

$$L := L_1,\ L_0 := \langle L^S \rangle, \text{ and } U_0 := \langle U^S \rangle.$$

Then $L_0 \trianglelefteq H$ by 1.2.1.3, so $L_0 \in \mathcal{H}^e$ by 1.1.3.1, and hence $L_0S \in \mathcal{H}^e$. Further $L_0S \not\leq M$, so $(S, L_0S) \in \mathcal{U}(L_0S)$ and hence $L_0S \in \Gamma^e$. Then $L_0S \in \Gamma_0^e$ by 2.4.3.2, so replacing H by L_0S, we may assume $H = L_0S$. Then from section B.6:

$$H = L_0S \text{ is a minimal parabolic and } L \text{ is a } \chi_0\text{-block.}$$

Lemma 2.4.4. *If $1 \neq S_0 \leq S$ with $S_0 \trianglelefteq H$, then $N_T(S_0) = S$.*

Proof. By 2.3.7.2, $N_G(S_0) \in \Gamma_0$ and $S \in Syl_2(N_G(S_0))$. In particular $S = N_T(S_0)$. □

Lemma 2.4.5. *(1) Hypotheses C.5.1 and C.5.2 are satisfied with S in the roles of both "T_H, R" for any subgroup M_0 of T with S a proper normal subgroup of M_0.*

(2) Assume $S \leq M_0 \leq T$ with $|M_0 : S| = 2$ and set $D := C_S(L_0)$. Then

(a) $Q = U_0D \in \mathcal{A}(S)$.

(b) For each $x \in M_0 - S$, $1 = D \cap D^x$ and $U_0^x \not\leq Q$.

(3) Assume either that L is an A_3-block, or that $L = L_0$ is an $L_2(2^n)$-block or an A_5-block. Then the hypotheses of Theorem C.6.1 are satisfied with T, S in the roles of "Λ, T_H".

(4) $U_0 = O_2(L_0)$.

Proof. We saw that $H = L_0S$ is a minimal parabolic, and the rest of Hypothesis C.5.1 is straightforward. As S is proper in M_0, Hypothesis C.5.2 follows from 2.4.4. Thus (1) holds.

Choose M_0 as in (2) and set $D_0 := C_{\text{Baum}(S)}(L_0)$. This is the additional hypothesis for C.5.6.7; and that result implies (4); and also says that $Q = U_0D_0$, $Q \in \mathcal{A}(S)$, and $D_0 \cap D_0^x = 1$ for each $x \in M_0 - S$. As $Q \in \mathcal{A}(S)$, $D_0 = C_S(L_0) = D$. By C.5.5, there exists $y \in M_0$ with $U_0^y \not\leq Q$. Then as $U_0 \trianglelefteq S$ and $|M_0 : S| = 2$, $M_0 - S = \{x_0 \in M_0 : U_0^{x_0} \not\leq Q\}$, so the proof of (2) is complete.

Finally assume the hypotheses of (3). The first three conditions in the hypothesis of Theorem C.6.1 are immediate, while condition (iv) follows from 2.4.4, establishing (3). □

Lemma 2.4.6. *$L_0 \trianglelefteq G_Q$.*

Proof. By 2.4.3.1, $S \in Syl_2(G_Q)$ with $G_Q \in \Gamma_0^e$. Hence we may apply 2.3.8.4 to G_Q, to conclude that G_Q is the product of $N_M(Q)$ with a product of χ_0-blocks. But using 1.2.4 and A.3.12, no larger χ_0-block contains an S-invariant product L_0 of χ_0-blocks, so we conclude $L_0 \trianglelefteq G_Q$. □

2.4.1. Shadows of groups of rank 2 with $L_2(2^n)$-blocks. In this subsection we continue the proof of Theorem 2.4.1 by eliminating the shadows of $L_3(2^n)$ and $Sp_4(2^n)$ extended by an outer automophism nontrivial on the Dynkin diagram. To be more precise, we will show that if L is an $L_2(2^n)$-block, then H essentially has the structure of a maximal parabolic of $L_3(2^n)$ or $Sp_4(2^n)$. Then we will show that $O^2(G) < G$ via transfer.

The main result of this subsection is:

THEOREM 2.4.7. *L is not an $L_2(2^n)$-block for $n > 1$.*

Throughout this subsection, G and H continue to be a counterexample to Theorem 2.4.1, with $H = L_0S$ and $L_0 = \langle L^S \rangle$. Moreover we also assume that H is a counterexample to Theorem 2.4.7, so that L is an $L_2(2^n)$-block with $n > 1$. Set $q := 2^n$. Fix a Hall $2'$-subgroup D of $L_0 \cap M$ normalizing $L_0 \cap S$; thus $L_0 \cap M = (L_0 \cap S)D$ is a Borel subgroup of L_0. Of course $D \neq 1$ as $n > 1$.

The proof divides into two cases: $s = 1$ and $s = 2$. Further the case where $s = 1$ is by far the more difficult, as that is where the shadows of $L_3(q)$ and $Sp_4(q)$ extended by outer automorphisms arise. Thus the treatment of that case involves a long series of lemmas.

In the remainder of this subsection, set

$$R := J(S).$$

The Case $s = 1$.

Until this case is complete, we assume that $s = 1$, so that $L_0 = L_1 = L$, and $H = LS$.

LEMMA 2.4.8. *(1) $|T : S| = 2$. Hence T normalizes S and R.*

(2) $O_2(L) = U = Q = O_2(H)$.

(3) $R = Baum(S) = UU^x \in Syl_2(L)$, $U \cap U^x = Z(R)$, and $\mathcal{A}(S) = \mathcal{A}(T) = \{U, U^x\}$ for each $x \in T - S$.

(4) D^x acts on L, and either

(a) $Z(L) = 1$ and $L \cong P^\infty$ for P a maximal parabolic in $L_3(q)$, or

(b) $Z(L) \cong E_{2^n}$, D^x is regular on $Z(L)^\#$, and $L \cong P^\infty$ for P a maximal parabolic in $Sp_4(q)$.

(5) $\langle T, D \rangle = TB$, where B is an abelian Hall $2'$-subgroup of $\langle T, D \rangle$ containing D, S normalizes RD, $R \trianglelefteq RB \trianglelefteq BT$, $C_R(B) = 1$, and T is the split extension of R by $N_T(B)$. If $x \in N_T(B) - S$, then $B = DD^x$.

(6) If $Z(L) \neq 1$, then $B = D \times D^x$; while if $Z(L) = 1$, then $Aut_B(Z(R)) = Aut_D(Z(R)) \cong D$ is regular on $Z(R)^\#$.

(7) U and U^x are the maximal elementary abelian subgroups of R.

PROOF. By 2.4.5.3, we have the hypotheses of Theorem C.6.1, with T, S in the roles of "Λ, T_H", so we may appeal to Theorem C.6.1. In particular, conclusion (a) of C.6.1.6 holds, since L is of type $L_2(2^n)$ for $n > 1$. Thus (1) holds and $R = J(S) = J(T)$. By C.6.1.1, Baum$(S) = R = QQ^x$ for each $x \in T - S$ and by C.6.1.3, $\{Q, Q^x\} = \mathcal{A}(S)$. As $R = QQ^x$ with $Q \in \mathcal{A}(S)$, $Q \cap Q^x = Z(R)$ using B.2.3.7. Since $Q^x \not\leq Q$, Q^x is an FF-offender on U by Thompson Factorization B.2.15, so as $H = LS$ with L an $L_2(2^n)$-block, $R/Q = QQ^x/Q$ is Sylow in LQ/Q

by B.4.2.1, and hence $R \in Syl_2(LQ)$. Then (3) will follow once we prove (2). However, we will first establish (5) and the assertion in (4) that D^x normalizes L.

As $L = O^2(H)$, $C_R(L) \leq Q$, so as Q is abelian and $R \leq LQ$, $C_R(L) \leq Z(R)$. Further by (1), we may apply 2.4.5.2 with T in the role of "M_0", to conclude that $Q = UC_S(L)$. Then as $U \leq L$ and $R \leq LQ$, $R = (R \cap L)C_R(L)$. As R is Sylow in LQ, $S \cap L = R \cap L$; then $M \cap L = (R \cap L)D$ is a Borel subgroup of L, and $R = (R \cap L)C_R(L)$ is D-invariant.

Now S normalizes the Borel subgroup $(R \cap L)D$ over $R \cap L$, and hence normalizes RD. Thus S also normalizes $(RD)^x = RD^x$. Also T permutes Q and Q^x, and so acts on $G_Q \cap G_Q^x =: Y$. We saw D normalizes R, so as $D = O^2(D)$, D normalizes the two members Q and Q^x of $\mathcal{A}(R)$; that is, $D \leq Y$, and hence also $D^x \leq Y$. By 2.4.6, G_Q normalizes $L_0 = L$; in particular D^x normalizes L, giving the first assertion in (4). Now $M \cap G_Q$ normalizes $M \cap L = (R \cap L)D$ as well as $Q = UC_R(L)$, and hence normalizes their product RD. Then as $D^x \leq M \cap Y$, D^x also normalizes RD. As $x^2 \in S$ normalizes RD and $(RD)^x = RD^x$,

$$RDD^x \trianglelefteq \langle RDD^x, S, x\rangle = RDD^xT = DD^xT.$$

Hence by a Frattini Argument, we may take x to act on a Hall $2'$-subgroup B of RDD^x containing D.

We can now obtain the conclusions of (5), except possibly for $C_R(B) = 1$: First D^x normalizes $RD \cap B = D$, so DD^x is a subgroup of B, and hence $DD^x = B$. Then T normalizes $RDD^x = RB$, while R is normalized by D and hence also by D^x, and further $\langle T, D\rangle = BT$. We saw $D \trianglelefteq DD^x = B$, so also $D^x \trianglelefteq B$, and then $D^x[D, D^x] \leq D^x$. As D^x is abelian this shows that no element of D^x induces an outer automorphisms on $L/U \cong L_2(2^n)$, so that $B = DD^x = D \times C_B(L/U)$. Then since $B = DD^x$ and D is abelian, it follows that B is abelian. By a Frattini Argument on $RB \trianglelefteq TB$, $T = RN_T(B)$. This extension splits once we show $C_R(B) = 1$.

Thus to complete the proof of (5), it remains to show that $C_R(B) = 1$. We saw that $R = (R \cap L)C_R(L)$, so $[R, D] = [R \cap L, D] = R \cap L$ since L is an $L_2(2^n)$-block; thus also $[R, D^x] = R \cap L^x$. Further we saw $Q \cap Q^x = Z(R) \geq C_R(L)$, so $R/Z(R) = [R/Z(R), D]$. Then also $R/Z(R) = [R/Z(R), D^x]$, so

$$R \cap L = [R \cap L, D^x] \leq [R, D^x] = R \cap L^x;$$

so as $(R \cap L)^x = R \cap L^x$, $R \cap L = R \cap L^x$. Therefore $R \cap L = [R, D^x]$, so $[R, B] = [R, DD^x] = R \cap L$. As $C_{R/C_R(L)}(D) = 1$, $C_R(B) \leq C_R(L) \leq Z(R)$, so $C_R(B) \trianglelefteq LRN_S(B) = LS = H$. Also x normalizes R and B, and hence also $C_R(B)$, so that $C_R(B) = 1$ by 2.4.4, completing the proof of (5).

Now by Coprime Action, $R = [R, B] = R \cap L$, so that $R \leq L$. As $Q \leq R$, $Q = O_2(L) = U$ by 2.4.5.1, so that (2) holds. This also completes the proof of (3) as mentioned earlier.

So it remains to complete the proof of (4) and establish (6) and (7). As L is an $L_2(q)$-block, L is indecomposable on U with $U/Z(L)$ the natural module for L/U. From the cohomology of that module in I.1.6, $m(Z(L)) \leq n$. Further $Z(L) = C_R(D)$ with D semiregular on $R/Z(L)$, so D^x is semiregular on $R/C_R(D^x)$. Thus as $C_R(B) = 1$, D^x is semiregular on $Z(L)^\#$, so as $m(Z(L)) \leq n$, either $Z(L) = 1$ or $m(Z(L)) = n$. In each case (using I.1.6 in the latter) the representation of L/U on U is determined up to equivalence, and as the Sylow group $R = UU^x$ of L splits over U, L also splits over U by Gaschütz's Theorem A.1.39. Therefore L is determined up to isomorphism in each case. The parabolics P in cases (a) and (b) of (4) exhibit such extensions, so this completes the proof of (4). Part (4) implies (7).

When $Z(L) \neq 1$ the fact that D^x is semiregular on $Z(L) = C_R(D)$ shows that $B = D \times D^x$. When $Z(L) = 1$, D is regular on $Z(R)^\#$, so $D \cong Aut_D(Z(R))$ is self-centralizing in $GL(Z(R))$; thus as B is abelian, $Aut_B(Z(R)) = Aut_D(Z(R))$. This completes the proof of (6), and hence also of 2.4.8. □

Remark 2.4.9. The cases (a) and (b) of 2.4.8.4 were treated separately in sections 3 and 4 of [**Asc78a**]. However many of the arguments for the two cases are parallel, so we give a common treatment here where possible.

Notation 2.4.10. During the remainder of the treatment of the case $s = 1$, x denotes an element of $T - S$. By 2.4.8.1, $|T : S| = 2$, so as S acts on L, H, U, the conjugates L^x, H^x, U^x are independent of the choice of x.

Lemma 2.4.11. *(1)* $G_Q = N_G(L)$.
(2) $G_Q = !\mathcal{M}(L)$.
(3) If $Z(L) \neq 1$ *then* $Z(L)$ *is a TI-subgroup of* G *with* $G_Q = N_G(Z(L))$.

Proof. Recall $G_Q \leq N_G(L)$ by 2.4.6; so as $Q = U = O_2(L)$ by 2.4.8.2, $N_G(L) \leq G_Q$, so (1) holds.

As $Q = O_2(LS)$ while $S \in Syl_2(G_Q)$ by 2.4.3.1, $\mathcal{U}^*_{G_Q}(L, 2) = \{Q\}$. Then as L is irreducible on $Q/Z(L)$ and indecomposable on Q, if $1 \neq V \in \mathcal{U}_{G_Q}(L, 2)$ then either $V = Q$ or $V \leq Z(L)$.

Let $X \in \mathcal{H}(L)$; to prove (2), we must show that $L \trianglelefteq X$, so assume otherwise. Let $P := O_2(X)$. Then $1 \neq P_0 := N_P(Q) \leq G_Q$, so $P_0 \in \mathcal{U}_{G_Q}(L, 2)$. Thus by the previous paragraph, either $P_0 = Q$ or $P_0 \leq Z(L)$. In either case $P_0 \leq Q$, so that $N_{PQ}(Q) = P_0Q = Q$, and then $PQ = Q$ so that $P = P_0$. If $P = Q$, then $X \leq N_G(Q) = G_Q$, contrary to assumption; hence $P \leq Z(L)$. This shows that (2) holds when $Z(L) = 1$. Thus for the rest of the proof, we may assume $Z(L) \neq 1$, since this is also the hypothesis of (3). In particular, case (b) of 2.4.8.4 holds.

Next we claim that $C_G(v) \leq G_Q$ for each $v \in Z(L)^\#$. Assume otherwise; then we may choose $C_G(v)$ in the role of "X" in the previous paragraph. As B is transitive on $Z(L)^\#$ by 2.4.8.4, we may assume $S \leq X$. Thus $H = LS \leq X$, so by 2.3.7.2, $X \in \Gamma_0$ and $S \in Syl_2(X)$. Then by 1.2.4, $L \leq K \in \mathcal{C}(X)$. As the Sylow 2-subgroup S of X normalizes L, but we are assuming L is not normal in X, $L < K$. Now $O_2(K) \leq O_2(X) = P \leq Z(L)$ by the previous paragraph, so $K = [K, L]$ centralizes $O_2(K)$. Also $m_2(K) \geq m_2(L) > 1$, so we conclude from 1.2.1.5 that K is quasisimple, and hence K is a component of X. Thus K is described in 2.3.9.7. Since $1 \neq v \in L \cap Z(X) \leq Z(K)$, $Z(K)$ is of even order, so $K/O_2(K)$ is $L_3(4)$ or A_6 and $Z(K) = O_2(K)$. If $K/Z(K)$ is A_6, then $K \cong SL_2(9)$ by I.2.2.1, a contradiction as $L \leq K$ with $m_2(L) \geq 4$. Thus $K/O_2(K) \cong L_3(4)$, $L/Q \cong L_2(4)$, and $Z(K) = Z(L) = P$ as L is irreducible on $Q/Z(K)$ and we saw $Z(K) \leq P \leq Z(L)$. In particular, $P \trianglelefteq H$. Further $Z(L) \cong E_4$ since $n = 2$ and case (b) of 2.4.8.4 holds. Observe since $K/Z(K) \cong L_3(4)$ that $L = N_K(Q)$, so as R is Sylow in L by 2.4.8.3, R is Sylow in K. Now consider $x \in T - S$ as in Notation 2.4.10. By parts (2) and (3) of 2.4.8, $\mathcal{A}(R) = \{Q, Q^x\}$, so $N_K(Q^x)$ is the maximal parabolic of K over R distinct from L. Therefore $N_K(Q^x)/Q^x \cong L_2(4)$ and $P = Z(K) = Z(N_K(Q^x))$. Hence $L^x = \langle R^{N_G(Q^x)} \rangle \geq N_K(Q^x)$, so we conclude

$L^x = N_K(Q^x)$ since $N_K(Q^x) \cong N_K(Q) = L$. Thus $Z(L) = P = Z(L^x)$, contrary to 2.4.4. This contradiction completes the proof that $C_G(v) \leq G_Q$.

In the remainder of the proof, X again denotes an arbitrary member of $\mathcal{H}(L)$ not normalizing L; thus $1 \neq O_2(X) = P \leq Z(L)$ by earlier remarks. Now $C_G(Z(L)) \leq C_G(v) \leq G_Q = N_G(L)$ by the previous paragraph, so $L \in \mathcal{C}(N_G(Z(L)))$. As $H \leq N_G(Z(L))$, $S \in Syl_2(N_G(Z(L)))$ by 2.3.7.2, so $L \trianglelefteq N_G(Z(L))$ by 1.2.1.3. Therefore $N_G(Z(L)) = N_G(L) = G_Q$ using (1). Also B is transitive on $Z(L)^\#$ and $C_G(v) \leq G_Q = N_G(Z(L))$, so $Z(L)$ is a TI-subgroup of G by I.6.1.1, completing the proof of (3). Then as $1 \neq O_2(X) \leq Z(L)$, $X \leq N_G(Z(L)) = G_Q$ by (3), contrary to assumption. This contradiction completes the proof of the lemma. □

We next repeat some arguments from sections 3 and 4 of [**Asc78a**], which force the 2-local structure of G to be essentially that of an extension of $L_3(2^n)$ or $Sp_4(2^n)$; this information is used later in transfer arguments to eliminate these shadows.

In fact, by 2.4.8 and 2.4.11, the hypotheses of section 3 or 4 in [**Asc78a**] are satisfied, in cases (a) or (b) of 2.4.8.4, respectively. Thus we could now appeal to Theorems 2 and 3 of [**Asc78a**]. However those results are not quite strong enough for our present purposes, and in any event we wish to keep our treatment as self-contained as possible, as discussed in the Introduction to Volume I under Background References. Thus we reproduce those arguments from [**Asc78a**] necessary to complete our proof.

LEMMA 2.4.12. *(1) H is the split extension of L by a cyclic subgroup F of S inducing field automorphisms on L/Q. Thus S is the split extension of R by F.*

(2) If f is an involution in F, then all involutions in fR are fused to f under R, $C_L(f)$ is an $L_2(q^{1/2})$-block (or S_4 or $\mathbf{Z}_2 \times S_4$ if $q^{1/2} = 2$), and either

(a) $Z(L) = 1$, and $C_R(f)$ is special of order $q^{3/2}$; in this case we say $C_R(f)$ is of type $L_3(q^{1/2})$.

(b) $Z(L) \cong E_q$, with $|C_{Z(L)}(f)| = q^{1/2}$ and $|C_R(f)| = q^2$; in this case we say $C_R(f)$ is of type $Sp_4(q^{1/2})$.

PROOF. Recall $H = LS$, while by parts (3) and (5) of 2.4.8, S is the split extension of $R \in Syl_2(L)$ by $N_S(B) =: F$. Thus $F \cap L = F \cap R = 1$, so that (1) holds.

Suppose f is an involution in F. As f induces a field automorphism on $\bar{L} := L/Q$, $q = r^2$, $C_{\bar{L}}(f) \cong L_2(r)$, and $C_{Q/Z(L)}(f)$ is the natural module for $C_{\bar{L}}(f)$. Indeed if $Z(L) \neq 1$, then $Z(L) \cong E_q$ by 2.4.8.4, so from I.1.6, Q is the largest indecomposable extension of a submodule centralized by $\bar{L}$ by a natural $\bar{L}$-module; hence $m(Z(L)) = 2m(C_{Z(L)}(f))$. Thus in any event $m(Q) = 2m(C_Q(f))$, so Q is transitive on the involutions in fQ. Then by a Frattini Argument, $C_{\bar{L}}(f) = \overline{C_L(f)}$, so $C_L(f)$ is as claimed in (2). Further by Exercise 2.8 in [**Asc94**], R is transitive on involutions in fR, completing the proof of (2). □

DEFINITION 2.4.13. Relaxing somewhat the usual definition in the literature, we define a *Suzuki 2-group* to be a 2-group I admitting a cyclic group of automorphisms transitive on its involutions, with $[I, I] = Z(I)$.

LEMMA 2.4.14. *Assume $t \in T - S$ with $t^2 \in R$. Then $\langle t \rangle R$ splits over R, R is transitive on the involutions in tR, and choosing t to be an involution, one of the following holds:*

(1) $Z(L) = 1$, $Z(R) = C_R(t)$, *R is transitive on* $t[R,t]$, *and* $[R,t]$ *is transitive on* $tZ(R)$*; in this case we say* $C_R(t)$ *is of type* $L_2(q)$.

(2) $Z(L) = 1$, *n is even, and* $C_R(t)$ *is a Suzuki* 2*-group of order* $q^{3/2}$ *with* $|\Omega_1(C_R(t))| = q^{1/2}$*; in this case we say* $C_R(t)$ *is of type* $U_3(q^{1/2})$.

(3) $Z(L) \cong E_q$, *and* $C_R(t)$ *is a Suzuki* 2*-group of order* q^2 *with* $|\Omega_1(C_R(t))| = q$*; in this case we say* $C_R(t)$ *is of type* $Sz(q)$.

PROOF. Since $t \in T - S$, t serves in the role of the element "x" in Notation 2.4.10; in particular, we may apply 2.4.8. As $RB \trianglelefteq TB$ by 2.4.8.5, by a Frattini Argument we may choose t to normalize B. Also by 2.4.8.5, $R \trianglelefteq RB$ and $C_R(B) = 1$, so as $t^2 \in R$, $[B, t^2] \leq B \cap R = 1$ and hence t is an involution. In particular $R\langle t\rangle$ splits over R.

We recall from Notation 2.4.10 that $Q^t = Q^x$ is independent of the choice of $x \in T - S$, so by 2.4.8.3, $m(R/Z(R)) = 2m(C_{R/Z(R)}(t))$ and $C_{R/Z(R)}(t) = [R/Z(R), t]$. Let R_t denote the preimage of $C_{R/Z(R)}(t)$, so that R_t contains $C_R(t)$. By 2.4.8.7, Q and Q^t are the maximal elementary abelian subgroups of R, so $Z(R) = \Omega_1(R_t)$, and hence $C_{Z(R)}(t) = \Omega_1(C_R(t))$.

Assume that $Z(L) \neq 1$. Then $Z(L) \cong E_q$ is a TI-subgroup of G by 2.4.11.3, while $|Z(R)| = 2^{2n} = |Z(L)|^2$ by 2.4.8.4, so $Z(R) = Z(L) \times Z(L)^t$. Thus by Exercise 2.8 in [**Asc94**], R is transitive on the involutions in tR, and $R_t = Z(R)C_R(t)$. As $B = D \times D^t$ by 2.4.8.6, $C_B(t)$ is a full diagonal subgroup of B, and so $C_B(t)$ is regular on $C_{Z(R)}(t)^\# = Z(C_R(t))^\#$. Further $C_R(t)$ is nonabelian, so that $[C_R(t), C_R(t)] = Z\big(C_R(t)\big)$; thus $C_R(t)$ is a Suzuki 2-group of order q^2, so that conclusion (3) holds.

Now assume instead that $Z(L) = 1$. Set $(TB)^* := TB/C_{TB}(Z(R))$. As t normalizes B and $B^* = D^*$ is regular on $Z(R)^\#$ by 2.4.8.6, either $t^* = 1$ or $m(Z(R)) = 2m(C_{Z(R)}(t))$, and in either case $C_{B^*}(t^*) = C_B(t)^*$ is regular on $C_{Z(R)}(t)^\#$. Assume first that $t^* = 1$. Then as $R_t/Z(R) = [R/Z(R), t]$, with $\Omega_1(R_t) = Z(R)$, t inverts an element r of order 4 in each coset of $Z(R)$ in R_t. So as r is of order 4, $C_R(t) = C_{R_t}(t) = Z(R)$, and conclusion (1) holds. Further $|R : C_R(t)| = |R_t|$, so R is transitive on tR_t and hence on the involutions in tR. Now assume instead that $t^* \neq 1$, so that $m(Z(R)) = 2m(C_{Z(R)}(t))$. By Exercise 2.8 in [**Asc94**], $|C_R(t)| = q^{3/2}$ and R is transitive on the involutions in tR. As $C_B(t)$ is transitive on $C_{Z(R)}(t)^\# = Z(C_R(t))^\#$, and $C_R(t)$ is nonabelian so that $[C_R(t), C_R(t)] = Z\big(C_R(t)\big)$, $C_R(t)$ is a Suzuki 2-group of order $q^{3/2}$. Thus conclusion (2) holds. □

NOTATION 2.4.15. In the remainder of our treatment of the case $s = 1$, we define Z as follows: If $Z(L) = 1$, set $Z := Z(R)$, while if $Z(L) \neq 1$ set $Z := Z(L)$.

LEMMA 2.4.16. *(1)* $Z \cong E_{2^n}$ *and* $Z \trianglelefteq S$.

(2) For $x \in T - S$, *either*

(a) $Z(L) = 1$ *and* $Z^x = Z = Z(R)$, *or*

(b) $Z(L) \neq 1$ *and* $Z(R) = Z \times Z^x$.

(3) $U = \langle (Z^x)^L \rangle = \langle Z^G \cap U \rangle$.

PROOF. Part (1) follows from 2.4.8.4. Next x normalizes $J(S) = R$, so conclusion (a) of (2) holds when $Z(L) = 1$ as $Z = Z(R)$ in that case. If $Z(L) \neq 1$ then $Z = Z(L)$ is a TI-subgroup of G by 2.4.11.3, and $Z \neq Z^x$ by 2.4.4, so conclusion (b) of (2) holds as $|Z(R)| = |Z|^2$.

If $Z(L) = 1$ then $Z = Z^x$ by (2); hence $(Z^x)^L = Z^L$ gives the partition of the natural module U by its 1-dimensional $\mathbf{F}_q$-subspaces, so (3) holds in this case. If $Z(L) \neq 1$ then $Z(R) = ZZ^x$ and $(Z(R)/Z)^L$ is the corresponding partition of U/Z, so again (3) holds as L is indecomposable on U. □

LEMMA 2.4.17. *(1) Either $R = C_T(Z)$; or case (1) of 2.4.14 holds, so that $Z(L) = 1$ and $C_T(Z) = R\langle t\rangle$ for some involution t in $T-S$ with $Z = Z(R) = C_R(t)$.*
(2) If $Z(L) = 1$ then $T/C_T(Z)$ is cyclic.

PROOF. By 2.4.12.1, $C_S(Z) = R$, since the field automorphisms in F do not centralize Z. Assume $C_T(Z) > R$. Then $C_T(Z) = R\langle u\rangle$ for some $u \in C_T(Z) - S$, so $u^2 \in R$ and hence 2.4.14 completes the proof of (1).

Assume $Z(L) = 1$, so that $Z = Z(R)$. By 2.4.8.6, $Aut_B(Z)$ is cyclic and regular on $Z^\#$, so $Aut_{GL(Z)}(Aut_B(Z))$ is the multiplicative group of $\mathbf{F}_q$ extended by $Aut(\mathbf{F}_q)$. Since $Aut_B(Z)$ is normal in $Aut_{BT}(Z)$ by 2.4.8.5, we conclude $Aut_T(Z)$ is cyclic, so that (2) holds. □

LEMMA 2.4.18. *(1) Z is a TI-subgroup of G.*
(2) If $Z(L) = 1$ then $N_G(Z) = M$.
(3) if $Z(L) \neq 1$ then $N_G(Z) = G_Q$.

PROOF. If $Z(L) \neq 1$ then (1) and (3) hold by 2.4.11.3. Thus we may assume $Z(L) = 1$, so $Z = Z(R)$ from Notation 2.4.15. Set $P := O_2(M)$. As T normalizes R by 2.4.8.1, there is an involution z in $Z \cap Z(T)$. As $F^*(M) = O_2(M)$, $z \in C_M(P) = Z(P)$. Then as $D \leq M$ and D is irreducible on Z, $Z \leq Z(P)$. It suffices to show that $Z \trianglelefteq M$: For then $M = N_G(Z)$ since $M \in \mathcal{M}$, so that (2) holds. Further as $M = !\mathcal{M}(T)$, $C_G(z) \leq M$, and hence as as D is transitive on $Z^\#$, Z is a TI-set in G by I.6.1.1, so that (1) also holds.

Thus it remains to show that $Z \trianglelefteq M$. If $R \leq P$, then as $R = J(T)$, also $R = J(P)$ by B.2.3, so that $Z = Z(J(P)) \trianglelefteq M$. Thus we assume that $R \not\leq P$. Now for $x \in T - S$, $R = UU^x$ by 2.4.8.3, so $U \not\leq P$. Then as $Z \leq P$ and D is irreducible on U/Z, $Z = U \cap P$, and then also $Z = U^x \cap P$. So since U and U^x are the maximal elementary subgroups of R by 2.4.8.7, $Z = \Omega_1(R \cap P)$. We now assume Z is not normal in M, and it remains to derive a contradiction. We saw $Z \leq Z(P)$, so that $Z < Z_P := \Omega_1(Z(P))$ and hence there is an involution $t \in Z_P - Z$. As $Z = \Omega_1(R \cap P)$, $t \notin R$, so as t centralizes Z, the second case of 2.4.17.1 holds. Therefore $Z = C_R(t)$ and t is described in case (1) of 2.4.14. But $[R, t] \leq [R, Z_P] \leq R \cap Z_P \leq C_R(t) = Z$, impossible as $[R, t] > Z$ in case (1) of 2.4.14. □

LEMMA 2.4.19. *R is the weak closure of Z in T.*

PROOF. By 2.4.16.3, $Q = U = \langle (Z^x)^L\rangle$, and $R = QQ^x$ by 2.4.8.3. Hence R is contained in the weak closure of Z in T. Thus we may assume that there is $g \in G$ with $Z^g \leq T$ but $Z^g \not\leq R$, and it remains to derive a contradiction. By 2.4.16.1, $|Z| = 2^n > 2 = |T : S| \geq |T : N_T(Z)|$, so that $N_{Z^g}(Z) \neq 1$. Then as Z is a TI-subgroup of G by 2.4.18.1, and $\langle Z, Z^g\rangle$ is a 2-group, $Z^g \leq C_T(Z)$ by I.6.2.1. As $Z^g \not\leq R$, there is an involution $t \in Z^g - S$ with $C_T(Z) = R\langle t\rangle$ by 2.4.17.1. Then t satisfies conclusion (1) of 2.4.14 with $C_R(t) = Z$. Hence as $|Z^g| > 2 = |C_T(Z) : R|$, $R \cap Z^g \neq 1$. But $R \cap Z^g \leq C_R(t) = Z$, so as Z is a TI-subgroup of G, $Z^g = Z \leq R$, contrary to $Z^g \not\leq R$. □

LEMMA 2.4.20. *Assume $Z(L) \neq 1$. Then for $x \in T - S$:*

(1) $B = D \times D^x$ is regular on $\Delta := Z(R) - (Z \cup Z^x)$.

(2) For $u \in \Delta$, u is 2-central in M and hence 2-central in G, $C_G(u) \leq M$, and $u^G \cap Z = \emptyset$.

(3) All involutions in R are fused to $u \in \Delta$ or $z \in Z^\#$.

(4) $R \trianglelefteq M$, so $R \trianglelefteq C_G(u)$ for $u \in \Delta$.

(5) For $z \in Z^\#$, Sylow 2-subgroups of $C_G(z)$ are in S^G.

(6) If $u \in \Delta$ and $X = \langle Z^G \cap X\rangle$ is a 2-subgroup of $C_G(u)$, then $X \leq R$.

PROOF. As $Z(L) \neq 1$, $E_{2^n} \cong Z(L) = Z$. By 2.4.11.3, Z is a TI-subgroup of G with $N_G(Z) = G_Q$, so for $z \in Z^\#$, $C_G(z) \leq G_Q$. Further $S \in Syl_2(G_Q)$ by 2.4.3, so (5) holds and z is not 2-central in G.

By 2.4.8.6, $B = D \times D^x$, while $Z(R) = Z \times Z^x$ by 2.4.16.2. By 2.4.8.4, D^x is regular on $Z^\#$, so as x interchanges D and D^x and Z and Z^x, D is regular on $(Z^x)^\#$. Thus $Z = C_{Z(R)}(D)$, completing the proof of (1). Next Q and Q^x are the maximal elementary abelian subgroups of R by 2.4.8.7, while all elements of Q are fused into $Z(R)$ under L, so (3) holds. Then as z is not 2-central in G, but $Z \times Z^x = Z(R) \trianglelefteq T$ since T normalizes R by 2.4.8.1, $u \in Z(T)$ for some $u \in \Delta$. So as $M = !\mathcal{M}(T)$, $G_u := C_G(u) \leq M$, and then (2) follows from the transitivity of D on Δ in (1).

Next we prove (4). Set $P := O_2(M)$. As $R = J(T)$ by 2.4.8.3, it suffices to show that $R \leq P$, since then $R = J(P)$ by B.2.3.3. As $F^*(M) = P$, $u \in C_M(P) = Z(P)$, so by (1), $Z(R) = \langle u^{BT}\rangle \leq Z(P)$. Let $W := \langle Z^G \cap P\rangle$. By 2.4.19, $W \leq R$, so as B is irreducible on $Q/Z(R)$, either $W = Z(R)$ or $W = R$. Since $W \trianglelefteq M$, (4) holds if $W = R$. If $W \neq R$ then $Z(R) = W \trianglelefteq M$ so that $M = N_G(Z(R))$ since $M \in \mathcal{M}$. But then as Z is a TI-subgroup of G, it follows from (1) and (2) that $M = N_M(Z)\langle x\rangle$. Now $N_G(Z) = G_Q = N_G(L)$ by 2.4.11, so $N_M(Z)$ normalizes $O_2(N_{M\cap L}(Z)) = R$. As x also normalizes R, we conclude (4) holds in this case also.

Finally assume the hypotheses of (6). Then $X \leq C_G(u) \leq M$ by (2), and as X is a 2-group, $X \leq T^m$ for some $m \in M$. Then $X \leq \langle Z^G \cap T^m\rangle = R^m$ by 2.4.19, so that $X \leq R$ by (4). □

In the remainder of the treatment of the case $s = 1$, we let z denote an involution of $Z^\#$. If $Z(L) \neq 1$, let u denote an element of the set Δ defined in 2.4.20.1.

LEMMA 2.4.21. *(1) R is the strong closure of Q in T.*

(2) $i^G \cap T \subseteq R$ for each involution i in R.

PROOF. By parts (2) and (7) of 2.4.8, all involutions in R are fused into Q, so (1) implies (2).

By 2.4.19, R is contained in the strong closure of Q in T. Hence we may assume that a is an involution in $T - R$ fused into Q, and it remains to derive a contradiction. If $Z(L) = 1$ then L is transitive on $Q^\#$, so $a = z^g$ for some $g \in G$. If $Z(L) \neq 1$ then by 2.4.20.3, either $a = z^g$, or $a = u^g$ for $u \in \Delta$. Set $I := C_R(a)$ and let $I \leq T^* \in Syl_2(C_G(a))$ and set $R^* := J(T^*)$.

We claim that if $Z(L) \neq 1$ then $a \in S$. Thus we assume $Z(L) \neq 1$ and $a \in T - S$, and it remains to derive a contradiction. By 2.4.14, I is of type $Sz(q)$, so the involutions of I lie in Δ rather than in Z or $Z^x = Z^a$, since $a \in T - S$. Assume first that $a = z^g$. By 2.4.20.5, $T^* \in S^G$, and by 2.4.12.1, T^*/R^* is cyclic, so $Z(I) = [I, I] \leq R^*$. Now we saw that involutions of $Z(I)$ lie in Δ, so we may assume that $u \in Z(I)$. Thus $Z(R^*) \leq C_G(u) \leq M$ by 2.4.20.2. By 2.4.16.2, $Z(R^*)$ is generated by a pair of conjugates of Z, so $Z(R^*) \leq R \leq S$ by 2.4.20.6. As $a \in Z(R^*)$, this contradicts our assumption that $a \notin S$. Therefore $a = u^g$, and so $T^* \in T^G$. Let $Q^* \in \mathcal{A}(T^*)$ and $S^* = N_{T^*}(Q^*)$. Then $|T^* : S^*| = 2$, so arguing much as before, $a \in Z(T^*) \leq Z(R^*)$ and $Z(I) = [I, I] \leq S^*$. Then as S^*/R^* is cyclic by 2.4.12.1, either $Z(I)$ is noncyclic so that $Z(I) \cap R^* \neq 1$, or $Z(I)$ is of order 2 so that $q = 4$. In the former case we obtain a contradiction as before, and in the latter T^*/R^* is of order at most 4 and hence abelian, so again $[I, I] \leq R^*$, for the same contradiction. This completes the proof of the claim.

We now summarize the remaining possibilities: If $Z(L) \neq 1$ then $a \in S = N_T(Z)$ by the claim, so that I is of type $Sp_4(q^{1/2})$ by 2.4.12.2. So assume that $Z(L) = 1$. Then $a = z^g$ and $T = N_T(Z)$, so again a normalizes Z. If $a \notin S$, then by 2.4.14, either $I = Z$ is of type $L_2(q)$, or I is of type $U_3(q^{1/2})$. Finally if $a \in S$, then I is of type $L_3(q^{1/2})$ by 2.4.12.2.

Assume that a centralizes Z. Then by the previous paragraph, $Z(L) = 1$, $a = z^g \in T - S$, and $I = Z$ is of type $L_2(q)$. Since Z is a TI-subgroup by 2.4.18.1 and $a = z^g$ centralizes Z, $[Z, Z^g] = 1$ by I.6.2.1. Thus $aZ \subseteq V := ZZ^g \cong E_{2^{2n}}$. However $[R, a]$ is transitive on aZ by 2.4.14. Thus for $r \in [R, a]$, $a^r \in aZ \subseteq V \leq C_G(V)$, so again by I.6.2.1, $Z^{gr} \leq C_G(V)$. Then as $m(V) = 2n = m_2(T)$, $Z^{gr} \leq V$, so $[R, a]$ normalizes $\langle Z^{g[R,a]} \rangle = V$. Notice $V \in \mathcal{A}(G) = Q^G$ in view of 2.4.8.3, and of course $Z \in Z^G \cap V$. Now $|[R, a]V| = q^3 = |R|$ and by 2.4.17.1, R is Sylow in $G_Q \cap C_G(Z)$, so that $[R, a]V = R^h$ for some $h \in G$. By 2.4.14, R is transitive on $a[R, a]$, so for $s \in R$, $a^s \in R^h$. Thus a^s is contained in some conjugate of Z contained in R^h, so as Z is a TI-subgroup of G, $Z^{gs} \leq R^h$. Then $V = \langle Z^{g[R,a]} \rangle \leq \langle Z^{gR} \rangle =: X$ is a subgroup of R^h normalized by R. It follows that R normalizes R^h: for if $X < R^h$, then $V = J(X)$, so that R normalizes $[R, a]V = R^h$. So as $R^h = J(T^h)$ is weakly closed in T^h, $R = R^h$. But then $a \in V \leq R^h = R$, contradicting our observation that $a \notin S$.

Therefore $[a, Z] \neq 1$, so from our earlier summary, I is of type $Sp_4(q^{1/2})$, $U_3(q^{1/2})$, or $L_3(q^{1/2})$. In each case $[I, I] = Z(I)$. Furthermore setting $Z_a := C_Z(a)$, either $Z_a \leq [I, I]$, or $q = 4$ and $I \cong \mathbf{Z}_2 \times D_8$ is of type $Sp_4(2)$.

Suppose first that $a = z^g$. Assume $Z(L) = 1$. Then by 2.4.17.2, $T^*/C_{T^*}(Z^g)$ is cyclic, and using the previous paragraph, $Z_a \leq [I, I] \leq C_{T^*}(Z^g)$. Thus as $1 \neq Z_a \leq Z$, $[Z^g, Z] = 1$ by I.6.2.1, contradicting $[a, Z] \neq 1$. Thus $Z(L) \neq 1$ so $T^* \in S^G$ by 2.4.20.5, and T^*/R^* is cyclic by 2.4.12.1. Hence $[I, I] \leq R^* = C_{T^*}(Z^g)$. Thus if $Z_a \leq [I, I]$, we get the same contradiction as above, so from the previous paragraph, $q = 4$ and $[I, I] =: \langle u \rangle \leq R^* = C_{T^*}(Z^g)$. Then $a \in Z^g \leq \langle Z^G \cap C_G(u) \rangle$, as $C_G(u) \leq M$ by 2.4.20.2, so $a \in R$ using 2.4.20.6. Again this contradicts $[a, Z] \neq 1$, so $a \notin z^G$.

Therefore $a = u^g$, so that $Z(L) \neq 1$ by our previous summary; and it also now follows from our remarks at the start of the proof that R is the weak closure of z in T. From our summary, $a \in S$ and I is of type $Sp_4(q^{1/2})$. We may assume $z \in I$. Then $I = \langle z^G \cap I \rangle \leq \langle z^G \cap C_G(a) \rangle =: Y$. Since $C_G(a) \leq M^g$ by 2.4.20.2, and R is the weak closure of z in T, we conclude from 2.4.20.4 that $z \in Y \leq R^g$. But then z is contained in a conjugate of Z in $R^g = R^*$, so as Z is a TI-subgroup of G, $Z \leq R^g \leq C_G(a)$, again contradicting $[a, Z] \neq 1$. This finally completes the proof of (1), and hence of 2.4.21. □

At this point, we have obtained strong control over the 2-local structure and 2-fusion of G, which we can use to obtain contradictions via transfer arguments.

LEMMA 2.4.22. *(1) T/R is not cyclic.*
(2) $R < S$.

PROOF. If $R = S$ then $|T : R| = 2$ by 2.4.8.1, so T splits over R by 2.4.14, and hence there is an involution $t \in T - R$. On the other hand if $R < S$, there is an involution t in $S - R$ by 2.4.12.1. Thus in any case there is an involution $t \in T - R$.

As T/R is cyclic if $S = R$, it remains to assume T/R is cyclic and derive a contradiction. By 2.4.21, $t^G \cap R = \emptyset$. Then by Generalized Thompson Transfer A.1.36.2, $t \notin O^2(G)$, contrary to the simplicity of G. □

By 2.4.22.2, $R < S$; so since S splits over R by 2.4.12.1, there is an involution $S - R$. It is convenient to use the notation s for this involution; there should be no confusion with the earlier numerical parameter "s", as in the branch of the argument for several pages before and after this point, that parameter has the value 1. Let $G_s := C_G(s)$, $L_s := C_L(s)$, etc.

We use the standard notation that for x an integer, x_2 denotes the 2-primary part of x.

LEMMA 2.4.23. *(1) Either L_s is an $L_2(2^{n/2})$-block with $U_s = U(L_s)$, or $q = 4$ and $L_s \cong S_4$ or $S_4 \times \mathbf{Z}_2$.*
(2) R_s is the strong closure of Q in T_s.
(3) $U_s = O_2(L_s)$ and $N_G(U_s) \leq G_Q$.
(4) $T = RT_s$, there exists $x \in T_s - S$, and $T_s \in Syl_2(G_s)$.
(5) Assume $Z(L) \neq 1$ and $q = 2^n > 4$. Set $K_s := \langle L_s, L_s^x\rangle$. Then $K_s \cong Sp_4(2^{n/2})$, $C_{T_s}(K_s) = \langle s\rangle$, and $T_s/\langle s\rangle R_s$ is cyclic of order $n_2 = |Out(K_s)|_2$.

PROOF. Part (1) follows from 2.4.12.2. By (1), $U_s = O_2(L_s)$. Part (2) follows from 2.4.21.1.

From (1) and the proof of 2.4.16.3, $U_s = \langle (Z_s^x)^G \cap U_s\rangle$. But $N_G(U_s)$ permutes $(Z_s^x)^G \cap U_s$ and Z is a TI-subgroup of G, so $N_G(U_s)$ permutes $(Z^x)^G \cap U$ and hence $N_G(U_s) \leq N_G(U) = G_Q$ by 2.4.16.3, and as $Q = U$ by 2.4.8.2. This completes the proof of (3).

By 2.4.12.1, S/R is cyclic, so $\langle s\rangle R \trianglelefteq T$. By 2.4.12.2, R is transitive on the involutions in sR, so by a Frattini Argument $T = RT_s$, and as $S \in Syl_2(G_Q)$ by 2.4.3.1, S_s is Sylow in $N_{G_s}(U_s)$ by (3). As $S < T$, there is $x \in T_s - S$ and by 2.4.8.7, U and U^x are the maximal elementary abelian subgroups of R, so $\mathcal{A}(R_s) = \{U_s, U_s^x\}$. Therefore $N_G(R_s) = N_G(U_s)\langle x\rangle$. So using (2), $N_G(T_s) \leq N_G(U_s)\langle x\rangle$. Thus as S_s is Sylow in $N_{G_s}(U_s)$ and $S_s\langle x\rangle = T_s$, $T_s \in Syl_2(G_s)$, so that (4) holds.

Assume the hypotheses of (5), and set $K_s := \langle L_x, L_x^s\rangle$. Let Θ be the set of subgroups of S_s invariant under L_s. From the action of S and L, $U_s\langle x\rangle$ is the unique maximal member of Θ, and if $Y \in \Theta$ with $U_s \not\leq Y$, then $Y \leq \langle s\rangle C_U(L)$. Therefore as $R_s = U_sU_s^x$ and $C_U(L) \cap C_U(L)^x = 1$, $\langle s\rangle$ is the largest subgroup of T_s invariant under L_s and L_s^x, and hence $\langle s\rangle = O_2(K_sT_s)$. As $q > 4$ by hypothesis, $L_s \in \mathcal{L}(G_s, S_s)$, so since $|T_s : S_s| = 2$ with $T_s \in Syl_2(G_s)$ by (4), $L_s \leq K \in \mathcal{C}(G_s)$ by 1.2.5. As x acts on $R_s \leq K$, x acts on K, so $K_s \leq K$. Then using (4) and A.1.6, $O_2(K) \leq O_2(KT_s) \leq O_2(K_sT_s) = \langle s\rangle \leq C_G(K)$. As $m_2(K) \geq m_2(L_s) > 1$, K is quasisimple by 1.2.1.5. By (1), L_s is an $L_2(q^{1/2})$-block with $Z_s = Z(L_s) \neq 1$, so as $R_s = U_sU_s^x$ is a Sylow 2-subgroup of L_s, we conclude by examination of the possibilities in Theorem C (A.2.3) that $K_s = K \cong Sp_4(q^{1/2})$, and x induces an outer automorphism on K nontrivial on the Dynkin diagram. Then $C_{T_s}(K_s) = O_2(K_sT_s) = \langle s\rangle$. Finally $Out(K_s)$ is cyclic and $R_s \in Syl_2(K_s)$, so $T_s/R_s\langle s\rangle$ is cyclic. Further (cf. 16.1.4 and its underlying reference) a Sylow 2-subgroup of $Out(K_s)$ is generated by the image of any 2-element nontrivial on the Dynkin diagram of K, so $|T_s : R_s\langle s\rangle| = n_2 = |Out(K)|_2$, completing the proof of (5). □

LEMMA 2.4.24. *Let $T_B := N_T(B)$. Then*

(1) T is the split extension of R by T_B.

(2) $Z(L) = 1$.

(3) $T_B = \langle x\rangle \times F$, where x is an involution such that $C_R(x) = Z$, $R\langle x\rangle = C_T(Z)$, and F is cyclic and induces field automorphisms on L/Q.

PROOF. Part (1) is one of the conclusions of 2.4.8.5. By 2.4.12.1, $T_B = F\langle x\rangle$, where $F := N_S(B)$ is cyclic and induces field automorphisms on L/Q, and $x \in T_B - S$. By 2.4.22.1, T_B is noncyclic. Choose $s \in F$.

Suppose first that $Z(L) = 1$. By 2.4.17.1, either $C_T(Z) = R$, or there exists some involution $x \in T_B - S$ with $Z = C_R(x)$ such that $C_T(Z) = R\langle x\rangle$. In the former case, T_B is cyclic by 2.4.17.2, contrary to the previous paragraph, so the latter must hold. Then $[x, F] \leq C_F(Z) = 1$, so $T_B = \langle x\rangle \times F$, establishing (3). Since (2) holds by assumption, the lemma holds in this case. Thus we may assume that $Z(L) \neq 1$ and it remains to derive a contradiction.

Suppose first that $n/2$ is odd. Then $|S : R| = 2$ since $R < S$ by 2.4.22, so $|T : R| = 4$ using 2.4.8.1. Hence $T_B \cong T/R \cong E_4$, since T_B is noncyclic, so there is an involution x in $T - S$ and by 2.4.14, $C_R(x)$ is of type $Sz(q)$, so $V := \Omega_1(C_R(x)) \cong E_q$ and $V^{\#} \subseteq \Delta$. It will suffice to show that V is the strong closure of u in a Sylow 2-subgroup T_x of $C_G(x)$ containing $C_T(x)$: For by 2.4.23.2, R_s is the strong closure of u in a Sylow 2-group of $C_G(s)$, and hence is nonabelian by 2.4.12.2. So as V is the strong closure of u in T_x, it follows that $s \notin x^G$. Further $x^G \cap R = \emptyset$ by 2.4.21.2, so as all involutions in $S - R$ are fused to s by 2.4.12.2, we conclude that $x^G \cap S = \emptyset$. Then $x \notin O^2(G)$ by Thompson Transfer, for the usual contradiction to the simplicity of G.

So it remains to show that V is strongly closed in T_x. Now conjugates of u generate R by 2.4.20; so by 2.4.21 and 2.4.20.4, R is the strong closure of u in $C_G(u)$. Therefore as $V = \Omega_1(C_R(x))$ and $V^{\#} \subseteq u^G$, V is strongly closed in T_x. As we mentioned, this completes the elimination of the case $n/2$ odd.

Therefore $n/2$ is even, so $q > 4$. Thus by 2.4.23.5, $T_s/R_s\langle s\rangle$ is cyclic of order $n_2 \geq 4$, and $n_2 = |Out(K_s)|_2$. Let $tR_s\langle s\rangle$ denote the involution of $T_s/R_s\langle s\rangle$; then this involution lies in the cyclic subgroup of index 2 in $T_s/R_s\langle s\rangle$ inducing field automorphisms, so any preimage t of $tR_s\langle s\rangle$ induces an involutory field automorphism on L_s/U_s. Thus t induces a field automorphism of order 4 on L/Q, so t is not an involution. Since $s \in T_B$, $T_B/\langle s\rangle \cong T/R\langle s\rangle \cong T_s/R_s\langle s\rangle$ using 2.4.23, so s is the unique involution in T_B. Also T_B is not quaternion since $T_B/\langle s\rangle$ is cyclic. Therefore T_B is cyclic, contrary to our earlier reduction. This contradiction completes the proof. □

We can now finally eliminate the case where the numerical parameter we denoted earlier by "s" has the value 1: Let $T_C := C_T(Z)$. By 2.4.24.2, $Z(L) = 1$. Then $Z = Z(R) \trianglelefteq T$, so $T_C \trianglelefteq T$. By 2.4.24.3, there is an involution $x \in T - S$ such that $Z = C_R(x)$, $T_C = R\langle x\rangle$, and $T/T_C \cong T_B/\langle x\rangle \cong F$ is cyclic. It will suffice to show that $s^G \cap T \subseteq S$, for the involution we have been denoting by s: For then

$$s^G \cap T_C \subseteq s^G \cap T_C \cap S = s^G \cap R = \emptyset$$

using 2.4.21.2. Then as T/T_C is cyclic, $s \notin O^2(G)$ by Generalized Thompson Transfer A.1.37.2, as usual contrary to the simplicity of G.

Thus it remains to show that $s^G \cap T \subseteq S$. By 2.4.23, R_s is the strong closure of Q in $T_s \in Syl_2(G_s)$. As $Z(L) = 1$, R_s is of type $L_3(2^{n/2})$ by 2.4.12. Finally by 2.4.14, for each involution $i \in T - S$, $C_R(i)$ is of type $L_2(2^n)$ or $U_3(2^{n/2})$, and in either case, $\Omega_1(C_R(i)) \leq Z$. To show that $s^G \cap T \subseteq S$, we must show that $i \notin s^G$ for each such i; so we assume that that $i \in s^G$, and it remains to derive a contradiction.

Assume first that $C_R(i)$ is of type $L_2(2^n)$. Then i centralizes Z of order 2^n, whereas for each $g \in G$ with $Z^g \cap R_s \neq 1$, $|C_{Z^g}(s)| = 2^{n/2}$, contrary to $i \in s^G$ and 2.4.21.1.

Therefore $R_i := C_R(i)$ is of type $U_3(2^{n/2})$. Set $Z_i := C_Z(i) = Z(R_i)$. Then $i^g = s$ for some $g \in G$, and for suitable $c \in G_s$, $R_i^{gc} \leq T_s$ as $T_s \in Syl_2(G_s)$ by 2.4.23.4. Then $Z_i^{gc} \leq R_s$ by 2.4.23.2. Interchanging U and U^x if necessary, we may assume that $Z_i^{gc} \leq U_s$. Indeed we claim $Z_i^{gc} = Z_s$: For assume otherwise. By 2.4.18.1, Z_i^{gc} and Z_s are TI-subgroups of G_s of order $q^{1/2}$, so $U_s = Z_s \times Z_i^{gc}$, and hence $R_i^{gc} \leq C_{T_s}(U_s/Z_s) = R_s\langle s\rangle$. Then $Z_i^{gc} = \Phi(R_i^{gc}) \leq \Phi(R_s\langle s\rangle) = Z_s$, a contradiction establishing the claim that $Z_i^{gc} = Z_s$.

By the claim, $R_i^{gc} \leq C_{T_s}(Z_s)$. But $R\langle x\rangle = C_T(Z)$ with $T/R\langle x\rangle$ cyclic, so $R\langle x, s\rangle = C_T(Z_s)$ as Z is a TI-subgroup. Thus $|C_{T_s}(Z_s) : R_s\langle s\rangle| \leq 2$, so as $|R_i| = q^{3/2} = |R_s|$, also $|C_{T_s}(Z_s) : R_i^{gc}\langle s\rangle| \leq 2$, and hence $|U_s\langle s\rangle : U_s\langle s\rangle \cap R_i^{gc}\langle s\rangle| \leq 2$. Now $U_s\langle s\rangle$ is elementary abelian of order $2q$, while $\Omega_1(R_i^{gc}\langle s\rangle) = Z_i^{gc}\langle s\rangle$ is elementary of order $2q^{1/2}$, so $2q \leq 4q^{1/2}$, and hence we conclude $q = 4$. Therefore $T_s = \langle s\rangle \times R_s\langle x\rangle$, with x an involution by 2.4.24.3, and $R_s = U_s U_s^x \cong D_8$, so $R_s\langle x\rangle \cong D_{16}$. This is impossible, as the group R_i of type $U_3(2)$ is isomorphic to Q_8, and $\mathbf{Z}_2 \times D_{16}$ contains no such subgroup.

This contradiction finally completes the treatment of the case $s = 1$ of Theorem 2.4.7.

The case $s = 2$.

So we turn to the case $s = 2$. Here we will produce members of Γ_0 other than $H = L_0 S$, which we use to obtain a contradiction.

As $s = 2$, $L_0 = L_1 L_2$ with $L = L_1$, and we set $U_i := U(L_i)$, so that $U_0 = \langle U^S\rangle = U_1 U_2$. By 2.4.5.1, Hypotheses C.5.1 and C.5.2 are satisfied with S in the roles of both "T_H" and "R", for any subgroup M_0 of T with $|M_0 : S| = 2$. Observe U_0, Baum(S) play the roles that "U, S" play in section C.5. Further as $|M_0 : S| = 2$, the hypotheses of C.5.6.7 are satisfied by 2.4.5.2.

Recall from the beginning of this subsection 2.4.1 that $R = J(S)$, and also that D is defined there; and from the opening few pages of this section 2.4 that $Q = O_2(H) = O_2(L_0 S)$. By 2.4.5.2, $Q = U_0 C \in \mathcal{A}(S)$, where $C := C_S(L_0)$, and $U_0^x \not\leq Q$. As $s = 2$, case (iii) of C.5.6.7 holds; hence there are two S-invariant members $\{Q, Q^x\}$ of $\mathcal{A}(S)$, and $QQ^x = R = $ Baum(S) since Baum(S) contains R, and RQ is Sylow in $L_0 Q$ by B.4.2.1.

We can now argue much as in the proof of 2.4.8.5, but using M_0, L_0 in the roles of "T, L", to show that $B := DD^x$ is abelian of odd order, omitting details except to point out where the argument differs slightly: Notice this time that D normalizes Q and the unique member Q^x of $\mathcal{A}(S)$ with $R = QQ^x$. Further $D^x = O^2(D^x)$, so D^x normalizes each of the two conjugates L_1 and L_2 of L in L_0S.

Now G_Q is an SQTK-group, so $m_p(G_Q) \leq 2$ for p a prime divisor of $|D|$; then as $m_p(D) = 2$ it follows that $B = D = D^x$.

Next $x \in T - S$ acts on B and hence on $C_R(B)$. As $B = D$ and L is an $L_2(2^n)$-block, $C_R(B) = C_R(L_0)$, and as Q is abelian, $C_R(L_0) \leq Z(R)$ so that $C_R(B) \trianglelefteq L_0N_S(B) = L_0S = H$. Hence $1 = C_R(B) = C_R(D)$ by 2.4.4. Then $C_{U_1}(L) = 1$ and $Q = CU_0 = U_0 \leq L_0$, so that $G_Q = N_G(L_0)$, just as in the proof of 2.4.11.1. Also as RQ is Sylow in L_0Q, $R = QQ^x$ is Sylow in L_0. Then $L_0 = L_1 \times L_2$ so $R = R_1 \times R_2$, where $R_i := R \cap L_i \in Syl_2(L_i)$ is of order q^3, and $D = D_1 \times D_2$, where $D_i := D \cap L_i$, and D_1 and D_2 are the subgroups of D maximal subject to $C_R(D_i) \neq 1$. Therefore as $N_S(D)$ interchanges D_1 and D_2, we may choose x in $M_0 - S$ so that x normalizes D_1 and D_2, and hence x acts on $C_R(D_{3-i}) = R_i$. As $L_2 = [L_2, Q^x]$, $x \notin N_G(L_2)$. Thus $L_2 < K := \langle L_2, L_2^x \rangle \leq C_G(R_1D_1)$, and $S_1 := \langle x \rangle N_S(R_1) = N_{M_0}(R_1)$ normalizes K. Observe that $|S : N_S(R_1)| = 2$ with $R = J(S) \leq N_S(R_1)$, $Q = O_2(H) \leq N_S(R_1)$, and $H \in \Gamma_0^e$. Thus $N_S(R_1) \in \beta$ by 2.3.8.5b. Then as $L_2N_S(R_1) \in \mathcal{H}^e$ and $L_2 \not\leq M$, from the definitions in Notation 2.3.4 and Notation 2.3.5, $(N_S(R_1), L_2N_S(R_1)) \in \mathcal{U}(KS_1)$, so that $KS_1 \in \Gamma$.

We claim next that $R = J(M_0)$: For suppose $A \in \mathcal{A}(M_0)$ with $A \not\leq R$. By 2.4.3, $S = N_T(Q)$, so as $R = J(S)$, there is an involution $a \in A - S$; hence $Q^a = Q^x$, since $M_0 = S\langle x \rangle = S\langle a \rangle$ and S acts on Q. If $R_1^a = R_2$ then $C_R(a) \cong R_1$ is of rank $2n$, while if $R_1^a = R_1$, then as $Q^a = Q^x$, $\Omega_1(C_{R_i}(a)) \leq Z(R_i)$, and so again $m_2\big(C_R(a)\big) \leq 2n$. Now S/R is contained in the wreath product of a cyclic group of field automorphisms of $L_2(2^n)$ by $\mathbf{Z}_2$, so that $m_2(S/R) \leq 2$; hence

$$4n \leq m(A) \leq m(M_0/S) + m(S/R) + m(A \cap R) \leq 1 + 2 + m(C_R(a)) \leq 3 + 2n < 4n$$

since $n \geq 2$. This contradiction establishes the claim that $R = J(M_0)$.

Next from the proof of C.5.6.7, $|\mathcal{A}(R)| = 4$, and $M_0 - N_T(R)$ induces a 4-subgroup on $\mathcal{A}(R)$ generated by a pair of commuting transpositions. Thus either $M_0 = N_T(R)$ and $Q^{N_T(R)} = \{Q, Q^x\}$ is of order 2, or $M_0 < N_T(R)$ with $Q^{N_T(R)} = \mathcal{A}(R)$ and $N_T(R)$ inducing D_8 on $\mathcal{A}(R)$.

Assume that the latter case holds. Now D acts on each member of $\mathcal{A}(R)$, so for each $y \in N_T(R)$, $D \leq G_Q^y = N_G(L_0^y)$, and by 1.2.2, $D \leq L_0^y$. It follows that $N_T(R)$ normalizes the intersection RD of the groups L_0 and L_0^y; hence $RD \trianglelefteq N_T(R)D$, so $N_T(R) = R(N_T(R) \cap N_T(D))$ by a Frattini Argument. Then arguing as above, $N_T(R)$ permutes the subgroups D_i maximal subject to $C_R(D_i) \neq 1$, and so permutes their fixed spaces $\{R_1, R_2\}$. Therefore $N_S(R_1)$ is of index 2 in a subgroup $S_2 \leq N_T(R)$ such that S_2 acts on R_1 and U_2. We have seen that $N_S(R_1) \in \beta$, so $S_2 \in \beta$ by 2.3.2.1. Next $R_1U_2 = QQ^s$ for $s \in S_2 - S$ with $\mathcal{A}(R_1U_2) = \{Q, Q^s\}$, so $N := N_G(R_1U_2) = (N_G(Q) \cap N_G(Q^s))S_2$. By 2.4.3.1, $Q \in \mathcal{S}_2^e(G)$, so by 1.1.4.1, $N \in \mathcal{H}^e$. Then as $L_2 \leq N$ with $L_2 \not\leq M$, $(S_2, N) \in \mathcal{U}(N)$, so $N \in \Gamma$. But $|S_2| = |S|$, so by 2.4.3.2, $N \in \Gamma_0^e$ and $S_2 \in Syl_2(N)$. Now $H_1 := \langle S_2, L_2 \rangle \leq N$ and as $S_2 \in Syl_2(N)$ and $N \in \mathcal{H}^e$, $H_1 \in \mathcal{H}^e$ by 1.1.4.4. Thus $(S_2, H_1) \in \mathcal{U}(H_1)$, so $H_1 \in \Gamma$; then $H_1 \in \Gamma_0^e$ by 2.4.3.2. Therefore as $H_1 \leq N_G(R_1)$, $S_2 \in Syl_2(N_G(R_1))$ by 2.3.7.2. This is impossible as $|N_T(R) : N_T(R_1)| = 2$ since $N_T(R)$ permutes

$\{R_1, R_2\}$ transitively, so that $|N_T(R_1)| \geq 2|S| = 2|S_2| > |S_2|$. This contradiction eliminates the case $M_0 < N_T(R)$.

Therefore $M_0 = N_T(R)$. Then as $N_T(M_0) \leq N_T(J(M_0)) = N_T(R) = M_0$, we conclude $M_0 = T$, and hence $|T| = 2|S|$. Recall $S_1 = \langle x\rangle N_S(R_1) = N_{M_0}(R_1)$; thus $|S_1| = |S| = |T|/2$. Then by 2.3.7.1, $H \in \Gamma_*$, and as we saw $KS_1 \in \Gamma$, similarly $KS_1 \in \Gamma_*$ with $S_1 \in Syl_2(KS_1)$.

As $L_2 \in \mathcal{L}(KS_1, N_S(R_1))$ and $|S_1 : N_S(R_1)| = 2$, $L_2 \leq K_2 \in \mathcal{C}(KS_1)$ by 1.2.5. By construction S_1 normalizes R_1, and K centralizes R_1D_1; indeed much as in the proof of 2.4.23.5, R_1 is the largest subgroup of S_1 invariant under L_2 and x, so that $R_1 = O_2(KS_1) \geq O_2(K_2)$. As K centralizes R_1, we conclude that $O_2(K_2) \leq Z(K_2)$. Then as $m_2(K_2) \geq m_2(L_2) > 1$, we conclude from 1.2.1.5 that K_2 is quasisimple, and hence is a component of KS_1. Thus K_2 is described in 2.3.9.7; so as $K_2 \cap M$ contains the $L_2(q)$-block L_2 and $C_{U_2}(L_2) = 1$, we conclude that $K = K_2$ and $K/O_2(K) \cong L_3(q)$. But now $m_p(KD_1) > 2$, for p a prime divisor of $q-1$, contradicting KD_1 an SQTK-group.

This contradiction shows that the case $s = 2$ cannot occur in Theorem 2.4.7. Hence the proof of Theorem 2.4.7 eliminating $L_2(2^n)$ blocks for $n > 1$ is at last complete.

2.4.2. The small examples and shadows of extensions of $L_4(3)$. In this subsection, we complete the proof of Theorem 2.4.1. Thus we continue the hypotheses and notation from the beginning of this section. By Theorem 2.4.7, the block L is of type A_3 or A_5, and in the latter case $L_0 \cap M$ is a Borel subgroup of L_0 as L_0S is a minimal parabolic. Therefore $Z(L) = 1$ by C.1.13.c.

Recall from the beginning of this section 2.4 that $Q := O_2(H)$. However in this new subsection, $J(S)$ is no longer denoted by R, but instead

$$R := \mathrm{Baum}(S).$$

Recall also from 2.3.8.4 that $L_i = [L_i, J(S)]$ for each i, so that R normalizes L_i by C.1.16.

LEMMA 2.4.25. *If L is an A_5-block, then $s = 1$.*

PROOF. Assume otherwise, so that $s = 2$. Recall we defined $R = \mathrm{Baum}(S)$ just above, and set $Q_i := O_2(L_iR)$, $I := C_R(L)$, $S_I := N_S(L)$, and $T_0 := N_T(S)$. By 2.4.5.1, Hypotheses C.5.1 and C.5.2 are satisfied with S in the role of both "T_H" and "R", for each subgroup M_0 of T_0 with S a proper normal subgroup of M_0. As R denotes $\mathrm{Baum}(S)$, U_0, R play the roles played by "U, S" in section C.5, while I plays the role of "D_1".

By C.5.4.3, $Q_2 = U_2 \times D_2$ where $D_2 := C_R(L_2)$ and $U_2 := O_2(L_2)$, and $R/Q_2 \cong E_4$ is generated by two transpositions in $L_2R/Q_2 \cong S_5$. Also from the proof of C.5.4.3, $RQ_2 = J(S)Q_2$ and for $A \in \mathcal{A}(S)$ with $A \not\leq Q_2$, $|U_2 : C_{U_2}(A)| = |A : (A \cap Q_2)|$. It follows that $[A, U_1] = 1$ so $[A, L] \leq C_L(U_1) = U_1$, and hence $A = U_1 \times (A \cap I)$. Thus I/Q_I is generated by two transpositions in $L_2I/Q_I \cong S_5$, where $Q_I := O_2(L_2I) = U_2 \times D_0$, and $D_0 := C_R(L_0)$. Thus $[U_2, I] \leq \Phi(I) \leq Q_I$, and as U_2 is the A_5-module for L_2, it follows that $\Phi := C_{\Phi(I)}(S_I) = \Phi_2 \times D_\Phi$ centralizes $O^{3'}(M \cap L_2)$, where $\Phi_2 := C_{U_2}(S_I) \cong \mathbf{Z}_2$ and $D_\Phi := C_{\Phi(I)\cap D_0}(S_I)$. Therefore $O^{3'}(M \cap L_0) = O^{3'}(M \cap L)O^{3'}(M \cap L_2)$ centralizes Φ. Observe also that $S_I = N_S(\Phi(I)) = N_S(\Phi)$.

By C.5.5, we may choose $x \in M_0$ with $U^x \not\leq Q$, and and so as $Q_1^t = Q_2$ with $Q = Q_1 \cap Q_2$, $U^x \not\leq Q_1$. By C.5.6.4, $\Phi(I)^x = \Phi(I)$. Then as x also acts on S, x acts on S_I and hence also on Φ. Let $G_I := N_G(\Phi)$, and set $S_0 := \langle S_I, x\rangle$. Then $S_0 \leq N_T(\Phi)$. As $J(S) \leq R$, $O_2(L_0S)J(S) \leq N_S(L) = S_I$, while $N_S(L)$ is of index 2 in S, so $S_I \in \beta$ by 2.3.8.5b. As $N_H(\Phi) \in \mathcal{H}^e$ by 1.1.3.2, and $N_H(\Phi)$ contains $L \not\leq M$, from the definitions in Notations 2.3.4 and 2.3.5, $(S_I, N_H(\Phi)) \in \mathcal{U}(G_I)$, and hence $G_I \in \Gamma$. By 1.1.6, the 2-local G_I satisfies the hypotheses of 1.1.5 in the role of "H".

As $U^x \not\leq Q_1 = O_2(LR)$, $U^x \not\leq O_2(LS_I)$. Therefore as $U^x \leq R \leq S_I$, while L^x is irreducible on U^x, $U^x \cap O_2(G_I) = 1$. Notice $LS_I \in \mathcal{H}^e$, so that $(S_I, LS_I) \in \mathcal{U}(LS_I)$, and hence $LS_I \in \Gamma$. Define $\mathcal{H}_1$ to consist of the subgroups H_1 satisfying:

$H_1 \in \mathcal{H}^e(LS_I) \cap G_I$, and

$H_1 = \langle L, S_1\rangle$ for some $S_1 \in Syl_2(H_1)$ containing S_I.

Then $\mathcal{H}_1$ is nonempty, since $LS_I \in \mathcal{H}_1$.

We next claim

(*) $L \in \mathcal{C}(H_1)$ for any $H_1 \in \mathcal{H}_1$.

It is clear that (*) holds if $S_1 = S_I$, so assume instead that $S_1 > S_I$. Then as $|S : S_I| = 2$, $|S_1| \geq |S|$. Since $H_1 \in \mathcal{H}^e$, $(S_I, H_1) \in \mathcal{U}(H_1)$ and $H_1 \in \Gamma$. Then by 2.4.3.2, $|S_1| = |S|$ and $H_1 \in \Gamma_0^e$. As $|S_1 : S_I| = 2$ and $L \in \mathcal{L}(H_1, S_I)$, $L \leq K \in \mathcal{C}(H_1)$ by 1.2.5. Then as $H_1 \in \Gamma_0^e$, K is a χ_0-block of H_1 by 2.3.8.4. Since no χ_0-block has a proper A_5-block, $K = L$, completing the verification of (*).

Now $L \leq G_I^\infty$, and by 1.2.1.1, G_I^∞ is a product of $\mathcal{C}$-components $K_1, \ldots, K_r$, with L inducing inner automorphisms on $K_i/O_2(K_i)$ for each i. However using 1.2.1.1, $C_{G_I^\infty}(G_I^\infty/O_2(G_I^\infty)) = O_2(G_I^\infty)$, so as $U \cap O_2(G_I) = 1$, $L \cap O_2(G_I^\infty) = 1$. Hence $Aut_U(K_i/O_2(K_i)) \neq 1$ for some $K_i \in \mathcal{C}(G_I)$. Choose notation so that the projection L_{K_i} of L on $K_i/O_2(K_i)$ is nontrivial iff $1 \leq i \leq t$. Since $L \cap O_2(G_I^\infty) = 1$, it follows that $L \leq L_{K_1} \cdots L_{K_t}$. Observe for $i \leq t$ that L_{K_i} has a quotient A_5, so that $m_3(L_{K_i}) \geq 1$.

We claim that $t = 1$: For $t \leq m_3(G_I) \leq 2$ since G_I is an SQTK-group, so that $t = 2$ if $t > 1$, and then the proof of 1.2.2 (which does not depend on conjugacy of the $\mathcal{C}$-components in the lemma) shows that $O^{3'}(G_I) = K_1K_2$. Since $O^{3'}(M \cap L_0)$ centralizes Φ, $O^{3'}(M \cap L_0) \leq O^{3'}(G_I) \leq K_1K_2$. Therefore for $i = 1$ or 2, there exists y of order 3 in $L_0 \cap K_i$ with $L = [L, y]$. Then $L = [L, y] \leq K_i$, so that $L \leq L_{K_i}$, and hence $L = L_{K_i}$ with $L_{K_j} = 1$ for $j \neq 1$, contrary to our assumption that $t = 2$. This contradiction establishes the claim that $t = 1$. Hence $L = L_{K_1} \leq K_1 =: K$. Since $U \cap O_2(G_I) = 1$, $m_2(K/O_2(K)) \geq m(U) = 4$, ruling out cases (c) and (d) of 1.2.1.4, and hence showing that $K/O_2(K)$ is quasisimple.

Suppose first that $F^*(K) = O_2(K)$. Now $S_I = N_S(L)$ normalizes L and hence normalizes K. Then $KS_I \in \mathcal{H}_1$ so $L \in \mathcal{C}(KS_I)$ by (*). Thus $L \in \mathcal{C}(K)$ and hence $L = K$, contrary to $U \cap O_2(G_I) = 1$.

Thus $F^*(K) > O_2(K)$, so as $K/O_2(K)$ is quasisimple, we conclude that K is quasisimple, and hence K is a component of G_I. Thus K is on the list of 1.1.5.3. Indeed as K contains the A_5-block L, we conclude from that list that K is either of Lie type and characteristic 2 of Lie rank at least 2, but not $L_3(2)$, or one of M_{22}, M_{23}, M_{24}, J_4, HS, He, or Ru. Let $S \leq T_I \in Syl_2(G_I)$; then T_I normalizes K by 1.2.1.3. Let $X \in \mathcal{H}^e \cap KT_I$, $S_I \leq S_1 \in Syl_2(X)$, and $Y := \langle L, S_1\rangle$. Then $S_1 \in Syl_2(Y)$, and $Y \in \mathcal{H}^e$ by 1.1.4.4, so $Y \in \mathcal{H}_1$. Then by (*), L is subnormal in Y, so $L \in \mathcal{L}(X, S_1)$. Thus we have shown that for each $X \in \mathcal{H}^e \cap KT_I$ and

$S_1 \in Syl_2(X)$ with $T_I \leq S_1$, we have $L \in \mathcal{L}(X, S_1)$. This is a contradiction, since from the 2-local structure of the groups K on our list, none contains an A_5-block L, such that for each overgroup X of LS_+ in K with $F^*(X) = O_2(X)$ and $S_+ \in Syl_2(N_K(L))$, $L \in \mathcal{L}(X, S_1)$ for $S_+ \leq S_1 \in Syl_2(X)$. This completes the proof of 2.4.25. □

By 2.4.7 and 2.4.25, either L is an A_5-block with $s = 1$, or L is an A_3-block with $s = 1$ or 2. So by 2.4.5.3, the hypotheses of Theorem C.6.1 are satisfied with T, S in the roles of "Λ, T_H". Similarly by 2.4.5.1, we can appeal to results from section C.5, with S, S, L_0, U_0, Baum(S) in the roles of "T_H, R, K, U, S".

We will first show that when $s = 1$ and L is an A_3-block, then G is a group in the conclusion of Theorem 2.4.1. Since G is a counterexample to Theorem 2.4.1, this will establish the following reduction:

LEMMA 2.4.26. *If L is an A_3-block, then $s = 2$.*

PROOF. Assume L is an A_3-block with $s = 1$. By Theorem C.6.1, $H \cong S_4$ or $\mathbf{Z}_2 \times S_4$.

Suppose first that $H \cong S_4$, so that case (b) of Theorem C.6.1.6 holds, and in particular T is dihedral or semidihedral. Then by I.4.3, G is $L_2(p)$, p a Fermat or Mersenne prime, A_6, $L_3(3)$, or M_{11}. As $M = !\mathcal{M}(T)$, G is not $L_2(7)$ or A_6. As $\delta \neq \emptyset$, G is not $L_2(5)$. This leaves the groups in Theorem 2.4.1, contradicting the choice of H, G as a counterexample.

Therefore $H \cong \mathbf{Z}_2 \times S_4$, so case (a) of Theorem C.6.1.6 holds. Then $|T : S| = 2$ and $J(S) = S = J(T)$. By C.6.1.1, $S = QQ^x$ for $x \in T - S$. Define y and z by $\langle y \rangle = Z(H)$ and $\langle z \rangle = \Phi(S)$; by 2.4.4, $S = C_T(y)$. Since $S = J(T)$ is weakly closed in T, by Burnside's Fusion Lemma A.1.35, $N_G(S)$ controls fusion in $Z(S)$, so $y \notin z^G$. Thus $y^x = yz$, and H is transitive on $yU - \{y\}$, so all involutions in yUU^x are in y^G, and all involutions in UU^x are in z^G.

Suppose first that $y^G \cap T \subseteq S$. Then $y^G \cap T \subseteq yUU^x$. Now T/UU^x is of order 4 and hence abelian, so by Generalized Thompson Tranfer A.1.37.2, $y \notin O^2(G)$, contradicting the simplicity of G.

Thus we may take $x \in y^G$; in particular, x is now an involution. Let $u \in U - \langle z \rangle$. Then $\langle u, x \rangle \cong D_{16}$, and we saw $[x, y] = z$, so $S_1 := \langle xy, u \rangle \cong SD_{16}$, with xy of order 4. Hence all involutions in S_1 are in UU^x and therefore lie in z^G. Therefore $y^G \cap S_1 = \emptyset$, so Thompson Transfer produces our usual contradiction to the simplicity of G, completing the proof. □

By 2.4.25 and 2.4.26, the structure of S is similar in the two remaining cases where L_0 is either an A_5-block or the product of two A_3-blocks; we summarize some of these common features in the next lemma:

LEMMA 2.4.27. *(1) $|T : S| = 2$ and $R = Baum(S) = J(S) = J(T)$.*

(2) $\mathcal{A}(T) = \{Q, Q^x, A_1, A_1^r\}$ for $x \in T - S$, $r \in S - R$, and $|A_1 : A_1 \cap Q| = 2$.

(3) Let $T_C := C_T(L_0)$. Then $\Phi(T_C) = 1$, $Q = T_C \times U_0$, and $T_C \cap T_C^x = 1$ for each $x \in T - S$.

(4) $R = T_C \times U_0 U_0^x$, with $L_0 = [L_0, U_0^x]$.

PROOF. Let $M_0 := N_T(S)$; by Theorem C.6.1, $|M_0 : S| = 2$. Thus by 2.4.5.2, the hypotheses of C.5.6.7 are satisfied. Further by C.6.1.1, $QQ^x = R = \text{Baum}(S) = J(S)$. By 2.4.25 and 2.4.26, L_0 is an A_5-block or the product of two A_3-blocks, so by C.6.1.4, $\mathcal{A}(R) = \mathcal{A}(S)$ is described in (2). Thus to complete the proof of (2), it remains to show that $\mathcal{A}(S) = \mathcal{A}(T)$, or equivalently to establish the assertion $J(S) = J(T)$ in (1).

As $T_C = C_T(L_0) \leq Q \in \mathcal{A}(S)$, $\Phi(T_C) = 1$. Further if L_0 is an A_5-block, then $Q = T_C \times U$ by C.5.4.3, and this holds when L_0 is a product of A_3-blocks as $S_4 = Aut(A_4)$. Also C.5.6.7 says $T_C \cap T_C^x = 1$ for $x \in M_0 - S$; hence (3) will also follow, once we have established the equality $|T : S| = 2$ in (1). Thus to prove (1)–(3), it remains to establish (1).

Suppose (1) fails. Since we saw that $R = J(S)$, either $|T : S| \neq 2$ or $J(S) \neq J(T)$; thus conclusion (a) of C.6.1.6 does not hold. By 2.4.26, L_0 is not an A_3-block, so conclusion (b) of C.6.1.6 does not hold. Hence conclusion (c) of C.6.1.6 holds. Define $A_1 \in \mathcal{A}(S)$ as in C.6.1.4 and set $W := A_1Q$ and $S_W := N_T(W)$. By conclusion (c) of C.6.1.6, $|S_W| \geq |S|$, and $Q^y = A_1$ for some $y \in S_W$, since $N_T(N_T(S))$ induces D_8 on $\mathcal{A}(S) = \mathcal{A}(N_T(S))$. Then $\mathcal{A}(W) = \{Q, Q^y\}$, so $G_W := N_G(W) = (G_Q \cap G_Q^y)S_W$. By 2.4.3.1, $G_Q \in \mathcal{H}^e$. As $G_Q \cap G_Q^y = N_{G_Q}(W)$, $G_Q \cap G_Q^y \in \mathcal{H}^e$ by 1.1.3.2; therefore $G_W = (G_Q \cap G_Q^y)S_W \in \mathcal{H}^e$. From the structure of L_0, $Q \leq J(S) = R = N_S(W)$, $|S : R| = 2$, and $N_H(W) \not\leq M$, so $G_W \not\leq M$. As $H \in \Gamma_0^e$, 2.3.8.5c says $C_{O_2(M)}(R) \leq R$. Then we conclude from 2.3.8.5b that $R \in \beta$. Then as usual $(R, G_W) \in \mathcal{U}(G_W)$, so $G_W \in \Gamma$. Hence as $|S_W| \geq |S|$, $G_W \in \Gamma_0$ by 2.4.3.2, so that $G_W \in \Gamma_0^e$. Thus G_W satisfies the hypotheses for H in this section. In particular as we showed that $Q = O_2(H)$ is abelian, by symmetry between H and G_W, $O_2(G_W)$ is abelian. This is a contradiction, as $W \leq O_2(G_W)$ and $W = A_1Q$ is nonabelian since $Q \in \mathcal{A}(S)$. This contradiction establishes (1), and completes the proof of (1)–(3).

By C.5.6.2, for each $x \in T - S$, $R = U_0^xQ$ and $[U_0, U_0^x] = U_0 \cap U_0^x$. Thus $L_0 = [L_0, U_0^x]$ and $U_0^x \cap Q \leq U_0$, so as $U_0U_0^x$ and T_C are normal in R, $R = T_C \times U_0U_0^x$. That is, (4) holds. □

REMARK 2.4.28. In the next lemma, we deal with the shadows of extensions of $L_4(3) \cong P\Omega_6^+(3)$ which are not contained in $PO_6^+(3)$. In this case, L is an A_5-block. The subcase where $C_T(L) \neq 1$ is quickly eliminated using 2.3.9.7: that subcase is the shadow of $Aut(L_4(3))$, which is not quasithin since an involution in $C_T(L)$ has centralizer $\mathbf{Z}_2 \times PO_5(3)$. The remaining cases we must treat correspond to the two extensions of $L_4(3)$ of degree 2 distinct from $PO_6^+(3)$, which are in fact quasithin. These subcases are eventually eliminated by using transfer to show G is not simple, but only after building much of the 2-local structure of such a shadow.

Shadows of extensions of $L_4(3)$ will also appear several more times in later reductions.

LEMMA 2.4.29. *L is an A_3-block. Hence $H = L_0S$ where L_0 is a product of two S-conjugates of L.*

PROOF. The second statement follows from the first in view of 2.4.26. We assume L is not an A_3-block, and derive a contradiction. Then L is an A_5-block, and $s = 1$ by 2.4.25. Set $T_C := C_T(L)$. By 2.4.27, $Q \leq J(T) = J(S) = \mathrm{Baum}(S) = R$, $\Phi(T_C) = 1$, $Q = T_C \times U$, and $R = T_C \times UU^x$. By C.5.4.3, $R/Q \cong E_4$ and $LR/Q \cong S_5 = Aut(A_5)$, so that $LS = LR$. Recall $L \cap M$ is a Borel subgroup of L.

Let $K := O^2(M \cap L)$ and $P := O_2(K)$. Then $P \cong Q_8^2$, and $S = PR = PUU^xT_C$ centralizes T_C, so $\Phi(S) = \Phi(UU^x)\Phi(P)[UU^x, P] \leq P$. Therefore $Z := Z(P) = \langle z \rangle = \Phi(S) \cap Z(S)$. Since S is of index 2 in T by 2.4.27.1, $z \in Z(T)$.

Set $G_Z := C_G(Z)$ and $\tilde{G}_Z := G_Z/Z$. Then $\tilde{P}\tilde{T}_C = J(\tilde{S})$ is x-invariant, so $PT_C \trianglelefteq \langle x, KS \rangle =: M_1$. Observe $T \leq M_1 \leq G_Z$ and $T_C Z = Z(PT_C) \trianglelefteq M_1$. By 2.4.27.3, $T_C \cap T_C^x = 1$, so as x normalizes $Z(PT_C) = ZT_C$, and T_C is of index 2 in $T_C Z$, $|T_C| \leq 2$ with $[x, T_C] = Z$ in case of equality. As $|ZT_C| \leq 4$ and $M_1 \leq N_{G_Z}(ZT_C)$, $O^2(M_1)$ centralizes ZT_C. As S centralizes T_C, $H = LS$ centralizes T_C.

Next $PT_C \cong Q_8^2$ or $Q_8^2 \times \mathbf{Z}_2$, so $Aut_{Aut(PT_C)}(PT_C/ZT_C) \cong O_4^+(2)$. Let $M_1^+ := M_1/C_{M_1}(PT_C/ZT_C)$. Then $M_1^+ \leq O_4^+(2)$ and $K^+R^+ \cong S_3 \times \mathbf{Z}_2$ with $U^+ = O_2(K^+R^+)$. As $U^x \not\leq O_2(KR)$ and $x \in M_1$, $U \not\leq O_2(M_1)$; then as $M_1^+ \leq O_4^+(2)$, $M_1^+ \cong O_4^+(2)$. In particular, M_1 is irreducible on PT_C/ZT_C.

Suppose first that $T_C \neq 1$. As $H = LS \leq C := C_G(T_C)$, we conclude from 2.3.7.2 that $C \in \Gamma_0$ and $S \in Syl_2(C)$. By 1.2.4, $L \leq L_C \in \mathcal{C}(C)$, and the embedding of L in L_C is described in A.3.14. From the previous paragraph, $O^2(M_1) \leq C$ but $U \not\leq O_2(M_1)$, so $U \not\leq O_2(C)$. Hence as L is irreducible on U, $O_2(C) = T_C \leq Z(C)$. Therefore as $m_2(L_C) \geq m_2(L) > 1$, L_C is quasisimple by 1.2.1.5, and so L_C is a component of C. But the list of A.3.14 contains no embedding $L_C > L$ with L an A_5-block.

Therefore $T_C = 1$, so $Q = T_C \times U = U$, and hence $U = O_2(N_G(U)) = F^*(N_G(U))$ by 2.4.3.1. In particular, $C_G(U) = U$, so $C_G(L) = C_U(L) = 1$. By 2.4.6, $L \trianglelefteq N_G(U)$, so as $LS = Aut(L)$, $H = LS = N_G(U)$.

As $T \leq M_1 \leq G_Z$ and $M = !\mathcal{M}(T)$, $M_1 \leq G_Z \leq M$. As G is of even type, $M \in \mathcal{H}^e$, so $Z \leq C_M(O_2(M)) \leq O_2(M) =: P_M$. Also $Q_8^2 \cong P = C_T(\tilde{P})$, so as $M_1^+ \cong O_4^+(2)$, $P = O_2(M_1)$. Therefore as $T \leq M_1 \leq M$, $P_M \leq P$ by A.1.6. Then as M_1 is irreducible on P/Z, P_M is either P or Z. As $M \in \mathcal{H}^e$, the latter is impossible, so $P = P_M$. Then as $Z = Z(P)$, $M \leq G_Z$, so that $M = G_Z$ as $M \in \mathcal{M}$. Since $M_1/P \cong O_4^+(2) \cong Out(P)$, $M = M_1 = G_Z = C_G(z)$. In particular, M is solvable.

Let $u \in Z(R) - Z$. As U is the S_5-module for H/U, we can adopt the notation of section B.3 to describe U, and choose $u = e_{1,2}$. Then $z = e_{1,2}e_{3,4} = uu^s$ for a suitable $s \in S - R$. Set $G_u := C_G(u)$, $H_u := C_H(u)$, etc. Then as $H/U \cong S_5$, $H_u \cong D_8 \times S_4$, so $R = S_u$ is of index 2 in S. Further $C_U\left(O^2(H_u)\right)^{\#} = \{u, e_{1,3,4,5}, e_{2,3,4,5}\}$ with $e_{1,3,4,5}$ and $e_{2,3,4,5}$ in z^L. As $\langle u, z \rangle = Z(R) \trianglelefteq T$ and $u^s = uz$, there is $x \in T_u - S$, and $T_u = \langle x \rangle R$ is of index 2 in T with $T_u = M_u$. As $T_u \trianglelefteq T$, $N_G(T_u) \leq M = !\mathcal{M}(T)$. Then as $T_u = M_u$, $T_u \in Syl_2(G_u)$, and in particular $u \notin z^G$. Also $H_u \not\leq M$ with $|T_u| = |S| = |T|/2$, so by 2.3.7.1, $G_u \in \Gamma_*$.

Suppose first that $F^*(G_u) = O_2(G_u)$, so that $G_u \in \Gamma_0^e$. Then we may apply the results of this section to G_u in the role of "H". By 2.3.8.4, $G_u = M_u K_1 \cdots K_t$ is a product of blocks K_i, where K_i is an A_5-block or A_3-block, since 2.4.7 eliminated the case where some K_i is an $L_2(2^n)$-block. Indeed as we saw $M_u = T_u$ is a 2-group, and $K_i \cap M$ is a Borel subgroup of K_i in 2.3.8.4, each K_i is in fact an A_3-block. Then as T_u is of order 2^7 and 2-rank 4 with $1 \neq u \in Z(G_u)$, $t = 1$. But now $K_1 = O^2(G_u) \cong A_4$ contains $O^2(H_u) \cong A_4$, so that $O^2(G_u) = O^2(H_u)$. As S_u is of index 2 in T_u, H_u is of index 2 in G_u, and hence is normal in G_u. Then as $U \leq O_2(H_u)$ and $x \in T_u$, $U^x \leq O_2(H_u)$, which is not the case.

Thus $F^*(G_u) \neq O_2(G_u)$. As $O^2(H_u) \cong A_4$, $O_2(O^2(H_u))$ centralizes $O(G_u)$ by A.1.26, so $z \in \langle u \rangle O_2(O^2(H_u)) \leq C_{G_u}(O(G_u))$; hence $O(G_u) = 1$, as z inverts $O(G_u)$ by 2.3.9.5. Therefore G_u has a component K, which must appear in 2.3.9.7. We further restrict the list of 2.3.9.7 using the facts that $M_u = C_{G_u}(z)$ is a 2-group of order 2^7 and rank 4, and $H_u = N_{G_u}(U) \cong D_8 \times S_4$, to conclude that $K \cong A_6$, $L_2(7)$, or $L_2(17)$. Next $O^2(N_{G_u}(U)) = O^2(H_u) \cong A_4$ and $O^2(N_K(U)) \cong A_4$ in each of the possibilities for K, so $O^2(H_u) \leq K$. Now $z \in \langle u \rangle O^2(H_u) \leq \langle u \rangle K$, but $z \neq u$, so T_u normalizes the component K, and hence $K \trianglelefteq G_u$ by 1.2.1.3. As $J(T_u) = R = UU^x$, K is not $L_2(17)$, and in the remaining two cases, x induces an outer automorphism on K interchanging the two 4-groups in $R \cap K \in Syl_2(K)$, so that $K = \langle O^2(H_u), O^2(H_u)^x \rangle$. Also $z = uu^s$ for $s \in S - R$, and $u^s \in O^2(H_u) \leq K$; so as K has one class of involutions, by a Frattini Argument,

$$G_u = KC_{G_u}(u^s) = KC_{G_u}(z) = KM_u = KT_u.$$

Let $D := C_R(O^2(H_u))$ and $U_D := D \cap U$. Then $D \cong D_8$, U_D is a 4-group, and $U_D - \langle u \rangle \subseteq z^L$ from an earlier remark. Hence as $C_G(z)$ is solvable, $C_{U_D}(K) = \langle u \rangle$. But if K is $L_2(7)$, then $C_{Aut(K)}(O^2(H_u)) = 1$, so we conclude that $K \cong A_6$ and $v \in U_D - \langle u \rangle$ induces a transposition on K. As $G_u = KT_u$ and K is simple, $B := C_{G_u}(K) = C_{T_u}(K) \leq C_{T_u}(O^2(H_u)) = D$, so $B = C_D(K)$ is of order 4, with $G_u/B \cong Aut(A_6)$, since x interchanges the two 4-groups in $R \cap K$. As $R = UU^x$, x also interchanges the two 4-groups in $R/(R \cap K) = D(R \cap K)/(R \cap K) \cong D$, and hence $B \cong \mathbf{Z}_4$, since $U_D \not\leq B$.

Let $I := O^2(M)$. Then $I = I_1 I_2$ with $I_i \cong SL_2(3)$ and $[I_1, I_2] = 1$. Further there exists $y \in T - S$ centralizing I_1 with $y^2 \in Z$: namely any y inducing an orthogonal transvection on $\tilde{P}$ centralizing I_1. Moreover each $t \in T - S$ with $t^2 \in Z$ is conjugate under M to y or ya, where $a \in I_1 \cap P$ is of order 4, and exactly one of y and ya is an involution. Thus M is transitive on the set $\mathcal{I}$ of involutions in $M - IS$, and either y or ya is a representative i for $\mathcal{I}$. Let $j := ia$; then $j^2 = z$. Observe $j^G \cap S = \emptyset$: For if $j^g \in S$ then $z^g = (j^g)^2 \in \Phi(S) \leq P$. But as $u \in P$ and $u \notin z^G$ while M is transitive on the involutions in $P - Z$, Z is weakly closed in P with respect to G; so $z = z^g$ and hence $g \in G_Z = M$, contradicting $IS \trianglelefteq M$.

As $j^G \cap S$ is empty but $G = O^2(G)$ with $|T : S| = 2$, we can apply Generalized Thompson Transfer A.1.37 to j in the role of "g", to see that $j^2 = z$ must have a G-conjugate in $T - S$; so $i = z^g$ for some $g \in G$. Now if $y = i$ then $SL_2(3) \cong I_1 \leq C_G(i) = M^g$, so $z \in O_2(I_1) \leq O_2(M^g) = P^g$. However we saw in the previous paragraph that $z^G \cap P = \{z\}$, so $z = z^g = i$, contradicting $i \notin S$. Therefore y is of order 4 and $i = ya$ centralizes bc, where $b \in I_2$ is of order 4 and inverted by y, and $O_2(I_1) = \langle a, c \rangle$. As $bc \in u^M$, we may assume $bc = u$, so that u centralizes i. Then $i \in T_u - S$ acts on $K \cong A_6$. As $S \geq U_D$ and $v \in U_D - \langle u \rangle$ induces a transposition on K, KS induces the S_6-subgroup of $Aut(K)$ on K, so as $i \notin S$, i does not induce an automorphism in S_6. Then as i is an involution, i induces an automorphism in $PGL_2(9)$ rather than M_{10}, and hence $C_K(i) \cong D_{10}$. This is impossible as $i \in z^G$ and $M = C_G(z)$ is a $\{2, 3\}$-group. The proof of 2.4.29 is complete. □

By 2.4.29, we have reduced to the case where L_0 is the product of two A_3-blocks. Henceforth we let s denote an element of $S - N_S(L)$. Thus $H = L_0 S$ and $L_0 = L \times L^s$. Let $U_1 := U$ and $U_2 := U^s$.

LEMMA 2.4.30. *(1)* $QQ^x = R = Baum(S) = J(S)$ *for* $x \in T - S$.
(2) $H \in \Gamma_*$.
(3) $\{Q, Q^x\}$ *are the* S*-invariant members of* $\mathcal{A}(R)$.
(4) $RL_0 = C_S(L_0) \times L_0 U_0^x$ *with* $\Phi(C_S(L_0)) = 1$ *and* $L_0 U_0^x \cong S_4 \times S_4$.
(5) R *is of index 2 in* $S = R\langle s \rangle$, *so* $|T : R| = 4$.

PROOF. By 2.4.27.1, $|T : S| = 2$, so (2) holds by 2.3.7.1. By 2.4.27, $R = J(S) = T_C \times U_0U_0^x$, where $T_C := C_S(L_0)$, and $Q = T_C \times U_0$, so $R = QQ^x$. Thus (1) holds, and (3) follows from 2.4.27.2. By 2.4.27.4, $R = T_C \times U_0U_0^x$ and $L_0 = [L_0, U_0^x]$, so (4) holds. Further as $L_1^s = L_2$, $H/Q \cong S_3$ wr $\mathbf{Z}_2$, and as $R = J(S)$ acts on L_1 and $Q \leq R$, R is Sylow in $RL_0 = N_H(L_1)$ of index 2 in S, so (5) holds. $\square$

REMARK 2.4.31. In the remainder of the proof of Theorem 2.4.1, we are again faced with a shadow of an extension of $L_4(3)$, but now approached from the point of view of a 2-local with two A_3-blocks. We will construct the centralizer of the involution z_2 defined below, as a tool for eventually obtaining a contradiction to the absence of an A_5-block in any member of Γ_0. In $P\Omega_6^+(3)$, z_2 is an involution whose commutator space on the orthogonal module is of dimension 2 and Witt index 0, and whose centralizer has a component $\Omega_4^-(3) \cong L_2(9) \cong A_6$.

Now let $\langle z_i \rangle = C_{U_i}(R)$. Then by 2.4.30, $\langle z_1, z_2 \rangle = \Phi(R) \trianglelefteq T$ and $L_{3-i} \leq C_G(z_i) =: G_i$, so $G_i \not\leq M$. Since $Q \leq R$, we conclude by 2.3.8.5c that $C_{O_2(M)}(R) \leq R$. Then since $|S : R| = 2$ and $R = J(S)$ by 2.4.30.1, the first sentence of 2.3.8.5b says $R \in \beta$. So since $L_i \not\leq M$, we conclude as usual from the definitions in Notation 2.3.4 and Notation 2.3.5 that $(R, L_{3-i}R) \in \mathcal{U}(G_i)$ and $G_i \in \Gamma$. Next $z_1^s = z_2$, so $z := z_1z_2$ generates $Z(T) \cap \Phi(R)$, and replacing x by xs if necessary, we may assume $x \in G_i$, for $i = 1$ and 2. Let $S_1 := R\langle x \rangle$. Then $|T : S_1| = 2 = |T : S|$, so by 2.3.7.1, $G_i \in \Gamma_*$ and $S_1 \in Syl_2(G_i)$.

Observe that $F^*(G_2) \neq O_2(G_2)$: For otherwise by 2.3.8.4 and 2.4.29, $G_2 = C_M(z_2)K_0$, where K_0 is the product of two A_3-blocks. But $R = J(S_1)$, so applying 2.4.30.4 to K_0S_1, $C_{\Phi(R)}(K_0) = 1$, contradicting $z_2 \in C_{\Phi(R)}(K_0)$.

Next $O_2(L) \cong E_4$ centralizes $O(G_2)$ by A.1.26, so $z \in \langle z_2 \rangle O_2(L) \leq C_{G_2}(O(G_2))$, and hence $O(G_2) = 1$ since z inverts $O(G_2)$ by 2.3.9.5. Thus as $F^*(G_2) \neq O_2(G_2)$, there exists a component K of G_2, and K is described in 2.3.9.7. By 2.3.9.6, $K = [K, z]$, so L is faithful on K since $z \in \langle z_2 \rangle L$.

Recall $S_1 \in Syl_2(G_2)$ and $|S_1 : R| = 2$ with $R = N_{S_1}(L)$; therefore $Aut_{S_1}(K) \in Syl_2(Aut_{G_2}(K))$ with $|Aut_{S_1}(K) : N_{Aut_{S_1}(K)}(Aut_L(K))| \leq 2$. Further we saw L is faithful on K, so $Aut_L(K) \cong A_4$. Inspecting the 2-locals of the automorphism groups of the groups K listed in 2.3.9.7 for such a subgroup, and recalling $O(G_2) = 1$, we conclude that K is one of A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(17)$, $L_3(3)$, or M_{11}. Moreover if L_K is the projection of L on K, then as $|S_1 \cap K : N_{S_1 \cap K}(L)| \leq 2$ (since L is irreducible on $O_2(L)$ of rank 2), $O_2(L_K) \leq N_K(L)$, and then $O_2(L_K) = [O_2(L_K), L] = O_2(L) = U$. As S_1 centralizes z and z_2, S_1 centralizes $z_1 = zz_2 \in U \leq K$, so S_1 acts on K and hence $K \trianglelefteq G_2$ by 1.2.1.3. If $K \cong A_5$ or A_7, then $U \trianglelefteq S_1$, contradicting $x \notin N_T(U)$. If K is A_8, then L is an A_4-subgroup moving 4 of the 8 points permuted by K, so z_1 is not 2-central in K, a contradiction. If K is $L_3(3)$, M_{11}, or $L_2(17)$, there is $x_K \in S_1 \cap K$ with $Q^{x_K} \neq Q$, so we may take $x = x_K$; but now $|Q^x : C_{Q^x}(Q)| = 2$, contradicting 2.4.30.1 which shows this index is 4. Therefore:

LEMMA 2.4.32. $G_2 \in \Gamma_*$ *and* $L \leq K \cong A_6$ *or* $L_2(7)$.

Next $z_1^{G_2} = z_1^K$ since A_6 and $L_3(2)$ have one class of involutions; so by a Frattini Argument, $G_2 = KC_{G_2}(z_1) = KC_{G_2}(z) = KM_2$, where $M_2 := M \cap G_2$. As $G_2 \in \Gamma_*$, $F^*(M_2) = O_2(M_2)$ by 2.3.9.4. Then as $C_{G_2}(K) \leq M_2$, $F^*(G_2) = KO_2(G_2)$. In particular:

LEMMA 2.4.33. *$K = E(G_2)$ and $F^*(G_2) = KO_2(G_2)$.*

Now suppose that $U_2 \le C_G(K)$. For $g \in L_2$, $z_2^g \in U_2 \le C_G(z_2) = G_2$, so K is a component of $C_{G_2}(z_2^g)$ by 2.4.33. By I.3.2 and 2.4.33, $K \le O_{2',E}(C_G(z_2^g)) = K^g$. We conclude $K = K^g$, and hence $K = E(C_G(u))$ for each $u \in U_2^\#$. Therefore $K^s = E(C_G(u))$ for each $u \in U_1^\# = U^\#$. Also x centralizes z_1 and hence normalizes $K^s = E(C_G(z_1))$, so $K^s = E(C_G(u^x))$ for each $u^x \in (U^x)^\#$. Further $L = [L, U_0^x]$ by 2.4.30.4, so as $U_2^x \le C_G(K)$, $L = [L, U^x]$. Thus using the structure of K in 2.4.32,

$$K = \langle C_K(u), C_K(u^x) : u \in U^\# \rangle \le N_G(K^s).$$

As z_2 centralizes K, z_1 centralizes K^s, so $K = [K, z_1] \le C_G(K^s)$, and hence $T = S_1\langle s \rangle$ normalizes $KK^s = K \times K^s$. Let $I := KK^sT$. Since I contains $L \not\le M = !\mathcal{M}(T)$, $O_2(I) = 1$. As G is quasithin, $m_{2,3}(KK^s) \le 2$, so $K \cong L_3(2)$ rather than A_6. As $O_2(I) = 1$, $m_2(T) \le m_2(Aut(KK^s)) = 4$, so $Q = U_0$ and $R \cong D_8 \times D_8$. It follows that $R \in Syl_2(KK^s)$ and $T = R\langle x, s \rangle$, with x an involution inducing an outer automorphism on K and K^s, and s an involution centralizing x. Then I has 5 classes of involutions, with representatives z, z_2, x, s, and sx. Now $O_2(G_2) \le C_{S_1}(K) \cong D_8$, so $O^2(G_2)$ centralizes $O_2(G_2)/\langle z_2 \rangle$ and z_2, and hence by Coprime Action also centralizes $O_2(G_2)$. Therefore as $F^*(C_{G_2}(K)) = O_2(G_2)$ using 2.4.33, we conclude that $C_{G_2}(K)$ is a 2-group, and hence $C_{G_2}(K) = C_{S_1}(K) = O_2(G_2)$. Thus $G_2/KO_2(G_2) \le Out(K)$ which is a 2-group, so G_2/K is a 2-group, and hence $K = O^2(G_2)$, so $m_3(G_2) = 1$.

Now $C_I(s) = \langle s \rangle \times K_s\langle x \rangle$ with $K_s \cong L_3(2)$, and the involutions in the subgroup K_s diagonally embedded in $K \times K^x$ are in z^G as $z = z_1z_2$; thus $s \notin z_2^G$, since the involutions in $K = G_2^\infty$ are in z_2^G. Similarly $sx \notin z_2^G$. Next $C_I(x) = \langle x, s \rangle (I_1 \times I_2)$ with $I_1 := C_K(x) \cong S_3$ and $I_1^s = I_2$. In particular as $m_3(G_2) = 1$, $x \notin z_2^G$. As $O(C_{G_2}(x)) \ne 1$, $F^*(C_G(x)) \ne O_2(C_G(x))$ by 1.1.3.2, so $x \notin z^G$.

But as $G = O^2(G)$, by Thompson Transfer, $x^G \cap S \ne \emptyset$. Therefore as we saw x is not conjugate to z or z_2, it must be conjugate to s. Arguing similarly with S replaced by $\langle sx \rangle UU^x$, we conclude $sx \in s^G$. So $x^G = s^G = (sx)^G$, and hence by the previous two paragraphs, s, z, and z_2 are representatives for the conjugacy classes of involutions of G. Thus s is in fact *extremal* in T: that is, $T_s := C_T(s) \in Syl_2(C_G(s))$. But each involution in $C_I(s)$ is fused in I to s, x, sx, or z, so $z_2^G \cap T_s = \emptyset$. This is impossible as $z_2 \in C_G(x)$ with x conjugate to s. This contradiction shows $U_2 \not\le C_G(K)$, and hence:

LEMMA 2.4.34. *$K = [K, U_2]$.*

Now $U_2 \le C_{G_2}(L)$. But if $K \cong L_3(2)$, then $C_{G_2}(L) = C_{G_2}(K)$ from the structure of $Aut(K)$, so U_2 centralizes K, contrary to 2.4.34. Therefore part (1) of the following lemma holds:

LEMMA 2.4.35. *(1) $K \cong A_6$, and some $u \in U_2 - \langle z_2 \rangle$ induces a transposition on K centralizing L.*

(2) The automorphism induced by x on K is not in S_6.

For if part (2) of 2.4.35 fails, then setting $(KS_1)^+ := KS_1/C_{KS_1}(K)$, $x^+ \in K^+R^+$, so $U_0^{x+} \in U_0^{+K}$. Then as $U_0^+ = O_2(L^+R^+)$ is weakly closed in R^+ with respect to K^+ from the structure of A_6, $U_0^+ = U_0^{x+}$, contrary to 2.4.30.4.

LEMMA 2.4.36. *(1) $R = R_K \times R_K^s$ with $R_K := R \cap K \in Syl_2(K) \cong D_8$.*

(2) $C_R(K) = C_{G_2}(K)$ *is cyclic of order 4, and* $G_2/C_R(K) \cong Aut(A_6)$.
(3) $C_T(L_0) = 1$ *and* $|T| = 2^8$.

PROOF. We claim that z_2 is the unique involution in $C_R(K)$. Assume the claim fails, and let $z_2 \neq r \in C_R(K)$ be an involution. Recall $R \leq G_2$.

Under this assumption, we establish a second claim: namely that $K \trianglelefteq G_r := C_G(r)$. First K is a component of $C_{G_r}(z_2)$ using 2.4.33, so by I.3.2, there is a 2-component K_r of G_r such that either $K \leq K_r$, or $K \leq K_r K_r^{z_2}$ with $K_r \neq K_r^{z_2}$—and in the latter case, $K_r/O_\infty(K_r) \cong K$. As $K \cong A_6$ by 2.4.35, the former case holds by 1.2.1.3. As K_r is a 2-component of G_r, $K_r \in \mathcal{C}(G_r)$ and $O_2(K_r) \leq Z(K_r)$. As $m_2(K_r) \geq m_2(K) > 1$ and $O_2(K_r) \leq Z(K_r)$, K_r is quasisimple by 1.2.1.5.

Now as $m_3(K_r) \geq m_3(K) = 2$, $K_r \trianglelefteq G_r$ using 1.2.1.3, so our second claim holds if $K = K_r$. Thus we may assume that $K < K_r$, and it remains to derive a contradiction. We verify the hypotheses of 1.1.5 for G_r in the role of "H": Let $C_R(r) \leq T_r \in Syl_2(G_r)$, and $T_r \leq T^g$, so that $z^g \in Z(T^g) \leq T_r$, and hence $z^g \in Z(T_r)$; thus z^g, T_r, M^g play the roles of "z, S,M". As $r \in O_2(G_r \cap M^g)$, trivially $C_{O_2(M^g)}(O_2(G_r \cap M^g)) \leq G_r$. This completes the verification of the hypotheses of 1.1.5. As $K \cong A_6$ is a component of $C_{K_r}(z_2)$, we conclude from inspection of the list of 1.1.5.3 that one of the following holds:

(i) z_2 induces a field automorphism on $K_r \cong Sp_4(4)$.
(ii) z_2 induces an outer automorphism on $K_r \cong L_4(2)$ or $L_5(2)$.
(iii) z_2 induces an inner automorphism on $K_r \cong HS$.

Recall that $|T : R| = 4$, while $|R : C_R(r)| \leq 2$ by 2.4.30, and $z_2 \in Z(R)$. Thus

$$|T_r : C_{T_r}(z_2)| \leq |T_r : C_R(r)| < |T : C_R(r)| \leq 8,$$

where the strict inequality holds since r is not 2-central in G, as $G_r \notin \mathcal{H}^e$. Since z_2 centralizes K but not K_r, we conclude (ii) holds, with $K_r \cong L_4(2) \cong A_8$. Now L is an A_4-subgroup of K_r fixing 4 of the 8 points permuted by K_r, so it centralizes an A_4-subgroup L_r of K_r. Then using A.3.18 and the fact that $z_1 = z_2^s \in O_2(L)$,

$$K_0 := \langle L_r, L_2 \rangle \leq O^{3'}(C_G(L)) \leq O^{3'}(G_1) = K^s.$$

Now $K^s \cong A_6$ with $z_2 \in L_2 \leq K^s$ and z_2 induces an outer automorphism on L_r. Thus $\langle z_2 \rangle L_r \cong S_4$, so $\langle z_2 \rangle L_r$ is a maximal subgroup of K^s. It follows that $K^s = K_0 \leq C_G(L)$, so $m_{2,3}(LK_0) = 3$, contradicting G quasithin. This contradiction establishes the second claim, namely that $K = K_r$ is a normal component of G_r for each involution $r \in C_R(K)$.

Set $E_r := \langle z_2, r \rangle$. Using 2.4.35.2, $C_{K^s S_1}(z_2)$ is a maximal subgroup of $K^s S_1$, which does not contain $C_{K^s S_1}(a)$ for any involution $a \notin z_2 C_G(K^s)$. Thus in the notation of Definition F.4.41, $K^s S_1 = \Gamma_{1,E_r}(K^s S_1)$, so $K^s \leq N_G(K)$ using the second claim. Then as $m_{2,3}(N_G(K)) \leq 2$ since G is quasithin, $K = K^s$. This is impossible as $z_1 \in K$ but $z_2 = z_1^s$ centralizes K. This contradiction completes the proof of the first claim that z_2 is the unique involution in $C_R(K)$.

By 2.4.35, $C_R(K)$ is of index 2 in $C_R(L) \cong C_Q(L_0) \times D_8$, so by the uniqueness of z_2, $C_R(K)$ is cyclic of order 4 and $C_Q(L_0) = 1$. Then $C_T(L_0) = C_Q(L_0) = 1$. Therefore $R \cong D_8 \times D_8$ by 2.4.30.4, so $|T| = 4|R| = 2^8$ by 2.4.30.5, completing the proof of (3).

As $R_K \cong D_8$ and $R_K \cap R_K^s \trianglelefteq S$ but $Z(R_K) = \langle z_1 \rangle \not\leq Z(S)$, we conclude $R_K \cap R_K^s = 1$. Thus $R \geq R_K R_K^s = R_K \times R_K^s$; so as $|R| = |R_K|^2$, $R = R_K \times R_K^s$, and (1) holds.

Let $\bar{G}_2 := G_2/C_{G_2}(K)$. By 2.4.35, $\bar{S}_1\bar{K} \cong Aut(A_6)$ and hence $\bar{G}_2 \cong Aut(A_6)$. In particular $|\bar{S}_1| = 2^5$, so as $C_R(K) \cong \mathbf{Z}_4$ and $|S_1| = |T|/2 = 2^7$, it follows that $C_R(K) \in Syl_2(C_{G_2}(K))$. Then by Cyclic Sylow 2-Subgroups A.1.38, $C_{G_2}(K) = O(G_2)C_R(K)$. Recall that $z = z_1z_2$ with $z_1 \in K$, so that $C_{G_2}(K) \leq C_G(z)$. However by 2.3.9.5, z inverts $O(G_2)$, so $O(G_2) = 1$, completing the proof of (2). □

LEMMA 2.4.37. $z_2^G \cap R = (z_2^G \cap R_K) \cup (z_2^G \cap R_K^s)$ *with* $|z_2^G \cap R_K| = 5$.

PROOF. Recall $A_1 \in \mathcal{A}(T)$ is defined in 2.4.27.2. Further by 2.4.27.2, T induces the 4-group

$$\langle (Q, Q^x), (A_1, A_1^r) \rangle$$

of permutations on $\mathcal{A}(T)$. Thus $y := x$ or xr acts on A_1, so $S_A := R\langle y \rangle$ is of index 2 in T and S_A normalizes A_1. As $z \in A_1$, $H_1 := N_G(A_1) \in \mathcal{H}^e$ by 1.1.4.3. Now $N_H(A_1)$ contains $L_2 \not\leq M$. Also $Q \leq R = J(S)$, so by 2.3.8.5c, $C_{O_2(M)}(R) \leq R$. Then $R \in \beta$ by 2.3.8.5b, so as usual $H_1 \in \Gamma$. Then as $|S_A| = |S|$, $H_1 \in \Gamma_*$ by 2.3.7.1, so we may apply the results of this section to H_1 in the role of "H". In particular we conclude from 2.4.29 [2] that H_1 induces $O_4^+(2)$ on A_1. Therefore for each $A \in \mathcal{A}(T)$, $A = A^1 \times A^2$ with $A^i \cong E_4$ and $A^{1\#} \cup A^{2\#} = z_2^G \cap A$. By 2.4.36.1, $R = R_K \times R_K^s$, so $A = (A \cap R_K) \times (A \cap R_K^s)$ with $A \cap R_K \cong A \cap R_K^s \cong E_4$. Thus as all involutions in R_K are in z_2^G, $A^1 = A \cap R_K$ and $A^2 = A \cap R_K^s$. Therefore as each involution in R is in a member of $\mathcal{A}(T)$, the lemma holds. □

We are now in a position to obtain a contradiction, and hence complete the proof of Theorem 2.4.1. By 2.4.36, $B := C_{G_2}(K) = C_R(K) \cong \mathbf{Z}_4$ and $R = R_K \times R_K^s$. Let $\bar{G}_2 := G_2/B$; then $\bar{R} = \bar{R}_K\langle \bar{u} \rangle$, where $u \in U - \langle z_2 \rangle$. By 2.4.35, $\bar{u}$ induces a transposition on $\bar{K}$, so $\bar{R} = \langle \bar{u} \rangle \times \bar{R}_K \cong \mathbf{Z}_2 \times D_8$ is Sylow in $\bar{R}\bar{K} \cong S_6$.

Next each involution in $\bar{R} - \bar{K}$ is either a transposition or of cycle type 2^3, and there are a total of 6 involutions in $\bar{R} - \bar{K}$. Further $u \in z_2^G$ and $\bar{u}$ is a transposition, so as x induces an outer automorphism on $\bar{R}\bar{K}$, $\bar{u}^x$ is of type 2^3. Thus $\Delta := z_2^G \cap (R - K)$ is of order $6m$, where $m := |z_2^G \cap uB|$. However by 2.4.37, Δ is s-conjugate to $z_2^G \cap R_K$ of order 5.

This contradiction finally completes the proof of Theorem 2.4.1.

2.5. Eliminating the shadows with Γ_0^e empty

The groups occurring in the conclusion of Theorem 2.1.1 have already appeared in Theorems 2.2.5 and 2.4.1, so from now on we are working toward a contradiction. We have also dealt with the most troublesome shadows, although a number of other shadows are still to appear.

By Theorem 2.4.1, we may assume Γ_0^e is empty: that is no member of Γ_0 is contained in $\mathcal{H}^e$. In 2.5.3, we will produce a component K of H, consider the various possibilities for K listed in 2.3.9.7, and analyze the structure of $C_S(\langle K^S \rangle)$, where $S \in Syl_2(H)$. Eventually we eliminate all configurations, completing the proof of Theorem 2.1.1.

We continue to assume that G is a counterexample to Theorem 2.1.1. Therefore as the groups in Theorem 2.4.1 are conclusions of Theorem 2.1.1, in the remainder of the section we assume that

$$\Gamma_0^e = \emptyset.$$

[2] As mentioned earlier, our use of 2.4.29 here to exclude A_5-blocks is essentially eliminating the shadow configuration.

In addition we define $\mathcal{T}$ to consist of the 4-tuples (H, S, T, z) such that $H \in \Gamma_0$, $S \in Syl_2(H \cap M)$, $T \in Syl_2(M)$ with $Z(T) \leq S < T$, and z is an involution in $Z(T)$. For each $H \in \Gamma_0$, there exists a tuple in $\mathcal{T}$ whose first entry is H, using 2.3.9.3. Throughout this section (H, S, T, z) denotes a member of $\mathcal{T}$.

LEMMA 2.5.1. *(1)* $\mathcal{H}^e(S) \subseteq M$.
(2) $H \cap M$ *is the unique maximal member of* $\mathcal{H}^e(S) \cap H$.
(3) $S \in Syl_2(H)$.

PROOF. By 2.3.8.1, $S \in \beta$ and $S \in Syl_2(H)$, so that (3) holds. Suppose there is $X \in \mathcal{H}^e(S)$ with $X \not\leq M$. Then from the definitions in Notation 2.3.4 and Notation 2.3.5, $(S, X) \in \mathcal{U}(X)$, so $X \in \Gamma$. Then by 2.3.7.4, $X \in \Gamma_0$, contrary to our assumption in this section that $\Gamma_0^e = \emptyset$. Thus (1) holds. By 2.3.9.4, $H \cap M \in \mathcal{H}^e$, so that (1) implies (2). □

From now on we use without comment the fact from 2.5.1.3, that S is Sylow in H.

LEMMA 2.5.2. *Suppose* L *is a component of* H *and set* $M_L := M \cap L$. *Then* z *induces an inner automorphism on* L, $L = [L, z] \not\leq M$, *and one of the following holds:*

(1) L *is a Bender group and* M_L *is a Borel subgroup of* L.

(2) $L \cong Sp_4(2^n)'$ *or* $L_3(2^n)$ *or* $L/O_2(L) \cong L_3(4)$. *Further* $N_S(L)$ *is nontrivial on the Dynkin diagram of* $L/Z(L)$, *and* M_L *is a Borel subgroup of* L.

(3) $L \cong L_3(3)$ *or* M_{11} *and* $M_L = C_L(z_L)$, *where* z_L *is the projection on* L *of* z.

(4) $L \cong L_2(p)$, $p > 7$ *a Mersenne or Fermat prime, and* $M_L = S \cap L$.

PROOF. Observe L is described in 2.3.9.7, and $L = [L, z] \not\leq M$ by 2.3.9.6. If $L \cong L_4(2)$, M_{22}, M_{23}, A_7, or $\hat{A}_7$, then from the description of M_L in 2.3.9.7, there is $H_1 \in \mathcal{H}^e(S) \cap H$ with $H_1 \cap L \not\leq M_L$, contradicting 2.5.1.2. Similarly if conclusion (b) of 2.3.9.7 holds, then by 2.5.1.2, S is nontrivial on the Dynkin diagram of $L/Z(L)$, and M_L is as described in (2)—in particular, observe we cannot have $L \cong A_6$ or $\hat{A}_6$ with z inducing a transposition, since S is nontrivial on the Dynkin diagram, while $z \in Z(S)$ as $(H, S, T, z) \in \mathcal{T}$. So when L is A_6 or $\hat{A}_6$, z induces an inner automorphism of L. Indeed as $z \in LC_S(L)$, and z inverts $O(H)$ by 2.3.9.5, L is not $\hat{A}_6$ for any action of z on L. If conclusion (a) of 2.3.9.7 holds, then by 2.5.1.2, M_L is a Borel subgroup of L, so that (1) holds. The remaining cases (d) and (e) of 2.3.9.7 appear as (3) and (4). Since we have eliminated the case where z induces an outer automorphism on $L/Z(L) \cong A_6$ or A_7, in each case z induces an inner automorphism on L by 2.3.9.7. □

Part (4) of the next result produces the component of H on which the remainder of the analysis in this section is based. Furthermore it eliminates case (1) of 2.5.2 where the component is a Bender group.

LEMMA 2.5.3. *Assume*

$$H = \bigcap_{i=1}^{k} N_G(B_i) \text{ for some 2-subgroups } B_1, \ldots, B_k \text{ of } H,$$

and let $(U, H_U) \in \mathcal{U}(H)$. *Set* $Q_U := O_2(H_U)$. *Then*

(1) If $O_2(H) \leq Q_U$, *then* $N_H(Q_U) \in \mathcal{H}^e$.

(2) If $Q_1 \in И_H(H_U, 2)$*, then* $(U, H_U Q_1) \in \mathcal{U}(H)$*.*

(3) If L *is a component of* H *which is a Bender group and* $И_H(H_U, 2) \subseteq Q_U$*, then* $Q_U \cap L \in Syl_2(L)$*.*

(4) There exists a component K *of* H *such that* K *is not a Bender group, and if* $H \in \Gamma^*$*,* Γ_**, then* $\langle K, S\rangle \in \Gamma^*$*,* Γ_**, respectively.*

PROOF. By 2.3.8.2, $Q_U \in \mathcal{S}_2^e(G)$, so $H_0 := N_G(Q_U) \in \mathcal{H}^e$. Assume $O_2(H) \le Q_U$. By hypothesis, $B_i \le H = \bigcap_{j=1}^k N_G(B_j)$, so $B_i \le O_2(H) \le Q_U$. Thus

$$N_H(Q_U) = \bigcap_{i=1}^k N_{H_0}(B_i) \in \mathcal{H}^e$$

by 1.1.3.3. Hence (1) holds.

Assume the hypotheses of (2), and let $Q_2 := Q_U Q_1$ and $H_2 := H_U Q_1$. As $F^*(H_U) = O_2(H_U) = Q_U$ since $H_U \in \mathcal{H}^e$, also $F^*(H_2) = Q_2 = O_2(H_2)$; so as $U \le H_U \le H_2$ with $U \in \beta$, $(U, H_2) \in \mathcal{U}(H)$, and hence (2) holds.

Assume the hypotheses of (3), and let $L_0 := \langle L^H\rangle$. First, $O_2(H) \in И_H(H_U, 2) \subseteq Q_U$ by hypothesis; so by (1), $N_H(Q_U) \in \mathcal{H}^e$, and then by 1.1.3.1,

$$F^*(N_{L_0}(Q_U)) = O_2(N_{L_0}(Q_U)). \qquad (*)$$

Set $P_U := L_0 C_H(L_0) \cap Q_U$, and let P_L, P_1 denote the projections of P_U on L, L_0, respectively. If $L < L_0 = LL^s$, let P_{L^s} be the projection of P_U on L^s. If $P_1 = 1$ then $Aut_{Q_U}(L_0) \cap Inn(L_0) = 1$, so as L is a Bender group, from the structure of $Aut(L_0)$, $O^2(F^*(C_{L_0}(Q_U))) \ne 1$, contrary to (*). Thus $P_1 \ne 1$ and $P_1 \in И_H(H_U, 2) \subseteq Q_U$. Similarly if $L < L_0$, $P_1 \le P_L P_{L^s} \in И_H(H_U, 2) \subseteq Q_U$, and as $P_1 \ne 1$, either $P_L \ne 1$ or $P_{L^s} \ne 1$. Further if $P_L = 1$, then Q_U acts on $P_1 = P_{L^s}$ and hence on L, and $Aut_{Q_U}(L) \cap Inn(L) = 1$ so again $O^2(F^*(C_L(Q_U)) \ne 1$, contrary to (*). Thus $P_L \ne 1$, and if $L < L_0$ also $P_{L^s} \ne 1$. Therefore as L is a Bender group, there is a unique Sylow 2-group P_0 of L_0 containing P_1, so $P_0 \in И_H(H_U, 2) \subseteq Q_U$ and hence $P_0 = Q_U \cap L_0$, establishing (3).

It remains to prove (4). Let L_+ be the product of all Bender-group components of H, with $L_+ := 1$ if no such components exist. Partially order $\mathcal{U}(H)$ by $(U_1, H_1) \le (U_2, H_2)$ if $U_1 \le U_2$ and $H_1 \le H_2$, and choose (U, H_U) maximal with respect to this order. Then by (2) and maximality of (U, H_U), $И_H(H_U, 2) \subseteq H_U$, and hence

$$И_H(H_U, 2) \subseteq Q_U \ \text{ and in particular } \ O_2(H) \le Q_U. \qquad (!)$$

Observe by (!) that we may apply (1) and (3).

Replacing (U, H_U) by a suitable conjugate under $H \cap M$, we may assume $S \cap H_U \in Syl_2(H_U \cap M)$. Set $Q_+ := S \cap L_+ \in Syl_2(L_+)$. Then $Q_+ = Q_U \cap L_+$ by (3), and so $H_U \le X := N_H(Q_+)$. When $L_+ \ne 1$, $M_+ := M \cap L_+ = N_{L_+}(Q_+)$ by 2.5.2.1. In any case by a Frattini Argument, $H = L_+ X$. Further $S \in Syl_2(X)$ since $S \in Syl_2(H)$ by 2.5.1.3. Also $(U, H_U) \in \mathcal{U}(X)$, so $X \in \Gamma$. As $(U, H_U) \in \mathcal{U}(X)$ is maximal with respect to our ordering and $S \le X$, it follows from parts (3) and (4) of 2.3.7 that $X \in \Gamma^*$, Γ_*, when $H \in \Gamma^*$, Γ_*, respectively. Moreover the components of X are the components of H not in L_+, so by definition of L_+, X has no Bender components. Thus replacing (H, S, T, z) by $(X, S, T, z) \in \mathcal{T}$, and adjoining Q_+ to $B_1, \ldots, B_k$, we may assume $L_+ = 1$; that is, that H has no Bender components.

Let $H \in \Gamma_*$, Γ^*; it remains to show that there is a component K of H with $\langle K, S\rangle \in \Gamma_*$, Γ^*, respectively.

We first consider the case where $E(H) \neq 1$; thus there is a component K of H. As $L_+ = 1$, K is not a Bender group, and so K is described in one of cases (2)–(4) of 2.5.2. Set $K_0 := \langle K^S \rangle$ and $R_U := K_0 C_H(K_0) \cap Q_U$, and let R_0 denote the projection of R_U on K_0.

We now argue as in the proof of (3) using (!) to conclude that $N_{K_0}(Q_U) \in \mathcal{H}^e$ and $R_0 \leq Q_U$. Further $z \in Q_U$, so by the initial statement in 2.5.2, we conclude that $R_U \not\leq C_H(K_0)$. Therefore $R_0 \neq 1$. Indeed since $O_2(N_{K_0}(R_0)) \in \mathcal{U}_H(H_U, 2)$, $O_2(N_{K_0}(R_0)) \leq Q_U \cap K_0 \leq R_0$, so that $R_0 = O_2(N_{K_0}(R_0))$. From the description of K in cases (2)–(4) of 2.5.2, $N_{K_0}(R_0) \in \mathcal{H}^e$. Thus if $N_{K_0}(R_0) \not\leq M$, we can argue as in case (ii) that (4) holds.

Therefore we may assume that $N_{K_0}(R_0) \leq M$. It follows that $O_2(M \cap K_0) \leq O_2(N_{K_0}(R_0)) = R_0$. Now from the description of K and $M \cap K$ in cases (2)–(4) of 2.5.2, either $O_2(M \cap K_0) \in Syl_2(K_0)$, or case (3) holds with $K = K_0 \cong M_{11}$ or $L_3(3)$ and $O_2(M \cap K_0) = C_K(z)$ is of index 2 in a Sylow 2-group of K. Hence $R_0 = O_2(M \cap K_0)$, and either $R_0 = S \cap K_0 \in Syl_2(K_0)$, or case (3) holds and $R_0 = O_2(M \cap K_0) = O_2(C_{K_0}(z))$. In any case $R_0 \trianglelefteq S$ and $H_U \leq N_H(Q_U) \leq N_H(R_0)$. Further $R_0^H = R_0^{K_0}$, either by Sylow's Theorem or as M_{11} and $L_3(3)$ have one class of involutions. Therefore by a Frattini Argument, $H = K_0 X_0$, where $X_0 := N_H(R_0)$. Now $(U, H_U) \in \mathcal{U}(X_0)$, so that $X_0 \in \Gamma$, and as usual $X_0 \in \Gamma^*$, Γ_*, when $H \in \Gamma^*$, Γ_*, respectively. Now $(X_0, S, T, z) \in \mathcal{T}$ and adjoining R_0 to $B_1, \ldots, B_k$, X_0 satisfies the hypotheses for H, so we conclude (4) holds by induction on the number of components of H.

We have reduced to the case where $E(H) = 1$, where to complete the proof we derive a contradiction.

As $F^*(H) \neq O_2(H)$ and $E(H) = 1$, $Y := O(H) \neq 1$. By 2.3.9.5, z inverts Y, so Y is abelian. By (!) and (1), $O_2(N_H(Q_U)) = Q_U$ and $N_H(Q_U) \in \mathcal{H}^e$. Then by our maximal choice of (U, H_U), $N_H(Q_U) = H_U$ and $U \in Syl_2(H_U)$ so $Q_U \leq U$. Then as $U \leq S$, $z \in Z(S) \leq C_H(Q_U) \leq C_{N_H(Q_U)}(Q_U) = Z(Q_U)$.

As $E(H) = 1$, $F^*(H) = F(H) = O_2(H)Y$. Further $O_2(H) \leq S \leq C_H(z)$, so $[z, H] \leq C_H(O_2(H))$, while as z inverts Y, $[z, H] \leq C_H(Y)$, and Y is abelian, so

$$[z, H] \leq C_H(F^*(H)) = Z(F^*(H)) = Z(O_2(H))Y.$$

Hence setting $O_2(H)\langle z \rangle =: D$, $DY \trianglelefteq H$, so by a Frattini Argument, $H = Y N_H(D)$. As $z \in D$, $D \in \mathcal{S}_2^e(G)$ by 1.1.4.3, so $N_G(D) \leq M$ by 2.5.1.1. Now $O_2(H) \leq Q_U$ by (!), and $z \in Z(Q_U)$ by the previous paragraph, so $D \leq Q_U$. Hence $D = Q_U \cap DY \trianglelefteq H_U$, so that $H_U \leq N_G(D) \leq M$, contradicting $H_U \not\leq M$. Therefore (4) is finally established, completing the proof of 2.5.3. □

In view of 2.5.3.4, we are led to define Γ^+ to consist of those $H \in \Gamma_0$ such that $H = \langle K, S \rangle$, for some component K of H and $S \in Syl_2(H \cap M)$, such that K is not a Bender group.

We verify that Γ^+ is nonempty: For given any $(H_0, S, T, z) \in \mathcal{T}$, we conclude from 2.3.9.1 that $H_1 := N_G(O_2(H_0)) \in \Gamma_0$, $S \in Syl_2(H_1)$, and if $H_0 \in \Gamma^*$, then also $H_1 \in \Gamma^*$. Now applying 2.5.3.4 to the 2-local H_1, we obtain a component K of H_1 such that K is not a Bender group, $H_2 := \langle K, S \rangle \in \Gamma_0$, and $H_2 \in \Gamma^*$ if $H_0 \in \Gamma^*$. Thus $H_2 \in \Gamma^+$, so Γ^+ is nonempty, and since we saw in section 1 that Γ^* is nonempty, also $\Gamma^+ \cap \Gamma^*$ is nonempty.

NOTATION 2.5.4. Let $\mathcal{T}^+$ consist of the tuples (H, S, T, z) in $\mathcal{T}$ such that $H \in \Gamma^+$. In the remainder of the section we pick $(H, S, T, z) \in \mathcal{T}^+$ and let $K \in \mathcal{C}(H)$ and

$K_0 := \langle K^S \rangle$. Set $S_K := S \cap K$, $S_{K_0} := S \cap K_0$, $S_C := C_S(K_0)$, and $\bar{H} := H/S_C$. Let $x \in N_T(S) - S$ with $x^2 \in S$.

As $H \in \Gamma^+$, K_0 is the product of at most two conjugates of the component K of H, and $H = K_0S$. Further K is not a Bender group, and $S \in Syl_2(H)$, so $S_K \in Syl_2(K)$, $S_{K_0} \in Syl_2(K_0)$, and $S_C = O_2(H) \in Syl_2(C_H(K_0))$. As $H \in \Gamma \subseteq \mathcal{H}$, $1 \neq S_C$. By 2.5.2, z induces an inner automorphism on K with $K = [K, z]$. Thus $z \in K_0S_C - S_C$, so z has nontrivial projection in $Z(S_K)$ and in $Z(S_{K_0})$.

We begin to generate information about S_C:

LEMMA 2.5.5. *(1) $S_C \cap S_C^x = 1$, so $S_C^x \cong S_C$ is isomorphic to a subgroup of $\bar{S}$.*
(2) $S_C S_C^x = S_C \times S_C^x$, so in particular $S_C^x \leq C_S(S_C)$.

PROOF. Recall $S_C = O_2(H) \trianglelefteq S$. Then as x normalizes S, S_C^x is also normal in S. As $x^2 \in S$, $S_0 := S_C \cap S_C^x \trianglelefteq S_1 := S\langle x \rangle$, and $S_0 \leq S_C$, so $S_0 \trianglelefteq K_0S = H$. Thus if $S_0 \neq 1$, then by 2.3.7.2, $N_G(S_0) \in \Gamma_0$ and $S \in Syl_2(N_G(S_0))$. This is a contradiction since $S < S_1 \leq N_G(S_0)$. So $S_0 = 1$, and hence (1) holds. Then as both S_C and S_C^x are normal in S, (1) implies (2). □

LEMMA 2.5.6. *If $1 \neq E \leq S_C$ with $E \trianglelefteq S$, then $G_E := N_G(E) \in \Gamma_0$, $S \in Syl_2(G_E)$, and $G_E \in \Gamma^*$ if $H \in \Gamma^*$. Further either*
(1) K is a component of G_E, or
(2) $K = K_0 \cong A_6$, $H/S_C \cong M_{10}$, and $K_E := \langle K^{G_E} \rangle \cong M_{11}$.

PROOF. As $E \leq S_C$ and $E \trianglelefteq S$, $H = K_0S \leq G_E$. Thus by parts (2) and (4) of 2.3.7, $G_E \in \mathcal{H}(H) \subseteq \Gamma_0$, $S \in Syl_2(G_E)$, and $G_E \in \Gamma^*$ if $H \in \Gamma^*$. Next by 1.2.4, $K \leq K_E \in \mathcal{C}(G_E)$. Then by 2.3.7.2, $\langle K_E, S \rangle \in \Gamma_0$, and $\langle K_E, S \rangle \notin \mathcal{H}^e$ by our assumption in this section that $\Gamma_0^e = \emptyset$. As $m_2(K_E/O_2(K_E)) \geq m_2(K) > 1$, $K_E/O_2(K_E)$ is quasisimple by 1.2.1.4. So as $\langle K_E, S \rangle \notin \mathcal{H}^e$, K_E is a component of G_E. Then K_E is described in 2.5.2, K is described in one of cases (2)–(4) of 2.5.2, and if $K < K_E$, then the embedding of K in K_E is described in A.3.12. We conclude that the lemma holds. □

We next show that K is essentially defined over $\mathbf{F}_2$:

LEMMA 2.5.7. *If $K/O_2(K) \cong L_3(2^n)$ or $Sp_4(2^n)$, then $n = 1$.*

PROOF. Assume that $n > 1$ and set $B := K \cap M$. By 1.2.1.3, $K_0 = K$, so that $H = KS$. By 2.5.2, some element s in S is nontrivial on the Dynkin diagram of $K/O_2(K)$ and B is a Borel subgroup of K. Let K_1 be a maximal parabolic of K over B, set $L_1 := K_1^\infty$ and $V := O_2(L_1)$.

We first observe that as case (2) of 2.5.2 holds, either $Z(K) = 1$, or $Z(K) = O_2(K)$ with $K/Z(K) \cong L_3(4)$. In the latter case, $\Phi\big(Z(K)\big) = 1$: for otherwise from the structure of the covering group in I.2.2.3a, $Z(S) \leq C_S(K) = S_C$; and as $x \in N_T(S)$, this is contrary to 2.5.5.1. By this observation and the structure of the covering group in I.2.2.3b when $Z(K) \neq 1$, in each case $\Phi(V) = 1$ and $V/C_V(L_1)$ is the natural module for $L_1/V \cong L_2(2^n)$.

Recall from Notation 2.5.4 that $S_K = S \cap K$ and $S_K \in Syl_2(K)$. Set $R := J(S)$ and $R_C := S_C \cap R = C_R(K)$. Observe since s is nontrivial on the Dynkin diagram of $K/O_2(K)$ that $S_K = VV^s$ and $\mathcal{A}(S_K) = \{V, V^s\}$ are the maximal elementary abelian subgroups of S_K.

We claim that $R = S_K R_C$: For let $A \in \mathcal{A}(S)$. Suppose first that $A \leq N_S(L_1)$. As $V/C_V(L_1)$ is the natural module for $L_1/V \cong L_2(2^n)$, either A centralizes V, or by B.2.7 and B.4.2.1, $Aut_A(V)$ is Sylow in $Aut_{AL_1}(V)$. In the former case $V \leq A$ since $A \in \mathcal{A}(S)$, so as V is self-centralizing in $Aut(K)$, $A = VC_A(K)$, where $C_A(K) \leq R_C$. In the latter case A induces an elementary abelian group of inner automorphisms on K not centralizing V, and hence A centralizes V^s, so by symmetry between V and V^s, $A = V^s C_A(K)$. Thus the claim holds if $R \leq N_S(L_1)$, so we may assume there is $a \in A - N_S(L_1)$. Then $m_2(C_{K/Z(K)}(a)) = n$, so $m(C_A(K)) \geq m(A) - (n+1)$. Hence as $A \in \mathcal{A}(S)$, and $n > 1$ by hypothesis, we conclude that

$$m(A) \geq m\big(VC_A(K)\big) \geq 2n + m\big(C_A(K)\big) \geq m(A) + n - 1 > m(A),$$

since we are assuming that $n > 1$. This contradiction completes the proof of the claim.

Next suppose that $\Phi(R_C) = 1$. Set $Q := O_2(L_1 S_C) = VS_C$. By the claim, $Q_R := Q \cap R = V(S_C \cap R) = VR_C$. Then $Q_R S_C = Q$ and $N_S(Q_R) = N_S(Q)$. Since $\mathcal{A}(S_K) = \{V, V^s\}$, and we are assuming that R_C is elementary abelian, $Q_R = VR_C \in \mathcal{A}(S)$, and $\mathcal{A}(S) = \{Q_R, Q_R^s\}$ is of order 2. Hence $|S : N_S(Q_R)| = 2$, and for $T_Q := N_T(S) \cap N_T(Q_R)$, $N_T(S) = T_Q\langle s\rangle$. Also $|T_Q| \geq |S|$, since $S < N_T(S)$ because $S < T$. As $RS_C = S_K S_C \leq L_1 S_C$ normalizes $O_2(L_1 S_C) = Q$, $RS_C \leq N_S(Q) = N_S(Q_R)$. Thus we have shown that $|S : N_S(Q_R)| = 2$ and both $J(S) = R$ and $S_C = O_2(H)$ lie in $N_S(Q_R)$. Also $C_S\big(N_S(Q_R)\big) \leq C_S(Q_R) \leq N_S(Q_R)$. Therefore applying 2.3.9.8 to $N_S(Q_R)$ in the role of "R", we conclude that $N_S(Q_R) \in \beta$. So as $N_S(Q_R) \leq T_Q$, $T_Q \in \beta$ by 2.3.2.1. We saw earlier that $N_S(Q_R) = N_S(Q)$. Further $N_H(Q) = L_1 N_S(Q)$, so $Q = O_2\big(N_H(Q)\big)$ and $N_H(Q) \in \mathcal{H}^e$. Also $N_H(Q) \not\leq M$ since $K_0 \cap M$ is a Borel subgroup of K_0. Therefore $\big(N_S(Q), N_H(Q)\big) \in \mathcal{U}\big(N_G(Q)\big)$, and hence $N_G(Q) \in \Gamma$. Then by 2.3.8.2, $Q = O_2\big(N_H(Q)\big) \in \mathcal{S}_2^e(G)$, so $N_G(Q) \in \mathcal{H}^e$. Since we saw above that $T_Q \in \beta$, $\big(T_Q, N_G(Q)\big) \in \mathcal{U}\big(N_G(Q)\big)$. However $|T_Q| \geq |S| \geq |U_1|$ for each $U_1 \in \mathcal{U}$ by 2.3.6. Hence by the maximality of $|U|$ and/or $|S|$ in the definitions of $H \in \Gamma^*$ or Γ_* in Notation 2.3.5, $N_G(Q) \in \Gamma_0$, and therefore $N_G(Q) \in \Gamma_0^e$, contrary to our assumption in this section that $\Gamma_0^e = \emptyset$. This contradiction shows that $\Phi(R_C) \neq 1$.

By 2.5.5.1, $R_C^x \cap S_C = 1$, while $R_C^x \leq R^x = R$; so as $R = S_K R_C$, R_C^x is isomorphic to a subgroup of $S_K/Z(K)$. Indeed we further claim that the members of $\mathcal{A}(S)$ are of the form $A_C \times A_K$ with $A_X \in \mathcal{A}(R_X)$ for each $X \in \{C, K\}$: If $Z(K) = 1$, then $R = R_C \times S_K$, so the second claim is clear in this case. Otherwise $K/O_2(K) \cong L_3(4)$, and as $\Phi\big(Z(K)\big) = 1$, from the structure of the covering group K in I.2.2.3b, each elementary subgroup of $S_K/Z(K)$ lifts to an elementary subgroup of S_K, completing the proof of the second claim.

Hence as $\Phi(R_C) \neq 1$ and R_C is isomorphic to a subgroup of $S_K/Z(K)$, which has exactly two maximal elementary subgroups $V/Z(K)$ and $V^s/Z(K)$, we conclude that $\mathcal{A}(S_C) = \{A_1, A_2\}$, where A_1 and A_2 are the two maximal elementary abelian subgroups of R_C.

Now suppose that $[V, V^x] \neq 1$. Then as $V \leq A \in \mathcal{A}(S)$, $m\big(V^x/C_{V^x}(V)\big) = n = m\big(R/C_R(V)\big)$. Similarly $m\big(V/C_V(V^x)\big) = m\big(R/C_R(V^x)\big)$, so $R = VV^x C_R(VV^x)$ with $\Phi\big(C_R(VV^x)\big) \leq R_C$, and as V and V^x are normal in R, $[V, V^x] = V \cap V^x = C_{V^x}(V) = V^x \cap Z(R)$. By symmetry, $\Phi(C_R(VV^x)) \leq R_C^x$, so $\Phi\big(C_R(VV^x)\big) = 1$ by 2.5.5.1. Further for $v \in V - Z(R)$, $m([v, R]) = n$ and $[v, R] \cap R_C = 1$; so for $u \in V^s - Z(R)$, $m([u, R]) = n$ and $[u, R] \cap R_C = 1$. Now for $w \in V^x - Z(R)$, since $R = S_K R_C$, $w = uc$ for some $u \in V^s - Z(R)$ and $c \in R_C$, so $[V, w] = [V, u]$

is of rank n, and hence $[R, w] = [V, u]$ and $[R, w] \cap R_C = [V, u] \cap R_C = 1$. Thus $[R_C, w] = 1$, so $[R_C, V^x] = 1$, and hence $\Phi(R_C) \leq \Phi(C_R(VV^x)) = 1$, contrary to an earlier reduction. This contradiction shows that $V^x \leq C_R(V) = VR_C$, and hence $V^x \leq VA_i$ for $i = 1$ or 2 using the second claim.

Next suppose that x normalizes $N_S(V)$. Set $I := \Omega_1\big(Z(T)\big)VV^x$. Then $I \trianglelefteq N_S(V) = N_S(V)^x$ using our assumption. Further as $J(S) = R = S_K R_C$, $\Omega_1\big(Z(T)\big) \leq VR_C$. Therefore as $V^x \leq VR_C$, $I \leq VR_C$ with $[VR_C, L_1] = V \leq I$, and hence

$$I \trianglelefteq L_1 N_S(V).$$

Also arguing as above using 2.3.9.8, $N_S(V) \in \beta$. As $\Omega_1\big(Z(T)\big) \leq I$, $I \in \mathcal{S}_2^e(G)$ by 1.1.4.3. Hence as $N_G(I)$ contains $L_1 \not\leq M$, $\big(N_S(V), N_G(I)\big) \in \mathcal{U}\big(N_G(I)\big)$ and thus $N_G(I) \in \Gamma^e$. However $S_1 := \langle N_S(V), x\rangle \leq N_T(I)$ with $|S_1| = |S|$, so again by 2.3.6, $|S_1| \geq |U_1|$ for each $U_1 \in \mathcal{U}$. Hence again from the maximality of $|U|$ and/or $|S|$ in the definitions of $H \in \Gamma^*$ or Γ_* in Notation 2.3.5, $N_G(I) \in \Gamma_0$. Then $N_G(I) \in \Gamma_0^e$, contrary to our assumption in this section that $\Gamma_0^e = \emptyset$.

Therefore x does not normalize $N_S(V)$. Set $W := N_S(V) \cap N_S(V)^x$ and $T_W := S\langle x\rangle$. As $|S : N_S(V)| = 2$ and $N_S(V) \neq N_S(V^x)$, $S/W \cong E_4$, $T_W/W \cong D_8$, and we can choose x with $s := x^2 \in S - N_S(V)$. Thus $(V^x, V^{x^{-1}}) = (V, V^s)^x$. Hence setting $D := [V, V^s]$, $D^x = [V^x, V^{x^{-1}}]$. We showed $[V, V^x] = 1$, and by symmetry between x and x^{-1}, $V^{x^{-1}}$ also centralizes V, so $\langle V^x, V^{x^{-1}}\rangle$ centralizes V. Thus conjugating by s,

$$\langle V^x, V^{x^{-1}}\rangle \leq C_S(VV^s) = R_C D.$$

Therefore $D^x \leq \Phi(R_C D) \leq R_C$. Also $D^x \trianglelefteq S$, so as K is not A_6 since $n > 1$, $K \trianglelefteq N_G(D^x)$ by 2.5.6.

Let p be a prime divisor of $2^n - 1$, and for $J \leq G$, let $\theta(J) := O^{p'}(J)$ if $p > 3$, and $\theta(J) := \langle j \in J : |j| = 3\rangle$ if $p = 3$. By A.3.18, either $K = \theta(N_G(D^x))$, or $p = 3$ and $\theta\big(N_G(D^x)\big) / O_{3'}\big(\theta(N_G(D^x))\big) \cong PGL_3(2^n)$. Thus, except possibly in the exceptional case, as $x^2 \in N_S(D)$ and $D^x \leq R_C$, we have $\theta(N_K(D)) \leq K^x \leq C_G(D)$, impossible as $[D, \theta\big(N_K(D)\big)] \neq 1$. Thus $K/Z(K) \cong L_3(2^n)$; 3 is the only prime divisor of $2^n - 1$, so that $n = 2$; and $K/Z(K) \cong L_3(4)$ and a subgroup X of order 3 in $N_K(D)$ induces outer automorphisms on K^x. Now $X \leq Y \in Syl_3\big(N_G(D) \cap N_G(D^x) \cap N_G(R)\big)$ with $Y = X(Y \cap K^x) \cong E_9$. By a Frattini Argument, we may assume x acts on Y. Now $R_C = C_R(X)$ from the structure of K, so as $R_C \cap R_C^x = 1$ by 2.5.5.1, $R_C^x = [R_C^x, X] \leq K$. Now Y normalizes R and K^x, so Y normalizes R_C^x; then as R_C^x is not elementary abelian, $R_C^x = S_K$. This is impossible, as X^x centralizes R_C^x, but is faithful on S_K. This contradiction completes the proof of 2.5.7. □

As a consequence of 2.5.7, the groups remaining in cases (2)–(4) of 2.5.2 have the following common features:

LEMMA 2.5.8. *(1) $Out(K)$ is a 2-group.*

(2) K is simple so $K_0 S_C = K_0 \times S_C$.

(3) Either S_K is dihedral of order at least 8 or S_K semidihedral of order 16.

(4) $Z(S) = (Z(S) \cap S_{K_0}) \times (Z(S) \cap S_C)$ and $Z(S) \cap S_{K_0} = \langle z_K\rangle$ is of order 2, where z_K is the projection on K_0 of z.

(5) For each 4-subgroup F of K, $N_K(F) \cong S_4$; and furthermore if $F \leq S_K$, then $C_{Aut_H(K)}(F) \leq Aut_S(K)$.

PROOF. First either K appears in case (3) or (4) of 2.5.2, or by 2.5.7, K appears in case (2) with $n = 1$. Now (1)–(3) and (5) follow by examination of those groups. Then $Z(S_K)$ is of order 2 by (3), so $Z(S) \cap K_0$ is of order 2. By 2.5.2, z induces a nontrivial inner automorphism on K_0, so $Z(S) \cap K_0 = \langle z_K \rangle$. Further $Z(\bar{S}) = Z(\bar{S}_{K_0})$, since S is nontrivial on the Dynkin diagram when $K = K_0 \cong A_6$ by 2.5.7. Then (2) completes the proof of (4). □

Just before establishing Notation 2.5.4, we verified that $\Gamma^* \cap \Gamma^+ \neq \emptyset$, and hence there is a member of $\mathcal{T}^+$ with first entry in this set. We now take advantage of this flexibility:

NOTATION 2.5.9. In the remainder of the section, we choose $(H, S, T, z) \in \mathcal{T}^+$ with $H \in \Gamma^*$. Let $\mathcal{U}^*(H)$ denote the pairs $(U, H_U) \in \mathcal{U}(H)$ with U of maximal order in $\mathcal{U}$. By definition of $\Gamma^*, \mathcal{U}^*(H) \neq \emptyset$.

LEMMA 2.5.10. *(1) If $(U, H_U) \in \mathcal{U}(H)$, then $N_G(O_2(H_U)) \in \mathcal{H}^e$.*
(2) If $(U, H_U) \in \mathcal{U}^*(H)$, then $U \in Syl_2(N_G(O_2(H_U)))$, so $U \in Syl_2(H_U)$. If also $U \leq S$ then $z \in Z(S) \leq Z(U)$.

PROOF. By 2.3.8.2, $N := N_G(O_2(H_U)) \in \mathcal{H}^e$, establishing (1) and showing $(U, N) \in \mathcal{U}(N)$. Then if $(U, H_U) \in \mathcal{U}^*(H)$, U is Sylow in H_U and N by 2.3.2.2 and maximality of $|U|$, so the first statement in (2) holds. Finally if $U \leq S$, then as $U \in Syl_2(N)$, $O_2(N) \leq U = S \cap N$, and so using (1) we conclude

$$z \in Z(S) \leq C_H(U) \leq C_H(O_2(N)) \leq O_2(N) \leq U,$$

completing the proof of (2). □

LEMMA 2.5.11. *(1) $Z(T) = \langle z \rangle$ is of order 2 and $Z(S) = \langle t, z \rangle = \langle t, t^x \rangle = \langle t, z_K \rangle \cong E_4$, where t is an involution in S_C and z_K is the projection of z on K_0.*
(2) $H = K_0 S \leq C_G(t) \in \Gamma^*$, with $S \in Syl_2(C_G(t))$. In particular, $t \notin z^G$.

PROOF. By 2.5.8.4, $Z(S) = \langle z_K \rangle \times Z_{S,C}$, where $Z_{S,C} := Z(S) \cap S_C$, and z_K is the projection on S_{K_0} of z. In the discussion following Notation 2.5.4 we observed $1 \neq O_2(H) = S_C$, so $Z_{S,C} \neq 1$. Then as $Z_{S,C}$ is of index 2 in $Z(S)$ while $Z_{S,C} \cap Z_{S,C}^x = 1$, we conclude from 2.5.5.1 that $\langle t \rangle := Z_{S,C}$ is of order 2 and $Z(S) = \langle t, t^x \rangle$. Now (2) follows from 2.5.6.1. Finally as $1 \neq z \in Z(T) \leq Z(S)$ from the definition of $\mathcal{T}$, $Z(T) = \langle z \rangle$ is of order 2, completing the proof of (1). □

For the remainder of the section, let t be defined as in 2.5.11, and set $G_t := C_G(t)$.

LEMMA 2.5.12. *Assume $K \trianglelefteq H$, and let $(U, H_U) \in \mathcal{U}^*(H)$ with $U \leq S$. Then*
(1) $H_U = N_H(E)$ and $U = N_S(E)$ for some 4-subgroup E of S_K.
(2) $O^2(H_U) \cong A_4$ and $E = O_2(O^2(H_U)) = C_K(E)$.
(3) The map $E \mapsto (N_S(E), N_H(E))$ is a bijection of the set of 4-subgroups of S_K with

$$\{(U', H_{U'}) \in \mathcal{U}^*(H) : U' \leq S\}.$$

In particular, $N_S(E) \in Syl_2(N_H(E))$.
(4) If Q_E is a 2-group with $z \in Q_E \trianglelefteq H_U$, then $N_G(Q_E) \in \Gamma$ and $U \in Syl_2(N_G(Q_E))$.

PROOF. By Notation 2.5.4, $H \in \Gamma^+$ so that $H = K_0S$ with K a component of H, $K_0 = \langle K^H \rangle$, and $K/O_2(K)$ is not a Bender group. Thus as $K \trianglelefteq H$ by hypothesis, $H = KS$ and $K = O^2(H)$. Further by 2.5.8.5, for each 4-subgroup F of K, $N_K(F) \cong S_4$, and if $F \leq S_K$ then $C_{Aut_H(K)}(F) \leq Aut_S(K)$. It follows as $H = KS$ with $S \in Syl_2(H)$ that if $F \leq S_K$ then $N_H(F) = N_K(F)C_S(F)$, and in particular $N_S(F) \in Syl_2(N_H(F))$.

Next as $(U, H_U) \in \mathcal{U}^*(H)$ by hypothesis, $U \in Syl_2(H_U)$ by 2.5.10.2. Hence $H_U = O^2(H_U)U \in \mathcal{H}^e$ with $O^2(H_U) \leq O^2(H) = K$. Set $E := \langle z_K^{H_U} \rangle$. Now $z_K \in Z(S) \leq Z(U)$ by 2.5.10.3, so by B.2.14, $E \leq O_2(H_U)$ and E is elementary abelian. In particular, $E \leq U$ as $U \in Syl_2(H_U)$. As $H_U \leq G_t$ by 2.5.11.2 and $H_U \not\leq M$ but $C_G(z) \leq M = !\mathcal{M}(T)$, we conclude $m(E) > 1$. Then as $O^2(H) \leq K$ and $m_2(K) = 2$, $E \cong E_4$. Now $H_U \leq N_H(E)$, and we saw in the previous paragraph that $N_H(E) = N_K(E)C_S(E)$, with $N_K(E) \cong S_4$ and $N_S(E) \in Syl_2(N_H(E))$. Since $H_U \not\leq M$, $A_4 \cong O^2(N_K(E)) = O^2(H_U)$ and $E = O_2(O^2(H_U))$, so that (2) holds. Further $N_H(E) \in \mathcal{H}^e$ and $U \leq N_S(E)$ so that $N_S(E) \in \beta$ by 2.3.2.1. Therefore $(N_S(E), N_H(E)) \in \mathcal{U}(H)$ and $N_S(E) \in \mathcal{U}$, so as $(U, H_U) \in \mathcal{U}^*(H)$, we conclude $N_S(E) = U \in Syl_2(H_U)$, and hence $N_H(E) = O^2(H_U)N_S(E) = H_U$. This completes the proof of (1). Further (3) follows from (1) since we saw that $N_S(E) \in Syl_2(N_H(E))$.

Now assume that $z \in Q_E \trianglelefteq H_U$ with Q_E a 2-group. Then as $z \in Q_E$, $N_G(Q_E) \in \mathcal{H}^e$ by 1.1.4.3. So as $U \in \mathcal{U}$, and $H_U \leq N_G(Q_E)$ with $H_U \not\leq M$, $(U, N_G(Q_E)) \in \mathcal{U}(N_G(Q_E))$ and $N_G(Q_E) \in \Gamma$. Then since $(U, H_U) \in \mathcal{U}^*(H)$ by hypothesis, we conclude $U \in Syl_2(N_G(Q_E))$ using 2.3.2.1. This completes the proof of (4), and hence of 2.5.12. □

LEMMA 2.5.13. *(1)* $|N_T(S) : S| = 2$, *and* $t^x = tz$ *for each* $x \in N_T(S) - S$.
(2) If $\langle z_K \rangle$ *char* S, *or more generally if* $[x, z_K] = 1$, *then* $z = z_K$ *and* $t^x = tz_K$.
(3) If $tz_K \in t^G$, *then* $z = z_K$ *and* $t^x = tz_K$.

PROOF. By 2.5.11.1, $Z(S) = \langle z, t \rangle \cong E_4$ with $\langle z \rangle = Z(T)$. By 2.5.11.2, $S \in Syl_2(G_t)$ and hence $S = C_T(t)$, so (1) follows. Then (1) implies (2). Further $z \notin t^G$ by 2.5.11.2, so (1) also implies (3). □

REMARK 2.5.14. There are extensions of $L_4(3) \cong P\Omega_6^+(3)$ by a 2-group, with involution centralizer $\mathbf{Z}_2 \times L_3(3)$ or $\mathbf{Z}_2 \times Aut(L_3(3))$, which are of even characteristic, and in which a Sylow 2-group is contained in a unique maximal subgroup. The first extension is even quasithin. The next lemma eliminates the shadows of such extensions.

LEMMA 2.5.15. *K is not M_{11} or $L_3(3)$.*

PROOF. Assume otherwise. Then case (3) of 2.5.2 holds, and $K = K_0 \trianglelefteq H$ by 1.2.1.3. As z induces inner automorphisms on K, $K_z := O^2(C_K(z)) \cong SL_2(3)$ from the structure of K.

By 2.5.11.2, $H = KS \leq G_t$, so by 2.5.6, $K \trianglelefteq G_t$. Then $K = O^{3'}(G_t)$ by A.3.18. By 2.5.11.1, $Z_S := Z(S) = \langle z, t \rangle \cong E_4$. Then as $K = O^{3'}(G_t)$, $K_z = O^{3'}(C_G(Z_S))$, so x acts on $Z(K_z) = \langle z_K \rangle$. Hence by 2.5.13.2, $z = z_K \in K$ and $t^x = tz$.

Next as $Aut(K_z)$ is induced in K_zS, we may choose $x \in C_T(K_z)$. Furthermore as $\langle z \rangle = C_K(K_z)$, $M_{11} = Aut(M_{11})$, and $|Aut(L_3(3) : L_3(3)| = 2$ with $C_{Aut(K)}(K_z) \cong \mathbf{Z}_4$ if $K \cong L_3(3)$, either:

(i) S induces inner automorphisms on K, and $C_S(K_z) = S_C \times \langle z \rangle$, or

(ii) $\bar{H} \cong Aut(L_3(3))$ and $C_S(K_z) = S_C\langle y\rangle$, where y induces an outer automorphism on K with $\bar{y}^2 = \bar{z}$.

Recall from Notation 2.5.9 that we may choose $(U, H_U) \in \mathcal{U}^*(H)$ with $U \leq S$. By 2.5.12.3, $H_U = N_H(E)$ for some 4-subgroup E of S_K and $U = N_S(E) \in Syl_2(H_U)$. Then as $O^2(H_U) \cong A_4$ by 2.5.12.2, $Q_E := O_2(H_U) = C_S(E)$. In case (i) S induces inner automorphisms on K, so $S = S_C \times S_K$, and hence as $E = C_K(E)$ by 2.5.12.2, $Q_E = S_C \times E$. On the other hand in case (ii), we compute that $e \in E - \langle z\rangle$ inverts y, so $Q_E = (S_C \times E)\langle f\rangle$, where $f = yk$ and k is one of the two elements of $O_2(K_z)$ of order 4 inverted by e.

Recall $x \in N_T(S) \cap C_T(K_z)$, so x normalizes $C_S(K_z)$, and hence

$$[e, x] \in S \cap C_T(K_z) = C_S(K_z). \qquad (*)$$

But if case (i) holds then $C_S(K_z) = S_C\langle z\rangle \leq Q_E$, and by the previous paragraph $S_C E = Q_E$, so $x \in N_G(Q_E)$. Then $U < N_S(Q_E)\langle x\rangle \leq N_G(Q_E)$, contradicting 2.5.12.4.

Therefore case (ii) holds. Here x normalizes $C_S(K_z) = S_C\langle y\rangle$, while $S_C \cap S_C^x = 1$ by 2.5.5.1, so as $t \in S_C$, S_C is cyclic of order 2 or 4.

Assume $S_C \cong \mathbf{Z}_4$. Then by 2.5.5.2, $S_C S_C^x = S_C \times S_C^x$, so as $\bar{y}$ and S_C are of order 4, $C_S(K_z) = S_C \times S_C^x$ is abelian. In particular y centralizes S_C, so since $S = S_C S_K\langle y\rangle$, $Z(S)$ contains $S_C \cong \mathbf{Z}_4$, contrary to 2.5.11.1.

Therefore $S_C = \langle t\rangle$, so $C_S(K_z) = \langle t, y\rangle$, and as $\bar{y}^2 = \bar{z}$, $y^2 = z$ or tz. Hence as we saw $t^x = tz$, while x normalizes $\Phi(S_C\langle y\rangle) = \langle y^2\rangle$, $y^2 = z$. Therefore as $H = KS$,

$$H = \langle t\rangle \times A,$$

where $A := K\langle y\rangle \cong Aut(L_3(3))$. Observe that $S_C\langle z\rangle = \langle t, z\rangle = Z(S)$ using 2.5.11.1.

Assume that $[e, x] \in \langle t, z\rangle$. Then as x acts on $Z(S) = \langle z\rangle$, x acts on $S_C E \trianglelefteq H_U$, so that $N_S(E) < N_S(E)\langle x\rangle \leq N_G(S_C E)$, again contrary to 2.5.12.4. Therefore $[e, x] \notin \langle t, z\rangle$.

Next A is transitive on involutions in $A - K$, and on E_8-subgroups of A, with representatives f and $F := \langle f, E\rangle$, respectively. Further we may choose notation so that $C_A(f) = \langle f\rangle \times C_K(f)$ with $C_K(f) = N_K(E) \cong S_4$. Now x acts on $C_S(K_z) = \langle t, y\rangle$, and we've seen that $[e, x] \in C_S(K_z) - \langle t, z\rangle$, so replacing y by a suitable element of $y\langle t, z\rangle$, we may take $e^x = ey$. Thus $ey \in A - K$ is an involution in $e^G = z^G$, so all involutions in $F^\#$ are in z^G. On the other hand, we saw that $tz = t^x \in t^G$, so all involutions in tK are in t^G, and in particular $te \in t^G$. Further

$$(te)^x = t^x e^x = tzey = tey^{-1},$$

with ey^{-1} an involution in $A - K$; so all involutions in $H - A$ are in t^G.

As F^A is the set of E_8-subgroups of A, and $Q_E = O_2(N_H(E)) = \langle t\rangle \times F$, Q_E^H is the set of E_{16}-subgroups of H. By 2.5.11.2, $G_t \in \Gamma^*$ and $S \in Syl_2(G_t)$. So $\langle t\rangle$ is Sylow in $C_{G_t}(K)$, and hence using Cyclic Sylow-2 Subgroups A.1.38 we conclude that $C_{G_t}(K) = O(G_t)\langle t\rangle$. We saw that $K \trianglelefteq G_t$ so $z \in K \leq C(O(G_t))$. Thus $O(G_t) = 1$ since z inverts $O(G_t)$ by 2.3.9.5. Hence $G_t = KS = H$. Therefore $C_G(t) = H$ is transitive on its E_{16}-subgroups with representative Q_E, so by A.1.7.1, $N_G(Q_E)$ is transitive on $t^G \cap Q_E = Q_E - F$ of order 8. Then $|N_G(Q_E) : N_{G_t}(Q_E)| = 8$, whereas $N_S(E) \in Syl_2(N_G(Q_E))$ by 2.5.12.4, and $N_S(E) \leq G_t$. Hence the proof of 2.5.15 is at last complete. $\square$

Observe that by 2.5.7 and 2.5.15, we have reduced the list of possibilities for K in 2.5.2 to:

LEMMA 2.5.16. *One of the following holds:*
(1) $K \cong L_2(p)$, $p > 7$ *a Mersenne or Fermat prime.*
(2) $K \cong L_3(2)$ *and* $N_H(K)/C_S(K) \cong Aut(L_3(2))$.
(3) $K \cong A_6$ *and* $N_H(K)/C_S(K) \cong M_{10}$, $PGL_2(9)$, *or* $Aut(A_6)$.

REMARK 2.5.17. All of these configurations appear in some shadow which is of even characteristic, and in which a Sylow 2-group is in a unique maximal 2-local. Usually the shadow is even quasithin. The group is not simple, but it takes some effort to demonstrate this and hence produce a contradiction.

The groups $L_2(p) \times L_2(p)$ extended by a 2-group interchanging the components are shadows realizing the configurations in (1) and (2), while $L_4(3) \cong P\Omega_6^+(2)$ extended by a suitable group of outer automorphisms realize the configurations in (3). The last case causes the most difficulties, and consequently is not eliminated until the final reduction.

LEMMA 2.5.18. *(1)* K *is a component of* G_t.
(2) $G_t = K_0 S C_{G_t}(K_0)$ *with* $C_{G_t}(K_0)S \leq M$.
(3) $C_{G_t}(K_0) \in \mathcal{H}^e$, *so* $O(G_t) = 1$.

PROOF. By 2.5.11.3, $H \leq G_t \in \Gamma^*$ and $S \in Syl_2(G_t)$. Thus if K is not a component of G_t, we may apply 2.5.6 with $\langle t \rangle$ in the role of "E", to conclude that $K = K_0 \cong A_6$ and $K_t := \langle K^{G_t} \rangle \cong M_{11}$. Since $H \in \Gamma^+ \cap \Gamma^*$, we conclude from parts (2) and (4) of 2.3.7 that $K_t S \in \Gamma^+ \cap \Gamma^*$, contrary to 2.5.15.

Thus (1) holds, so as $S \in Syl_2(G_t)$, $K_0 \trianglelefteq G_t$, and by 2.5.8.1, $G_t = K_0 S C_{G_t}(K_0)$. Then $C_{G_t}(K_0) \leq C_{G_t}(z_K) \leq C_{G_t}(z) \leq M$, proving (2). By 2.3.9.4, $G_t \cap M \in \mathcal{H}^e$, so (2) implies (3). □

LEMMA 2.5.19. *Assume* i *is an involution in* $C_S(K)$ *such that* K *is not a component of* $C_G(i)$. *Then*
(1) $K = K_0$.
(2) $C_S(i) \cap C_S(K) = \langle t, i \rangle$.
(3) There exists a component K_i *of* $C_G(i)$ *such that either:*
(I) $K_i \neq K_i^t$, $K = C_{K_i K_i^t}(t)^\infty$, *and* $K_i \cong K \cong L_2(p)$, $p \geq 7$, *or*
(II) $K = C_{K_i}(t)^\infty$, *and one of the following holds:*
(a) $K \cong L_3(2)$, *and* t *induces a field automorphism on* $K_i \cong L_3(4)$ *or* $L_3(4)/\mathbf{Z}_2$.
(b) $K \cong L_3(2)$, *and* t *induces an outer automorphism on* $K_i \cong J_2$.
(c) $K \cong A_6$ *and* $K_i \cong Sp_4(4)$, $L_5(2)$, HS, *or* A_8.
(4) Either $z = z_K \in K$ *and* $tz \in t^G$, *or* $K_i \cong A_8$ *and* t *induces a transposition on* K_i.

PROOF. Let $G_i := C_G(i)$ and $R := G_i \cap C_S(K)$. As $t \in Z(S) \cap S_C$, $\langle t, i \rangle \leq R$ by our hypothesis on i. As K is not a component of G_i, $i \neq t$ by 2.5.18. Therefore $i \notin Z(S)$, or otherwise i centralizes $\langle K^S \rangle = K_0$, whereas $Z(S) \cap S_C = \langle t \rangle$ by 2.5.11.1. By 2.5.18, $C_{G_t}(K_0) \leq M$ and $S \in Syl_2(G_t)$, so conjugating in $C_{G_t}(K_0)$ we may assume $C_S(\langle i, K_0 \rangle) \in Syl_2(C_G(\langle t, i, K_0 \rangle))$.

Next K is a component of $C_{G_i}(t)$ in view of 2.5.18, so by I.3.2 there is $K_i \in \mathcal{C}(G_i)$ with $K_i/O(K_i)$ quasisimple, such that for $K_+ := \langle K^{O_{2',E}(G_i)} \rangle$, either

(i) $K_+ = K_iK_i^t$, $K_i \neq K_i^t$, $K_i/O_{2',2}(K_i) \cong K$, and $K = C_{K_+}(t)^\infty$, or

(ii) $K_+ = K_i = [K_i, t]$ and K is a component of $C_{K_i}(t)$.

Set $R_0 := C_R(K_+)$. In case (ii) as $K_i/O(K_i)$ is quasisimple, $O_2(K_i) \leq Z(K_i)$, so as $m_2(K_i) \geq m_2(K) > 1$, K_i is quasisimple by 1.2.1.5. Similarly if (i) holds, then $O(K_i) = 1$ by 1.2.1.3, so that K_i is quasisimple. Thus in any case K_i is a component of G_i.

Let $g \in G$ with $T_i := C_{T^g}(i) \in Syl_2(G_i)$; then applying 1.1.6 to the 2-local G_i, the hypotheses of 1.1.5 are satisfied with G_i, M^g, z^g in the roles of "H, M, z". Therefore K_i is described in 1.1.5.3.

Suppose for the moment that case (i) holds. Then by 1.2.1.3 applied to K_i, K is not A_6, so by 2.5.16, K is $L_2(p)$ for $p \geq 7$ a Fermat or Mersenne prime. Then as $K_i/Z(K_i) \cong K$ in (i), $K_i \cong K$ by 1.1.5.3. Therefore all involutions in tK_+ are conjugate, and hence $tz_K \in t^G$, so $z = z_K$ by 2.5.13.3 and hence $tz \in t^G$. Therefore conclusion (I) of (3) and the first alternative in (4) hold in case (i). Thus in case (i), it remains only to verify (1) and (2). Observe also in this case that $N_R(K_i)$ centralizes the full diagonal subgroup K of K_+, so $R_0 = N_R(K_i)$ and $R = \langle t\rangle \times R_0$.

Next suppose for the moment that case (ii) holds. Comparing the groups in 2.5.16 to the components of centralizers of involutions in $Aut(K_i/Z(K_i))$ for groups K_i on the list of 1.1.5.3, we conclude that one of the following holds:

(α) $K \cong L_3(2)$, and t induces a field automorphism on $K_i/Z(K_i) \cong L_3(4)$.

(β) $K \cong L_3(2)$, and t induces an outer automorphism on $K_i/Z(K_i) \cong J_2$.

(γ) $K \cong A_6$, and t induces one of: an inner automorphism on $K_i/Z(K_i) \cong HS$, an outer automorphism on $K_i \cong L_4(2)$ or $L_5(2)$, or a field automorphism on $K_i \cong Sp_4(4)$.

Thus to prove that conclusion (II) of (3) holds in case (ii), it remains to show that $|Z(K_i)| \leq 2$ if (α) holds, and to show that $Z(K_i) = 1$ when (β) holds, or when $K_i/Z(K_i) \cong HS$ and (γ) holds.

Notice also when (ii) holds that from the structure of $C_{Aut(K_i/Z(K_i))}(t)$ for the groups in (α)–(γ), either $R_0 = C_R(K_i)$ is of index 2 in R, or else $K_i/Z(K_i) \cong HS$—and in the latter case some $r \in R$ induces an outer automorphism on K_i, with $|R : R_0| = 4$, and $C_{K_i}(R_1)^\infty \cong A_8$ for some subgroup R_1 of index 2 in R.

In the next few paragraphs, we will reduce the proof of 2.5.19 to the proof of (2). So until that reduction is complete, suppose that (2) holds; that is that $R = \langle i, t\rangle \cong E_4$.

We first deduce (1) from (2), so suppose that (1) fails. Thus $K_0 = KK^u$ for some $u \in S - N_S(K)$. Therefore i also acts on K^u, and hence also on $S \cap K^u$, so that $|C_{\langle i\rangle(S\cap K^u)}(i)| > 2$. Since $S \cap K^u \leq C_S(K)$ and $t \notin \langle i\rangle(S \cap K^u)$ because t centralizes K_0, $|R| > 4$, contrary to assumption. This contradiction shows that (2) implies (1).

As remarked earlier, (1) and (2) suffice to prove the entire result when case (i) holds. Thus to complete the proof of the sufficiency of (2), we may now assume that case (ii) holds, and it remains to establish (3) and (4). Recall that at the start of the proof we chose $C_S(\langle i, K_0\rangle) \in Syl_2\big(C_G(\langle i, t, K_0\rangle)\big)$, so as $K = K_0$ by (1), $R \in Syl_2\big(C_G(\langle t, i, K\rangle)\big)$.

As $K \trianglelefteq H$, $N_S(C_S(i))$ acts on $C_S(i) \cap C_S(K) = R$. We saw $i \notin Z(S)$, so $C_S(i) < N_S(C_S(i))$. Then as $N_S(C_S(i))$ acts on $R = \langle i, t\rangle$, $it \in i^{N_S(C_S(i))}$. But by A.3.18, $K_i = O^{3'}(E(G_i))$, so $i \notin t^G$ by 2.5.18, and hence as $it \in i^G$, also $it \notin t^G$. As $K \leq K_i = [K_i, t]$ and $R \in Syl_2(C_G(\langle t, i, K\rangle))$, $\langle i\rangle = C_{O_2(K_iC_S(i))}(t)$.

Therefore if $\langle i\rangle < O_2(K_iC_S(i))$, then $it \in t^{O_2(K_iC_S(i))}$, contradicting $it \notin t^G$. Thus $\langle i\rangle = O_2(K_iC_S(i))$.

Suppose case (α) or (β) holds. If $O_2(K_i) = 1$ then (3) holds, and from the structure of $Aut(K_i)$, K_i is transitive on involutions in tK_i, so $tz_K \in t^G$, and hence $z = z_K$ by 2.5.13.3, establishing (4). Thus we may assume that $O_2(K_i) \neq 1$, so $\langle i\rangle = O_2(K_i)$ from the previous paragraph. If (β) holds, then from the embedding of $K_i/\langle i\rangle$ in $G_2(4)$, t acts faithfully on some root subgroup $Q/\langle i\rangle$, with $Q \cong Q_8$, so that $ti \in t^Q$, contrary to a remark in the previous paragraph. Thus (α) holds, with $K_i/\langle i\rangle \cong L_3(4)$, so (3) holds in this case. Further the field automorphism t normalizes each maximal parabolic P of K_i over $C_S(i) \cap K_i$. From the structure of the covering group in I.2.2.3b, $V := O_2(P)$ is an indecomposable P-module such that $V/\langle i\rangle$ is the natural module for $P/V \cong L_2(4)$. Now t centralizes X of order 3 in P, and $V = [V,X] \times \langle i\rangle$ with

$$C_{[V,X]}(t) = [V,X,t] \leq O^2(C_{K_i}(t)) = K.$$

It follows that $tz_K \in t^G$, so $z = z_K$ by 2.5.13.3, and hence (4) holds.

Thus it remains to consider the case where (γ) holds. If $K_i \cong A_8$, then the lemma holds, since there we do not assert that $tz_K \in t^G$. If $K_i \cong L_5(2)$ or $Sp_4(4)$, then K_i is transitive on involutions in tK_i, so that $tz_K \in t^G$, and hence $z = z_K$ by 2.5.13.3, so the lemma holds. Thus we have reduced to the case $K_i/O_2(K_i) \cong HS$. Assume first that $Z(K_i) \neq 1$. Then as before, $\langle i\rangle = Z(K_i) = O_2(K_iC_S(i))$, so as we are assuming $\langle t, i\rangle = R$ and t is inner on K_i in (γ), $t \in K_iC_{K_iC_S(i)}(K_i) = K_i$. Thus $t \in C_{K_i}(K)$ so t is not 2-central in K_i. However, an element of the covering group K_i projecting on a non-2-central involution of HS is of order 4 by I.2.2.5b. This contradiction shows that K_i is HS, so that (3) holds. Furthermore if u is the projection on K_i of t, then $uz_K \in u^{K_i}$ and $iuz_K \in (iu)^{K_i}$. Therefore as $t = u$ or iu, $tz_K \in t^{K_i}$, and again (4) follows from 2.5.13.3. This completes the proof of the reduction of the proof of the lemma to the proof of (2).

We have shown that it suffices to prove that $R = \langle i, t\rangle$. Thus we assume that $\langle i, t\rangle < R$, and derive a contradiction. Choose i so that $R = C_S(i) \cap C_S(K)$ is maximal subject to K not being a component of G_i. Further if $i \in Z(C_S(K))$ then $R = C_S(K)$, and we choose i so that $C_S(i)$ is maximal subject to the constraint that $R = C_S(K)$.

Recall we showed soon after stating (i) that that assumption implies $|R : R_0| = 2$. Inspecting the groups in cases (α)–(γ) of (ii), we check that either $|R : R_0| = 2$, or $K_i/Z(K_i) \cong HS$ and $|R : R_0| = 4$. When $|R : R_0| = 2$ we set $R_2 := R_0$, and when $|R : R_0| = 4$ we let R_2 be the subgroup R_1 of index 2 in R with $C_{K_i}(R_1)^\infty \cong A_8$ discussed earlier. Thus in either case, $i \in R_0 \leq R_2$ and $|R : R_2| = 2$.

We next claim that $K < K_0$ and $i \in Z(N_S(K))$. Thus we assume that at least one of the two assertions of the claim fails, and derive a contradiction. As $i \notin Z(S)$ there is $s \in N_S(C_S(i)) - C_S(i)$ with $s^2 \in C_S(i)$. Furthermore we observe when $K < K_0$ that $C_S(i)$ normalizes K: For otherwise i centralizes some $u \in C_S(i) - N_S(K)$ and $K_+ \neq K_+^u$. But in all cases appearing in (i) and (ii), $m_3(K_+) = 2$; therefore as K_+ and K_+^u are products of components of G_i, $m_3(K_+K_+^u) > 2$, impossible as G_i is an SQTK-group. Thus in any case, $C_S(i)$ normalizes K, and hence $C_S(i)$ normalizes $C_S(K)$ and $N_S(K)$.

During the remainder of the proof of the claim, we choose the element $s \in N_S(C_S(i)) - C_S(i)$ with $s^2 \in C_S(i)$ as follows:

(A) If $R < C_S(K)$ choose $s \in C_S(K)$.

(B) If $R = C_S(K)$, choose $s \in N_S(K)$; we check this choice is possible: When $K = K_0$ this is trivial, while when $K < K_0$, by assumption $i \notin Z(N_S(K))$, so again the choice is possible.

In either (A) or (B), $s \in N_S(K)$. Hence as $s \in N_S(C_S(i))$, s normalizes $C_S(i) \cap C_S(K) = R$.

In case (A) set $W := R\langle s\rangle$, and in case (B) set $W := C_S(i)\langle s\rangle$. In either case, $W = C_W(i)\langle s\rangle$. Furthermore $s^2 \in C_W(i)$: As $s^2 \in C_S(i)$, this is immediate from the definition of W in case (B), while in case (A) we chose $s \in C_S(K)$, so that $s^2 \in C_S(i) \cap C_S(K) = R = C_W(i)$.

We now show that $R_2 \trianglelefteq C_W(i)$: In case (A), this holds as R_2 is of index 2 in $R = C_W(i)$, so assume case (B) holds. Then $C_W(i) = C_S(i)$ normalizes $C_S(i) \cap C_S(K_+) = R_0$, so the claim holds when $|R : R_0| = 2$, since in that case $R_2 = R_0$. Thus we may assume $|R : R_0| = 4$ and $K_i/Z(K_i) \cong HS$, so that R_2 is the subgroup R_1 of R_0 with a component A_8 in its centralizer. But $C_S(i)$ acts on the 4-group R/R_0, and hence also on the unique subgroup R_1/R_0 of order 2 with $K < E(C_{K_i}(R_1))$. So indeed $R_2 \trianglelefteq C_W(i)$.

As $R_2 \trianglelefteq C_W(i)$ and $s^2 \in C_W(i)$, $W = C_W(i)\langle s\rangle$ normalizes $R_2 \cap R_2^s$. Assume $R_2 \cap R_2^s \neq 1$; then $C_{R_2\cap R_2^s}(W) \neq 1$. Let r be an involution in $C_{R_2\cap R_2^s}(W)$; from the definition of R_2, K is not a component of $C_G(r)$. In case (A), $R < W \leq C_S(r) \cap C_S(K)$, contrary to the maximality of R. In (B), $R = C_S(K) \leq C_S(i) < W \leq C_S(r)$, contrary to the maximality of $C_S(i)$ in our choice of i, R under the constraint that $R = C_S(K)$. Therefore $R_2 \cap R_2^s = 1$, so as $|R : R_2| = 2$, $|R| = 4$, contrary to our assumption that $R \neq \langle i, t\rangle$. This finally completes the proof of the claim.

By the claim, $K_0 = KK^u$ for $u \in S - N_S(K)$ and $i \in Z(N_S(K))$. Therefore by 1.2.1.3, K is described in case (1) or (2) of 2.5.16, so $K \cong L_2(p)$ for $p \geq 7$ a Fermat or Mersenne prime. In case (i) we showed that $K_i \cong K$, so K_+ is the direct product of two t-conjugates of a copy of K. In case (ii), $K \cong L_3(2)$, so (α) or (β) holds.

Let j be an involution in $R_0 = C_R(K_+)$, $G_j := C_G(j)$, $L_0 := \langle K_i^{O_{2',E}(G_j)}\rangle$, and $L_+ := \langle K_+^{O_{2',E}(G_j)}\rangle$. Then $K < K_+ \leq G_j$, so as K is not subnormal in K_+, K is not a component of G_j. Indeed we claim that $K_+ \trianglelefteq G_j$. As K_i is a component of $C_{G_j}(i)$, we may apply the initial arguments of the proof of 2.5.19 to j, i, K_i in the roles of "i, t, K". We conclude that there is a component L of G_j such that either $L = L_0$ is i-invariant, or $L < L_0 = LL^i$ with $C_{L_0}(i)^\infty$ a component of $C_{G_j}(i)$ isomorphic to $K_i \cong L_2(p)$ for suitable p. It follows that $L_+ = L_0L_0^t$. Similarly in case (i) where $K \cong K_i$, if $L = L_0$ we may apply 1.1.5 to conclude that L is $L_3(4)$ or J_2 of 3-rank 2.

If $K_+ = L_+$, then we conclude from A.3.18 in case (ii) or from 1.2.2 in case (i) that $L_+ = O^{3'}(E(G_j)) \trianglelefteq G_j$. Thus to establish the claim that $K_+ \trianglelefteq G_j$, it will suffice to show that $K_+ = L_+$.

Suppose that case (ii) holds. Then K_i is described in (α) or (β), so that 1.2.1.3. rules out the case $L < L_0$. Thus $L_0 = L$, and $L = [L, t]$ as t acts on K_i. Then our earlier argument applied to t, j, K in the roles of "t, i, K" shows that L is $L_3(4)$ or J_2. But then as K_i is a component of $C_L(i)$, $L = K_i$. Then as $L = L_0 = K_i$, $L_+ = LL^t = K_iK_i^t = K_+$, as desired.

So assume that case (i) holds. Suppose first that $L < L_0$. We saw that $L \cong K_i \cong L_2(p)$ for a suitable prime p, so $L_0 = L \times L^i$ with $K_i = C_{L_0}(i)$ a full diagonal subgroup of L_0. By 1.2.2, $L_0 = O^{3'}(G_j)$, so t acts on L_0 and then on $C_{L_0}(i) = K_i$, contrary to our assumption that case (i) holds. Thus $L = L_0$, and by an earlier remark, $L = [L, i]$ is $L_3(4)$ or J_2. But then t acts on L by 1.2.1.3, so $K_+ = K_i K_i^t \leq L$, a contradiction as $L_3(4)$ and J_2 contain no such subgroup. This completes our proof that $K_+ \trianglelefteq G_j$.

We showed that in case (ii), that $K_i/Z(K_i)$ is not HS; hence in either case (i) or (ii), $|R : R_0| = 2$, so $R = R_0 \times \langle t\rangle$. As $i \in Z(N_S(K))$ and $K^u\langle t\rangle$ centralizes K, $S \cap K^u\langle t\rangle \leq C_S(\langle i, K\rangle) = R$. Therefore $S \cap K^u\langle t\rangle = S_0 \times \langle t\rangle$, where $S_0 := R_0 \cap (S \cap K^u)\langle t\rangle$. Thus $S_0\langle t\rangle$ is Sylow in $K^u\langle t\rangle$, so from the structure of $Aut(K) \cong PGL_2(p)$ for $p \geq 7$ a Fermat or Mersenne prime, and using the second claim,

$$K^u = \langle C_{K^u}(j) : j \text{ an involution of } S_0\rangle \leq N_G(K_+).$$

Therefore $K^u \leq (N_G(K_+) \cap C_G(K))^\infty \leq C_G(K_+)$ from the structure of $C_{Aut(K_+)}(K)$. But now as $m_3(K_+) = 2$ in cases (i) and (ii), $m_{2,3}(K_+K^u) > 2$, contradicting G quasithin. This contradiction completes the proof of (2), which we saw suffices to establish 2.5.19. □

LEMMA 2.5.20. $K = K_0$.

PROOF. Assume $K < K_0$. By 2.5.16 and 1.2.1.3, $K \cong L_2(p)$ with $p \geq 7$ a Fermat or Mersenne prime, and $K_0 = KK^u$ for $u \in S - N_S(K)$. By 2.5.19.1, K is a component of $C_G(i)$ for each $i \in C_S(K)$.

We claim that $K_0 = O^{3'}(N_G(K^u))$. For let i be an involution in $K^u \cap S = S_K^u$. Then as $K^u \cong L_2(p)$ has one class of involutions, by a Frattini Argument, $N_G(K^u) = K^u I_i$ where $I_i := C_G(i) \cap N_G(K^u)$. Further we just saw that K is a component of $C_G(i)$, and hence K is a component of I_i. As $K \cong L_2(p)$ has no outer automorphism of order 3, $O^{3'}(N_{I_i}(K)) = KO^{3'}(C_{I_i}(K)) = KO^{3'}(C_{I_i}(K_0))$. As G is quasithin and $m_{2,3}(K_0) = 2$, $O^{3'}(C_{I_i}(K_0)) = 1$, so $O^{3'}(N_{I_i}(K)) = K$ and hence $K = O^{3'}(I_i)$ as K is subnormal in I_i. Thus $O^{3'}(N_G(K^u)) = K^u O^{3'}(I_i) = K^u K$, establishing the claim.

Then as u interchanges K and K^u, also $K_0 = O^{3'}(N_G(K))$, so that $K^u = O^{3'}(C_G(K))$ and hence $N_G(K) = N_G(K^u)$. Thus $C_G(i) \cap N_G(K) = C_G(i) \cap N_G(K^u) = I_i$, so that $O^{3'}(C_G(i)) \cap N_G(K)) = O^{3'}(I_i) = K$. We saw K is subnormal in $C_G(i)$, so

$$O^{3'}(C_G(i)) = K,$$

and hence $C_G(i) \leq N_G(K) = N_G(K^u)$. Thus if there is an involution $i \in K^u \cap K^{ug}$, then $K = O^{3'}(C_G(i)) = K^g$, so $g \in N_G(K) = N_G(K^u)$; that is, K^u is *tightly embedded* in G. Then as S_K is nonabelian, I.7.5 says that distinct conjugates of S_K in T commute. Suppose $S_K^g \leq T$ with $S_K \neq S_K^g \neq S_K^u$. Then $S_K^g \leq C_G(S_K S_K^u) \leq N_G(K^u) = N_G(K_0)$ since K^u is tightly embedded. Then since the center of a Sylow 2-subgroup of $Aut_S(K)$ is elementary abelian, $\Phi(S_K^g) \leq \Phi(\ C_T(S_K S_K^u) \cap N_G(K_0)\ ) \leq C_T(K_0)$, and then $KK^u = K_0 \leq O^{3'}(C_G(\Phi(S_K^g))) = K^{ug}$, a contradiction. Therefore $\{S_K, S_K^u\} = S_K^G \cap T$, so T permutes the set Δ of groups $O^{3'}(C_G(j))$ for j an involution in $S_K \cup S_K^u$. We showed $\Delta = \{K, K^u\}$, so T acts on K_0. Therefore $H = K_0 S \leq K_0 T \leq M = !\mathcal{M}(T)$, contradicting $H \not\leq M$. This completes the proof of 2.5.20. □

We now eliminate all possibilities for K remaining in 2.5.16 except for the one corresponding to the most stubborn remaining shadow discussed earlier:

LEMMA 2.5.21. $\bar{H} \cong Aut(A_6)$.

PROOF. First $K_0 = K$ by 2.5.20, so $H = KS$. Assume $\bar{H}$ is not $Aut(A_6)$. Then by 2.5.16, either $K \cong L_2(p)$ for $p \geq 7$ a Fermat or Mersenne prime, or $\bar{H} \cong PGL_2(9)$ or M_{10}. Therefore either $\bar{S}$ is dihedral, or $\bar{H} \cong M_{10}$ and $\bar{S} \cong SD_{16}$. Hence by 2.5.5.1, S_C is cyclic or dihedral, unless possibly $S_C \cong Q_8$ or SD_{16} when $\bar{H} \cong M_{10}$. In each case $|\bar{H} : \bar{K}| \leq 2$.

Assume that S_C is of order 2, so that $S_C = \langle t \rangle$. As $|\bar{H} : \bar{K}| \leq 2$, $|S : S_K| \leq 4$, so S/S_K is abelian and hence $[S, S] \leq S_K$. Also $\bar{S}$ is dihedral or semidihedral of order at least 8, so $\Omega_1([\bar{S}, \bar{S}]) = \langle \bar{z}_K \rangle$. Therefore $\Omega_1([S, S]) = \langle z_K \rangle$. Then x centralizes z_K, so by 2.5.13.2, $z = z_K$ and $t^x = tz$. Thus all involutions in K are in z^G, and all involutions in tK are in t^G. Choose $(U, H_U) \in \mathcal{U}^*(H)$. Then by 2.5.12, there is a 4-subgroup E of S_K such that $U = N_S(E)$ and $H_U = N_H(E)$. Since $S_C = \langle t \rangle$ is of order 2, $N_H(E) = N_H(F) \cong \mathbf{Z}_2 \times S_4$, where $F := ES_C = O_2(N_H(E)) = O_2(H_U) \cong E_8$. In particular $N_S(F) = U \in Syl_2(N_G(F))$ by 2.5.10.2. If $F^x \in F^S$, then by a Frattini Argument, we may take $x \in N_T(F)$, contradicting $N_S(F) \in Syl_2(N_G(F))$. Thus $F^x \notin F^S$.

Assume first that $S \not\leq KS_C$. $H = KS$ is transitive on E_8-subgroups of KS_C, so $F^x \not\leq KS_C$. But all involutions in M_{10} are in $E(M_{10})$, so if $K \cong A_6$ then $\bar{H} \cong PGL_2(9)$. Thus $\bar{H} \cong PGL_2(q)$ for q a Fermat or Mersenne prime or 9. But x acts on $Z(S) = \langle z, t \rangle \leq KS_C$, so as $F^x \not\leq KS_C$ by the previous paragraph, $e^x \notin KS_C$ for $e \in E - \langle z \rangle$. As $e^x \notin KS_C$ and $\bar{H} \cong PGL_2(q)$, $O(C_K(e^x)) \neq 1$, so since K is a component of G_t by 2.5.18, $1 \neq O(C_K(e^x)) \leq O(C_G(\langle e^x, t \rangle))$. Hence $C_G(e^x) \notin \mathcal{H}^e$ by 1.1.3.2, contradicting $e^x \in z^G$.

Therefore $S \leq KS_C$, so $H = K \times S_C$, and hence $S = S_K \times S_C$. This rules out cases (2) and (3) of 2.5.16 in which S is nontrivial on the Dynkin diagram of K, so $K \cong L_2(p)$ for $p > 7$ a Fermat or Mersenne prime. We saw earlier that $tE \subseteq t^G$, so there is $g \in G$ with $t^g \in F - \langle t, z \rangle$. As $S_C = \langle t \rangle$ is of order 2, $C_{G_t}(K) = O(C_{G_t}(K))S_C$ by Cyclic Sylow 2-Subgroups A.1.38. By 2.5.18, $G_t = KSC_{G_t}(K)$ and $O(C_{G_t}(K)) = O(G_t) = 1$, so $G_t = KS_C = H$. Thus $F \leq G_t^g = H^g = K^g S_C^g$, so $Aut_{K^g}(F) \cong S_3$, and hence $\langle Aut_K(F), Aut_{K^g}(F) \rangle$ is the parabolic in $GL(F)$ stabilizing $\langle z^G \cap F \rangle = K \cap F = E$. As this group is transitive on $F - E$ of order 4 and $S \leq G_t$, we conclude $|N_G(F) : N_S(F)|_2 \geq 4$, contradicting our earlier remark that $N_S(F) \in Syl_2(N_G(F))$. Therefore $|S_C| > 2$.

Suppose next that S_C is abelian. From remarks at the start of the proof, either S_C is cyclic, or possibly $S_C \cong E_4$ when $\bar{H} \cong M_{10}$. By 2.5.11.1, $Z(S) \cap S_C = \langle t \rangle$, so $S_C \not\leq Z(S)$, and hence $S \not\leq KS_C$. Indeed as $|\bar{S} : \bar{S}_K| \leq 2$, $|S : S_K S_C| = 2$ and $C_S(S_C) = S_K S_C$. Thus conjugating by x, also $|S : C_S(S_C^x)| = 2$, so $|\bar{S} : C_{\bar{S}}(\bar{S}_C^x)| \leq 2$. Hence as $\bar{S}$ is dihedral or semidihedral of order at least 16, while $S_C^x \cong \bar{S}_C^x$ is abelian of order at least 4 by 2.5.5.1, we conclude $\bar{S}_C$ is cyclic and $\bar{S}_C^x \leq \bar{K}$. Since $S_C S_C^x = S_C \times S_C^x$ by 2.5.5.1.2 we conclude $S_C \times S_C^x \leq S_C \times Y$, where Y is the cyclic subgroup of index 2 in S_K, and $C_S(S_C^x) = S_C \times Y$. This is impossible, as

$$C_S(S_C^x) = C_S(S_C)^x \cong C_S(S_C) = S_C \times S_K,$$

and S_K is nonabelian.

This contradiction shows that S_C is nonabelian. So again by our initial remarks, either S_C is dihedral of order at least 8, or $H/S_C \cong M_{10}$ and $S_C \cong Q_8$ or SD_{16}.

Set $S_0 = C_S(S_C)$. In any case, $\langle t\rangle = Z(S_C)$ and $S_K \le S_0$, so as $|\bar{S} : \bar{S}_K| \le 2$, $|S_0 : S_K| \le 4$ and hence $z_K \in [S_K, S_K] \le [S_0, S_0] \le S_K$. Let Y be the cyclic subgroup of index 2 in S_K. Then $\Omega_1(Y) = \langle z_K\rangle$ and $[\bar{S}, \bar{S}] \le \bar{Y}$, so $[S_0, S_0] \le Y$ and hence $\Omega_1([S_0, S_0]) = \langle z_K\rangle$. However $S_C^x \le C_S(S_C)$ by 2.5.5.2, and hence $[S_C^x, S_C^x] \le [S_0, S_0]$, so $t^x = z_K$ and $z = tz_K \ne z_K$. Therefore $tz_K \notin t^G$ in view of 2.5.11.2.

We next show that K is a component of $C_G(i)$ for each involution $i \in S_C$. We assume i is a counterexample and derive a contradiction: As $z \ne z_K$, 2.5.19.4 says $K \cong A_6$ and $K < K_i \trianglelefteq C_G(i)$ with $K_i \cong A_8$ and t induces a transposition on K_i. But then $C_{K_i}(t) \cong S_6$, whereas $S \in Syl_2(G_t)$ by 2.5.11.2, and no element of S induces an outer automorphism in S_6 on K since $\bar{H} \cong PGL_2(9)$ or M_{10}. This contradiction shows K is a component of $C_G(i)$.

Next we claim that $K \trianglelefteq C_G(i)$ for each involution i of $C_{G_t}(K)$: For assume $u \in C_G(i)$ with $K \ne K^u$. Then $\langle K, K^u\rangle = K \times K^u$ as K is a component of $C_G(i)$, and $i \ne t$ by 2.5.20. Now $\langle i, t\rangle$ is not Sylow in $K^u\langle i, t\rangle \cap G_t$, so $\langle i, t\rangle$ is not Sylow in $C_G(\langle i, t\rangle K)$. On the other hand as S is Sylow in G_t, we may assume $C_{S_C}(i) \in Syl_2(C_{G_t}(K\langle t\rangle))$, a contradiction as S_C is dihedral, semidihedral or quaternion. This contradiction establishes the claim that K is normal in $C_G(i)$ for each involution i of S_C.

Now assume S_C is not Q_8; in this part of the proof we eliminate the shadows of subgroups of $PSL_2(p)$ wr $\mathbf{Z}_2$. By our earlier remarks, either S_C is dihedral of order at least 8, or $H/S_C \cong M_{10}$ and $S_C \cong SD_{16}$. Recall from 2.5.5.1 that $S_C \cap S_C^x = 1$, so $K \ne K^x$ and $S_C^x \cong \bar{S}_C^x$. Since $K \cong L_2(q)$ for q a Fermat or Mersenne prime or 9, we compute from the possibilities for $\bar{H} \le Aut(K)$ that

$$K = \langle C_K(j) : j \text{ an involution of } S_C^x\rangle,$$

so that $K \le N_G(K^x)$ by the claim in the previous paragraph. By symmetry K^x acts on K, so $[K, K^x] = 1$, so K is not A_6 since G is quasithin. Thus $K \cong L_2(p)$ for $p \ge 7$ a Fermat or Mersenne prime. Let $K_+ := KK^x$, $M_+ := N_G(K_+)$, and $S_+ := S\langle x\rangle$.

Next $S_+ \le MK_+$, and as we saw that $t^x = z_K \in K$, $t \in K^x$. Then as $K \cong L_2(p)$ has one class of involutions, by a Frattini Argument, $M_+ = K_+N_{M_+}(\langle t, t^x\rangle)$. Then as $S \in Syl_2(G_t)$, $S_+ \in Syl_2(M_+)$. Also $C_S(K_+) = S_C \cap S_C^x = 1$, and hence $C_{S_+}(K_+) = 1$ so $C_G(K_+)) = O(C_G(K_+))$. As $K = K_0$, $G_t = KSC_{G_t}(K)$ and $O(G_t) = 1$ by 2.5.18. Then as $K^x \le C_G(K) \le G_{t^x}$ since $t^x \in K$, K^x is normal in $C_G(K)$. Thus $C_G(K) = K^xC_G(K_+) = K^xO(C_G(K_+)) = K^xO(N_G(K))$. Then $C_{G_t}(K) = C_{K^x}(K)O(N_G(K)) = S_K^xO(N_G(K))$, so

$$G_t = KSO(N_G(K)) = KSO(G_t) = KS = H \le M_+.$$

In particular $K = O^2(G_t)$.

We claim that $t^G \cap M_+$ is the set $\mathcal{I}$ of involutions in $K \cup K^x$. We saw earlier that $z_K = t^x$ and K has one class of involutions, so $\mathcal{I} \subseteq t^G \cap M_+$. Furthermore we saw that $z = tz_K = tt^x$, so that the diagonal involutions in K_+ are in z^G, and hence these involutions are not in t^G by 2.5.11.3. Thus if the claim fails, there is $i := t^g \in S_+ - \mathcal{I}$, such that either i induces an outer automorphism on K or K^x, or $K^x = K^i$. In the latter case, $C_{K_+}(i) =: K_i \cong K$, so $K_i = K^g$ since $K = O^2(G_t)$; this is impossible as the involutions in K_i are in z^G, while those in K are in t^G. Thus we may assume that i induces an outer automorphism on K.

Suppose first that i either centralizes K^x or induces an outer automorphism on K^x. If i induces an outer automorphism on K^x, then $K^x\langle i\rangle \cong PGL_2(p)$, so in either situation, i centralizes an E_{q^2}-subgroup of K_+, where q is an odd prime divisor of $p+\epsilon$, $p \equiv \epsilon \mod 4$, and $\epsilon = \pm 1$. This is impossible, as $K = O^2(G_t) \cong L_2(p)$ is of q-rank 1. Therefore i induces a nontrivial inner automorphism on K^x. Then $t \in C_{K^x}(i) \cong D_{p-\epsilon}$ centralizes $C_K(i) \cong D_{p+\epsilon}$, so

$$t \in \Phi(\ C_{K^x}(i)) \cap C_G(C_K(i))\ ) \leq C_{G_i}(K^g),$$

since the centralizer in $Aut(K^g)$ of a $D_{p+\epsilon}$-subgroup of K^g is of order 2. Then as $K = O^2(G_t)$, $K^g = K$, a contradiction as i centralizes K^g but not K. This establishes the claim that $\mathcal{I} = t^G \cap M_+$.

We've shown that $t^G \cap M_+ = \mathcal{I}$, so $t^G \cap M_+ = t^{M_+}$. We also showed that $G_t \leq M_+$, so by 7.3 in [**Asc94**], t fixes a unique point in the representation of G by right multiplication on G/M_+. Therefore as T is nilpotent (cf. the proof of 2.2.2), $T \leq M_+$. Further M_+ is the unique fixed point of each member of $S_K^\#$, so $S_K \cup S_K^x$ is strongly closed in T with respect to G. Thus the hypotheses of 3.4 in [**Asc75**] are satisfied with S_K, S_K^x, M_+ in the roles of "A_1, A_2, H", so that result says $G = M_+$, contradicting G simple.

We have reduced to the case where $S_C \cong Q_8$. In particular $\bar{H} \cong M_{10}$. Now $S_C S_C^x = S_C \times S_C^x$ by 2.5.5.2. Since $S_K \cong D_8$, $\bar{S}_C^x \not\leq \bar{K}$, so $\bar{H} = \bar{K}\bar{S}_C^x$. Thus $[\bar{S}_C^x, \bar{S}_K]$ is the image of the cyclic subgroup Y of index 2 in S_K. Then as $S_C^x \trianglelefteq S$, $Y = [S_C^x, S_K] \leq S_C^x$, so $S_C^x = Y\langle v\rangle$ for $v \in S_C^x - K$. Then as $[S_C, S_C^x] = 1$ and v induces an outer automorphism on K with $v^2 = z_K \in Y \leq K$, it follows that $H = S_C \times S_C^x K$, so $S = S_C \times S_C^x S_K$ with $S_C^x S_K$ a Sylow 2-subgroup of M_{10}. Since $z_K = t^x$, $C_S(S_C^x) = S_C \times Z(S_C^x) = S_C\langle t^x\rangle$, and hence

$$|C_S(S_C^x)| = 16 < 32 = |C_S(S_C)|,$$

a contradiction as x acts on S. This finally completes the proof of 2.5.21. □

In view of 2.5.21, it only remains to eliminate the case $\bar{H} \cong Aut(A_6)$. In particular $K \cong A_6$ and $S_K \cong D_8$.

LEMMA 2.5.22. *(1) If $z^g \in S$ for some $g \in G$, then $K = [K, z^g]$, and z^g induces an automorphism in S_6 on K.*

(2) $H = G_t$ and $C_H(z) = S$.

PROOF. Assume $z^g \in S$ for some $g \in G$. Then as $C_{G_t}(z^g) \in \mathcal{H}^e$ by 1.1.3.2, $C_K(z^g) \in \mathcal{H}^e$ using 1.1.3.1. Further $K = K_0 \trianglelefteq G_t$ by 2.5.20, so since $\bar{H} \cong Aut(A_6)$ by 2.5.21, (1) follows.

Let $C := C_{G_t}(K)$. By 2.5.18, $G_t = KSC$ and $C \in \mathcal{H}^e$. Thus $R := O_2(G_t) \leq S_C$ and $R \trianglelefteq S$. By 2.5.5.1, $S_C \cap S_C^x = 1$, so $R \cong \bar{R}^x \trianglelefteq \bar{S}$. As $\bar{S}$ is Sylow in $\bar{H} \cong Aut(A_6)$, it follows that either

(i) $\bar{R}$ is abelian and $m(\bar{R}) \leq 2$, or

(ii) $[\bar{R}, \bar{R}] =: \bar{Y}$ is the cyclic subgroup of index 2 in $\bar{S}_K$, and either $m(\bar{R}/\bar{Y}) \leq 2$ or $\bar{R} = \bar{S}$.

We conclude that $Aut(R)$ is a $\{2,3\}$-group, and hence C is a $\{2,3\}$-group. However as H is an SQTK-group, C is a $3'$-group, so as $F^*(C) = O_2(C)$, C is a 2-group. Thus $G_t = KSC = KS = H$, so $C_H(z) = S$ as $K \cong A_6$. □

LEMMA 2.5.23. *$\mathcal{U}^*(H) = \{(N_S(E_i), N_H(E_i)) : i = 1, 2\}$, where E_1 and E_2 are the 4-subgroups of S_K, and $N_S(E_i) \in Syl_2(N_H(E_i))$.*

PROOF. This follows from 2.5.12.3. □

LEMMA 2.5.24. $\Phi(S_C) \neq 1$.

PROOF. Assume $\Phi(S_C) = 1$, define E_1 and E_2 as in 2.5.23, and set $Q_i := O_2(N_H(E_i))$. Now $\bar{S}/\bar{S}_K \cong E_4$ since $\bar{H} \cong Aut(A_6)$ by 2.5.21, so that $\Phi(S) \leq S_K S_C$. Let Y denote the cyclic subgroup of S_K of index 2. Then $Y \leq [S_K, S] \leq [S,S] \leq \Phi(S)$. Since $\bar{Y} = \Phi(\bar{S}) \geq \overline{\Phi(S)}$, $\Phi(S) \leq Y \times S_C$. Then using the Dedekind Modular Law, $\Phi(S) = Y \times \Phi_C$, where $\Phi_C := \Phi(S) \cap S_C$. In particular as $\Phi(S_C) = 1$, $\Phi(\Phi(S)) = \Phi(Y) = \langle z_K \rangle$, so by 2.5.13.2, $z = z_K \in K$ and $t^x = tz_K = tz$.

Next $C_S(Y) = S_1$, where $\bar{S}_1$ is the modular subgroup M_{16} (see p. 107 in [**Asc86a**]) of $\bar{S}$. Thus

$$S_+ := \Omega_1(C_S(\Phi(S))) = \Omega_1(C_{S_1}(\Phi_C)) \text{ is either } S_C\langle z \rangle \text{ or } S_0\langle z \rangle$$

where S_0 is the preimage in S of the subgroup generated by the transposition in $\bar{H} \cong Aut(A_6)$ centralizing $\bar{Y}$. Thus as $S_C \cap S_C^x = 1$ by 2.5.5.1 while x acts on S_+, we conclude as usual that $m(S_C) \leq 2$, with $S_+ = S_0\langle z \rangle = S_C \times S_C^x$ in case of equality.

Suppose the latter case holds. Then $m(S_C) = 2$, and S_C^x contains an element inducing a transposition on K. Thus $\mathcal{A}(S) = \{Q_1, Q_2\}$, and $Q_i = S_C S_C^x E_i \cong E_{32}$. Further S is transitive on $\mathcal{A}(S)$, so by a Frattini Argument, we may take $x \in N_T(S) \cap N_T(Q_i)$ for each i, and hence $N_S(E_i) < N_S(E_i)\langle x \rangle$, so $N_S(E_i) \notin Syl_2(N_G(Q_i))$. But by 2.5.23, $(N_S(E_i), N_H(E_i)) \in \mathcal{U}^*(H)$, whereas 2.5.10.2 says $N_S(E_i)$ is Sylow in $N_G(Q_i)$. This contradiction eliminates the case $m(S_C) = 2$.

Therefore $m(S_C) = 1$, so as we are assuming S_C is elementary abelian, in fact $S_C = \langle t \rangle$ is of order 2. Suppose first that $E_1^x \leq KS_C$. Then x acts on $S_- := E_1 S_C (E_1 S_C)^x$. If x does not normalize $E_1 S_C$ then $\mathcal{A}(S_-) = \{E_1 S_C, E_2 S_C\}$, so S is transitive on $\mathcal{A}(S_-)$, and again by a Frattini Argument we may replace x by $x' \in N_T(S) \cap N_T(S_C E_1)$, and assume x acts on $E_1 S_C$. Thus x acts on $E_1 S_C$ and hence on $C_S(E_1 S_C) = Q_1$, allowing us to obtain a contradiction as in the previous paragraph.

Thus $E_1^x \not\leq S_C S_K$. We showed $z = z_K$, so $E_1^\# \subseteq z^G$. Therefore by 2.5.22.1, e^x induces a transposition on $K \cong A_6$ for some $e \in E_1 - \langle z \rangle$. Now some conjugate v of e^x in $S_K e^x$ centralizes S_K, so $Q_i = S_C \times E_i\langle v \rangle \cong E_{16}$, and S is transitive on $\mathcal{A}(S) = \{Q_1, Q_2\}$, so by a Frattini Argument we may choose $x \in N_T(S) \cap N_T(Q_i)$, leading to the same contradiction as in the two previous paragraphs. □

LEMMA 2.5.25. $t^x = z_K$ *and* $z = tz_K$.

PROOF. Assume otherwise. Then by 2.5.11.1, $z = z_K$, $t^x = tz_K$, and $\langle t \rangle = Z(S) \cap S_C$, so $\langle t^x \rangle = Z(S) \cap S_C^x$. But $S_K \cap S_C^x$ is normal in S, so if $1 \neq S_K \cap S_C^x$ then $1 \neq Z(S) \cap S_K \cap S_C^x$, contradicting $t^x = tz_K$. Hence $S_K \cap S_C^x = 1$. Thus $[S_K, S_C^x] \leq S_K \cap S_C^x = 1$, so $S_C^x \leq C_S(S_K) =: S_0$, and hence S_C^x is isomorphic by 2.5.5.1 to a subgroup of $\bar{S}_0 \cong E_4$, whereas S_C is not elementary abelian by 2.5.24. □

LEMMA 2.5.26. $m_2(S_C) = 1$.

PROOF. Assume $m_2(S_C) > 1$. In the first few paragraphs of the proof, we will establish the claim that K is a component of $C_G(i)$ for each involution $i \in S_C$. Assume otherwise; by 2.5.18, $i \neq t$, and by 2.5.19.2, $C_{S_C}(i) = \langle i, t \rangle$. Further $z \neq z_K$ by 2.5.25, so by 2.5.19.4, $K \leq K_i \trianglelefteq C_G(i)$ where $K_i \cong A_8$, and t induces a transposition on K_i. As $C_{S_C}(i) = \langle i, t\rangle$, S_C is dihedral or semidihedral by a lemma of Suzuki (cf. Exercise 8.6 in [**Asc86a**]), so as S_C is not elementary abelian by 2.5.24, $|S_C| \geq 8$. Using 2.5.5.2, $S_C^x \leq C_S(S_C) \leq C_S(i)$. However, as $C_{S_C}(i) = \langle i, t\rangle$, $K_iC_S(i) \cong \langle i\rangle \times S_8$. Therefore a Sylow 2-subgroup of $K_iC_S(i) \cap C_G(t)$ is isomorphic to $E_8 \times D_8$, which contains no D_{16} or SD_{16}-subgroup, so $S_C \cong D_8$. Hence $|S| = 2^8$ since $\bar{H} \cong Aut(A_6)$ by 2.5.21.

Let V denote the cyclic subgroup order 4 in S_C. By 2.5.22.2 $G_t = KS$, so $V \trianglelefteq G_t$, and thus V is a TI-set in G. Hence as V is not elementary abelian, $\langle V^G \cap T\rangle$ is abelian by I.7.5.

Assume $V^g \leq T$ for some $g \in G$. Then by the previous paragraph, $V^g \leq C_T(V) = C_S(V)$ and hence $\Phi(V^g) \leq \Phi(S) \leq S_KS_C$ since $\bar{S}/\bar{S}_K \cong E_4$. Now no involution in $\bar{S}_K - \langle\bar{z}\rangle$ is a square in $\bar{S}$, so no involution in $S_KS_C - \langle z\rangle S_C$ is a square in S. Hence

$$\langle t^g\rangle = \Phi(V^g) \leq \Omega_1(C_{\langle z\rangle S_C}(V)) = \Omega_1(V\langle z\rangle) = \langle t, z\rangle.$$

Therefore $t^g \in t^G \cap \langle t, z\rangle$, so that t^g is t or t^x by 2.5.11. Hence V^g is either V or V^x.

Since $V^G \cap T = \{V, V^x\}$, $VV^x \trianglelefteq T$, so $\Omega_1(VV^x) = \langle t, t^x\rangle \trianglelefteq T$. Then as $S \in Syl_2(G_t)$ by 2.5.11.2, $|T| = 2|S| = 2^9$.

Let $H_0 := K_i\langle i, t\rangle$, $T_i \in Syl_2(H_0)$, and $T_i \leq T^g$ for suitable $g \in G$. As K_i is a component of $C_G(i)$, $H_0 \not\leq M^g$ by 1.1.3.2. As $H_0 \cong \mathbf{Z}_2 \times S_8$, $H_0 = \langle H_1, H_2\rangle$, where H_1 and H_2 are the maximal 2-locals of H_0 over T_i; thus we may assume $H_1 \not\leq M^g$. As $|T_i| = 2^8 = |T|/2$, $T_i^{g^{-1}} \in \beta$ by 2.3.10, so $(T_i^{g^{-1}}, H_1^{g^{-1}}) \in \mathcal{U}(H_1^{g^{-1}})$ and $H_1^{g^{-1}} \in \Gamma$ from the definitions in Notation 2.3.4 and Notation 2.3.5. Then by 2.3.7.1, $H_1^{g^{-1}} \in \Gamma_0^e$, contrary to the hypothesis of this section. This contradiction finally completes the proof of the claim.

By the claim, K is a component of $C_G(i)$ for each involution $i \in S_C$. Further $K \trianglelefteq C_G(i)$ by 1.2.1.3. Recall $t^x = z_K$ and $S_CS_C^x = S_C \times S_C^x$, so for any $i \in S_C$ distinct from t, $i^x \notin t^xS_C = z_KS_C$. Therefore from the 2-local structure of $Aut(A_6)$, $C_K(i^x) \not\leq S_K$. Hence as $S = C_{KS}(t^x)$ is a maximal subgroup of KS,

$$KS = \langle C_{KS}(t^x), C_{KS}(i^x)\rangle \leq N_G(K^x)$$

using the claim. By symmetry, K^x acts on K, and $K \neq K^x$ as t^x centralizes K^x but not K. Therefore $[K, K^x] = 1$, a contradiction as $m_{2,3}(KK^x) \leq 2$ since G is quasithin. This contradiction completes the proof of 2.5.26. □

LEMMA 2.5.27. *$S_C \cong \mathbf{Z}_4$, $\mathbf{Z}_8$, or Q_8.*

PROOF. By 2.5.26, $m_2(S_C) = 1$; by 2.5.24, S_C is not elementary abelian; and by 2.5.5.1, $S_C \cong S_C^x$ is isomorphic to a subgroup of $\bar{S}$. Thus the lemma holds as the three groups listed in the lemma are the only subgroups X of $\bar{S} \in Syl_2(Aut(A_6))$ of 2-rank 1 with $\Phi(X) \neq 1$. □

We are now ready to complete the proof of Theorem 2.1.1.

By 2.5.27 there is a cyclic subgroup V of S_C of order 4 normal in S. Let Y be cyclic of order 4 in S_K, and S_0 the preimage in S of the subgroup generated by the transposition in $C_{\bar{S}}(\bar{S}_K)$. As $V \trianglelefteq S$, $V^x \trianglelefteq S$, so $\bar{V}^x \trianglelefteq \bar{S}$ and hence $\bar{V}^x/\langle\bar{z}\rangle \leq Z(\bar{S}/\langle\bar{z}\rangle) = \bar{Y}\bar{S}_0/\langle\bar{z}\rangle$. Therefore $\bar{V}^x\bar{S}_0 = \bar{Y}\bar{S}_0$. Let E be a 4-subgroup of S_K and $e \in E - \langle z_K\rangle$. As $\bar{V}^x\bar{S}_0 = \bar{Y}\bar{S}_0$ and $\bar{e}$ inverts $\bar{Y}$, $\bar{e}$ inverts $\bar{V}^x$, and hence e inverts V^x and centralizes S_C. Therefore e^x inverts V and centralizes S_C^x, so $e^x \notin S_K$ as S_K centralizes V.

As $S_K \trianglelefteq S$ and x acts on S, $S_K \cap S_K^x \trianglelefteq S$. However $t^x = z_K$ by 2.5.25, so

$$Z(S) \cap S_K \cap S_K^x \leq Z(S_K) \cap Z(S_K^x) = \langle t^x \rangle \cap \langle t \rangle = 1,$$

and hence $S_K \cap S_K^x = 1$. Thus $[S_K, e^x] \leq [S_K, S_K^x] \leq S_K \cap S_K^x = 1$, so $\bar{e}^x \in \Omega_1(C_{\bar{S}}(\bar{S}_K)) = \bar{S}_0\langle \bar{z} \rangle$. Hence as $e \in K$ centralizes S_C,

$$\bar{S}_C^x \leq C_{\bar{S}}(\bar{e}^x) = C_{\bar{S}}(\bar{S}_0) = \bar{S}_0 \times \bar{S}_K \cong \mathbf{Z}_2 \times D_8.$$

Thus as $S_C^x \cong \bar{S}_C^x$ by 2.5.5.1, and $\mathbf{Z}_2 \times D_8$ contains no Q_8 or $\mathbf{Z}_8$ subgroups, we conclude from 2.5.27 that $S_C = V \cong \mathbf{Z}_4$, and hence $|S| = 2^7$.

Next $A := E \times E^x = \langle t, t^x, e, e^x \rangle \cong E_{16}$, and

$$N_H(A) = \langle e^x, V \rangle \times N_K(E) \cong D_8 \times S_4,$$

as e^x inverts V and centralizes $N_K(E)$. It follows that $N_{H^x}(A) \cong D_8 \times S_4$ and $I := \langle N_H(A), x \rangle$ acts on A. Now $N_S(A) = N_S(E) \in \mathcal{U}^*(H)$ by 2.5.23, so $(N_S(A), N_H(A)) \in \mathcal{U}(I) \subseteq \mathcal{U}(N_G(A))$ from the definitions in Notation 2.3.4. As $T \cap I$ contains $\langle N_S(A), x \rangle$ of order $2^7 = |S|$ where $S \in Syl_2(H)$ for $H \in \Gamma^*$, and U has maximal order in $\mathcal{U}$, from the maximality of these groups in the definition of Γ^* in Notation 2.3.5, also $N_G(A) \in \Gamma^* \subseteq \Gamma_0$. This is impossible: for $z \in A$, so that $A \in \mathcal{S}_2^e(G)$ by 1.1.4.2; hence $N_G(A) \in \mathcal{H}^e$, so that $N_G(A) \in \Gamma_0^e$, contradicting our hypothesis in this section that $\Gamma_0^e = \emptyset$.

This contradiction completes the proof of Theorem 2.1.1.

CHAPTER 3

Determining the cases for $L \in \mathcal{L}_f^*(G,T)$

By Theorem 2.1.1, we may assume in the remainder of the proof of our Main Theorem that the Sylow 2-subgroup T of our QTKE-group G is contained in at least two distinct maximal 2-local subgroups. Thus we may implement the Thompson amalgam strategy described in the outline in the Introduction to Volume II: We choose $M \in \mathcal{M}(T)$ to contain a uniqueness subgroup of the sort considered in 1.4.1, and choose a 2-local subgroup H not contained in M. Indeed we may choose H minimal subject to this constraint:

Definition 3.0.1. $\mathcal{H}_*(T,M)$ denotes the members of $\mathcal{H}(T)$ which are minimal subject to not being contained in M.

In this chapter, we establish two important technical results, and define and begin to analyze the Fundamental Setup, which will occupy us for most of the proof of the Main Theorem.

We begin in section 3.1 by proving Theorem 3.1.1 and various corollaries of that result. Theorem 3.1.1 ensures that suitable pairs of subgroups are contained in a common 2-local subgroup of G. We appeal to this theorem and its corollaries many times during the proof of the Main Theorem, but most particularly in applying Stellmacher's *qrc*-lemma D.1.5, and in proving the main result of section 3.3.

In section 3.2 we define the Fundamental Setup and use the *qrc*-lemma to determine the cases that can arise there. A discussion of this important part of the proof can be found in the introduction to section 3.2.

Finally in section 3.3, we prove that if L is in $\mathcal{L}^*(G,T)$ or $\Xi^*(G,T)$ with $M := !\mathcal{M}(\langle L,T\rangle)$ as in 1.4.1, then $N_G(T) \leq M$. We use this result often, most frequently via its important consequence that each $H \in \mathcal{H}_*(T,M)$ is a minimal parabolic in the sense of Definition B.6.1.

3.1. Common normal subgroups, and the *qrc*-lemma for QTKE-groups

In this section we assume G is a simple QTKE-group, $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, and $M \in \mathcal{M}(T)$. We derive various consequences for QTKE-groups from Theorem C.5.8 of Volume I, in one case by applying the result in conjunction with the *qrc*-lemma D.1.5. We begin with a restatement of Theorem C.5.8.

Theorem 3.1.1. *Assume that $M_0, H \in \mathcal{H}(T)$, T is in a unique maximal subgroup of H, and $1 \neq R \leq T$ with $R \in Syl_2(O^2(H)R)$ and $R \trianglelefteq M_0$. Then there is $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq \langle M_0, H\rangle$.*

Proof. We verify the hypotheses of Theorem C.5.8, most particularly Hypothesis C.5.1: As $H \in \mathcal{H}(T)$, $F^*(H) = O_2(H)$ by 1.1.4.6, and as G is a $QTKE$-group, $m_3(H) \leq 2$. By the hypotheses of Theorem 3.1.1, T is in a unique maximal subgroup of H—completing the verification of C.5.1.1. Again by those hypotheses,

$R \trianglelefteq M_0$ and $R \in Syl_2(O^2(H)R)$, so C.5.1.2 holds. Thus Hypothesis C.5.1 is indeed satisfied, while by the hypotheses of this section, $T \in Syl_2(G)$ and G is a simple QTKE-group, supplying the remaining hypotheses of Theorem C.5.8. Of course the conclusion of C.5.8 is the existence of a nontrivial normal subgroup of $\langle M_0, H\rangle$ contained in R, so Theorem 3.1.1 is established. □

We sometimes use the following easy observation:

LEMMA 3.1.2. *If $T \leq Y \leq H \in \mathcal{H}(T)$, then also $Y \in \mathcal{H}(T) \subseteq \mathcal{H}^e$.*

PROOF. As $H \in \mathcal{H}$, $O_2(H) \neq 1$. Further $T \in Syl_2(Y)$, so $1 \neq O_2(H) \leq O_2(Y)$ by A.1.6, and hence also $Y \in \mathcal{H}$. Finally $Y \in \mathcal{H}^e$ by 1.1.4.6. □

In view of Theorem 2.1.1, we may assume that our fixed $M \in \mathcal{M}(T)$ is not the unique maximal 2-local subgroup of G containing T, so that $\mathcal{H}_*(T,M)$ is nonempty. During the remainder of our proof of our Main Theorem, we typically implement the Thompson amalgam strategy exploiting the interaction of M with some member of $\mathcal{H}_*(T,M)$.

Recall also from Definition B.6.2 that a subgroup X of G is in $\mathcal{U}_G(T)$ if T is contained in a unique maximal subgroup of X; and X is in $\hat{\mathcal{U}}_G(T)$ if $X \in \mathcal{U}_G(T)$ and T is not normal in X. In the terminology of Definition B.6.1, the members of $\hat{\mathcal{U}}_G(T)$ are called *minimal parabolics*.

As mentioned in the Introduction to Volume II and at the start of this chapter, once we have established Theorem 3.3.1 in the final section of this chapter, part (2) of the next lemma will ensure that members of $\mathcal{H}_*(T,M)$ are minimal parabolics for suitable choices of M.

LEMMA 3.1.3. *Assume $H \in \mathcal{H}_*(T,M)$. Then*

(1) $H \cap M$ is the unique maximal subgroup of H containing T. That is, $\mathcal{H}_*(T,M) \subseteq \mathcal{U}_G(T)$.

(2) If $N_G(T) \leq M$ or H is not 2-closed, then $H \in \hat{\mathcal{U}}_G(T)$. Thus H is a minimal parabolic, and so is described in B.6.8, and in E.2.2 if H is nonsolvable.

PROOF. Since $H \not\leq M$, $T \leq H \cap M < H$. If $T \leq Y < H$, then by 3.1.2, $Y \in \mathcal{H}(T)$; thus $Y \leq H \cap M$ by the minimality of H in the definition of $\mathcal{H}_*(T,M)$, so that (1) holds. If $N_G(T) \leq M$ or H is not 2-closed, then T is not normal in H, so (2) holds. □

LEMMA 3.1.4. *Assume that $H \leq G$ and V is an elementary abelian 2-subgroup of $H \cap M$ such that V is a TI-set under M with $N_G(V) \leq M$ and $H \leq N_G(U)$ for some $1 < U \leq V$. Then*

(1) $H \cap M = N_H(V)$.

(2) $H \not\leq M$ iff $H \not\leq N_G(V)$, in which case $H \cap M = N_H(V) < H$.

PROOF. As we assume $N_G(V) \leq M$, $N_H(V) \leq H \cap M$. Conversely as V is a TI-set in M, $N_M(U) \leq N_M(V)$. Then as $H \leq N_G(U)$ by hypothesis, $H \cap M = H \cap N_M(U) \leq N_H(V)$, so that (1) holds. Then (2) follows. □

Usually we will apply Theorem 3.1.1 under one of the hypotheses in Hypothesis 3.1.5—which will hold in the Fundamental Setup (3.2.1).

Recall from Definition B.2.11 the set $\mathcal{R}_2(M_0)$ of 2-reduced modules for M_0 from the Introduction to Volume II, and see the discussion in chapter B of Volume I.

HYPOTHESIS 3.1.5. *$T \leq M_0 \leq M$, $H \in \mathcal{H}_*(T, M)$, and $V \in \mathcal{R}_2(M_0)$ such that $R := O_2(M_0) = C_T(V)$. Further either*

(I) $H \cap M \leq N_G(O^2(M_0))$, or

(II) $H \cap M \leq N_G(V)$.

Observe that Hypothesis 3.1.5 includes the hypotheses of Theorem 3.1.1, other than the condition that $R \in Syl_2(O^2(H)R)$: For example, T is in a unique maximal subgroup of H by 3.1.3.1.

The next result is a corollary to Stellmacher's qrc-lemma D.1.5 using Theorem 3.1.1.

THEOREM 3.1.6. *Assume Hypothesis 3.1.5. Then one of the following holds:*

(1) There exists $1 \neq R_0 \leq R$ such that $R_0 \trianglelefteq \langle M_0, H\rangle$.

(2) $V \not\leq O_2(H)$ and $\hat{q}(M_0/C_{M_0}(V), V) \leq 2$. If in addition V is a TI-set under M, then in fact $\hat{q}(M_0/C_{M_0}(V), V) < 2$.

(3) $q(M_0/C_{M_0}(V), V) \leq 2$.

PROOF. Assume that conclusion (1) does not hold. We verify Hypothesis D.1.1, with M_0, H in the roles of "G_1, G_2": By Hypothesis 3.1.5, T lies in both M_0 and H—so it is Sylow in both, since it is Sylow in G. By 3.1.5, $V \in \mathcal{R}_2(M_0)$ and $H \in \mathcal{H}_*(T, M)$, so that $H \cap M$ is the unique maximal overgroup of T in H by 3.1.3.1, giving (1) of D.1.1. By 3.1.5, $R = O_2(M_0) = C_T(V)$, which is (2) of D.1.1. Finally, our assumption that (1) fails is (3) of D.1.1. Thus we may apply the qrc-Lemma D.1.5, to see (on combining its conclusions (2) and (4) in conclusion (ii) below) that one of the following holds:

(i) $V \not\leq O_2(H)$.
(ii) $q(M_0/C_{M_0}(V), V) \leq 2$.
(iii) V is a dual FF-module.
(iv) $R \cap O_2(H) \trianglelefteq H$, and $U := \langle V^H\rangle$ is elementary abelian.

Observe in case (ii) that conclusion (3) of Theorem 3.1.6 holds, so we may assume that (ii) fails, and it remains to treat cases (i), (iii), and (iv).

Suppose case (iii) holds and let V^* be the dual of V as an M_0-module. Then V^* is a faithful $\mathbf{F}_2$-module for $Aut_{M_0}(V^*) \cong Aut_{M_0}(V)$, so $O_2(Aut_{M_0}(V^*)) = 1$ since $V \in \mathcal{R}_2(M_0)$. As (iii) holds, $J^* := J(Aut_{M_0}(V^*), V^*) \neq 1$. Also M_0 is an SQTK-group using our QTKE-hypothesis, and hence so is the preimage in M_0 of J^*. Therefore Hypothesis B.5.3 is satisfied with J^*, V^* in the role of "G, V", so we may apply B.5.13 to see that conclusion (3) again holds, completing the treatment of case (iii).

As we are assuming that (ii) fails, $q(M_0/C_{M_0}(V), V) > 1$, so we may apply D.1.2. By (2) and (3) of D.1.2,

$$J(T) = J(R) \not\leq O_2(H).$$

By (4) of D.1.2, H is a minimal parabolic in the sense of Definition B.6.1, and is described in B.6.8.

In case (i), we argue that conclusion (2) holds: We will apply E.2.13, so we need to verify that Hypothesis E.2.8 is satisfied with $H \cap M$ in the role of "M", and that $F^*(H) = O_2(H)$. We just saw that H is a minimal parabolic in the sense of Definition B.6.1, and is described in B.6.8. As $H \in \mathcal{H}(T)$, using our QTKE-hypothesis and 1.1.4.6, H is an SQTK-group with $F^*(H) = O_2(H)$. By

Hypothesis 3.1.5, $V \in \mathcal{R}_2(M_0)$, so V is elementary abelian, normal in T, and contained in $\Omega_1\big(Z(O_2(M_0))\big)$. Further $T \leq M_0 \leq M$ so that $O_2(M) \leq O_2(M_0)$ by A.1.6; and $M \in \mathcal{M}(T) \subseteq \mathcal{H}^e$ since G is of even characteristic. Therefore $V \leq C_M(O_2(M)) \leq O_2(M)$, and hence $V \leq O_2(H \cap M)$. Finally $V \not\leq O_2(H)$ in case (i), and $O_2(H) = \ker_{H\cap M}(H)$ by B.6.8.5. This completes the verification of the hypothesis of E.2.13. Hence we conclude from E.2.13.3, that $\hat{q}(Aut_H(V), V) \leq 2$. Therefore since T is Sylow in both H and M_0, $\hat{q}(M_0/C_{M_0}(V), V) \leq 2$. Further if V is a TI-set under M, then we have the hypotheses for E.2.15, so that result shows that $\hat{q}(Aut_H(V), V) < 2$, and hence $\hat{q}(M_0/C_{M_0}(V), V) < 2$. Thus (2) holds, as claimed.

Thus we may assume that cases (i)–(iii) do not hold. In particular, case (iv) holds; and as (i) fails, now $V \leq O_2(H)$. By our observation following Hypothesis 3.1.5, it suffices to prove that $R \in Syl_2(O^2(H)R)$, since then Theorem 3.1.1 shows that conclusion (1) of Theorem 3.1.6 holds.

Set $Q_H := O_2(H)$, $K := O^2(H)$, and $H^* := H/Q_H$. As case (iv) holds, $Q := R \cap Q_H \trianglelefteq H$, so as $C_T(V) = R$ and $V \leq Q_H$ by the previous paragraph, $V \leq Z(Q)$. Therefore $U \leq Z(Q)$.

We saw earlier that $J(T) = J(R) \not\leq Q_H$, and H is a minimal parabolic described in B.6.8. Now by Hypothesis 3.1.5, $Q_H \leq T \leq M_0 \leq N_G(R)$, so $[Q_H, J(R)] \leq Q_H \cap R = Q$, and hence $[K, J(R)]J(R)$ centralizes Q_H/Q. Next $[K, J(R)]J(R)$ is normal in $KT = H$, but $J(R) \not\leq Q_H$, so $K \leq [K, J(R)]J(R)$ by B.6.8.4, and then K centralizes Q_H/Q. Therefore $[O_2(K), K] \leq Q$.

If K centralizes U then K centralizes V, so $C_T(V) = R$ is Sylow in $C_G(V)$ and hence R is Sylow in KR, which as we observed earlier suffices to complete the proof. Thus we may assume that K does not centralize U. Then $C_H(U) \leq \ker_{H\cap M}(H)$ and $C_T(U) = C_{Q_H}(U)$ by B.6.8.6.

As $J(R) \not\leq Q_H$, there is some $A \in \mathcal{A}(R)$ with $A^* \neq 1$. As $A \leq R$ and $U \leq Z(Q)$, $A \cap Q_H = A \cap Q \leq C_A(U)$, so $A \cap Q_H = C_A(U)$ by the previous paragraph. Then as $A \in \mathcal{A}(R)$, $r_{A^*,U} \leq 1$ by B.2.4.1. Now U might not be in $\mathcal{R}_2(H)$, but each nontrivial H-chief section W on U is an irreducible for $H/C_H(W)$, so that $O_2(H/C_H(W)) = 1$. Furthermore $C_H(W) \leq \ker_{H\cap M}(H)$ and $C_T(W) = C_{Q_H}(W)$ by B.6.8.6, so $m(A^*) = m(Aut_A(W))$ and hence $r_{Aut_A(W),W} \leq r_{A^*,U} \leq 1$. Therefore W is an FF-module for $Aut_H(W)$. Hence by B.6.9 and E.2.3, $m(W/C_W(A^*)) = m(A^*)$, $K = K_1$ or K_1K_2, and $[W, K_i]$ is the natural module for $K_i^* \cong L_2(2^n)$, A_3, or A_5. Furthermore as $m(U/C_U(A)) \leq m(A^*) = m(W/C_W(A^*))$, we conclude K_i has a unique noncentral chief factor $\tilde{U}_i$ on U, where $\tilde{U}_i = U_i/C_{U_i}(K_i)$ is the natural module for K_i^*, and $[U, K_i] = U_i$.

Set $B := H \cap M$ and observe that B is solvable: This is clear if H is solvable, while if H is not solvable then by E.2.2 and the previous paragraph, $B^* \cap K^*$ is a Borel subgroup of K^*, and in particular B is solvable. By Hypothesis 3.1.5, either (I) holds and B normalizes $L := O^2(M_0)$, or (II) holds and B normalizes V. In case (I), let $D := C_B(L/O_2(L))$, and in case (II), let $D := C_B(V)$. Then B normalizes D in either case.

We claim that R is Sylow in D, and $D \trianglelefteq B$: In case (II), $R = C_T(V)$ is Sylow in $C_G(V)$, and hence also in $C_B(V) = D$. As B normalizes V in (II), $D \trianglelefteq B$. In case (I), we apply parts (4) and (5) of A.4.2 with L, M_0 in the roles of "X, M", to see that $R = O_2(M_0)$ is Sylow in $C_{M_0}(L/O_2(L))$. Hence R is also Sylow in $C_B(L/O_2(L)) = D$. As B normalizes L in (I), $D \trianglelefteq B$.

Let Y denote a Hall $2'$-subgroup of B. As $D \trianglelefteq B$ by the previous paragraph, $Y_D := Y \cap D$ is also Hall in D, so $D = Y_D R = RY_D$. Further $Y \leq B \leq N_G(D)$, so

$$YR = YY_DR = YD = DY = RY_DY = RY.$$

Then R is Sylow in the group YR, and Y normalizes $O_2(YR) \leq R$.

We claim that $T \cap K \leq R$; this is the crucial step in showing that R is Sylow in RK, and hence in completing the proof. Since T is transitive on the groups K_i, it suffices to show that $T_i := T \cap K_i \leq R$ for some i. Let $Q_i := O_2(K_i)$, $T_0 := N_T(K_i)$, $Y_i := Y \cap K_i$, and $\overline{K_iT_0} := K_iT_0/O_2(K_iT_0)$. Then $A \leq T_0$ by B.1.5.4, and as $A \not\leq Q_H$, while K_i^* is quasisimple or of order 3, we may choose i so that $K_i = [K_i, A]$. Next

$$P_i := [Q_i, K_i] \leq Q_i \leq Q \leq R. \qquad (*)$$

But if $K_i^* \cong A_3$ then $P_i = Q_i \in Syl_2(K_i)$ since $K_i = O^2(K_i)$, so that $T_i = P_i \leq R$ by (*), as claimed.

Suppose next that $\tilde{U}_i$ is the natural module for $K_i^* \cong L_2(2^n)$ with $n > 1$. Then by B.4.2.1, the FF^*-offender $\bar{A}$ is Sylow in $\bar{K}_i$, so that $T_i \leq J(R)Q_i$ with $J(R) \leq O_2(Y_iT_0)$. Thus $J(R) \leq O_2(YT_0)$, so

$$J(R) \leq O_2(YT_0) \cap YR \leq O_2(YR) \leq R,$$

so Y acts on $J(O_2(YR)) = J(R)$ using B.2.3.3, and hence again using (*),

$$T_i = [J(R), Y_i]P_i \leq RP_i \leq R.$$

Finally if U_i is the natural module for $K_i \cong A_5$, then by B.3.2.4, the FF^*-offender $\bar{A}$ is generated by one or two transpositions. Thus $[A, T_i] \leq R \cap K_i =: R_i$, so as $[A, T_i] \not\leq Q_i$, (*) says

$$T_i = \langle R_i^{Y_i} \rangle P_i = R_iP_i \leq R.$$

We have established the claim that $T \cap K \leq R$. Since T is Sylow in H and $K \trianglelefteq H$, $T \cap K$ is Sylow in K, so R is Sylow in RK, completing the proof of Theorem 3.1.6. □

The next result is another corollary of Theorem 3.1.1, in the same spirit as Theorem 3.1.6. Recall that Z is $\Omega_1(Z(T))$, and the Baumann subgroup of T from Definition B.2.2 is $\mathrm{Baum}(T) = C_T\big(\Omega_1(Z(J(T)))\big)$.

LEMMA 3.1.7. *Assume Hypothesis 3.1.5, with $J(T) \leq R$. Then either*

(1) $Z \leq Z(H)$ and $Z(M_0) = 1$, or

(2) There is $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq \langle M_0, H \rangle$.

PROOF. By hypothesis $J(T) \leq R = C_T(V)$. Then $J(T) = J(R)$ and $S := \mathrm{Baum}(T) = \mathrm{Baum}(R)$ by B.2.3.5 with V in the role of "U". Therefore if $J(T) \trianglelefteq H$, then (2) holds with $J(T)$ in the role of "R_0". Thus we may assume $J(T)$ is not normal in H, so H is not 2-closed. Hence $H \in \hat{\mathcal{U}}_G(T)$ and H is described in B.6.8 by 3.1.3.

Suppose $Z \leq Z(H)$. If $Z(M_0) = 1$ then conclusion (1) holds, so we may assume $Z(M_0) \neq 1$. By Hypothesis 3.1.5, $M_0 \in \mathcal{H}(T)$, and hence $M_0 \in \mathcal{H}^e$ by 1.1.4.6. Therefore $Z(M_0)$ is a 2-group, so $\Omega_1(Z(M_0)) \leq Z \leq Z(H)$, and hence conclusion (2) holds with $\Omega_1(Z(M_0))$ in the role of "R_0".

Thus we may assume that $Z \not\leq Z(H)$. Let $U_H := \langle Z^H \rangle$ and $K := O^2(H)$. As $H = KT$, $K \not\leq C_H(Z)$, so $K \not\leq C_H(U_H)$. We saw in the previous paragraph that

$H \in \mathcal{H}^e$, so $U_H \in \mathcal{R}_2(H)$ by B.2.14. As $K \not\leq C_H(U_H)$, $C_H(U_H) \leq \ker_{M\cap H}(H)$ by B.6.8.6, and $C_H(U_H)$ is 2-closed by B.6.8.5. So as $J(T)$ is not normal in H, $J(T) \not\leq C_H(U_H)$. Hence by E.2.3, $K = K_1 \cdots K_s$, with $s = 1$ or 2, T permutes the K_i transitively, $K_1/C_{K_1}(U_H) \cong L_2(2^n)$, A_3, or A_5, $S = \mathrm{Baum}(T) = \mathrm{Baum}(R)$ acts on K_i, and either S is Sylow in K_iS, or $[U_H, K_i]$ is the A_5-module for $K_i/O_2(K_i)$. In the latter case, by E.2.3.3, S is of index 2 in a Sylow 2-group S_i of SK_i and $S_i \leq \langle S^{K_i\cap M}\rangle$. Then by an argument near the end of the proof of 3.1.6, $S_i \leq R$. So in either case, $R \cap K \in Syl_2(K)$, and hence $R \in Syl_2(KR)$. As we observed after Hypotheses 3.1.5, this is sufficient to establish the hypotheses of Theorem 3.1.1. Hence conclusion (2) holds by that result, completing the proof. □

Finally we extend Theorems 3.1.6 and 3.1.7, by bringing uniqueness subgroups into the picture:

THEOREM 3.1.8. *Assume $L_0 = O^2(L_0) \trianglelefteq M$ with $M = !\mathcal{M}(L_0T)$, and $V \in \mathcal{R}_2(L_0T)$ such that $O_2(L_0T) = C_T(V)$. Then*

(1) $\hat{q}(L_0T/C_{L_0T}(V), V) \leq 2$.

(2) Either

(i) $q(L_0T/C_{L_0T}(V), V) \leq 2$, or

(ii) For each $H \in \mathcal{H}_*(T,M)$, $V \not\leq O_2(H)$. If in addition V is a TI-set under M, then $\hat{q}(L_0T/C_{L_0T}(V), V) < 2$.

(3) Either:

(i) $J(T) \not\leq C_T(V)$, so V is an FF-module for $L_0T/C_{L_0T}(V)$, or

(ii) $J(T) \leq C_T(V)$, $Z \leq Z(H)$ for each $H \in \mathcal{H}_*(T,M)$, and $Z(L_0T) = 1$.

PROOF. Set $M_0 := L_0T$, and consider any $H \in \mathcal{H}_*(T,M)$. Observe that case (I) of Hypothesis 3.1.5 holds. Further as $M = !\mathcal{M}(M_0)$ and $H \not\leq M$, $O_2(\langle M_0, H\rangle) = 1$. In particular, neither conclusion (1) of Theorem 3.1.6, nor conclusion (2) of 3.1.7 holds. Therefore since $\hat{q}(Aut_{L_0T}(V), V) \leq q(Aut_{L_0T}(V), V)$ from the definitions B.1.1 and B.4.1, we conclude from Theorem 3.1.6 that conclusions (1) and (2) of Theorem 3.1.8 hold.

If $J(T) \not\leq C_T(V)$, then conclusion (i) of (3) holds by B.2.7. On the other hand, if $J(T) \leq C_T(V)$, then by the previous paragraph, conclusion (1) of 3.1.7 holds, so conclusion (ii) of (3) is satisfied. □

In certain situations we will require a refinement of the *qrc*-Lemma making use of information in D.1.3 and definition D.2.1.

LEMMA 3.1.9. *Assume case (II) of Hypothesis 3.1.5 holds, with $H \in \mathcal{H}_*(T,M)$. Further assume:*

(a) $q\big(M_0/C_{M_0}(V), V\big) = 2$.

(b) $M = !\mathcal{M}(M_0)$.

(c) $V \leq O_2(H)$.

(d) V is not a dual FF-module for M_0.

Set $U_H := \langle V^H\rangle$ and $Z := \Omega_1(Z(T))$. Then U_H is elementary abelian, and

(1) H has exactly two noncentral chief factors U_1 and U_2 on U_H.

(2) There exists $A \in \mathcal{A}(T) = \mathcal{A}(C_T(V))$ with $A \not\leq O_2(H)$, and for each such A chosen with $AO_2(H)/O_2(H)$ minimal, A is quadratic on U_H.

(3) For A as in (2), set $B := A \cap O_2(H)$. Then $B = C_A(U_i)$,

$$2m(A/B) = m(U_H/C_{U_H}(A)) = 2m(B/C_B(U_H)),$$

$$2m(B/C_B(V^h)) = m(V^h/C_{V^h}(B))$$

for each $h \in H$ *with* $[V^h, B] \neq 1$, $m(A/B) = m(U_i/C_{U_i}(A))$, *and* $C_{U_H}(A) = C_{U_H}(B)$.

(4) Define

$$m := \min\{m(D) : D \in \mathcal{Q}(Aut_M(V), V)\}.$$

Then $m(A/B) \geq m$.

(5) Assume $O^2(C_M(Z)) \leq C_M(V)$. *Then* $H/C_H(U_i) \cong S_3$, S_3 *wr* $\mathbf{Z}_2$, S_5, *or* S_5 *wr* $\mathbf{Z}_2$, *with* U_i *the direct sum of the natural modules* $[U_i, F]$, *as* F *varies over the* S_3*-factors or* S_5*-factors of* $H/C_H(U_i)$. *Further* $J(H)C_H(U_i)/C_H(U_i) \cong S_3$, $S_3 \times S_3$, S_5, *or* $S_5 \times S_5$, *respectively.*

(6) Assume that each $\{2,3\}'$*-subgroup of* $C_M(Z)$ *permuting with* T *centralizes* V, $m \geq 2$, *and each subgroup of order* 3 *in* $C_M(Z)$ *has at least three noncentral chief factors on* V. *Then* $H/C_H(U_i) \cong S_3$ *wr* $\mathbf{Z}_2$.

PROOF. Observe that hypothesis (a) implies:

(a') V is not an FF-module for M_0.

We will first show that (a') and (b)–(d) lead to the hypotheses of the qrc-lemma D.1.5.

Set $R := C_T(V)$. By (a'), $J(T) \leq C_T(V) = R$. Thus the hypothesis of Theorem 3.1.7 holds, and by B.2.3.3, $J(T) = J(R)$.

Next by (b), there is no $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq \langle M_0, H\rangle$. Thus conclusion (1) of Theorem 3.1.7 holds, so that $Z \leq Z(H)$, and in particular $H \cap M \leq C_G(Z)$. Further $J(T)$ is not normal in H, so we conclude from 3.1.3.2 that H is a minimal parabolic in the sense of Definition B.6.1. Also (as at the start of the proof of Theorem 3.1.6) Hypothesis D.1.1 holds with M_0, H in the roles of "G_1, G_2". Thus we can appeal to results in section D.1, and in particular to the qrc-lemma D.1.5.

Observe that (c) rules out conclusion (1) of D.1.5, and (a') and (d) rule out conclusions (2) and (3), respectively. We rule out conclusion (5) of D.1.5 just as in the proof of 3.1.6, using (c) to eliminate case (i) in that proof. Thus conclusion (4) of D.1.5 holds, so U_H is abelian, and H has more than one noncentral chief factor on U_H. This last condition together with (c) and (a') are the hypotheses of D.1.3. Furthermore (a') gives the hypothesis of D.1.2, so by part (4) of that result, H is a minimal parabolic in the sense of Definition B.6.1, and is described in B.6.8.

Next (a) supplies the hypothesis of part (3) of D.1.3. Then (1) follows from D.1.3.3. We saw earlier that $J(T) = J(R)$, so by D.1.3.2 there is $A \in \mathcal{A}(T)$ with $A \not\leq O_2(H)$ and A quadratic on U_H. Indeed from the proof of D.1.3.2, our choice of $A \in \mathcal{A}(T) - \mathcal{A}(O_2(H))$ with $AO_2(H)/O_2(H)$ minimal guarantees that A is quadratic on U_H, and that $B := A \cap O_2(H) = C_A(U_i)$ for $i = 1, 2$. Thus (2) holds, and D.1.3 establishes the remaining assertions of (3).

By (3), $m(V^h/C_{V^h}(B)) = 2m(B/C_B(V^h))$, and B is quadratic on V^h by (2), so $Aut_B(V^h) \in \mathcal{Q}(Aut_{M^h}(V^h), V^h)$ by (a). Thus

$$m \leq m(B/C_B(V^h)) \leq m(B/C_B(U_H)) = m(A/B),$$

establishing (4).

Set $H^* := H/C_H(U_i)$. As H is irreducible on U_i, $O_2(H^*) = 1$, so $U_i \in \mathcal{R}_2(H)$. As $B = C_A(U_i)$ and $m(A/B) = m(U_i/C_{U_i}(A))$ by (3), $A^* \cong A/B$ is an FF*-offender on U_i. Therefore by B.6.9, $H = YT$ where $Y := J(H, V)$, $Y^* = Y_1^* \times \cdots \times Y_s^*$, and $U_i = U_{i,1} \oplus \cdots \oplus U_{i,s}$ with $U_{i,j}$ the natural module for $Y_j^* \cong L_2(2^n)$ or S_{2^k+1}. By

A.1.31.1, $s \leq 2$; by E.2.3.2, if Y_j^* is a symmetric group, then Y_j^* is S_3 or S_5; and in any case $H \cap M$ is the product of T with the preimages of the Borel subgroups over $T^* \cap Y_j^*$ in Y_j^*. Further if $s = 2$, then as U_i is irreducible under H, $\{U_{i,1}, U_{i,2}\}$ is permuted transitively by T.

Assume for the moment that $Y_j^* \cong L_2(2^n)$ with $n > 1$ and some $U_{i,j}$ the natural module. Then by B.4.2.1, A^* is Sylow either in Y^* or in some Y_j^*. Now A^* is also an FF*-offender on U_{3-i}, and B.4.2 says that the only other possible FF*-module for Y_j^* is the A_5-module when $n = 2$, whereas the FF*-offenders on that module are not Sylow in Y_j^*. Thus in any case U_1 is Y-isomorphic to U_2.

Let $K := O^2(H)$, and W an H-submodule of U_H maximal subject to $U_0 := [U_H, K] \not\leq W$. Set $U_H^+ := U_H/W$. Thus $U_0^+ \neq 0$, H is irreducible on U_0^+, and $C_{U_H^+}(K) = 0$. As $U_H = \langle V^H \rangle$, $U_H^+ = \langle V^{+H} \rangle$, so $V_0^+ := C_{V^+}(T) \neq 0$. As $C_{U_H^+}(K) = 0$, $V_0^+ \leq U_0^+$ using Gaschütz's Theorem A.1.39. As H is irreducible on U_0^+, we may take $U_1 = U_0^+$. Further

$$0 \neq V_0^+ \leq C_{U_1}(J(R)^*), \tag{$*$}$$

and as case (II) of Hypothesis 3.1.5 holds,

$$H \cap M \text{ acts on } V^+. \tag{$**$}$$

Let X denote a Cartan subgroup of $Y_j \cap M$.

Suppose that $Y_j^* \cong L_2(2^n)$ with $n > 1$ and $U_{1,j}$ the natural module. Then as $J(R)^* \in Syl_2(Y^*)$, we conclude from (*) and (**) that

$$V_j^+ := V^+ \cap U_{1,j} = C_{U_{1,j}}(J(R)^*) \tag{!}$$

is the $J(R)^*$-invariant 1-dimensional $\mathbf{F}_{2^n}$-subspace of $U_{1,j}$. In particular X acts faithfully on V. This is a contradiction to the hypotheses of (5), and under the hypotheses of (6), $O^3(X) = 1$ so $n = 2$. But now V_j^+ is the only noncentral chief factor for X on V^+, and the image of $[V \cap W, X]$ in $U_{2,j}$ is contained in $C_{U_{2,j}}(J(R)^*)$, so X has a single noncentral chief factor on $V \cap W$. Thus X has just two noncentral chief factors on V, contrary to the hypotheses of (6).

We have completed the proof of (5), so we may assume the hypotheses of (6) with $U_{i,j}$ the natural module for $Y_j^* \cong S_3$ or S_5. By (4) and the hypothesis of (6), $m(A^*) \geq m \geq 2$, so H^* is not S_3. Thus we may assume $Y_j^* \cong S_5$. Then from the description of FF^*-offenders in B.3.2.4, $O^2((H \cap M)^*) = [O^2(H \cap M)^*, J(R)^*]$, so as $H \cap M$ acts on V and $J(R)$ centralizes V, X centralizes V, contrary to the hypotheses of (6). This completes the proof of (6). □

3.2. The Fundamental Setup, and the case division for $\mathcal{L}_f^*(G,T)$

The bulk of the proof of the Main Theorem involves the analysis of various possibilities for $L \in \mathcal{L}_f^*(G,T)$. In this section we establish a formal setting for treating these subgroups, and provide the list of groups L and internal modules V which can arise in that setting. In the language of the Introduction to Volume II, this gives a solution to the First Main Problem—reducing from an arbitrary choice for L, V to the much shorter list arising in what we call below our Fundamental Setup (FSU).

In this section we assume G is a simple QTKE-group, $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, and $M \in \mathcal{M}(T)$. The notation $Irr_+(X, V)$ and $Irr_+(X, V, Y)$ appears in Definition A.1.40. We will be primarily interested in

HYPOTHESIS 3.2.1 (Fundamental Setup (FSU)). *G is a simple QTKE-group, $T \in Syl_2(G)$, $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple, $L_0 := \langle L^T \rangle$, $M := N_G(L_0)$, and $V_\circ \in Irr_+(L_0, R_2(L_0T), T)$. Set $V := \langle V_\circ^T \rangle$, $V_M := \langle V^M \rangle$, $M_V := N_M(V)$, $\bar{M}_V := M_V/C_{M_V}(V)$, and $\tilde{V}_M := V_M/C_{V_M}(L_0)$.*

In our first lemma we apply results from section D.3 to subgroups $M \in \mathcal{M}(T)$ such that M is the normalizer of one of the uniqueness subgroups constructed in chapter 1. We will also see in 3.2.3 that case (i) of 3.2.2 includes the Fundamental Setup, as the similar notation in the lemmas suggests.

LEMMA 3.2.2. *Assume there is $M_+ = O^2(M_+) \trianglelefteq M$ such that either*

(i) $M_+ = \langle L^T \rangle$ for some $L \in \mathcal{L}_f(G,T)$ with $L/O_2(L)$ quasisimple, or

(ii) $M_+ = O_{2,p}(M_+)$ for some odd prime p, with T irreducible on $M_+/O_{2,\Phi}(M_+)$.

Let $V_\circ \in Irr_+\big(M_+, R_2(M_+T), T\big)$ and set $V_M := \langle V_\circ^M \rangle$, $V := \langle V_\circ^T \rangle$, and $\tilde{V}_M := V_M/C_{V_M}(M_+)$. Then

(1) $C_{M_+}(V_M) \leq O_{2,\Phi}(M_+)$.

(2) $V_M \in \mathcal{R}_2(M)$.

(3) $V_M = [V_M, M_+]$, $\tilde{V}_M$ is a semisimple M_+-module, and M is transitive on the M_+-homogeneous components of $\tilde{V}_M$.

(4) $C_{V_M}(M_+) = \langle C_{V_\circ}(M_+)^M \rangle = \langle C_V(M_+)^M \rangle$.

(5) If $C_{V_\circ}(M_+) = 0$, then $V_\circ$ is a TI-set under M.

(6) If $C_{V_M}(M_+) \neq 0$ and $M = !\mathcal{M}(M_+T)$, then $M_+ = [M_+, J(T)]$ and V is an FF-module for M_+T.

(7) Hypothesis D.3.1 is satisfied with $Aut_M(V_M)$, $Aut_{M_+}(V_M)$, $V_\circ$ in the roles of "M, M_+, V".

(8) $V \in \mathcal{R}_2(M_+T)$ and $O_2(M_+T) = C_T(V)$.

(9) Assume $M = !\mathcal{M}(M_+T)$. Then the hypothesis of Theorem 3.1.8 is satisfied with M_+ in the role of "L_0", and D.3.10 applies.

PROOF. By A.1.11, $R_2(M_+T) \leq R_2(M)$. Now it is straightforward to verify that Hypothesis D.3.2 is satisfied with M, T, M_+, $R_2(M)$, 1, $V_\circ$ in the roles of "$\dot{M}$, $\dot{T}$, $\dot{M}_+$, Q_+, Q_-, V". Notice that V, V_M play the roles of "V_T, V_M" in Hypothesis D.3.2 and lemma D.3.4. Now (1) and (7) follow from parts (2) and (1) of D.3.3.

By (7), we may apply D.3.4 to $Aut_M(V_M)$; then conclusions (1)–(4) and (6) of D.3.4 imply conclusions (2)–(5) of 3.2.2.

Set $M_0 := M_+T$ and $R := O_2(M_0)$. By D.3.4.1, $O_2(M_0/C_{M_0}(V)) = 1$, so $V \in \mathcal{R}_2(M_0)$ and hence $R \leq C_T(V)$. By D.3.4.2, $C_{M_+}(V) \leq O_{2,\Phi}(M_+)$, so as $M_+ = O^2(M_0)$, $C_{M_0}(V) \leq RO_{2,\Phi}(M_+)$ and hence $R = C_T(V)$, completing the proof of (8).

Now assume that $M = !\mathcal{M}(M_+T)$. Then (9) follows from (8), so it remains to prove (6); thus we assume that $C_{V_M}(M_+) \neq 0$. Then $Z_0 := C_Z(M_+T) \neq 0$ and $Z_0 \leq Z(M_0)$. By (9) we may apply Theorem 3.1.8.3 to conclude that $J(T) \not\leq C_T(V)$. From the structure of M_+ in cases (i) and (ii) of the lemma, $\Phi(M_+/O_2(M_+))$ is the largest M_0-invariant proper subgroup of $M_+/O_2(M_+)$, so we conclude that $M_+ = [M_+, J(T)]O_2(M_+)$. Then as $M_+ = O^2(M_+)$, also $M_+ = [M_+, J(T)]$, completing the proof of (6), and hence of 3.2.2. □

LEMMA 3.2.3. *Assume $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple, and let $L_0 := \langle L^T \rangle$. Then $M := N_G(L_0) \in \mathcal{M}(T)$, $M = !\mathcal{M}(L_0T)$, and for each member I of*

$Irr_+(L_0, R_2(L_0T))$ there exists $V_\circ \in Irr_+(L_0, R_2(L_0T), T)$ with $V_\circ/C_{V_\circ}(L_0)$ L_0-isomorphic to $I/C_I(L_0)$. In particular L and $V := \langle V_\circ^T \rangle$ satisfy the Fundamental Setup (3.2.1).

PROOF. By 1.2.7.3, $M = !\mathcal{M}(L_0T)$. By A.1.42.2, there exists a member $V_\circ$ of $Irr_+(L_0, R_2(L_0T), T)$ with $V_\circ/C_{V_\circ}(L_0)$ isomorphic as L_0-module to $I/C_I(L_0)$. Hence the lemma holds. □

REMARK 3.2.4. Given $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple, lemma 3.2.3 shows that we can choose V so that L and V satisfy the Fundamental Setup. Then by 3.2.2.7, we may apply the results of section D.3 to analyze V, V_M, and $Aut_{L_0}(V_M)$. By 3.2.2, we may also appeal to Theorem 3.1.8, and in view of 3.2.2.4, 3.2.2.6 supplies extra information when $C_V(L) \neq 0$.

In the next few lemmas, we determine the list of modules V and V_M that can arise in the Fundamental Setup for the various possible $L \in \mathcal{L}_f^*(G,T)$. The first result 3.2.5 below gives us a qualitative description of what goes on in the case $L = L_0$, including a fairly complete description of the case where $V_\circ < V$. Then 3.2.8 gives more detailed information when $L = L_0$ but $V_\circ = V$.

Recall that $V_M := \langle V_\circ^M \rangle$ and that V plays the role of "V_T" played in lemma D.3.4. Also recall that in the FSU, $\bar{M}_V$ denotes $N_M(V)/C_M(V)$.

THEOREM 3.2.5. *Assume the Fundamental Setup (3.2.1), with $L = L_0$. Then $\hat{q}(\bar{L}\bar{T}, V) \leq 2 \geq \hat{q}(Aut_M(V_M), V_M)$, and one of the following holds:*

(1) $V_\circ = V = V_M$; that is, $V_\circ \trianglelefteq M$.

(2) $V_\circ = V \trianglelefteq T$, $C_{V_\circ}(L) = 0$, and V is a TI-set under M.

(3) $\bar{L} \cong SL_3(2^n)$ or $Sp_4(2^n)$ for some n, A_6, $L_4(2)$, or $L_5(2)$; $C_{V_\circ}(L) = 0$ and either $V_\circ$ is a natural module for $\bar{L}$ or $V_\circ$ is a 4-dimensional module for $\bar{L} \cong A_7$; and $V_M = V = V_\circ \oplus V_\circ^t$ with $t \in T - N_T(V_\circ)$, and $V_\circ^t$ not $\mathbf{F}_2L$-isomorphic to $V_\circ$.

PROOF. As discussed in Remark 3.2.4, we may apply 3.2.2, Theorem 3.1.8, and results in section D.3. Recall that in our setup, $V_\circ$ and V play the roles of "V" and "V_T" in Hypothesis D.3.2 and lemma D.3.4.

Set $\hat{q} := \hat{q}(\bar{L}\bar{T}, V)$ and $q := \hat{q}(Aut_M(V_M), V_M)$. As $L = L_0$ by hypothesis, conclusion (1) of Theorem 3.1.8 gives $\hat{q} \leq 2$.

Next we will show that $q \leq 2$ by an appeal to Theorem 3.1.6. Set $R := C_T(V_M)$, so that $R \in Syl_2(C_M(V_M))$. We first verify that for any $H \in \mathcal{H}_*(T,M)$, Hypothesis 3.1.5 is satisfied with $M_0 := N_M(R)$ and V_M in the role of "V": First as $V_M \trianglelefteq M$, hypothesis (II) of 3.1.5 is satisfied. By a Frattini Argument, $M = C_M(V_M)M_0$, so $Aut_M(V_M) \cong Aut_{M_0}(V_M)$, and hence as $V_M \in \mathcal{R}_2(M)$ by 3.2.2.2, also $V_M \in \mathcal{R}_2(M_0)$. As $R \trianglelefteq M_0$, $R \leq O_2(M_0)$. As $V_M \in \mathcal{R}_2(M_0)$, $O_2(M_0) \leq C_M(V_M)$, so as R is Sylow in $C_M(V_M)$, $R = O_2(M_0)$. This completes the verification of Hypothesis 3.1.5.

Next $V \leq V_M$, so $R \leq C_T(V)$, while $C_T(V) \trianglelefteq LT$ by 3.2.2.8. Thus $R = C_T(V) \cap C_M(V_M) \trianglelefteq LT$, so as $M = !\mathcal{M}(LT)$, $M = !\mathcal{M}(M_0)$. Therefore conclusion (1) of Theorem 3.1.6 is not satisfied, so one of conclusions (2) or (3) holds, and in either case, $q \leq 2$ as desired.

We have shown that $\hat{q} \leq 2 \geq q$, so it remains to show that one of conclusions (1)–(3) holds. Suppose first that $C_{V_\circ}(L) \neq 0$. Then by 3.2.2.6, $L = [L, J(T)]$, so that (in the language of Definition B.1.3) $Aut_L(V_M) \leq J(Aut_M(V_M), V_M)$ by B.2.7. Thus we have the hypotheses for D.3.20, which gives conclusion (1). Therefore we may assume that $C_{V_\circ}(L) = 0$. Then by 3.2.2.5, $V_\circ$ is a TI-set under M. If $V_\circ = V$ then conclusion (2) holds, so we may assume that $V_\circ < V$. As $\hat{q} \le 2 \ge q$, the hypotheses of Theorem D.3.10 are satisfied; therefore as we have reduced to the case where $V_\circ < V$, conclusion (2) of Theorem D.3.10 holds. But this is precisely conclusion (3) of Theorem 3.2.5, so the proof is complete. □

The notation $\hat{\mathcal{Q}}(X, W)$ appears in Definition D.2.1.

THEOREM 3.2.6. *Assume the Fundamental Setup (3.2.1) with $L < L_0$. Set $M^* := M/C_M(V_M)$, $U := [V_M, L]$, and let $t \in T - N_T(L)$. Then $\hat{q}(\bar{L}_0\bar{T}, V) \le 2 \ge \hat{q}(M^*, V_M)$, and one of the following holds:*

(1) $L^* \cong L_2(2^n)$ and $V_\circ = V = V_M$ is the $\Omega_4^+(2^n)$-module for L_0^*.

(2) $L^* \cong L_3(2)$ and $V_\circ = V = V_M$ is the tensor product of natural modules for L^* and L^{*t}.

(3) Each of the following holds:

(a) $\tilde{V}_M = \tilde{U} \oplus \tilde{U}^t$, where $U = [V_M, L] \le C_{V_M}(L^t)$.

(b) Each $A \in \hat{\mathcal{Q}}_*(M^*, V_M)$ acts on U, so $\hat{q}(Aut_{L_0T}(U), U) \le 2$.

(c) One of the following holds:

(i) $U = V_\circ$ and $V = V_M$.

(ii) $Aut_M(L^*) \cong Aut(L_3(2))$, $V = V_M$, $U = V_\circ \oplus V_\circ^s$ for $s \in N_T(L) - LO_2(LN_T(L))$, and $m(V_\circ) = 3$.

(iii) $L^* \cong L_3(2)$, U is the sum of four isomorphic natural modules for L^*, and $O^2(C_{M^*}(L_0^*)) \cong \mathbf{Z}_5$ or E_{25}.

PROOF. Proceeding as in the proof of Theorem 3.2.5, and recalling the discussion in Remark 3.2.4, we verify Hypothesis 3.1.5 for $M_0 := N_M(R)$ where $R := C_T(V_M)$, and apply Theorems 3.1.6 and 3.1.8 as before to conclude

$$\hat{q}(\bar{L}_0\bar{T}, V) \le 2 \ge \hat{q}(M^*, V_M).$$

Recall from the remark before that result that we may reduce case (3) to case (1) by a new choice of V. If $V < V_M$, then conclusion (2) of D.3.21 holds, so that conclusion (3) of 3.2.6 holds, with case (iii) of part (c) of (3) satisfied.

So we may suppose instead that $V = V_M$, as in conclusion (1) of D.3.21. Assume first that $V_\circ < V$. In particular we have the hypotheses of D.3.6, and conclusions (1) and (2) of that result give parts (a) and (b) of conclusion (3) of 3.2.6, while the two alternatives in part (3) of D.3.6 are cases (i) and (ii) of part (c) of conclusion (3) of 3.2.6.

Thus the Theorem holds when $V_\circ < V$, so assume instead that $V_\circ = V$. Then we have the hypotheses of D.3.7, and its conclusions (1) and (2) give the corresponding conclusions of 3.2.6. The proof is complete. □

We often need to know that V is a TI-set under M. The previous two results say that this is almost always the case:

LEMMA 3.2.7. *Assume the Fundamental Setup (3.2.1). Then either*

(1) V is a TI-set under M, or

(2) $\bar{L} \cong L_3(2)$, $L < L_0$, and subcase (3.c.iii) of Theorem 3.2.6 holds.

PROOF. Suppose V is not a TI-set under M. Then in particular V is not normal in M, so that $V < V_M$. Therefore $L < L_0$, since if $L = L_0$ then either

$V = V_M$ or V is a TI-set under M, by 3.2.5. Thus L_0 and V are described in 3.2.6, where $V < V_M$ occurs only in subcase (3.c.iii). □

With 3.2.6 in hand, we return to the case in the Fundamental Setup where $L = L_0$, and we obtain more information in the subcase where $V = V_\circ$. As in the proof of the Main Theorem, we divide our analysis into the case where V is an FF-module and the case where V is not an FF-module.

LEMMA 3.2.8. *Assume the Fundamental Setup (3.2.1) with $L = L_0$ and $V_\circ = V$. Assume further that V is an FF-module for $Aut_{GL(V)}(\bar{L})$. Then one of the following holds:*

(1) $\bar{L} \cong L_2(2^n)$ and $\tilde{V}$ is the natural module.

(2) $\bar{L} \cong SL_3(2^n)$, and either V is a natural module or V is a 4-dimensional module for $L_3(2)$.

(3) $\bar{L} \cong Sp_4(2^n)$ and $\tilde{V}$ is a natural module.

(4) $\bar{L} \cong G_2(2^n)'$ and $\tilde{V}$ is the natural module.

(5) $\bar{L} \cong A_5$ or A_7, and V is the natural module.

(6) $\bar{L} \cong A_6$ and $\tilde{V}$ is a natural module.

(7) $\bar{L}\bar{T} \cong A_7$ and $m(V) = 4$.

(8) $\bar{L} \cong \hat{A}_6$ and $m(V) = 6$.

(9) $\bar{L}\bar{T} \cong L_n(2)$, $n = 4$ or 5, and V is a natural module.

(10) $\bar{L} \cong L_4(2)$ and $\tilde{V}$ is the 6-dimensional orthogonal module.

(11) $\bar{L}\bar{T} \cong L_5(2)$ and $m(V) = 10$.

PROOF. This is a consequence of Theorem B.4.2, using the 1-cohomology of those modules listed in I.1.6. □

PROPOSITION 3.2.9. *Assume the Fundamental Setup FSU (3.2.1), with $L = L_0$ and $V_\circ = V$. Further assume V is not an FF-module for $Aut_{GL(V)}(\bar{L})$. Set $q := q(\bar{L}\bar{T}, V)$ and $\hat{q} := \hat{q}(\bar{L}\bar{T}, V)$. Then one of the following holds:*

(1) $\bar{L} \cong L_2(2^{2n})$, $n > 1$, V is the $\Omega_4^-(2^n)$-module, and $q = \hat{q} \geq 3/2$, or $q \geq 4/3$ if $n = 2$.

(2) $\bar{L} \cong U_3(2^n)$, V is a natural module, and $q = \hat{q} = 2$.

(3) $\bar{L} \cong Sz(2^n)$, V is a natural module, and $q = \hat{q} = 2$.

(4) $\bar{L} \cong (S)L_3(2^{2n})$, $m(V) = 9n$, $q > 2$, and $\hat{q} = 5/4$. Further $\bar{T}$ is trivial on the Dynkin diagram of $\bar{L}$.

(5) $\bar{L}\bar{T} \cong Aut(M_{12})$, $m(V) = 10$, $q > 2$, and $\hat{q} > 1$.

(6) $\bar{L} \cong \hat{M}_{22}$, $m(V) = 12$, and $\hat{q} > 1$.

(7) $\bar{L} \cong M_{22}$, $m(V) = 10$, $q \geq 2$, $\hat{q} > 1$, and $q > 2$ if V is the cocode module.

(8) $\bar{L} \cong M_{23}$, $m(V) = 11$, $q > 2$, and $\hat{q} > 1$.

(9) $\bar{L} \cong M_{24}$, $m(V) = 11$, $q > 2$, and $\hat{q} > 1$.

PROOF. By hypothesis, V is not an FF-module for $\bar{L}\bar{T}$, so $J(T) \leq C_T(V)$ by B.2.7; hence we conclude $C_V(L) = 0$ from 3.2.2.6. Then as $V \in Irr_+(L, R_2(LT))$, L is irreducible on V. By 3.2.5, $\hat{q} \leq 2$. Then the result follows from the list in B.4.5, plus the following remarks: The cases in B.4.5 where $\bar{L}$ is A_7 or $G_2(2)'$ do not arise here because of our hypothesis that V is not an FF-module for $Aut_{GL(V)}(\bar{L})$. If $\bar{L} \cong (S)L_3(2^{2n})$ and $m(V) = 9n$, then V may be regarded as an $\mathbf{F}_{2^n}$-module, and $\mathbf{F}_{2^{2n}} \otimes_{\mathbf{F}_{2^n}} V = N \otimes N^\sigma$, where N is the natural $\mathbf{F}_{2^{2n}}$-module for $SL_3(2^{2n})$ and σ is the involutory field automorphism of $\mathbf{F}_{2^{2n}}$. Hence V is not invariant under an automophism nontrivial on the Dynkin diagram. Finally we eliminate the cases in part (iii) of B.4.5, via an appeal to Theorem 3.1.8.2: For in these cases, $q > 2 = \hat{q}$ in the notation of B.4.5. As $q > 2$, case (i) of 3.1.8.2 does not hold. But V is a TI-set under M by 3.2.7, so as $\hat{q} = 2$, case (ii) of 3.1.8.2 does not hold either, a contradiction. □

In our final result on the Fundamental Setup, we collect some useful properties that hold when $J(T) \leq C_T(V)$—and hence in particular under the hypotheses of 3.2.9 where V is not an FF-module.

Recall that $J_1(T)$ appears in Definition B.2.2, Further $n(X)$ appears in E.1.6, $r(G, V_+)$ in E.3.3, and $W_0(T, V_+)$ in E.3.13.

PROPOSITION 3.2.10. *Assume the Fundamental Setup (3.2.1). Set $V_+ := V$, except in case (3.c.iii) of 3.2.6, where we take $V_+ := V_M$. Assume $J(T) \leq C_T(V_+)$. Then*

(1) $N_G(J(T)) \leq M$.

(2) $N_M(V_+)$ controls fusion in V_+.

(3) For each $U \leq V_+$, $N_G(U)$ is transitive on $\{V_+^g : U \leq V_+^g\}$.

(4) For each $U \leq V_+$, $|N_G(U) : N_M(U)|$ is odd.

(5) If $U \leq V_+$ with $\langle V_+^{N_G(U)} \rangle$ abelian, then $[V_+, V_+^g] = 1$ for all $g \in G$ with $U \leq V_+^g$.

(6) Suppose $U \leq V_+$ with $V_+ \leq O_2(N_G(U))$, and either

(a) $[V_+, W_0(T, V_+)] = 1$, or

(b) V_+ is not an FF-module for $Aut_{L_0T}(V_+)$.

Then $[V_+, V_+^g] = 1$ for each $g \in G$ with $U \leq V_+^g$.

(7) If $J_1(T) \leq C_T(V_+)$ and $r(G, V_+) > 1$, then $n(H) > 1$ for each $H \in \mathcal{H}_*(T, M)$.

(8) If $J(T) \leq S \in \mathcal{S}_2(G)$, then $J(T) = J(S)$ and so $N_G(S) \leq M$.

(9) $C_Z(L_0) = 1 = C_{V_+}(L_0)$.

PROOF. By 3.2.3, $M = !\mathcal{M}(L_0T)$. We have $C_T(V_+) \leq C_T(V) = O_2(L_0T)$ by 3.2.2.8. Hence as $J(T) \leq C_T(V_+)$ by hypothesis, using B.2.3.3,

$$J(T) = J(C_T(V_+)) = J(O_2(L_0T)) \trianglelefteq L_0T,$$

so that (1) holds. Notice the same argument establishes (8). Further $Z(L_0T) = 1$ by Theorem 3.1.8.3, so (9) follows.

Observe that V_+ is a TI-set under M: This holds in case (3.c.iii) of 3.2.6 as $V_+ = V_M$ is normal in M in that case, and in the remaining case $V_+ = V$ is a TI-set under M by 3.2.7.

Also $V_+ \leq E := \Omega_1(Z(J(T)))$. As $J(T)$ is weakly closed in T, by Burnside's Fusion Lemma A.1.35, $N_G(J(T))$ controls fusion in E and hence in V_+. Thus as V_+ is a TI-subgroup under M, (1) implies (2). Then (2) implies (3) using A.1.7.1.

Let $U \leq V_+$ and $S \in Syl_2(N_M(U))$. As $J(T) \leq C_G(V_+)$ by hypothesis, we may assume $J(T) \leq S$. Then $N_G(S) \leq M$ by (8), so $S \in Syl_2(N_G(U))$, establishing (4). Assume the hypotheses of (5), and let $U \leq V_+^g$. By (3), we may take $g \in N_G(U)$; then as $\langle V_+^{N_G(U)} \rangle$ is abelian by hypothesis, $[V_+, V_+^g] = 1$—so that (5) is established.

Assume the hypotheses of (6). Then $V_+ \leq O_2(N_G(U))$, so $\langle V_+^{N_G(U)} \rangle \leq W_0(T, V_+)$. Hence if $[V_+, W_0(T, V_+)] = 1$ as in (6a), then $\langle V_+^{N_G(U)} \rangle \leq C_T(V_+)$, so $\langle V_+^{N_G(U)} \rangle$ is abelian, and thus (5) implies (6) in this case. Now assume the hypothesis of (6b). We may take $g \in N_G(U)$ by (3), so

$$\langle V_+, V_+^g \rangle \le O_2(N_G(U)) \le S \cap S^g \le N_M(U) \cap N_M(U^g) \le N_M(V_+) \cap N_M(V_+^g),$$

where the last inclusion holds since V_+ is a TI-set under M. Reversing the roles of V_+ and V_+^g if necessary, we may assume that $m(V_+^g/C_{V_+^g}(V_+)) \ge m(V_+/C_{V_+}(V_+^g))$. Thus as $Aut_{L_0T}(V_+)$ is not an FF-module by hypothesis, $[V_+, V_+^g] = 1$. This completes the proof of (6).

As L_0T normalizes $O_2(L_0T) \cap C_M(V_+) = C_T(V_+)$, $M = !\mathcal{M}(N_{N_M(V_+)}(C_T(V_+)))$. Thus Hypothesis E.6.1 is satisfied with V_+ in the role of "V", so part (7) follows from E.6.26 with 1 in the role of "j". □

Sometimes in arguments where we can pin down the structure of a pair in the FSU (especially when we can show L is a block), we encounter the following situation:

Lemma 3.2.11. *Assume the Fundamental Setup (3.2.1). Assume further that $V = O_2(L_0T)$. Then $O_2(M) = V = C_G(V)$ and $M = M_V$. If further $\bar{M}_V = \bar{L}_0\bar{T}$, then $M_V = M = L_0T$.*

Proof. By A.1.6, $O_2(M) \le O_2(L_0T) = V \le O_2(L_0) \le O_2(M)$, so that $O_2(M) = V$, and in particular $M = M_V$ as $M \in \mathcal{M}$. Now as $F^*(M) = O_2(M)$, $C_G(V) \le Z(O_2(M)) \le V$, so that $C_G(V) = V$. The result follows. □

Our last two results of the section involve the collection $\Xi(G,T)$ of Definition 1.3.1, and appearing in case (ii) of the hypothesis of 3.2.2.

Definition 3.2.12. Define $\Xi_-(G,T)$ to consist of those $X \in \Xi(G,T)$ such that either

(a) X is a $\{2,3\}$-group, or
(b) $X/O_2(X)$ is a 5-group and $Aut_G(X/O_2(X))$ a $\{2,5\}$-group.

Set $\Xi_+(G,T) := \Xi(G,T) - \Xi_-(G,T)$.

Lemma 3.2.13. *$\Xi_f^*(G,T) \subseteq \Xi_-(G,T)$.*

Proof. Assume $X \in \Xi_f^*(G,T)$. Then $X/O_2(X) \cong E_{p^2}$ or p^{1+2} for some odd prime p, and T is irreducible on $X/O_{2,\Phi}(X)$. By 1.3.7, $M = !\mathcal{M}(XT)$, where $M := N_G(X)$. Let $(XT)^* := XT/C_{XT}(R_2(XT))$. By A.4.11, $V := [R_2(XT), X] \ne 1$, so as T is irreducible on $X/O_{2,\Phi}(X)$, $C_X(V) \le O_{2,\Phi}(X)$. Thus as $R := O_2(XT)$ centralizes $R_2(XT)$, $X^* = F^*(X^*T^*)$, so as X^* is faithful on V, also X^*T^* is faithful on V. Hence $C_T(V) = R$ and $V \in \mathcal{R}_2(XT)$. Therefore the hypotheses of Theorem 3.1.8 are satisfied with X in the role of "L_0", so $\hat{q} := \hat{q}(X^*T^*, V) \le 2$ by 3.1.8.1. As $\hat{q} \le 2$, D.2.13 says $p = 3$ or 5. We may assume by way of contradiction that $X \notin \Xi_-(G,T)$, so $p = 5$ and $Aut_G(X/O_2(X))$ is not a $\{2,5\}$-group. By D.2.17 and D.2.12, $X^* = X_1^* \times \cdots \times X_s^*$ and $V = V_1 \oplus \cdots \oplus V_s$, where $X_i^* \cong \mathbf{Z}_5$, $V_i := [V, X_i]$ is of rank 4, and $s \le 2$. As $m_5(X/O_{2,\Phi}(X)) = 2$, $s = 2$. As $T \in Syl_2(N_G(X))$, $R \in Syl_2(C_G(X/O_2(X)))$ by A.4.2.5; so by a Frattini Argument, $Aut_G(X/O_2(X)) = Aut_H(X/O_2(X))$, where $H := N_G(X) \cap N_G(R)$. Thus $Aut_H(X/O_2(X))$ is not a $\{2,5\}$-group, so $Aut_H(X^*)$ is not a $\{2,5\}$-group. As R centralizes $R_2(XT)$, $R_2(XT) \le \Omega_1(Z(R))$. Then as $V \le R_2(XT)$,

$$C_{XT}(\Omega_1(Z(R))) \le C_{XT}(V) \le RO_{2,\Phi}(X),$$

so $\Omega_1(Z(R))$ is 2-reduced. Therefore $R_2(XT) = \Omega_1(Z(R))$, so H acts on

$$[\Omega_1(Z(R)), X] = [R_2(XT), X] = V,$$

and hence $O^2(H)$ acts on V_i and X_i^*. This is a contradiction as $Aut_H(X^*)$ is not a $\{2, 5\}$-group, but $Aut(\mathbf{Z}_5)$ is a 2-group. □

Lemma 3.2.13 allows us to establish a result about those $L \in \mathcal{L}(G, T)$ such that $L/O_2(L)$ is not quasisimple. Recall from chapter 1 that $\Xi_p(L)$ is $O^2(X_p)$ where X_p is the preimage of $\Omega_1\big(O_p(L/O_2(L))\big)$.

LEMMA 3.2.14. *If $L \in \mathcal{L}(G, T)$ and $L/O_2(L)$ is not quasisimple, then $O_\infty(L)$ centralizes $R_2(LT)$.*

PROOF. We assume L is a counterexample, and it remains to derive a contradiction.

By 1.2.1.4, $L/O_{2,F}(L) \cong SL_2(q)$ for some prime $q > 3$, and T normalizes L by 1.2.1.3. Set $V := R_2(LT)$; by hypothesis $[V, L] \neq 1$ so $L \in \mathcal{L}_f(G, T)$.

Let $L \leq K \in \mathcal{L}^*(G, T)$; then $K \in \mathcal{L}_f^*(G, T)$ by 1.2.9. In the cases in A.3.12 where "$B/O_2(B)$" is not quasisimple, either $O_\infty(L) \leq O_\infty(K)$ in case (21) or (22), or $K/O_2(K) \cong (S)L_3(r)$ for some prime $r > 3$ in case (9). In the latter case by 3.2.3, K is listed in one of 3.2.5, 3.2.8, or 3.2.9, but of course $(S)L_3(r)$ for a prime $r > 3$ does not appear on any of those lists. Thus $O_\infty(L) \leq O_\infty(K)$, so replacing L by K, we may assume $L \in \mathcal{L}_f^*(G, T)$.

Let $\pi := \pi(O_{2,F}(L)/O_2(L))$, $p \in \pi$, and $X := \Xi_p(L)$. Since $L \in \mathcal{L}_f^*(G, T)$, $X \in \Xi_{rad}^*(G, T)$ by the definition in chapter 1, so $X \in \Xi^*(G, T)$ by 1.3.8. As $Aut_L(X/O_2(X))$ contains $SL_2(q)$ for $q > 3$, $X \notin \Xi_-(G, T)$, so X centralizes V by 3.2.13. Hence

$$Y := \prod_{p \in \pi} \Xi_p(L) \leq C_L(V).$$

Let $I_p := O^{p'}(O_\infty(L))$. If I_p centralizes V for each $p \in \pi$, then $O_{2,F}(L) \leq O_2(L)Y \leq C_L(V)$, so $O_\infty(L)$ centralizes V as $L/O_{2,F}(L) \cong SL_2(q)$ and V is 2-reduced. Thus as L is a counterexample, there is $p \in \pi$ such that $I := I_p$ does not centralize V, so $I \neq X_p$ and hence case (d) of 1.2.1.4 holds and $I/O_2(I) \cong \mathbf{Z}_{p^e}^2$ for some $e > 1$. As case (d) of 1.2.1.4 holds, $L/O_{2,F}(L) \cong SL_2(5)$. Since $e > 1$, we conclude from A.1.30 that $p > 5$.

Set $R := C_T(V)$. As $V = R_2(LT)$, $O_2(LT) \leq R$. As $L/O_{2,F}(L) \cong SL_2(5)$, $O_2(LT) = O_2(IT)$, and then as $[I, V] \neq 1$, $R = O_2(IT)$ and $V \in \mathcal{R}_2(IT)$. As $X \in \Xi^*(G, T)$, $M := N_G(X) = !\mathcal{M}(XT)$ by 1.3.7. As $L \in \mathcal{L}^*(G, T)$, $L \trianglelefteq M$, so as I char L, $I \trianglelefteq M$. Thus for each $H \in \mathcal{H}_*(T, M)$, $H \cap M$ normalizes I, so case (I) of Hypothesis 3.1.5 is satisfied with IT in the role of "M_0". As $M = !\mathcal{M}(XT)$, $O_2(\langle IT, H\rangle) = 1$, so conclusion (2) or (3) of Theorem 3.1.6 holds. In either case $\hat{q}(IT/C_{IT}, V) \leq 2$. As $p > 5$ and $[V, I] \neq 1$, this contradicts D.2.13. □

3.3. Normalizers of uniqueness groups contain $N_G(T)$

The bulk of the proof of the Main Theorem analyzes the situation where $\mathcal{L}_f(G, T)$ is nonempty, leading (as we saw in 3.2.3) to the Fundamental Setup (3.2.1) and the extended analysis of the cases arising there. The very restricted situation where $\mathcal{L}_f(G, T)$ is empty will be treated only at the end of the proof after that analysis.

In this section, in Theorem 3.3.1 we establish an important property of maximal 2-locals containing T and suitable uniqueness subgroups. Theorem 3.3.1 will be used repeatedly in our analysis of the cases arising from the Fundamental Setup.

It turns out that case (2) of Theorem 3.3.1 is not actually required to prove the Main Theorem, contary to what we expected when we proved the result. However as the proof for this case is short, we have retained its statement and proof here.

THEOREM 3.3.1. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, $M \in \mathcal{M}(T)$, and either*

(1) $L \in \mathcal{L}^*(G,T)$ with $L/O_2(L)$ quasisimple and $L \leq M$, or

(2) $X \in \Xi^*(G,T)$ with $X \leq M$.

Then $N_G(T) \leq M$.

We first record an elementary but important consequence of Theorems 2.1.1 and 3.3.1, that we will use repeatedly in the remainder of the paper: In the Fundamental Setup, the members of $\mathcal{H}_*(T,M)$ are minimal parabolics.

COROLLARY 3.3.2. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}^*(G,T)$ with $L/O_2(L)$ quasisimple. Set $M := N_G(\langle L^T \rangle)$. Then*

(1) $M = !\mathcal{M}(\langle L,T \rangle)$.

(2) $|\mathcal{M}(T)| > 1$, so $\mathcal{H}_*(T,M) \neq \emptyset$.

(3) $N_G(T) \leq M$.

(4) For each $H \in \mathcal{H}_*(T,M)$, $H \cap M$ is the unique maximal subgroup of H containing T, and $H \in \hat{\mathcal{U}}_G(T)$ so that H is a minimal parabolic described in B.6.8, and in E.2.2 when H is nonsolvable.

PROOF. Part (1) follows from 1.2.7. Part (2) holds since 2-locals of odd index in the groups G in the conclusion of Theorem 2.1.1 are solvable, so that $\mathcal{L}(G,T)$ is empty. Part (3) follows from Theorem 3.3.1. Finally (4) follows from (3) and 3.1.3. □

REMARK 3.3.3. In the simple QTKE-groups G, $N_G(T) \leq M$ under the hypotheses of Theorem 3.3.1. However there is an almost simple shadow where this assertion fails: In the extension G of $\Omega_8^+(2)$ by a graph automorphism of order 3, there is a maximal parabolic L of $E(G)$ which is an A_8-block and is a member of $\mathcal{L}^*(G,T)$, but which is not invariant under an element of order 3 in $N_G(T)$ inducing the triality outer automorphism on $E(G)$. This extension is of even characteristic, but it is neither simple nor quasithin. However it is difficult to verify these global properties just from the point of view of the 2-local L, so that the shadow of this group causes difficulties in the proof of 3.3.21.f. Also the proof of 3.3.24 is complicated by the shadow of the non-maximal parabolic $L_3(2)/2^{3+6}$ in this same extension G.

Case (2) of the hypothesis of Theorem 3.3.1 will be eliminated fairly early in the argument in 3.3.10.3. Thus the bulk of the proof is devoted to case (1) of the hypothesis.

NOTATION 3.3.4. In case (1) of the hypothesis of Theorem 3.3.1, where $L \in \mathcal{L}^*(G,T)$ with $L/O_2(L)$ quasisimple, set $M_+ := L_0 := \langle L^T \rangle$. In case (2) of that hypothesis, where $X \in \Xi^*(G,T)$, set $M_+ := X$. As $N_G(T)$ is 2-closed and hence solvable, $N_G(T) = TD$, where D is a Hall $2'$-subgroup of $N_G(T)$.

We recall that M_+T is a uniqueness subgroup in the language of chapter 1:

LEMMA 3.3.5. *(1)* $M = N_G(M_+)$.
(2) $M = !\mathcal{M}(M_+T)$.
(3) $F^*(M_+T) = O_2(M_+T)$.

PROOF. Parts (1) and (2) are a consequence of 1.2.7.3 and 1.3.7. By definition $M_+T \in \mathcal{H}(T)$, so (3) follows from 1.1.4.6. □

Throughout this section, we assume we are working in a counterexample to Theorem 3.3.1, so that $N_G(T) \not\leq M$. Our arguments typically derive a contradiction by violating one of the consequences of 3.3.5.2 in the following lemma:

LEMMA 3.3.6. *(a)* $D \not\leq M$.
(b) $O_2(\langle M_+T, D\rangle) = 1$. *Thus if* $1 \neq X \trianglelefteq M_+T$, *then* $D \not\leq N_G(X)$.
(c) No nontrivial characteristic subgroup of T *is normal in* M_+T.
(d) Assume case (1) of Theorem 3.3.1 holds with $L/O_{2,Z}(L)$ *of Lie type and Lie rank* 2 *in characteristic* 2. *Then* T *acts on* L *unless possibly* $L/O_2(L) \cong L_3(2)$; *and if* T *acts on* L, *then* (LT, T) *is an* MS*-pair in the sense of Definition C.1.31.*

PROOF. Part (a) holds as $T \leq M$, but $TD = N_G(T) \not\leq M$. Then (b) follows from (a) and 3.3.5.2, and (c) follows from (b).

Assume the hypothesis of (d). Then unless $L/O_2(L) \cong L_3(2)$, T acts on L by 1.2.1.3. Assume T acts on L. Then (LT, T) satisfies hypothesis (MS1) in Definition C.1.31 by 3.3.5.3, hypothesis (MS2) is satisfied as T is Sylow in LT, and hypothesis (MS3) holds by (c). □

Set $Z := \Omega_1(Z(T))$, $V := \langle Z^{M_+}\rangle = \langle Z^{M_+T}\rangle$, $\overline{M_+T} := M_+T/C_{M_+T}(V)$, and $\tilde{V} := V/C_V(M_+)$.

LEMMA 3.3.7. *(1)* $C_{M_+T}(V) \leq O_{2,\Phi}(M_+T)$ *and* $C_T(V) = O_2(M_+T)$.
(2) $J(T) \not\leq C_T(V)$, *so* V *is a failure of factorization module for* $\bar{M}_+\bar{T}$.
(3) $V \in \mathcal{R}_2(M_+T)$, *so* $O_2(\bar{M}_+\bar{T}) = 1$.
(4) $[V, M_+] = [Z, M_+]$ *and* $V = [V, M_+]C_Z(M_+)$.

PROOF. Since $F^*(M_+T) = O_2(M_+T)$ by 3.3.5.3, part (3) is a consequence of B.2.14. As $V = \langle Z^{M_+}\rangle$, $V = [V, M_+]Z$, so that $V = [V, M_+]C_Z(M_+)$ using Gaschütz's Theorem A.1.39. If $\bar{M}_+ = 1$, then $V = Z$ and $M_+T \leq C_G(Z)$, contrary to 3.3.6.c. Thus $\bar{M}_+ \neq 1$, so (1) follows from (3) and 1.4.1.5 with M_+ in the role of "L_0". If $J(T) \leq C_T(V)$, then by B.2.3.3, $J(T) = J(C_T(V)) = J(O_2(M_+T)) \trianglelefteq M_+T$, contrary to 3.3.6.c. Thus $J(T) \not\leq C_T(V)$, so V is an FF-module for $\bar{M}_+\bar{T}$ by B.2.7. □

We now use 3.3.7 to determine a list of possibilities for $\bar{M}_+$ and V, which we will eliminate during the remainder of the proof. Notice if case (2) of the hypothesis of Theorem 3.3.1 holds, then conclusion (1) of the next lemma holds with $\bar{L}_i \cong \mathbf{Z}_3$.

LEMMA 3.3.8. *One of the following holds:*

(1) $\bar{M}_+ = \bar{L}_1 \times \bar{L}_2$ *with* $\bar{L}_i \cong L_2(2^n)$, $L_3(2)$, *or* $\mathbf{Z}_3$, *and* $\bar{L}_1^t = \bar{L}_2$ *for some* $t \in T - N_T(L_1)$. *Further* $[\tilde{V}, M_+] = \tilde{V}_1 \oplus \tilde{V}_2$, *where* $\tilde{V}_i := [\tilde{V}, L_i]$, *and either* $\tilde{V}_i$ *is the natural module for* $\bar{L}_i$, *the* A_5*-module for* $\bar{L}_i \cong A_5$, *or the sum of two isomorphic natural modules for* $\bar{L}_i \cong L_3(2)$.

(2) $\bar{M}_+ \cong L_2(2^n)$ *with* $n > 1$, *and* $[\tilde{V}, M_+]$ *is the natural module for* $\bar{M}_+$.

(3) $\bar{M}_+ \cong A_5$ *or* A_7, *and* $[V, M_+]$ *is the natural module for* $\bar{M}_+$.

(4) $\bar{M}_+ \cong SL_3(2^n)$, $Sp_4(2^n)'$, *or* $G_2(2^n)'$, *and* $[\tilde{V}, M_+]$ *is either the natural module for* $\bar{M}_+$ *or the sum of two isomorphic natural modules for* $\bar{M}_+ \cong SL_3(2^n)$.

(5) $\bar{M}_+ \cong A_7$, *and* $[V, M_+]$ *is of rank* 4.

(6) $\bar{M}_+ \cong \hat{A}_6$, *and* $[V, M_+]$ *is of rank* 6.

(7) $\bar{M}_+ \cong L_4(2)$ *or* $L_5(2)$, *and the possiblities for* $[V, M_+]$ *are listed in Theorem B.5.1.1.*

PROOF. By 3.3.7.2, V is an FF-module for $\bar{M}_+\bar{T}$, and by 3.3.7.3, $O_2(\bar{M}_+\bar{T}) = 1$. Hence the action of $\bar{J} := J(\bar{M}_+\bar{T}, V)$ on $[V, \bar{J}]$ is described in Theorem B.5.6.

In case (2) of Theorem 3.3.1, M_+T is a minimal parabolic, and using 3.3.7.1, $\bar{M}_+$ is noncyclic, so conclusion (1) of the lemma holds by B.6.9. Thus we may assume case (1) holds. Therefore $F^*(\bar{M}_+\bar{T}) = \bar{M}_+ = \bar{L}$ or $\bar{L}\bar{L}^t$ for $t \in T - N_T(L)$. Therefore as $1 \neq \bar{J} \trianglelefteq \bar{M}_+\bar{T}$, $\bar{M}_+ = F^*(\bar{J})$. Further if $L < M_+$, then $\bar{L} \cong L_2(2^n)$, $Sz(2^n)$, $L_2(p)$ or J_1 by 1.2.1.3. Therefore conclusion (1) of the lemma holds by B.5.6.

Thus we may assume that $L = M_+$, so that $\bar{L} = F^*(\bar{J}) = F^*(\bar{M}_+\bar{T})$ is quasisimple. Hence the action of L on V is described in Theorem B.5.1. The conclusions of the lemma include cases (ii), (iii), and (iv) of B.5.1.1 in which $[\tilde{V}, L]$ is reducible, so we may assume $[\tilde{V}, L]$ is irreducible. Hence by B.5.1 the possibilities for the action of $\bar{L}\bar{T}$ on $[\tilde{V}, L]$ are listed in Theorem B.4.2, and again our conclusions contain all those cases. □

LEMMA 3.3.9. $C_{M_+}(Z) = C_{M_+}(Z \cap [V, M_+])$.

PROOF. Since $Z = (Z \cap [V, M_+])C_Z(M_+)$ by 3.3.7.4, the lemma follows. □

We now begin to eliminate cases from 3.3.8:

LEMMA 3.3.10. *(1) If* $H \in \mathcal{H}(T)$ *and* T *is contained in a unique maximal subgroup of* H, *then* $O_2(\langle H, D\rangle) \neq 1$.

(2) $\bar{M}_+$ *is not* $L_2(2^n)$, *eliminating case (2) of 3.3.8 and the* A_5*-subcase of case (3) of 3.3.8.*

(3) If case (1) of 3.3.8 holds, then $\bar{L}_i \cong L_3(2)$.

(4) Case (1) of the hypothesis of Theorem 3.3.1 holds.

PROOF. Part (1) follows from Theorem 3.1.1, with TD, T in the roles of "M_0, R". In particular if T lies in a unique maximal subgroup of M_+T, then (1) contradicts 3.3.6.b. Parts (2) and (3) follow from this observation. Finally, as we observed earlier, if case (2) of the hypothesis of Theorem 3.3.1 holds, then conclusion (1) of 3.3.8 holds with $\bar{L}_i \cong \mathbf{Z}_3$. Thus (3) implies (4). □

REMARK 3.3.11. By 3.3.10.4, case (1) of Notation 3.3.4 holds. Therefore $M_+ = \langle L^T\rangle$, where $L \in \mathcal{L}^*(G,T)$ with $L/O_2(L)$ quasisimple. Thus L has this meaning from now on.

LEMMA 3.3.12. *Suppose* $Y \in \mathcal{L}(L,T)$ *and* $O_2(H) \neq 1$ *where* $H := \langle Y, TD\rangle$. *Then*

(1) $Y \leq K \in \mathcal{C}(H)$.

(2) $K \trianglelefteq H$.

(3) One of the following holds:

(a) $D \leq N_G(Y)$, *or*

(b) $Y/O_2(Y) \cong L_2(4)$, $K/O_2(K) \cong J_1$, $D = (K \cap D)N_D(Y)$, *and* $|D : N_D(Y)| = 7$. *Further* T *induces inner automorphisms on* $Y/O_2(Y)$.

(c) $Y/O_2(Y) \cong A_6$, $K/O_2(K) \cong U_3(5)$, *and* D *of order* 3 *induces an outer automorphism on* $K/O_2(K)$ *centralizing a subgroup isomorphic to the double covering of* S_5 *which is not* $GL_2(5)$.

PROOF. Part (1) follows from 1.2.4 applied with Y, H in the roles of "B, H".

By 3.3.6.b, $Y < L$. Applying 1.2.4 with Y, L in the roles of "B, H", and comparing the embeddings described in A.3.12 to the list of possibilities for L in 3.3.8, we conclude that $Y/O_2(Y)$ is $L_2(2^n)$, $L_3(2)$, A_6, or $L_4(2)$. Furthermore $\bar{L}$ is not $L_3(2)$, so we conclude from 3.3.10.3 and 3.3.8 that $M_+ = L$. Now by 1.2.8.1, T normalizes Y, and then T also normalizes K. Thus (2) follows from 1.2.1.3.

Assume that conclusion (a) of (3) fails; we must show that conclusion (b) or (c) of (3) holds. By (2), $Y < K$. Then $Y/O_2(Y)$ is described in the previous paragraph, and the possible proper overgroups K of Y are described in A.3.12.

Set $H^* := H/C_H(K/O_2(K))$, and let Y_H be the preimage of Y^* in H. We claim that $Y \trianglelefteq N_H(Y^*)$: By hypothesis, $Y \in \mathcal{L}(L,T)$, so Y is the unique member of $\mathcal{C}(O_2(K)Y)$. Then as $YO_2(K) \trianglelefteq Y_H$, $Y \in \mathcal{C}(Y_H)$ by A.3.3.2. Therefore as T acts on Y, $Y \trianglelefteq Y_H$ by 1.2.1.3, establishing the claim.

By assumption, $D \not\leq N_H(Y)$, so by the claim:

$$N_{D^*}(Y^*) = N_D(Y)^*, \quad \text{so} \quad D^* \not\leq N_{H^*}(Y^*). \tag{$*$}$$

In particular, $D^* \neq 1$. Similarly $C_D(K^*) \leq C_D(Y^*) < D$. Next $T_K := T \cap K \in Syl_2(K)$ and $1 \neq D^* \leq N_{H^*}(T_K^*)$, so

$$N_{H^*}(T_K^*) \geq T_K^* D^* > T_K^*. \tag{$**$}$$

Assume that K^* is sporadic; that is, K appears in one of cases (11)–(20) of A.3.12. Then $Out(K^*)$ is a 2-group, so $D^* \leq K^*$, and we conclude from (**) that $K^* \cong J_1$ or J_2. In the latter case, $Y^* \cong A_5/2^{1+4}$ is uniquely determined by A.3.12, and $D^* \leq Y^*$, contrary to (*). In the former case, $N_{H^*}(T_K^*) = N_H(T)^*$ is a Frobenius group of order 21, and T^* induces inner automorphisms on $Y^* \cong A_5$, so that $|D^* : N_{D^*}(Y^*)| = 7$. Thus $D = (D \cap K)N_D(Y)$ and $|D : N_D(Y)| = 7$ by (*). Then since the multiplier of J_1 is trivial by I.1.3, $K/O_2(K) \cong J_1$, so case (b) of conclusion (3) holds.

Thus we may assume K^* satisfies one of cases (2), (4)–(9), (21), or (22) of A.3.12. In cases (4)–(7), $Out(K^*)$ is a 2-group, so that $D^* \leq K^*$, and (**) supplies a contradiction. In case (2), K^* is of Lie type and Lie rank 2 in characteristic 2, with $Y^* = P^{*\infty}$ for some T-invariant maximal parabolic P^* of K^*. Thus as there are exactly two such parabolics,

$$D^* \leq O^2(N_{H^*}(T_K^*)) \leq N_{H^*}(P^*) \leq N_{H^*}(Y^*),$$

again contrary to (*).

In cases (21) and (22), T_K^* is contained in a unique complement K_1^* to $O(K^*)$ in K^*, with $K_1^* \cong SL_2(p)$ for an odd prime $p > 3$. By the uniqueness of K_1^*, $Y^* \leq K_1^*$ and D^* acts on K_1^*, so that $Y^* < K_1^*$ by (*). So replacing K by the $\mathcal{C}$-component K_1 of the preimage of K_1^*, we reduce the treatment of these cases to the elimination of the subcase of case (8) where $H^* \cong L_2(p)$ for some prime $p \equiv \pm 3 \mod 8$ and $Y^* \cong L_2(5)$. Then as $D^* \neq 1$ normalizes T^*, $A_4 \cong N_{H^*}(T^*) = T^*D^* \leq Y^*T^* \leq N_{H^*}(Y^*)$, again contrary to (*). In the remaining subcase of (8), $K^* \cong L_2(p^2)$ for an odd prime p. Here T_K^* is dihedral of order greater than 4 and self-centralizing in $Aut(K^*)$, so that $N_{H^*}(T_K^*)$ is a 2-group, and then $D^* = 1$, contrary to (*).

Thus case (9) of A.3.12 holds, with $K^* \cong L_3^\epsilon(p)$. If $Y^* \cong SL_2(p)$, $Y^* = C_{K^*}(Z(T_K^*))^\infty$ is D^*-invariant, again contrary to (*).

In the remaining subcase of (9), $K^* \cong U_3(5)$, with $Y^* \cong A_6$. Here $X^* = O^2(C_{Aut(K^*)}(T_K^*))$ is of order 3 and induces outer automorphisms on K^* with $C_{K^*}(X^*)$ the double covering of S_5 which is not $GL_2(5)$. We conclude $D^* = X^*$. Finally $K/O_2(K)$ is not $SU_3(5)$ by A.3.18. Therefore $K/O_2(K) \cong U_3(5)$, so case (c) of conclusion (3) holds.

This completes the treatment of the cases appearing in A.3.12, and hence completes the proof of the lemma. □

LEMMA 3.3.13. *If $H \in \mathcal{H}(T)$ with $H/O_2(H) \cong S_3$ wr $\mathbf{Z}_2$, then $D \leq N_G(H)$.*

PROOF. Let $H_0 := \langle H, D\rangle$; by 3.3.10.1, $O_2(H_0) \neq 1$. Set $Y := O^2(H)$ and notice $Y \in \Xi(G,T)$. If D normalizes Y, then D normalizes $YT = H$ and the lemma holds, so we assume that D does not act on Y. Therefore Y is not normal in H_0, so by 1.3.4, $Y < K_0 := \langle K^T\rangle$ for some $K \in \mathcal{C}(H_0)$, and K_0 is a normal subgroup of H_0 described in cases (1)–(4) of 1.3.4 with 3 in the role of "p". Let $(K_0TD)^* := K_0TD/C_{K_0TD}(K_0/O_2(K_0))$. Notice that $O_2(K_0) \leq O_2(H_0) \leq O_2(H)$ using A.1.6, so that $N_{D^*}(Y^*) = N_D(Y^*)$. Hence D^* does not act on Y^* and in particular $D^* \neq 1$, so that

(*) T^* is not self-normalizing in $K^*T^*D^*$.

Further $H^*/O_2(H^*) \cong H/O_2(H) \cong O_4^+(2)$, so

(**) $T^*/O_2(Y^*T^*) \cong D_8$.

Inspecting the list in 1.3.4 for cases in which (*) and (**) are satisfied, we conclude that case (1) of 1.3.4 holds, with $K^* \cong L_2(2^n)$ for $2^n \equiv 1 \mod 3$, and H^* is contained in the T-invariant Borel subgroup B^* of K_0^*. As D^* acts on T^*, D^* acts on B^* and hence also on the characteristic subgroup Y^* of B^*, contrary to an earlier remark. This completes the proof. □

LEMMA 3.3.14. *$L = M_+ \trianglelefteq M$, eliminating case (1) of 3.3.8.*

PROOF. Assume otherwise. Then by 3.3.10.3, $\bar{L}_i \cong L_3(2)$. Therefore $M_+T = \langle H_1, H_2\rangle$, where $H_i := \langle H_{i,1}, T\rangle$ and $\bar{H}_{i,1}$, $i = 1, 2$, are the maximal parabolics of $\bar{L}_1$ over $\bar{T} \cap \bar{L}_1$. Notice that $H_i/O_2(H_i) \cong S_3$ wr $\mathbf{Z}_2$, so by 3.3.13, D normalizes H_i. But then D normalizes $M_+T = \langle H_1, H_2\rangle$, contrary to 3.3.6.b. □

Our next lemma puts us in a position to exploit an argument much like that in the proof of 3.3.14, to eliminate many cases where L is generated by a pair of members of $\mathcal{L}(L,T)$.

LEMMA 3.3.15. *Suppose $LT = \langle Y_1, Y_2, T\rangle$ with $Y_j \in \mathcal{L}(L,T)$. Set $H_j := \langle Y_j, TD\rangle$, and assume $O_2(H_j) \neq 1$ for $j = 1$ and 2. Then for $i = 1$ or 2: D does not normalize Y_i, $Y_i/O_2(Y_i) \cong L_2(4)$ or A_6, $Y_i < K \in \mathcal{C}(H_i)$ such that $K/O_2(K) \cong J_1$ or $U_3(5)$, respectively, $K \trianglelefteq H_i$, and $D \not\leq M$. When $K/O_2(K) \cong J_1$, T induces inner automorphisms on $Y_i/O_2(Y_i)$ and $K \cap D \not\leq M$.*

PROOF. Notice Y_j, H_j satisfy the hypotheses of 3.3.12 in the roles of "Y, H", so we can appeal to that lemma. Suppose D normalizes both Y_1 and Y_2. Then D normalizes $\langle Y_1, Y_2, T\rangle = LT$, contradicting 3.3.6.b. Thus D does not normalize some Y_i, so the pair Y_i, H_i is described in case (b) or (c) of 3.3.12.3. Further $D \not\leq M$ by 3.3.6.a, and when $K/O_2(K) \cong J_1$, $K \cap D \not\leq M$ by 3.3.12. □

LEMMA 3.3.16. *$\bar{L}$ is not $SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ with $n > 1$.*

PROOF. Assume otherwise. Let $T_L := T \cap L$ and $\bar{M}_i$, $i = 1, 2$, be the maximal parabolics of $\bar{L}$ containing $\bar{T}_L$. Set $Y_i := M_i^\infty$; then $Y_i \in \mathcal{L}(L,T)$ with $Y_i/O_2(Y_i) \cong L_2(2^n)$ and $LT = \langle Y_1, Y_2, T\rangle$. By 3.3.10.1, $H_i := \langle Y_i, TD\rangle \in \mathcal{H}(T)$. Thus by 3.3.15, we may assume that D does not normalize $Y_1 =: Y$, $n = 2$, and $Y < K \in \mathcal{C}(H_1)$ with $K \trianglelefteq H_1$, $K/O_2(K) \cong J_1$, $K \cap D \not\leq M$, and T induces inner automorphisms on $Y/O_2(Y)$. Set $H_1^* := H_1/C_{H_1}(K/O_2(K))$. Then $Y^* \cong L_2(4)$, so $O_2(Y^*) = 1$.

By 3.3.6.d, (LT, T) is an MS-pair, and so we may apply the Meierfrankenfeld-Stellmacher result Theorem C.1.32. Since $n = 2$, $L/O_2(L)$ is $SL_3(4)$ or $Sp_4(4)$ or $G_2(4)$. By Theorem C.1.32, $L/O_2(L)$ is not $G_2(4)$, and if $L/O_2(L) \cong Sp_4(4)$, then L is an $Sp_4(4)$-block.

As T induces inner automorphisms on $Y/O_2(Y)$, T induces inner automorphisms on $L/O_2(L)$ from the structure of $Aut(L/O_2(L))$. From the structure of $L/O_2(L)$, $X := C_{D\cap L}(Y/O_2(Y))$ is of order 3, and as X normalizes T, $Q := [X,T]$ is a 2-group. Now $X \leq D \leq H_1$, and we saw that $K \trianglelefteq H_1$. As $[X^*, Y^*] \leq O_2(Y^*) = 1$ and $C_{Aut(K^*)}(Y^*)$ is of order 2 since $K^* \cong J_1$, we conclude $X^* = 1$. Therefore $Q^* = [X^*, T^*] = 1$, so $Q = [X, O_2(KT)] = O_2(O^2(XO_2(KT))) \trianglelefteq KT$. But if $L/O_2(L) \cong SL_3(4)$, then $O_2(L)X = O_{2,Z}(L) \trianglelefteq LT$, so that $Q = [O_2(L), X]$ is also normal in LT, and hence $K \leq N_G(Q) \leq M = !\mathcal{M}(LT)$, contradicting $K \cap D \not\leq M$.

Therefore L is an $Sp_4(4)$-block. Now $O_2(L)$ is of order at most 2^{10} using the value for 1-cohomology of the natural module in I.1.6; thus Q is of order at most

$$|O_2(Y) : O_2(L)||O_2(L)| \leq 2^6 \cdot 2^{10} = 2^{16}.$$

Therefore as 19 divides the order of J_1 but not of $L_{16}(2)$, K centralizes Q. This is impossible as $Y \leq K$ and Y is nontrivial $QO_2(L)/O_2(L)$. This contradiction completes the proof. □

LEMMA 3.3.17. *If $\bar{L} \cong A_7$, then $m([V,L]) = 6$, eliminating case (5) of 3.3.8.*

PROOF. Assume the lemma fails. Then by 3.3.8, $m([V,L]) = 4$. We work with two of the three proper subgroups in $\mathcal{L}(L,T)$. First, let $M_1 := C_L(Z)^\infty$. By 3.3.9, $C_L(Z) = C_L(Z \cap [V,L])$, so $\bar{M}_1 = C_{\bar{L}}(Z) \cong L_3(2)$. Then $1 \neq Z \leq O_2(\langle TD, M_1\rangle)$. Second, there is $M_2 \in \mathcal{L}(L,T)$ with $\bar{M}_2\bar{T} \cong S_5$, so by 3.3.10.1, $O_2(\langle M_2, TD\rangle) \neq 1$. As $LT = \langle M_1, M_2, T\rangle$ and $M_iT/O_2(M_iT)$ is not isomorphic to $L_2(4)$ or A_6, 3.3.15 supplies a contradiction. □

LEMMA 3.3.18. *If $\bar{L} \cong L_n(2)$ with $n = 4$ or 5, then $[V,L]$ is not the direct sum of isomorphic natural modules.*

PROOF. Assume otherwise; then $[V,L] = V_1 \oplus \cdots \oplus V_r$, where the V_i are isomorphic natural modules for $\bar{L}$. Therefore T induces inner automorphisms on $L/O_2(L)$, and in particular normalizes each parabolic of L containing $T \cap L$.

Let $Y_1 := C_L(Z)^\infty$, and recall $C_L(Z) = C_L(Z \cap [V,L])$ by 3.3.9. As the natural submodules V_i are isomorphic, $C_L(Z \cap [V,L])$ is the parabolic stabilizing a vector in each V_i, so that $\bar{Y}_1 \cong L_{n-1}(2)/E_{2^{n-1}}$, and hence $Y_1 \in \mathcal{L}(L,T)$.

Let W_1 be the T-invariant 3-subspace of V_1, and set $Y_2 := N_L(W_1)^\infty$. Then $\bar{Y}_2 \cong L_3(2)/E_8$ or $L_3(2)/E_{64}$ for $n = 4$ or 5, respectively, so $Y_2 \in \mathcal{L}(L,T)$. If some nontrivial characteristic subgroup of T were normal in Y_2T, then $O_2(\langle Y_2T, D\rangle) \neq 1$; so as $L = \langle Y_1, Y_2, T\rangle$, and $Y_2/O_2(Y_2) \cong L_3(2)$ rather than $L_2(4)$ or A_6, we have a contradiction to 3.3.15. It follows that (Y_2T, T) is an MS-pair in the sense of

Definition C.1.31. As $Y_2/O_2(Y_2) \cong L_3(2)$, case (5) of Theorem C.1.32 holds, so that Y_2T is described in C.1.34. Since T is Sylow in Y_2T, case (5) of C.1.34 does not hold, so that one of cases (1)–(4) of C.1.34 holds.

Let $Q := [O_2(Y_2T), Y_2]$ and $U := Z(Q)$. By B.2.14, $Z \leq \Omega := \Omega_1(Z(O_2(Y_2T)))$, so $[Y_2, Z] \leq Q \cap \Omega = U$ and hence $W_1 \leq [Z, Y_2] \leq U$ and Y_2 acts on UZ. Then by 12.8 in [**Asc86a**], $UZ = UZ_0$, where $Z_0 := C_Z(Y_2)$, so $Z = Z_0(Z \cap U)$. On the other hand $C_{V_i}(Y_2) = 1$ for each i, so $C_V(Y_2) = 1$ and hence $Z_0 = C_Z(L)$ by 3.3.7.4. Then as $M = !\mathcal{M}(LT)$, $M = !\mathcal{M}(C_G(z_0))$ for each $z_0 \in Z_0^\#$.

Assume that case (4) of C.1.34 holds. Then $U = U_0 \oplus U_1$, where $U_0 := C_U(Y_2T)$ is of rank 2 and U_1 is a natural module for $Y_2T/O_2(Y_2T) \cong L_3(2)$. Thus $U \cap Z = U_0 \oplus Z_1$, where $Z_1 := U_1 \cap Z$ is of order 2, so as $Z = Z_0(U \cap Z)$, $|Z : Z_0| = 2$. Further $m(Z) \geq m(U \cap Z) = 3$, so for each $d \in D$, $Z_0 \cap Z_0^d \neq 1$. Finally by an earlier remark,

$$M^d = !\mathcal{M}(C_G(z^d)) = M \text{ for some } z \in Z_0^\# \text{ with } z^d \in Z_0.$$

Thus $D \leq N_G(M) = M$ as $M \in \mathcal{M}$, contradicting 3.3.6.a. Hence case (4) of C.1.34 is eliminated.

Next $\bar{Y}_2$ has $m := 1$ or 2 noncentral 2-chief factors in $O_2(\bar{Y}_2)$, for $n = 4$ or 5, respectively, and Y_2 has $r \geq 1$ noncentral 2-chief factors in $[V, L] \leq O_2(L)$. Therefore Y_2 is not an $L_3(2)$-block, eliminating case (1) of C.1.34. Next the chief factor(s) for Y_2 in $O_2(\bar{Y}_2)$ are isomorphic to $W_1 \leq U$, so case (3) of C.1.34 is also eliminated, since there the noncentral 2-chief factors of Y_2 other than U lie in Q/U and are dual to U. Thus case (2) of C.1.34 holds, so $Q = U = U_1 \oplus U_2$ is the sum of two isomorphic natural modules U_i, and in particular Y_2 has exactly two noncentral 2-chief factors. Thus $m + r \leq 2$, so as $m \geq 1 \leq r$, it follows that $m = r = 1$, and therefore $n = 4$ and $V = V_1 = [O_2(L), L]$. We may choose notation so that $W_1 \leq U_1$. As $V = [O_2(L), L]$, L is an $L_4(2)$-block, so $P := O_2(Y_1) \cong D_8^3$, $P/Z(P) = P_1/Z(P) \oplus P_2/Z(P)$ is the sum of two nonisomorphic natural modules $P_i/Z(P)$ for Y_1/P, and we may choose notation so that $V_1 = P_1$. Thus as we saw that D normalizes Y_1, D normalizes $O_2(Y_1) = P$, and hence $D = O^2(D)$ also normalizes P_1. Then as $P_1 = V_1 \trianglelefteq LT$, $D \leq N_G(P_1) \leq M = !\mathcal{M}(LT)$, contradicting 3.3.6.b. This completes the proof. □

LEMMA 3.3.19. *$\bar{L}$ is not $L_5(2)$.*

PROOF. Assume otherwise, and let $Y := C_L(Z)^\infty$. As $V = \langle Z^L \rangle$, part (4) of Theorem B.5.1 shows that $V = [V, L] \oplus C_Z(L)$. Since 3.3.18 eliminates case (iv) of B.5.1.1, either case (iii) of that result holds with $[V, L]$ the sum of the natural module and its dual, or case (i) there holds, with $[V, L]$ irreducible. In the latter case by Theorem B.4.2 and 3.3.18, $[V, L]$ is a 10-dimensional irreducible.

Assume first that $[V, L]$ is the sum of the natural module and its dual. Then by B.5.1.6, $\bar{Y} \cong L_3(2)/2^{1+6}$, so $Y \in \mathcal{L}(L, T)$. By 3.3.12.3, D acts on Y, and then also on $J(O_2(YT))$. But again by B.5.1.6 (notice we can apply B.2.10 with $O_2(YT)$ in the role of "R"), we see that $J(O_2(YT)) \leq C_T(V) = O_2(LT)$, so $J(O_2(YT)) = J(O_2(LT))$ by B.2.3.3. Hence $D \leq N_G(J(O_2(LT))) \leq M = !\mathcal{M}(LT)$, contradicting 3.3.6.b.

Therefore $[V, L]$ is irreducible of dimension 10, and in particular is the exterior square of a natural module. So this time (see e.g. K.3.2.3) Y is the parabolic determined by the stabilizer of a 2-space in that natural module; again $Y/O_2(Y) \cong L_3(2)$ so $Y \in \mathcal{L}(L, T)$ and as before D normalizes Y by 3.3.12.3. Now $O_2(\bar{Y}\bar{T})$ does not contain the unipotent radical of the maximal parabolic determined by the end node stabilizing a 4-space in the natural module. Thus by B.4.2.11 (again for more detail see K.3.2.3), $J(O_2(YT)) \leq C_T(V)$, so again $J(O_2(YT)) = J(O_2(LT))$, for the same contradiction. The proof is complete. □

The next technical result has the same flavor as 3.3.12, and will be used in a similar way. In particular it will help to eliminate the shadows discussed earlier.

LEMMA 3.3.20. *Assume $X = O^2(X)$ is T-invariant with $XT/O_2(XT) \cong S_3$, and D does not normalize $R := O_2(XT)$. Let $Y := \langle X^D \rangle$, and let γ denote the number of noncentral 2-chief factors for X. Then*

(1) $\langle XT, D\rangle \in \mathcal{H}(T)$ and $Y \trianglelefteq \langle XT, D\rangle = YTD$.

(2) $YT/O_2(YT) \cong L_2(p)$ for a prime $p \equiv \pm 11 \mod 24$.

(3) $O_2(X) \leq O_2(Y)$, $XT/O_2(YT) \cong D_{12}$, and $|D : N_D(X)| = 3$.

(4) $\gamma \geq 3$.

(5) If $\gamma \leq 4$, then:

(a) Y has a unique noncentral 2-chief factor W, $m(W) \geq 10$, and $|T| \geq 2^{12}$.

(b) $\Phi(O_2(X)) \leq Z(Y)$.

(c) If $Z(YT) \neq 1$, then $|T| > 2^{12}$.

PROOF. Let $B := \langle XT, D\rangle$. As D does not act on R, $R \neq 1$. Thus $XT \in \mathcal{H}(T)$ and T is maximal in XT as $XT/R \cong S_3$. Therefore $B \in \mathcal{H}(T)$ by 3.3.10.1. Also $X^{TD} = X^D$ so $Y \trianglelefteq B$, establishing (1).

Notice using A.1.6 that $O_2(B) \leq O_2(XT) = R$. Let B_0 be maximal subject to $B_0 \trianglelefteq B$ and $XT \cap B_0 \leq R$. Then $XT \cap B_0 = R \cap B_0 = T \cap B_0 =: T_0$ contains $O_2(B)$ and is invariant under XT and D, so $T_0 \trianglelefteq B$. Thus $T_0 = O_2(B)$. As D does not act on R by hypothesis, $T_0 < R$.

Set $B^* := B/B_0$. As $T_0 = XT \cap B_0 < R$, $R^* \neq 1 \neq X^*$. Then as $XT/R \cong S_3$, $|T^*| = 2|R^*| > 2$.

Let B_1^* be a minimal normal subgroup of B^*. By maximality of B_0, $XT \cap B_1 \not\leq R = O_2(XT)$, so $X^* \cap B_1^*$ is not a 2-group. So as $|X : O_2(X)| = 3$ and $X = O^2(X)$, $X^* \leq B_1^*$. Then by minimality of B_1^*, $B_1^* = \langle X^{*D}\rangle = Y^*$. In particular Y^* is the unique minimal normal subgroup of B^*, so $Y^* = F^*(B^*)$; hence T^* is faithful on Y^*.

Suppose Y^* is solvable. Then $Y^* \cong E_{3^n}$ as Y^* is a minimal normal subgroup of B^*. As $Y^* = \langle X^{*D}\rangle$, and D acts on T with X^* a simple T-submodule of Y^*, Y^* is a semisimple T-module. Therefore as T^* is faithful on Y^*, $\Phi(T^*) = 1$, and as $|T^*| > 2$, $m(T^*) > 1$. Then by (1) and (2) of A.1.31, $m(T^*) = 2$ and $m(C_{Y^*}(t^*)) \leq 1$ for each $t^* \in T^{*\#}$, so that $n = 2$ or 3. Further if $n = 3$, then as $B = YTD$ by (1), $T^*D^* \cong A_4$ is irreducible on Y^*, contrary to A.1.31.3. Thus $n = 2$, so $T^*D^* \leq GL_2(3)$. Then as $\Phi(T^*) = 1$ and D^* is a subgroup of $GL_2(3)$ of odd order normalizing T^*, $D^* = 1$. Hence $Y^* = \langle X^{*D^*}\rangle = X^*$, contradicting $n = 2$.

So Y^* is not solvable, and hence $Y^* = F^*(B^*) = Y_1^* \times \cdots \times Y_s^*$ is the direct product of isomorphic simple groups Y_i^* permuted transitively by TD. Then (1.a) of Theorem A (A.2.1) holds, so $m_q(Y^*) \leq m_q(B) \leq 2$ for each odd prime q, so that $s \leq 2$ and Y^* is an SQTK-group. Thus as $D = O^2(D)$, D normalizes each Y_i^*, so T is transitive on the Y_i^*. Therefore if T acts on Y_1^*, then $s = 1$ and Y^* is simple. As $Y^* = \langle X^{*D^*}\rangle$ and Y^* is not solvable, $D^* \neq 1$.

As D does not act on X, there is $g \in D - N_G(X)$. Set $G_1 := XT$, $G_2 := X^gT$, and $G_0 := \langle G_1, G_2\rangle$. As $XT/R \cong S_3$ and D acts on T, (G_0, G_1, G_2) is a Goldschmidt triple as in Definition F.6.1. Thus if g does not act on $R = O_2(XT)$, $O_2(XT) \neq O_2(X^gT)$, so $G_0^+ := G_0/O_{3'}(G_0)$ is described in Theorem F.6.18 by F.6.11.2.

Suppose for each $g \in D - N_G(X)$ that the group G_0^+ defined by g satisfies case (1) or (2) of F.6.18. Then $O_2(G_0) = R \cap R^g$ is normalized by XT for all $g \in D$, so

$$R_D := \bigcap_{d \in D} R^d = \bigcap_{d \in D} (R \cap R^d)$$

is invariant under XT and D, and hence $R_D \leq O_2(B) = T_0$. Therefore as $T_0 \leq R$, $R_D = T_0$. Also $\Phi(T) \leq R \cap R^d$ since $T/(R \cap R_d) \cong \mathbf{Z}_2$ or E_4 in cases (1) and (2) of F.6.18, so $\Phi(T) \leq T_0$ and hence $\Phi(T^*) = 1$. Thus T^* acts on each Y_i^* as $T^* \cap Y_i^* \neq 1$, so $s = 1$ and $Y^* = F^*(B^*)$ is a simple SQTK-group by earlier remarks. As $\Phi(T^*) = 1$, we conclude from Theorem C (A.2.3) that $Y^* \cong L_2(2^n)$, J_1, or $L_2(p)$ for a prime $p \equiv \pm 3 \mod 8$. As $X^*T^*/R^* \cong S_3$, the first two cases are eliminated. In the third case $B^* = Y^*$ as $Y^* = F^*(B^*)$ and $\Phi(T^*) = 1$. Thus $X^*T^* \cong D_{12}$, and $N_{B^*}(T^*) \cong A_4$. Then from the list of maximal subgroups of B^* in Dickson's Theorem A.1.3, $B^* = Y^*T^* = \langle X^*T^*, X^{*g}T^*\rangle$, contrary to our assumption that each $g \in D - N_G(X)$ defines a group G_0^+ satisfying case (1) of (2) of F.6.18.

Therefore we may choose $g \in D - N_G(X)$ so that G_0^+ satisfies one of the remaining cases (3)–(13) of F.6.18. In particular inspecting those cases, $1 \neq G_0^{+\infty} = E(G_0^+)$ is quasisimple. Then as $O_{3'}(G_0)$ is solvable by F.6.11.1, we conclude from 1.2.1.1 that $K_0 := G_0^\infty$ is the unique member of $\mathcal{C}(G_0)$, and $K_0^+ = E(G_0^+)$. Hence $K_0 \in \mathcal{L}(G,T)$. By 1.2.4, $K_0 \leq K \in \mathcal{C}(B)$, and $K \trianglelefteq B$ as T acts on K_0. As $T \cap B_0 = O_2(B_0)$, $K^* \neq 1$, so as Y^* is the unique minimal normal subgroup of B^*, $K^* = Y^* = F^*(B^*)$ is simple.

Assume for the moment that $K_0^* < K^*$. Set $T_K := T \cap K \in Syl_2(K)$. We compare the possiblities for K_0^+ described in F.6.18 to the embeddings described in A.3.12, to obtain a list of possiblities for K^*. Cases (2), (3), (15), (16), and (22) of A.3.12 do not arise, since there the candidate "$B/O_{3'}(B)$" for K_0^+ does not appear in F.6.18; this also eliminates the subcase of (8) with $K^* \cong L_2(p)$ for $p \equiv \pm 3 \mod 8$ and $K_0^* \cong A_5$. In cases (4)–(7), (11)–(14), and (17)–(21), and also in the remaining subcase of (8) where $K^* \cong L_2(p^2)$, $Aut(K^*)$ is a 2-group, so $B^* = K^*T^*$ since $K^* = Y^* = F^*(B^*)$. Furthermore in each case T_K^* is self-normalizing in $Aut(K^*)$, so $N_{B^*}(T^*) = T^*$ in these cases.

Next assume we are in the subcase of (9) where $K^* \cong U_3(5)$ and $K_0^* \cong A_6$. As in the proof of 3.3.12, D^* induces a group of outer automorphisms of order 3 on K^* centralizing T_K^*, and as D^* normalizes T^*, T^* induces inner automorphisms on K^* so that $B^* = K^*D^*$ and $T_K^* = T^*$. Now as D centralizes $T_K^* = T^* \in Syl_2(B^*)$, D centralizes $O_2(X^*T^*)$, so D normalizes the preimage S in B of $O_2(X^*T^*)$, and hence as $O_{2,Z}(K)$ is 2-closed, D normalizes $O^{2'}(S) = O_2(XT) = R$, contrary to the hypothesis of the lemma.

Finally in the remaining subcase of (9) and in (10), $K^* \cong L_3^\epsilon(p)$ with $K_0^* \cong SL_2(p)$ or $SL_2(p)/E_{p^2}$ for an odd prime p, since $K^* = Y^*$ is simple.

Thus we have shown that one of the following holds:

(a) $K_0^* = K^*$.
(b) $N_{B^*}(T^*) = T^*$.

(c) $K_0^* < K^* \cong L_3^\epsilon(p)$ for some odd prime p.

In case (b), $D^* = 1$, contrary to an earlier remark. Suppose case (c) holds. Then from the structure of $N_{Aut(K^*)}(T^*)$, $D^* \leq D_0^*$, where D_0^* is a cyclic subgroup of K^* of order dividing $p - \epsilon$ centralizing T^*. Further we saw that $K_0^* \cong SL_2(p)$ or $SL_2(p)/E_{p^2}$. But now $[K_0^*, D^*] \leq [K_0^*, D_0^*] \leq O(K_0^*)$, contradicting $G_0^* = \langle X^*T^*, X^{*g}T^* \rangle$.

Therefore case (a) holds, with $K_0^* = K^* = Y^* = F^*(B^*)$, and D^* acts on $Y^*T^* = G_0^*$, so $G_0^* \trianglelefteq B^* = Y^*T^*D^* \leq Aut(K^*)$. Recall G_0^+ satisfies one of cases (3)–(13) of F.6.18, but does not satisfy (b). As $F^*(G^*) = K^*$ is simple and K_0^+ is quasisimple, $K^* \cong K_0^+/Z(K_0^+)$. Examining F.6.18 for groups with $T^* < N_{G^*}(T^*)$, we conclude case (4) or (10) of F.6.18 holds. However $G_2 = G_1^g$, so $G_2^* \cong G_1^*$, ruling out case (10) of F.6.18, since $G_1^+ Z(K_0^+)/Z(K_0^+) \cong G_1^* \cong G_2^* \cong G_2^+ Z(K_0^+)/Z(K_0^+)$. This leaves case (4) of F.6.18, so we conclude that $G_0^+ = K_0^+ \cong L_2(p)$, $p \equiv \pm 11 \mod 24$, and $X^+T^+ \cong D_{12}$. As G_0^+ is simple, $G_0^+ \cong G_0^* = K^*$. Further $Aut(K^*)$ is a 2-group, so $B^* = G_0^* D^* = K^* \cong G_0^+$.

Next there is $t \in T \cap K$ with $X^* = [X^*, t^*]$, so $X = [X, t] \leq K$, and hence $Y = \langle X^D \rangle \leq K$ as $K \trianglelefteq B$. By (1), $Y \trianglelefteq B = YTD$, so since $K \in \mathcal{C}(B)$ with $K^* = B^* \cong G_0^+ \cong L_2(p)$, we conclude from 1.2.1.4 that either (2) holds, or $Y/O_2(Y) \cong SL_2(p)/E_{p^2}$. However in the latter case, by a Frattini Argument, $Y = O_p(Y)Y_0$, where $Y_0 := N_Y(T_1)$ and $T_1 := T \cap O_\infty(Y)$. But then XT and D act on T_1, so $T_1 \leq O_2(B)$, whereas $T_1 \not\leq O_2(Y)$. Thus (2) is established.

We saw that $X^*T^* \cong D_{12}$, and from (2), $N_{B^*}(T^*) \cong A_4$, so (3) follows. Further we observed earlier that $B^* \cong G_0^+$, so $B^* = \langle X^*T^*, X^{*g}T^* \rangle$ for $g \in D$ with $g^* \neq 1$.

Let W be a noncentral 2-chief factor of Y, $n := m(W)$ and $\alpha := m([W, X^*])/2$. Then α is the number of noncentral chief factors for X^* on W, so $\alpha \leq \gamma$. As $B^* = \langle X^*T^*, X^{*g}T^* \rangle$, $C_W(X) \cap C_W(X^g) = 0$, so $n \leq 2m([W, X^*]) = 4\alpha$. On the other hand, a Borel subgroup of B^* is a Frobenius group of order $p(p-1)/2$, so $n \geq (p-1)/2$ and hence $p \leq 2n + 1 \leq 8\alpha + 1$. Thus either $\alpha > 4$ or $p \leq 33$, and in the latter case as $p \equiv \pm 11 \mod 24$, $p = 11$ or 13. As neither 11 nor 13 divides the order of $GL_9(2)$, we conclude that $n \geq 10$ and hence $\alpha \geq n/4 > 2$. Thus as $\gamma \geq \alpha$, (4) holds.

It remains to prove (5), so assume that $\gamma \leq 4$. Then $\alpha \leq 4$, so by the previous paragraph, W is the unique noncentral 2-chief factor for Y, and $m(W) \geq 10$. Then as $|T^*| = 4$, $|T| \geq 2^{12}$, with equality only if $p = 11$ and $W = O_2(YT)$, so that $Z(YT) = 1$. Therefore parts (a) and (c) of (5) hold. Finally $W = U/U_0$ where $U := [O_2(Y), Y]$ and $U_0 := C_U(Y)$, and as $O_2(X) \leq O_2(Y)$ by (3), $O_2(X) = [O_2(X), X] \leq U$. Then as U/U_0 is elementary abelian and $X \leq Y$, $\Phi(O_2(X)) \leq U_0 \leq Z(Y)$, establishing part (b) of (5). This completes the proof. □

In the next lemma, we eliminate the first occurrence of the shadow of $\Omega_8^+(2)$ extended by triality.

Proposition 3.3.21. *$\bar{L}$ is not $L_4(2)$, eliminating case (7) of 3.3.8.*

Proof. Assume otherwise. Arguing as in the proof of 3.3.19 via appeals to Theorems B.5.1, B.4.2, and 3.3.18, we conclude:

(a) Either

(1) $[V, L] = U_1 \oplus U_2$, where U_1 is a natural submodule of V and U_2 is the dual of U_1, or

(2) $[\tilde{V}, L]$ is the 6-dimensional orthogonal module for $\bar{L}$.

Next by 3.3.9, and appealing to B.5.1.6 in case (a1):

(b) In case (a1), $C_{\bar{L}}(Z) \cong S_3/2^{1+4}$.

(c) In case (a2), $C_{\bar{L}}(Z) \cong (S_3 \times S_3)/E_{16}$.

Let $R := O_2(C_L(Z)T)$. Then $\bar{R}$ is the unipotent radical of the parabolic $C_{\bar{L}}(Z)$ of $\bar{L}$, so $N_M(R) \leq N_M(O^2(C_L(Z)))$. By B.5.1.6 and B.4.2.10, $J(R) \leq C_T(V) = O_2(LT)$, so that $J(R) = J(O_2(LT))$ by B.2.3.3, and hence $N_G(R) \leq N_G(J(R)) \leq M$ as $M = !\mathcal{M}(LT)$. Therefore as we just showed that $O^2(C_L(Z))$ is normal in $N_M(R)$:

(d) $J(R) = J(O_2(LT))$, and $O^2(C_L(Z)) \trianglelefteq N_G(R) \leq M$. Thus D does not act on R, and hence does not act on $O^2(C_L(Z))$.

Next we show:

(e) $C_Z(L) = 1$, so $Z \leq [V,L] = V$. Further when (a2) holds, L is irreducible on V.

For if $C_Z(L) \neq 1$, then $C_G(Z) \leq C_G(C_Z(L)) \leq M = !\mathcal{M}(LT)$, so $O^{3'}(C_G(Z)) = O^{3'}(C_M(Z)) = O^2(C_L(Z))$ is D-invariant, contrary to (d). Then since $V = [V,L]C_Z(L)$ by 3.3.7.4, $V = [V,L]$.

Our final technical result requires a lengthier proof:

(f) T is nontrivial on the Dynkin diagram of $\bar{L}$.

Assume that T is trivial on the Dynkin diagram of $\bar{L}$. Then $\bar{T} \leq \bar{P}_i \leq \bar{L}$, for $i = 1,2$, with $\bar{P}_i \cong L_3(2)/E_8$. Let $Y_i := P_i^\infty$, so that $Y_i \in \mathcal{L}(L,T)$. Then $LT = \langle Y_1, Y_2, T\rangle$.

We now repeat some of the proof of 3.3.18: By 3.3.15 we may assume there is no nontrivial characteristic subgroup of T normal in YT for $Y := Y_1$, so the MS-pair (YT,T) is described in C.1.34. As T is Sylow in G, case (5) of C.1.34 does not hold. By (a) and (e), $m(C_Z(Y)) \leq 1$, so case (4) does not hold. Next Y has a nontrivial 2-chief factor on $O_2(\bar{Y})$ and two on $[V,L]$ from (a1) and (a2), eliminating cases (1) and (2) of C.1.34 where there are at most two such factors. Therefore case (3) of C.1.34 holds. Set $Q := [O_2(YT),Y]$ and $U := Z(Q)$; then U is a natural module for $Y/O_2(Y)$ and Q/U the sum of two copies of the dual of U. In particular, Y has exactly three noncentral 2-chief factors. Then $C_Q(Y) = 1$, eliminating case (a1) where $C_{[V,Y]}(Y) \neq 1$ and $[V,Y] \leq Q$. Thus case (a2) holds and L is an A_8-block.

As $\bar{T} \leq \bar{L}$, $LT = O_2(LT)L$. By (e), $C_T(L) = 1$, so by C.1.13.b and B.3.3, either $V = O_2(LT)$ or $O_2(LT)$ is the 7-dimensional quotient of the permutation module for $\bar{L}$. But in the latter case, as $T = O_2(LT)(L \cap T)$, $J(T) \leq C_T(V)$ by B.3.2.4, contradicting 3.3.7.2.

Thus $O_2(LT) = V$, so $T \leq L$ and $|T| = 2^{12}$. Let L_i, $i = 1,2$, be the rank-1 parabolics of $C_L(Z)$ over T, and set $X_i := O^2(L_i)$, and $R_i := O_2(L_i)$. By (d), D does not act on R, so as $R = R_1 \cap R_2$, D does not act on R_i for some i, say $i = 1$. We now apply 3.3.20 to X_1 in the role of "X": Let $Y := \langle X_1^D\rangle$, and observe that the number γ of noncentral 2-chief factors of X_1 is four, and $Z \leq Z(YT)$. Thus as $|T| = 2^{12}$, part (c) of 3.3.20.5 supplies a contradiction, which establishes (f).

We now complete the proof of lemma 3.3.21.

Let P be the parabolic of L with $P/O_2(P) \cong S_3 \times S_3$, and set $H := PT$. Then by (f), $H/O_2(H) \cong S_3$ wr $\mathbf{Z}_2$, so by 3.3.13, $D \leq N_G(H)$. However in case (a2), $J(O_2(H)) \leq C_T(V)$ by B.3.2, so that $J(O_2(H)) = J(O_2(LT))$ by B.2.3.3; hence D normalizes $J(O_2(LT))$, contradicting 3.3.6.b. Therefore case (a1) must hold.

We have $Z \leq [V,L] = V$ by (e), and $T \not\leq LO_2(LT)$ by (f), so $V = W \oplus W^t$ for $t \in T - LO_2(LT)$ with $W := U_1$ the natural module for $\bar{L}$ and W^t dual to W. In particular $Z \cong \mathbf{Z}_2$ is D-invariant, and we saw $D \leq N_G(H)$, so D normalizes

$$U := \langle Z^H \rangle = (U \cap W) \oplus (U \cap W)^t,$$

with $U \cap W \cong E_4$. Now H acts as $O_4^+(2)$ on U, so $Aut_H(U)$ is self normalizing in $GL(U)$ and $Aut_T(U)$ is self normalizing in $Aut_H(U)$; thus we conclude $[U,D] = 1$. Hence $[H,D] \leq C_H(U) = O_2(H)$; in particular D centralizes $T/O_2(H)$, so D acts on $S := T \cap LO_2(LT)$, and hence on $Z_W := C_W(S)$, since $Z_W \leq U$ and D centralizes U.

Let $L_W := C_L(Z_W)^\infty$. Then $L_W/O_2(L_W) \cong L_3(2)$, and L_W has noncentral chief factors on each of W/Z_W, W^t, and $O_2(\bar{L}_W)$. We will now apply earlier arguments to see that (L_WS, S) cannot be an MS-pair; then since (MS1) and (MS2) hold, we can conclude (MS3) does not hold. So suppose (MS3) does hold: then we may apply C.1.32, and as before one of cases (1)–(4) of C.1.34 holds. Since we saw there are at least three noncentral 2-chief factors, cases (1) and (2) of C.1.34 are eliminated. As $Z_W \leq W = [W, L_W]$ is a nonsplit extension of a natural quotient over a trivial submodule, case (3) of C.1.34 does not hold. We've seen $m(Z) = 1$, so as $|T : S| = 2$, $m(Z(S)) \leq 2$, and hence case (4) of C.1.34 does not hold. This contradiction shows that (MS3) fails, so there is $1 \neq C$ char S with $C \trianglelefteq L_WS$. But then $C \trianglelefteq \langle L_W, T \rangle = LT$, while D normalizes S and hence also C, contradicting 3.3.6.b. □

LEMMA 3.3.22. *$\bar{L}$ is not A_7, eliminating case (3) of 3.3.8.*

PROOF. If $\bar{L} \cong A_7$ then by 3.3.8 and 3.3.17, $[V,L]$ is the natural module for $\bar{L}$. We adopt the notational conventions of section B.3; that is we regard $\bar{L}\bar{T} \cong S_7$ as the group of permutations on $\Omega := \{1, \dots, 7\}$, $[V,L]$ as the set of even subsets of Ω, and take $\bar{T}$ to have orbits $\{1,2,3,4\}$, $\{5,6\}$, $\{7\}$ on Ω. Set $\theta := \Omega - \{7\}$; then

$$Z_V := Z \cap [V,L] = \langle e_{5,6}, e_\theta \rangle.$$

Let $L_\theta := C_L(e_\theta)^\infty$. Observe $\bar{L}_\theta \cong A_6$ and $R := O_2(LT) = O_2(L_\theta T)$, with $C(G,R) \leq M$ by 1.4.1.1.

Consider any $z \in C_Z(L)e_\theta$, and set $G_z := C_G(z)$ and $M_z := C_M(z)$. Then $z \in Z$, so that $G_z \in \mathcal{H}^e$ by 1.1.4.6. As $L_\theta \trianglelefteq M_z$, $R \in \mathcal{B}_2(G_z)$ and $R \in Syl_2(\langle R^{M_z} \rangle)$ by A.4.2.7, so as $C(G,R) \leq M$, it follows that Hypothesis C.2.3 is satisfied with G_z, M_z in the roles of "H, M_H". Further by 1.2.4, $L_\theta \leq K \in \mathcal{C}(G_z)$. Now $F^*(K) = O_2(K)$ by 1.1.3.1, and $m_3(L_\theta) = 2$, so $K/O_2(K)$ is quasisimple by 1.2.1.4 and T acts on K by 1.2.1.3. Assume $L_\theta < K$, so that $K \not\leq N_G(L) = M$. Then C.2.7 supplies a contradiction, as in none of the cases listed there does there exist a T-invariant $L_\theta \in \mathcal{C}(M \cap K)$ with $L_\theta/O_2(L_\theta) \cong A_6$. Hence $L_\theta = K \trianglelefteq G_z$. Thus by A.3.18

$$L_\theta = O^{3'}(C_G(z)) \quad \text{for each} \quad z \in C_Z(L)e_\theta. \qquad (*)$$

Similarly for $z \in C_Z(L)$, $C_G(z) \leq M$ as $M = !\mathcal{M}(LT)$, so by A.3.18:

$$L = O^{3'}(C_G(z)) \quad \text{for each} \quad z \in C_Z(L). \qquad (**)$$

Now as $C_{[V,L]}(L) = 0$, 3.3.7.4 says that $V = [V,L] \oplus C_Z(L)$ and $Z = Z_V \oplus C_Z(L)$. We claim that $C_Z(L) = 1$, so that $V = [V,L]$ and $Z = Z_V$. Assume otherwise. Then $m(Z) > 2$, and $Z_\theta := \langle C_Z(L), e_\theta \rangle$ is a hyperplane of Z, so for each $d \in D$, $1 \neq Z_\theta \cap Z_\theta^d$. Hence we may choose $z \in Z_\theta^\#$ with $z^d \in Z_\theta$. First suppose $z \in$

$C_Z(L)$. By (**), $O^{3'}(C_G(z^d)) = L^d$ and $L^d \neq L_\theta$ since $|L| > |L_\theta|$. Therefore by (*), $z^d \notin C_Z(L)e_\theta = Z_\theta - C_Z(L)$, and hence $z^d \in C_Z(L)$, so again using (**), $L = O^{3'}(C_G(z^d)) = L^d$. Thus $d \in N_G(L) = M$ by 1.4.1 in this case. In the remaining case, $z \in Z_\theta - C_Z(L) = C_Z(L)e_\theta$, where by (*), $O^{3'}(C_G(z)) = L_\theta$, and hence $O^{3'}(C_G(z^d)) = L_\theta^d \neq L$. Therefore by (**), $z^d \in Z_\theta - C_Z(L) = C_Z(L)e_\theta$, and then $L_\theta = O^{3'}(C_G(z^d))$ by (*), and hence $L_\theta = L_\theta^d$. Thus d normalizes $L_\theta T$ and hence also $O_2(L_\theta T) = O_2(LT)$, so again $d \in M$. Therefore $D \leq M$, contrary to 3.3.6.a, establishing the claim.

Next $C_D(Z) \leq C_D(e_\theta)$, and $C_D(e_\theta)$ normalizes $O^{3'}(C_G(e_\theta)) = L_\theta$ using (*), and hence also normalizes $O_2(L_\theta T) = R$. Therefore $C_D(Z) \leq N_G(R) \leq M$ as $C(G,R) \leq M$, so $C_D(Z) < D$ as $D \not\leq M$. As Z is of rank 2, we conclude $|D : D \cap M| = 3$, with D transitive on $Z^\#$. In particular there is $d \in D$ with $e_{5,6}^d = e_\theta$. Let $L_{5,6} := C_L(e_{5,6})^\infty$. Thus $\bar{L}_{5,6}\bar{T} \cong \mathbf{Z}_2 \times S_5$, and $L_{5,6}^d \leq O^{3'}(C_G(e_\theta)) = L_\theta$. This is impossible, as $T = T^d$ acts on $L_{5,6}^d$ and L_θ, whereas there is no T-invariant subgroup of $L_\theta/O_{2,Z}(L_\theta) \cong A_6$ isomorphic to A_5.

We have shown that $\bar{L}$ is not A_7. Thus case (3) of 3.3.8 does not hold by 3.3.10.2. This completes the proof of 3.3.22. □

Notice that at this point, cases (1), (2), (3), (5), and (7) of 3.3.8 have been eliminated by 3.3.14, 3.3.10.2, 3.3.22, 3.3.17, and 3.3.21. Thus leaves case (6) of 3.3.8, where $\bar{L} \cong \hat{A}_6$, and case (4) of 3.3.8, where $\bar{L} \cong L_3(2)$, A_6, or $U_3(3)$ by 3.3.16. In each of these cases, $L/O_{2,Z}(L)$ is of Lie type and Lie rank 2 in characteristic 2, and T normalizes L. Threfore by 3.3.6.d, (LT,T) is an MS-pair in the sense of Definition C.1.31. Thus we may apply C.1.32 to LT to conclude that either L is a block, or $\bar{L} \cong L_3(2)$ is described in C.1.34. We first investigate the latter possibility in more detail:

LEMMA 3.3.23. *If $\bar{L}$ is $L_3(2)$, then either*

(1) L is an $L_3(2)$-block, and D acts on the preimage T_0 in T of $Z(\bar{T})$, or

(2) L has two or three noncentral 2-chief factors, and D does not act on $O_2(C_L(Z)T)$.

PROOF. As in earlier arguments we conclude that one of cases (1)–(4) of C.1.34 holds. In particular $[V,L]$ is a sum of $r \leq 2$ isomorphic natural modules, so by 3.3.7.4, $V = [V,L] \oplus Z_L$ and $Z = (Z \cap [V,L]) \oplus Z_L$, where $Z \cap [V,L]$ has rank r.

Suppose case (4) of C.1.34 holds; we argue as in the proof of 3.3.18, although many details are now easier: As $M = !\mathcal{M}(LT)$, $M = !\mathcal{M}(C_G(z))$ for each $z \in Z_L^\#$, and in case (4) of C.1.34, $m(Z_L) \geq 2$ and $r = 1$ so Z_L is a hyperplane of Z, leading to the same contradiction as in the proof of 3.3.18.

Thus we are in case (m) of C.1.34 for some $1 \leq m \leq 3$, where L has m noncentral 2-chief factors. This gives the first statements of (1) and (2). Next in each case of C.1.34, $T \leq LO_2(LT)$. Set $X := O^2(C_L(Z))$ and $R := O_2(XT)$. Now LR, R also satisfy (MS1) and (MS2), but if $m = 2$ or 3, then (LR,R) is not an MS-pair as the corresponding cases of C.1.34 exclude this choice of R. Therefore (MS3) must fail for R, so there is a nontrivial characteristic subgroup C of R normal in LR, and hence normal in LT as $R \trianglelefteq T$. Thus $N_G(R) \leq N_G(C) \leq M = !\mathcal{M}(LT)$, so D does not act on R as $D \not\leq M$ by 3.3.6.a, proving the second statement in (2).

Finally if $m = 1$, let P_i, $i = 1, 2$, denote the maximal parabolics of LT over T. Then P_i has just two noncentral 2-chief factors, so D acts on $O_2(P_i)$ by 3.3.20.4. Thus D acts on $T_0 := O_2(P_1) \cap O_2(P_2)$, completing the proof of the lemma. □

In the proof of the next lemma, we encounter the shadow of the non-maximal parabolic in $\mathbf{Z}_3/\Omega_8^+(2)$, and we eliminate this shadow using 3.3.20.

LEMMA 3.3.24. *L is a block of type A_6, $\hat{A}_6$, $G_2(2)$, or $L_3(2)$.*

PROOF. We observed earlier that either L is a block of type A_6, $\hat{A}_6$, or $G_2(2)$, or $\bar{L} \cong L_3(2)$. Thus appealing to 3.3.23, we only need to eliminate the cases arising in 3.3.23.2, where L has $k := 2$ or 3 noncentral 2-chief factors.

Let $Q := [O_2(LT), L]$. When $k = 2$, C.1.34.2 says that Q is the direct sum of two isomorphic natural modules for $L/O_2(L)$; then LT acts on at least one of the three natural submodules V_0 of Q, and we set $Z_0 := Z \cap V_0$. When $k = 3$, $V_0 := Z(Q)$ is a natural $L/O_2(L)$ module, and Q/V_0 is the direct sum of two copies of the dual of V_0. In this case we again set $Z_0 := Z \cap V_0$. Thus in either case Z_0 is of rank 1 and $V_0 = \langle Z_0^L \rangle = [Z_0, L]$ is an LT-invariant natural $L/O_2(L)$-module.

Set $R := O_2(C_L(Z)T)$, $X := O^2(C_L(Z))$, and $Y := \langle X^D \rangle$. Then X has $k + 1 \leq 4$ noncentral 2-chief factors. By 3.3.23.2, D does not act on R, so we can apply 3.3.20.5 to conclude that Y has a unique noncentral 2-chief factor W, and that $Z_0 \leq \Phi(O_2(X)) \leq Z(Y)$. Set $\widetilde{YT} := YT/Z_0$, $R_Y := O_2(YT)$ and $U := \langle V_0^Y \rangle$. As X is irreducible on $\tilde{V}_0$, we may apply G.2.2.1 with V_0, Z_0, YT in the roles of "V, V_1, H", to conclude that $\tilde{U} \leq \Omega_1(Z(\tilde{R}_Y))$, so $\Phi(U) \leq Z_0$. As $V_0 = [V_0, X]$, $U = [U, Y]$, so by uniqueness of W, $W = U/U_0$ where $U_0 := C_U(Y)$. By 3.3.20.3, $O_2(X) \leq R_Y$, so as $X = O^2(X)$, $O_2(X) = [O_2(X), X] \leq [R_Y, Y] = U$. Then $\Phi(O_2(X)) \leq Z_0$, eliminating the case $k = 2$, for there $\Phi(O_2(X)) = C_Q(L)$ is of rank 2. Thus $k = 3$, and here we compute that $Q/(O_2(X) \cap Q) \cong E_4$ and $[O_2(X), a] \not\leq Z_0$ for each $a \in Q - O_2(X)$. Therefore setting $(YT)^* := YT/R_Y$, $Q^* \cong E_4$. This is impossible, since by 3.3.20.3, $X^*T^* \cong D_{12}$, whereas $Q^* \trianglelefteq X^*T^*$. □

LEMMA 3.3.25. *(1) L is a block of type A_6, $G_2(2)$, or $L_3(2)$.*

(2) Assume $C_T(L) \neq 1$ and $\bar{L}$ is not $L_3(2)$, and let $X := O^2(C_L(Z))$ and $R := O_2(XT)$. Then D acts on X and R, but does not act on any nontrivial D-invariant subgroup of R normal in LT.

(3) If $C_T(L) = 1$, then either $V = O_2(LT)$, or L is an A_6-block.

PROOF. Let $X := O^2(C_L(Z))$ and $R := O_2(XT)$. Inspecting the cases listed in 3.3.24, $XT/R \cong S_3$.

We first prove (2), so suppose $C_T(L) \neq 1$ and $\bar{L}$ is not $L_3(2)$. Then $C_Z(L) \neq 1$, so as usual $C_G(Z) \leq C_G(C_Z(L)) \leq M = !\mathcal{M}(LT)$, and then $C_G(Z) = C_M(Z)$. As $\bar{L}$ is not $L_3(2)$, $m_3(L) = 2$ and so by A.3.18, L is the subgroup $\theta(M)$ generated by all elements of M of order 3. Therefore $X = \theta(C_G(Z))$, so D acts on X and hence also on R. Then the final statement of (2) follows from 3.3.6.b.

In view of 3.3.24, to prove (1) we may assume $\bar{L} \cong \hat{A}_6$, and it remains to derive a contradiction. By B.4.2, $J(R) \leq C_T(V) = O_2(LT)$, so that $J(R) = J(O_2(LT))$ by B.2.3.3. Therefore $C_T(L) = 1$ by (2). Then as the $\hat{A}_6$-module has trivial 1-cohomology by I.1.6, $V = O_2(LT)$ by C.1.13.b. But again using B.4.2, there is a unique member $\bar{A}$ of $\mathcal{P}(\bar{T}, V)$, $m(\bar{A}) = 2$, and $C_V(\bar{A}) = C_V(\bar{a})$ for each $\bar{a} \in \bar{A}^{\#}$. Therefore by B.2.21, there is a unique member $A \in \mathcal{A}(T)$ with $[A, V] \neq 1$, and hence $\mathcal{A}(T) = \{A, V\}$ is of order 2. Therefore as D is of odd order, D acts on V, contrary to 3.3.6.b. So (1) is estabished.

Finally we prove (3), so we assume that $C_T(L) = 1$, and $V < O_2(LT)$. By (1), we may assume $\bar{L}$ is $L_3(2)$ or $U_3(3)$, and it remains to derive a contradiction. As $C_T(L) = 1$, $Q := O_2(LT)$ is elementary abelian by C.1.13.a. Further by C.1.13.b, B.4.8, and B.4.6, Q is the indecomposable module with natural irreducible submodule V and trivial quotient, of rank 4 or 7, respectively. By 3.3.7.2, V is an FF-module, so by B.4.6.13, $\bar{L}$ is not $U_3(3)$. Thus $\bar{L} \cong L_3(2)$, and by B.4.8.3, there is a unique member $\bar{A}$ of $\mathcal{P}(\bar{T}, Q)$. As $C_Q(\bar{A}) = C_Q(\bar{a})$ for each $\bar{a} \in \bar{A}^\#$ and $Q = C_{LT}(Q)$, we may apply B.2.21 to obtain the same contradiction as earlier. This completes the proof of (3). □

Observe now that as L is a block by 3.3.25.1, Hypothesis C.6.2 is satisfied with L, T, T, TD in the roles of "L, R, T_H, Λ", For example, if $1 \neq R_0 \leq T$ with $R_0 \trianglelefteq LT$, then $D \not\leq N_G(R_0)$ by 3.3.6.b, which verifies part (3) of Hypothesis C.6.2. As Hypothesis C.6.2 is satisfied, we can apply C.6.3 to conclude:

LEMMA 3.3.26. *There exists $d \in D - M$ with $V^d \not\leq O_2(LT)$.*

In the remainder of the section, let d be defined as in 3.3.26. Set $Q_L := O_2(LT)$ and $T_C := C_T(L)$.

LEMMA 3.3.27. *Assume $T_C = C_T(L) \neq 1$. Then*

(1) $T_C \cap T_C^d = 1$.
(2) $\Phi(T_C) = 1$.
(3) Either $T_C^d \leq Q_L$ or $T_C \leq Q_L^d$.

PROOF. As L centralizes $T_C \trianglelefteq T$ and D acts on T, also $T_C^d \trianglelefteq T$, and then $T_C \cap T_C^d$ is normal in LT and in L^dT. Thus if $T_C \cap T_C^d \neq 1$, then

$$M^d = !\mathcal{M}(L^dT) = !\mathcal{M}(N_G(T_C \cap T_C^d)) = !\mathcal{M}(LT) = M,$$

contradicting our choice of $d \in D - M$ in 3.3.26, and so establishing (1). Then applying (1) to d^2 in the role of "d", $T_C \cap T_C^{d^2} = 1$, so also $T_C^{d^{-1}} \cap T_C^d = 1$.

Now as L is a block, $\Phi(Q_L) \leq T_C$ by C.1.13.a. Suppose (2) fails, so that $\Phi(T_C) \neq 1$. If $T_C^d \leq Q_L$, then $1 \neq \Phi(T_C^d) \leq \Phi(Q_L) \leq T_C$, contradicting (1); therefore $T_C^d \not\leq Q_L$, so by symmetry $T_C \not\leq Q_L^d$, and thus (3) fails. Hence (3) implies (2), so it remains to assume that (3) fails, and to derive a contradiction. Thus $T_C^d \not\leq Q_L$ and $T_C \not\leq Q_L^d$, so also $T_C^{d^{-1}} \not\leq Q_L$.

Suppose for the moment that $\bar{L}$ is $L_3(2)$. Then by 3.3.23.1, D acts on the preimage T_0 in T of $Z(\bar{T})$. Therefore as $\bar{T}_0$ is of order 2 and $T_C^d \not\leq Q_L$, $\bar{T}_C^d = \bar{T}_0$.

Now suppose that $\bar{L}$ is not $L_3(2)$. Then by 3.3.25.2, D acts on $X := O^2(C_L(Z))$ and on $R := O_2(XT)$. Therefore as $T_C \trianglelefteq XT$, $T_C^d \trianglelefteq XT$, and as T_C centralizes X, $1 \neq \bar{T}_C^d$ centralizes $\bar{X}$. Now $\bar{L} \cong A_6$ or $G_2(2)'$ by 3.3.25.1, and $\bar{T}$ is trivial on the Dynkin diagram of $\bar{L}$ if $\bar{L} \cong A_6$ since $\bar{L}$ is an A_6-block. Inspecting $Aut(\bar{L})$, we find that $C_{Aut(\bar{L})}(\bar{X}) = 1$ unless $\bar{L}\bar{T} \cong S_6$, whereas we saw $\bar{T}_C^d \neq 1$ centralizes $\bar{X}$. Therefore $\bar{L}\bar{T} \cong S_6$ and $\bar{T}_C^d = Z(\bar{X}\bar{T}) = \bar{T}_0$ is of order 2.

Thus $\bar{L}\bar{T} \cong S_6$ or $L_3(2)$ and $\bar{T}_C^d = \bar{T}_0$ is of order 2. As $T_C^d \trianglelefteq T$, $1 \neq [V, T_0] = [V, T_C^d] \leq T_C^d$. Similarly $[V, T_0] = [V, T_C^{d^{-1}}] \leq T_C^{d^{-1}}$, so $1 \neq [V, T_0] \leq T_C^d \cap T_C^{d^{-1}}$, contrary to the final remark in paragraph one. □

LEMMA 3.3.28. *If $\bar{L}$ is $L_3(2)$ or $U_3(3)$, then $Q_L = V \times T_C$ and $\Phi(Q_L) = 1$. Indeed if $\bar{L}$ is $U_3(3)$, then $T_C = 1$ and $Q_L = V$.*

PROOF. Assume that $\bar{L}$ is $L_3(2)$ or $U_3(3)$ and set $T_C := C_T(L)$. By 3.3.26, there is $d \in D - M$ with $V^d \not\leq Q_L$.

Suppose first that $\bar{L} \cong L_3(2)$. As case (1) of 3.3.23 holds, D acts on the preimage T_0 in T of $Z(\bar{T})$. Then as $|\bar{T}_0| = 2$, $T_0 = V^d Q_L$ and $m(T_0/C_{T_0}(V)) = 1$, so $m(Q_L/C_{Q_L}(V^d)) = 1 = m(V/C_V(V^d))$, and hence $Q_L = VC_{Q_L}(V^d)$. Now if $Q_L/C_T(L)$ is the unique nonsplit extension of V with a 1-dimensional submodule described in B.4.8, then the fixed points of $\bar{T}_0$ are contained in $VC_T(L)$, contrary to $Q_L = VC_{Q_L}(V^d)$ with $\bar{T}_0 = \bar{V}^d$. Therefore $Q_L = V \times T_C$, so as $\Phi(T_C) = 1$ by 3.3.27.2, the lemma holds in this case.

Thus we may assume $\bar{L} \cong U_3(3)$. Notice that if $T_C = 1$, then $V = O_2(LT)$ by 3.3.25.3, so that the lemma holds. Therefore we may assume that $T_C \neq 1$, and it remains to derive a contradiction.

Set $X := O^2(C_L(Z))$ and $R := O_2(XT)$. By 3.3.25.2, D acts on R and X. Then V^d is elementary abelian and normal in the parabolic subgroup XT, so using B.4.6, $m(\bar{V}^d) = 2$ or 3, and hence $m(V/C_V(V^d)) = 3$. Then by symmetry between V and V^d, $m(V^d/C_{V^d}(V)) = 3$. Thus $m(\bar{V}^d) = 3$ so as $\bar{V}^d \trianglelefteq \bar{X}$, $\bar{V}^d = C_{\bar{R}}(\bar{V}^d)$ is the unique FF-offender on V in $\bar{R}$ by B.4.6.13. Therefore $C_R(V^d) \leq V^d Q_L$, so $C_R(V^d) = V^d C_{Q_L}(V^d)$. Also $|C_R(V^d)| = |C_R(V)| = |Q_L|$, so $|Q_L : C_{Q_L}(V^d)| = |\bar{V}^d|$. Then as $|\bar{V}^d| = |V : C_V(V^d)|$, $Q_L = C_{Q_L}(V^d)V$. However in the unique nonsplit extension of $V/C_V(L)$ over a 1-dimensional submodule described in B.4.6, the fixed points of $\bar{V}^d$ are contained in $V/C_V(L)$. Thus as $Q_L = VC_{Q_L}(V^d)$, $Q_L = VT_C$. Then since $\Phi(T_C) = 1$ by 3.3.27.2, $\Phi(Q_L) = 1$.

Again by B.4.6.13, $\bar{V}^d$ is the unique member of $\mathcal{P}(\bar{R}, V)$, and $C_V(\bar{V}^d) = C_V(\bar{a})$ for each $\bar{a} \in \bar{V}^d - \bar{L}$. Therefore as $Q_L = VC_T(V^d)$ and $m(\bar{V}^d) = m(V/C_V(V^d))$, B.2.21 applied with Q_L in the role of "V" says Q_L^d is the unique member of $\mathcal{A}(R)$ with $[Q_L, Q_L^d] \neq 1$, so $\mathcal{A}(R)$ is of order 2. Then as D of odd order acts on R, D normalizes Q_L, contrary to 3.3.6.b. This completes the proof. □

LEMMA 3.3.29. *$\bar{L}$ is not $L_3(2)$.*

PROOF. Assume $\bar{L}$ is $L_3(2)$. By 3.3.23.1, D acts on the preimage T_0 in T of $Z(\bar{T})$. Thus as $D \not\leq M$ by 3.3.6.a and $M = !\mathcal{M}(LT)$, no D-invariant subgroup of T_0 is normal in LT. Hence $J(T_0) \not\leq Q_L$ by B.2.3.3, so there is $A \in \mathcal{A}(T_0)$ with $A \not\leq Q_L$. Then as $|\bar{T}_0| = 2$, $T_0 = \langle a \rangle Q_L$ for $a \in A - Q_L$. Now $\Phi(Q_L) = 1$ by 3.3.28, so $C_{Q_L}(A) = C_{Q_L}(a)$. Therefore by B.2.21, $\mathcal{A}(T_0) = \{A, Q_L\}$ is of order 2. Thus as D is of odd order, D acts on Q_L, so that $D \leq M = !\mathcal{M}(LT)$, contrary to $D \not\leq M$. □

LEMMA 3.3.30. *L is an A_6-block.*

PROOF. Assume otherwise. Then by 3.3.25.1 and 3.3.29, L is a $G_2(2)$-block, and it remains to derive a contradiction. By 3.3.28, $T_C = 1$ and $V = Q_L$, while by 3.3.7.2, V is an FF-module for $\bar{L}\bar{T}$, so $V \cong E_{64}$ is the natural module for $LT/V \cong G_2(2)$.

Define $\bar{A}_1$ as in B.4.6. Then by B.4.6, $m(\bar{A}_1) = 3$, $\mathcal{P}(\bar{L}\bar{T}, V) = \bar{A}_1^{\bar{L}}$, and $C_V(\bar{A}_1) = C_V(\bar{a})$ is of rank 3 for each $\bar{a} \in \bar{A}_1 - \bar{L}$. Let A_0 be the preimage in M of A_1; by B.2.21 there is a unique member A of $\mathcal{A}(A_0)$ with image $\bar{A}_1$. Hence $\mathcal{A}(A_0) = \{V, A\}$. By Burnside's Fusion Lemma A.1.35, $N_{\bar{L}\bar{T}}(\bar{T}) = \bar{T}$ is transitive on the members of $\bar{A}_1^{\bar{L}}$ normal in $\bar{T}$, so that $\bar{A}_1$ is the only such member. Thus $\{A, V\} = \{B \in \mathcal{A}(T) : B \trianglelefteq T\}$ is D-invariant, so as usual D acts on V. Then $D \leq N_G(V) \leq M$, contrary to 3.3.6.a. □

By 3.3.30, $\bar{L}\bar{T} \cong A_6$ or S_6, so we can represent $\bar{L}\bar{T}$ on $\Omega := \{1, \ldots, 6\}$ so that $\bar{T}$ has orbits $\{1,2,3,4\}$ and $\{5,6\}$, and permutes the set of pairs $\{\{1,2\},\{3,4\}\}$. Further we adopt the notation of section B.3.

LEMMA 3.3.31. $T_C = 1$.

PROOF. Assume otherwise; then in particular, $C_Z(L) \neq 1$. By 3.3.25.2, D acts on $Y := O^2(C_L(Z))$ and on $R := O_2(YT)$. Then by 3.3.26, there is $d \in D - M$ with $V^d \not\leq Q_L$. As $V^d \trianglelefteq YT$, either $\bar{V}^d = \langle (5,6) \rangle$, or $\bar{V}^d$ contains $\langle (1,2)(3,4), (1,3)(2,4) \rangle$. The latter is impossible, since as $V^d \trianglelefteq T$, V^d acts quadratically on V. Thus $\bar{V}^d = \langle (5,6) \rangle$, and in particular $LT/Q_L \cong S_6$ rather than A_6.

By Sylow's Theorem, D acts on some B of order 3 in Y, and so D acts on $C_R(B)$. Now for $v \in C_V(B) - Z$, $V = \langle v^T \rangle$, so $v \notin Q_L^d$ since $V \not\leq Q_L^d$. Therefore by symmetry, $v^d \notin Q_L$, and thus $\bar{v}^d = (5,6)$.

Next $|C_R(B) : C_{Q_L}(B)| = 2$ and $C_R(B) = \langle v^d \rangle C_{Q_L}(B)$ since $YT/Q_L \cong S_4 \times \mathbf{Z}_2$. As $Q_L = C_R(V)$, $C_{Q_L}(B) = C_{Q_L}(VB) = C_R(VB)$. Conjugating by d, $|C_R(B) : C_R(V^dB)| = 2$, so as $C_R(B) = \langle v^d \rangle C_{Q_L}(V^d)$, $|C_{Q_L}(B) : C_{Q_L}(V^d)| = 2$. Then as $[C_V(B)/C_V(L), v^d] \neq 1$, $C_{Q_L}(B) = C_V(B) C_{Q_L}(BV^d)$, so as $T_C \leq C_{Q_L}(B)$ $T_C \leq C_{Q_L}(BV^d)$ and hence V^d centralizes T_C. Thus $C_R(B) = \langle v^d \rangle C_V(B) C_{Q_L}(BV^d)$. Finally by Coprime Action, $Q_L = VC_{Q_L}(B)$, so $Q_L = VC_{Q_L}(BV^d)$.

Set $S := Q_L V^d$. As $C_{\bar{T}}(\bar{B}) = \bar{V}^d = \bar{S}$, $C_T(B) = C_S(B) = C_R(B)$ and $[V^d, B] \leq [Q_L, B] = [V, B]$. So by symmetry, $[V, B] \leq [V^d, B] = [V, B]^d$, and hence $[V, B] = [V^d, B] = [V, B]^d$ as these groups have the same order. Thus d acts on $C_T(B)[V, B] = \langle v^d \rangle C_V(B) C_{Q_L}(BV^d)[V, B] = \langle v^d \rangle V C_{Q_L}(BV^d) = V^d Q_L = S$.

By 3.3.7.4, $V = [V, L] C_Z(L)$, so that $Z = ([V, L] \cap Z) C_Z(L)$. Therefore $|Z : C_Z(L)| = |(Z \cap [V, L]) : C_{[V,L]}(L)| = 2$. We saw $C_Z(L) \neq 1$, so as $T_C \cap T_C^d = 1$ by 3.3.27.1, $Z \cong E_4$ and $C_Z(L) \cong \mathbf{Z}_2$.

Suppose $\Phi(Q_L) = 1$. Then as d normalizes S and $\bar{S} = \bar{V}^d$ is of order 2, $\mathcal{A}(S) = \{Q_L, Q_L^d\}$, so as d is of odd order, $d \in N_G(Q_L) \leq M = !\mathcal{M}(LT)$, contrary to our choice of $d \in D - M$. Thus $\Phi(Q_L) \neq 1$. So as $\Phi(T_C) = 1$ by 3.3.27.2, $T_C V < Q_L$. As we saw $Q_L = VC_{Q_L}(V^dB)$, we may choose $u \in C_{Q_L}(V^dB) - T_C V$.

Now $|Q_L : T_C V| \leq 2$ by C.1.13.b and B.3.1, so $Q_L = \langle u \rangle T_C V$ and $T = \langle u \rangle (T \cap L) T_C V^d$. Also $\Phi(T_C) = 1$, T_C commutes with L by definition, and we saw V^d centralizes T_C. Therefore as $T = \langle u \rangle (T \cap L) T_C V^d$, $1 \neq C_{T_C}(u) = Z \cap T_C \leq C_Z(L)$, so as $C_Z(L)$ is of order 2, $C_{T_C}(u)$ is of order 2. As $u^2 \in VT_C \leq C_G(T_C)$ and T_C is elementary abelian, it follows that $m(T_C) \leq 2$.

Assume first that $T_C \cong \mathbf{Z}_2$. Then as $\Phi(Q_L) \neq 1$ while $\Phi(Q_L) \leq T_C$ by C.1.13.a, u^2 generates T_C. Recall we chose u to centralize V^d and V^d centralizes T_C. Therefore $Z(S) = C_V(V^d) T_C \langle u \rangle$, with $C_V(V^d) T_C$ elementary, so that $\Phi(Z(S)) = T_C$ is d-invariant, contradicting 3.3.27.1.

Thus $T_C \cong E_4$, so $\langle u \rangle T_C \cong D_8$. Hence $S = S_1 \times S_2 \times E$, where $S_i \cong D_8$ and $E \cong E_4$. But then as d is of odd order, the Krull-Schmidt Theorem A.1.15, says d acts on $Z(S)S_i$ for $i = 1$ and 2, so d centralizes $\Phi(Z(S)S_i)$ of order 2, and hence also centralizes $\Phi(S)$. This contradicts 3.3.27.1, since $T_C \cap \Phi(S) \neq 1$. □

LEMMA 3.3.32. *(1) Either* $Q_L = V$ *is irreducible, or* $Q_L \cong E_{32}$ *is the quotient of the permutation module on* Ω *modulo* $\langle e_\Omega \rangle$*, denoted by "*$\tilde{U}$*" in section B.3.*

(2) $\bar{L}\bar{T} \cong S_6$.

(3) D *acts on the preimage* T_0 *in* T *of* $\bar{A}_2 := \langle (1,2)(3,4), (5,6) \rangle$.

PROOF. As $T_C = 1$ by 3.3.31, (1) follows from C.1.13 and B.3.1. Let P_1 be the stabilizer in LT of $\{5,6\}$, and P_2 the stabilizer of the partition $\{\{1,2\},\{3,4\},\{5,6\}\}$; set $R_i := O_2(P_i)$, and $X_i := O^2(P_i)$. Then P_1 and P_2 are the maximal parabolics of LT over T, P_1 has two noncentral 2-chief factors, P_2 has three noncentral 2-chief factors, and $O_2(X_2)$ is nonabelian with $Z(X_2) = 1$. Then P_1 does not satisfy conclusion (4) of 3.3.20 and P_2 does not satisfy conclusion (5c) of 3.3.20, so D acts on R_1 and R_2. Thus D acts on $T_1 := R_1 \cap R_2$.

If $\bar{L}\bar{T} \cong S_6$, then $T_0 = T_1$ and the lemma holds, so we may assume $\bar{L}\bar{T} \cong A_6$. Thus $\bar{T}_1 = \langle (3,4)(5,6) \rangle$. But then $\mathcal{P}(\bar{T}_1, Q_L)$ is empty by B.3.4.1, so $J(T_1) \le C_{LT}(Q_L) = Q_L$. Then as Q_L is elementary abelian by (1), $J(T_1) = Q_L \trianglelefteq LT$, and hence $D \le N_G(Q_L) \le M$, contrary to 3.3.6.a. Thus the lemma is established. □

We can now obtain a contradiction, and complete the proof of Theorem 3.3.1.

In view of 3.3.32.1, Q_L is either the natural module for $\bar{L}$ denoted by "$\tilde{U}_0$" in B.3.2, or the quotient denoted "$\tilde{U}$" of the permutation module. Define $\bar{A}_1 := \langle (5,6) \rangle$, and $\bar{A}_2$ as in 3.3.32.2. By 3.3.32.3, D acts on the preimage T_0 of $\bar{A}_2$ in T, and as $D \not\le M$ by 3.3.6.a, D acts on no nontrivial subgroup of T_0 normal in LT. In particular $J(T_0) \not\le Q_L$ by B.2.3.3, so there is $A \in \mathcal{A}(T_0)$ with $A \not\le Q_L$. By B.3.2, $\bar{A} = \bar{A}_i$ for $i = 1$ or 2. By inspection, $C_{Q_L}(\bar{A}) = C_{Q_L}(\bar{a})$ for some $\bar{a} \in \bar{A}$, so by B.2.21 there is at most one member of $\mathcal{A}(T_0)$ projecting on $\bar{A}_i$; if such a member exists, we denote it by A_i. Thus $\mathcal{A}(T_0) \subseteq \{Q_L, A_1, A_2\}$. Therefore as D acts on $\mathcal{A}(T_0)$ but not on Q_L, and D is of odd order, D_L is transitive on $\mathcal{A}(T_0)$ of order 3. Further D is transitive on the 2-subsets of $\mathcal{A}(T_0)$. This is impossible as $|A_1 Q_L| < |A_2 Q_L|$.

This contradiction completes the proof of Theorem 3.3.1.

CHAPTER 4

Pushing up in QTKE-groups

Recall that in chapter C of Volume I, we proved "local" pushing up theorems in SQTK-groups. In this Chapter we use those local theorems to prove "global" pushing up theorems in QTKE-groups. Let L, V be a pair in the Fundamental Setup (3.2.1), $L_0 := \langle L^T \rangle$, and $M := N_G(L_0)$. We use L_0T and our pushing up theorems to show that large classes of subgroups must be contained in M.

For example, in Theorem 4.2.13 we use the fact that L_0T is a uniqueness subgroup to prove roughly that if the pair L, V in the FSU is not too "small", then each subgroup I of L_0 which covers L_0 modulo $O_2(L_0T)$ with $O_2(I) \neq 1$ is also a uniqueness subgroup. Then we use Theorem 4.2.13 to prove Theorem 4.4.3, which shows that for suitable subgroups B of odd order centralizing V, $N_G(B) \leq M$. As a corollary, we see in Theorem 4.4.14 that for $H \in \mathcal{H}_*(T, M)$ with $n(H) > 1$, a Hall $2'$-subgroup of $H \cap M$ must act faithfully on V. This gives the inequality $n(H) \leq n'(N_M(V)/C_M(V))$, (cf. E.3.38) which is used crucially in many places in this work.

4.1. Some general machinery for pushing up

Our eventual goal is to show roughly in most cases of the FSU that if $\mathcal{I}$ is the set of subgroups I of L_0T covering L_0 modulo $O_2(L_0T)$ with $O_2(I) \neq 1$, then each member of $\mathcal{I}$ is also a uniqueness subgroup. If some member of $\mathcal{I}$ fails to be a uniqueness subgroup, then we study a maximal counterexample I using the theory of pushing up from chapter C of Volume I. Our starting point is 1.2.7.3, which says that L_0T is a uniqueness subgroup. We develop some fairly general machinery to implement this approach. So in this section we assume the following hypothesis (which we will see in 4.2.2 holds in the FSU):

HYPOTHESIS 4.1.1. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, $M \in \mathcal{M}(T)$, and $M_+ = O^2(M_+) \trianglelefteq M$. Further assume that $M = !\mathcal{M}(I)$ for each subgroup I of M such that*

$$M_+C_T(M_+/O_2(M_+)) \leq I \quad \text{and} \quad M = C_M(M_+/O_2(M_+))I.$$

Let $\Sigma(M_+)$ consist of those subgroups M_- of M containing $M_+C_M(M_+/O_2(M_+))$.

LEMMA 4.1.2. *Let $R_+ \in Syl_2(C_M(M_+/O_2(M_+)))$. Then $M = !\mathcal{M}(N_M(R_+))$.*

PROOF. By hypothesis T is Sylow in M, so as $M_+ \trianglelefteq M$, we may assume $R_+ = C_T(M_+/O_2(M_+))$. Also $M_+ = O^2(M_+)$, so by A.4.2, $M_+R_+ \leq N_G(R_+)$. Now $M = C_M(M_+/O_2(M_+))N_M(R_+)$ by a Frattini Argument. So by Hypothesis 4.1.1 with $N_M(R_+)$ in the role of "I", $M = !\mathcal{M}(N_M(R_+))$. □

Next we define some more technical notation. We will study overgroups of M_+ which (in contrast to the subgroups I in 4.1.1) need not cover *all* of M modulo

$C_M(M_+/O_2(M_+))$, but just cover M_- modulo $C_M(M_+/O_2(M_+))$ for some $M_- \in \Sigma(M_+)$. For example in the FSU, take $M_+ := L_0$, and $M_- := L_0 C_M(L_0/O_2(L_0))T$, or more generally $M_- \in \Sigma(M)$ with $M_- \leq L_0 C_M(L_0/O_2(L_0))T$ and $L^T = L^{M_-}$.

In the remainder of the section pick $M_- \in \Sigma(M_+)$ and define $\eta = \eta(M_+, M_-)$ to be the set of all subgroups I of M_- such that $IC_M(M_+/O_2(M_+)) = M_-$ and $M_+ \leq IO_2(M_+)$ with $O_2(I) \neq 1$. We wish to show that each $I \in \eta$ is a uniqueness subgroup; thus we consider the set of counterexamples to this conclusion, and define $\mu = \mu(M_+, M_-)$ to consist of those $I \in \eta$ such that $\mathcal{H}(I, M) \neq \emptyset$, where

$$\mathcal{H}(I, M) := \{H \in \mathcal{H}(I) : H \not\leq M\}.$$

Finally define a relation $\lesssim$ on η by $I_1 \lesssim I_2$ if $O_2(I_1) \leq O_2(I_2)$ and $I_1 \cap M_+ \leq I_2 \cap M_+$. Let $\mu^* = \mu^*(M_+, M_-)$ consist of those $I \in \mu$ such that $O_2(I)$ is not properly contained in $O_2(I_1)$ for any $I_1 \in \mu$ such that $I \lesssim I_1$.

We begin to study this set μ^* of "maximal" members of μ.

Lemma 4.1.3. *Let $I \in \eta$, $I \leq I_0 \leq M_-$, and $I_1 \leq I_0$ with $1 \neq O_2(I_1)$. Assume $I_0 = I_1 C_{I_0}(M_+/O_2(M_+))$ and $M_+ \cap I_0 \leq I_1 O_2(M_+)$. Then*

(1) $I_1 \in \eta$.

(2) If $I \in \mu^*$, $I \lesssim I_1$, and $O_2(I) < O_2(I_1)$, then $M = !\mathcal{M}(I_1)$.

Proof. By hypothesis $I \in \eta$ and $I \leq I_0 \leq M_-$, so from the definition of η,

$$M_- = IC_M(M_+/O_2(M_+)) \leq I_0 C_M(M_+/O_2(M_+)) \leq M_-, \qquad (*)$$

and hence all inequalities in (*) are equalities. Again from the definition of η, $M_+ \leq IO_2(M_+) \leq I_0 O_2(M_+)$.

Next as $I_0 = I_1 C_{I_0}(M_+/O_2(M_+))$ by hypothesis, and (*) is an equality,

$$M_- = I_0 C_M(M_+/O_2(M_+)) = I_1 C_M(M_+/O_2(M_+)) \leq M_-,$$

and again this inequality is an equality. As $M_+ \leq I_0 O_2(M_+)$ and $M_+ \cap I_0 \leq I_1 O_2(M_+)$, $M_+ = (I_0 \cap M_+)O_2(M_+) \leq I_1 O_2(M_+)$. Then as $O_2(I_1) \neq 1$ by hypothesis, $I_1 \in \eta$, and hence (1) holds.

Assume the hypothesis of (2). If $M \neq !\mathcal{M}(I_1)$, then $\mathcal{H}(I_1, M) \neq \emptyset$, so that $I_1 \in \mu$. As $I \lesssim I_1$ and $O_2(I) < O_2(I_1)$, this contradicts $I \in \mu^*$, establishing (2). □

The next two results are used to establish Hypothesis C.2.8 in various situations; see 4.2.4 for one such application. Hypothesis C.2.8 allows us to apply the pushing up results in chapter C of Volume I.

Lemma 4.1.4. *Suppose $I \in \mu^*$, and let $R := O_2(I)$ and $H \in \mathcal{H}(I, M)$. Set $H_+ := O^2(M_+ \cap H)$. Then*

(1) $R \leq C_M(M_+/O_2(M_+))$.

(2) $C(G, R) \leq M$.

(3) $M_+ = H_+ O_2(M_+)$ and $H_+ \trianglelefteq H \cap M$.

(4) $R \in Syl_2(C_H(H_+/O_2(H_+))) \cap Syl_2(C_{H\cap M}(H_+/O_2(H_+)))$.

(5) $R = O_2(N_H(R))$ so that $R \in \mathcal{B}_2(H)$, and $O_2(H) \leq O_2(H \cap M) \leq R$.

(6) $F^*(H \cap M) = O_2(H \cap M)$.

Proof. Let $I_+ := O^2(M_+ \cap I)$. As $M_+ \leq IO_2(M_+)$ by definition of η, while $M_+ = O^2(M_+)$ by Hypothesis 4.1.1, $M_+ = I_+ O_2(M_+)$. Therefore (1) follows from A.4.3.1, with M_+, I_+ in the roles of "X, Y". Also (3) follows as $I_+ \leq H_+$.

Set $M_1 := N_{M_-}(R)$, and pick $R_+ \in Syl_2(C_M(M_+/O_2(M_+))$ so that $R_1 := N_{R_+}(R) \in Syl_2(C_{M_1}(M_+/O_2(M_+)))$. If $R = R_+$, then (2) holds by 4.1.2, so we may assume that $R < R_+$, and hence $R < R_1$. We will verify the hypotheses of 4.1.3, with M_1, $N_{M_1}(R_1)$ in the roles of "I_0, I_1". First $I \leq M_1$, and $O_2(M_1) \neq 1 \neq O_2(N_{M_1}(R_1))$, since $1 \neq O_2(I) = R \leq O_2(M_1) \cap O_2(N_{M_1}(R_1))$. By a Frattini Argument,

$$M_1 = N_{M_1}(R_1)C_{M_1}(M_+/O_2(M_+)).$$

Finally M_+ acts on R_+ by A.4.2.4, and hence $M_+ \cap M_1 = N_{M_+}(R) \leq N_{M_1}(R_1)$, completing the verification of the hypotheses of 4.1.3. Thus $N_{M_1}(R_1) \in \eta$ by 4.1.3. Also $[N_{M_+}(R), R_1] \leq O_2(M_+) \cap M_1 \leq R_1$ as $R_1 \in Syl_2(C_{M_1}(M_+/O_2(M_+)))$, so $I \cap M_+ \leq N_{M_+}(R) \leq N_{M_+}(R_1)$. By construction $O_2(I) = R < R_1 \leq O_2(N_{M_1}(R_1))$, so $I \lesssim N_{M_1}(R_1)$. Therefore as $I \in \mu^*$ by hypothesis, $M = !\mathcal{M}(N_{M_1}(R_1))$ by 4.1.3.2. Then as $M_1 \leq N_G(R)$, (2) follows.

A similar argument shows $R \in Syl_2(C_{H \cap M}(H_+/O_2(H_+)))$: Assume that

$$R < R_H \in Syl_2(C_{H \cap M}(H_+/O_2(H_+))).$$

As $C_M(M_+/O_2(M_+)) \leq M_-$, R_H is also Sylow in $C_{H \cap M_-}(M_+/O_2(M_+))$. Set $H_1 := N_{H \cap M_-}(R)$ and choose R_H so that $R_1 := N_{R_H}(R) \in Syl_2(C_{H_1}(M_+/O_2(M_+))$. By a Frattini Argument, $H_1 = N_{H_1}(R_1)C_{H_1}(M_+/O_2(M_+))$. By (3),

$$M_+ \cap H = H_+O_2(M_+ \cap H) \leq H_+R_H,$$

and by A.4.2.4, H_+ acts on R_H, so

$$M_+ \cap H_1 = N_{M_+ \cap H_1}(R) \leq N_{H_1}(R_1).$$

Hence applying 4.1.3.1 to H_1, $N_{H_1}(R_1)$ in the roles of "I_0, I_1", we conclude $N_{H_1}(R_1) \in \eta$. By construction, $H_1 \leq H \not\leq M$, so $\mathcal{H}(N_{H_1}(R_1), M) \neq \emptyset$, and hence $N_{H_1}(R_1) \in \mu$. Also by construction, $O_2(I) = R < R_1 \leq O_2(N_{H_1}(R_1))$ and arguing as above, $I \lesssim N_{H_1}(R_1)$. This contradicts our hypothesis that $I \in \mu^*$, completing the proof that $R \in Syl_2(C_{H \cap M}(H_+/O_2(H_+)))$. Then (4) follows using (2).

As $H_+ \trianglelefteq H \cap M$, $R \in \mathcal{B}_2(H \cap M)$ by C.1.2.4. By (2), $N_H(R) \leq H \cap M$, so $R \in \mathcal{B}_2(H)$ by C.1.2.3. By C.2.1.2, both $O_2(H)$ and $O_2(H \cap M)$ lie in $R \leq H \cap M$, so in fact $O_2(H) \leq O_2(H \cap M) \leq R$, completing the proof of (5).

Let $H \leq H_1 \in \mathcal{M}$. Then $H_1 \in \mathcal{H}(I, M)$, so all results proved for H also apply to H_1. In particular by (5), $O_2(H_1 \cap M) \leq R \leq H \cap M$, and hence $O_2(H_1 \cap M) \leq O_2(H \cap M)$. Now if $F^*(H_1 \cap M) = O_2(H_1 \cap M)$, then

$$C_{H \cap M}(O_2(H \cap M)) \leq C_{H_1 \cap M}(O_2(H_1 \cap M)) \leq O_2(H_1 \cap M) \leq O_2(H \cap M),$$

so (6) holds. That is, if (6) holds for H_1, then it also holds for H, so we may assume $H = H_1 \in \mathcal{M}$. Now $C_G(O_2(H)) \leq N_G(O_2(H)) = H$, while $O_2(H) \leq O_2(H \cap M)$ by (5). Thus $C_{O_2(M)}(O_2(H \cap M)) \leq C_M(O_2(H)) \leq H \cap M$, so $H \cap M \in \mathcal{H}^e$ by 1.1.4.5, proving (6). This completes the proof of 4.1.4. □

LEMMA 4.1.5. *Let $R_+ \in Syl_2(C_M(M_+/O_2(M_+))$, and assume*

$$1 \neq V = [V, M_+] \leq \Omega_1(Z(R_+)).$$

Suppose $I \in \mu^*$ and $R := O_2(I) \leq R_+$. Then

(1) $V \leq Z(R)$.

(2) If $V = [\Omega_1(Z(R_+)), M_+]$, then $N_G(V) \leq M$.

(3) Let $H \in \mathcal{H}(I, M)$, and set $H_+ := O^2(M_+ \cap H)$. Then $V = [V, H_+]$.

PROOF. Notice that the pair I,R satisfies the hypotheses of 4.1.4 for any $H \in \mathcal{H}(I,M)$. Since $I \in \mu$, there is $H_1 \in \mathcal{M}(I) - \{M\}$. By 4.1.4.5, $O_2(H_1) \leq O_2(H_1 \cap M) \leq R$, while $R \leq R_+ \leq C_G(V)$. Then $V \leq C_G(O_2(H_1)) \leq H_1$ as $H_1 \in \mathcal{M}$, so as $F^*(H_1 \cap M) = O_2(H_1 \cap M)$ by 4.1.4.6, $V \leq C_{H_1 \cap M}(O_2(H_1 \cap M)) \leq O_2(H_1 \cap M) \leq R$. Hence $V \leq Z(R)$, proving (1).

Next $N_M(R_+)$ acts on R_+ and M_+, and hence also on $[\Omega_1(Z(R_+)), M_+]$, so (2) follows from 4.1.2. Let $H \in \mathcal{H}(I,M)$. By 4.1.4.3, $M_+ = H_+O_2(M_+)$, so as $O_2(M_+) \leq R_+ \leq C_M(V)$, $V = [V, M_+] = [V, H_+O_2(M_+)] = [V, H_+]$, establishing (3). □

4.2. Pushing up in the Fundamental Setup

In this section, we apply the machinery of the previous section in the context of our Fundamental Setup (3.2.1). Recall from the discussion in Remark 3.2.4 that under the following assumption, the FSU holds for some $V \in \mathcal{R}_2(\langle L, T\rangle)$:

HYPOTHESIS 4.2.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, $M \in \mathcal{M}(T)$; and $L \in \mathcal{L}^*_f(G,T) \cap M$ with $L/O_2(L)$ quasisimple.*

LEMMA 4.2.2. *Hypothesis 4.1.1 holds with $M_+ := \langle L^T\rangle$.*

PROOF. By 1.2.1.3, $M_+ \trianglelefteq M$, and by 1.2.7.3, $M = !\mathcal{M}(M_+T)$. Further by 1.4.1.2 $O_2(M_+T) = C_T(M_+/O_2(M_+))$ is Sylow in $C_M(M_+/O_2(M_+))$, so any subgroup satisfying the hypotheses on "I" in Hypothesis 4.1.1 contains a Sylow 2-group of M, and hence conjugating in M we may assume $T \leq I$. But then $M_+T \leq I$, so that $M = !\mathcal{M}(I)$, and so Hypothesis 4.1.1 is satisfied. □

HYPOTHESIS 4.2.3. *Assume Hypothesis 4.2.1, and set*

$$M_+ := \langle L^T\rangle \quad \textit{and} \quad R_+ := C_T(M_+/O_2(M_+)).$$

Further assume $M_- \leq M$ with $M_+C_M(M_+/O_2(M_+)) \leq M_-$ and $L^T = L^{M_-}$, $I \in \mu^*(M_+, M_-)$, and $R := O_2(I) \leq R_+$.

LEMMA 4.2.4. *Assume Hypothesis 4.2.3 and $H \in \mathcal{H}(I,M)$. Set $M_H := H \cap M$, $L_H := (L \cap H)^\infty$, $M_0 := \langle L_H^{M_H}\rangle$, and $V := [\Omega_1(Z(R_+)), M_+]$. Then*

(1) The hypotheses of 4.1.4 and 4.1.5 are satisfied, with $M_0 = O^2(M_+ \cap H)$ in the role of "H_+".

(2) Hypothesis C.2.8 is satisfied.

(3) $R_+ = O_2(M_+T) = C_T(V)$.

PROOF. By construction $V \leq Z(R_+)$, so that $R_+ \leq C_T(V)$. As $L/O_2(L)$ is quasisimple and $[L,V] \neq 1$, $C_{M_+}(V) \leq O_{2,Z}(M_+)$, so $C_T(V) \leq R_+$, establishing (3).

By hypothesis, $H \in \mathcal{H}$, so $O_2(H) \neq 1$ and H is an SQTK-group. Of course $R \leq H \cap M = M_H$. By 4.2.2 and Hypothesis 4.2.3, the hypotheses of 4.1.4 are satisfied, so $F^*(M_H) = O_2(M_H)$ by 4.1.4.6. Thus part (1) of Hypothesis C.2.8 is established.

By 4.1.4.3, $L = L_HO_2(L)$, so $L_H \in \mathcal{C}(M_H)$. Using Hypothesis 4.2.3, $L^M = L^{M_-} = L^I \subseteq L^{M_H} \subseteq L^M$, so that $O^2(M_+ \cap H) = \langle L_H^{M_H}\rangle = M_0$. Hence M_0 plays the role of "H_+" in 4.1.4. Now part (2) of Hypothesis C.2.8 holds by 4.1.4.

Since $R_2(M_+T) \leq \Omega_1(Z(O_2(M_+T))) = \Omega_1(Z(R_+))$, $V \neq 1$ by 1.2.10. Since $R \leq R_+$ by Hypothesis 4.2.3, the hypotheses of 4.1.5 are satisfied. In particular,

(1) holds, and $N_H(V) \leq M_H$ and $V = [V, M_0]$ by 4.1.5. As $V \leq Z(R_+)$ and $O_2(M_0R) \leq R_+$, $V \leq Z(O_2(M_0R))$. Thus part (3) of Hypothesis C.2.8 holds, completing the verification of that Hypothesis, and establishing (2). □

THEOREM 4.2.5. *Assume Hypothesis 4.2.3 and $H \in \mathcal{M}(I) - \{M\}$. Then*

$$O_{2,F^*}(H) \not\leq M.$$

The proof of Theorem 4.2.5 involves a short series of reductions, culminating in 4.2.10. Until it is complete, assume I, H afford a counterexample; that is, assume $O_{2,F^*}(H) \leq M$.

By 4.2.4, the quintuple H, $M_H := H \cap M$, $L_H := (L \cap H)^\infty$, R, $V := [\Omega_1(Z(R_+)), M_+]$ satisfies Hypothesis C.2.8, so we can apply results in the latter part of section C.2 to this quintuple.

LEMMA 4.2.6. *(1) $M_+ = L$.*
(2) $H_+ := L_H \in \mathcal{C}(H)$, and $M_H = H \cap M = N_H(L_H)$ is of index 2 in H.

PROOF. As we are assuming $O_{2,F^*}(H) \leq M$, we may apply C.2.13. Since $M \neq H \in \mathcal{M}$ we have $M_H < H$, so case (1) of C.2.13 does not hold. Thus case (2) of C.2.13 holds, so that (2) holds. By Hypothesis 4.2.3, $L^I = L^M$, while $L^I \subseteq L^{M_H} = \{L\}$ by (2), so (1) holds. □

We now reverse the roles of H, M—applying suitable results on pushing up to M instead of H.

Set $Q := O_2(M_H)$. By assumption $O_{2,F^*}(H) \leq M$, so $Q = O_2(H)$ by A.4.4.1. Now as $H \in \mathcal{M}$, $H = N_G(O_2(H))$, and $C(M, Q) = M_H$ by A.4.4.2. Thus $Q \in \mathcal{B}_2(M)$ and Q is Sylow in $\langle Q^{M_H} \rangle = Q$, so the triple Q, M_H, M satisfies Hypothesis C.2.3 in the roles of "R, M_H, H". Therefore we can apply the results from Section C.2 based on Hypothesis C.2.3 to this triple. Further as $Q \in \mathcal{B}_2(M)$,

$$O_2(M) \leq Q$$

by C.2.1.2.

LEMMA 4.2.7. *(1) $L = L_H \in \mathcal{C}(H)$.*
(2) $L^H = \{L, L^h\}$ for each $h \in H - M$.

PROOF. By 4.2.6.1, $M_+ = L \in \mathcal{C}(M)$. By 4.2.4, we may apply 4.1.4. Then by 4.1.4.3, $L = L_H O_2(L)$, so as $O_2(L) \leq O_2(M) \leq Q \leq H$, $L = L_H$. Thus (1) holds, and then (2) follows from 4.2.6.2. □

In the remainder of the proof of Theorem 4.2.5, let h denote an element of $H - M$. Set $H_0 := \langle L^H \rangle$. Then $H_0 \leq N_H(L) = M_H \leq M$ using 4.2.6.2 and 4.2.7.1. As $H_0 \trianglelefteq H$ and $H \in \mathcal{M}$, we have:

LEMMA 4.2.8. *$H = N_G(H_0)$.*

LEMMA 4.2.9. *$O_{2,F}(M) \leq H$.*

PROOF. Recall that Q, M satisfy Hypothesis C.2.3 in the roles of "R, H". We may assume that $O_{2,F}(M) \not\leq H$, so by C.2.6, there is a subnormal A_4-block Y of M with $Y \not\leq H$. As $m_3(M) \leq 2$, $H_0 \leq O^2(M) \leq N_M(Y)$, so as $Aut(Y/O_2(Y))$ is a 2-group, $[Y, H_0] \leq O_2(Y) \leq O_2(M) \leq Q$. But then Y acts on $O^2(H_0Q) = H_0$, so $Y \leq H$ by 4.2.8. This contradicts $Y \not\leq H$, completing the proof. □

By A.4.4.3, $O_{2,F^*}(M) \not\leq H$, so in view of 4.2.9, there is $K \in \mathcal{C}(M)$ with $K/O_2(K)$ quasisimple and $K \not\leq H$.

LEMMA 4.2.10. *(1)* $L^h \leq K \cap H < K$.
(2) $m_p(K) = 1$ *for each odd prime* $p \in \pi(L)$*, and* $K \trianglelefteq M$*.*

PROOF. First by 4.2.6.1, $M = N_G(L)$ since $M \in \mathcal{M}$. Then $L^h \leq C_H(L/O_2(L))$ by 4.2.7 and 1.2.1.2, and hence $L^h \leq C_M(L/O_2(L))$. Similarly $L \neq K$ as $K \not\leq H$, so $K \leq C_M(L/O_2(L))$ by 1.2.1.2. Hence by 1.2.1.1, $KL^h \leq \langle\, \mathcal{C}(C_M(L/O_2(L)))\,\rangle =: K_0 \trianglelefteq M$.

Let $p \in \pi(L)$ be an odd prime. As M is an SQTK-group, $m_p(M) \leq 2$, so as $K_0 \leq C_M(L/O_2(L))$, $m_p(K_0) \leq 1$. Thus $L^h \leq O^{p'}(K_0) =: K_1$, and $K_1 \in \mathcal{C}(K_0)$. If $K \neq K_1$ then K acts on $LL^h = H_0$, so that $K \leq N_G(H_0) = H$ by 4.2.8, contradicting $K \not\leq H$. Therefore $L^h \leq K_1 = K$, and then (1) holds as $K \not\leq H$. Further as $K = K_1$, $m_p(K) = 1$ and $K = O^{p'}(K_0)$ by earlier observations, so (2) holds as $K_0 \trianglelefteq M$. □

We are now in a position to complete the proof of Theorem 4.2.5. First $K \trianglelefteq M$ by 4.2.10.2, so Q acts on K. Set $(KQ)^* := KQ/C_{KQ}(K/O_2(K))$ and $J := L^h$. Then K^* and the action of Q^* on K^* are described in C.2.7. Now $J \leq K \cap M_H$ by 4.2.10.1, while by 4.2.6.2 and 4.2.7, $M_H = N_H(L) = N_H(J)$. Hence $J^* \trianglelefteq (K \cap M_H)^*$. As J^* is not solvable, inspecting the list of possibilities in C.2.7.3, cases (a)–(d) and (f) are eliminated, as are the cases in (h) where the parabolic is solvable. The condition in 4.2.10.2 that $m_p(K) = 1$ for each odd prime $p \in \pi(J^*)$ then eliminates the remaining cases. This contradiction completes the proof of Theorem 4.2.5.

NOTATION 4.2.11. Assume Hypothesis 4.2.1, set $M_+ := \langle L^T \rangle$, and let $\mathcal{I}$ be the set of subgroups I of M such that

$$L \leq IO_2(\langle L, T\rangle),\ L^T = L^I, \text{ and } O_2(I) \neq 1.$$

LEMMA 4.2.12. *Assume Hypothesis 4.2.1,* $I \in \mathcal{I}$*, and* $H \in \mathcal{M}(I) - \{M\}$*. Let* $O_2(I) \leq R_+ \in Syl_2(C_M(M_+/O_2(M_+)))$*. Then*

(1) $M_- := M_+C_M(M_+/O_2(M_+))I \in \Sigma(M_+)$ *and* $I \in \mu(M_+, M_-)$*.*

(2) Assume $I \in \mu^*$ *and set* $L_H := (L \cap I)^\infty$*. Then* $M_+ = L$*,* $L_H \in \mathcal{C}(H \cap M)$ *is normal in* $H \cap M$*,* $[\Omega_1(Z(R_+)), L_H] = [\Omega_1(Z(R_+)), L] = [R_2(LT), L]$*, and* $L_H \leq K \in \mathcal{C}(H)$ *with* $K \not\leq M$*,* $K/O_2(K)$ *quasisimple, and* K *is described in one of cases (1)–(9) of Theorem C.4.8.*

PROOF. Set $R := O_2(I)$. Since $T \in Syl_2(G)$, we may assume that $R \leq T \cap I \in Syl_2(I)$. By 4.2.2, Hypothesis 4.1.1 is satisfied. By construction, $M_- \in \Sigma(M_+)$. By definition of $I \in \mathcal{I}$ in Notation 4.2.11, $L^I = L^T$, $1 \neq R$, and $L \leq IR_+$, where $R_+ := O_2(\langle L, T\rangle)$. By A.4.2.4, $R_+ = C_T(M_+/O_2(M_+))$. As $L^T = L^I$, $M_+ \leq IR_+$, and hence $M_+ \leq IO_2(M_+)$, so $R = O_2(I) \leq C_T(M_+/O_2(M_+)) \leq R_+$ and $M_- = C_M(M_+/O_2(M_+))I$. Thus $I \in \eta$, and as $H \in \mathcal{M}(I) - \{M\}$, $I \in \mu$. That is, (1) is established.

Assume $I \in \mu^*$ and set $V_+ = [\Omega_1(Z(R_+)), M_+]$, $M_H := M \cap H$, $L_H := (L \cap H)^\infty$, and $M_0 := O^2(M_+ \cap H)$. As Hypothesis 4.2.3 holds, by 4.2.4 we may apply 4.1.4 and 4.1.5. By 4.1.5.3, $V_+ = [V_+, M_0]$. Also by 4.2.4, $M_0 = \langle L_H^{M_H} \rangle$ and the quintuple H, L_H, M_H, R, V_+ satisfies Hypothesis C.2.8.

We now appeal to Theorem C.4.8. By Theorem C.4.8, $L_H \trianglelefteq M_H$, so $L = L_0 \trianglelefteq M$ since $L^T = L^I$. As $O_{2,F^*}(H) \not\leq M$ by 4.2.5, one of cases (1)–(9) of Theorem C.4.8 holds. By Theorem C.4.8, $L_H \leq K \in \mathcal{C}(H)$ with $K \not\leq M$ and $K/O_2(K)$ quasisimple. As $L/O_2(L)$ is quasisimple, $\Omega_1(Z(R_+)) = R_2(LT)$, so $V_+ = [R_2(LT), L]$. This completes the proof of (2). □

Now we come to a fundamental result, showing that many subgroups of LT covering $L/O_2(L)$ are uniqueness subgroups, whenever V is not on a short list of FF-modules.

THEOREM 4.2.13. *Assume Hypothesis 4.2.1 and let $I \in \mathcal{I}$. Then either $M = !\mathcal{M}(I)$; or $L \trianglelefteq M$, $V := [R_2(LT), L]$ is an FF-module for $LT/O_2(LT)$, and one of the following holds:*

(1) $L/O_2(L) \cong L_2(2^n)$.

(2) $L/O_2(L) \cong L_3(2)$ or $L_4(2)$, and $V/C_V(L)$ is either the sum of isomorphic natural modules, or the 6-dimensional orthogonal module for $L_4(2)$.

(3) $O^2(I \cap L)$ is an A_6-block or an exceptional A_7-block.

(4) $O^2(I \cap L)$ is a block of type $\hat{A}_6$, and for each $z \in C_V(T)^\#$, $V \not\leq O_2(C_G(z))$.

(5) $O^2(I \cap L)$ is a block of type $G_2(2)$, and if $m(V) = 6$ and V_3 is the $(T \cap I)$-invariant subspace of V of rank 3, then $C_G(V_3) \not\leq M$.

PROOF. Assume $I \in \mathcal{I}$, $H \in \mathcal{M}(I) - \{M\}$, and set $R := O_2(I)$. Since $T \in Syl_2(G)$, we may assume that $R \leq T \cap I \in Syl_2(I)$. Define M_- as in 4.2.12; by 4.2.12.1, $I \in \mu$.

Let $I \lesssim I_1 \in \mu$. Then $I_1 \in \mathcal{I}$, and if I_1 satisfies one of the conclusions (1)–(5) of the Theorem, then so does I since $I \cap M_+ \leq I_1 \cap M_+$. Thus we may assume $I \in \mu^*$. Hence Hypothesis 4.2.3 is satisfied. Similarly let $I_2 := (T \cap I)(M_+ \cap I)$. Then $I = I_2 C_I(M_+/O_2(M_+))$, so the hypotheses of 4.1.3 are satisfied with I, I_2 in the roles of "I_0, I_1", and hence $I_2 \in \eta$ by that lemma. Then by construction, $I_2 \in \mu^*$, so replacing I by I_2, we may assume $I \leq M_+T$.

Set $M_H := M \cap H$ and $L_H := (L \cap H)^\infty$. As $I \in \mu^*$, 4.2.12.2 says $M_+ = L \trianglelefteq M$, $V = [\Omega_1(Z(R_+)), L_H] \leq L_H$, $L_H \leq K \in \mathcal{C}(H)$ with $K \not\leq M$ and $K/O_2(K)$ quasisimple, and one of cases (1)–(9) of Theorem C.4.8 holds. We first eliminate case (9): for in that case, K is the double cover of A_8 with $Z(K) = Z(L_H)$; but then $1 \neq Z(L_H) = C_V(L_H) = C_V(L)$ is LT-invariant, so that $K \leq M = !\mathcal{M}(LT)$, contrary to $K \not\leq M$. Among the remaining cases, only case (6) is not included among the conclusions of Theorem 4.2.13—although in cases (5) and (7) of C.4.8, we still need to show that the extra constraints in conclusions (4) and (5) of Theorem 4.2.13 hold. We will eliminate case (6) of C.4.8 later.

In case (5) of C.4.8, L_H is a block of type $\hat{A}_6$ with $m(V) = 6$ and $K \cong M_{24}$ or He. Therefore for each $z \in C_V(T \cap L)^\#$, $V \not\leq C_K(z)$, so that conclusion (4) of Theorem 4.2.13 holds.

Assume that case (7) of C.4.8 holds, so that L_H is a $G_2(2)$-block and $K \cong Ru$. We may assume that $m(V) = 6$, and it remains to show that $C_K(V_3) \not\leq M \cap K$. To see this, we will use facts about the 2-locals of $K \cong Ru$ found in chapter J of Volume I. Observe that $M \cap K = N_K(L_H)$ with $(M \cap K)/V \cong G_2(2)$. Let V_1 be the $(T \cap L_H)$-invariant subspace of V of rank 1; then $M_1 := C_{M \cap K}(V_1)$ is of order $3 \cdot 2^{12}$, so $3 \in \pi(C_K(V_1))$ and hence V_1 is 2-central in K by J.2.7.4 and J.2.9.1. Let $K_1 := C_K(V_1)$, $Q_1 := O_2(K_1)$, and $X_1 \in Syl_3(M_1)$. From (Ru2) in the definition of groups of type Ru in chapter J of Volume I, $K_1^* := K_1/Q_1 \cong S_5$, and from J.2.3, $C_{Q_1}(X_1) \cong Q_8$. Let $v \in C_V(X_1) - V_1$; it follows that v^* is of order 2 in $C_{M_1^*}(X_1^*)$, so $M_1^* \cong D_{12}$. Hence $P_1 := Q_1 \cap M_1$ is of order 2^{10} with $[O_2(M_1), X_1] \leq P_1$ and $|C_{P_1}(X_1)| = 4$. Then $V_3 \leq \Phi([O_2(M_1), X_1]) \leq \Omega_1(Q_1)$, and $\Omega_1(Q_1)$ is the group denoted by "U" in (Ru2). Thus by J.2.2.3, $C_{Q_1}(X_1) \leq C_{Q_1}(U) \leq C_{Q_1}(V_3)$. Hence as $|C_{Q_1}(X_1)| = 8 > |C_{P_1}(X_1)|$, $C_K(V_3) \not\leq M$, as claimed.

Thus to complete the proof of Theorem 4.2.13, we may assume that case (6) of Theorem C.4.8 holds, and it remains to derive a contradiction. Then L_H is a block of type M_{24} or $L_5(2)$, and $K \cong J_4$. In particular, K is a component of the maximal 2-local H, and so centralizes $O_2(H) \neq 1$. As $Out(J_4) = 1$, $H = K \times C_H(K)$, with $O_2(H) \leq C_H(K)$. Hence $I = L_H N_{T\cap K}(L_H) \times C_I(K)$, and setting $R_C := C_R(K)$, $R = O_2(I) = V \times R_C$. As V is self-centralizing in K, $R_C = C_R(K) = C_R(L_H)$. By 4.1.4.5, $O_2(H) \leq R$, so $O_2(H) \leq R_C$.

Recall we reduced in the first two paragraphs of the proof to the case where $I \leq LT$. Thus as $I = L_H N_{T\cap K}(L_H) \times C_I(K)$, $I = L_H(N_{T\cap K}(L_H)) \times R_C$. Let $S := N_{T\cap K}(L_H) \times R_C$ and r an involution in $Z(R_C)$; thus $S \in Syl_2(I)$ and $r \in Z(S)$. Next $O^2(I) = L_H \leq K \leq C_H(r)$ as $r \in R_C$, and hence r centralizes $O^2(I)S = I$, so without loss $H \in \mathcal{M}(C_G(r))$. Then in particular K is a component of $C_G(r)$.

From the structure of L_H in case (6) of Theorem C.4.8, there is X of order 3 in L_H with $C_V(X) \neq 1$. Let $K_X := C_K(X)^\infty$ and $G_X := C_G(X)$. Then K_X is quasisimple with $Z(K_X) \cong \mathbf{Z}_6$ and $K_X/Z(K_X) \cong M_{22}$. Thus K_X is also a component of $C_{G_X}(r)$, and hence by I.3.2, $K_X \leq L_X \in \mathcal{C}(G_X)$ with $\bar{L}_X := L_X/O(L_X)$ quasisimple. We claim $K_X = L_X$, so assume that $K_X < L_X$. Then as $K_X \in \mathcal{C}(C_{G_X}(r))$, r is faithful on L_X, and in particular on the quasisimple quotient $\bar{L}_X$. Now case (1.a) of Theorem A (A.2.1) holds since $\bar{L}_X$ is quasisimple, so $\bar{L}_X$ is quasithin. Then inspecting the list of groups in Theorem B (A.2.2), we find that none possesses an involutory automorphism r whose centralizer has a component $\bar{K}_X$ which is a covering of M_{22}. This contradiction establishes the claim that $K_X = L_X \in \mathcal{C}(G_X)$.

Recall from Hypothesis 4.2.3 that $R \leq R_+ = C_T(M_+/O_2(M_+)) = O_2(\langle L, T\rangle)$, and set $R_1 := N_{R_+}(R)$ and $R_1^* := R_1/R$. Recall also from our application of 4.2.12.2 early in the proof that $L \trianglelefteq M$, $V \leq L_H$, and $V = [R_2(LT), L]$, so V is T-invariant. If L_H is an $L_5(2)$-block, then by Theorem C.4.8, V is one of the 10-dimensional modules for L_H/V, so as V is T-invariant, T induces inner automorphisms on $L/O_2(L)$. Of course T induces inner automorphisms on $L/O_2(L)$ if L_H is an M_{24}-block as $Out(M_{24}) = 1$. Thus $LT = LR_+$, so as $I < LT$ (since $M = !\mathcal{M}(LT)$), $R = O_2(I) < R_+$ and hence $R < R_1$. By 4.1.4.4, $R = R_+ \cap H$, so as $C_G(r) \leq H$ we have $R = C_{R_1}(r)$. As $R = V \times R_C$, we can choose $r \in R_C$ so that $rV \in C_{R/V}(R_1)$. Hence the map $\chi : x^* \mapsto [r, x]$ is an L_H-isomorphism of R_1^* with V: Since $V \leq \Omega_1(Z(R_+))$, the map is a homomorphism by a standard commutator formula 8.5.4 in [**Asc86a**]; then injectivity follows from $R = C_{R_1}(r)$, and surjectivity as L_H is irreducible on V. Now there is $v \in C_V(X) - C_V(K_X)$, and for $s \in \chi^{-1}(v) \cap G_X$, $r^s = rv$. As M_{22} is not involved in the groups in A.3.8.2, $K_X \trianglelefteq G_X$, so as $[r, K_X] = 1$, also $[r^s, K_X] = 1$ and hence $[v, K_X] = 1$, contrary to the choice of v. This contradiction completes the proof of Theorem 4.2.13. □

4.3. Pushing up $L_2(2^n)$

In the first exceptional case of Theorem 4.2.13 where $L \trianglelefteq M$ and $L/O_2(L) \cong L_2(2^n)$ for $n > 1$, it is possible to obtain a result weaker than Theorem 4.2.13, but still stronger than $M = !\mathcal{M}(LT)$: Namely in Theorem 4.3.2, we show in this case that at least L is also a uniqueness subgroup. Theorem 4.3.2 will be used in the Generic Case of the proof of the Main Theorem. Therefore:

Throughout this section we assume Hypothesis 4.2.1, with $L/O_2(L) \cong L_2(2^n)$, and $L \trianglelefteq M$.

LEMMA 4.3.1. *Let S be a 2-subgroup of M, $T_H \in Syl_2(N_M(S))$, and assume that $S \cap L \in Syl_2(L)$ and $M = !\mathcal{M}(LT_H)$. Then $N_G(S) \leq M$.*

PROOF. Assume otherwise, and pick S to be a counterexample to 4.3.1 such that T_H is of maximal order subject to this constraint. We may assume $T_H \leq T$. We claim that $T_H \in Syl_2(N_G(S))$. If $T_H = T$ this is clear, so we may assume that $T_H < T$, and hence $T_H < N_T(T_H)$. As $S \leq T_H$, $T_H \cap L = S \cap L \in Syl_2(L)$ and by hypothesis $M = !\mathcal{M}(LT_H)$, so by maximality of $|T_H|$, $N_G(T_H) \leq M$. Hence if $T_H \leq T_S \in Syl_2(N_G(S))$, then $N_{T_S}(T_H) \leq T_S \cap M \leq N_M(S)$, so $T_H = T_S$ as claimed.

Observe next that Hypothesis C.5.1 of chapter C of Volume I is satisfied with LT_H, $N_G(S)$, S in the roles of "H, M_0, R". Further we may assume that Hypothesis C.5.2 is satisfied, or otherwise $O_2(\langle LT_H, N_G(S)\rangle) \neq 1$, so that $N_G(S) \leq M = !\mathcal{M}(LT_H)$, as desired. Thus we may apply C.5.6.6, and obtain a contradiction to $L \trianglelefteq M$. This completes the proof. $\square$

THEOREM 4.3.2. $M = !\mathcal{M}(L)$.

The proof of Theorem 4.3.2 involves a series of reductions, culminating in 4.3.16.

Assume the Theorem fails, and pick I so that $L \leq I \leq LO_2(LT)$ and I is maximal subject to $\mathcal{M}(I) \neq \{M\}$. Set $R := O_2(I)$ and $R_+ := O_2(LT)$, so that

$$I = LR \text{ and } R = I \cap R_+.$$

Set $V := [\Omega_1(Z(R_+)), L]$. Choose $H \in \mathcal{M}(I) - \{M\}$, and set $M_H := H \cap M$.

Define $\mathcal{I}$ as in Notation 4.2.11 and observe $I \in \mathcal{I}$. Set $M_- := LC_M(L/O_2(L))$; by 4.2.12.1, $M_- \in \Sigma(L)$ and $I \in \mu$. Then by maximality of I, $I \in \mu^*$, so Hypothesis 4.2.3 is satisfied and hence by 4.2.4, the quintuple H, L, M_H, R, V satisfies Hypothesis C.2.8. By 4.2.12.2, $L \leq K \in \mathcal{C}(H)$, with $K \not\leq M$, $K/O_2(K)$ quasisimple, and K appears in one of cases (1)–(9) of Theorem C.4.8. As $L/O_2(L) \cong L_2(2^n)$, case (1) of Theorem C.4.8 holds, so that either $V/C_V(L)$ is the natural module for $L/O_2(L)$, or $n = 2$ and V is the A_5-module. Furthermore M_H acts on K by Theorem C.4.8. By 1.2.1.5, either $F^*(K) = O_2(K)$, or K is quasisimple and hence a component of H. Therefore K is described in either Theorem C.4.1 or Theorem C.3.1, respectively. Set $M_K := M \cap K$.

Recall from 4.2.4.3 that $R_+ = O_2(LT)$, and $R_+ = C_T(L/O_2(L))$ by 1.4.1.2. Without loss $S := T \cap H \in Syl_2(M_H)$, and we choose $H \in \mathcal{M}(I) - \{M\}$ so that S is maximal. As $L \leq H$ and $M = !\mathcal{M}(LT)$, $T \not\leq H$, so $S < T$, and hence also $S < N_T(S)$.

LEMMA 4.3.3. *(a) If $S < X \leq T$, then $M = !\mathcal{M}(LX)$.*
(b) $N_G(S) \leq M$, so $S \in Syl_2(H)$ and $H = N_G(K)$.

PROOF. As $I = LR \leq LS$, maximality of S implies (a). Then as $S < N_T(S)$, (a) and 4.3.1 imply $N_G(S) \leq M$. Therefore as $S \in Syl_2(M_H)$, $S \in Syl_2(H)$. As we saw earlier that K is M_H-invariant, $K \trianglelefteq H$ by 1.2.1.3, so $H = N_G(K)$ as $H \in \mathcal{M}$. □

LEMMA 4.3.4. $R = S \cap R_+$. *In particular, S normalizes R and $R = O_2(IS)$.*

PROOF. As $I \leq L(S \cap R_+)$, this follows from maximality of I. □

We next choose an element $t \in N_T(S) - S$ with $t^2 \in S$. If $R < R_+$, then $R_+ \not\leq S$ by 4.3.4, so in this case we may choose t so that also $t \in R_+$ and $t^2 \in S \cap R_+ = R$. By convention, t will denote such an element throughout the proof.

As $t \in N_T(S)$, t normalizes $S \cap R_+ = R$. Further $t \notin S$, so by 4.3.3.a:

LEMMA 4.3.5. $M = !\mathcal{M}(LS\langle t\rangle)$.

LEMMA 4.3.6. $F^*(K) = O_2(K)$.

PROOF. Assume otherwise. Then from our remarks following the statement of Theorem 4.3.2, K is a component of H described in Theorem C.3.1. As $L/O_2(L) \cong L_2(2^n)$, we conclude that either

(i) $K/Z(K)$ is of Lie type and Lie rank 2 over $\mathbf{F}_{2^n}$, and M_K is a maximal parabolic of K, or

(ii) $K/Z(K) \cong M_{22}$ or M_{23}, and L is an $L_2(4)$-block.

Let $\overline{KS} := KS/C_{KS}(K)$ and $S_K := S \cap K$. Now $L \leq K \leq C_H(O_2(H))$ as K is a component of H, and $1 \neq O_2(H) \leq R$ by 4.1.4.5, so $1 \neq R_0 := C_R(L)$. Recall from 4.3.4 and our choice of t that $S\langle t\rangle$ acts on R and L and hence also on R_0, so $N_G(R_0) \leq M = !\mathcal{M}(LS\langle t\rangle)$ by 4.3.5. Then $[K, R_0] \neq 1$ as $K \not\leq M$, so $1 \neq \bar{R}_0 \leq C_{\bar{R}}(\bar{L})$. Inspecting the automorphism groups of the groups in (i) and (ii) (e.g., 16.1.4 and 16.1.5) for such a 2-local subgroup, we conclude $K/Z(K) \cong Sp_4(2^n)$. Indeed $Z(K) = 1$ since the multiplier of $Sp_4(2^n)$ for $n > 1$ is trivial by I.1.3. Furthermore $V = O_2(L)$ is the maximal nonsplit extension of the natural module for $L/O_2(L)$ over a trivial module by I.1.6, and $C_V(L)$ is a root subgroup of K. Since $Aut(K)$ fuses the two K-classes of root subgroups, we may regard $C_V(L)$ as a short root subgroup of K, and take $Z \leq C_V(S_K)$ to be a long root subgroup of K.

Set $G_Z := N_G(Z)$. As $Z = [C_V(T \cap L), N_L(T \cap L)]$ and T acts on L and V, T acts on Z; hence $F^*(G_Z) = O_2(G_Z) =: Q_Z$ by 1.1.4. Let $K_2 := N_K(Z)^\infty$ where $N_K(Z)$ is the maximal parabolic of K containing S_K and distinct from $N_K(C_V(L))$. As $L \leq M$ but $K = \langle L, K_2\rangle \not\leq M$, $K_2 \not\leq M$. Further $T \not\leq N_G(K_2)$, or otherwise T normalizes $\langle L, K_2\rangle = K$, and hence $T \leq H$ by 4.3.3.b, contrary to our observation just before 4.3.3. We will now analyze G_Z, and eventually obtain a contradiction by showing that $T \leq N_G(K_2)$.

First, a Cartan subgroup Y of the Borel group $M_K \cap N_K(Z)$ of K decomposes as $Y = Y_1 \times Y_2$, where $Y_1 := C_Y(K_2/O_2(K_2))$ and $Y_2 := Y \cap K_2^\infty$ are cyclic of order $2^n - 1$, Y_1 is regular on $Z^\#$, and $N_K(Z) = Y_1K_2^\infty$.

Next by 1.2.1.1, K_2 is contained in the product $L_1 \cdots L_r$ of those members L_i of $\mathcal{C}(C_G(Z))$ with $L_i = [L_i, K_2]$. If $r > 1$, then for a prime divisor p of $2^n - 1$, $m_p(L_1 \cdots L_rY_1) > 2$, contradicting G_Z an SQTK-group. Therefore $K_2 \leq L_1 =: K_Z \in \mathcal{C}(C_G(Z))$. Recall from the remarks after (i) and (ii) above that $K \cong Sp_4(2^n)$ is simple, so that K_2 contains a Levi complement isomorphic to $L_2(2^n)$, and in particular $K_Z/O_{2,F}(K_Z)$ is not $SL_2(p)$ for any odd prime p. This rules out cases (c) and (d) in 1.2.1.4, so that $K_Z/O_2(K_Z)$ is quasisimple. Furthermore as $Y_1 \leq G_Z$ is faithful on Z, $K_Z \trianglelefteq G_Z$ by 1.2.2. Similarly as $m_p(K_Z Y_1) \leq 2$, we conclude from A.3.18 that $m_p(K_Z) = 1$ for each prime divisor p of $2^n - 1$—unless possibly $p = 3$ (so that n is even), and a subgroup of Y_1 of order 3 induces a diagonal automorphism on $K_Z/O_2(K_Z) \cong L_3^\epsilon(q)$, with $q \equiv \epsilon \mod 3$. (If case (3b) of A.3.18 were to hold, then $m_3(Y_1 K_2 O_{2,3}(K_Z)) = 3$.)

Set $U := \langle C_V(L)^{G_Z}\rangle$. Now T acts on V and L, and hence on $C_V(L)$, so as $C_V(L) \neq 1$, $C_V(LT) \neq 1$. Then as $G_Z \in \mathcal{H}^e$, $C_V(LT) \leq \Omega_1(Z(Q_Z))$, so as Y is irreducible on $C_V(L)$ and $O_2(K_2) = \langle C_V(L)^{K_2}\rangle$,

$$O_2(K_2) \leq \langle C_V(L)^{G_Z}\rangle = U \leq \Omega_1(Z(Q_Z)). \qquad (*)$$

In particular U is generated by G_Z-conjugates of elements of $Z(T)$, so $U \in \mathcal{R}_2(G_Z)$ by B.2.14.

Let $G_Z^* := G_Z/C_{G_Z}(U)$. As $K_Z/O_2(K_Z)$ is quasisimple, so is K_Z^*. As $V/C_V(L)$ is the natural module for $L/O_2(L) \cong L_2(2^n)$, $C_T(C_V(L)Z) = C_T(V)(T \cap L)$ with $C_T(V)(T \cap L)/C_T(V) \cong E_{2^n}$, and in fact $C_T(V)(T \cap L) = C_T(V)O_2(K_2)$. Further $O_2(K_2) \leq Q_Z$ by (*); and also $[Q_Z, V] \leq Q_Z \cap V = O_2(K_2) \cap V \leq C_V(T \cap L)$, so that $Q_Z \leq C_T(V)(T \cap L)$. Hence

$$m(O_2(K_2)/C_{O_2(K_2)}(V)) = n = m(Q_Z/C_{Q_Z}(V)) \text{ and } Q_Z = O_2(K_2)C_{Q_Z}(V). \qquad (**)$$

By (*), $O_2(K_2) \leq U$, so as $m(V/V \cap O_2(K_2)) = n$ with $C_V(U) \leq C_V(O_2(K_2)) = V \cap O_2(K_2)$, $m(V^*) = n$. By (*) and (**), $m(U/C_U(V)) \leq m(Q_Z/C_{Q_Z}(V)) = n$. Therefore U is a failure of factorization module for K_Z^* with FF*-offender V^*. In particular $K_Z/O_2(K_Z)$ is not $L_3^\epsilon(q)$ with $q \equiv \epsilon \mod 3$, since in that event as U is an FF-module, Theorem B.5.6.1 says $K_Z^* \cong SL_3(q)$, whereas $SL_3(q)$ is not isomorphic to $L_3(q)$ when $q \equiv 1 \mod 3$. This eliminates the exceptional case in our discussion above, so we conclude that $m_p(K_Z) = 1$ for each p dividing $2^n - 1$. Therefore by inspection of the lists in Theorems B.5.1 and B.4.2, $K_Z^* = K_2^*$, and U/Z is the natural module for $K_2^* \cong L_2(2^n)$ or the orthogonal module for $L_2(4)$. Thus as $O_2(K_2)/Z$ is the natural module for K_2^*, $U = O_2(K_2)$ by (*), and as $Q_Z = O_2(K_2)C_{Q_Z}(V)$ by (**), we conclude $[V, Q_Z] = [V, U] \leq U$. Then as $K_2 = \langle V^{K_2}\rangle$, $[K_2, Q_Z] = U \leq K_2$,

$$K_Z = \langle K_2^{K_Z}\rangle \leq \langle K_2^{K_2 Q_Z}\rangle = K_2,$$

and hence $K_2 = K_Z$ is normalized by T, contrary to our earlier observation that $T \not\leq N_G(K_2)$. This contradiction completes the proof of 4.3.6. □

By 4.3.6, $F^*(K) = O_2(K)$; so as we observed following the statement of Theorem 4.3.2, K is described in Theorem C.4.1, and as $L/O_2(L) \cong L_2(2^n)$, one of cases (1)–(3) of Theorem C.4.1 holds.

LEMMA 4.3.7. *K is not a block.*

PROOF. Assume otherwise. Inspecting cases (1)–(3) of Theorem C.4.1, we conclude that either K is an $SL_3(2^n)$-block, or $n = 2$ and K is an A_7-block or an $Sp_4(4)$-block. Set $U := U(K)$ in the notation of Definition C.1.7. Now S normalizes K by 4.3.3.b, so as t normalizes S, S also normalizes U^t. Therefore if $U^t \leq O_2(KS)$, then as $[O_2(KS), K] \leq U \leq UU^t$, $UU^t \trianglelefteq KS\langle t\rangle$, forcing $K \leq M$ by 4.3.5, contrary to $K \not\leq M$. Hence $K = [K, U^t]$. Recall also that $V = [\Omega_1(Z(R_+)), L]$ is T-invariant, so $V = V^t$. As $R = O_2(LS)$ by 4.3.4, while $S \in Syl_2(H)$ by 4.3.3.b, $O_2(KR) \leq R$.

Suppose first that K is an $Sp_4(4)$-block. Then $Z_K := C_U(K) \leq V$ using I.2.3.3, and U/V is the natural $L_2(4)$-module for $L/O_2(L)$. So as $V = V^t$, U^t/V is also the natural module, with $C_U(K) \leq V \leq U \cap U^t < U$, impossible as $O_2(L)O_2(K)/O_2(K)$ is a non-split extension of a trivial submodule by a natural module, so that there is no natural L-submodule.

Suppose next that K is an A_7-block. Then by C.4.1.1, S induces a transposition on $L/O_2(L)$, so that $LS/O_2(LS) \cong S_5 = Aut(L/O_2(L))$, and hence $T = SR_+$. Hence as $R \leq S < T$, $R < R_+$ and so $R < N_{R_+}(R)$. We claim that K, R, S, R_+, KS satisfy Hypothesis C.6.2, and the hypotheses of C.6.4, in the roles of "L, R, T_H, Λ, H". Most requirements are either immediate or have been established earlier—except possibly for C.6.2 and C.6.4.II (recall the latter result uses C.6.3 and in particular verifies its hypotheses), which we now verify: If $1 \neq R_0 \leq R$ satisfies $R_0 \trianglelefteq KS$, then by 4.3.3.a, $N_T(R_0) = S$ as $K \not\leq M$; so by 4.3.4 $N_{R_+}(R_0) = R < N_{R_+}(R)$, completing the verification of those hypotheses. As $T = SR_+$, we conclude from C.6.4.10 that $e_{1,2} \in Z(T)$. Then as $e_{1,2}$ centralizes L, $C_G(e_{1,2}) \leq M = !\mathcal{M}(LT)$. Now $v := e_{3,4}$ is in V, and there is $k \in K$ with $e_{1,2}^k = v$. Then $R_+ \leq C_G(V) \leq C_G(v) \leq M^k$, so R_+ acts on L^k. But then R_+ acts on $K = \langle L, L^k \rangle$, so $T = SR_+ \leq N_G(K) = H$, which we saw earlier is not the case.

Therefore K is an $SL_3(2^n)$-block. Thus case (3) of C.4.1 holds, so L is the stabilizer of the line V of U, so that $[U, L] = V$. Therefore as t acts on V and L, also $[U^t, L] = V \leq U$. This is impossible as we saw $K = [K, U^t]$, whereas $K/O_2(K)$ admits no involutory automorphism centralizing $LO_2(K)/O_2(K)$. This contradiction completes the proof of 4.3.7. □

LEMMA 4.3.8. *$K/O_2(K) \cong SL_3(2^n)$, (KR, R) is an MS-pair described in one of cases (2)–(4) of Theorem C.1.34, and $S \in Syl_2(H)$.*

PROOF. Recall that K is described in one of cases (1)–(3) of Theorem C.4.1. As $L/O_2(L) \cong L_2(2^n)$ and K is not a block by 4.3.7, conclusion (3) of C.4.1 holds, so that $K/O_2(K) \cong SL_3(2^n)$, and one of cases (1)–(4) of C.1.34 holds. Further 4.3.7 rules out case (1) where K is an $SL_3(2^n)$-block. By 4.3.3, $S \in Syl_2(H)$. □

LEMMA 4.3.9. *$C_S(K) = 1$.*

PROOF. Let $U := \Omega_1(Z(O_2(KS)))$; as $K \trianglelefteq H$, $[U, K] \leq O_2(K)$. By 4.3.8, K is described in one of cases (2)–(4) of C.1.34, so that $[U, K]$ is the sum of one or two isomorphic natural modules for $K/O_2(K)$. So as the natural module has trivial 1-cohomology by I.1.6 since $n > 1$, we conclude that $U = C_U(K) \oplus [U, K]$. Further L stabilizes an $\mathbf{F}_{2^n}$-line in the natural summands of $[U, L]$ by C.4.1, so $C_{[U,K]}(L) = 0$. Thus $C_U(K) = C_U(L)$, so $C_Z(L) = C_Z(K)$, where $Z := \Omega_1(Z(S))$. But $N_T(S)$ normalizes $C_Z(L)$, so if $C_Z(L) \neq 1$ then $N_G(C_Z(L)) \leq M$ by 4.3.5. Therefore as $C_Z(K) = C_Z(L)$ and $K \not\leq M$, $C_Z(K) = 1$, establishing the lemma. □

LEMMA 4.3.10. *K satisfies conclusion (3) of Theorem C.1.34.*

PROOF. By 4.3.8, one of conclusions (2)–(4) of Theorem C.1.34 holds, and as $C_S(K) = 1$ by 4.3.9, conclusion (4) does not hold. Thus we may assume conclusion (2) holds, and it remains to derive a contradiction. Then $U = O_2(K)$ is the sum of two isomorphic natural modules. As $C_S(K) = 1$, we may apply C.1.36, to conclude that $\mathcal{A}(S) = \{U, A\}$ is of order 2 with $V = U \cap A$ of rank $4n$. We now obtain a contradiction similar to that in the $L_3(2^n)$-case of 4.3.7: Again $U^t \not\leq O_2(KS)$ using 4.3.5 and $V = [U, L]$ by C.4.1. As $U^t \not\leq O_2(KS)$ and $\mathcal{A}(S) = \{U, A\}$, $U^t = A$, while as $[U, L] = V$ is t-invariant, also $V = [L, U^t]$. This is a contradiction as $[A/U, L] = A/U \neq 1$. □

Set $Q := [O_2(K), K]$ and $U := Z(Q)$. By 4.3.10, conclusion (3) of C.1.34 holds; that is, U is the natural module for $K/O_2(K)$ and Q/U is the direct sum of two copies of the dual of U. In particular, S is trivial on the Dynkin diagram of $K/O_2(K)$, and hence normalizes both maximal parabolics over $S \cap K$.

Set $S_L := S \cap L$ and $Z_S := C_V(S_L)$. Set $G_Z := N_G(Z_S)$. By C.1.34, V is an $\mathbf{F}_{2^n}$-line in U, so Z_S an $\mathbf{F}_{2^n}$-point. As $S_L = T \cap L$ and V are T-invariant, Z_S is T-invariant.

Set $K_2 := C_K(Z_S)^\infty$, $R_2 := O_2(K_2S)$, and let Y be a Hall $2'$-subgroup of $O_{2,2'}(N_K(Z_S))$. Thus Y is cyclic of order $2^n - 1$, with $[K_2, Y] \leq O_2(K_2)$, and Y faithful on Z_S. Further Y is fixed point free on the natural module U for $K/O_2(K)$, so as we saw above just after 4.3.10 that the composition factors of Q are natural and dual, $Q = [Q, Y]$. Appealing to 4.3.9, we conclude from C.1.35.3 that:

Lemma 4.3.11. *$Q = O_2(KS)$ so $O_2(KS) = [O_2(KS), Y]$.*

Next by 1.2.1.1, K_2 is contained in the product $L_1 \cdots L_s$ of those members L_i of $\mathcal{C}(G_Z)$ such that K_2 projects nontrivially on $L_i/O_2(L_i)$. Therefore for each prime divisor p of $2^n - 1$, p divides the order of L_i. But if $s > 1$, then as Y is faithful on Z_S, and $Y = O^2(Y)$ acts on each L_i by 1.2.1.3, $m_p(YL_1L_2) > 2$, contradicting G_ZY an SQTK-group. Thus $s = 1$. Set $K_Z := L_1$. A similar argument shows K_Z is the unique member of $\mathcal{C}(G_Z)$ of order divisible by p, so that $K_Z \trianglelefteq G_Z$. If $p = 3$ and K_Z appears in case (3b) of A.3.18, then $m_3(YK_2O_{2,Z}(K_Z)) = 3$, contradicting G_Z an SQTK-group. Therefore we may appeal to A.3.18 to obtain:

Lemma 4.3.12. *(1) $K_2 \leq K_Z \in \mathcal{C}(G_Z)$ and $K_Z \trianglelefteq G_Z$.*
(2) For p a prime divisor of $2^n - 1$, either $m_p(K_Z) = 1$, or $p = 3$ and a subgroup of order 3 in Y induces a diagonal automorphism on $K_Z/O_2(K_Z) \cong L_3^\epsilon(q)$ for $q \equiv \epsilon$ mod 3.

If T normalizes K_2, then T acts on $\langle L, K_2 \rangle = K$, contradicting $M = !\mathcal{M}(LT)$. This shows:

Lemma 4.3.13. *$K_2 < K_Z$.*

Lemma 4.3.14. *(1) $N_G(R_2) \leq N_H(K_2)$.*
(2) $R_2 = O_2(N_{L_1T}(R_2))$.
(3) $O_2(K_ZT) \leq R_2$ and $K_2 < O_2(K_Z)K_2$.

Proof. Suppose $H_1 \in \mathcal{M}(KS)$. Then as $I \leq KS$ and $K \not\leq M$, $H_1 \in \mathcal{M}(I) - \{M\}$, so the reductions of this section also apply to H_1. In particular by 4.3.3, $H_1 = N_G(K) = H$; that is, $H = !\mathcal{M}(KS)$.

Next K_2 is the maximal parabolic over $S \cap K$ stabilizing the point Z_S of the natural module U. Now (KR_2, R_2) satisfies (MS1) and (MS2) of Definition C.1.31. If (KR_2, R_2) satisfies (MS3), C.1.34 would apply to R_2, whereas here $R_2 = O_2(C_{KS}(Z_S))$ which is explicitly excluded in case (3) of C.1.34, which holds by 4.3.10. Thus (MS3) fails, so there is a nontrivial characteristic subgroup C of R_2 normal in KS, and hence $N_G(R_2) \leq N_G(C) \leq H = !\mathcal{M}(KS)$. Then

$N_G(R_2) = N_H(R_2)$ acts on the parabolic K_2 of K, since we saw after 4.3.6 that $K \trianglelefteq H$, so (1) holds.

Next using A.4.2.4, R_2 is Sylow in $Syl_2(C_H(K_2/O_2(K_2))$, Now $K_2 \trianglelefteq G_Z \cap H$, so by C.1.2.4, $R_2 \in \mathcal{B}_2(N_{K_Z T \cap H}(R_2))$. Therefore (2) follows from C.1.2.3. By (2) and C.2.1, $O_2(K_Z T) \le R_2$, so by (1) $K_2 = O^2(K_2 O_2(K_Z T))$. Then 4.3.13 completes the proof of (3). □

Set $G_0 := L_1 R_2 Y$ and $G_0^* := G_0/C_{G_0}(L_1/O_2(L_1))$. By 4.3.14.3, $O_2(K_Z R_2) \le R_2$. As Y acts on R_2, $O_2(K_Z R_2) \in Syl_2(C_{G_0}(K_Z/O_2(K_Z)))$, so $N_{G_0}(R_2)^* = N_{G_0^*}(R_2^*)$ by a Frattini Argument. Thus $K_2^* \trianglelefteq N_{G_0^*}(R_2^*)$; so in view of 4.3.13 and 4.3.14:

Lemma 4.3.15. $R_2^* \neq 1$.

Now $K_Z^*/Z(K_Z^*)$ is a group appearing in Theorem C (A.2.3), satisfying the restrictions on prime divisors of $2^n - 1$ in 4.3.12.2.

Inspecting the automorphism groups of those groups for a proper 2-local subgroup $N_{K_Z^*}(R_2^*)$ with a normal subgroup K_2^* such that $K_2^*/O_2(K_2^*) \cong L_2(2^n)$, we conclude:

Lemma 4.3.16. *One of the following holds:*

(1) $K_Z/O_2(K_Z) \cong L_2(2^{2^i n})$ *for some* $i \ge 1$.

(2) $K_Z/O_2(K_Z) \cong (S)U_3(2^n)$.

(3) $n = 2$ *and* $K_Z/O_2(K_Z) \cong L_3(5)$ *or* J_1.

(4) $n = 2$, $K_Z/O_2(K_Z) \cong L_3(4)$ *or* $U_3(5)$, *and* Y *induces outer automorphisms on* $K_Z/O_2(K_Z)$.

We are now in a position to complete the proof of Theorem 4.3.2.

Assume that one of cases (1)–(3) of 4.3.16 holds and let p be a prime divisor of $2^n - 1$. As Y^* centralizes $K_2^*/O_2(K_2^*)$ and hence K_2^*, but the groups in those cases do not admit an automorphism of order p centralizing K_2^*, we conclude that $Y^* = 1$. By 4.3.11, $O_2(KS) = [O_2(KS), Y]$, so as $R_2/O_2(KS) = [R_2/O_2(KS), Y]$, also $R_2 = [R_2, Y]$. Then since $Y^* = 1$, $R_2^* = 1$, contrary to 4.3.15.

Thus case (4) of 4.3.16 holds. Choose X of order 5 in K_2. Recall that K has three noncentral 2-chief factors, U and two copies of the dual of U on Q/U. Thus K_2 has four noncentral 2-chief factors, and each is a natural module for K_2R_2/R_2. Therefore X has four nontrivial chief factors on R_2. As $G_Z \in \mathcal{H}(T)$ and $K_Z \trianglelefteq G_Z$, $F^*(K_Z) = O_2(K_Z)$, so at least one of those chief factors is in $O_2(K_Z)$.

Suppose that $K_Z/O_2(K_Z) \cong U_3(5)$. Then $X = Z(P)$ for some $P \in Syl_5(K_Z)$, and $P \cong 5^{1+2}$. Thus from the representation theory of extraspecial groups, X has five nontrivial chief factors on any faithful P-chief factor in $O_2(K_Z)$. But $O_2(K_Z) \le R_2$ by 4.3.14.3, and we saw that X has just four nontrivial chief factors on R_2, with at least one in $O_2(K_Z)$.

Therefore $K_Z/O_2(K_Z) \cong L_3(4)$. Therefore $K_Z/O_2(K_Z) \cong L_3(4)$. Let X be a subgroup of order 3 in $O_{2,Z}(K)$. Then X is faithful on Z_S, so $X \le G_Z$ but $X \not\le K_Z$, and hence $XK_Z/O_2(K_Z) \cong PGL_3(4)$ by A.3.18. Further X centralizes $K_2/O_2(K_2)$, and from the structure of $[O_2(K), K]$ in C.1.34.3, there are four nontrivial K_2-chief factors in $O_2(K)$, all natural modules for $K_2/O_2(K_2) \cong L_2(4)$, and $C_{R_2}(X)/C_{R_2}(K_2X)$ is a natural module for $K_2/O_2(K_2)$. It follows from B.4.14 that each nontrivial K_ZX-chief factor W in $O_2(K_Z)$ is the adjoint module for $K_Z/O_2(K_Z)$, and $C_W(X)/C_W(XK_2)$ is an indecomposable of $\mathbf{F}_4$-dimension 4 for

$K_2/O_2(K_2)$, contrary to $C_{R_2}(X)/C_{R_2}(K_2X)$ the natural module for $K_2/O_2(K_2)$. This contradiction completes the proof of Theorem 4.3.2.

THEOREM 4.3.17. *If $S \leq T$ with $S \cap L \in Syl_2(L)$, then $N_G(S) \leq M$.*

PROOF. By Theorem 4.3.2, $M = !\mathcal{M}(L)$, so the assertion follows from 4.3.1. □

4.4. Controlling suitable odd locals

In this section, we apply Theorem 4.2.13 to force the normalizers of suitable subgroups of odd order to lie in M. The main results are Theorem 4.4.3 and its corollary Theorem 4.4.14.

During most of this section, we assume:

HYPOTHESIS 4.4.1. *(1) Hypothesis 4.2.1 holds. Set $M_+ := \langle L^T \rangle$ and $R_+ := O_2(M_+T) = C_T(M_+/O_2(M_+))$.*

(2) $1 \neq B \leq C_M(M_+/O_2(M_+))$, with B abelian of odd order and $BT_+ = T_+B$ for some $T_+ \leq T$ with $L^T = L^{T_+}$.

(3) $1 \neq V_B = [V_B, M_+] \leq C_M(B)$ with V_B an M_+T-submodule of $\Omega_1(Z(R_+))$.

REMARK 4.4.2. Observe that if $L \trianglelefteq M$, then it is unnecessary to assume the existence of T_+. For example, we could then take $T_+ = 1$. Thus if Hypothesis 4.2.1 holds with $L \trianglelefteq M$ and $V \in \mathcal{R}_2(LT)$ with $[V, L] \neq 1$, then appealing to 1.4.1.4, Hypothesis 4.4.1 is satisfied for each nontrivial abelian subgroup B of $C_M(V)$ of odd order with V in the role of "V_B".

In this section we prove:

THEOREM 4.4.3. *Assume Hypothesis 4.4.1. Then either*

(1) $N_G(B) \leq M$; or

(2) $L \trianglelefteq M$, $L/O_2(L)$ is isomorphic to $L_2(2^n)$, $L_3(2)$, $L_4(2)$, A_6, A_7, $\hat{A}_6$, or $U_3(3)$, and one of the following holds:

(i) V_B is an FF-module for $LT/C_{LT}(V_B)$. Further:

(a) If $L/O_2(L) \cong L_n(2)$, then either V_B is the sum of one or more isomorphic natural modules for $L/O_2(L)$, or V_B is the 6-dimensional orthogonal module for $L/O_2(L) \cong L_4(2)$.

(b) If $L/O_2(L) \cong \hat{A}_6$, then for each $z \in C_{V_B}(T \cap L)^\#$, $V_B \not\leq O_2(C_G(z))$.

(c) If $L/O_2(L) \cong U_3(3)$ and $m(V_B) = 6$, then $C_G(V_3) \not\leq M$, for V_3 the $(T \cap L)$-invariant subspace of V_B of rank 3.

(ii) $L/O_2(L) \cong L_2(2^{2n})$, and V_B is the $\Omega_4^-(2^n)$-module.

(iii) $L/O_2(L) \cong L_3(2)$, and V_B is the core of a 7-dimensional permutation module for $L/O_2(L)$.

Set $G_B := N_G(B)$, $M_B := N_M(B)$, $L_B := C_{M_+}(B)^\infty$, and $T_B := N_{T_+}(B)$. Making a new choice of T_+ if necessary, we may assume $T_B \in Syl_2(M_B)$. As G is simple, $G_B < G$, so G_B is a quasithin $\mathcal{K}$-group.

Before working with a counterexample to Theorem 4.4.3, we first prove two preliminary lemmas which assume only parts (1) and (2) of Hypothesis 4.4.1.

LEMMA 4.4.4. *Assume parts (1) and (2) of Hypothesis 4.4.1. Then $T_+ = [O_2(T_+B), B]T_B$.*

PROOF. Let $X := T_+B$, $Q := O_2(X)$ and $X^* := X/Q$. Then $F(X^*)$ is of odd order, so as B^* is an abelian Hall $2'$-subgroup of X, $B^* \leq C_{X^*}(F(X^*)) \leq F(X^*)$, so $B^* = F(X^*)$. Thus $BQ \trianglelefteq X$, so by a Frattini Argument (using the transitivity of a solvable group on its Hall subgroups in P. Hall's Theorem, 18.5 in [**Asc86a**]), $X = QN_X(B) = QT_BB$, so that $T_+ = QT_B$. Also $Q = C_Q(B)[Q,B]$ by Coprime Action, with $C_Q(B) \leq T_B$, so $T_+ = [Q,B]T_B$. □

LEMMA 4.4.5. *Assume parts (1) and (2) of Hypothesis 4.4.1. Then* $M_+ = L_BO_2(M_+)$.

PROOF. By 4.4.1.2, $[M_+, B] \leq O_2(M_+)$, so M_+ acts on $BO_2(M_+)$; hence by a Frattini Argument, $M_+ = O_2(M_+)C_{M_+}(B)$. Now M_+ is perfect by Hypothesis 4.2.1 in 4.4.1.1, so $M_+ = O_2(M_+)C_{M_+}(B)^\infty = O_2(M_+)L_B$. □

In the remainder of this section, we assume we are in a counterexample to Theorem 4.4.3; in particular, $G_B \not\leq M$.

LEMMA 4.4.6. *(1)* $M = !\mathcal{M}(L_BT_B)$.
(2) If $L \trianglelefteq M$ *then* $M = !\mathcal{M}(L_B)$.
(3) $N_G(V_B) \leq M$.

PROOF. Set $I := L_BT_B$ and $V_L := [R_2(LT), L]$. Observe that (cf. Notation 4.2.11) $I \in \mathcal{I}$: By 4.4.5, $L \leq IR_+$; $L^T = L^{T_+} = L^{T_B}$ by 4.4.1.2 and 4.4.4 (since $[O_2(T_+B), B] \leq R_+$); and $1 \neq V_B \leq O_2(I)$ by 4.4.1.3. Thus if (1) fails, then by Theorem 4.2.13, $L \trianglelefteq M$, and $L_B/O_2(L_B) \cong L/O_2(L)$ appears on the list of Theorem 4.2.13. Further 4.2.13 says that V_L is an FF-module for $Aut_{LT}(V_L)$, so the LT-submodule V_B is an FF-module for $Aut_{LT}(V_B)$ by B.1.5. Suppose $L/O_2(L) \cong L_n(2)$ for $n = 3$ or 4. Then case (2) of 4.2.13 holds, so either V_L is the sum of one or more isomorphic natural modules, or V_L is the 6-dimensional orthogonal module for $L_4(2)$. Therefore the submodule V_B satisfies the same constraints, so conclusion (i.a) of case (2) of Theorem 4.4.3 holds. Similarly if conclusion (4) or (5) of 4.2.13 holds, then $V_B = V_L$ and conclusion (i.b) or (i.c) of part (2) of Theorem 4.4.3 holds. In the remaining cases in Theorem 4.2.13, subcase (i) of case (2) of Theorem 4.4.3 imposes no further restriction on V_B; hence subcase (i) of case (2) in 4.4.3 holds. This contradicts our assumption that we are in a counterexample to Theorem 4.4.3, so we conclude that (1) holds. Under the hypothesis of (2), $L^T = L$, so by Remark 4.4.2, we may take $T_+ = 1$ and $I := L_B$; thus (2) follows from (1). Finally (1) implies (3), completing the proof of 4.4.6. □

LEMMA 4.4.7. *(1)* $O_2(G_B) = 1$.
(2) M_B *is a maximal* 2*-local subgroup of* G_B.

PROOF. By 4.4.6.1, $M = !\mathcal{M}(M_B)$. Hence (2) holds, and as $G_B \not\leq M$, (2) implies (1). □

LEMMA 4.4.8. $O(G_B) \leq M_B$.

PROOF. By Hypothesis 4.4.1 and 4.4.5, $1 \neq V_B = [V_B, L_B]$. As L_B is perfect, $m(V_B) \geq 3$, and in case of equality, L_B acts irreducibly as $L_3(2)$ on V_B, so $V_B \cap Z^*(G_B) = 1$. Therefore applying A.1.28 with G_B in the role of "H", we conclude that $m_p(O_p(G_B)) \leq 2$ for each odd prime p. Thus by A.1.26, $V_B = [V_B, L_B] \leq C_G(O_p(G_B))$. Hence $V_B \leq C_{V_BO(G_B)}(F(V_BO(G_B))) \leq F(V_BO(G_B))$, so $V_B = O_2(V_BO(G_B))$ and thus $O(G_B) \leq N_G(V_B) \leq M$ by 4.4.6.3. □

LEMMA 4.4.9. *If K is a component of G_B, then $|K^{G_B}| \leq 2$, and in case of equality, $K \cong L_2(2^n)$, $Sz(2^n)$, $L_2(p^e)$, for some prime $p > 3$ and $e \leq 2$, J_1, or $SU_3(8)$.*

PROOF. Since we saw that G_B is a QTK-group, this follows from (1) and (2) of A.3.8; notice we use 4.4.7.1 to guarantee $O_2(K) = 1$, and I.1.3 to see that the Schur multiplier of $SU_3(8)$ is trivial, and in the remaining cases the multiplier of $K/Z(K)$ is a 2-group, so that K is simple. □

By 4.4.8, V_B centralizes $O(G_B)$, and by 4.4.7.1, $O_2(G_B) = 1$, so V_B is faithful on $E(G_B)$. Thus there is a component K of G_B with $[K, V_B] \neq 1$. Set $K_0 := \langle K^{M_B} \rangle$ and $M_K := M \cap K$. Recall that G_B is a quasithin $\mathcal{K}$-group, and hence so is K by (a) or (b) of (1) in Theorem A (A.2.1), so that $K/Z(K)$ is described in Theorem B (A.2.2).

LEMMA 4.4.10. *(1) $K \not\leq M_B$.*
(2) $V_B \leq K_0$.
(3) $C_{G_B}(K_0) = O(G_B)$.

PROOF. As $[K, V_B] \neq 1$ and $V_B \leq O_2(M_B)$, (1) holds. As $L_B = O^2(L_B)$, L_B acts on K by 4.4.9, so $1 \neq V_B = [V_B, L_B]$ acts on K. Indeed as $Out(K)$ is 2-nilpotent for each K in Theorem B, V_B induces inner automorphisms on K_0, so that $V_B \leq K_0H$ where $H := C_{G_B}(K_0)$. Then the projection of V_B on H is an M_B-invariant 2-group Q. If $Q \neq 1$, then by 4.4.7.2, $M_B = N_{G_B}(Q)$; but then $K \leq C_{G_B}(Q) \leq M_B$ contrary to (1). Thus $Q = 1$, giving (2). Now $H \leq C_{G_B}(V_B) \leq M_B$ by 4.4.6.3. Set $S := T_B \cap H$. As t_b IS Sylow in M_B, and $H \trianglelefteq M_B$, S is Sylow in H, $S \trianglelefteq T_B$, and

$$[S, L_B] \leq C_{L_B}(V_B) \cap H \leq O_2(L_B) \cap H \leq O_2(H) \leq O_2(G_B) = 1,$$

in view of 4.4.7.1. Thus $L_BT_B \leq N_G(S)$, so if $S \neq 1$ then $N_G(S) \leq M$ by 4.4.6.1; as S centralizes K, this contradicts (1). Thus the Sylow 2-group S of H is trivial, so (3) holds. □

LEMMA 4.4.11. *(1) $K = K_0 \trianglelefteq G_B$.*
(2) $L_B \leq M_K$.

PROOF. Observe $Out(K_0)$ is solvable, since $|K^{G_B}| \leq 2$ by 4.4.9 and the Schreier property is satisfied for the groups in Theorem B. Also $C_{G_B}(K_0)$ is solvable by 4.4.10.3. Hence $L_B = L_B^{\infty} \leq K_0$. Thus (2) will follow from (1).

Assume K is not normal in G_B. By 4.4.9, $K_0 = K_1K_2$ where $K_1 := K$ and $K_2 := K^s$ for $s \in G_B - N_{G_B}(K)$, and K is a simple Bender group, $L_2(p^e)$, J_1, or $SU_3(8)$. But then K has no nonsolvable 2-local M_K with $O_2(M_K)$ not in the center of M_K, contradicting $L_B \leq M \cap K_0$. This establishes (1). □

LEMMA 4.4.12. *$K/Z(K)$ is not of Lie type and characteristic 2.*

PROOF. Assume otherwise. By 4.4.11.1 and 4.4.10.3, $O(G_B) = C_G(K)$, so T_B is faithful on K. By 4.4.10.2, $V_B \leq K$, so $Q_B := O_2(M_B) \cap K \not\leq Z(K)$. Therefore as $K/Z(K)$ is of Lie type and characteristic 2 by hypothesis, M_B acts on some proper parabolic of K (e.g. using the Borel-Tits Theorem 3.1.3 in [**GLS98**]). Hence by 4.4.7.2, M_K is a maximal M_B-invariant parabolic of K. Furthermore from Theorem B, $K/Z(K)$ either has Lie rank at most 2, or is $L_4(2)$ or $L_5(2)$ or $Sp_6(2)$, so as we chose $T_B \in Syl_2(M_B)$, T_B is transitive on each orbit of M_B on parabolics of K containing $T_B \cap K$, and hence M_K is a maximal T_B-invariant parabolic.

As L_B is a nonsolvable subgroup of M_K, K is of Lie rank at least 2, and M_K is of Lie rank at least 1. Assume that K is of Lie rank exactly 2. Then as M_K is a proper parabolic of rank at least 1, it must be of rank exactly 1, and hence is a maximal parabolic. Also $L_B = M_K^\infty$ as $M_K^\infty/O_2(M_K)^\infty$ is quasisimple. Then as $V_B \leq Z(O_2(L_B))$ and $V_B = [V_B, L_B]$ we conclude by inspection of the parabolics of the rank 2 groups that $M_+/O_2(M_+) \cong L_B/O_2(L_B) \cong L_2(2^n)$, and either V_B is an FF-module, or (when K is unitary) V_B is the $\Omega_4^-(2^{n/2})$-module for $L_B/O_2(L_B)$. These are cases (i) and (ii) of conclusion (2) in Theorem 4.4.3, and in case (i) there are no further restrictions on V_B since $L/O_2(L) \cong L_2(2^n)$. This contradicts the choice of B as a counterexample to Theorem 4.4.3.

Therefore K is of Lie rank at least 3, so as we saw from Theorem B, $K \cong L_4(2)$, $L_5(2)$, or $Sp_6(2)$. Thus $M_+/O_2(M_+) \cong L_B/O_2(L_B) \cong L_3(2)$, $L_4(2)$, or A_6, and either V_B is an FF-module, which is a natural module in the first two cases, or $K \cong Sp_6(2)$, $L_B/O_2(L_B) \cong L_3(2)$, and $V_B = O_2(L_B)$ is the core of a 7-dimensional permutation module for $L_B/O_2(L_B)$. But then case (i) or (iii) of Theorem 4.4.3.2 holds, contrary to the choice of B as a counterexample, and completing the proof of 4.4.12. □

We are now in a position to complete the proof of Theorem 4.4.3.

By 4.4.12, $K/Z(K)$ is not of Lie type and characteristic 2. By 4.4.10.2, $V_B \leq K$.

Assume first that $m(V_B) \leq 4$. Then inspecting the list of quasisimple subgroups of $GL_4(2)$, $L_B/O_2(L_B)$ is one of $L_2(4)$, $L_3(2)$, $L_4(2)$, A_6, or A_7, with V_B an FF-module, or an A_5-module for $L_2(4)$. Further if $L_B/O_2(L_B) \cong L_3(2)$ or $L_4(2)$, then either V_B is a natural module for $L_B/O_2(L_B)$, so condition in (a) of subcase (i) of case (2) of Theorem 4.4.3 is satisfied, or $m(V_B) = 4$ and $L_B/O_2(L_B) \cong L_3(2)$. The former case contradicts our assumption that B is a counterexample, so we may assume the latter holds. Then as $V_B = [V_B, L_B]$, $Z_B := C_{V_B}(L_B)$ is of rank 1 and V_B/Z_B is a natural module. By 4.4.6.1, $M_K T_B = C_{KT_B}(Z_B)$, so $L_B \trianglelefteq C_K(Z_B)$. Examining involution centralizers in the groups appearing in Theorem B for such a normal subgroup, we conclude $K \cong M_{23}$; but there L_B is not normal in $N_K(V_B) \cong A_7/E_{16}$.

Thus we may assume that $m(V_B) > 4$, and hence $m_2(K) > 4$. Then from the list of Theorem B, $K/Z(K)$ is not $L_2(p^e)$, $L_3^\epsilon(p)$, $PSp_4(p)$, $L_4^\epsilon(p)$, $G_2(p)$, A_7, A_9, a Mathieu group other than M_{24}, a Janko group other than J_4, HS, or Mc.

Since $K/Z(K)$ is not of Lie type and characteristic 2 by 4.4.12, we conclude from Theorem B that $K/Z(K)$ is M_{24}, J_4, He, and Ru. Since the multipliers of these groups are 2-groups by I.1.3, while $O_2(K) = 1$ by 4.4.7.1, it follows that K is simple. Again by 4.4.6.1, $M_K T_B$ is the unique maximal 2-local subgroup of KT_B containing $L_B T_B$. Inspecting the maximal 2-locals of $Aut(K)$ for a nonsolvable 2-local $M_K T_B$ such that $L_B \trianglelefteq M_K T_B$ and $1 \neq V_B = [V_B, L_B] \leq Z(O_2(L_B))$, we conclude one of the following holds:

(a) $K \cong J_4$ and L_B is a block of type M_{24} or $L_5(2)$.
(b) K is M_{24} or He, and L_B is a block of type $\hat{A}_6$.
(c) K is Ru and L_B is a block of type $G_2(2)$.
(d) $K \cong Ru$ and $L_B/O_2(L_B) \cong L_3(2)$.
(e) $K \cong M_{24}$, and $L_B/O_2(L_B) \cong L_4(2)$ or $L_3(2)$.
(f) $K \cong J_4$ and $L_B/O_2(L_B) \cong L_3(2)$.

In cases (d)–(f), V_B is a natural module for $L_B/O_2(L_B)$, so that subcase (i) of case (2) of Theorem 4.4.3 holds, contrary to our assumption that B affords a counterexample to Theorem 4.4.3. Hence it only remains to dispose of cases (a)–(c).

Assume first that case (b) holds. Then from the structure of $K \cong M_{24}$ or He, $V_B \not\leq O_2(C_K(z))$ for each $z \in C_{V_B}(T \cap L)^{\#}$. Hence $V_B \not\leq O_2(C_G(z))$, so condition in (b) of subcase (i) of case (2) in Theorem 4.4.3 holds, again contrary to our choice of a counterexample. Similarly if case (c) holds then from the structure of Ru (cf. the case corresponding to Ru in the proof of Theorem 4.2.13, using facts from chapter J) of Volume I, $C_K(V_3) \not\leq M_K$. Thus condition (c) of subcase (i) of case (2) in Theorem 4.4.3.2 holds, for the same contradiction.

Therefore we may assume case (a) holds. Set $Z_B := C_{V_B}(T_B)$ and $G_Z := C_G(Z_B)$. Observe that Z_B is of order 2 and $K_Z := C_K(Z_B)^\infty \cong \hat{M}_{22}/2^{1+12}$. Arguing as in the last paragraph of the proof of Theorem 4.2.13, T induces inner automorphisms on $L/O_2(L)$, and hence $LT = LR_+$; therefore as $V_B \leq Z(R_+)$, $Z_B \leq Z(T)$, so $T \leq G_Z$. By 1.2.1.1, K_Z is contained in the product of the members of $\mathcal{C}(G_Z)$ on which it has nontrivial projection. Since $m_3(K_Z) = 2$ and G_Z is an SQTK-group, there is just one such member, so that $K_Z \leq L_Z \in \mathcal{C}(G_Z)$, and from 1.2.1.4, $L_Z/O_2(L_Z)$ is a quasisimple group described in Theorem C. Set

$$(L_Z B)^* := L_Z B/C_{L_Z B}(L_Z/O_2(L_Z)).$$

Then $K_Z^* \in \mathcal{C}(C_{L_Z^*}(B^*))$ with $K_Z^*/O_2(K_Z^*) \cong \hat{M}_{22}$ or M_{22}. Inspecting the p-locals (for odd primes p) of the groups in Theorem C, we conclude that either $K_Z^* = L_Z^*$ or $L_Z^* \cong J_4$ and $B^* = Z(K_Z^*)$ is of order 3. In the latter case, $K_Z \leq I_Z \leq L_Z$ with $I_Z \in \mathcal{L}(G,T)$ and $I_Z^* \cong \hat{M}_{22}/2^{1+12}$. Thus replacing L_Z by I_Z in this case, and replacing the condition that $L_Z \in \mathcal{C}(G_Z)$ by $L_Z \in \mathcal{L}(G,T)$, we may assume $L_Z = K_Z O_2(L_Z)$.

Thus in either case, $L_Z \in \mathcal{L}(G,T)$ with $L_Z = K_Z O_2(L_Z)$ and $[L_Z, B] \leq O_2(L_Z)$. Let $X := \langle B^T \rangle$; then $X = O^2(X) = O^2(XT)$. As $[L,B] \leq O_2(L)$, $[L,X] \leq O_2(L) \leq T \leq N_G(X)$, so that $X = O^2(XO_2(L)) \trianglelefteq LTX$, and hence $N_G(X) \leq M = !\mathcal{M}(LT)$. Similarly as $[L_Z, B] \leq O_2(L_Z)$, $L_Z \leq N_G(X)$, and hence $K_Z \leq L_Z T \leq N_G(X)$. Now $K = \langle L_B, K_Z \rangle \leq N_G(X) \leq M$, contradicting 4.4.10.1.

This final contradiction completes the proof of Theorem 4.4.3.

We interject a lemma which is often used in applying Theorem 4.4.3. Recall the notation $n(H)$ in Definition E.1.6.

LEMMA 4.4.13. *Assume that G is a simple QTKE-group, $H \in \mathcal{H}$ with $n(H) > 1$, $S \in Syl_2(H)$, and S is contained in a unique maximal subgroup M_H of H. Then $M_H \cap O^2(H)$ is 2-closed, and if we let B denote a Hall $2'$-subgroup of M_H, then:*

(1) If A is an elementary abelian p-subgroup of B with $AS = SA$, then $H = \langle M_H, N_H(A) \rangle$. In particular $N_H(A) \not\leq M_H$.

(2) Assume that $M \in \mathcal{M}(S)$, $M_H = M \cap H$, and $M_+ = O^2(M_+) \trianglelefteq M$. Then $C_B(M_+/O_2(M_+))S = SC_B(M_+/O_2(M_+))$.

PROOF. As $n(H) > 0$, S is not normal in H, so as M_H is the unique maximal subgroup of H over S, H is a minimal parabolic in the sense of Definition B.6.1. As $n := n(H) > 1$, E.2.2 then says that $K_0 := O^2(H) = \langle K^S \rangle$ for some $K \in \mathcal{C}(H)$ with $K/O_2(K)$ a Bender group over $\mathbf{F}_{2^n}$, $(S)L_3(2^n)$, or $Sp_4(2^n)$, and in the latter two cases S is nontrivial on the Dynkin diagram of $K/O_2(K)$. Set $H^* := H/O_2(H)$ and $M_0 := M_H \cap K_0$. By E.2.2, M_0 is the Borel subgroup of K_0 over $S \cap K_0$. In particular, M_0 is 2-closed, and a Hall $2'$-subgroup B of M_0 is abelian of p-rank at most 2 for each odd prime p.

In proving (1), we may take $A \neq 1$. Then $1 \leq m_p(A) \leq m_p(B) \leq 2$ for each $p \in \pi(A)$. It will suffice to show $N_{H^*}(A^*) \not\leq M_H^*$, since then as M_H is a maximal subgroup of H, $H = \langle M_H, N_H(A) \rangle$, so that (1) holds.

Suppose first that $m_p(A) = m_p(B)$ for some p. Then $A = \Omega_1(O_p(B))$ and so $N_H(B) \leq N_H(A)$. But as B^* is a Cartan subgroup of K_0^*, $N_{K_0^*}(B^*) \not\leq M_0^*$, and this suffices as we just observed.

So assume $m_p(B) = 2$ and $m_p(A) = 1$. Then by E.2.2, one of the following holds:

(i) $K < K_0$ and $K^* \cong L_2(2^n)$ or $Sz(2^n)$.

(ii) $K^* \cong Sp_4(2^n)$.

(iii) $K^* \cong (S)L_3(2^n)$.

In cases (i) and (ii), there is an element in $K_0^* - M_0^*$ inverting B^*, so $N_{K_0^*}(A^*) \not\leq M_0^*$, which suffices to establish (1) in this case as we indicated. Thus we may assume case (iii) holds, so some $t \in S$ acts nontrivially on the Dynkin diagram of K^*, and by a Frattini Argument we may take $t \in N_S(B)$. Then as $AS = SA$, A is t-invariant. Let $U^* := N_{H^*}(B^*)$, $\tilde{U} := U^*/B^*$, and $\tilde{W}$ the image of $N_{K^*}(B^*)$ in $\tilde{U}$. Then $\tilde{W} \cong S_3$ is the Weyl group of K^* and $\tilde{t} = \tilde{s}\tilde{w}$, where $\tilde{w}$ is an involution in $\tilde{W}$, and $\tilde{s} \in C_{\tilde{U}}(\tilde{W})$. Pick preimages w^* and s^* of $\tilde{w}$ and $\tilde{s}$. As $\tilde{W}$ acts indecomposably on $\Omega_1(O_p(B))$, $\tilde{s}$ inverts or centralizes B^*, so s^* and t^* act on A^*, and hence $w \in N_H(A) - M_H$ completing the proof of (1).

So we may assume the hypotheses of (2). Let $D := C_B(M_+/O_2(M_+))$ and $Q := O_2(BS)$. Then, as in the proof of 4.4.4, a Frattini Argument gives $S = QN_S(B)$. Now as $M_+ \trianglelefteq M$, $N_S(B)$ acts on M_+ and hence also on $D = C_B(M_+/O_2(M_+))$. Therefore $DN_S(B)$ is a subgroup of G acting on Q, and hence $DN_S(B)Q = DS$ is a subgroup of G, completing the proof of (2). □

Usually we use Theorem 4.4.3 via an appeal to the following corollary:

THEOREM 4.4.14. *Assume Hypothesis 4.2.1, and let $M_+ := \langle L^T \rangle$, $V_0 \in \mathcal{R}_2(M_+T)$, and $H \in \mathcal{H}_*(T, M)$. Assume*

(a) $V := [V_0, M_+] \neq 1$, $V_0 = \langle C_{V_0}(T)^{M_+} \rangle$, and V is not an FF-module for $M_+T/C_{M_+T}(V)$.

(b) $n(H) > 1$.

Then one of the following holds:

(1) $O^2(H) \cap M$ is 2-closed, and a Hall $2'$-subgroup of $H \cap M$ is faithful on $M_+/O_2(M_+)$.

(2) $M_+/O_2(M_+) \cong L_2(2^{2n})$, and V is the $\Omega_4^-(2^n)$-module.

(3) $M_+/O_2(M_+) \cong L_3(2)$, and V is the core of a 7-dimensional permutation module for $M_+/O_2(M_+)$.

PROOF. Let $Z := \Omega_1(Z(T))$ and $K := O^2(H)$. We observed in Remark 3.2.4 that Hypothesis 4.2.1 allows us to apply Theorem 3.1.8. As V is not an FF-module, $J(T) \leq C_T(V)$ by B.2.7, so $H \leq C_G(Z)$, by 3.1.8.3. Similarly by 3.3.2.4, H is a minimal parabolic described in E.2.2. Since $n(H) > 1$ by hypothesis, E.2.2 shows that $K/O_2(K)$ is of Lie type in characteristic 2 and of Lie rank at most 2, and $K \cap M$ is a Borel subgroup of K, so in particular $K \cap M$ is 2-closed. Let B_H be a Hall $2'$-subgroup of $H \cap M$; thus B_H is abelian of odd order.

Assume (1) fails. Then $B := C_{B_H}(M_+/O_2(M_+)) \neq 1$. Observe that we have the hypotheses of 4.4.13 with T, B_H, B in the roles of "S, B, A", so $BT = TB$ by 4.4.13.2. Hence parts (1) and (2) of Hypothesis 4.4.1 are satisfied, with T in the role of "T_+". Thus by 4.4.5, $M_+ = L_B O_2(M_+)$, where $L_B := C_{M_+}(B)^\infty$.

Next since $H \leq C_G(Z)$, $C_{V_0}(T) = Z \cap V_0 \leq C_G(B)$, so $V_0 = \langle (Z \cap V_0)^{M_+} \rangle = \langle (Z \cap V_0)^{L_B} \rangle \leq C_G(B)$ by (a). Therefore part (3) of Hypothesis 4.4.1 is also satisfied, with V in the role of "V_B", so that we may apply Theorem 4.4.3. By (a), V is not an FF-module for $L_B/O_2(L_B)$, which rules out subcase (i) of case (2) of Theorem 4.4.3. By 4.4.13.1, $N_H(B) \not\leq M$, ruling out case (1) of Theorem 4.4.3. Thus subcase (ii) or (iii) of case (2) of Theorem 4.4.3 must hold, and these are conclusions (2) and (3) of Theorem 4.4.14. □

Part 2

The treatment of the Generic Case

Part 1 has set the stage for the proof of the Main Theorem by supplying information about the structure of 2-locals, establishing the Fundamental Setup (3.2.1), and proving that in the FSU, the members of $\mathcal{H}_*(T, M)$ are minimal parabolics. We now begin the analysis of the various possibilites for $L \in \mathcal{L}_f^*(G, T)$ and $V \in \mathcal{R}_2(L_0T)$ arising in the FSU. Recall the FSU includes the hypotheses that G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G, T)$ with $L/O_2(L)$ quasisimple and V a suitable member of $\mathcal{R}_2(LT)$.

In Part 2, we consider the Generic Case of our Main Theorem. This is the case where $L/O_2(L) \cong L_2(2^n)$ with $L \trianglelefteq M$ and $n(H) > 1$ for some $H \in \mathcal{H}_*(T, M)$. We show in Theorem 5.2.3 of chapter 5 that in the Generic Case, (modulo the sporadic exception M_{23} and the "$\mathbf{F}_2$-case") G is one of the generic conclusions in our Main Theorem: namely G is of Lie type of Lie rank 2 and characteristic 2. In chapter 6 we consider the remaining case where $n(H) = 1$ for each $H \in \mathcal{H}_*(T, M)$, and show in that case that $n = 2$ and V is the A_5-module. The case where V is the A_5-module is treated in Part 5 on groups over $\mathbf{F}_2$, since the A_5-module is the module for $\Omega_4^-(2)$.

Thus once we have dealt with the groups $L_2(p)$ and the Bender groups in Theorem 2.1.1, and the groups of Lie type in characteristic 2 of Lie rank 2 in Theorem 5.2.3, we will have handled all the infinite families of groups appearing as conclusions in the Main Theorem.

CHAPTER 5

The Generic Case: $L_2(2^n)$ in $\mathcal{L}_f$ and $n(H) > 1$

In this chapter we assume the following hypothesis:

HYPOTHESIS 5.0.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_2(2^n)$ and $L \trianglelefteq M \in \mathcal{M}(T)$.*

As L is nonsolvable, $n \geq 2$. Further $M = !\mathcal{M}(LT)$ by 1.2.7.3 and $M = N_G(L)$. Set

$$Z := \Omega_1(Z(T)).$$

From the results of section 1.2, there exists $V \in \mathcal{R}_2(LT)$ with $[V, L] \neq 1$; choose such a V and set $\overline{LT} := LT/C_{LT}(V)$. By 3.2.3 it is possible to choose V so that the pair L, V satisfies the hypotheses of the Fundamental Setup (3.2.1). However occasionally we need information about other members of $\mathcal{R}_2(LT)$, so usually in this chapter we do not assume V satisfies the hypotheses of the FSU. Later, when appropriate, we sometimes specialize to that case.

By Theorem 2.1.1, $\mathcal{H}_*(T, M)$ is nonempty.

In the initial section 5.1, we determine the possibilities for V and provide restrictions on members of $\mathcal{H}_*(T, M)$. The following section begins the proof of Theorem 5.2.3, which supplies very strong information when $n(H) > 1$ for some $H \in \mathcal{H}_*(T, M)$. Indeed in the FSU, if V is not the A_5-module, then either G is of Lie type and Lie rank 2 over a field of characteristic 2, or G is M_{23}; hence we refer to this situation as the Generic Case . The final section 5.3 completes the proof of Theorem 5.2.3.

Our primary tool for proving Theorem 5.2.3 is the main theorem of the "Green Book" of Delgado-Goldschmidt-Stellmacher [**DGS85**], which gives a local description of weak BN-pairs of rank 2. To apply the Green Book, we must achieve the setup of Hypothesis F.1.1. There are two major obstacles to verifying this hypothesis: Let D be a Hall $2'$-subgroup of $N_L(T \cap L)$, and $K := O^2(H)$. We must first show that D acts on K, unless the exceptional case in part (1) of Theorem 5.2.3 holds. Second, we must construct a normal subgroup S of T such that S is Sylow in SL and SK, and so that there exists an S-invariant subgroup K_1 of K such that $K_1/O_2(K_1)$ a Bender group. Now $K/O_2(K)$ is of Lie type in characteristic 2 of Lie rank 1 or 2. If K is of Lie rank 1, we take $K_1 := K$; if K is of Lie rank 2, we choose K_1 to be a rank one parabolic of K. In either case, we take S to be $O_2(H \cap M)$, unless $K/O_2(K) \cong L_3(4)$, which provides a final obstruction that we deal with in Theorem 5.1.14.

After producing our weak BN-pair and identifying it up to isomorphism of amalgams using the Green Book, we still need to identify G. To do so we appeal to Theorem F.4.31 as a recognition theorem; ultimately Theorem F.4.31 depends upon the Tits-Weiss classification of Moufang generalized polygons, although the Fong-Seitz classification of split BN-pairs of rank 2 would also suffice. There is also an obstacle to applying this recognition theorem: the case where $K \notin \mathcal{L}^*(G,T)$, leading to M_{23}. This case is dealt with in Theorem 5.2.10.

5.1. Preliminary analysis of the $L_2(2^n)$ case

5.1.1. General analysis of V and H. Since this is the first case in the FSU which we analyze, we begin with a lemma summarizing some of the basic tools (developed in Volume I and earlier chapters of Volume II) to deal with the FSU. We thank Ulrich Meierfrankenfeld for several improvements to the proofs in this section.

LEMMA 5.1.1. *(1) $C_T(V) = O_2(LT)$.*

(2) Each $H \in \mathcal{H}_*(T,M)$ is a minimal parabolic described in B.6.8, and in E.2.2 if $n(H) > 1$.

(3) For each $H \in \mathcal{H}_*(T,M)$, case (I) of Hypothesis 3.1.5 is satisfied with LT in the role of "M_0".

(4) LT is a minimal parabolic.

PROOF. Part (1) follows from 1.4.1.4, (2) follows from 3.3.2.4, (3) follows from (1) and the fact that $L \trianglelefteq M$, and (4) is well known and easy. □

We begin by discussing the possibilities for V:

LEMMA 5.1.2. *One of the following holds:*

(1) $J(T) \leq C_M(V)$, so $J(T)$ and $Baum(T)$ are normal in LT and $M = !\mathcal{M}(N_G(J(T))) = !\mathcal{M}\big(N_G(Baum(T))\big)$.

(2) $[V, J(T)] \neq 1$ and $V/C_V(L)$ is the natural module for $\bar{L}$.

(3) $[V, J(T)] \neq 1$, $n = 2$, and $V = C_V(LT) \oplus [V,L]$ with $[V,L]$ the S_5-module for $\bar{L}\bar{T} \cong S_5$.

PROOF. By 5.1.1.1, $C_T(V) = O_2(LT)$. Thus if $J(T) \leq C_M(V)$, then $J(T) = J(O_2(LT))$ and $\mathrm{Baum}(T)) = \mathrm{Baum}(O_2(LT))$ by B.2.3, so LT acts on $J(T)$ and $\mathrm{Baum}(T)$. However by 1.2.7.3, $M = !\mathcal{M}(LT)$, so (1) holds in this case. So assume $[V, J(T)] \neq 1$. Then V is an FF-module for $\bar{L}\bar{T}$ by B.2.7, so by B.5.1.1, $I := [V,L] \in Irr_+(L,V)$, and by B.5.1.5, $V = I + C_V(L)$. By B.4.2, either $I/C_I(L)$ is the natural module, or $n = 2$ and $I/C_I(L)$ is the A_5-module. In the former case (2) holds as $V = I + C_V(L)$, and in the latter (3) holds by B.5.1.4. □

LEMMA 5.1.3. *One of the following holds:*

(1) V is the direct sum of two natural modules for $\bar{L}$.

(2) $n = 2$ and V is the direct sum of two S_5-modules for $\bar{L}\bar{T} \cong S_5$.

(3) $[V,L]/C_{[V,L]}(L)$ is the natural module for $\bar{L}$.

(4) n is even and V is the $O_4^-(2^{n/2})$-module for $\bar{L}$.

(5) $V = [V,L] \oplus C_V(LT)$, and $[V,L]$ is the S_5-module for $\bar{L}\bar{T} \cong S_5$.

REMARK 5.1.4. Recall that the A_5-module and the $O_4^-(2)$-module are the same. Notice however that in case (4) we may have $\bar{L}\bar{T} \cong A_5$, which is not allowed in (5). On the other hand in case (5) we may have $C_V(L) \neq 1$, which is not allowed in (4).

PROOF. If $[V, J(T)] \neq 1$ then (3) or (5) holds by 5.1.2. Thus we may assume $[V, J(T)] = 1$, so that $C_V(L) = 1$ by 3.1.8.3.

Next $\hat{q}(\bar{L}\bar{T}, V) \leq 2$ by 3.1.8.1. Hence in the language of Definition D.2.1, there is $\bar{A} \in \hat{\mathcal{Q}}(\bar{T}, V)$. Recall that we are not yet assuming the FSU, so we will work with the results of section D.3 rather than those of section 3.2 based on the FSU. By A.1.42.2, there is $I \in Irr_+(L, V, T)$. Now Hypothesis D.3.1 is satisfied with $\bar{L}\bar{T}$, $\bar{L}$, I, $V_M := \langle I^T \rangle$ in the roles of "M, M_+, V, V_M". Hence we may apply D.3.10 to conclude that $I \trianglelefteq LT$.

Suppose first that $I < [V, L]$, and choose an LT-submodule V_1 of V with $[V, L] \not\leq V_1 \geq I$. As $\bar{L} = F^*(\bar{L}\bar{T})$ is simple, $\bar{L}$—and hence also $\bar{A}$—is faithful on V_1 and on $\tilde{V} := V/V_1$. Thus

$$2 \geq r_{\bar{A},V} \geq r_{\bar{A},V_1} + r_{\bar{A},\tilde{V}}$$

in the language of Definition B.1.1. On the other hand, by B.6.9.1, $r_{\bar{A},W} \geq 1$ for each faithful $\bar{L}\bar{A}$-module W, so $r_{\bar{A},V_1} = r_{\bar{A},\tilde{V}} = 1$. Then by another application of B.6.9, V_1 and $\tilde{V}$ have unique noncentral chief factors, and either both factors are natural, or $n = 2$ and at least one is an A_5-module. Now if a factor is natural, then $\bar{A} \in Syl_2(\bar{L})$, while if a factor is an A_5-module, then $\bar{A} \not\leq \bar{L}$. So if one factor is an A_5-module, then both are A_5-modules; then as A_5-modules have trivial 1-cohomology by I.1.6, and we saw $C_V(L) = 1$, (2) holds. This leaves the case where both factors are natural modules. Here we choose V_1 maximal subject to $[\tilde{V}, L] \neq 1$, so as $\tilde{V}$ is an FF-module, $\tilde{V}$ is natural by B.5.1.5. Also V_1 is an FF-module, so $V_1/C_{V_1}(L)$ is natural by B.5.1.5; hence as $C_V(L) = 1$, both $V_1 = I$ and V/I are natural. Further as $r_{\bar{A},V} = 2$ with $m(V/C_V(\bar{i})) = 2n = 2m(\bar{L})$ for each involution $\bar{i} \in \bar{L}$, $\bar{A} \in Syl_2(\bar{L})$ with $C_V(\bar{A})) = C_V(\bar{a}) = [V, \bar{a}]$ for each $\bar{a} \in \bar{A}^{\#}$. Therefore V is semisimple by Theorem G.1.3, and hence (1) holds.

Thus we may assume that $I = [V, L]$, and therefore that LT is irreducible on $W := [V, L]/C_{[V,L]}(L)$. Then as $\hat{q}(\bar{L}\bar{T}, V) \leq 2$, it follows from B.4.2 and B.4.5 that either W is the natural module, or n is even and W is the orthogonal module. In the first case (3) holds, so assume the second holds. Then $H^1(\bar{L}, W) = 0$ by I.1.6, so as $C_V(L) = 1$, V is irreducible and hence (4) holds. This completes the proof. $\square$

Recall that by Theorem 2.1.1, there is $H \in \mathcal{H}_*(T, M)$.

LEMMA 5.1.5. *Let $H \in \mathcal{H}_*(T, M)$ and D_L a Hall $2'$-subgroup of $N_L(T \cap L)$. Then*

(1) $H \cap M$ acts on $T \cap L$ and on $O^2(D_L T)$, and

(2) if $n(H) > 1$, then $H \cap M$ is solvable, and some Hall $2'$-subgroup of $H \cap M$ acts on D_L.

PROOF. Let $T_L := T \cap L$ and $B := N_L(T_L)$. Since $L/O_2(L) \cong L_2(2^n)$, B is the unique maximal subgroup of L containing T_L. But as $M = !\mathcal{M}(LT)$ and $H \not\leq M$, $L \not\leq H$, so $H \cap L \leq B$; hence $H \cap M$ acts on $O_2(H \cap L) = T_L$ and on $N_L(T_L) = B$. Thus (1) holds.

Assume $n(H) > 1$. Then $H \cap M$ is solvable by E.2.2, so as $H \cap M$ acts on B and B is solvable, $(H \cap M)B$ is solvable. Therefore by Hall's Theorem, a Hall $2'$-subgroup D_H of $H \cap M$ is contained in a Hall $2'$-subgroup D of $(H \cap M)B$, and $D \cap B$ is a Hall $2'$-subgroup of B. By Hall's Theorem there is $t \in T_L$ with $(D \cap B)^t = D_L$, so as $T_L \leq H$, the Hall $2'$-subgroup D_H^t of $H \cap M$ acts on D_L, completing the proof of (2). $\square$

LEMMA 5.1.6. *Let $H \in \mathcal{H}_*(T, M)$, D_L a Hall $2'$-subgroup of $N_L(T \cap L)$, and assume $O_2(\langle D_L, H \rangle) = 1$. Then n is even and one of the following holds:*

(1) $n = 2$, V is the direct sum of two natural modules for $\bar{L}$, and $[Z, H] = 1$.

(2) $n = 2$ *or* 4, $[V, L]$ *is the natural module for* $\bar{L}$, *and* $[Z, H] = 1$.

(3) $n = 2$, $[V, L]$ *is the* S_5*-module for* $\bar{L}\bar{T} \cong S_5$, *and* $Z(H) = 1$.

(4) $n \equiv 0 \mod 4$, V *is the* $\Omega_4^-(2^{n/2})$*-module for* $\bar{L}$, *and* $[Z, H] = 1$. *Furthermore if we take* D_ϵ *to be the subgroup of* D_L *of order* $2^{n/2} - \epsilon$, $\epsilon = \pm 1$, *and* $X_\epsilon := \langle D_\epsilon, H\rangle$, *then* $Z \leq Z(X_-)$ *and either* $O_2(X_+) \neq 1$, *or* $n = 4$ *or* 8.

PROOF. Let $X := \langle D_L, H\rangle$. Then by hypothesis, $O_2(X) = 1$. Recall from the start of the chapter that $Z = \Omega_1(Z(T))$, and set $V_D := \langle Z^{D_L}\rangle$ and $V_Z := \langle Z^L\rangle$. Observe that $V_Z \in \mathcal{R}_2(LT)$ and $V_D \in \mathcal{R}_2(TD_L)$ by B.2.14. In each case of 5.1.3,

$$V = \langle (Z \cap V)^L\rangle \leq V_Z.$$

Suppose first that $T \trianglelefteq TD_L$. Then applying Theorem 3.1.1 with TD_L, T in the roles of "M_0, R", we contradict $O_2(X) = 1$. Therefore $T \not\trianglelefteq TD_L$.

Since $\bar{L} \cong L_2(2^n)$, it follows that n is even, and also that $\bar{L}\bar{T} = \bar{L}\bar{S}$ where $S \leq T$, $\bar{S} \neq 1$, $\bar{L} \cap \bar{S} = 1$, and $\bar{S}$ acts faithfully as field automorphisms of $\bar{L}$.

As $V_Z \in \mathcal{R}_2(LT)$, we can apply 5.1.2 and 5.1.3 to V_Z in the role of "V". For example by 5.1.2 and 3.1.8.3, either

(i) $[Z, H] = 1 = C_{V_Z}(L)$, or

(ii) $[V_Z, J(T)] \neq 1$, and either $V_Z/C_{V_Z}(L)$ is the natural module for $\bar{L}$, or $[V_Z, L]$ is the S_5-module for $\bar{L}\bar{T} \cong S_5$.

To complete the proof, we consider each of the possibilities for V arising in 5.1.3.

Suppose first that V is described in case (1) of 5.1.3. As the overgroup V_Z of V is also described in one of the cases in 5.1.3, we conclude that $V = V_Z$. By the previous paragraph, $[Z, H] = 1$. From the structure of V, $V_D \leq C_V(T \cap L)$ which is of rank $2n$ in V of rank $4n$, D_L is faithful on V_D so that $m(V_D) \geq n$, with

$$(T \cap L)C_T(V) = O_2(TD_L) = C_T(V_D) = C_{TD_L}(V_D),$$

and $T/C_T(V_D)$ is cyclic. Thus as $H \cap M$ normalizes TD_L by 5.1.5.1, Hypothesis 3.1.5 is satisfied by TD_L, V_D in the roles of "M_0, V". As $O_2(X) = 1$, we conclude from 3.1.6 that $\hat{q}(TD_L/O_2(TD_L), V_D) \leq 2$. Hence as $T/C_T(V_D)$ is cyclic and $m(V_D) \geq n$, we conclude that $n = 2$, so that conclusion (1) holds.

Similarly if V appears in case (3) of 5.1.3, we conclude as in the previous paragraph that V_Z appears in case (1) or (3) of 5.1.3, that Hypothesis 3.1.5 is satisfied with TD_L, V_D in the roles of "M_0, V", and that $\hat{q}(TD_L/O_2(TD_L), V_D) \leq 2$. Hence either $n = 2$, or possibly $n = 4$ in case V_Z satisfies conclusion (3) of 5.1.3—since $m(V_D/C_{V_D}(t)) = n/2$ for $t \in T - C_T(V_D)$ with $t^2 \in C_T(V_D)$ when V_Z satisfies that conclusion. Further $J(T) \leq C_T(V_D)$ by B.4.2.1, so $[H, Z] = 1 = C_Z(L)$ by Theorem 3.1.7, which completes the proof that conclusion (2) holds in this case.

Suppose next that V appears in case (2) or (5) of 5.1.3, or in case (4) with $n = 2$. These are the cases where $n = 2$ and L has an A_5-submodule on V, and hence also on V_Z, so that V_Z must also satisfy one of these three conclusions. Therefore $D_L \leq C_G(Z)$. Recall $H \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ by 1.1.4.6, so if $Z(H) \neq 1$ then $Z \cap Z(H) \neq 1$. Thus as $O_2(X) = 1$, $Z(H) = 1$, so that case (ii) holds; therefore V_Z satisfies conclusion (3), and hence so does V.

This leaves the case where V satisfies case (4) of 5.1.3 with $n > 2$. Thus $V = V_Z$ as before, and hence (ii) does not hold, leaving case (i) where $[Z, H] = 1 = C_Z(L)$. Now V is a 4-dimensional FL-module, where $F := \mathbf{F}_{2^{n/2}}$, and $Z = C_U(T)$ where U is the 1-dimensional F-subspace of V stabilized by $\bar{S} := \bar{T} \cap \bar{L}$. Further setting $A := N_{GL(V)}(\bar{L})$, A is the split extension of $\bar{L}$ by $\langle\sigma\rangle$ where σ is a field automorphism.

Also if s is the involution in $\langle\sigma\rangle$, then $C_A(U) = \bar{S}\langle s\rangle D_-$ and $U = \langle Z^{D_+}\rangle$, so $U = V_D$. In particular $[D_-, Z] = 1$, so $Z \leq Z(X_-)$. If $n \equiv 2 \mod 4$, then $\bar{T} \leq \bar{S}\langle s\rangle$, so $Z = U$ is D_+-invariant; hence $X = \langle H, D_L\rangle \leq N_G(Z)$, contrary to $O_2(X) = 1$. Thus $n \equiv 0 \mod 4$. Finally D_+ is faithful on V_D, so applying 3.1.6 with TD_+, V_D in the roles of "M_0, V" as before, either $O_2(X_+) \neq 1$ or $\hat{q}(D_+T/O_2(D_+T), V_D) \leq 2$. In the latter case, as $T/C_T(V_D)$ is cyclic and $m(V_D/C_{V_D}(t)) \geq n/4$ for $t \in T - C_T(V_D)$, $n = 4$ or 8. Thus (4) holds. □

LEMMA 5.1.7. *(1)* $N_G(Baum(T)) \leq M$.
(2) Let $H \in \mathcal{H}_*(T, M)$ *and set* $K := O^2(H)$. *Assume* $[Z, H] \neq 1$. *Then*

(i) $L = [L, J(T)]$.
(ii) $K = [K, J(T)]$.
(iii) Either $O_2(\langle N_L(T \cap L), H\rangle) \neq 1$, *or* $[V, L]$ *is the* S_5*-module for* $\bar{L}\bar{T} \cong S_5$, *and* $Z(H) = 1$.

PROOF. We first prove (1). Let $S := \mathrm{Baum}(T)$. If $J(T) \leq C_T(V)$, then (1) follows from 5.1.2. Thus we may assume $J(T) \not\leq C_T(V)$, so by 5.1.2, either $V/C_V(L)$ is the natural module for $\bar{L}$ or $[V, L]$ is the A_5-module. In the former case, $S \cap L \in Syl_2(L)$ by E.2.3.2, so (1) follows from 4.3.17.

Therefore we may assume that $[V, L]$ is the A_5-module. As $[V, J(T)] \neq 1$, we conclude from E.2.3 that $\bar{L}\bar{T} \cong S_5$, $\bar{S} = \overline{J(T)} \cong E_4$ is generated by the two transvections in $\bar{T}$, and $\langle Z^L\rangle = [V, L] \oplus C_Z(L)$. We may assume $V = [V, L]$.

Assume that $N_G(S) \not\leq M$; then no nontrivial characteristic subgroup of S is normal in LT as $M = !\mathcal{M}(LT)$. Hence by E.2.3.3, L is an A_5-block, so $V = O_2(L) \trianglelefteq M$. Let $Q := O_2(LS)$. It follows using C.1.13.b that $Q = V \times Q_C$, where $Q_C := C_S(L)$.

For any $1 \neq S_0 \leq S$ normalized by LT, we have $N_G(S_0) \leq M = !\mathcal{M}(LT)$, so $N_G(S) \not\leq N_G(S_0)$ by our assumption. Thus Hypothesis C.6.2 is satisfied with L, S, T, $N_G(S)$ in the roles of "L, R, T_H, Λ". Therefore by C.6.3.1 there is $g \in N_G(S)$ with $V^g \not\leq Q$. As $V \trianglelefteq M$, $g \notin M$.

Suppose that $Q_C \not\leq Q^g$. Since $[Q_C, V^g] \cap [V, V^g] \leq Q_C \cap V = 1$, from the action of S on V and hence on V^g, we conclude that Q_C and V induce distinct transvections on V^g. Thus as $|S : Q^g| = 4$, $S = Q_C V Q^g$. Let $x \in [Q_C, V^g]^\#$; then $x \in Q_C \leq C_G(L)$, so as $M = !\mathcal{M}(L)$ by Theorem 4.3.2, $C_G(x) \leq M$, so $V \leq O_2(C_G(x))$. Since Q_C induces a transvection on the A_5-module V^g for L^g, $C_{L^gS}(x)Q_CQ^g/Q_CQ^g \cong S_3$, so $V \leq O_2(C_{L^gS}(x)Q_CQ^g) = Q_CQ^g$, contrary to V and Q_C inducing distinct transvections on V^g.

Therefore $Q_C \leq Q^g$. Hence

$$\Phi(Q_C) \leq \Phi(Q^g) = \Phi(Q_C^g V^g) = \Phi(Q_C^g),$$

so $\Phi(Q_C) = \Phi(Q_C^g)$. Thus as $\Phi(Q_C) \trianglelefteq LT$ and $g \notin M = !\mathcal{M}(LT)$, $\Phi(Q_C) = 1 = \Phi(Q)$.

Next we claim we can choose g so that $S = QQ^g$. If not then $Q \cap Q^g$ is a hyperplane of Q and Q^g centralized by Q^g, so Q^g induces a transvection on Q and hence $S = Q^gQ^{gt}$ for $t \in T - SO_2(LT)$. Thus as $g \in N_G(S)$, $S = QQ^{gtg^{-1}}$, establishing the claim.

As $S = QQ^g$ with $\Phi(Q) = 1$ and $Q_C \leq Q^g$, $S = Q_C \times D_1 \times D_2$, where $D_1 \cong D_2$ is dihedral of order 8. By the Krull-Schmidt Theorem A.1.15, $N_G(S)$

permutes $\{D_1Z(S), D_2Z(S)\}$. Then $O^2(N_G(S))$ acts on $D_iZ(S)$, and indeed centralizes $D_iZ(S)/Z(S)$ as $D_iZ(S)/Z(S)$ is of order 4 and contains a unique coset of $Z(S)$ containing elements of order 4. Thus $O^2(N_G(S))$ acts on Q, and hence $O^2(N_G(S)) \leq M = !\mathcal{M}(N_G(Q))$. But then $N_G(S) = O^2(N_G(S))T \leq M$, contrary to assumption. This contradiction completes the proof of (1).

As (1) is established, we may assume the hypotheses of (2). Thus $[Z, H] \neq 1$, so $J(T) \not\leq C_T(V)$ by 3.1.8.3, and then part (i) of (2) holds by B.6.8.6.d. Therefore by 5.1.2, either $[V, L]$ is the S_5-module for $\bar{L}\bar{T} \cong S_5$, or $V/C_V(L)$ is the natural module for $\bar{L}$. Set $U := \langle Z^H \rangle$, so that $U \in \mathcal{R}_2(H)$ by B.2.14. By (1), $S \neq \mathrm{Baum}(O_2(H))$. Then as $[Z, H] \neq 1$, $J(T) \not\leq C_T(U)$ by B.6.8.3.d, and (ii) follows. Finally if $O_2(\langle N_L(T \cap L), H \rangle) = 1$, we may apply 5.1.6; as $[Z, H] \neq 1$, conclusion (3) of 5.1.6 holds, completing the proof of (iii). □

5.1.2. Further analysis when $n(H) > 1$. Recall that in this Part we focus on the "generic" situation, where $n(H) > 1$ for some $H \in \mathcal{H}_*(T, M)$. Later in Theorem 6.2.20, we will reduce the case where $n(H) = 1$ for each $H \in \mathcal{H}_*(T, M)$ to $n = 2$ with $\bar{L} = L_2(4) \cong A_5$ acting on $[Z, L]$ as the sum of at most two A_5-modules. That situation is treated later in those Parts dedicated to groups defined over $\mathbf{F}_2$.

So in the remainder of this section we assume the following hypothesis:

HYPOTHESIS 5.1.8. *Hypothesis 5.0.1 holds, and there is $H \in \mathcal{H}_*(T, M)$ with $n(H) > 1$. Set $K := O^2(H)$, $M_H := M \cap H$, and $M_K := M \cap K$.*

NOTATION 5.1.9. By 5.1.5.2, we may choose a Hall $2'$-subgroup B of M_H, and a B-invariant Hall $2'$-subgroup D_L of $N_L(T \cap L)$. This notation is fixed throughout the remainder of the section.

Observe $M_H = BT = TB$ since $T \in Syl_2(M_H)$. Further B and T normalize $N_L(T \cap L) = D_L(T \cap L)$ by 5.1.5.1, so D_LBT is a subgroup of G.

Our goal (oversimplifying somewhat) is to show in the following section that (LTB, D_LTB, D_LH) forms a weak BN-pair of rank 2 in the sense of [**DGS85**], as in our Definition F.1.7. Indeed we already encounter such rank 2 amalgams in this section.

The next few results study the structure of K and the embedding of K in members X of $\mathcal{H}(H)$, and show that usually $D_L \cap X$ acts on K. This last type of result is important, since to achieve Hypothesis F.1.1 and show (LTB, TD_LB, HD_L) is a weak BN-pair of rank 2, we need to show D_L acts on K.

LEMMA 5.1.10. *Let $k := n(H)$ and $H^* := H/O_2(H)$. Then K^* is a group of Lie type over $\mathbf{F}_{2^k}$ of Lie rank 1 or 2, M_K^* is a Borel subgroup of K^*, and B^* is a Cartan subgroup of K^*. More specifically, $K = \langle K_1^T \rangle$ for some $K_1 \in \mathcal{C}(H)$, and one of the following holds:*

(1) $K_1 < K$ and $K_1^* \cong L_2(2^k)$ or $Sz(2^k)$.

(2) $K_1 = K$ and K^* is a Bender group over $\mathbf{F}_{2^k}$.

(3) $K_1 = K$, $K^* \cong (S)L_3(2^k)$ or $Sp_4(2^k)$, and T is nontrivial on the Dynkin diagram of K^*.

PROOF. As $n(H) > 1$, this follows from E.2.2. □

From now on, whenever we assume Hypothesis 5.1.8, we also take $K_1 \in \mathcal{C}(H)$.

LEMMA 5.1.11. *Let $S := O_2(M_H)$ and $H^* := H/O_2(H)$. Then*

(1) $S \cap K \in Syl_2(K)$.
(2) $S \cap L \in Syl_2(L)$.
(3) If K^* *is of Lie rank* 2*, then either*

(i) S *acts on both rank one parabolics of* K^**, or*
(ii) K^*S^* *is* $L_3(4)$ *extended by a graph automorphism.*

PROOF. Note that $O_2(H) \leq S$ by A.1.6. By 5.1.10, M_K^* is 2-closed and $O_2(M_K^*) \in Syl_2(K^*)$, so (1) follows. By 5.1.5, B acts on $T \cap L$, and hence $T \cap L \leq O_2(BT) = O_2(M_H) = S$, so $S \cap L \in Syl_2(L)$, proving (2).

Note by 5.1.10 that B^* is a Cartan subgroup of K^*. Thus by inspection of the groups $L_2(2^k) \times L_2(2^k)$, $Sz(2^k) \times Sz(2^k)$, $(S)L_3(2^k)$, and $Sp_4(2^k)$ of Lie rank 2 listed in 5.1.10, either $C_{T^*}(B^*) = 1$—so that (i) holds; or $K^* \cong L_3(4)$, and $C_{T^*}(B^*)$ is of order 2 and induces a graph automorphism on K^*, giving (ii). Hence (3) holds. □

LEMMA 5.1.12. *For each* $X \in \mathcal{H}(H)$*,* K_1 *lies in a unique* $\hat{K}_1(X) \in \mathcal{C}(X)$*,* $K \leq \hat{K}(X) := \langle \hat{K}_1(X)^T \rangle$*, and one of the following holds:*

(1) $K = \hat{K}(X)$.
(2) $K_1 < K$*,* $K_1/O_2(K_1) \cong L_2(4)$*, and* $\hat{K}_1(X)/O_2(\hat{K}_1(X)) \cong J_1$ *or* $L_2(p)$*,* p *prime with* $p^2 \equiv 1 \mod 5$ *and* $p \equiv \pm 3 \mod 8$.
(3) $K/O_2(K) \cong Sz(2^k)$ *and* $\hat{K}(X)/O_2(\hat{K}(X)) \cong {}^2F_4(2^k)$.
(4) $K/O_2(K) \cong L_2(2^k)$ *and* $\hat{K}(X)/O_2(\hat{K}(X))$ *is of Lie type and characteristic* 2 *and Lie rank* 2.
(5) $K/O_2(K) \cong L_2(4)$ *and* $K < \hat{K}(X)$ *with* $\hat{K}(X)/O_2(\hat{K}(X))$ *not of Lie type and characteristic* 2*. The possible embeddings are listed in A.3.14.*

PROOF. By 1.2.4, K_1 lies in a unique $\hat{K}_1(X) \in \mathcal{C}(X)$, and the embedding is described in A.3.12. If $K_1 < K$, then (1) or (2) holds by 1.2.8.2, so we may assume $K_1 = K$, and hence $\hat{K}_1(X) = \hat{K}(X)$ by 1.2.8.1. We may assume (1) does not hold, so that $K < \hat{K}(X)$.

As $K_1 = K$, $K/O_2(K)$ satisfies conclusion (2) or (3) of 5.1.10. In conclusion (3) of 5.1.10 as $k \geq 2$, $K/O_2(K) \cong L_3(4)$ by 1.2.8.4, and then $\hat{K}(X)/O_2(\hat{K}(X)) \cong M_{23}$ by A.3.12. However this case is impossible as T is nontrivial on the Dynkin diagram of $K/O_2(K)$, whereas this is not the case for the embedding in M_{23}.

Thus we may assume conclusion (2) of 5.1.10 holds. By 1.2.8.4, $K/O_2(K)$ is not unitary, while if $K/O_2(K)$ is a Suzuki group, then (3) holds by A.3.12. Thus we may assume $K/O_2(K) \cong L_2(2^k)$. Then by A.3.12 and A.3.14, (4) or (5) holds. □

LEMMA 5.1.13. *Let* $X \in \mathcal{H}(H)$*, define* $\hat{K} := \hat{K}(X)$ *as in 5.1.12, and set* $D := D_L \cap X$*. Then either* $D \leq N_G(K)$*, or the following hold:*

(1) $K/O_2(K) \cong L_2(4)$.
(2) $L/O_2(L) \cong L_2(4)$.
(3) V *is the sum of at most two copies of the* A_5*-module.*
(4) $\hat{K} \leq C_G(Z)$.
(5) $\hat{K}/O_2(\hat{K}) \cong A_7$*,* J_2*, or* M_{23}.
(6) $\hat{K}(C_G(Z)) = O^{3'}(C_G(Z))$*, and either* $\hat{K} = \hat{K}(C_G(Z))$ *or*

$$\hat{K}/O_2(\hat{K}) \cong A_7 \quad \textit{with} \quad \hat{K}(C_G(Z))/O_2(\hat{K}(C_G(Z))) \cong M_{23}.$$

PROOF. We may assume D does not act on K, so in particular, $D \neq 1$. As $\hat{K} \trianglelefteq X$ by 1.2.1, D acts on $\hat{K}$ but not on K, so $K < \hat{K}$ and the possibilities for the embedding of K in $\hat{K}$ are described in 5.1.12.

If $\hat{K}/O_2(\hat{K})$ is of Lie type of characteristic 2 and Lie rank 2, then $K = P^\infty$, where $P/O_2(P)$ is one of the two maximal parabolics of $\hat{K}/O_2(\hat{K})$ containing $(T \cap \hat{K})/O_2(\hat{K})$. Then as D permutes with T, and T acts on P, also D acts on P, and hence also on K, contrary to assumption.

Therefore we may assume that case (2) or (5) of 5.1.12 holds. Let $D_c := C_D(\hat{K}/O_2(\hat{K}))$. Then $[D_c, K] \leq [D_c, \hat{K}] \leq O_2(\hat{K}) \leq O_2(KT)$, so D_c acts on $O^2(KO_2(KT)) = K$. Thus $D_c < D$.

Set $(\hat{K}TD)^* := \hat{K}TD/C_{\hat{K}TD}(\hat{K}/O_2(\hat{K}))$; then $1 \neq D^* \leq (\hat{K}TD)^* \leq Aut(\hat{K}^*)$. If D^* acts on K^* with preimage K_+, then D acts on $K = K_+^\infty$, contrary to our assumption; thus we may also assume that D^* does not act on K^*, and in particular that $D^* \not\leq B^*$ and so $D^* \neq 1$.

Suppose that case (2) of 5.1.12 holds. The case $\hat{K}_1^* \cong L_2(p)$ can be handled as in the case $\hat{K}^* \cong L_2(p)$ below, so take $\hat{K}_1^* \cong J_1$. As $K_1 < K$, $B \cong E_9$ is a Sylow 3-subgroup of $N_{\hat{K}}(T \cap \hat{K})$. Recall B normalizes D, so we may embed B^*D^* in a Hall $2'$-subgroup $E^* \cong (Frob_{21})^2$ of $N_{\hat{K}^*}(T^* \cap \hat{K}^*)$. Now D^* is cyclic as $D \leq D_L$. Also D permutes with T, so D^* is invariant under $N_{T^*}(E^*)$. But $N_{T^*}(E^*) = \langle t^* \rangle$ is of order 2, where t^* interchanges the two components of $\hat{K}^*$, so D^* is diagonally embedded in $\hat{K}^*$. Then as D^* is cyclic and B^*-invariant, $O_7(D^*) = 1$. So $D^* \leq B^*$, contradicting an earlier reduction. Therefore case (5) of 5.1.12 holds, establishing (1).

By (1), $B \cong B^*$ is of order 3. It remains to consider the corresponding possibilities for $\hat{K}^*$ in A.3.14. Furthermore the possibilities of Lie type in characteristic 2 in case (1) of A.3.14 were eliminated earlier.

Suppose first that $\hat{K}^*$ is not quasisimple. Then by 1.2.1.4, $\hat{K}^*/O(\hat{K}^*) \cong SL_2(p)$ for some odd prime p. Let R be the preimage in T of $O_{2',2}(\hat{K}^*)$. As $DT = TD$, D^* centralizes R^*, and so acts on $C_{\hat{K}}(R^*)^\infty =: K_R$; notice $K_R < \hat{K}$ as $K_R/O_2(\hat{K})) \cong SL_2(p)$. Similarly $K \leq K_R$ and T acts on K_R; so as $K_R/O_2(K_R)$ is quasisimple, D acts on K by induction on the order of $\hat{K}$, contrary to assumption. Thus we may assume $\hat{K}^*$ is quasisimple.

Suppose $\hat{K}^* \cong L_2(p)$ for some odd prime p. Recall in this case that $p \equiv \pm 3 \mod 8$, so that $B^*T^* \cong A_4$; so as B^* acts on $1 \neq D^* \leq Aut(\hat{K})^*$ and $D^*T^* = T^*D^*$, we conclude $D^* = B^*$, contrary to an earlier reduction. As mentioned earlier, this argument suffices also when $K_1 < K$, where $B^*T^* \cong A_4$ wr $\mathbf{Z}_2$.

Suppose $\hat{K}^* \cong (S)L_3^\epsilon(5)$. Then $K^* = E(C_{\hat{K}^*}(Z(T^*))$, and as D^* is cyclic and permutes with T^*, we conclude from the structure of $Aut(\hat{K}^*)$ that either $D^* \leq C_{\hat{K}^*}(Z(T^*)) \leq N_{\hat{K}^*}(K^*)$, or $\hat{K}^* \cong L_3(5)$ and D^*T^* is the normalizer in $\hat{K}^*T^*$ of the normal 4-subgroup E^* of $T^* \cap \hat{K}^*$. In the former case we contradict our assumption that D^* does not act on K^*; in the latter, $B^* \leq N_{K^*}(D^*T^*) = T^*$, contradicting B^* of order 3. Similarly if $\hat{K}^* \cong L_2(25)$ then as D^* permutes with T^*B^*, from the structure of $Aut(\hat{K}^*)$, $D^*T^* = B^*T^* \leq K^*T^*$, a contradiction.

Next suppose that $|D^*| = |D : D_c|$ is not a power of 3. Then as $DT = TD$, and $\hat{K}^*$ is not of Lie type and characteristic 2, A.3.15 says that $\hat{K}^* \cong J_1$, $L_2(q^e)$, $L_3^\delta(q)$, for q a suitable odd prime and $e \leq 2$. Then comparing these groups to our list of embeddings of A_5 in A.3.14, we conclude $\hat{K}^* \cong J_1$. As $D \not\leq N_G(K)$ is cyclic, we conclude that $D^* = [D^*, B^*]$ is of order 7; hence as $D \leq D_L = N_L(T \cap L)$, $n = 3m$ for some m. In particular as B does not centralize D, B induces a group of field automorphisms of order 3 on $L/O_2(L)$. Further $D \cap \hat{K} =: D_7$ is the subgroup of D_L of order 7. If all noncentral 2-chief factors of L on V are natural, then $C_D(Z) = 1$. If not, then by 5.1.3, m is even so that $m = 2s$ for some s, and the unique noncentral chief factor is orthogonal; so as 7 divides $2^{3s} - 1 = 2^{n/2} - 1$, $[Z, D_7] \neq 1$. Hence in any case $[Z, D_7] \neq 1$, so as $D_7 \leq \hat{K}$, $[Z, \hat{K}] \neq 1$. Thus $\langle Z^{\hat{K}} \rangle \in \mathcal{R}_2(\hat{K})$ by B.2.14, so that $\hat{K} \in \mathcal{L}_f(G, T)$. Then $\hat{K} \in \mathcal{L}_f^*(G, T)$ by 1.2.8.4. Now by 3.2.3, a suitable module for $\hat{K}$ satisfies the FSU. As J_1 does not appear among the possibilities for "$\bar{L}$" given in 3.2.6–3.2.9, this is a contradiction.

Thus D^* is a 3-group, and we have seen $D^* \not\leq B^*$, so B^*D^* is a 3-group of order at least 9 permuting with T^*. Inspecting the possibilities for $\hat{K}$ in the remaining cases of A.3.14, we conclude that $\hat{K}/O_2(\hat{K}) \cong A_7$, $\hat{A}_7$, J_2, or M_{23}, and D^* is of order 3 and inverted by some $t \in \hat{K} \cap T$. (There are groups of order 9 in J_4 containing B^* and permuting with T^*, but each such group acts on K^*). Since $D^* \not\leq B^*$ and B acts on the cyclic group D, $\hat{K}/O_2(\hat{K})$ is not $\hat{A}_7$, establishing (5).

Next $\hat{K} = O^{3'}(X)$ by A.3.18, so $BO_3(D) \leq \hat{K}$. Hence as D^* is a 3-group, $D = O_3(D) \times D_c$, with $O_3(D) =: D_3$ of order 3 and $D_c = O^3(D)$.

Now $\hat{K} \leq \tilde{K} \in \mathcal{L}^*(G, T)$ and $D_3 \leq \hat{K} \leq \tilde{K}$ with $D_3 \not\leq N_{\tilde{K}}(K)$. Therefore $\tilde{K}$ satisfies the hypotheses of $\hat{K}$, and hence replacing $\hat{K}$ by $\tilde{K}$ if necessary, we may assume $\tilde{K} = \hat{K} \in \mathcal{L}^*(G, T)$.

We next prove (4) by contradiction, so we assume that $\hat{K} \not\leq C_G(Z)$ and choose V so that $Z \leq V$; this argument will require several paragraphs. By 5.1.7.1, Baum(T) is not normal in $\hat{K}T$, so $\hat{K} = [\hat{K}, J(T)]$ using B.6.8.6.d. Set $U := [\langle Z^{\hat{K}} \rangle, \hat{K}]$, so that $U \in \mathcal{R}_2(\hat{K}T)$ by B.2.14 and U is an FF-module for $\hat{K}T$ by B.2.7. Then as M_{23} and J_2 do not have FF-modules by B.4.2, $\hat{K}/O_2(\hat{K}) \cong A_7$. Hence as B^*D^* is of order 9, $B^*D^*T^*$ is the stabilizer of a partition of type $3, 4$ in the 7-set permuted by $\hat{K}^*T^*$, and K^*T^* is the stabilizer of a partition of type $2, 5$. By B.5.1 and B.4.2, U is irreducible of dimension 4 or 6, with $\langle Z^{\hat{K}} \rangle = UZ = U \times C_Z(\hat{K})$. Then from the action of $\hat{K}$ on U, $[Z \cap U, K] \neq 1$, so by 3.1.8.3, $L = [L, J(T)]$. Therefore by 5.1.2, $V/C_V(L)$ is the natural module or the A_5-module for $\bar{L}$.

Suppose first that $V/C_V(L)$ is the A_5-module. Then $D_L = D = D_3 \leq C_G(Z)$. But if $m(U) = 6$, then $B = C_{BD}(Z)$, contradicting $B^* \not\leq D^*$. Hence $m(U) = 4$. However from the description of FF^*-offenders in B.4.2.7, $N_{\hat{K}^*}(J(T))$ is the stabilizer in $\hat{K}^*$ of a partition of type $3, 4$, so $J(T) \trianglelefteq BDT$; while as $[V, L]$ is the S_5-module, $J(T)$ is not normal in DT.

Therefore $V/C_V(L)$ is the natural module. Then $J(T) \leq (T \cap L)O_2(LT)$ by B.4.2.1, so that $J(T) \trianglelefteq DT$. If $m(U) = 6$, then $J(T)$ is not normal in D_3T using the discussion of FF^*-offenders in B.3.2.4; hence $m(U) = 4$.

As $V/C_V(L)$ is the natural module, $[Z, D_3] \neq 1$ and $C_Z(D_3) = C_Z(L)$. Then as $m(U) = 4$ and $[Z, D_3] \neq 1$, with $UZ = U \times C_Z(\hat{K})$, $C_Z(D_3) = C_U(\hat{K}) = C_Z(\hat{K})$. Therefore $C_Z(L) = C_Z(\hat{K})$, so $C_Z(L) = C_Z(\hat{K}) = 1$ as $H = KT \not\leq M = !\mathcal{M}(LT)$. Then $Z \leq U$, so $C_Z(K) \leq C_U(K) = 1$. Next by C.1.28, either there is a nontrivial characteristic subgroup C of Baum(T) normal in both LT and KT, or one of L or K is a block. As $M = !\mathcal{M}(LT)$ but $K \not\leq M$, L or K is a block.

Suppose first that K is a block. Then so is $\hat{K}$, and of the four subgroups of BD_3 of order 3, B has three noncentral chief factors on $O_2(BD_3T)$ and all others have two such factors. Thus D_3 has at most three noncentral chief factors on $O_2(BD_3T)$, so L is a $L_2(4)$-block. But then $D_L = D_3$ has exactly three noncentral chief factors, so $D = B$, contrary to $D^* \not\leq B^*$.

Consequently L is a block. But if $n = 2$, then as $C_Z(L) = 1$, T is of order at most 2^7, so K is also a block, the case we just eliminated. Hence $n > 2$. Further as K is not a block, we saw that there is a $C \trianglelefteq KT$; then as $C \trianglelefteq D_LT$, $\hat{K}T = \langle KT, D_3\rangle \leq N_G(C)$—so that $D_L \leq N_G(C) \leq N_G(\hat{K}) = !\mathcal{M}(\hat{K}T)$ by 1.2.7.3, since we chose $\hat{K} \in \mathcal{L}^*(G,T)$. Now D_3 is inverted by $t \in T \cap \hat{K}$, so t induces a nontrivial field automorphism on $L/O_2(L)$, and hence n is even. Then the subgroup D_- of D_L of order $2^{n/2}+1$ satisfies $D_- = [D_-, t] \leq D_L \cap \hat{K}$ as $t \in \hat{K}$. As $\hat{K}/O_2(\hat{K}) \cong A_7$, this forces $D_- = D_3$. But then $n = 2$, a case we eliminated at the start of the paragraph. This contradiction shows that $\hat{K} \leq C_G(Z)$, establishing (4).

We have established (1), (4), and (5) and also showed $\hat{K} = O^{3'}(X)$. As we could take $X = C_G(Z)$, it follows that (6) holds: for $A_7/E_{2^4} < M_{23}$ is the only proper inclusion in A.3.12 among the groups in (5).

As $K \leq \hat{K} \leq C_G(Z)$ by (4), as usual $C_Z(L) = 1$ using 1.2.7.3. Hence 5.1.3 says either V is the $O_4^-(2^{n/2})$-module and indeed $n/2$ must be odd, or V is the sum of two S_5-modules. In the latter case, (2) and (3) hold. In the former case, the subgroup D_- of D_L of order $2^{n/2}+1$ centralizes Z. Now in each of the possibilities for $\hat{K}$ in (5), D_3 is inverted by $t \in T \cap \hat{K}$. Then the final few sentences in the proof of (4) show that $n = 2$. This completes the proof of (2) and (3) and hence of the lemma. □

5.1.3. More detailed analysis of the case $\mathbf{K/O_2(K) = L_3(4)}$. The remainder of the section is devoted to an analysis of the subcase of 5.1.10.3 where $K/O_2(K) \cong L_3(4)$. This case is the remaining major obstruction to applying the Green Book [**DGS85**] and beginning the identification of G as a rank 2 group of Lie type and characteristic 2 in Theorem 5.2.3 of the next section.

THEOREM 5.1.14. *Let $H^* := H/O_2(H)$ and assume $K^* \cong L_3(4)$. Then*

(1) $K \in \mathcal{L}^*(G,T)$, so $N_G(K) = !\mathcal{M}(H)$ but $K \notin \mathcal{L}_f^*(G,T)$.

(2) $[Z,H] = 1$ and $C_G(z) \leq N_G(K)$ for each $z \in Z^\#$.

(3) $C_Z(L) = 1$.

(4) $n = 2$, V is the sum of one or two copies of the S_5-module for $\bar{L}\bar{T} \cong S_5$, and $D_L = B$.

(5) $C_G(K/O_2(K))$ is a solvable $3'$-group.

In the remainder of this section assume the hypotheses of Theorem 5.1.14, and set $H^* := H/O_2(H)$. We will prove Theorem 5.1.14 by a series of reductions.

Note that B has order 3, since $K^* \cong L_3(4)$, and B^* is a Cartan subgroup of K^*. By 5.1.12, $K \in \mathcal{L}^*(G,T)$. In particular $N_G(K) = !\mathcal{M}(H)$ by 1.2.7.3. On the other hand, $K \notin \mathcal{L}_f^*(G,T)$: For if $K \in \mathcal{L}_f^*(G,T)$ then by 3.2.3, there exist $V_K \in \mathcal{R}_2(KT)$ such that the pair K, V satisfies the FSU. By 5.1.10.3, T is nontrivial on the Dynkin diagram of K^*, so case (4) of 3.2.9 in the FSU is excluded, while $L_3(4)$ (as opposed to $SL_3(4)$) does not arise anywhere else in 3.2.8 or 3.2.9. This contradiction establishes conclusion (1) of Theorem 5.1.14.

Now as $K \notin \mathcal{L}^*_f(G,T)$, K centralizes $R_2(KT)$ by 1.2.10, so that $H = KT$ centralizes Z. Then the remaining statement in conclusion (2) follows as $N_G(K) = !\mathcal{M}(H)$; and conclusion (2) implies conclusion (3) as $H \not\leq M = \mathcal{M}(LT)$.

Thus it only remains to prove parts (4) and (5) of Theorem 5.1.14. Moreover throughout the remainder of the proof we can and will appeal to the first three parts of Theorem 5.1.14.

Set $M_+ := N_G(K)$; by 5.1.14.1, $M_+ \in \mathcal{M}(T)$. If n is even, define D_ϵ for $\epsilon = \pm 1$ as in Lemma 5.1.6.

LEMMA 5.1.15. *One of the following holds:*

(1) $D_L \leq M_+$.

(2) $n = 2$ *and* V *is the direct sum of two natural modules for* $\bar{L}$.

(3) $n = 2$ *or* 4 *and* $[V, L]$ *is the natural module for* $\bar{L}$.

(4) $n = 4$ *or* 8, V *is the* $\Omega_4^-(2^{n/2})$*-module for* $\bar{L}$, *and* $D_- \leq M_+$.

PROOF. First if $D \leq D_L$ and $O_2(\langle D, H\rangle) \neq 1$, then by 5.1.14.1, $D \leq M_+$. However we may assume conclusion (1) does not hold, so $D_L \not\leq M_+$ and hence $O_2(\langle D_L, H\rangle) = 1$. Cases (1) and (2) of 5.1.6 appear as cases (2) and (3) of 5.1.15. Case (3) of 5.1.6 cannot occur since there $Z(H) = 1$, contrary to 5.1.14.2. Finally in case (4) of 5.1.6, $O_2(\langle D_-, H\rangle) \neq 1$, so $D_- \leq M_+$. Thus as $D_+D_- = D_L \not\leq M_+$, $O_2(\langle D_+, H\rangle) = 1$, so $n = 4$ or 8 by 5.1.6.4. Hence 5.1.15.4 holds. □

We now begin to make use of the local classification of weak BN-pairs of rank 2 in the Green Book [**DGS85**]. We recognize weak BN-pairs of rank 2 by verifying Hypothesis F.1.1.

LEMMA 5.1.16. *Let* $C_K := C_G(K/O_2(K))$. *Then*

(1) C_K *is a* $3'$*-group.*

(2) If C_K *is not solvable, then* $C_K^\infty/O_2(C_K^\infty) \cong Sz(2^k)$ *for some odd* $k \geq 3$, $C_K^\infty \not\leq M$, *and* $D_L \not\leq M_+$.

PROOF. Part (1) follows as H is an SQTK-group. Thus it remains to prove (2), so we assume $C_K^\infty \neq 1$. Hence by 1.2.1, there exists $K_+ \in \mathcal{C}(C_K)$. Then any such K_+ satisfies $K_+/O_2(K_+) \cong Sz(2^k)$ for some odd $k \geq 3$. Further $m_5(K_+) = 1 = m_5(K)$, while $m_5(M_+) \leq 2$ as M_+ is an SQTK-group, so $K_+ = C_K^\infty$ by 1.2.1.1, establishing the first assertion of (2). Further $M_+ = N_G(K_+)$ since we saw $M_+ \in \mathcal{M}$. Let B_+ be a Borel subgroup of K_+; then $B_+ \leq N_G(T) \leq M = N_G(L)$ using Theorem 3.3.1. Now if $K_+ \leq M$, then $[K_+, L] \leq O_2(L)$, so that L normalizes $O^2(K_+O_2(L)) = K$ and hence $L \leq N_G(K_+) = M_+$ contradicting $M = !\mathcal{M}(LT)$. Thus $K_+ \not\leq M$, proving the second statement of (2).

To complete the proof of (2), we suppose by way of contradiction that $D_L \leq M_+ = N_G(K_+)$. We claim that under this assumption, Hypothesis F.1.1 is satisfied with L, K_+, T in the roles of "L_1, L_2, S". Let $G_+ := \langle LT, H\rangle$. As $K_+ \not\leq M = !\mathcal{M}(LT)$, $O_2(G_+) = 1$, establishing hypothesis (e) of F.1.1. We have seen that $B_+ \leq M = N_G(L)$, and we are assuming $D_L \leq N_G(K_+)$, so hypothesis (d) of F.1.1 holds. The remaining conditions in F.1.1 are easy to verify, in particular since we take S to be the Sylow 2-subgroup T of G; therefore Hypothesis F.1.1 is satisfied as claimed. We conclude from F.1.9 that $\alpha := (LTB_+, D_LTB_+, D_LTK_+)$ is a weak BN-pair of rank 2. Indeed $T \trianglelefteq B_+T$, so by F.1.12.I, α is of type ${}^2F_4(2^k)$, with $n = k$—as this is the only type where a parabolic possesses an $Sz(2^k)$

composition factor. By F.1.12.II, $T \leq K_+$. But then $T \leq K_+ \leq C_K$, contradicting $T \cap K \not\leq C_K$. □

Notice now that to complete the proof of Theorem 5.1.14, it suffices to prove part (4) of 5.1.14: Namely we have already established the first three parts of Theorem 5.1.14. Further if part (4) holds then $D_L = B \leq M_+$, which by 5.1.16 forces C_K to be a solvable $3'$-group, establishing part (5) of 5.1.14.

LEMMA 5.1.17. *n is even.*

PROOF. Assume n is odd, so in fact $n \geq 3$ as $n > 1$. Let $F := \mathbf{F}_{2^n}$. Then T induces inner automorphisms on $\bar{L}$, so $\bar{T} \leq \bar{L}$. By 5.1.15, $D_L \leq M_+$; then as D_L is a $\{2,3\}'$-group acting on T and $K/O_2(K) \cong L_3(4)$, we conclude that $D_L \leq C_K := C_G(K/O_2(K))$.

We now specialize our choice of V to be the module "V" in the Fundamental Setup (3.2.1) for L, as we may by 3.2.3. As $L/O_2(L) \cong L_2(2^n)$, case (1) or (2) of Theorem 3.2.5 holds, so L is irreducible on $V/C_V(L)$ and V is a TI-set under M. Since n is odd, $V/C_V(L)$ is the the natural module for $\bar{L}$ by 5.1.3; then as $C_Z(L) = 1$ by 5.1.14.3, V is a natural module. Let $Z_1 := Z \cap V$. Notice as $\bar{T} \leq \bar{L}$, Z_1 is the 1-dimensional F-subspace of V stabilized by T. In particular Z_1 is a TI-set under $N_M(V)$, so as V is a TI-set under M, Z_1 is a TI-set under M.

Observe also that L is not a block: For if it were, then as $C_Z(L) = 1$, $C_T(D_L) = 1$, contradicting $D_L \leq C_K$. Also C_K is a solvable $3'$-group by 5.1.16, since we saw $D_L \leq M_+$.

Let $S := \mathrm{Baum}(T)$, and recall from 5.1.7.1 that $N_G(S) \leq M$.

We claim Z_1 is a TI-set in G. For let $Z_0 := \langle Z^{C_K} \rangle$; then $Z_0 \in \mathcal{R}_2(C_K T)$ by B.2.14. As C_K is a solvable $3'$-group, by Solvable Thompson Factorization B.2.16, $[Z_0, J(T)] = 1$, so that $S = \mathrm{Baum}(C_T(Z_0))$ using B.2.3. Now by a Frattini Argument, $C_K = C_{C_K}(Z_0)N_{C_K}(S)$. Then as $Z_1 \leq Z_0$ while $N_{C_K}(S) \leq M$ and Z_1 is a TI-set under M, Z_1 is a TI-set under C_K. Now $n \neq 6$ since n is odd, so by Zsigmondy's Theorem [**Zsi92**], there is a *Zsigmondy prime divisor* p of $2^n - 1$, namely such that a suitable element of order p is irreducible on Z_1. Let $P \in Syl_p(C_K)$. As $D_L \leq C_K = C_{C_K}(Z_0)N_{C_K}(S)$ with $N_{C_K}(S) \leq M$, we may choose P so that $P = C_P(Z_0)(P \cap M)$ and $P_L := P \cap D_L \in Syl_p(D_L)$. By the choice of p, $P \cap M = P_L \times C_{P \cap M}(Z_1)$, so $P = P_L C_P(Z_1)$, and P is irreducible on Z_1. Therefore Z_1 is a TI-set under $N_{M_+}(P)$. Further by a Frattini Argument, $M_+ = C_K N_{M_+}(P)$, so as Z_1 is a TI-set under C_K, Z_1 is a TI-set under M_+. Finally by 5.1.14.2, $C_G(z) \leq M_+$ for each $z \in Z_1^\#$, so as $D_L \leq M_+$ is transitive on $Z_1^\#$, Z_1 is a TI-set under G by I.6.1.1, and hence the claim holds.

Let $G_1 := N_G(Z_1)$ and $\tilde{G}_1 := G_1/Z_1$. Recall by 5.1.14.2 that $H \leq C_G(Z_1)$, so $G_1 \leq M_+$ by 5.1.14.1.

Consider any H_1 with $HD_L \leq H_1 \leq G_1$, and set $Q_1 := O_2(H_1)$ and $U := \langle V^{H_1} \rangle$. Observe that Hypothesis G.2.1 is satisfied with Z_1 and H_1 in the roles of "V_1" and "H". Therefore $\tilde{U} \leq Z(\tilde{Q}_1)$ and $\Phi(U) \leq Z_1$ by G.2.2.

Suppose by way of contradiction that $\Phi(U) \neq 1$. Then $U = \langle V^{H_1} \rangle$ is not elementary abelian, so $U \not\leq C_T(V)$. Thus $\bar{U} \neq 1$, and hence the hypotheses of G.2.3 are satisfied. Therefore $\bar{U} \in Syl_2(\bar{L})$ by G.2.3.1. Set $I := \langle U^L \rangle$ and $W := O_2(I)$. By G.2.3.4, there exists an I-series

$$1 = W_0 \leq W_1 \leq W_2 \leq W_3 = W,$$

where $W_1 = V$, $W_2 = U \cap U^l$, for some $l \in L - G_1$, and W/W_2 is the sum of r natural modules for $L/O_2(L)$ and some $0 \leq r$, with $(U \cap W)/W_2 = C_{W/W_2}(\bar{U})$. In particular $W = [W, D_L]W_2$. But $D_L \leq C_K$ and C_K is a solvable $3'$-group, so by A.1.26.2, $[W, D_L] \leq O_2(C_K) \leq O_2(M_+) \leq O_2(H_1) = Q_1$ using A.1.6. Thus as $W_2 \leq U \leq Q_1$,

$$W \leq Q_1 \leq C_G(\tilde{U}).$$

Therefore as $Z_1 \leq W_2 \leq U \cap W$ and $(U \cap W)/W_2 = C_{W/W_2}(\bar{U})$, it follows that $W \leq U$. But in G.2.3.6, $(U \cap W)/W_2$ is a proper direct summand of W/W_2 if $r > 0$, so we conclude $W = W_2$ and thus $[O_2(I), I] \leq W_2$. Then as $L \leq I$ and $[W_2, L] = V$, we conclude $V = [O_2(L), L]$, so that L is an $L_2(2^n)$-block, contrary to an earlier observation.

This contradiction shows that U is elementary abelian. Applying this result to G_1 in the role of "H_1", we conclude that $\langle V^{G_1} \rangle$ is abelian. But L is transitive on $V^\#$ and Z_1 is a TI-set in G, so (cf. A.1.7.1) G_1 is transitive on $\{V^g : Z_1 \cap V^g \neq 1\}$, and hence as $\langle V^{G_1} \rangle$ is abelian, $[V, V^g] = 1$ whenever $Z_1 \cap V^g \neq 1$. This verifies part (a) of Hypothesis F.8.1 with Z_1, HD_L in the roles of "V_1, H".

During the remainder of the proof take $H_1 := HD_L$. Then part (b) of Hypothesis F.8.1 is part of Hypothesis G.2.1 verified earlier. Next using 3.1.4.1, $C_{H_1}(\tilde{V}) \leq N_{H_1}(V) = H_1 \cap M = TBD_L$. As V is the natural module for $\bar{L}$, $C_{N_{GL(V)}(\bar{L})}(\tilde{V}) \cong Z_{2^n-1}$, so as D_L is a Hall subgroup of TBD_L and D_L is faithful on $\tilde{V}$, we conclude $C_{H_1}(\tilde{V}) = C_{TB}(V)$. Therefore $\ker_{C_{H_1}(\tilde{V})}(H_1) \leq \ker_{TB}(H_1) = Q_1$, so part (c) of F.8.1 holds. Finally part (d) holds as $H \not\leq M = !\mathcal{M}(LT)$. Thus we have verified Hypothesis F.8.1, so we can apply the results of section F.8.

Define b, γ, etc. as in section F.8. By F.8.5.1, $b \geq 3$ is odd, so G_γ is a conjugate of H_1 and hence as $D_L \leq C_K$,

$$\hat{G}_\gamma := G_\gamma/O_2(G_\gamma) \cong H_1^+ := H_1/Q_1 = KT/Q_1 \times D_LQ_1/Q_1$$

with KT/Q_1 an extension of $L_3(4)$ and $D_LQ_1/Q_1 \cong D_L \cong \mathbf{Z}_{2^n-1}$.

As $D_L^+ \trianglelefteq H_1^+$ and $\tilde{V} = [\tilde{V}, D_L]$, $\tilde{U} = [\tilde{U}, D_L]$. Thus each KD_L-irreducible is the sum of n K-irreducibles $\tilde{I}$, as $\mathbf{F}_4$ is a splitting field for K^* and n is odd. We claim $m(H_1^+, \tilde{U}) \geq 9$: For if y is an involution in H^+ with $m([\tilde{U}, y]) < 9$, then as $m(\tilde{I}) \geq 9$, y^+ acts on $\tilde{I}$. Then by H.4.7, either $m([\tilde{I}, y]) \geq 4$, or $m(\tilde{I}) = 9$ and $m([\tilde{I}, y]) = 3$. So $\tilde{I}_D := \langle \tilde{I}^{D_L} \rangle$ is the sum of $n \geq 3$ conjugates of $\tilde{I}$, so $m([\tilde{I}_D, y]) = m([\tilde{I}, y])n \geq 9$, proving the claim. In particular $\tilde{U}$ is not an FF-module for H_1^+ by B.4.2.

Recall from section F.8 that $Q_1 = C_{H_1}(\tilde{U})$, there is $g_b \in G$ with $\gamma = \gamma_1 g_b$, $A_1 := Z^{g_b}$, $D_\gamma := C_{U_\gamma}(\tilde{U})$, and $D_{H_1} := C_U(U_\gamma/A_1)$.

Suppose U_γ centralizes $\tilde{U}$, so that $U_\gamma = D_\gamma$. By F.8.7.7, $[D_{H_1}, U_\gamma] = 1$. By F.8.7.5, $[V, U_\gamma] \neq 1$, so $[Z_1^l, U_\gamma] \neq 1$ for some $l \in L$. If $1 \neq Z_1^l \cap D_{H_1}$, then

$$U_\gamma \leq O^{2'}(C_G(Z_1^l \cap D_{H_1})) \leq C_G(Z_1^l)$$

as Z_1 is a TI-set in G in the center of $T \in Syl_2(G)$. Of course this contradicts the choice of Z_1^l, so we conclude that $1 = Z_1^l \cap D_{H_1}$, and hence Z_1^l is isomorphic to a subgroup of $\hat{G}_\gamma$. Therefore

$$4 = m_2(\hat{G}_\gamma) \geq m(Z_1) = n,$$

so as n is odd, $n = 3$. As we are assuming $D_\gamma = U_\gamma$, $[U_\gamma, V] \leq Z_1$ by F.8.7.6; so for $1 \neq y \in Z_1^l$, $m([U_\gamma, y]) \leq m(Z_1) = 3$, contradicting $m(H_1^+, \tilde{U}) \geq 9$.

This contradiction shows that $D_\gamma < U_\gamma$. Therefore there is $\beta \in \Gamma(\gamma)$ with $V_\beta \not\leq Q_1$, and $d(\beta, \gamma_1) = b$ by minimality of b. Thus we have symmetry between γ_0, γ_1, γ and β, γ, γ_1; so reversing the roles of these triples if necessary, we may assume that $m(U_\gamma^+) = m(U_\gamma/D_\gamma) \geq m(U/D_{H_1})$. Thus if $\tilde{D}_{H_1} \leq C_{\tilde{U}}(U_\gamma)$, then $\tilde{U}$ is an FF-module for H_1^+, contrary to an earlier observation. Therefore $[D_{H_1}, U_\gamma] \neq 1$, so there is $g \in G$ with $Z_1^g = Z_\gamma$ (so that $V^g \leq U_\gamma$) and $[D_{H_1}, V^g] \neq 1$. By F.8.7.6, $[D_{H_1}, U_\gamma] \leq A_1$, so D_{H_1} acts on V^g; then since V is the natural module for $\bar{L}$ and n is odd,

$$m(D_{H_1}/C_{D_{H_1}}(V^g)) \leq m_2(\bar{L}\bar{T}) = n = m(V^g/C_{V^g}(D_{H_1})).$$

Also $V^g \cap Q_1 \leq D_\gamma \leq C_G(D_{H_1})$ by F.8.7.7. Thus

$$4 = m_2(H_1^+) \geq m(V^{g+}) \geq m(V^g/C_{V^g}(D_{H_1})) = n,$$

so $n = 3$ and

$$m(\tilde{U}/C_{\tilde{U}}(V^g)) \leq m(U/D_{H_1}) + m(D_{H_1}/C_{D_{H_1}}(V^g)) \leq m(U_\gamma^+) + 3 \leq 7,$$

contradicting $m(H_1^+, \tilde{U}) \geq 9$. This contradiction completes the proof of 5.1.17. $\square$

As n is even by 5.1.17, there is a unique subgroup D_3 of order 3 in D_L.

LEMMA 5.1.18. *If $D_3 \leq M_+$, then $D_3 = B$, so that $[Z, D_3] = 1$.*

PROOF. Notice the final statement follows from the first, as $B \leq H \leq C_G(Z)$ by 5.1.14.2.

Assume $D_3 \leq M_+$. It suffices to assume $D_3 \neq B$ and establish a contradiction. If D_3 induces inner automorphisms on K^* then $D_3 \leq K$ by 5.1.16.1. Then as BT is the largest solvable subgroup of KT containing T, $D_3 \leq BT$ and hence $D_3 = B$, contrary to assumption. Therefore D_3 induces outer automorphisms on K^*, and $K^*D_3^* \cong PGL_3(4)$. Set $D := D_L \cap M_+$ and $S := O_2(DBT)$. Arguing as in 5.1.11, $T \cap L \leq S$ and hence $S \cap L \in Syl_2(L)$; similarly $S \cap K \in Syl_2(K)$. From the structure of $Aut(L_3(4))$, $C_{T^*}(B^*D_3^*) = 1$, so $S = (T \cap K)C_S(K^*) = (S \cap K)O_2(KS)$. Let P_2 be a rank-1 parabolic of K over $S \cap K$, and set $K_2 := O^2(P_2)$. Then SD acts on K_2, and as $K \not\leq M$ with T nontrivial on the Dynkin diagram of $K/O_2(K)$, $K_2 \not\leq M$. Thus $O_2(G_0) = 1$, where $G_0 := \langle LS, K_2 \rangle$, since $M = !\mathcal{M}(L)$ by Theorem 4.3.2.

Suppose that $D_L \leq M_+ = N_G(K)$. Then $D_L = D$ acts on K_2, so that $N_L(S \cap L) = D(S \cap L)$ acts on K_2. Now it is easy to verify the remainder of Hypothesis F.1.1 with K_2, L in the roles of "L_1, L_2": For example as $O_2(M) \leq S \geq O_2(M_+)$, $L_iSBD \in \mathcal{H}^e$ by 1.1.4.5. So by F.1.9, $\alpha := (K_2SD, BSD, BSL)$ is a weak BN-pair of rank 2. Further by construction $S \trianglelefteq SBD$, so α is described in F.1.12. Since $K_2/O_2(K_2) \cong L_2(4)$ and $L/O_2(L) \cong L_2(2^n)$ with n even, it follows from F.1.12 that α is the amalgam of a (possibly twisted) group of Lie type over $\mathbf{F}_4$. Then as K_2 centralizes Z, α is the amalgam of $G_2(4)$ or $U_4(4)$. But now K_2 has only two noncentral chief factors, which is incompatible with the embedding of K_2 in K with $F^*(K) = O_2(K)$.

Therefore $D_L \not\leq M_+$, so one of the last three cases of 5.1.15 must hold. However by hypothesis, $D_3 \leq M_+$, so $D_L > D_3$ and hence $n > 2$. Thus either case (3) of 5.1.15 holds with $n = 4$, or case (4) holds with $n = 8$—since in that case $D_- \leq M_+$, so that $D_L = D_3D_- \leq M_+$ if $n = 4$. Similarly in either case, $D_5 \not\leq M_+$, where D_5 is the subgroup of D_L of order 5, since in case (3) , $D_L = D_3D_5$, while in case (4), $D_L = D_3D_5D_-$ with $D_- \leq M_+$.

Recall $S \cap L \in Syl_2(L)$, so $SD_5 \trianglelefteq TD_5$, and hence S_0D_5 is a subgroup of G for each subgroup S_0 of T containing S. As B acts on D_5 and S, it acts on SD_5. As $S \cap K \in Syl_2(K)$ and $S \trianglelefteq T$, $1 \neq O_2(\langle N_G(S), H\rangle)$ by Theorem 3.1.1. Then $N_G(S) \leq M_+$ by 5.1.14.1, and hence $D_5 \not\leq N_G(S)$.

Let $X := \langle SBD_5, K_2\rangle$. Suppose first that $O_2(X) = 1$. We just saw D_5 does not act on S, so $SD_5/O_2(SD_5) \cong D_{10}$ or $Sz(2)$. Therefore Hypothesis F.1.1 is satisfied with K_2, SD_5 in the roles of "L_1, L_2". Thus $\beta := (K_2S, BS, BSD_5)$ is a weak BN-pair of rank 2 by F.1.9, and as S is self-normalizing in SD_5, β is on the list of F.1.12. But D_{10} or $Sz(2)$ occur as factors of $L_i/O_2(L_i)$ only in the amalgams of ${}^2F_4(2)'$ and ${}^2F_4(2)$, where the rank-1 parabolic over S other than K_2 in those amalgams is solvable, a contradiction as K_2 is not solvable.

This contradiction shows that $O_2(X) \neq 1$. Set $T_0 := N_T(K_2)$. We saw earlier that T acts on SD_5 and similarly T acts on SB. Thus T acts on SBD_5, so T_0 acts on X, and hence on $O_2(X)$. Embed $T_0 \leq T_1 \in Syl_2(XT_0)$; as $|T : T_0| = 2$, $|T_1 : T_0| \leq 2$. As $O_2(KT_0) \leq T_0$ and $K \notin \mathcal{L}_f(G,T)$ by 5.1.14.1, $[Z(T_0), K] = 1$ using B.2.14; hence $N_G(T_0) \leq M_+$ by 5.1.14.1. Also by 4.3.17, $N_G(T_0) \leq M$, so $T_1 \leq M \cap M_+$. Thus if $T_0 < T_1$ we may take $T_1 = T$. However if $T_1 = T$, then $KT = \langle K_2, T\rangle \leq XT_0 \in \mathcal{H}$, so that $D_5 \leq X \leq M_+$ using 5.1.14.1, contrary to an earlier reduction. Hence $T_0 \in Syl_2(X)$.

We claim D_5 acts on K_2; assume otherwise. As $K_2 \in \mathcal{L}(X, T_0)$ and $T_0 \in Syl_2(X)$, $K_2 < K_X \in \mathcal{C}(X)$ by 1.2.4, with the embedding described in A.3.14. Let $Y := K_XT_0D_5$ and $Y^* := Y/C_Y(K_X/O_2(K_X))$. Arguing as in the beginning of the proof of 5.1.13, $C_{D_5}(K_X/O_2(K_X))$ normalizes K_2; so as we are assuming $D_5 \not\leq N_G(K_2)$, $D_5^* \neq 1$. As $S \leq T_0$, D_5 permutes with T_0 and so D_5T_0 is a subgroup of G by an earlier remark; therefore K_X^* appears on the list of A.3.15. Comparing that list to the list of A.3.14, we conclude that case (3) of A.3.14 holds with $K_X^* \cong L_2(p)$, $p^2 \equiv 1 \mod 5$ and $p \equiv \pm 3 \mod 8$. But B acts on D_L, so that $[B, D_5] = 1$. Then D_5^* permutes with the subgroup $(T_0 \cap K_X)^*B^* \cong A_4$ of K_X^*, which is not the case in $Aut(L_2(p))$.

This contradiction establishes our claim that D_5 acts on K_2. By symmetry, D_5 also acts on $K_3 := O^2(P_3)$, where P_3 is the second rank one parabolic of K over $T \cap K$. Therefore D_5 acts on $K = \langle K_2, K_3\rangle$, a contradiction as we showed $D_5 \not\leq M_+$. This completes the proof of 5.1.18. □

From this point on, we assume H is a counterexample to Theorem 5.1.14. Under this assumption we show:

LEMMA 5.1.19. *One of the following holds:*

(1) $D_3 \not\leq M_+$, and either

(i) $n = 2$, and V is the direct sum of two natural modules for $\bar{L}$, or

(ii) $n = 2$ or 4, and $[V, L]$ is a natural module for $\bar{L}$.

(2) $n = 4$ or 8, V is the $\Omega_4^-(2^{n/2})$-module for $\bar{L}$, and $D_3 \not\leq M_+$.

(3) $n \equiv 2 \mod 4$, $n > 2$, 3 does not divide n, $D_3 = B \leq M_+$, and V is the $\Omega_4^-(2^{n/2})$-module for $\bar{L}$.

PROOF. First suppose $D_3 \leq M_+$. Then by 5.1.18, $[Z, D_3] = 1$ and $D_3 = B$. This forces one of cases (2), (4), or (5) of 5.1.3 to hold, with $n \equiv 2 \mod 4$ in (4).

Assume first that $n = 2$, so that $D_L = D_3 = B$. As $C_Z(L) = 1$ by part (3) of Theorem 5.1.14, V is the sum of at most two copies of the S_5-module, so part (4) of Theorem 5.1.14 holds. Hence by our remark after 5.1.16, Theorem 5.1.14 holds, contrary to our assumption that H is a counterexample to that Theorem.

So $n > 2$, and then case (4) of 5.1.3 holds, with $n \equiv 2 \mod 4$. Thus $D_L \leq M_+$ by 5.1.15. Further 3 does not divide n, or otherwise D_L contains a cyclic subgroup of order 9, which must be faithful on K^* as $D_3 = B$ is faithful. However this is impossible as $Aut(K^*)$ has no cyclic subgroup of order 9 permuting with T^*. So conclusion (3) holds when $D_3 \leq M_+$.

Therefore we may assume $D_3 \not\leq M_+$. Then one of the last three cases of 5.1.15 must hold. Cases (2) and (3) give conclusion (1), and case (4) gives conclusion (2). □

LEMMA 5.1.20. *$D_3 \not\leq M_+$, so $O_2(\langle H, D_3\rangle) = 1$.*

PROOF. If $D_3 \not\leq M_+$, then $O_2(\langle H, D_3\rangle) = 1$ by 5.1.13.1. Thus it suffices to assume $D_3 \leq M_+$, and derive a contradiction. As $D_3 \leq M_+$, case (3) of 5.1.19 holds; thus $n \equiv 2 \mod 4$, $n > 2$, and 3 does not divide n, so $n \geq 10$. Set $S := (T \cap L)O_2(LT)$; then $S \in Syl_2(LS)$. Also $S = O_2(D_3T)$, so as $D_3 \leq M_+$, $S \in Syl_2(KS)$.

Next as case (3) of 5.1.19 holds, $J(T) \trianglelefteq LT$ by 5.1.2, so $J(T) \leq O_2(LT) \leq S$ and hence $J(T) = J(S)$ by B.2.3.3. As $K \not\leq M = !\mathcal{M}(LT)$, $J(S)$ is not normal in KS. By B.5.1 and B.4.2, K^*S^* has no FF-modules, so as $m_2(K^*S^*) = 4$, E.5.4 says $E := \Omega_1(Z(J_4(S))) \trianglelefteq KS$. Therefore as $K \not\leq M$ and $M = !\mathcal{M}(L)$, $J_4(S) \not\leq O_2(LS) = C_S(V)$. By E.5.5, there is $\bar{A} \in \mathcal{A}^2(\bar{S})$ with $m(V/C_V(\bar{A})) - m(\bar{A}) \leq 4$. But by construction $\bar{S} \leq \bar{L}$, so by H.1.1.3 applied with $n/2$ in the role of "n",

$$n/2 \leq m(V/C_V(\bar{A})) - m(\bar{A}) \leq 4.$$

Thus $n \leq 8$, whereas we saw earlier that $n \geq 10$. This contradiction completes the proof. □

By 5.1.20, $D_3 \not\leq M_+$. So by 5.1.19, case (1) or (2) of 5.1.19 holds. In particular, $n = 2$, 4, or 8. However by 5.1.14.1 we may apply Theorem 3.3.1 to K, to conclude $N_G(T) \leq M_+$; hence $D_3 \not\leq N_G(T)$. Therefore $\bar{L}\bar{T} \cong Aut(L_2(2^n))$.

By B.2.14, $V_Z := \langle Z^L\rangle \in \mathcal{R}_2(LT)$, so we can apply the results of this section to V_Z in the role of "V". In particular as $\bar{L}\bar{T} = Aut(\bar{L})$, from the structure of the modules in case (1) or (2) of 5.1.19, either Z is of order 2, in which case we set $Z_1 := Z$; or V_Z is the sum of two natural modules for $\bar{L} \cong L_2(4)$, where we take $Z_1 := Z \cap V_1$ for some $V_1 \in Irr_+(L, V_Z)$. Thus in any case Z_1 is of order 2, and $V_2 := \langle Z_1^{D_3}\rangle \cong E_4$. Set $G_1 := C_G(Z_1)$, $G_2 := N_G(V_2)$, and consider any H_1 with $H \leq H_1 \leq G_1$. Set $U := \langle V_2^{H_1}\rangle$, $Q_1 := O_2(H_1)$, $\tilde{G}_1 := G_1/Z_1$, and $L_2 := \langle D_3^T\rangle = D_3[O_2(D_3T), D_3]$. Observe Hypothesis G.2.1 is satisfied with L_2, V_2, Z_1, H_1 in the roles of "L, V, V_1, H", so by G.2.2 we have:

LEMMA 5.1.21. *$\tilde{U} \leq Z(\tilde{Q}_1)$ and $\Phi(U) \leq Z_1$.*

LEMMA 5.1.22. *(1) $C_G(V_2) = C_T(V_2)B \leq M$.*
(2) $n = 2$ or 4, and $[V, L]$ is the sum of at most two natural modules for $\bar{L}$.
(3) $[V_2, O_2(K)] = Z_1$ and $D_3O_2(C_G(V_2)) \trianglelefteq G_2$.

PROOF. Notice (1) implies (2), since if case (2) of 5.1.19 holds, then $1 \neq C_{D_L}(V_2)$ is a $3'$-group.

If K normalizes V_2, then by 5.1.14.1, $D_3 \leq G_2 \leq M_+$, contradicting 5.1.20. Thus $[K, V_2] \neq 1$. Set $Q_K := O_2(K)$. Then $V_2 \not\leq Z(Q_K)$, for otherwise $K \in$

$\mathcal{L}_f(G,T)$ using 1.2.10, contrary to 5.1.14.1. Thus 5.1.21 says $[V_2, Q_K] = Z_1$, proving the first assertion of (3). Hence as $V_2 = [V_2, L_2]$, $L_2 = [L_2, Q_K]$. Now $K \trianglelefteq G_1$ by 5.1.14.2, so $C_G(V_2) \leq G_1 \leq N_G(Q_K)$, and hence $C_{Q_K}(V_2) \leq O_2(G_2)$. Therefore $P := \langle C_{Q_K}(V_2)^{G_2}\rangle \leq O_2(G_2)$, and $[C_G(V_2), Q_K] \leq C_{Q_K}(V_2) \leq P$. Then $L_2 = [L_2, Q_K] \leq C_G(C_G(V_2)/P)$, so as $G_2 = L_2TC_G(V_2)$, $L_2P \trianglelefteq G_2$. Then as $P \leq O_2(G_2) \leq T \leq N_G(L_2)$, $L_2 = O^2(L_2P) \trianglelefteq G_2$. Now since $L_2 = D_3O_2(L_2)$ with $O_2(L_2) = C_{L_2}(V_2)$, $D_3O_2(C_G(V_2)) \trianglelefteq G_2$. Therefore (3) holds, and it remains to establish (1).

Now B acts on D_3 and $B \leq K \leq C_G(Z_1)$, so B centralizes $\langle Z_1^{D_3}\rangle = V_2$. On the other hand as G_2 is an SQTK-group, $m_3(G_2) \leq 2$, so by (3), $m_3(C_G(V_2)) = 1$. Further $C_G(V_2) = C_{G_1}(V_2)$, with $G_1 \leq M_+$. As C_K is a $3'$-group by 5.1.16.1, either $O^{3'}(M_+) = K$, or $O^{3'}(M_+)/O_{3'}(O^{3'}(M_+)) \cong PGL_3(4)$. In particular as Sylow 3-groups of $PGL_3(4)$ are of exponent 3 and $m_3(C_G(V_2)) = 1$, $B \in Syl_3(C_G(V_2))$. Therefore as $B \leq K$ and $C_G(V_2) \leq G_1 \leq N_G(K)$, $Y := O^{3'}(C_G(V_2)) \leq K$. Then as BT is the unique maximal subgroup of KT containing BT, and $[K, V_2] \neq 1$, we conclude $Y = O^{3'}(TB)$. Thus to complete the proof of (1) and hence of the lemma, it remains to show $X := O^{\{2,3\}}(C_G(V_2)) = 1$. As X is BT-invariant and $Aut_{BT}(K/O_2(K))$ is maximal in $Aut_{KT}(K/O_2(K))$, $X \leq C_K$. Therefore $\langle H, D_3\rangle \leq N_G(X)$, so if $X \neq 1$, then by 5.1.14.1, $D_3 \leq N_G(X) \leq M_+$, contradicting 5.1.20. This establishes (1), and completes the proof of 5.1.22. □

LEMMA 5.1.23. *$\langle V_2^{G_1}\rangle$ is abelian.*

PROOF. We specialize to the case $H_1 = G_1$, and recall Hypothesis G.2.1 is satisfied with L_2, V_2, Z_1, G_1 in the roles of "L, V, V_1, H". Our proof is by contradiction, so we assume that U is nonabelian. Then $[V_2, U] = Z_1$ using 5.1.21, so $L_2 = [L_2, U]$, and hence the hypotheses of G.2.3 are also satisfied. So setting $I := \langle U^{G_2}\rangle$, G.2.3 gives us an I-series

$$1 = S_0 \leq S_1 \leq S_2 \leq S_3 = S := O_2(I)$$

such that $S_1 = V_2$, $S_2 = U \cap U^g$ for $g \in D_3 - G_1$, $[S_2, I] \leq S_1 = V_2$, and S/S_2 is the sum of natural modules for $I/S \cong L_2(2)$ with $(U \cap S)/S_2 = C_{S/S_2}(U)$. As L_2 has at least two noncentral chief factors on V and one on $(S \cap L)/C_{S\cap L}(V)$, $m := m((U \cap S)/S_2) > 1$.

Let $G_1^* := G_1/C_{G_1}(\tilde{U})$, $W := U \cap S$, and $A := U^g \cap S$. Observe

$$\tilde{S}_2 = \widetilde{A \cap U} \leq C_{\tilde{U}}(A)$$

and $[U, a] \not\leq S_2$ for each $a \in A - S_2$. Thus as $Z_1 \leq S_2$, $S_2 = C_A(\tilde{U})$. Therefore as $m(U/(U \cap S)) = 1$ since $I/S \cong L_2(2)$,

$$m(A^*) = m(A/S_2) = m((U \cap S)/S_2) = m = m(\tilde{U}/\tilde{S}_2) - 1 \geq m(\tilde{U}/C_{\tilde{U}}(A^*)) - 1,$$

so $A^* \in \hat{\mathcal{Q}}_r(G_1^*, \tilde{U})$, where $r := (m+1)/m < 2$ as $m > 1$. Let $C_1 := C_{G_1}(K/O_2(K))$; we apply D.2.13 to G_1^* in the role of "G". By 5.1.16.1, C_1 is a $3'$-group, so as $r_{A^*,\tilde{U}} \leq r < 2$, D.2.13 says that $[F(C_1^*), A^*] = 1$. But as $G_1 \leq N_G(K)$, $F^*(G_1^*) = K^*F^*(C_1^*)$, so either A^* is faithful on K^*, or by 5.1.16.2, A^* acts nontrivially on a component $X^* \cong Sz(2^k)$ of C_1^*. Let $Y := K$ in the first case, and $Y := X$ in the second. By A.1.42.2 there is $\tilde{W} \in Irr_+(\tilde{U}, Y^*, T^*)$; set $\tilde{U}_T := \langle \tilde{W}^T\rangle$. As $Y^* = [Y^*, A^*]$, $C_A(U_T) < A$. Then by D.2.7,

$$\hat{q} := \hat{q}(Aut_{YT}(\tilde{U}_T), \tilde{U}_T) \leq r < 2.$$

Observe that Hypothesis D.3.1 is satisfied, with Y^*T^*, Y^*, $\tilde{U}_T$, $\tilde{W}$ in the roles of "M, M_+, V_M, V". So as $\hat{q} < 2$, we conclude from D.3.8 that $Y^* \not\cong Sz(2^k)$; hence $Y = K$. By construction $\tilde{U}_T$ plays the role of both "V_T" and "V_M" in Hypothesis D.3.2 and lemma D.3.4, so the hypotheses of D.3.10 are satisfied. Thus we conclude from D.3.10 that $\tilde{W} = \tilde{U}_T$. Then B.4.2 and B.4.5 show that $\hat{q} > 2$, keeping in mind that K^* is $L_3(4)$ rather than $SL_3(4)$, and $\dim(\tilde{W}) \neq 9$ as T is nontrivial on the Dynkin diagram of K^*. This contradiction completes the proof of 5.1.23. □

We are now in a position to obtain a contradiction which will establish Theorem 5.1.14. We specialize to the case $H_1 = H$. As L_2 is transitive on $V_2^\#$ and Z_1 is of order 2, G_1 is transitive on $\{V_2^g : Z_1 \leq V_2^g\}$ by A.1.7.1. So by 5.1.23, $[V_2, V_2^g] = 1$ whenever $Z_1 \leq V_2^g$. Also $C_H(\tilde{U}) = O_2(H)$, since otherwise by Coprime Action, K centralizes V_2, contrary to 5.1.22.1 as $K \not\leq M$. Further as $D_3 \leq L_2$, $O_2(\langle L_2T, H\rangle) = 1$ by 5.1.20. Hence Hypothesis F.8.1 is satisfied with Z_1 , V_2, L_2 in the roles of "V_1, V, L". As Z_1 is of order 2, Hypothesis F.9.8 is satisfied with V_2 in the role of "V_+" by Remark F.9.9). Therefore by F.9.16.3 $q(H^*, \tilde{U}) \leq 2$. However we observe that the argument at the end of the proof of 5.1.23, with H^*, $\tilde{U}$ in the roles of "G_1^*, $\tilde{U}_T$", shows that $q(H^*, \tilde{U}) > 2$.

The proof of Theorem 5.1.14 is complete.

5.2. Using weak BN-pairs and the Green Book

In this section, we continue to assume Hypothesis 5.1.8—in particular, $n(H) > 1$.

We work toward the goal of constructing a weak BN-pair of rank 2. This will be accomplished by establishing Hypothesis F.1.1. In our construction, L plays the role of "L_1" in Hypothesis F.1.1, and we choose L_2 to be a suitable subgroup of K. To be precise, if $K_1/O_2(K_1)$ is a Bender group in 5.1.10, we let $L_2 := K_1$. Otherwise $K/O_2(K) \cong (S)L_3(2^n)$ or $Sp_4(2^n)$, in which case we let P_+ be a maximal parabolic of K over $T \cap K$, and take $L_2 \in \mathcal{C}(P_+)$. Notice in either case that $T \cap L_2 \in Syl_2(L_2)$. Further $K = \langle L_2^T \rangle$ and $H \not\leq M$, so that $L_2 \not\leq M$.

In any case, $L_2/O_2(L_2)$ is a group of Lie type of Lie rank 1, and of course $L/O_2(L) \cong L_2(2^n)$ in this chapter. Next set $S := O_2(M_H) = O_2(BT)$. By 5.1.11, $S \cap K \in Syl_2(K)$, and $S \cap L \in Syl_2(L)$. Then as $S \cap K = T \cap K$, $S \cap L_2 \in Syl_2(L_2)$ by a remark in the previous paragraph. Further by 5.1.11.3:

LEMMA 5.2.1. *If $K/O_2(K)$ is not $L_3(4)$ then S acts on L_2.*

Next the Cartan group B of K lies in M, and so normalizes L; therefore to achieve condition (d) of F.1.1, we need to show that D_L acts on L_2. To show D_L acts on L_2, we first show that—modulo an exceptional case where we view L as defined over $\mathbf{F}_2$—D_L acts on K. Then we deduce that D_L acts on L_2. Eventually it turns out that $L_2 = K$.

LEMMA 5.2.2. *Either*

(1) $D_L \leq N_G(K)$, or

(2) $K/O_2(K) \cong L/O_2(L) \cong L_2(4)$, V is the sum of at most two copies of the A_5-module, and $K \leq K_Z := O^{3'}(C_G(Z))$, with $K_Z/O_2(K_Z) \cong A_7$, J_2, or M_{23}.

PROOF. Assume that neither (1) nor (2) holds. In particular $D_L \not\leq B$ as (1) fails. For $D \leq D_L$ let $X_D := \langle D, H\rangle$. Let $\mathcal{D}$ consist of those $D \leq D_L$ such that $O_2(X_D) = 1$. If $D \in \mathcal{D}$ then $D \not\leq N_G(K)$ as $O_2(K) \neq 1$ and $K \trianglelefteq H$. If $O_2(X_D) \neq 1$, then by 5.1.13, either $D \leq N_G(K)$ or the various conclusions of 5.1.13 hold, and the latter contradicts our assumption that (2) fails. Thus $O_2(X_D) \neq 1$ iff $D \leq N_G(K)$ iff $D \notin \mathcal{D}$. Finally if Δ is a collection of subgroups generating D_L, then as (1) fails, $D \not\leq N_G(K)$ for some $D \in \Delta$, so that $\Delta \cap \mathcal{D} \neq \emptyset$.

In particular $D_L \in \mathcal{D}$. We conclude from 5.1.6 that one of the four cases of 5.1.6 holds. Now in the first three cases of 5.1.6, $n = 2$ or 4. If case (4) holds, then we may take Δ to consist of D_- and D_+. However $1 \neq Z \leq O_2(X_{D_-})$ in that case by 5.1.6, so that $D_+ \in \mathcal{D}$. Therefore $n = 4$ or 8 by 5.1.6.4.

So in any case, we have $n = 2$, 4, or 8. Next let D_p denote the subgroup of D_L of order p. When $n = 2$, $D_3 = D_L$ so $D_3 \in \mathcal{D}$. When $n = 4$, $D_L = \langle D_3, D_5\rangle$, so $D_p \in \mathcal{D}$ for $p = 3$ or 5. Finally when $n = 8$, $D_+ = \langle D_3, D_5\rangle$, and we saw D_- acts on K, so again $D_p \in \mathcal{D}$ for $p = 3$ or 5. Thus in each case, $D_p \in \mathcal{D}$ for $p = 3$ or 5; choose p with this property during the remainder of the proof.

As $D_L \not\leq B$, $K/O_2(K)$ is not $L_3(4)$ by part (4) of Theorem 5.1.14. Hence S acts on L_2 by 5.2.1, and, as we observed at the beginning of this section, $S \cap K \in Syl_2(K)$ and $S \cap L \in Syl_2(L)$. Recall B normalizes $O_2(BT) = S$ and L_2. Set $G_0 := \langle D_p, L_2S\rangle$.

We first suppose that $O_2(G_0) = 1$. This gives part (e) of Hypothesis F.1.1, with D_pS and L_2 in the roles of "L_1" and "L_2". Part (f) follows from 1.1.4.5, as M and H are in $\mathcal{H}^e$ and S contains $O_2(H)$ and $O_2(M)$. To check part (c), we only need to prove that S is not normal in D_pS, since then $D_pS/O_2(D_pS) \cong L_2(2)$, D_{10}, or $Sz(2)$. But if $S \trianglelefteq SD_p$, then as $S \trianglelefteq T$ and $S \cap K \in Syl_2(K)$, Theorem 3.1.1 says $1 \neq O_2(\langle D_pT, H\rangle) \leq O_2(H) \leq O_2(BT) = S \leq G_0$ using A.1.6, contrary to our assumption that $O_2(G_0) = 1$. The remaining parts of Hypothesis F.1.1 are easily verified.

Now by F.1.9, $\alpha := (D_pSB_2, SB_2, SL_2)$ is a weak BN-pair of rank 2, where $B_2 := B \cap L_2$. Indeed since S is self-normalizing in SD_p, α is described in F.1.12. As we saw in 5.1.18, when $p = 5$ the amalgams in F.1.12 have solvable parabolics, and so are ruled out as L_2 is not solvable. So $p = 3$ and $D_3S/O_2(D_3S) \cong L_2(2)$; then as L_2 is not solvable, we conclude that α is of type J_2, $Aut(J_2)$, ${}^3D_4(2)$, or $U_4(2)$. In each case, $Z(S)$ is of order 2, and is centralized by one of the parabolics in the amalgam.

Suppose first that α has type $U_4(2)$. Then D_3S is the solvable parabolic centralizing $Z(S)$, with $[O_2(SD_3), D_3] \cong Q_8^2$, and L_2 is an A_5-block with $O_2(L_2) = F^*(L_2S)$. Thus $O_2(L_2)$ is the unique 2-chief factor for L_2S, so $K = L_2$. Also $C_S(L_2) = 1$, so $Z(H) = 1$. As $Z(H) = 1$, from the discussion above we are in case (3) of 5.1.6, so that $[V, L]$ is the A_5-module for $L/O_2(L)$; in particular $n = 2$ and $D_L = D_3 \not\leq B$. As $[D_3, O_2(D_3S)] \cong Q_8^2$, L also is an A_5-block. But then as $D_3 < D_3B$ and $S = O_2(BT)$, $1 \neq C_{BD_3}(L) \leq O(LTB)$, contradicting $F^*(LTB) = O_2(LTB)$).

Thus we may suppose α is of type J_2, $Aut(J_2)$, or ${}^3D_4(2)$. In each case L_2S is the parabolic centralizing $Z(S)$, so as $Z = \Omega_1(Z(T)) \leq Z(S)$ and $Z(S)$ is of order 2, we conclude $Z(S) = Z$ centralizes $\langle L_2, T\rangle = H$. Again in each case $Q := O_2(L_2S)$ is extraspecial and L_2 is irreducible on Q/Z; so as $H \in \mathcal{H}^e$, $Q = O_2(H)$ using A.1.6. Arguing as above, as Q/Z is the unique noncentral factor for L_2 and $Z \leq \Phi(Q)$, again $K = L_2$. Then $B \leq K = L_2$, so as $S = O_2(BT)$, α is not the $Aut(J_2)$-amalgam. Now either α is of type ${}^3D_4(2)$ and $O^{2'}(Aut(K)) = Inn(K)$, or α is of type J_2 and $O^{2'}(Aut(K)) = Aut(K) \cong S_5/E_{16}$. So either $T = S \leq K$; or α is the J_2-amalgam, $|T : S| = 2$, and (D_3TB, TB, TK) is a weak BN-pair extending α, and hence is the $Aut(J_2)$-amalgam. Therefore if α is of type J_2, then T is a Sylow 2-subgroup of either J_2 or $Aut(J_2)$, so $m_2(T) = 4$ and T has no normal E_{16}-subgroup. This is impossible as $m(V) \geq 4$ from 5.1.6. Thus α is of type ${}^3D_4(2)$ with $S = T$, so $K/O_2(K) \cong L_2(8)$, B is of order 7, and T is a Sylow 2-subgroup of ${}^3D_4(2)$. We are free to choose V to be $\langle Z^L \rangle$; thus $Z \leq V$, so $V_2 := \langle Z^{D_3} \rangle \leq V$. From the structure of α, $V_2 \leq C_T(B) \cong D_8$. As B acts on L and Z, B acts on $\langle Z^L \rangle = V$. Therefore $V_2 = C_V(B)$ and in particular $[B, V] \neq 1$, so B is faithful on $L/O_2(L)$. This is impossible as $n = 2$, 4, or 8 and B acts on $S \cap L = T \cap L$ with $|B| = 7$.

This contradiction shows that $O_2(G_0) \neq 1$. Let $T_0 := N_T(L_2)$. As TB acts on D_pS and S, and T_0B acts on L_2, T_0B acts on G_0; hence $O_2(G_0T_0B) \neq 1$. Thus as $O_2(X_{D_p}) = 1$ and $D_p \leq G_0$, $H \not\leq G_0T_0$; hence $T_0 < T$ and $L_2 < K$. Therefore either case (1) of 5.1.10 holds with $L_2 = K_1 < K$, or case (3) holds with $L_2 < K_1 = K$. In either case $L_2 < K$, $T_0 < T$, and $K = \langle L_2, L_2^t \rangle$ for $t \in T - T_0$. Furthermore as T acts on D_pT_0, $(L_2^t)^{D_pT_0} = (L_2^{D_pT_0})^t$, so as $D_p \not\leq N_G(K)$ it follows that $D_p \not\leq N_G(L_2)$.

Embed T_0 in $T_1 \in Syl_2(G_0T_0B)$. As $|T : T_0| = 2$, $|T_1 : T_0| \leq 2$. As $S \leq T_0$, $N_G(T_0) \leq M$ by 4.3.17; hence T_1 acts on $T_0 \cap L$, and then as $D_p(T_0 \cap L) \trianglelefteq N_M(T_0 \cap L)$, T_1 acts on D_pT_0.

By Theorem 3.1.1, applied with T_0, $N_G(T_0)$, H in the roles of "R, M_0, H", we conclude $O_2(X) \neq 1$, where $X := \langle N_G(T_0), H \rangle$. Now $K_1 \in \mathcal{L}(X, T)$ and $T \in Syl_2(X)$, so by 1.2.4, $K_1 \leq K_X \in \mathcal{C}(X)$, and we set $K_+ := \langle K_X^T \rangle$. Recalling that $L_2 < K$, we conclude from A.3.12 and 1.2.8 that either $K = K_+ \trianglelefteq X$, or $K_1/O_2(K_1) \cong L_2(4)$ and $K_1 < K_X$, with $K_X \neq K_X^t$ for $t \in T - T_0$ and $K_X/O_2(K_X) \cong J_1$ or $L_2(p)$. In any case, $T \cap K_+ = S \cap K_+ = T_1 \cap K_+$.

Suppose that $T_0 < T_1$. Set $H_0 := \langle L_2, T_1 \rangle$ and $K_0 := \langle L_2^{T_1} \rangle$. As $T, T_1 \in Syl_2(X)$, and $T \cap K_+ = T_1 \cap K_+$, $K = \langle L_2^T \rangle = \langle L_2^{T_1} \rangle$ from the structure of K_+T. Thus $K \in \mathcal{L}(H_0, T_1)$, $K = \langle L_2^{T_1} \rangle \leq G_0T_0B$, and applying 5.1.13 with G_0T_0B, T_1, H_0 in the roles of "X, T, H", we conclude that either $K/O_2(K) \cong L_2(4)$, or D_p acts on K. The first case is impossible as we saw $L_2 < K$, and the second is impossible as we chose p so that D_p does not act on K.

This contradiction shows that $T_1 = T_0 \in Syl_2(G_0T_0)$. Now we can repeat parts of the proof of 5.1.13 with G_0T_0B, L_2T_0, D_p in the roles of "X, H, D" to obtain a contradiction: We know $G_0T_0B \in \mathcal{H}(L_2T_0)$ and $D_p \not\leq N_G(L_2)$ from earlier reductions. Then $L_2 < \hat{L}_2 \in \mathcal{C}(G_0T_0)$ using 1.2.4, and arguing as in 5.1.12 with L_2 in the role of "K_1", one of conclusions (2)–(5) of that result must hold. Indeed as L_2 is normalized by the Sylow group T_0, conclusion (2) of that result cannot arise. Then the argument in the second paragraph of the proof of 5.1.13 shows $\hat{L}_2/O_2(\hat{L})$ is not of Lie type in characteristic 2 of Lie rank 2, so that conclusions (3) and (4) of 5.1.12 are ruled out. Hence we are reduced to case (5) of 5.1.12, and in particular, $L_2/O_2(L_2) \cong L_2(4)$, with the embedding $L_2 < \hat{L}_2$ described in A.3.14. We saw $K/O_2(K) \not\cong L_3(4)$, so by 5.1.10, $K/O_2(K)$ is $Sp_4(4)$ or $L_2(4) \times L_2(4)$, and in either case $B \cong E_9$. Next proceeding as in the proof of 5.1.13 with D_p in the role of "D", we obtain $p = 3$; notice that here $\hat{L}_2/O_2(\hat{L}_2)$ is not J_1, since here $p = 3$ or 5 rather than 7. Since B acts on D_3 by 5.1.5.2, B centralizes D_3. But also $B \leq N_G(L_2)$ so $D_3 \not\leq B$; hence $M \geq D_3B \cong E_{27}$, a contradiction as M is an SQTK-group. This completes the proof of 5.2.2. □

We now state the main result of this chapter:

THEOREM 5.2.3. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}^*_f(G,T)$ with $L/O_2(L) \cong L_2(2^n)$ and $L \trianglelefteq M \in \mathcal{M}(T)$. In addition assume $H \in \mathcal{H}_*(T,M)$ with $n(H) > 1$, let $K := O^2(H)$, $Z := \Omega_1(Z(T)$, and $V \in \mathcal{R}_2(LT)$ with $[V,L] \neq 1$. Then one of the following holds:*

(1) $n = 2$, V is the sum of at most two copies of the A_5-module for $L/O_2(L) \cong A_5$, and $K \leq K_Z \in \mathcal{C}(C_G(Z))$. Further either $K/O_2(K) \cong L_2(4)$ with $K_Z/O_2(K_Z) \cong A_7$, J_2, or M_{23}, or $K = K_Z$ and $K/O_2(K) \cong L_3(4)$.

(2) $G \cong M_{23}$.

(3) G is a group of Lie type of characteristic 2 and Lie rank 2, and if G is $U_5(q)$ then $q = 4$.

Note that conclusions (2) and (3) of Theorem 5.2.3 are also conclusions in our Main Theorem. Thus once Theorem 5.2.3 is proved, whenever $L \in \mathcal{L}^*_f(G,T)$ is T-invariant with $L/O_2(L) \cong L_2(2^n)$, we will be able to assume that either conclusion (1) of Theorem 5.2.3 holds, or $n(H) = 1$ for each $H \in \mathcal{H}_*(T, N_G(L))$. The treatment of these two remaining cases is begun in the following chapter 6, and eventually completed in Part 5, devoted to those $L \in \mathcal{L}^*_f(G,T)$ with $L/O_2(L)$ defined over $\mathbf{F}_2$.

5.2.1. Determining the possible amalgams. The proof of Theorem 5.2.3 will not be completed until the final section 5.3 of this chapter. In this subsection, we will produce a weak BN-pair α, and use the Green Book [**DGS85**] to identify α up to isomorphism of amalgams. This leaves two problems: First, show that the subgroup G_0 generated by the parabolics of α is indeed a group of Lie type. Second, show that $G_0 = G$. In one exceptional case, G_0 is proper in G; the second subsection will give a complete treatment of that branch of the argument, culminating in the identification of G as M_{23}.

Assume the hypotheses of Theorem 5.2.3. Notice that Hypothesis 5.1.8 holds, since in Theorem 5.2.3 we assume $n(H) > 1$. During the proof of Theorem 5.2.3, write D for D_L.

Notice that if $K/O_2(K) \cong L_3(4)$, then conclusion (1) of Theorem 5.2.3 holds by Theorem 5.1.14. Thus we may assume during the remainder of the proof of Theorem 5.2.3 that $K/O_2(K)$ is not $L_3(4)$. Therefore by 5.1.11.3, S acts on the rank one parabolics of K, and hence on the group L_2 defined at the start of the section.

Next if $D \not\leq N_G(K)$, then conclusion (2) of 5.2.2 is satisfied, so again conclusion (1) of Theorem 5.2.3 holds. Thus we may also assume during the remainder of the proof that D acts on K; we will show under this assumption that conclusion (2) or (3) of Theorem 5.2.3 holds. The following consequences of these observations are important in producing our weak BN-pair:

LEMMA 5.2.4. *(1) $D \leq N_G(K)$.*

(2) $D \leq N_G(B)$ and $B \leq N_G(D)$.

(3) $B \leq N_G(S)$, $D \leq N_G(S \cap L_2)$, and $DS = SD$.

(4) DSB acts on L_2.

PROOF. By construction in Notation 5.1.9, $B \leq N_G(D)$. Part (1) holds by assumption, and says D acts on $M_K := M \cap K = (S \cap K)B$. Thus D acts on $DB \cap (S \cap K)B = B$, completing the proof of (2). Further as the Borel subgroup M_K is 2-closed by 5.1.10, D acts on $S \cap K$. As D acts on $S \cap K$ and there are at most two rank one parabolics of K over $S \cap K$, D acts on each such parabolic. So as $L_2 = P^\infty$ for one of these parabolics, D acts on L_2 and hence also on $S \cap L_2$.

By definition of S, $S = O_2(BT)$, so B acts on S. As $N_L(S \cap L) = (S \cap L)D$, $DS = SD$, completing the proof of (3). As B acts on SD, DSB is a group. By 5.2.1, S acts on L_2, while by definition B is a Cartan subgroup acting on L_2. This completes the proof of (4). □

We now verify that Hypothesis F.1.1 is satisfied with L, L_2, S in the roles of "L_1", "L_2", "S". Set $B_2 := B \cap L_2$, $G_1 := LSB_2$, $G_2 := DSL_2$, and $G_{1,2} := G_1 \cap G_2$. As $L \trianglelefteq M$ and B_2 normalizes S by 5.2.4.3, G_1 is a subgroup of G with $L \trianglelefteq G_1$. Again using 5.2.4, G_2 is a subgroup of G with $L_2 \trianglelefteq G_2$. Thus $L_i = G_i^\infty$ as DSB is solvable. Notice conditions (a), (b), and (c) of F.1.1 follow from remarks at the beginning of the section, together with the fact that S acts on L_2. Further condition (d) of F.1.1 holds as $N_{L_j}(S \cap L_j) \leq DSB \leq G_i$, and we saw $L_i \trianglelefteq G_i$. Condition (f) follows from 1.1.4.5, since $G_1 \leq M$, $G_2 \leq N_G(K)$, and S contains $O_2(M)$ and $O_2(H)$, and hence contains $O_2(N_G(K))$ using A.1.6. Finally we establish (e) of F.1.1 in the following lemma:

LEMMA 5.2.5. $O_2(\langle G_1, G_2 \rangle) = 1$.

PROOF. Let $G_0 := \langle G_1, G_2 \rangle$. By 4.3.2, $M = !\mathcal{M}(L)$, so as $L_2 \not\leq M$, $O_2(G_0) = 1$. □

We now use the Green Book [**DGS85**] (via an appeal to F.1.12) to determine the possible amalgams that can arise; these will subsequently lead us to the "generic" quasithin groups in conclusion (3) of Theorem 5.2.3, and to M_{23} in conclusion (2) of 5.2.3.

PROPOSITION 5.2.6. *$\alpha := (G_1, G_{1,2}, G_2)$ is a weak BN-pair of rank 2. Further $L_2 = K = G_2^\infty$, with $O_2(G_i) = O_2(L_i)$ for $i = 1$ and 2, and one of the following holds:*

(1) α is the $L_3(2^n)$-amalgam and L and K are $L_2(2^n)$-blocks.

(2) α is the $Sp_4(2^n)$-amalgam and L and K are $L_2(2^n)$-blocks.

(3) α is the $G_2(q)$-amalgam for $q = 2^n$, $L/O_2(L) \cong K/O_2(K) \cong L_2(q)$, $O_2(K) \cong q^{1+4}$, and $|O_2(L)| = q^5$.

(4) α is the ${}^3D_4(q)$-amalgam for $q = 2^n$, $L/O_2(L) \cong L_2(q)$, $|O_2(L)| = q^{11}$, $K/O_2(K) \cong L_2(q^3)$, and $O_2(K) \cong q^{1+8}$.

(5) α is the ${}^2F_4(q)$-amalgam for $q = 2^n$, $L/O_2(L) \cong L_2(q)$, $|O_2(L)| = q^{11}$, $K/O_2(K) \cong Sz(q)$, and $|O_2(K)| = q^{10}$.

(6) $n > 2$ is even, α is the $U_4(q)$-amalgam for $q = 2^{n/2}$ or its extension of degree 2, L is an $O_4^-(q)$-block, $K/O_2(K) \cong L_2(q)$, and $O_2(K) \cong q^{1+4}$.

(7) $n = 4$, α is the $U_5(4)$-amalgam, $L/O_2(L) \cong L_2(16)$, $|O_2(L)| = 2^{16}$, $K/O_2(K) \cong SU_3(4)$, and $O_2(K) \cong 4^{1+6}$.

Moreover $O_2(KT) = O_2(KS)$, and either

(a) $S \leq L_i$ and $O_2(L_iS) = O_2(L_i)$ for $i = 1$ and 2, or

(b) α is an extension of the $U_4(q)$ amalgam of degree 2 and $O_2(KS)$ is the extension of $O_2(K)$ by an involution t such that $C_K(t) \cong P^\infty$ for P a maximal parabolic of $Sp_4(q)$.

PROOF. We have already verified Hypothesis F.1.1, so by F.1.9, α is a weak BN-pair of rank 2. By 5.2.4.2, $B_2 \leq N_G(S)$, so that we may apply F.1.12 to determine α. As L_1 and L_2 are not solvable, cases (8)–(13) of F.1.12.I are ruled out. Together with F.1.12.II, this shows that $S \leq L_i$ and hence also $O_2(L_i) = O_2(G_i)$ for $i = 1$ and 2, unless possibly α is the extension of the $U_4(q)$ amalgam of degree 2. In the latter case by F.4.29.5, (II.i) fails only weakly, in the sense that $O_2(L) = O_2(LS)$ and $|S : S \cap L| = |S : S \cap L_2| = |O_2(L_2S) : O_2(L_2)| = 2$. Further by F.4.29.4, $O_2(G_i) = O_2(L_i)$. Now the remaining amalgams in cases (1)–(7) of F.1.12.I are those given in 5.2.6; notice that the numbering convention for L_1 and L_2 in F.1.12 differs in some cases from that used here in 5.2.6. We are using the facts that $L/O_2(L) \cong L_2(2^n)$ and $1 \neq [Z, L]$.

We next show that $L_2 = K$; that is, we eliminate cases (1) and (3) of 5.1.10. First suppose $L_2 = K_1 < K$. Then for $t \in T - N_T(K_1)$, $O_2(L_2S)$ contains $S \cap L_2^t$ with $O_2(L_2) \cap L_2^t \leq O_2(L_2^t)$ and $|S \cap L_2^t : O_2(L_2^t)| > 2$; therefore $|O_2(L_2S) : O_2(L_2)| > 2$, contrary to an earlier observation. So we may suppose instead that $K/O_2(K)$ is $(S)L_3(2^k)$ or $Sp_4(2^k)$. We recall in this case that $L_2 = P_+^\infty$ for a maximal parabolic P_+ of K. Thus $L_2/O_2(L_2) \cong L_2(2^k)$, $O_2(L_2)O_2(K)/O_2(K)$ has a natural chief factor, and there is at least one more noncentral 2-chief factor for L_2 in $O_2(K)$. Thus L_2 has at least two noncentral 2-chief factors, so that α is not the $L_3(q)$ or $Sp_4(q)$-amalgam. As $L_2/O_2(L_2) \cong L_2(2^k)$, rather than $Sz(q)$ or $SU_3(q)$, α is not the ${}^2F_4(q)$ or $U_5(4)$-amalgam. If α is the amalgam for $G_2(q)$ or ${}^3D_4(q)$, then L_2D has just one noncentral 2-chief factor, and that factor is *not* natural. This leaves the $U_4(q)$-amalgam, where L_2 has two natural 2-chief factors on the Frattini quotient of $O_2(L_2)$, but L_2D is irreducible on the Frattini quotient. However D acts on $O_2(K)$ and hence on the 2-chief factor for $O_2(L_2)$ in $O_2(K)$. This contradiction shows that $L_2 = K$, completing the proof of the claim.

Recall $S = O_2(M_H)$, so that $O_2(KT) \leq S$ by A.1.6, and hence $O_2(KT) = O_2(KS)$. By F.1.12.II, $O_2(L) = O_2(LS)$, and either $O_2(K) = O_2(KS)$ or α is the extension of the $U_4(q)$-amalgam of degree 2. In the latter case by F.4.29.5, $O_2(KS) = O_2(K)\langle t\rangle$, where t induces a graph-field automorphism on $U_4(q)$, and hence $C_{U_4(q)}(t) \cong Sp_4(q)$, so that $C_K(t)$ is as claimed. This completes the proof of 5.2.6. □

In the remainder of this section, if α is the extension of degree 2 of the $U_4(q)$-amalgam, we replace α by its subamalgam of index 2. Thus in effect, we are replacing $S = O_2(BT)$ by $S \cap L$ of index 2 in S. Subject to this convention:

LEMMA 5.2.7. *(1) $\alpha := (G_1, G_{1,2}, G_2)$ is the amalgam of $L_3(q)$, $Sp_4(q)$, $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, $U_4(q)$, with $q > 2$, or $U_5(4)$.*

(2) $G_i = L_iBD$ and $G_{1,2} = SBD$, where $L_1 = L$, $L_2 = K$; and $S = T \cap L_1 = T \cap L_2 = O_2(G_{1,2})$.

(3) $O_2(L_i) = O_2(G_iT)$.

PROOF. Parts (1) and (2) are immediate from 5.2.6 and the convention for $U_4(q)$. Let $S_0 := O_2(BT)$. By 5.2.6, $O_2(G_iS_0) = O_2(L_i)$ for $i = 1, 2$. Further as $B \leq G_i$, $O_2(G_iT) \leq O_2(BT) = S_0$ using A.1.6, so $O_2(G_iT) \leq O_2(G_iS_0) = O_2(L_i)$ and hence $O_2(G_iT) = O_2(L_i)$, establishing (3). □

Recall the notion of a completion of an amalgam from Definition F.1.6. Let $G(\alpha)$ denote the simple group of Lie type for which there is a completion $\xi : \alpha \to G(\alpha)$; that is, α is an amalgam of type $G(\alpha)$. To establish conclusion (3) of Theorem 5.2.3, we must show that $G \cong G(\alpha)$. Let $2m(\alpha)$ be the order of the Weyl group of $G(\alpha)$.

LEMMA 5.2.8. *Either*

(1) $K \in \mathcal{L}^*(G, T)$, *or*

(2) α *is the* $L_3(4)$*-amalgam, and* $K < \hat{K} \in \mathcal{L}^*(G, T)$ *with* $\hat{K}$ *an exceptional* A_7*-block.*

PROOF. Assume $K < \hat{K} \in \mathcal{L}^*(G, T)$ and let $Q := O_2(K)$ and $\hat{K}^* := \hat{K}/O_2(\hat{K})$. Then K/Q is not $SU_3(4)$, since in that event $K \in \mathcal{L}^*(G, T)$ by 1.2.8.4. Thus we may assume α is not of type $U_5(4)$.

Recall $\hat{K} \in \mathcal{H}^e$ by 1.1.3.1, so $1 \neq [O_2(\hat{K}), K] \leq K \cap O_2(\hat{K})$. If $K/Q \cong Sz(q)$, then α appears in case (5) of 5.2.6, and $\hat{K}^* \cong {}^2F_4(q)$ by 1.2.4 and A.3.12. But then K is isomorphic to its image K^* in $\hat{K}^*$, so $K \cap O_2(\hat{K}) = 1$, contrary to our earlier observation. Thus we have eliminated the case where α is the ${}^2F_4(q)$-amalgam.

If α is the $L_3(q)$ or $Sp_4(q)$ amalgam, then K is an $L_2(q)$-block, so it has a unique noncentral 2-chief factor, and hence the same holds for $\hat{K}$, with $Q \leq O_2(\hat{K})$. By 5.2.6, $Q = O_2(KT)$, so $Q = O_2(\hat{K})$. Therefore $K^* \cong L_2(q)$ is a T-invariant quasisimple subgroup of $\hat{K}^*$, so by A.3.12, $q = 4$; and then by A.3.14, $\hat{K}^*$ is A_7, $\hat{A}_7$, J_1, $L_2(25)$, or $L_2(p)$, $p \equiv \pm 3 \mod 8$ and $p^2 \equiv 1 \mod 5$. As α is of type $L_3(4)$ or $Sp_4(4)$, Q is an extension of a natural module for $K/Q \cong L_2(4)$ and $m(Q) = 4$ or 6. As $\hat{K} \in \mathcal{H}^e$ and $\hat{K}^*$ is quasisimple, $\hat{K}^*$ is faithful on Q, so that $\hat{K}^* \leq GL(Q)$. Comparing the possiblities for K^* listed above to those in G.7.3, we conclude from G.7.3 that $\hat{K}^* \cong A_7$, and then as $m(Q) = 4$ or 6, $\hat{K}$ is an A_7-block or an exceptional A_7-block. In the former case, the noncentral chief factor for K on Q is not the $L_2(4)$-module, so the latter case holds, forcing α to be the $L_3(4)$-amalgam. Thus (2) holds in this case.

Suppose α is the $U_4(q)$-amalgam. From 5.2.6, $K/Q \cong L_2(q)$ for $q = 2^{n/2} > 2$ and Q is special of order q^{1+4} with K trivial on $Z(Q)$. Further by 5.2.6, either $Q = O_2(KT)$, or $O_2(KT) = Q\langle t \rangle$ where t is an involution with $C_Q(t) \cong E_{q^3}$.

We claim $Q \trianglelefteq \hat{K}$, so assume otherwise. Suppose first that $Q \leq R := O_2(\hat{K})$. Then as $R \leq O_2(KT)$, and $Q < R$ by assumption, $R = O_2(KT) = Q\langle t \rangle$. But now $Z(Q) = Z(R) \trianglelefteq \hat{K}$, and $Q/Z(Q) = J(R/Z(Q))$, so $Q \trianglelefteq \hat{K}$, contrary to assumption. Thus $Q \not\leq R$, so as K has two natural chief factors on $Q/Z(Q)$ and $[R, K] \neq 1$, we conclude $(Q \cap R)Z(Q)/Z(Q)$ is one of these chief factors. Thus $Z(Q) = [Q \cap R, Q] \leq R$ and $Q \cap R \cong E_{q^3}$. Again as $R \leq O_2(KT)$, either $R = Q \cap R$ or $|R : Q \cap R| = 2$. In the latter case $Q \cap R = [Q, t] = C_Q(t)$, so $R = (Q \cap R)\langle t \rangle$.

In any case K^* is an $L_2(q)$-block with $|O_2(K^*)| = q^2$. The only possibilities for such an embedding in A.3.12 are that $\hat{K}^* \cong (S)L_3(q)$, or $q = 4$ and $\hat{K}^* \cong M_{22}$, $\hat{M}_{22}$, or M_{23}. The last three cases are impossible, as those groups are of order divisible by 11, a prime not dividing the order of $GL_7(2)$. Thus $\hat{K}^* \cong SL_3(q)$ and $[R, \hat{K}]$ is the natural module for $\hat{K}^*$, so $[R, \hat{K}] = [R, K] = Q \cap K$. However as α is the $U_4(q)$-amalgam, $J(T) = O_2(L)$ is normal in LT, so $N_G(J(T)) \leq M = !\mathcal{M}(LT)$. From the action of $\hat{K}$ on R, $K_1 := N_{\hat{K}}(J(T))$ is the scond maximal parabolic of $\hat{K}$ over $\hat{K} \cap T$. Thus as $T \cap L = T \cap K$ by 5.2.7.2, $K_1^\infty = [K_1^\infty, K \cap T] \leq L$, and then as $|L| = |K_1^\infty|$, $L \leq \hat{K}$, contradicting $M = !\mathcal{M}(LT)$. This contradiction completes the proof of the claim.

Finally we treat the case where α is the $U_4(q)$-amalgam and $Q \trianglelefteq \hat{K}$, along with the remaining two cases where α is the amalgam of $G_2(q)$ or ${}^3D_4(q)$. In these last two cases Q is special and K is irreducible on $Q/Z(Q)$, so as in the earlier cases of the $L_3(q)$ and $Sp_4(q)$ amalgams, there is a unique noncentral 2-chief factor under the extension of K by a Cartan subgroup, and again we get $O_2(\hat{K}) = Q$. Thus in each of our three cases, $Q \trianglelefteq \hat{K}$, so $\hat{K} \in \mathcal{C}(N_G(Q))$ by 1.2.7 as $\hat{K} \in \mathcal{L}^*(G,T)$. Further $K^* \cong L_2(q)$ when α is the amalgam for $U_4(q)$ or $G_2(q)$, and $K^* \cong L_2(q^3)$ when α is the ${}^3D_4(q)$-amalgam. As above, A.3.12 gives a proper extension with "$O_2(B) = 1$" only when "B" is $L_2(4)$. This eliminates the ${}^3D_4(q)$ amalgam, and forces α to be the amalgam of $U_4(4)$ or $G_2(4)$. Therefore $Q \cong 4^{1+4}$ and there is X of order 3 in $C_{D_L B}(K/Q)$ with $Q/\Phi(Q) = [Q/\Phi(Q), X]$, so X acts on $\hat{K} \in \mathcal{C}(N_G(Q))$ by 1.2.1.3. But as in our application of A.3.12 above, $\hat{K}^* \cong A_7$, $\hat{A}_7$, J_1, $L_2(25)$, or $L_2(p)$ for $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$, and X centralizes $A_5 \cong K^* \leq \hat{K}^*$, so we conclude from the structure of $Aut(\hat{K}^*)$ that X centralizes $\hat{K}^*$. Thus $\hat{K}^*$ is not A_7, for otherwise $m_3(\hat{K}X) = 3$, contradicting $N_G(\hat{K})$ an SQTK-group. Further as $Q/\Phi(Q) = [Q/\Phi(Q), X]$ is of rank 8, and 8 is not divisible by 3, $\hat{K}^*$ is not $\hat{A}_7$. Finally G.7.2 eliminates the remaining possiblities for $\hat{K}^*$. This completes the proof of 5.2.8. □

Conclusions (1) and (2) of 5.2.8 will lead to conclusions (3) and (2) of Theorem 5.2.3, respectively, so we adopt notation reflecting the groups arising in those conclusions. Namely we define G to be of *type* $X_r(q)$ if α is the $X_r(q)$-amalgam and $K \in \mathcal{L}^*(G,T)$. Define G to be of *type* M_{23} if α is the $L_3(4)$-amalgam and $K \notin \mathcal{L}^*(G,T)$. Thus in this language, we can summarize what we have accomplished in 5.2.6 and 5.2.8:

THEOREM 5.2.9. *One of the following holds:*
(1) G is of type $L_3(q)$, $Sp_4(q)$, $G_2(q)$, ${}^3D_4(q)$, or ${}^2F_4(q)$, for some even $q > 2$.

(2) $n > 2$ is even and G is of type $U_4(2^{n/2})$.
(3) G is of type $U_5(4)$.
(4) G is of type M_{23}.

5.2.2. Characterizing $\mathbf{M_{23}}$. The remainder of this section is devoted to a proof that:

THEOREM 5.2.10. *If G is of type M_{23} then G is isomorphic to M_{23}.*

The proof of Theorem 5.2.10 involves a short series of reductions. Assume G is of type M_{23}. Then by 5.2.8, α is the $L_3(4)$-amalgam and $K < \hat{K} \in \mathcal{L}^*(G,T)$ with $\hat{K}$ an exceptional A_7-block. Let $M_2 := \hat{K}$, $M_1 := M$, and $M_{1,2} := M_1 \cap M_2$. Set $V_i := O_2(M_i)$, $V := V_1$, and $U := V_2$. Then $V \cong U \cong E_{16}$ with $M_2/U \cong A_7$. Hence we can represent M_2/U on $\Omega = \{1, \ldots, 7\}$ so that T has orbits $\{1,2,3,4\}$, $\{6,7\}$, and $\{5\}$ on Ω. Indeed:

LEMMA 5.2.11. *(1) H is the global stabilizer in M_2 of $\{6,7\}$.*
(2) $M_{1,2}$ is the global stabilizer in M_2 of $\{5,6,7\}$.
(3) $M/V \cong \Gamma L_2(4)$.
(4) $M_2 \in \mathcal{M}(T)$.

(5) $|T : S| = 2$.

PROOF. Let $M_2^* := M_2/U$. There is a unique T^*-invariant subgroup $K_T^* \cong A_5$ of M_2^*, and $K_T^* T^*$ is the global stabilizer in M_2^* of $\{6, 7\}$, so (1) holds. Then VU/U is the 4-group with fixed-point set $\{5, 6, 7\}$ and $N_{M_2}(VU) = N_{M_2}(V) = M_{1,2}$ as $M \in \mathcal{M}(T)$, so (2) holds.

Let $M_2 \leq M_0 \in \mathcal{M}(T)$. By 5.2.8, $M_2 \in \mathcal{L}^*(G, T)$, so $M_2 \trianglelefteq M_0$ by 1.2.7.3. Then $U = O_2(M_2) \leq O_2(M_0)$, so as $T \leq M_2$, $U = O_2(M_0)$ by A.1.6. As $M_0 \in \mathcal{H}^e$, $M_0/U \leq GL(U)$, so as M_2/U is self-normalizing in $GL(U)$, $M_0 = M_2$, proving (4).

As $V = O_2(LT)$, $O_2(M) = V = C_G(V)$ by 3.2.11, so $M/V \leq GL(V)$. Next $UV \in Syl_2(L)$, so by a Frattini Argument, $M = LN_M(UV) \geq LN_M(U) = LM_{1,2}$ using (4). From the structure of M_2, $M_{1,2}/V$ is isomorphic to a Borel group of $\Gamma L_2(4)$, so $LM_{1,2}/V = N_{GL(V)}(L/V)$ as $N_{GL(V)}(L/V) \cong \Gamma L_2(4)$. Then as $L \trianglelefteq M$, (3) holds, and (3) implies (5). □

LEMMA 5.2.12. *(1)* $Z(T) = \langle z \rangle$ *is of order 2.*
(2) $C_G(z) = C_{M_2}(z)$ *is an* $L_3(2)$*-block.*
(3) M_2 *is transitive on* $U^\#$.
(4) U *is a TI-set in* G.

PROOF. Parts (1) and (3) are easy consequences of the fact that M_2 is an exceptional A_7-block containing T. As another consequence, $Y := C_{M_2}(z)$ is an $L_3(2)$-block. Let $G_z := C_G(z)$ and $G_z^* := G_z/\langle z \rangle$. As $T \leq G_z$, $F^*(G_z) = O_2(G_z)$ by 1.1.4.6, so $F^*(G_z^*) = O_2(G_z)^*$ by A.1.8. Thus as $U = O_2(Y) \geq O_2(G_z)$ by A.1.6, and Y is irreducible on U^*, $U = O_2(G_z)$. Thus $G_z \leq N_G(U) = M_2$ using 5.2.11.4. Therefore (2) holds. Then (2), (3), and I.6.1.1 imply (4). □

LEMMA 5.2.13. *G has one conjugacy class of involutions.*

PROOF. All involutions of V are conjugate under M and hence fused into $U \cap V$. Similarly all involutions in U are conjugate under M_2, so as U and V are the maximal elementary abelian subgroups of UV, all involutions in UV are fused in G. From the structure of M_2, each involution in M_2 is fused into UV in M_2. So the lemma holds, as M_2 contains a Sylow 2-group T of G. □

LEMMA 5.2.14. *(1) G is transitive on its elements of order* 3 *which centralize involutions.*
(2) All elements of order 3 *in* $M_1 \cup M_2$ *are conjugate in* G.

PROOF. By 5.2.12.2, $C_G(z)$ has one class of elements of order 3, so 5.2.13 implies (1). Next M_2 has two classes of elements of order 3, those with either 1 or 2 cycles of length 3 on Ω. The first class centralizes an involution in M_2/U and hence has centralizer of even order. The second class centralizes an involution in U. Thus (1) implies all elements of order 3 in M_2 are conjugate in G. Then as $M_{1,2}$ contains a Sylow 3-group of M_1 and M_2, (2) holds. □

LEMMA 5.2.15. *Let* $X \in Syl_3(C_M(L/O_2(L))$. *Then* $N_G(X) = N_M(X) \cong \Gamma L_2(4)$.

PROOF. First $N_M(X) \cong \Gamma L_2(4)$. On the other hand by 5.2.14, X is conjugate to $Y \leq C_G(z)$ and $C_G(Y\langle z \rangle) = Y \times C_U(Y) \cong \mathbf{Z}_3 \times E_4$. Let $G_Y := C_G(Y)$ and $G_Y^* := G_Y/Y$. Then $C_{G_Y^*}(z^*) \cong E_4$, and as $C_M(X)$ is not 2-closed, neither is G_Y^*. Thus by Exercise 16.6.8 in [**Asc86a**], $G_Y^* \cong A_5$. Therefore $|C_M(X)| = |G_Y|$, so $C_M(X) = C_G(X)$. Then as $|N_M(X) : C_M(X)| = 2 = |Aut(X)|$, the lemma follows. □

Recall the definition of the subgroups G_1 and G_2 in our amalgam α from the previous subsection, and let $G_0 := \langle G_1, G_2 \rangle$.

LEMMA 5.2.16. *(1)* $G_0 \cong L_3(4)$.
(2) G_0T *is* G_0 *extended by a field automorphism.*

PROOF. Notice in the $L_3(4)$-amalgam that we have $B = D = BD$. Thus to prove (1), it suffices by F.4.26 to show that there exist involutions $s_i \in N_{L_i}(B)$, such that $|s_1s_2| \leq 3$. Then (2) follows from (1), since T acts on G_i, with $|T : S| = 2$ by 5.2.11.5, and $O_2(L_iT) = O_2(L_i)$ by 5.2.7.3. Thus it remains to exhibit involutions $s_i \in N_{L_i}(B)$, with $|s_1s_2| \leq 3$.

Notice that each involution $s_i \in N_{L_i}(B)$ inverts B. Now $B \leq M_1$, so by 5.2.14.2, B is conjugate to the subgroup X defined in 5.2.15. Therefore as s_1 inverts B, $N_G(B) = (B \times L_B)\langle s_1 \rangle$, where $L_B \cong A_5$, s_1 inverts B, and s_1 induces a transposition on L_B. But s_2 also inverts B, so replacing s_1 by a suitable member of Bs_1, we may assume $s_1s_2 \in L_B$. Thus s_1 and s_2 are distinct transpositions in $L_B\langle s_1 \rangle \cong S_5$, so $|s_1s_2| = 2$ or 3, completing the proof. □

We now define some notation to use in our identification of G with M_{23}. Let $\bar{G} := M_{23}$ act on $\Theta := \{1, \ldots, 23\}$. Then (cf. chapter 6 in [**Asc94**]) we may take our 7-set Ω to be a block in the Steiner system $(\Theta, \mathcal{C})$ on Θ preserved by $\bar{G}$, so that $N_{\bar{G}}(\Omega) = \bar{M}_2$ is the split extension of $\bar{U} = \bar{G}_\Omega \cong E_{16}$ by A_7, and $\bar{M}_2$ is an exceptional A_7-block.

LEMMA 5.2.17. *There is a permutation equivalence* $\zeta : M_2 \to \bar{M}_2$ *of* M_2 *and* $\bar{M}_2$ *on* Ω.

PROOF. As B is of order 3 in $K \cap M_{1,2}$, it follows from parts (1) and (2) of 5.2.11 that we may choose B to act on Ω as $\langle (1, 2, 3) \rangle$. Then as $C_U(B) = 1$, $N_{M_2}(B) \cong \mathbf{Z}_2/(\mathbf{Z}_3 \times A_4)$ has Sylow 2-groups of order 8. Thus T splits over U, so M_2 splits over U by Gaschütz's Theorem A.1.39. Thus there is an isomorphism $\zeta : M_2 \to \bar{M}_2$, and adjusting by a suitable inner automorphism, this map is a permutation equivalence. □

For the remainder of this section, define ζ as in 5.2.17.

Let $\Gamma := \Theta^2$ be the set of unordered pairs of elements from Θ and fix $\bar{x} := \{6, 7\}$ and $\bar{y} := \{5, 6\}$ in Γ. From chapter 6 of [**Asc94**]:

LEMMA 5.2.18. *(1)* $\bar{G}_{\bar{x}}$ *is the extension of* $L_3(4)$ *by a field automorphism.*
(2) $\Theta - \{6, 7\}$ *is a projective plane over* $\mathbf{F}_4$ *with lines* $\{C - \{6, 7\} : \{6, 7\} \subseteq C \in \mathcal{C}\}$*, and* $\bar{G}_{\bar{x}}$ *preserves this structure.*
(3) The global stabilizer $\bar{I}$ *of* $\{4, 5, 6, 7\}$ *in* $\bar{G}$ *is the global stabilizer in* $\bar{M}_2$ *of* $\{4, 5, 6, 7\}$.

PROOF. In [**Asc94**], the Steiner system $(\Theta, \mathcal{C})$ is constructed so that (1) and (2) hold. As each 4-point subset of Θ is contained in a unique block of the Steiner system, (3) holds. □

Regard Γ as a a graph by decreeing that $a, b \in \Gamma$ are adjacent if $|a \cap b| = 1$. We wish to show $G \cong \bar{G}$. To do so, we write G_x for G_0T and essentially show there is a graph structure on $\Gamma_G := G/G_x$ isomorphic to the graph Γ, such that the representations of $\bar{G}$ on Γ (which is in turn $\bar{G}$-isomorphic to the analogous graph on $\Gamma_{\bar{G}} := \bar{G}/\bar{G}_x$) and G on Γ_G are equivalent. This leads us to write x for G_x regarded as a point of Γ_G—namely the coset of G_x containing the identity. Thus G_x is indeed the stabilizer of the point $x \in \Gamma_G$.

Let I denote the global stabilizer in M_2 of $\{4, 5, 6, 7\}$. Notice that the representation of M_2 on $\Omega \subseteq \Theta$ induces a representation of M_2 on $\Omega^2 \subseteq \Gamma$; this is the representation implicit in the next lemma.

LEMMA 5.2.19. *(1) $G_x \cap M_2 = H$ is the stabilizer in M_2 of $\bar{x} = \{6, 7\} \in \Gamma$, and the stabilizer in M_2 of $x = G_x \in \Gamma_G$.*

(2) The representation of M_2 on $xM_2 \subseteq \Gamma_G$ is equivalent to its representation on $\Omega^2 \subseteq \Gamma$.

(3) $\zeta : M_2 \to \bar{M}_2$ restricts to an isomorphism $\zeta : I \to \bar{I}$.

(4) $\zeta(I_{\bar{x}}) = \bar{I}_{\bar{x}}$ and $\zeta(I_{\bar{x},\bar{y}}) = \bar{I}_{\bar{x},\bar{y}}$.

PROOF. By 5.2.11.1, H is the stabilizer of $\bar{x} = \{6, 7\} \in \Gamma$, and hence is a maximal subgroup of M_2. Therefore $H = G_x \cap M_2$, and thus H is also the stabilizer in M_2 of the coset $G_x \in \Gamma_G$, which we are denoting by x. Therefore (1) holds. Then (1) implies (2), while 5.2.17 and the definition of I and $\bar{I}$ imply (3) and (4). □

Using the equivalence of 5.2.19.2, the point $\bar{y} = \{5, 6\} \in \Gamma$ corresponds to a point $y \in \Gamma_G$; namely the coset $y = G_x t$, where $t \in I$ has cycle $(\bar{x}, \bar{y})$ on Γ. Such a t exists, as I is the global stabilizer of $\{4, 5, 6, 7\}$ in M_2, and hence induces the full symmetric group on that subset. The coset y is independent of t by 5.2.19.1.

Recall as in [**Asc94**] that $I(\{x, y\})$ denotes the global stabilizer in I of $\{x, y\}$.

LEMMA 5.2.20. *(1) $G_{x,y} = L$.*

(2) I_x is the stabilizer in M_2 of the partition $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$ of Ω, and $I_x/U \cong S_3 \times \mathbf{Z}_2$.

(3) $I_{x,y} = UB$ and $\zeta(I(\{x, y\})) = \bar{I}(\{\bar{x}, \bar{y}\})$.

(4) $G_x = \langle G_{x,y}, I_x \rangle$.

(5) There is an isomorphism $\beta : G_x \to \bar{G}_{\bar{x}}$ agreeing with ζ on I_x, such that $\beta(G_{x,y}) = \bar{G}_{\bar{x},\bar{y}}$.

PROOF. By 5.2.11.2, $M_{1,2}$ is the global stabilizer in M_2 of $\{5, 6, 7\}$, so there is $t \in (M_{1,2})_4 \leq I \cap M$ with cycle $(\bar{x}, \bar{y})$. Then by a remark preceding this lemma, t has cycle (x, y). As $L \leq G_0 T = G_x$, L fixes x, so that $L = L^t$ fixes $xt = y$, and then $L \leq G_{x,y}$. But LT and G_0 are the only maximal subgroups of G_x containing L, and by 5.2.19.2, $T \not\leq G_{x,y} \not\geq K$. So (1) holds.

Parts (2) and (3) are easy calculations given 5.2.19. As observed earlier, LT and G_0 are the only maximal subgroups of G_x containing L and $G_{x,y} = L$ by (1). So as $I_x \not\leq G_0 \cap M_2 = K$ and $I_x \not\leq M_{1,2}$, (4) holds.

By 5.2.16 and 5.2.18.1, there is an isomorphism $\beta : G_x \to \bar{G}_{\bar{x}}$, which we may take to map T to $\bar{T} := \zeta(T)$, B to $\bar{B} := \zeta(B)$, and K and L to the parabolics $\bar{K} := \zeta(K)$ and $\bar{L} := \bar{G}_{\bar{x},\bar{y}}$ of $O^2(\bar{G}_{\bar{x}})$. Now by (2) and (3), $I_x = UB\langle t, r \rangle$, where $t := (1, 2)(6, 7)$ and $r := (4, 5)(6, 7)$ on Ω. In particular $I_x = UN_{KT}(B)$, so

$$\beta(I_x) = \beta(U) N_{\beta(K)\beta(T)}(\beta(B)) = \bar{U} N_{\bar{K}\bar{T}}(\bar{B}) = \bar{I}_x.$$

Finally let $\gamma := \zeta^{-1} \circ \beta$, regarded as an automorphism of I_x, so that $\gamma \in Aut(I_x)$. Notice $|N_{GL(U)}(Aut_{I_x}(U)) : Aut_{I_x}(U)| = 2$ and $U = C_{Aut(I_x)}(U)$, so $|Aut_I(I_x) : Inn(I_x)| = 2$. Then as $|N_I(I_x) : I_x| = 2$, $Aut(I_x) = Aut_I(I_x)$. Indeed as $\beta(T) = \bar{T} = \zeta(T)$, $\gamma(t) \in O^2(I_x)t$, so $\gamma \in Inn(I_x)$. Thus adjusting β by the inner automorphism of G_x which acts on I_x as γ^{-1}, we may choose $\beta = \zeta$ on I_x, proving (5). $\square$

LEMMA 5.2.21. *$G = \langle M, M_2 \rangle = \langle G_x, I \rangle$.*

PROOF. Let $Y := \langle M, M_2 \rangle$. If $Y < G$, then by induction on the order of G, $Y \cong M_{23}$. In particular, Y has one class of involutions; while by (1) and (2) of 5.2.12, $N_G(T) \leq C_G(z) \leq Y$. Thus Y is a strongly embedded subgroup of G (see I.8.1), so by 7.6 in [**Asc94**], Y has a subgroup of odd order transitive on the involutions in Y. Now Y has

$$i := 3 \cdot 5 \cdot 11 \cdot 23$$

involutions, but no subgroup of odd order divisible by i. This contradiction shows $G = \langle M, M_2 \rangle$. But $M = LM_{1,2}$ and $M_2 = \langle K, I \rangle$, so

$$G = \langle M, M_2 \rangle = \langle LT, K, I \rangle = \langle G_x, I \rangle,$$

completing the proof. $\square$

LEMMA 5.2.22. *$I = \langle I(\{x, y\}), I_x \rangle$.*

PROOF. Notice I_x contains the kernel UB of the action of I on $\Lambda := \{4, 5, 6, 7\}$. Further I_x contains elements inducing $(4, 5)$ and $(6, 7)$ on Λ, while $I(\{x, y\})$ contains an element inducing $(5, 6)$. So as the symmetric group on Λ is generated by these three transpositions, the lemma holds. $\square$

We are now in a position to complete the proof of Theorem 5.2.10, by appealing to the theory of uniqueness systems in section 37 of [**Asc94**]. Namely write Γ_G for the graph on $\Gamma_G = G/G_x$ with edge set $(x, y)G = (G_x, G_x t)G$, and let Γ_I be the subgraph with vertex set xI and edge set $(x, y)I$. By 5.2.19.2, Γ_I is isomorphic to the subgraph $\Gamma_{\bar{I}} := \{4, 5, 6, 7\}^2$ of Γ.

Observe that $\mathcal{U} := (G, I, \Gamma_G, \Gamma_I)$ is a uniqueness system in the sense of [**Asc94**]. Namely by 5.2.21, $G = \langle G_x, I \rangle$; by 5.2.20.4, $G_x = \langle G_{x,y}, I_x \rangle$; and by 5.2.22, $I = \langle I(\{x, y\}), I_x \rangle$. This verifies the defining conditions for uniqueness systems (see (U) on page 198 of [**Asc94**]). Similarly $\bar{\mathcal{U}} := (\bar{G}, \bar{I}, \Gamma, \Gamma_{\bar{I}})$ is a uniqueness system.

Now by 5.2.19 and 5.2.20, $\beta : G_x \to \bar{G}_{\bar{x}}$ and $\zeta : I \to \bar{I}$ define a similarity of uniqueness systems, as defined on page 199 of [**Asc94**]. Next we will apply Theorem 37.10 in [**Asc94**], to prove this similarity is an equivalence.

In applying Theorem 37.10, we take L in the role of the group "K" in the Theorem, and take $t, h \in I$ to be elements acting on Ω as

$$t := (1, 2)(5, 7), \ h := (1, 2)(5, 6).$$

Then $t, h \in M_{1,2} \leq N_G(L)$, and by construction t has cycle (x, y), $t^h = (1, 2)(6, 7) \in K \leq G_x$, and $\zeta(h) \in \bar{M}_{1,2} \leq N_{\bar{G}}(\bar{L})$, so that hypothesis (2) of Theorem 37.10 holds. Next $L = G_{x,y}$ by 5.2.20.1, so trivially $G_{x,y} = \langle L_y, I_{x,y} \rangle$, which is hypothesis (3) of 37.10. Finally $L \cap I = BU$, and from the structure of the $L_2(4)$-block L, we check that $C_{Aut(L)}(BU) = 1$; this verifies hypothesis (1) of 37.10.

Therefore $\mathcal{U}$ is equivalent to $\bar{\mathcal{U}}$. It remains to check that $\Gamma_{\bar{I}}$ is a *base* for $\bar{\mathcal{U}}$ in the sense of p.200 of [**Asc94**]: for then as $\bar{G} = M_{23}$ is simple, Exercise 13.1 in [**Asc94**] says $G \cong \bar{G}$, completing the proof of Theorem 5.2.10.

Recall from page 200 of [**Asc94**] that $\Gamma_{\bar{I}}$ is a base for $\bar{\mathcal{U}}$ if each cycle in the graph Γ is in the closure of the conjugates of cycles of $\Gamma_{\bar{I}}$. But each triangle in Γ is conjugate to one of:

$$\{6,7\},\{5,6\},\{5,7\} \text{ or } \{6,7\},\{5,7\},\{4,7\},$$

which are triangles of $\Gamma_{\bar{I}}$. So it remains to show Γ is *triangulable* in the sense of section 34 of [**Asc94**]; that is, that each cycle of Γ is in the closure of the triangles, or equivalently the graph Γ is simply connected. This is the crucial advantage of working with Γ as opposed to Γ_G; one can calculate in Γ to check it is triangulable.

As Γ is of diameter 2, by Lemma 34.5 in [**Asc94**], it suffices to show each r-gon is in the closure of the triangles, for $r \leq 5$. For $r = 2, 3$ this holds trivially, and we now check the cases with $r = 4$ and 5, using the 4-transitivity of M_{23} on Θ.

It follows from 34.6 in [**Asc94**] that 4-gons are in the closure of triangles: Namely a pair of points at distance 2 are conjugate to $\{1,2\}$ and $\{3,4\}$, whose common neighbors are $\{1,3\}$, $\{1,4\}$, $\{2,4\}$, $\{2,3\}$—forming a square in Γ, which is in particular connected. Finally it follows from Lemma 34.8 in [**Asc94**] that 5-gons are in the closure of the triangles: for if x_0, x_1, x_2, x_3 is a path in Γ with $d(x_0, x_2) = d(x_0, x_3) = d(x_1, x_3) = 2$, then up to conjugation under $\bar{G}$, $x_0 = \{1,2\}$, $x_1 = \{2,3\}$, $x_2 = \{3,4\}$, and $x_3 = \{4,a\}$ for some $a \in \Theta - \{1,2,3,4\}$. Then as x_0, x_2, and x_3 are all connected to $\{2,4\}$, it follows that $x_0^\perp \cap x_2^\perp \cap x_3^\perp \neq \emptyset$, in the language of [**Asc94**].

Thus the proof of Theorem 5.2.10 is complete.

5.3. Identifying rank 2 Lie-type groups

In this section, we complete the proof of Theorem 5.2.3. Recall the definition of groups of type $X_r(q)$ and type M_{23} appearing before the statement of Theorem 5.2.9. If G is of type M_{23}, then conclusion (2) of Theorem 5.2.3 holds by Theorem 5.2.10. Therefore by Theorem 5.2.9, we may assume that one of conclusions (1)–(3) of Theorem 5.2.9 holds. Thus G is of *type* $X_r(q)$ for some even $q > 2$ and some X_r of Lie rank 2. Recall from 5.2.7 that $\alpha = (G_1, G_{1,2}, G_2)$ is an $X_r(q)$-amalgam, where $G_i = L_iBD$, $G_{1,2} = SBD$, and $S = T \cap L_1 = T \cap L_2 = O_2(G_{1,2})$. We write $G(\alpha)$ for the corresponding group $X_r(q)$ of Lie type defining the amalgam. To establish Theorem 5.2.3, we must show $G \cong G(\alpha)$.

Set $M_i := N_G(L_i)$ and $M_{1,2} := M_1 \cap M_2$, and let $\gamma := (M_1, M_{1,2}, M_2)$ be the corresponding amalgam.

LEMMA 5.3.1. *(1) $L_i \in \mathcal{L}(G,T)$ and $M_i = !\mathcal{M}(L_iT)$ with $M_1 \neq M_2$.*
(2) $F^*(M_i) = O_2(M_i) = O_2(L_i)$.
(3) $N_G(S) = M_{1,2} = N_{M_i}(S)$.
(4) $M_i = L_iM_{1,2}$.

PROOF. By the hypothesis of Theorem 5.2.3, $L_1 = L \in \mathcal{L}^*(G,T)$, and by 5.2.7.2, $L_2 = O^2(H)$ so that $L_2 \not\leq M_1$. By 5.2.8 and our assumption that G is not of type M_{23}, $L_2 = K \in \mathcal{L}^*(G,T)$. Thus (1) holds by 1.2.7. Hence $F^*(M_i) = O_2(M_i)$ by 1.1.4.6. By 5.2.7.3, $O_2(G_iT) = O_2(L_i)$, so $O_2(M_i) = O_2(L_i)$ using A.1.6, completing the proof of (2).

To prove (3), it will suffice to show $N_G(S) \leq M_i$ for $i = 1$ and 2. For then $N_G(S) \leq M_{1,2}$. On the other hand $N_{M_i}(S)$ is maximal in M_i, and M_1 and M_2 are distinct maximal 2-locals by (1), so $M_{1,2} = N_{M_i}(S)$ and hence (3) holds.

Thus it remains to show $N_G(S) \leq M_i$. But as $S \in Syl_2(L_i)$ and T is in a unique maximal subgroup of L_iT, we conclude from Theorem 3.1.1 that $O_2(\langle N_G(S), L_i\rangle) \neq 1$. Therefore $N_G(S) \leq M_i = !\mathcal{M}(L_iT)$ by (1). Thus (3) is established. Then (4) follows from (3) via a Frattini Argument. □

LEMMA 5.3.2. *γ is an $M(\alpha)$-extension of α (in the sense of Definition F.4.3), for some extension $M(\alpha)$ of $G(\alpha)$.*

PROOF. Let $M_0 := \langle M_1, M_2\rangle$. We first verify that γ satisfies Hypothesis A of the Green Book [**DGS85**], with L_i in the role of "P_i^{**}".

By 5.3.1.1, $O_2(M_0) = 1$. By 5.3.1.2, $F^*(M_i) = O_2(M_i) = O_2(L_i)$, so condition (ii) of Hypothesis A holds. Then as $O_2(M_i) = O_2(L_i)$, condition (i) holds by 5.3.1.4. Condition (iii) follows from 5.3.1.3, and the list of possibilities for L_i in 5.2.6. This completes the verification of Hypothesis A.

As Hypothesis A holds, and $q > 2$ by 5.2.7.1, case (a) of Theorem A in the Green Book [**DGS85**] holds, so that γ is an extension of the Lie amalgam α of $G(\alpha)$. That is, γ determines subgroups $M_i(\alpha) \cong M_i$ of $Aut(G(\alpha))$, with corresponding completion $M(\alpha) := \langle M_1(\alpha), M_2(\alpha)\rangle \leq Aut(G(\alpha))$. So the lemma holds. □

Let $Z_S := Z(S)$ and $Z_i := Z(L_i)$.

LEMMA 5.3.3. *Either*

(1) The hypotheses of Theorem F.4.31 are satisfied, with G in the role of "M", or

(2) $G(\alpha) \cong L_3(q)$, and $C_G(z) \not\leq M_{1,2}$ for each involution $z \in Z_S$.

PROOF. By 5.3.2, γ is an extension of the Lie amalgam α, so that $M(\alpha)$ plays the role of "$\bar{M}$" in Theorem F.4.31. Hypothesis (d) of F.4.31 holds for G in the role of "M", as $T \leq M_{1,2}$ and $T \in Syl_2(G)$. Hypothesis (e) holds as G is simple. Hypothesis (a) follows from the fact that L_iT is a uniqueness subgroup by 5.3.1.1. Further BD is transitive on $Z_i^\#$. Thus if $Z_i \neq 1$, each involution in Z_i is conjugate under M_i to some $z \in Z(L_iT)$, and therefore $C_G(z) \leq M_i$ using 5.3.1.1. Similarly if $G(\alpha) \cong L_3(q)$, then BD is transitive on $Z_S^\#$, so if $C_G(z_0) \leq M_1$ for some $z_0 \in Z_S^\#$, then $C_G(z) \leq M_1$ for all $z \in Z_S^\#$. Hence the first statement in Hypothesis (c) holds, and either Hypothesis (b) holds, or conclusion (2) of 5.3.3 holds. If $G(\alpha)$ is $Sp_4(q)$, then each involution z in Z_S is fused into $Z(T)$ under BD, and hence $C_G(z) \in \mathcal{H}^e$ by 1.1.4.6. This completes the verification of Hypothesis (c). Therefore either the hypotheses of F.4.31 are satisfied, so that conclusion (1) of 5.3.3 holds, or conclusion (2) of 5.3.3 holds. □

THEOREM 5.3.4. *Either*

(1) $G \cong G(\alpha)$, or

(2) $G(\alpha) \cong L_3(q)$, and $C_G(z) \not\leq M_{1,2}$ for each involution $z \in Z_S$.

PROOF. If 5.3.3.1 holds, we may apply Theorem F.4.31 to conclude $G \cong M(\alpha)$. Since G is simple, we must in fact have $M(\alpha) \cong G(\alpha)$. □

By Theorems 5.2.9, 5.2.10, and 5.3.4, Theorem 5.2.3 holds unless possibly G is of type $L_3(q)$ and conclusion (2) of 5.3.4 holds. We will finish by showing (in 5.3.7 below) that the latter case leads to a contradiction.

Thus in the remainder of this section, we assume G is of type $L_3(q)$ and conclusion (2) of 5.3.4 holds.

Pick $z \in Z^\#$ and set $G_z := C_G(z)$. Set $V_i := O_2(L_i)$ and observe $S = V_1V_2 = J(T)$ using 5.3.2 and F.4.29.6. Similarly by F.4.29.2, if $t \in T - S$, then t induces a field automorphism on L_i, so $[Z_S, t] \neq 1$; that is, $S = C_T(Z_S)$.

LEMMA 5.3.5. $N_G(Z_S) = M_{1,2}$.

PROOF. Set $G_Z := C_G(Z_S)$. Then $S = V_1V_2 \in Syl_2(G_Z)$, as we just observed. As $T \leq N_G(Z_S)$, $F^*(G_Z) = O_2(G_Z)$ by 1.1.4.6, and therefore also $F^*(G_Z/Z_S) = O_2(G_Z/Z_S)$ by A.1.8. Hence as S/Z_S is abelian, $S/Z_S = O_2(G_Z/Z_S)$, so $S = O_2(G_Z)$. Then as $\mathcal{A}(S) = \{V_1, V_2\}$ with $V_i \trianglelefteq T$, $N_G(Z_S) \leq N_G(V_i) = M_i$ as $M_i \in \mathcal{M}$ by 5.3.1.1. On the other hand by 5.3.1.3, $M_{1,2} = N_G(S) \leq N_G(Z_S)$. □

LEMMA 5.3.6. *(1) V_i is weakly closed in T with respect to G.*
(2) $Z_S^G \cap V_i = Z_S^{L_i}$ is of order $q+1$.

PROOF. We saw $\mathcal{A}(T) = \{V_1, V_2\}$ and $V_i \trianglelefteq T$; in particular, $V_1^G \cap T \subseteq \mathcal{A}(T)$, so to establish (1) we only need to show $V_2 \notin V_1^G$. But if $V_2 \in V_1^G$ then as $V_i \trianglelefteq T$ and $N_G(T)$ controls fusion of normal subgroups of T by Burnside's Fusion Lemma, V_2 is in fact conjugate to V_1 in $N_G(T)$, and hence in $O^2(N_G(T))$ as T normalizes V_1 and V_2. This is impossible as $|\mathcal{A}(S)| = 2$, establishing (1). By (1) and Burnside's Fusion Lemma, M_i controls fusion in V_i, so (2) follows. □

LEMMA 5.3.7. $G_z \leq M_{1,2}$.

PROOF. As $Z_S \trianglelefteq T$ and $M_{1,2}$ is transitive on $Z_S^\#$, we may take $z \in Z(T)$. Therefore $F^*(G_z) = O_2(G_z) =: R$ by 1.1.4.6. Next unless $q = 4$ and $M_i/V_i \cong S_5$ for $i = 1$ and 2, $S = V_1V_2 = O_2(C_{M_i}(z))$ for each i. Assume for the moment that the exceptional case does not hold. Then as $S \in Syl_2(C_G(Z_S))$ by 5.3.5, $R \leq S$ by A.1.6, so $Z_S = Z(S) \leq \Omega_1(C_{G_z}(R)) = \Omega_1(Z(R)) =: Z_R$. If $Z_S = Z_R$ then $R \leq N_G(Z_S) \leq M_{1,2}$ by 5.3.5, and the lemma holds; so assume instead that $Z_S < Z_R$.

Let $\hat{G}$ denote our target group $G(\alpha) \cong L_3(q)$ and $\hat{M}_i$ the subgroups $M_i(\alpha)$ in 5.3.2. Recall we have a corresponding isomorphism of amalgams $\beta : \hat{\gamma} := (\hat{M}_1, \hat{M}_{1,2}\hat{M}_2) \to \gamma$. Thus $S \cong \hat{S}$ is isomorphic to a Sylow 2-group of $L_3(q)$, so V_1 and V_2 are the maximal elementary abelian subgroups of S. Therefore $V_i \cap Z_R > Z_S$ for $i = 1$ or 2, so that $R \leq C_S(V_i \cap Z_R) = V_i$. Then $V_i \leq C_{G_z}(R) \leq R$, so $V_i = R$. But then by 5.3.6.1, $G_z \leq N_G(V_i) = M_i$, so that $G_z = G_z \cap M_i \leq M_{1,2}$ using β, and the lemma holds.

It remains to treat the exceptional case where $q = 4$ and $M_i/V_i \cong S_5$ for $i = 1$ and 2. Let $\bar{G}_z := G_z/\langle z \rangle$, so that $F^*(\bar{G}_z) = O_2(\bar{G}_z)$ by A.1.8. Now M_i is determined up to isomorphism, so in particular T is isomorphic to a Sylow 2-subgroup of M_{22}. Therefore $J(\bar{T}) = \bar{Q} \cong E_{16}$ with $Q \cong Q_8^2$ and $C_T(Q) \leq Q$. Let $V_z := \langle Z_S^{G_z} \rangle$. As $\bar{Z}_S \leq Z(\bar{T})$, $\bar{V}_z$ is elementary abelian by B.2.14, so $\Phi(V_z) \leq \langle z \rangle$.

Suppose first that V_z is abelian, and therefore elementary abelian. Then $V_z \leq C_T(Z_S) = S$ using an earlier observation. As V_1 and V_2 are the maximal elementary abelian subgroups of S, $V_z \leq V_i$ for $i = 1$ or 2. If $V_z = Z_S$, then the lemma holds by 5.3.5, so we may assume $Z_S < V_z \leq V_i$. But $V_i = C_G(A)$ for each hyperplane A of V_i through Z_S, so $V_z = A$ or V_i, and in any case $V_i \trianglelefteq G_z$. Hence the lemma holds, again since $G_z \cap M_i \leq M_{1,2}$.

Thus we may suppose instead that V_z is not abelian. Now if $V_z \leq S$, then $Z_S = Z(V_z) \trianglelefteq G_z$, and the lemma holds by 5.3.5. Hence there is $v \in V_z - S$; we will see this leads to a contradiction. Now from the structure of M_1, $E_4 \cong [v, S/Z_S] \leq (V_z \cap S)/Z_S$, so $m(\bar{V}_z) \geq 4$. Therefore as $E_{16} \cong \bar{Q} = J(\bar{T})$, we must have $V_z = Q$. Next as $C_T(Q) \leq Q$, $G_z/Q \leq Out(Q) \cong O_4^+(2)$, so $|G_z : T| = 3$ or 9. As

$$|G_z : T| \geq |\bar{Z}_S^{G_z}| \geq m(\bar{V}_z) = 4,$$

$|G_z : T| = 9$. As $m_2(Q) = 3$ and $m(V_i) = 4$, $V_i \not\leq Q$; indeed $[V_i, v]Z_S \leq Q$ and $[V_i, Q] \leq V_i$, so that V_iQ/Q has order 2 and induces an involution of type a_2 on $\bar{Q}$, so it centralizes a nontrivial element in $O^2(G_z/Q) \cong E_9$. Therefore $O^2(N_{G_z}(V_iQ)) \neq 1$. However by 5.3.6.1, V_i is weakly closed in V_iQ; so $O^2(N_{G_z}(V_iQ)) \leq O^2(G_z \cap M_i) = 1$, contradicting the previous remark. This contradiction completes the proof of 5.3.7. □

Observe that 5.3.7 contradicts our assumption that 5.3.4.2 holds. So the proof of Theorem 5.2.3 is complete.

CHAPTER 6

Reducing $\mathbf{L_2(2^n)}$ to $\mathbf{n = 2}$ and $\mathbf{V}$ orthogonal

In this chapter, we continue our analysis of simple QTKE-groups G for which there exists a T-invariant $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_2(2^n)$. Recall that we began this analysis in chapter 5. In particular in Theorem 5.2.3 we showed under these hypotheses, and the hypothesis that $n(H) > 1$ for some $H \in \mathcal{H}_*(T,M)$, that either

(I) G is M_{23} or a group of Lie type of characteristic 2 and Lie rank 2, or

(II) the conclusion of 5.2.3.1 holds; in particular $n = 2$ and $[R_2(LT), L]$ is the sum of at most two A_5-modules.

In Theorem 6.2.20 of this chapter, we complete the reduction to the situation where $n = 2$ and $[R_2(LT), L]$ is the sum of A_5-modules by considering the remaining case where $n(H) = 1$ for each $H \in \mathcal{H}_*(T,M)$, and $[R_2(LT), L]$ is not the sum of A_5-modules when $n = 2$. Section 6.1 carries out the reduction to the subcase $n = 2$. Then section 6.2 shows that the only quasithin example to arise in this subcase is M_{22}.

This reduction allows us thereafter to regard $L/O_2(L) \cong L_2(4)$ as $\Omega_4^-(2)$. We treat that case in Part 5, which deals with the situation where there exists $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ a group of Lie type group defined over $\mathbf{F}_2$.

6.1. Reducing $\mathbf{L_2(2^n)}$ to $\mathbf{L_2(4)}$

As mentioned above, we wish to complete the reduction to the situation where $n = 2$ and $[R_2(LT), L]$ is the sum of A_5-modules, under the hypothesis of chapter 5. By Theorem 5.2.3, we may assume Hypothesis 5.1.8 fails. Thus in this section, we assume the following hypothesis:

Hypothesis 6.1.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_2(2^n)$, $L \trianglelefteq M \in \mathcal{M}(T)$, and $V \in \mathcal{R}_2(LT)$ with $[V,L] \neq 1$. In addition, assume*

(1) $[V,L]$ is not the sum of one or two copies of the A_5-module for $L/O_2(L) \cong A_5$.

(2) $n(H) = 1$ for each $H \in \mathcal{H}_*(T,M)$.

Remark 6.1.2. Notice Hypothesis 6.1.1.1 has the effect of excluding cases (2) and (5) of 5.1.3 plus case (4) of 5.1.3 when $n = 2$. Thus either case (1) or (3) of 5.1.3 holds, or $n > 2$ and case (4) of 5.1.3 holds. Similarly 6.1.1.1 excludes case (3) of 5.1.2; therefore by 5.1.2, either case (3) of 5.1.3 holds, or $J(T) \leq C_T(V) = O_2(LT)$, so that $J(T) \trianglelefteq LT$ and hence $M = !\mathcal{M}(N_G(J(T)))$.

Throughout this section, define $Z := \Omega_1(Z(T))$, $V_L := [V,L]$, and $T_L := T \cap L$. Set $M_V := N_M(V)$, and $\bar{M}_V := M_V/C_M(V)$.

In contrast to the previous chapter, we find now when $n(H) = 1$ for each $H \in \mathcal{H}_*(T, M)$ that weak-closure methods are frequently effective.

LEMMA 6.1.3. *(1) If V is a TI-set under M, then Hypothesis E.6.1 holds.*
(2) Either

(I) $r(G, V) = 1$, or
(II) $J_1(T) \not\leq C_T(V)$.

PROOF. Part (1) of Hypothesis E.6.1 follows from Hypothesis 6.1.1. We saw $C_T(V) = O_2(LT)$, so that $M = !\mathcal{M}(LT) = !\mathcal{M}(N_{M_V}(C_T(V)))$, giving part (3) of Hypothesis E.6.1. This establishes (1). Further $n(H) = 1$ for all $H \in \mathcal{H}_*(T, M)$ by Hypothesis 6.1.1.2, so the hypotheses of E.6.26 are satisfied with "j" equal to 1. Therefore (2) follows from E.6.26. □

6.1.1. Initial reductions.

In this subsection, we establish various reductions culminating in the two cases of Proposition 6.1.15; eliminating the first of those cases is then the goal of the second subsection.

LEMMA 6.1.4. *$V_L/C_{V_L}(L)$ is the natural module for $\bar{L}$.*

PROOF. Assume that the lemma fails. This assumption excludes case (3) of 5.1.3, so by Remark 6.1.2 and 5.1.3, either

(A) $n > 2$ is even and V is the $O_4^-(2^{n/2})$-module, or

(B) V is the sum of two copies of the natural module.

Similarly by Remark 6.1.2 and 5.1.2, $J(T) \trianglelefteq LT$ and $M = !\mathcal{M}(N_{LT}(J(T)))$. Thus $[Z, H] = 1$ for each $H \in \mathcal{H}_*(T, M)$ by 5.1.7. Further by Hypothesis 6.1.1.2, $n(H) = 1$. Enlarging V if necessary, we may take $V = R_2(LT)$.

Assume that there is $A \in \mathcal{A}_1(T)$ with $\bar{A} \neq 1$. Then by B.2.4.1,

$$m(V/C_V(A)) \leq m(\bar{A}) + 1 \leq m_2(\bar{L}\bar{T}) + 1 = n + 1. \qquad (*)$$

But in case (B), $m(V/C_V(A)) \geq 2n > n+1$ since $n > 1$, contrary to (*), so case (A) holds. Hence by H.1.1.2 with $n/2$ in the role of "n", $n = 4$, and $\bar{A}$ is of rank 1 and generated by an orthogonal transvection. Further for $t \in T - C_T(V)$, $m(V/C_V(t)) \geq 2n$ in case (A), and $m(V/C_V(t)) \geq n$ in case (B) by H.1.1.1. Therefore we have shown that either:

(i) $n = 4$, V is the $O_4^-(4)$-module, and if $\overline{J_1(T)} \neq 1$ then $\overline{J_1(T)}$ is generated by an orthogonal transvection, or

(ii) $m(\bar{L}\bar{T}, V) > 2$ and $J_1(T) \leq C_T(V) = O_2(LT)$, so that $J_1(T) \trianglelefteq LT$.

Suppose first that case (ii) holds. Then $r(G, V) = 1$ by 6.1.3.2. Now if Hypothesis E.6.1 is satisfied, then since $m(\bar{L}\bar{T}, V) > 2$ in case (ii), $r(G, V) > 1$ by Theorem E.6.3, a contradiction. Thus V is not a TI-set in M by 6.1.3.1. Therefore as $L \trianglelefteq M$, L is not irreducible on V, so case (B) holds where $V = V_1 \oplus V_2$ is the sum of two natural modules V_1 and V_2. Further we may choose V_1 to be T-invariant (cf. the proof of A.1.42.1). As L is irreducible on V_i, V_i is a TI-set under M. As $r(G, V) = 1$, there is a hyperplane W of V with $C_G(W) \not\leq N_G(V)$. Set $U_i := W \cap V_i$. Then $C_G(W) \leq C_G(U_i)$ and $m(V_i/U_i) \leq m(V/W) = 1$, so $U_i \neq 1$ as $m(V_i) \geq 4$. Thus if $C_G(W) \leq M$, then as V_1 and V_2 are TI-sets in M, $C_G(W)$ normalizes $V_1 \oplus V_2 = V$, contrary to our choice of W. Therefore $C_G(W) \not\leq M$, so $C_G(U_1) \not\leq M$. But as $V_1 \trianglelefteq LT$, $N_G(V_1) \leq M$, so $r(G, V_1) = 1$. As V_1 is a TI-set in

M, Hypothesis E.6.1 holds by 6.1.3.1 applied to V_1 in the role of "V". However as L is transitive on the hyperplanes of V_1, and the stabilizer in LT of a hyperplane contains a Sylow 2-subgroup of LT, we may take $T \leq N_G(U_1)$. Thus $C_G(U_1) \leq M$ by E.6.13, contrary to an earlier observation.

This contradiction shows that case (i) holds. The elimination of case (i) will be lengthier. As L is irreducible on V, V is a TI-set in M, so that by 6.1.3.1, Hypothesis E.6.1 is satisfied, and we may appeal to results in section E.6.

We first claim $r(G, V) > 1$. If not, there is a hyperplane U of V with $C_G(U) \not\leq M$, and by E.6.13, U is not T-invariant. Thus U contains the subspace U_0 orthogonal to a nonsingular $\mathbf{F}_4$-point of the orthogonal space V. Therefore U contains a 2-central involution. As $V = R_2(LT)$, $V = \Omega_1(Z(Q))$, where $Q := O_2(LT)$. Finally $C_V(N_T(U)) \leq U$, so as $C_{\bar{L}\bar{T}}(U) = 1$,

$$\Omega_1(Z(N_T(U))) = C_{\Omega_1(Z(Q))}(N_T(U)) = C_V(N_T(U)) \leq U,$$

contrary to E.6.10.4, establishing the claim that $r(G, V) > 1$.

Let $M_1 \in \mathcal{M}(C_G(Z))$, and set $Q_1 := O_2(M_1)$, so that $M_1 = N_G(Q_1)$. As $H \leq C_G(Z)$ for $H \in \mathcal{H}_*(T, M)$, $M \neq M_1$. Suppose that $O_{2,F^*}(M_1) \leq M$. Then

$$Q_1 = O_2(M \cap M_1) = O_2(N_M(Q_1))$$

by A.4.4.1, so that $Q_1 \in \mathcal{B}_2(M)$. By A.4.4.2, $C(M, Q_1) = M \cap M_1$, so Hypothesis C.2.3 is satisfied, with M, $M \cap M_1$, Q_1 in the roles of "H, M_H, R". Now since V is the orthogonal module and $n > 2$, L is not a χ-block; so for L in the role of "K", the conclusions of C.2.7 do not hold, and hence $L \leq M \cap M_1$. But then $M_1 = !\mathcal{M}(LT) = M$, contradicting $M_1 \neq M$. This contradiction shows that $O_{2,F^*}(M_1) \not\leq M$.

Next $Z \leq R_2(LT) = V$ by B.2.14, so $Z = C_V(T)$. Let $X := O^{5'}(N_L(T_L))$. Then $X/O_2(X) \cong \mathbf{Z}_5$ and $XT \leq C_G(Z) \leq M_1$, from the structure of $O_4^-(4) \cong L_2(16)$ and its action on V. Let $S := O_2(XT)$, so that $S = T_L Q$. Then $J_1(S) \leq C_S(V)$ since case (i) holds. Define

$$\mathcal{H}_S := \{M_S \leq M_1 : S \in Syl_2(M_S) \text{ and } T \leq N_G(M_S)\}.$$

As $r(G, V) > 1$, E.6.26 says $M_S \leq M$ for each $M_S \in \mathcal{H}_S$ with $n(M_S) = 1$.

Now $O_2(M_1) = Q_1 \leq O_2(XT) = S$ by A.1.6. Then S is Sylow in $SO_{2,F}(M_1)$, so that $SO_{2,F}(M_1) \in \mathcal{H}_S$—and since $n(O_{2,F}(M_1)) = 1$ by E.1.13, $O_{2,F}(M_1) \leq M$ by the previous paragraph. We saw $O_{2,F^*}(M_1) \not\leq M$, so there is $K_1 \in \mathcal{C}(M_1)$ with $K_1 \not\leq M$, and $K_1/O_2(K_1)$ quasisimple. Let $K_0 := \langle K_1^T \rangle$ and observe that $X = O^2(X)$, so X normalizes K_1 by 1.2.1.3.

Next as $K_1 \not\leq M$, there is $H_S \in \mathcal{H}_*(T, M) \cap K_0 T$. Now $n(H_S) = 1$ by Hypothesis 6.1.1.2. Thus if $S \in Syl_2(O^2(H_S)S)$, then $O^2(H_S)S \in \mathcal{H}_S$, and hence $H_S \leq M$ by an earlier remark. Therefore S is not Sylow in $O^2(H_S)S$, and hence S is not Sylow in $K_0 S$. But if X normalizes $T \cap K_0 \in Syl_2(K_0)$, then $T \cap K_0 \leq O_2(XT) = S$; thus we conclude $X \not\leq N_G(T \cap K_0)$. In particular, $[X, K_1] \not\leq O_2(K_1)$, so a Sylow 5-subgroup X_5 of X acts faithfully on $K_1/O_2(K_1)$. Then as $X_5 T = T X_5$, this quotient is described in A.3.15. In cases (5)–(7) of A.3.15, X normalizes $T \cap K_0$, contrary to an earlier observation. As $X/O_2(X)$ is of order 5, cases (2) and (4) are ruled out. So it follows from A.3.15 that either

(a) $K_1/O_2(K_1) \cong L_2(p^e)$ and $(M \cap K_1)/O_2(K_1) \cong D_{p^e - \epsilon}$, or

(b) $K_1/O_2(K_1) \cong L_3^\delta(p)$, and there is an X-invariant $K_2 \in \mathcal{L}(G,T) \cap K_1$ with $K_2O_2(K_1)/O_2(K_1) \cong SL_2(p)$.

In case (b), if the projection of X_5 on K_1 centralizes $K_2/O_2(K_2)$, then from the structure of $L_3^\delta(p)$, X_5 centralizes a Sylow 2-group of $K_1/O_2(K_1)$, which is not the case as X does not normalize $T \cap K_0$. Thus the projection is inverted in $T \cap K_2$, so as $X \trianglelefteq XT$, $X \leq K_2$. Similarly in case (a) the projection is inverted in $T \cap K_1$, so $X \leq K_1$. Now $L \cap M_1$ contains T_L and so is contained in a Borel subgroup of L, and hence $X \trianglelefteq M_1 \cap M$. Thus in case (b), $K_2 \not\leq M$ as $X < K_2$. In this case, we replace K_1 by K_2, reducing to the case where $K_1 \in \mathcal{L}(G,T)$, $K_1 \not\leq M$, and $K_1/O_2(K_1) \cong L_2(p^e)$ as in case (a). (We no longer require $K_1 \in \mathcal{C}(M_1)$). As $X \leq K_1$ is normalized by T, $K_0 = \langle K_1^T \rangle = K_1$.

Let $K_1^*T^* := K_1T/O_2(K_1T)$. Recall by Remark 6.1.2 that $M = !\mathcal{M}(N_G(J(T))$ and $J(T) = J(O_2(LT)) \trianglelefteq XT$. Thus $J(T)$ is not normal in K_1T as $K_1 \not\leq M$, so there is $A \in \mathcal{A}(T)$ with $A^* \neq 1$. As $J(T) \trianglelefteq XT$, $A^* \leq J(T)^* \leq O_2(X^*T^*)$. But from the structure of $Aut(L_2(p^e))$, each nontrivial elementary abelian 2-subgroup of $O_2(X^*T^*)$ is fused under K_1^* to a subgroup of T^* not in $O_2(X^*T^*)$, contrary to $J(T)^* \leq O_2(X^*T^*)$. This contradiction finally completes the proof of 6.1.4. □

LEMMA 6.1.5. *$\mathcal{H}_*(T,M) \subseteq C_G(Z)$.*

PROOF. Assume that $H \in \mathcal{H}_*(T,M)$ with $[H,Z] \neq 1$, and let $K := O^2(H)$. Let D_L be a Hall $2'$-subgroup of $N_L(T_L)$. Enlarging V if necessary, we may take $V = R_2(LT)$, so $Z \leq V$. By 5.1.7.2, $K = [K,J(T)]$ and $L = [L,J(T)]$.

Let $\tilde{V}_L := V_L/C_{V_L}(L)$ and $Z_L := Z \cap V_L$. As $\tilde{V}_L$ is the natural module for $\bar{L}$ by 6.1.4, $\overline{J(T)} = \bar{T}_L$ by B.4.2.1. Hence $J(T) \leq T_LQ$ where $Q := O_2(LT)$, so D_L normalizes $J(T_LQ) = J(T)$. Also $V_L = [Z_L, L]$ and $C_{LT}(\tilde{V}_L) = C_{LT}(V_L) = Q$. Let $S := \mathrm{Baum}(T)$. As $L = [L,J(T)]$, and $\tilde{V}_L$ is the natural module, E.2.3.2 says $S \in Syl_2(LS)$ and hence $S \cap L \in Syl_2(L)$. As $\overline{J(T)} = \bar{T}_L$ and $T_LQ = C_T(C_V(T_LQ))$, also $S = \mathrm{Baum}(T_LQ)$, so that D_L normalizes S.

As $\tilde{V}_L$ is the natural module for $\bar{L}$, the normalizer N of $\bar{L} \cong SL_2(2^n)$ in $GL(\tilde{V}_L)$ is $\Gamma L_2(2^n)$, with $C_N(\bar{L}) \cong \mathbf{Z}_{2^n-1}$, and $O^2(C_N(\tilde{Z}_L))$ is the product of $\bar{T}$ with a diagonal subgroup of $C_N(\bar{L}) \times \bar{L}$ isomorphic to $\mathbf{Z}_{2^n-1}$. Therefore $C_Z := C_M(Z_L)$ acts on T_L and on $[Z_L, L] = V_L$, and $O^2(\bar{C}_Z/\bar{T}_L)$ is a subgroup of $\mathbf{Z}_{2^n-1}$.

Let $U_H := \langle Z^H \rangle$ and set $\hat{H} := H/C_H(U_H)$. Observe $U_H \in \mathcal{R}_2(H)$ by B.2.14. By Hypothesis 6.1.1, $n(H) = 1$. Recall by 3.3.2.4 that we may apply results of section B.6 to H. So as $K = [K,J(T)]$ and $[H,Z] \neq 1$, H appears in case (2) of E.2.3, with $\hat{H} \cong S_3$ or S_3 wr $\mathbf{Z}_2$ and $S \in Syl_2(KS)$. By parts (a) and (b) of B.6.8.6, $C_T(U_H) \trianglelefteq H$.

We claim $C_H(U_H) = O_2(H)$, so assume otherwise. By B.6.8.6.a, $C_H(U_H) \leq O_{2,\Phi}(H)$, so by B.6.8.2, $H/O_2(H) \cong D_8/3^{1+2}$. Thus there is a T-invariant subgroup $Y = O^2(Y)$ of $O_{2,\Phi}(K)$ with $Y = [J(T), Y]$ and $|Y : O_2(Y)| = 3$, and Y centralizes U_H by assumption. Then by B.6.8.2, $Y \leq O_{2,\Phi}(K) \leq M$, so as Y centralizes U_H and $Z \leq U_H$, Y centralizes Z and normalizes $[Z,L] = V_L$. If Y centralizes V_L then $[Y,L] \leq C_L(V_L) = O_2(L)$, so that LT normalizes $O^2(YO_2(L)) = Y$, and hence $N_G(Y) \leq M = !\mathcal{M}(LT)$. As $K \leq N_G(Y)$, this contradicts $K \not\leq M$. Hence $\bar{Y} \neq 1$, and as $Y \leq C_Z$, we conclude from paragraph three that $\overline{J(T)} = \bar{T}_L \trianglelefteq \bar{T}\bar{Y}$. This contradicts $Y = [Y,J(T)]$, and so completes the proof that $C_H(U_H) = O_2(H)$. It follows that $H = J(H)T$ with $H/O_2(H) \cong S_3$ or S_3 wr $\mathbf{Z}_2$, and in particular that $H \cap M = T$.

Let $X := \langle D_L, H\rangle$. Then $X \in \mathcal{H}(T)$ by 5.1.7.2.iii, as V_L is not the S_5-module. Set $U := \langle Z^X\rangle$, $Q_X := O_2(X)$ and $X^* := X/C_X(U)$. As $\tilde{V}_L$ is the natural module and $Z \leq V$, for $d \in D_L^{\#}$ we have $C_Z(d) = C_Z(L) < Z$, so that D_L is faithful on U. Thus $C_{D_LT}(U) = C_T(U)$. Also $C_H(U) \leq C_H(U_H) = O_2(H)$ from an earlier reduction. Thus $C_T(U) = C_H(U)$, so $C_T(U)$ is normal in $X = \langle D_L, H\rangle$. Finally $Q_X \leq C_T(U)$ as $U \in \mathcal{R}_2(X)$, so $Q_X = C_T(U)$ is Sylow in $C_X(U)$.

We next show that D_L^* does not act on K^*, so we assume that $D_L^* \leq N_{X^*}(K^*)$, and derive a contradiction during the next few paragraphs. First D_L acts on the preimage $KC_X(U)$ of K^*. Recall D_L acts on S, so that D_L normalizes $[C_U(S), K^*] = [C_U(S), K] =: U_K$. We saw that $S \in Syl_2(SK)$, so that $U_K \in \mathcal{R}_2(SK)$ by B.2.14. As $K = [K, J(T)]$, we may apply E.2.3.2 to U_K to conclude $K^*S^* = H_1^* \times \cdots \times H_s^*$ and $U_K = U_1 \oplus \cdots \oplus U_s$ with $s \leq 2$, $H_i^* \cong S_3$, and $U_i := [U_K, H_i] \cong E_4$. As $s \leq 2$, D_L normalizes H_i^* and U_i. Therefore D_L acts on $C_{U_i}(S) \cong \mathbf{Z}_2$, so D_L centralizes K^*S^* and U_K. Then as T normalizes K and $C_Z(D_L) = C_Z(L)$,

$$1 < Z \cap U_K \leq C_Z(D_L) = C_Z(L),$$

so that $C_X(U) \leq C_X(Z \cap U_K) \leq M = !\mathcal{M}(LT)$. Thus $C_X(U) \leq C_Z \leq N_G(T_L) \cap N_G(V_L)$ using paragraph three. Set $X_0 := O^2(C_X(U))$ and $C := C_{X_0}(\tilde{V}_L)$.

Suppose for the moment that there exists an odd prime divisor p of $|X_0|$ coprime to $2^n - 1$. Then as $O^2(\bar{C}_Z/\bar{T}_L)$ is a subgroup of $\mathbf{Z}_{2^n-1}$ by paragraph three, $O^{p'}(X_0) \leq C$. In this case set $X_1 := O^{p'}(X_0)$; then X_1 char $X_0 \trianglelefteq X$, so that $X_1 \trianglelefteq X$. Now suppose instead that q is any prime divisor of $2^n - 1$. Then $m_q(M) \leq 2$ as M is an SQTK-group, so as D_L is faithful on U, $m_q(X_0) \leq 1$. Thus if all odd prime divisors of $|X_0|$ divide 2^n-1, and C is not a 2-group, then for some odd prime p, $X_1 := O^{p'}(O_{2,p}(C)) \neq 1$, and X_0 has cyclic Sylow p-groups, so again X_1 char X_0, and $X_1 \trianglelefteq X$.

We have shown that if C is not a 2-group, then there is $1 \neq X_1 = O^2(X_1) \leq C$ with $X_1 \trianglelefteq X$. Thus $[L, X_1] \leq C_L([\tilde{V}, L]) = O_2(L)$, so that LT normalizes $O^2(O_2(L)X_1) = X_1$. But then $X \leq N_G(X_1) \leq M = !\mathcal{M}(LT)$, contradicting $H \not\leq M$. We conclude that C is a 2-group, and so $C_{X_0T}(\tilde{V}_L) = C_T(\tilde{V}_L)C = Q$ from paragraph two. Then as we saw that $C_X(U)$ normalizes V_L and T_L, X_0 normalizes $\mathrm{Baum}(T_LQ) = S$. Therefore as D_L acts on S and KX_0, D_L acts on $\langle S^{KX_0}\rangle = \langle S^K\rangle$, and hence on $O^2(\langle S^K\rangle) = K$.

Let $K_1 := O^2(K \cap H_1)$. We saw that H appears in case (2) of E.2.3, so S acts on K_1 with S Sylow in SK_1 and $SK_1/O_2(SK_1) \cong S_3$. As D_L normalizes H_1, D_L normalizes K_1S. Thus parts (a)–(d) of Hypothesis F.1.1 hold with LS, K_1S in the roles of "L_1", "L_2". By Theorem 4.3.2, $M = !\mathcal{M}(LS)$, so $O_2(\langle LS, K_1\rangle) = 1$, giving part (e). Finally as $LS \trianglelefteq LT$, $LS \in \mathcal{H}^e$ by 1.1.3.1, and similarly $K_1S \in \mathcal{H}^e$, giving part (f). Thus $\alpha := (LS, SD_L, K_1D_LS)$ is a weak BN-pair of rank 2 by F.1.9. Indeed as $N_{L_2}(S) \leq S$, α is described in F.1.12. Then α is not of type $L_3(q)$ since $n(K_1) = 1 < n(L)$. In all other cases of F.1.12, one of LS or K_1S centralizes $Z(S) \geq Z$, which is not the case. This contradiction shows that D_L^* does not act on K^*.

Recall that $H = J(H)T$, and U_H is an FF-module for $H/O_2(H) \cong S_3$ or S_3 wr $\mathbf{Z}_2$. Thus U is also an FF-module for X^*. By Theorem B.5.6, $J(X)^* = L_1^* \times \cdots \times L_s^*$ is a direct product of $s \leq 2$ subgroups L_i^* permuted by H, with either $L_i^* \cong L_2(2)$ or $F^*(L_i^*)$ quasisimple. In particular as $s \leq 2$, $O^2(X)$ normalizes each L_i^*. Choose numbering so that $L_0^* := L_1^* \cdots L_r^*$ is the product of those factors L_i^* upon which some X-conjugate of K projects nontrivially; in particular $K^* = [K^*, J(T)^*] \leq L_0^*$, $1 \leq r \leq 2$, and by construction $L_0^* \trianglelefteq X^*$. Thus $X^* = \langle K^*, D_L^* T^* \rangle = L_0^* D_L^* T^*$ and D_L acts on each L_i^*.

Now for $1 \leq i \leq r$, $[U, L_i^*]$ is an FF-module for L_i^*, and we claim L_i^* is on the following list: $L_k(2)$, $k = 2, 3, 4, 5$; S_k, $k = 5, 6, 7, 8$; A_k, $k = 6, 7, 8$; $\hat{A}_6$, or $G_2(2)$. For no L_i^* can be isomorphic to $L_2(2^m)$, $SL_3(2^m)$, $Sp_4(2^m)$, or $G_2(2^m)$ with $m > 1$, acting on the natural module, since in those cases $J(T)^*$ induces inner automorphisms on L_i^*, whereas T acts on the solvable group K and $K = [K, J(T)]$. Thus the claim follows from B.5.6 and B.4.2. Furthermore L_i^* is not isomorphic to $L_2(2)$ for all $i \leq r$, since D_L does not normalize K^* by a previous reduction.

As $D_L T = T D_L$ and the groups L_i^* do not appear in A.3.15, we conclude $O^3(D_L^*)$ centralizes L_i^*. So as D_L^* does not normalize $K^* \leq L_0^*$, $O^3(D_L^*) < D_L^*$. As $L/O_2(L) \cong L_2(2^n)$, it follows that 3 divides $2^n - 1$, so that n is even. As $Out(L_i^*)$ is a 2-group for each L_i^*, D_L induces inner automorphisms on L_0^*. Then as D_L is cyclic and L_i^* has no element of order 9, $D_L^*/C_{D_L^*}(L_1^* \cdots L_r^*)$ is of order 3.

Set $D_0 := O^2(D_L T)$ and let A_i^* be the projection of D_0^* on L_i^*. By the previous paragraph, $1 \neq A_i^*$ for some i, and $A_i^* = O_2(A_i^*) B^*$ for B^* of order 3. As D_0 is invariant under the Sylow group T, we conclude by inspection of the possibilities for L_i^* listed above that $A_i^* = O^2(P^*)$, where P^* is either a rank one parabolic over $T^* \cap L_i^*$, or a subgroup isomorphic to S_3 or S_4 containing $T^* \cap L_i^*$ in case $O_2(L_i^*) \cong A_7$. Let L_i denote the preimage of L_i^*. In each case $A_i^* = [T \cap L_i, A_i^*]$, so $O^{3'}(D_0) = [O^{3'}(D_0), T \cap L_i] \leq L_i$. It follows as D_L is cyclic that $A_i^* \neq 1$ for a unique i, and $T \cap L_i$ centralizes a subgroup of index 3 in $D_0/O_2(D_0)$. We conclude from the structure of $Aut(L/O_2(L))$ that $n = 2$; hence $D_L = O_3(D_L) \leq L_i$ and $D_0 T/O_2(D_0 T) \cong S_3$. We may choose notation so that $i = 1$.

As T acts on D_0, T acts on L_1, so as $O^2(X)$ normalizes each L_i, $L_1 \trianglelefteq X$. Recall by definition that the projection A^* of K^* on L_1^* is nontrivial. As A^* is T-invariant with $A^*/O_2(A^*) \cong \mathbf{Z}_3$ or E_9, arguing as in the previous paragraph, we conclude that $A^* = [A^*, T \cap L_1]$. Then as T acts on K, $A^* \cap K^* \neq 1$, so as T is irreducible on $K/O_2(K)$, $K^* = A^* \leq L_1^*$. Now as X acts on L_1, and D_L and K are contained in L_1, $X = \langle D_L, KT \rangle = L_1 T$.

Assume L_1^* is $L_2(2)$ or S_5. Then there is a unique T^*-invariant nontrivial solvable subgroup $Y^* = O^2(Y^*)$ of L_1^*. Hence $K^* = Y^* = D_0^*$, impossible as D_L^* does not act on K^*. Therefore L_1^* is $L_k(2)$, $3 \leq k \leq 5$, S_k or A_k, $6 \leq k \leq 8$, $\hat{A}_6$, or $G_2(2)$.

Suppose that $H/O_2(H) \cong S_3$ wr $\mathbf{Z}_2$. Then as $K^* \leq L_1^*$ and $X = L_1 T$, $X^* \cong Aut(L_k(2))$, $k = 4$ or 5, and K^* a rank-2 parabolic determined by a pair of non-adjacent nodes. As T normalizes D_0^*, with $D_0^*/O_2(D_0^*)$ of order 3, $k = 4$. Then as $[K_j, Z] \neq 1$ for $j = 1$ and 2, Theorems B.5.1 and B.4.2 show that $[U, L_1]$ is the sum of the natural module and its dual. But then $J(T)^* = O_2(K^*)$, contrary to $K = [K, J(T)]$.

This contradiction shows that $H/O_2(H) \cong S_3$. Recall also $D_0 T/O_2(D_0 T) \cong S_3$. Now $X = \langle H, D_0 T \rangle$, so that $O^2(X^*)$ is generated by K^* and D_0^*. We conclude $O^2(L_1^*)$ is $L_3(2)$, $U_3(3)$, A_6, A_7, or $\hat{A}_6$. Further neither D_0 nor K centralizes Z, so we conclude $X^* \cong S_7$ and $[U, L_1^*]$ is the natural module for X^*. From the description of offenders in B.3.2.4, $J(T)^*$ is generated by the three transpositions in T^*, so as $J(T) \trianglelefteq D_0 T$, it follows that D_0^* permutes these transpositions transitively, and hence $C_Z(D_0T) \cap [U, L_1]$ is a vector of weight 6, so that $C_{X^*}(C_Z(D_0T)) \cong S_6$. Now $C_Z(D_0T) = C_Z(D_L) = C_Z(L)$, so $C_X(C_Z(D_0T)) \leq M = !\mathcal{M}(LT)$. But this is impossible as $D_0 \trianglelefteq X \cap M$, completing the proof of 6.1.5. □

LEMMA 6.1.6. *(1) $C_Z(L) = 1$, and hence $C_T(L) = 1$.*
(2) V_L is the natural module for $\bar{L}$, and $V = V_L$ if $L = [L, J(T)]$.
(3) $V_L = [R_2(LT), L]$.

PROOF. If $C_Z(L) \neq 1$, then $C_G(Z) \leq C_G(C_Z(L)) \leq M = !\mathcal{M}(LT)$. But then for $H \in \mathcal{H}_*(T, M)$, $H \leq M$ by 6.1.5, contrary to $H \not\leq M$. This contradiction establishes (1). Then 6.1.4 and (1) imply V_L is the natural module for $\bar{L}$. The final statement of (2) follows as $V = C_V(L)[V, L]$ by E.2.3.2. Finally $V \leq R_2(LT)$, so $V_L \leq [R_2(LT), L]$. On the other hand, applying (2) to $R_2(LT)$ in the role of "V", L is irreducible on $[R_2(LT), L]$, so (3) holds. □

Now replacing V by V_L if necessary, we assume throughout the rest of this section that

$$V = V_L.$$

Thus by 6.1.6.2, V is the natural module for $\bar{L} \cong L_2(2^n)$. Since $L \trianglelefteq M$, and L is irreducible on V, using 6.1.3.1 we have:

LEMMA 6.1.7. *(1) V is a TI-set under M. Thus if $1 \neq U \leq V$, then $N_M(U) \leq N_M(V) = M_V$.*
(2) Hypothesis E.6.1 holds, so we may apply results from section E.6.

Using 3.1.4.1, 6.1.7, and 6.1.5 we have:

LEMMA 6.1.8. *If $H \leq N_G(U)$ for $1 \neq U \leq V$, then $H \cap M = N_H(V)$. In particular $H \cap M = N_H(V)$ for each $H \in \mathcal{H}_*(T, M)$.*

Let $Z_S := C_V(T_L)$, so that Z_S is a 1-dimensional $\mathbf{F}_{2^n}$-subspace of the natural module V. Let $S := C_T(Z_S)$.

LEMMA 6.1.9. *(1) $S = T_L O_2(LT)$ and $S \in Syl_2(C_G(Z_S))$.*
(2) $N_G(S) \leq M$.
(3) $F^*(N_G(Z_S)) = O_2(N_G(Z_S))$.
(4) $V \leq O_2(C_G(Z_S))$ and $V/Z_S \leq Z(S/Z_S)$.
(5) $N_G(Z_S) = C_G(Z_S)N_M(Z_S) = C_G(Z_S)N_{M_V}(Z_S)$.
(6) $J(T) = J(S)$ and $Baum(T) = Baum(S)$.

PROOF. As $T \leq N_G(Z_S)$, (3) holds by 1.1.4.6, and also $C_T(Z_S) = S \in Syl_2(C_G(Z_S))$. As V is the natural module for $\bar{L}$, the remaining assertion of (1) holds, and also $V/Z_S \leq Z(S/Z_S)$. Then an application of G.2.2.1, with $N_G(Z_S)$ in the role of "H", establishes the remaining assertion of (4).

Now using (1), we may apply a Frattini Argument to conclude that $N_G(Z_S) = C_G(Z_S)(N_G(Z_S) \cap N_G(S))$. Thus (5) will follow from (2) since V is a TI-set in M by 6.1.7; so it remains to prove (2) and (6).

If $J(T) \leq C_T(V)$, then in particular $J(T) \leq S$. On the other hand, if $J(T)$ does not centralize V, then as V is the natural module for $\bar{L}$, $J(T) \leq S$ by B.4.2.1. Therefore as $S = C_T(Z_S)$, (6) follows from B.2.3.5. Finally Theorem 4.3.17 implies (2). □

LEMMA 6.1.10. *(1)* $r(G,V) \geq n$.
(2) $s(G,V) = m(Aut_M(V), V) = n$.
(3) Suppose that V^g *normalizes but does not centralize* V *for some* $g \in G$. *Then* $m(V^g/C_{V^g}(V)) = n$.

PROOF. As V is the natural module for $\bar{L}$, $m(Aut_M(V), V) = n$. By 6.1.7.2, V satisfies Hypothesis E.6.1. Thus if $n > 2$, (1) and (2) hold by Theorem E.6.3. So assume $n = 2$, and let $U \leq V$ with $m(V/U) = 1$. As V is the natural module, L is transitive on $\mathbf{F}_2$-hyperplanes of V, so we may choose $U \trianglelefteq T$. Then E.6.13 says $C_G(U) \leq M$. Thus in any case, $r(G,V) \geq n = m(Aut_M(V), V)$, so that (1) and (2) are established.

Assume the hypotheses of (3), and set $U := C_{V^g}(V)$. As $V^g \leq N_G(V)$,

$$m(V^g/U) \leq m_2(LT/C_{LT}(V)) = n.$$

On the other hand as $V \not\leq C_G(V^g)$, $m(V^g/U) \geq s(G,V) = n$ by E.3.7 and (2), establishing (3). □

LEMMA 6.1.11. *Suppose* $V^g \leq T$ *with* $1 \neq [V, V^g] \leq V \cap V^g$. *Then* $Z_S = [V, V^g] = V \cap V^g$ *and* $V^g \in V^{C_G(Z_S)}$.

PROOF. Let $A := V^g$. By 6.1.10.3, $m(A/C_A(V)) = m(V/C_V(A)) = n$, so that $\bar{A}$ is an FF^*-offender on V. Therefore by B.4.2.1, $\bar{A} \in Syl_2(\bar{L})$ and $Z_S = [A, V] = C_V(A)$. As V normalizes A by hypothesis, we have symmetry between A and V, so $Z_S = C_A(V)$. Therefore $Z_S^{g^{-1}} = C_V(V^{g^{-1}})$ is a 1-dimensional $\mathbf{F}_{2^n}$ subspace of V, and hence $Z_S^{g^{-1}} = Z_S^h$ for some $h \in L$ by transitivity of L on such subspaces. Thus $V^g = V^{hg}$ with $hg \in N_G(Z_S)$, so $V^g \in V^{C_G(Z_S)}$ by 6.1.9.5. □

LEMMA 6.1.12. *(1) Either* $N_G(W_0(T,V)) \not\leq M$ *or* $C_G(C_1(T,V)) \not\leq M$.
(2) If $n > 2$, *then* $Z_S \leq C_1(T,V)$.
(3) $W_0(T,V) \leq S$.

PROOF. By Hypotheses 6.1.1, $n(H) = 1$ for each $H \in \mathcal{H}_*(T,M)$. Hence as $H \not\leq M$, part (1) follows from 6.1.10.2 and E.3.19. Assume $A \leq V^g \cap T$, with $w := m(V^g/A)$ satisfying $n - w \geq 2$. By 6.1.10, $n = s(G,V)$, so by E.3.10, either $\bar{A} = 1$ or $\bar{A} \in \mathcal{A}_2(\bar{T}, V)$. In either case, $\bar{A} \leq \bar{T}_L$, so that $A \leq S$ by 6.1.9.1. Since $n \geq 2$, (3) follows from this observation in the case $w = 0$. If $n > 2$, (2) follows from the observation in the case $w = 1$. □

LEMMA 6.1.13. *Let* $U \leq V$ *with* $m(V/U) = n$. *Then one of the following holds:*
(1) $C_G(U) \leq N_G(V)$.
(2) $U \in Z_S^L$.
(3) n *is even, and* $U = C_V(t)$ *for some* $t \in M$ *inducing an involutory field automorphism on* $\bar{L}$.

PROOF. If U does not satisfy either (2) or (3), then $C_M(U) = C_M(V)$. Then as $r(G,V) \geq n > 1$ by 6.1.10.1, (1) holds by E.6.12. □

LEMMA 6.1.14. *Assume* n *is even and* $U = C_V(t)$ *for some* $t \in T$ *inducing an involutory field automorphism on* $\bar{L}$. *Choose notation so that* $T_U := N_T(U) \in Syl_2(N_M(U))$. *Then*
(1) $R := Q\langle t\rangle \in Syl_2(C_G(U))$, $N_G(J(R)) \leq M$, *and* $T_U \in Syl_2(N_G(U))$.
(2) $N_G(U)$ *and* $C_G(U)$ *are in* $\mathcal{H}^e$.

(3) $W_0(R,V) \leq C_T(V)$*, and if* $n > 2$ *then* $W_1(R,V) \leq C_T(V)$.

(4) $V_U := \langle V^{N_G(U)} \rangle$ *is elementary abelian, and* $V_U/U \in \mathcal{R}_2(N_G(U)/U)$*; further* $[O_2(N_G(U)), V_U] \leq U$.

(5) Assume further that $n > 2$*, and* $V^g \leq C_G(U)$ *is* V*-invariant with* $[V, V^g] \neq 1$*. Then* $C_G(Z_S) \not\leq M$.

PROOF. Observe that $R := C_T(U) = Q\langle t \rangle$, where $Q := C_T(V)$, and U and V/U are the natural module for $N_L(U)/O_2(N_L(U)) \cong L_2(2^{n/2})$. Now $\mathcal{A}(R) = \mathcal{A}(Q)$, so $J(R) = J(Q) \trianglelefteq LT$, and hence $N_G(R) \leq N_G(J(R)) \leq M = !\mathcal{M}(LT)$, so $R \in Syl_2(C_G(U))$. Similarly $T_U := N_T(U) \in Syl_2(N_G(U))$ since $J(T_U) = J(Q)$. Thus (1) holds. As $U \cap Z \neq 1$, $F^*(N_G(U)) = O_2(N_G(U))$ by 1.1.4.3. Then $C_G(U) \in \mathcal{H}^e$ by 1.1.3.1, so (2) holds.

Next by 6.1.12, $W_i := W_i(R,V) \leq C_R(Z_S) \leq Q$ for $i = 0$ when $n \geq 2$, and for $i = 1$ when $n > 2$. Thus (3) holds.

Let $V_U := \langle V^{N_G(U)} \rangle$ and $N_G(U)^* := N_G(U)/C_G(V_U)$. We may apply G.2.2 with U, V, $O^2(C_L(U))$, T_U, $N_G(U)$ in the roles of "V_1, V, L, T, H". ByG.2.2.4, $V_U/U \in \mathcal{R}_2(N_G(U)/U)$. By G.2.2.1, $V_U \leq O_2(C_G(U))$ and $[O_2(N_G(U)), V_U] \leq U$. Then $V_U \leq O_2(C_G(U)) \leq R$ using (1), so that $V_U \leq W_0(R,V) = W_0$. Therefore as $W_0 \leq C_T(V)$ by (3), $V_U = \langle V^{N_G(U)} \rangle$ is elementary abelian. This establishes (4).

Now assume the hypotheses of (5). First $m(V/C_V(V^g)) = n$, by applying 6.1.10.3 with the roles of V, V^g reversed. Then as $U \leq C_V(V^g)$ with $m(U) = m(V/U) = n$, we conclude $U = C_V(V^g)$. As $n > 2$, $L_U := O^2(N_L(U)) \in \mathcal{L}(N_G(U), T_U)$. As $T_U \in Syl_2(N_G(U))$ by (1), $L_U \leq K \in \mathcal{C}(N_G(U))$ by 1.2.4. As $[U, L_U] = U$, $C_K(U) \leq O_\infty(K)$. By (1) and a Frattini Argument, $KR = C_{KR}(U)N_{KR}(J(R)) = C_K(U)(K \cap M)R$. Now $L_U = L_U^\infty \trianglelefteq K \cap M$, and $K/O_\infty(K)$ is simple by A.3.3.1, so $K = L_U C_K(U)$. Thus if $C_K(U) \leq M$, then $K \leq M$, so that $K = K^\infty = L_U$. On the other hand, if $C_K(U) \not\leq M$, then also $O_\infty(K) \not\leq M$.

By (3) and E.3.16, $N_G(W_0) \leq M \geq C_G(C_1(R,V))$. Each solvable subgroup X of $C_G(U)$ containing R satisfies $n(X) = 1$ by E.1.13, and so is contained in M by E.3.19. This eliminates the exceptional case $O_\infty(K) \not\leq M$ of the previous paragraph, so that $L_U = K$. Since T_U normalizes $L_U \in \mathcal{C}(N_G(U))$, and is Sylow in $N_G(U)$ by (1), $N_G(U)$ normalizes L_U by 1.2.1.3. Then as $O_2(L_U) \leq Q \leq C_G(V)$, $O_2(L_U) \leq C_G(V_U)$.

Recall V_U is elementary abelian by (4). As V is the direct sum of two copies of the natural module U for $L_U/O_2(L_U)$, and $L_U \trianglelefteq N_G(U)$, V_U is the sum and hence the direct sum of copies of the natural module for $L_U/O_2(L_U)$. Next as $V^g \leq C_G(U)$, $V^g \leq R^h$ for some $h \in C_G(U)$, so by (3)

$$V^g \leq W_0(R^h, V) \leq Q^h \leq O_2(T_U^h L_U).$$

Thus $[V^g, L_U] \leq [O_2(T_U^h L_U), L_U] \leq O_2(L_U) \leq C_G(V_U)$. Thus L_U normalizes $Z_1 := [V^g C_G(V_U), V] = [V^g, V]$. We saw earlier that $U = C_V(V^g)$ with $m(V/U) = n$. Then as $n > 2$, $V \leq S^g$, so that in fact $S^g = VO_2(L^g S^g)$. Hence $Z_1 = [V, V^g] = [S^g, V^g] = Z_S^g$.

We finally assume that $C_G(Z_S) \leq M$. Then $N_G(Z_S) \leq M$ by 6.1.9.5, so

$$L_U \leq N_G(Z_1) = N_G(Z_S^g) = N_{M^g}(Z_S^g) \leq N_{M^g}(V^g),$$

since V is a TI-set in$M$ by 6.1.7. This is impossible, as the L_U^*-submodule $V \cap V^g = Z_1 = Z_S^g$ of rank n in V_U is natural by an earlier remark, whereas $Aut_M(Z_S)$ is solvable. This contradiction establishes (5), and so completes the proof of the lemma. □

PROPOSITION 6.1.15. *Either*

(1) $C_G(Z_S) \not\leq M$, *or*

(2) $n = 2$, *and either* $N_G(W_0(T,V)) \not\leq M$ *or* $W_1(T,V) \not\leq S$.

PROOF. Set $W_0 := W_0(T,V)$. Suppose first that $N_G(W_0) \not\leq M$. Then as $M = !\mathcal{M}(LT)$, $W_0 \not\leq O_2(LT) = C_T(V)$ by E.3.16.1, so there is $V^g \leq T \leq N_G(V)$ which does not centralize V. Set $U := C_{V^g}(V)$; then $m(V^g/U) = n$ by 6.1.10.3, so that 6.1.13 applies to U with V^g is in the role of "V". If V acts on V^g, then by 6.1.11, $V^g \in V^{C_G(Z_S)}$, while $V^M \leq O_2(L) \leq C_G(V)$, so (1) holds. Therefore we may assume $V \not\leq N_G(V^g)$. In particular $C_G(U) \not\leq N_G(V^g)$, so that case (1) of 6.1.13 does not hold. If case (2) of 6.1.13 holds, then again (1) holds. If case (3) of 6.1.13 holds with $n > 2$, then $v \in V - C_V(V^g)$ induces a field automorphism on V^g with $U = C_{V^g}(v)$ and V is V^g-invariant with $1 \neq [V, V^g]$, so by 6.1.14.5, (1) holds yet again. Finally if $n = 2$, then (2) holds as we are assuming that $N_G(W_0) \not\leq M$.

Thus we may instead assume that $N_G(W_0) \leq M$. Therefore by 6.1.12.1, $C_G(C_1(T,V)) \not\leq M$. Thus if $Z_S \leq C_1(T,V)$, then (1) holds. On the other hand if $Z_S \not\leq C_1(T,V)$ then $n = 2$ by 6.1.12.2, and also $W_1(T,V) \not\leq S$, so (2) holds. □

6.1.2. Reducing to $C_G(Z_S) \leq M$ and $n = 2$.

In this subsection, we consider the first case of 6.1.15, where $C_G(Z_S) \not\leq M$. Our object is to establish a contradiction and so eliminate that case; this is accomplished in Theorem 6.1.27. In the following chapter, we show that in the second case, G is isomorphic to M_{22}.

Hence in this subsection, we assume:

HYPOTHESIS 6.1.16. $C_G(Z_S) \not\leq M$, *where* $Z_S := C_V(T_L)$.

Let $I := C_G(Z_S)$ and

$$\mathcal{H}_S := \{H \in \mathcal{H}(T) : H \not\leq M \text{ and } O^2(H) \leq I\}.$$

In particular $IT \in \mathcal{H}_S$, so that $\mathcal{H}_S$ is nonempty.

Let H denote some arbitrary member of $\mathcal{H}_S$. As $O^2(H) \leq I$, $H = O^2(H)T \leq IT \leq N_G(Z_S)$. Set $U_H := \langle V^H \rangle$, $H_S := C_H(Z_S)$, $Q_H := O_2(H_S)$, and $\tilde{H} := H/Z_S$.

Notice that $U_{IT} = \langle V^I \rangle$, $(IT)_S = I$, and $Q_{IT} = O_2(I)$. Also a Hall $2'$-subgroup D_L of $N_L(T_L)$ normalizes Z_S and hence I, but $D_L \cap I = 1$. Then as $N_G(Z_S)$ is an SQTK-group,

$$m_p(D_L I) \leq 2 \text{ for each odd prime } p.$$

LEMMA 6.1.17. *(1)* $V \leq Q_H$, $S \in Syl_2(H_S)$, $F^*(H_S) = O_2(H_S) = Q_H$, *and* $H_S \trianglelefteq H = H_S T$.

(2) $\tilde{U}_H \in \mathcal{R}_2(\tilde{H}_S)$, *so* $\tilde{U}_H \leq Z(\tilde{Q}_H)$.

(3) $Q_H = C_{H_S}(\tilde{U}_H)$.

(4) For $s \in S - C_S(V)$ *and* $Z_S \leq Y \leq V$, $[V,s] = Z_S$ *and* $m([Y,s]) = m(Y/Z_S)$.

(5) If $Z_S \leq Y \leq V$ *with* $|V : Y| = 2$, *and* $\bar{S}_0$ *is a noncyclic subgroup of* $\bar{S}$, *then* $Z_S = [Y, S_0]$.

PROOF. As $H \in \mathcal{H}(T)$, $F^*(H) = O_2(H)$ by 1.1.4.6. We saw $H \leq N_G(Z_S)$, so that $H_S = C_H(Z_S) \trianglelefteq H$; then $F^*(H_S) = O_2(H_S) = Q_H$ by 1.1.3.1, and $S = C_T(Z_S) \in Syl_2(H_S)$. Recall also $T \leq H \leq IT$, so that $H = T(H \cap I) = TH_S$. As

V is the natural module for $\bar{L}$, $[S,V] = Z_S$; therefore $\tilde{V}$ is central in $\tilde{S} \in Syl_2(\tilde{H}_S)$, and hence $\tilde{U}_H = \langle \tilde{V}^H \rangle \in \mathcal{R}_2(\tilde{H}_S)$ by B.2.14. This establishes (2).

Next $C_H(\tilde{U}_H) \leq N_H(V) \leq M$, and further $X := O^2(C_{H_S}(\tilde{U}_H)) \leq C_M(V) \leq C_M(L/O_2(L))$, so that LT normalizes $O^2(XO_2(L)) = X$. Hence if $X \neq 1$, then $H \leq N_G(X) \leq M = !\mathcal{M}(LT)$, contradicting $H \not\leq M$. Therefore $X = 1$, so $C_{H_S}(\tilde{U}_H) \leq O_2(H_S) = Q_H$; then (3) follows from (2). Parts (4) and (5) follow from the fact that V is the natural module for $\bar{L}$. □

Let $G_1 := LT$, $G_2 := H$, and $G_0 := \langle G_1, G_2 \rangle$. Notice Hypothesis F.7.6 is satisfied: in particular $O_2(G_0) = 1$ as $G_2 \not\leq M = !\mathcal{M}(G_1)$. Form the coset geometry $\Gamma := \Gamma(G_0; G_1, G_2)$ as in Definition F.7.2, and adopt the notation in section F.7. In particular for $i = 1, 2$ write γ_{i-1} for G_i regarded as a vertex of Γ, let $b := b(\Gamma, V)$, and pick $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$ and $V \not\leq G_\gamma^{(1)}$. Without loss, γ_1 is on the geodesic

$$\gamma_0, \gamma_1, \cdots, \gamma_b := \gamma.$$

Observe in particular that U_H plays the role played by "V_{γ_1}" in section F.7. For $\alpha := \gamma_0 x \in \Gamma_0$ let $V_\alpha := V^x$. For $\beta := \gamma_1 y \in \Gamma_1$ let $Z_\beta := Z_S^y$ and $U_\beta = U_H^y$.

Notice that by 6.1.17.1 and F.7.7.2, $V \leq Q_H \leq G_{\gamma_1}^{(1)}$, so that by F.7.9.3:

LEMMA 6.1.18. $b > 1$.

LEMMA 6.1.19. *Suppose there exists $H \in \mathcal{H}_S \cap \mathcal{H}_*(T, M)$ with b odd. Then $n = 2$ and $\langle V^I \rangle$ is nonabelian.*

PROOF. Assume b is odd. By 6.1.18, $b > 1$, so $b \geq 3$. Then U_H is elementary abelian by F.7.11.4.

Further by F.7.11.5, $U_H \leq G_\gamma$ and $U_\gamma \leq H$, so applying 6.1.12.3 to suitable Sylow 2-subgroups of G_γ and H, we obtain:

$$U_H \leq C_{G_\gamma}(Z_\gamma), \text{ and } U_\gamma \leq C_{G_{\gamma_1}}(Z_S) = H_S. \tag{!}$$

Observe that the hypotheses of F.7.13 are satisfied: We just verified hypothesis (a) of F.7.13, and hypothesis (c) holds by 6.1.1.2 as $H \in \mathcal{H}_*(T, M)$. Also as $H \in \mathcal{H}_*(T, M)$, $H \cap M$ is the unique maximal subgroup of H containing T by 3.3.2.4. Finally $H \cap M = N_H(V)$ by 6.1.8, so hypothesis (b) of F.7.13 holds. Applying F.7.13 to $A := U_H$, we conclude there is $\alpha \in \Gamma(\gamma)$ with $B := N_A(V_\alpha)$ of index 2 in A. Write $E := V_\alpha$. If $[E, B] = 1$, then as $s(G, V) > 1$ by 6.1.10.2, for each $h \in H$

$$E \leq C_G(B) \leq C_G(B \cap V^h) \leq C_G(V^h).$$

But then $[E, A] = 1$, contrary to $B < A$. Therefore $[E, B] \neq 1$. So as $A \leq C_{G_\gamma}(Z_\gamma)$ by (!), $[E, B] = Z_\gamma$ by 6.1.17.4.

Suppose that $E \leq Q_H$. Then $[A, E] \leq Z_S$ by 6.1.17.2, so that $Z_\gamma = [B, E] \leq [A, E] \leq Z_S$. Hence $Z_S = Z_\gamma$, as these groups are conjugate and so have the same order. This is impossible, as $V \leq O_2(C_G(Z_S))$ by 6.1.17.2, while $V \not\leq O_2(G_\gamma)$ by choice of γ, and $G_\gamma \leq N_G(Z_\gamma)$.

Therefore $E \not\leq Q_H$, so since $U_\gamma \leq H_S$ by (!), also $E \not\leq O_2(H)$. But as $H \in \mathcal{H}_*(T, M)$, by 3.3.2.4 we may apply B.6.8.5 to conclude that $O_2(H) = O^{2'}(G_1^{(1)})$, so that $E \not\leq G_{\gamma_1}^{(1)}$. Thus $d(\alpha, \gamma_1) = b$ with $\alpha, \gamma, \cdots, \gamma_1$ a geodesic, so we have symmetry between γ and γ_1. Using this symmetry, and applying F.7.13 to E in the role of "A", we conclude there is $\delta \in \Gamma(\gamma_1)$ such that $F := N_E(V_\delta)$ is a hyperplane of E.

Then applying the subsequent arguments with F in the role of "B", $[F, V_\delta] = Z_S$ and $V_\delta \not\leq Q_\gamma$, so replacing γ_0 by δ, we may assume that $\delta = \gamma_0$ and $V_\delta = V$.

Let $V_B := V \cap B = N_V(E)$. Notice V_B is of index at most 2 in V, as B is of index 2 in U_H. Then $[V_B, F] \leq Z_S \cap Z_\gamma$, with V_B, F of index 2 in V, E. Therefore $[V_B, F]$ is of index at most 2 in Z_S and Z_γ by (!) and 6.1.17.4. Further if $[V_B, F] = Z_S$, then $Z_S = [V_B, F] = Z_\gamma$, which we saw earlier is not the case. Hence $[V_B, F]$ is of index 2 in both Z_S and Z_γ, so $|V : V_B| = 2$ by 6.1.17. Therefore by 6.1.17.5, $n = 2$, and $\langle z \rangle := [V_B, F] = Z_S \cap Z_\gamma$ is of order 2. Thus we have established the first assertion of 6.1.19.

As D_L is transitive on $Z_S^\#$, z is 2-central in LT, so we may assume $T \leq G_z$. Thus $H \leq G_z$. As D_L is transitive on $Z_S^\#$, and L is transitive on $V^\#$, we conclude from A.1.7.1 that $G_z := C_G(z)$ is transitive on the G-conjugates of Z_S and V containing z. Then $Z_\gamma = Z_S^g$ for $g \in G_z$. Similarly if $V \leq O_2(G_z)$, then $E \leq O_2(G_z)$ as $E \in V^{G_z}$; but then $E \leq O_2(G_z) \leq O_2(H)$, contrary to an earlier reduction. We conclude $V \not\leq O_2(G_z)$.

Let $W_0 := \langle V^I \rangle$; to complete the proof, we assume W_0 is abelian and it remains to derive a contradiction. Let $Q_z := \langle Z_S^{G_z} \rangle$. By 1.1.4.6, $F^*(G_z) = O_2(G_z)$. As $n = 2$, $[Z_S, T] \leq \langle z \rangle$, so $Q_z \leq O_2(G_z)$ by B.2.14 applied in $\hat{G}_z := G_z/\langle z \rangle$, and hence $Q_z \leq T$. Let $W := W_0 \cap Q_z$; as $I \leq G_z$, $I \leq N_G(W)$. Set $I^* := I/C_I(\hat{W})$. Now $Q_z \leq T \leq N_G(V)$, so $Q_z \leq \ker_{G_z}(N_{G_z}(V))$. Then as $E \in V^{G_z}$, Q_z acts on E, and in particular W acts on E. We have seen that $E \leq I \leq N_G(W)$, so that $[W, E] \leq W \cap E$. Next as $V_B \leq W_0$, $[V_B, E] \leq W_0$. But $Z_\gamma \leq Q_z$ as $Z_\gamma \in Z_S^{G_z}$, and $Z_\gamma = [V_B, E]$ by (!) and 6.1.17.4, so $Z_\gamma \leq W \cap E$. Finally if $Z_\gamma < E \cap W$, then $m(E/(E \cap W)) \leq 1$ since $n = 2$. Then as $V \leq W_0$ and W_0 is abelian by assumption, $V \leq C_G(E \cap W) \leq C_G(E)$ by 6.1.10.2, contrary to $[V_B, F] = \langle z \rangle$. Thus $[E, W] \leq E \cap W = Z_\gamma$, so $[E^*, \hat{W}] \leq \hat{Z}_\gamma$ of order 2, and hence E^* is trivial or induces a group of transvections on $\hat{W}$ with center $\hat{Z}_\gamma = \hat{Z}_S^g$.

Note that $C_I(\hat{W}) \leq N_G(Z_S^g) \leq N_G(O_2(I^g))$, so that

$$O_2(I^g) \cap C_I(\hat{W}) \leq O_2(C_I(\hat{W})) \leq O_2(I). \qquad (*)$$

Then as $E \leq U_\gamma \leq O_2(I^g)$, but we saw $E \not\leq O_2(H)$, we conclude from (*) that E does not centralize $\hat{W}$, so that $E^* \neq 1$. As W_0 is abelian, $Z_\gamma \leq C_I(\tilde{W})$, so we conclude $1 \leq m(E^*) \leq m(E/Z_\gamma) = n = 2$.

Let $P := \langle E^I \rangle$. As E centralizes Z_S but $N_E(V) = F < E$, $P \not\leq M$ by 6.1.7.1. As $E \leq O_2(I^g)$ and we saw $C_I(\hat{W})$ acts on $O_2(I^g)$, it follows from (*) that

$$[E, C_I(\hat{W})] \leq O_2(I^g) \cap C_I(\hat{W}) \leq O_2(I),$$

so we conclude that $C_P(\hat{W}) \leq O_{2,Z}(P)$. Let P_0 denote the preimage in P of $O_2(P^*)$. Then $P_0 \leq O_{2,Z,2}(P) = O_{2,Z}(P)$, so that $P_0 = O_2(P)C_P(\hat{W})$, and hence $O_2(P^*) = O_2(P)^*$. On the other hand, by 6.1.17.2, $O_2(P) \leq O_2(I) \leq C_I(\hat{W}_0) \leq C_I(\hat{W})$, so $O_2(P^*) = O_2(P)^* = 1$, and then $\hat{W} \in \mathcal{R}_2(P^*)$. Thus as E^* induces a group of transvections on $\hat{W}$ with center $\hat{Z}_\gamma$ of order 2, we see from G.6.4 that P^* is the direct product of subgroups X_i^* isomorphic to S_m or $L_k(2)$ for suitable m and k. So either $X_i^* \cong L_2(2) \cong S_3$, or X_i^* is nonsolvable, in which case as the preimage X_i is normal in P and P is subnormal in $N_G(Z_S)$, $X_i^\infty \in \mathcal{C}(N_G(Z_S))$. In that case, as $D_L \cong \mathbf{Z}_3$ and $D_L \cap I = 1$, we conclude from A.3.18 that $m_3(X_i^*) = 1$. Therefore

$X_i^* \cong S_3$, S_5 or $L_3(2)$. In particular now $O_{2,Z}(P) = O_2(P)$ as the multiplier of these groups is a 2-group. Thus $P^* = P/O_2(P)$.

Note that $O^2(I) \leq N_I(X_i^*)$ by G.6.4.3. Next if X_i^* is not S_3, then D_L normalizes $O^2(X_i) = X_i^\infty$ by 1.2.1.3. On the other hand, if $X_i^* \cong S_3$, then for $d \in D_L$, $O^2(X_i)^d \leq O^2(I) \leq N_I(X_i^*)$. Then recalling that $m_3(I) \leq 2$, either $O^2(X_i) = O^2(X_i)^d$, or else $X_iO^2(X_i^d)/O_2(X_iO^2(X_i)^d) \cong S_3 \times \mathbf{Z}_3$ and $O^2(X_i)O^2(X_i)^d = O^{3'}(I) =: J$. In the latter case, $I/C_I(J/O_2(J)) \cong S_3 \times S_3$ or S_3 wr $\mathbf{Z}_2$, whose outer automorphism groups are 2-groups, so the former must hold. Thus in any case, D_L and $O^2(I)$ act on each X_i. So as $m_3(ID_L) \leq 2$, $P = X_1$ and $O^2(P) = O^{3'}(I)$. If $P^* \cong S_5$, then the T-invariant Borel subgroup of P is not contained in M—for otherwise, $TP \in \mathcal{H}_*(T, M)$ with $n(PT) > 1$, contrary to 6.1.1.2. If P^* is $L_3(2)$ then T induces inner automorphisms on P^* by G.6.4.2a. Thus in each case there exists a TD_L-invariant parabolic subgroup P_1 of P, with $P_1 \not\leq M$ and $TP_1/O_2(P_1) \cong S_3$. Then $\theta := (LT, D_LT, P_1D_LT)$ satisfies Hypothesis F.1.1, and so by F.1.9 defines a weak BN-pair. Moreover the hypotheses of F.1.12 are satisfied by P_1D_LT, so that θ is described in one of the cases of F.1.12.I. Since $L/O_2(L) \cong L_2(4)$ and $P_1/O_2(P_1) \cong L_2(2)$, the only possibility there is the $U_4(2)$-amalgam, which cannot occur here, since in that amalgam V is the A_5-module for $L/O_2(L)$. This contradiction completes the proof of 6.1.19. □

Let $U := \langle V^I \rangle$ and recall $\tilde{H} = H/Z_S$.

LEMMA 6.1.20. *(1) $U \leq O_2(I)$ and $\tilde{U} \leq Z(O_2(\tilde{I}))$.*
(2) U is nonabelian.
(3) For $x \in U - Z(U)$, $[U, x] = Z_S$.
(4) $U/C_U(V) \cong E_{2^n}$. Further for $g \in I$ with $[V, V^g] \neq 1$, $U = VV^gC_U(VV^g)$, and $\{V, V^g\}$ is the set of maximal elementary abelian subgroups of VV^g.

PROOF. Pick $H \in \mathcal{H}_*(T, M)$. If b is odd, then (2) holds by 6.1.19. On the other hand, if b is even, then $1 \neq [V, V_\gamma] \leq V \cap V_\gamma$ by F.7.11.2, so that $V_\gamma \leq N_G(V) \leq M$, and we may take $V_\gamma \leq T$. Then by 6.1.11, $Z_S = [V, V_\gamma]$ and $V_\gamma \in V^I$. So (2) is established in this case also.

Part (1) folows from 6.1.17.2 applied to IT in the role of "H". For $x \in U - Z(U)$, x does not centralize all I-conjugates of V; so replacing x by a suitable I-conjugate, we may assume $[x, V] \neq 1$. Then as $x \in O_2(I) \leq S$, $[x, V] = Z_S$ by 6.1.17.4, so (3) holds. By (2) we may choose $g \in I$ with $[V, V^g] \neq 1$; by (1), $V^g \leq N_S(V)$. Then by 6.1.10, $m(V^g/C_{V^g}(V)) = n = m(S/C_S(V))$, so $S = V^gC_S(V)$, and hence also $U = V^gC_U(V)$. Then we conclude that (4) holds from the symmetry between V and V^g. □

For the remainder of the section, we choose $H := N_G(Z_S)$; in contrast to our earlier convention, this "H" is not in $\mathcal{H}_S$. We also pick $g \in I$ with $[V, V^g] \neq 1$; such a g exists by 6.1.20.2. As $N_L(Z_S)$ is irreducible on V/Z_S, Hypothesis G.2.1 is satisfied with Z_S in the role of "V_1". Recall from section G.2 that the condition U nonabelian in 6.1.20.2 is equivalent to $\bar{U} \neq 1$. Thus we have the hypotheses of G.2.3, so we can appeal to that lemma.

LEMMA 6.1.21. *Let $l \in L - H$, and set $L_1 := \langle U, U^l \rangle$, $R := O_2(L_1)$, and $E := U \cap U^l$. Then*
(1) $L_1 = \langle U^{M_V} \rangle \trianglelefteq M_V$ and $L_1 = LU$.
(2) $R = C_U(V)C_{U^l}(V)$ and $UR \in Syl_2(L_1)$.

(3) $\Phi(E) = 1$, $E/V \leq Z(L_1/V)$, *and* $E = \ker_U(M_V) \trianglelefteq M_V$.

(4) $\Phi(R) \leq E$, *and* $R/E = C_U(V)/E \times C_{U^l}(V)/E$ *is the sum of natural modules for* L_1/R *with* $C_U(V)/E = C_{R/E}(U)$.

(5) $M_V \leq N_G(R)$; *in particular,* $R \leq O_2(M_V)$.

PROOF. As we just observed, we may apply G.2.3 with Z_S in the role of "V_1"; in that application, L_1, R, E play the roles of "I, S, S_2".

Now $L_1 = LU$ by G.2.3.2 and $LU = LO_2(LU)$ by G.2.3.1, so $O_2(L_1) = L_1 \cap O_2(LT)$. Hence $U \cap O_2(L_1) = C_U(V)$, so that $C_U(V)$ plays the role of "W". Then (2) follows from parts (3) and (1) of G.2.3, while (4) follows from G.2.3.6. By G.2.3.5, $E/V \leq Z(L_1/V)$ and $\Phi(E) = 1$. Thus it remains to establish the first statement of (1), the last statement of (3), and (5).

Now $U = \langle V^{C_G(Z_S)} \rangle$, so as $N_G(Z_S) = N_{M_V}(Z_S)C_G(Z_S)$ by 6.1.9.4, $N_G(Z_S)$ acts on U. Next $Z_S^{M_V} = Z_S^L$, so that $M_V = N_{M_V}(Z_S)L \leq N_{M_V}(U)L$. Then as $L_1 = LU$, $L_1 \trianglelefteq M_V$, completing the proof of (1). Similarly $\ker_U(M_V) = \ker_U(L_1) \leq U \cap U^l = E$ and $E \trianglelefteq L_1$ by G.2.3.4, so $E = \ker_U(L_1)$, completing the proof of (3). Finally (5) holds as $R = O_2(L_1)$ and $L_1 \trianglelefteq M_V$ by (1). □

During the remainder of the section, R and E are as defined in 6.1.21.

LEMMA 6.1.22. $E < R$.

PROOF. Assume that $R = E$. In particular $R \leq U$, and hence $R = C_U(V)$ by 6.1.21.2. By 6.1.20.2, we may choose $g \in I$ with $[V, V^g] \neq 1$; then $U = VV^gC_U(VV^g)$ by 6.1.20.4. Also $C_U(VV^g) = C_R(V^g) = C_E(V^g)$. By 6.1.21.3, $\Phi(E) = 1$, while by 6.1.20.4, the maximal elementary abelian subgroups of VV^g are V and V^g, so the maximal elementary abelian subgroups of U are $R = C_U(V)$ and $R^g = C_U(V^g)$. By 6.1.21.5, LT acts on R, so T normalizes both members of $\mathcal{A}(U)$, and hence both R and R^g are normal in $O^2(I)C_T(Z_S) = I$. But then $I \leq N_G(R) \leq M = !\mathcal{M}(LT)$, contradicting Hypothesis 6.1.16. This completes the proof. □

LEMMA 6.1.23. *If* $S_0 \leq S$ *with* $RU \leq S_0$, *then* $N_G(S_0) \leq M$.

PROOF. By 6.1.21, RU is Sylow in $L_1 = LU$, so that $S_0 \cap L$ is Sylow in L. Thus the assertion follows from Theorem 4.3.17. □

Recall $H = N_G(Z_S)$. Let $H^* := H/C_H(\tilde{U})$ and set $q := 2^n$. By 6.1.21.4 and 6.1.22, R/E is the sum of $s \geq 1$ natural modules for $L_1/R \cong L_2(q)$.

LEMMA 6.1.24. *(1)* $C_U(V) = C_R(\tilde{U})$.

(2) $R^* \cong E_{q^s}$, *and* $R^* = [R^*, D]$ *for each* $1 \neq D \leq D_L$.

(3) $[R^*, F(I^*)] = 1$.

(4) $O_2(I^*) = 1$.

PROOF. By 6.1.20.4, $U = V^gC_U(V)$. Also by 6.1.21.4, $C_{R/E}(U) = C_U(V)/E$, so that $[U, r] \not\leq E$ for $r \in R - C_U(V)$; as $\tilde{U}$ is abelian by 6.1.17.2, we conclude (1) holds. By 6.1.21.4, $R/E \cong E_{q^{2s}}$ is the sum of s natural modules for L_1/R with $C_U(V)/E$ the centralizer in R/E of U, so

$$R^* = C_{U^l}(V)^* \cong C_{U^l}(V)/E = [R^*, D] \cong E_{q^s}$$

for each $1 \neq D \leq D_L$. That is, (2) holds.

By 6.1.17.2, $\tilde{U} \in \mathcal{R}_2(I)$. Hence $O_2(I^*) = 1$, which proves (4), and also shows that $F(I^*) \leq O(I^*)$. Then as $R^* = [R^*, D_L]$ by (2), (3) follows from A.1.26. □

By 6.1.24.2, $R^* \neq 1$ as $s \geq 1$. By 6.1.24.4, R^* is faithful on $F^*(I^*)$. Thus by 6.1.24.3, R^* is faithful on $E(I^*)$, so there is $K \in \mathcal{C}(I)$ with $K/O_2(K)$ quasisimple and $[K^*, R^*] \neq 1$. As $|K^H| \leq 2$ by 1.2.1.3, D_L acts on K; further $D_L \cap I = 1$. So as $R^* = [R^*, D_L]$ by 6.1.24.2, R also acts on each member of K^H, and hence $[K^*, R^*] = K^*$. Let $M_K := M \cap K$, and $S_K := S \cap K$; then $S_K \in Syl_2(K)$ as $S \in Syl_2(I)$.

We claim that $K \not\leq M$, so that $M_K^* < K^*$ as $C_H(\tilde{U}) \leq N_G(V) \leq M$: For otherwise $K \leq C_M(Z_S) \leq M_V \leq N_G(R)$ using 6.1.7.1 and 6.1.21.5, contradicting $[K^*, R^*] = K^*$.

LEMMA 6.1.25. *(1)* $n = 2$.
(2) $K^* \cong L_2(p)$, $p \equiv \pm 3 \mod 8$, $p \geq 11$.
(3) $s = 1$, *so that* R/E *is the natural module for* L_1/R.

PROOF. First D_L normalizes $S \in Syl_2(I)$, and hence also normalizes $S_K^* \in Syl_2(K^*)$. If $D_K := C_{D_L}(K^*) \neq 1$, then as we saw R^* acts on K^*, $R^* = [R^*, D_K] \leq C_{I^*}(K^*)$ by 6.1.24.2, contrary to the choice of K. Thus D_L is faithful on K^*. Therefore either

(A) D_L is a 3-group, and hence of order 3 with $n = 2$, or

(B) $K^*/Z(K^*)$ is described in A.3.15 with $Z(K^*)$ of odd order by 6.1.24.4.

Assume for the moment that (B) holds. As D_L acts on S_K^*, it follows from A.3.15 that one of the following holds:

(a) K^* is of Lie type and characteristic 2.

(b) K^* is J_1 and $n = 3$ as D_L has order 7.

(c) K^* is $(S)L_3^\epsilon(p)$ and $D_L^* \cap K^* = 1$.

However in case (c), using the description in A.3.15.3, D_L centralizes S_K^*. As $R^* = [R^*, D_L]$ and $Out(K^*) \cong S_3$, R^* induces inner automorphisms on K^*, impossible as $1 \neq R^* = [R^*, D_L]$ and D_L centralizes S_K^*. This eliminates case (c).

Now assume for the moment that (A) holds. We check the list of Theorem C (A.2.3) for groups $K^*/Z(K^*)$ in which the normalizer of S_K^* in $Aut(K^*/Z(K^*))$ contains a subgroup of order 3, and conclude that either K^* is of Lie type and characteristic 2, or K^* is $L_2(p)$ with $p \equiv \pm 3 \mod 8$ or J_2. The case where $K^* \cong J_2$ is ruled out by A.3.18 as $D_L \cap I = 1$.

Next suppose (A) or (B) holds and K^* is of Lie type over $\mathbf{F}_{2^k}$. Then as D_L acts on S_K^*, either $k > 1$, or K^* is ${}^3D_4(2)$ and D_L is of order 7—so that $n = 3$. In any case, D_L acts on a Borel subgroup B^* of K^* containing S_K^*. Further either K^* is of Lie rank 1, in which case we set $K_1 := K$, or K^* is of Lie rank 2. In the latter case, as $K \not\leq M$, either

(i) $D_L T$ acts on a maximal parabolic P^* of K with preimage P satisfying $K_1 := O^{2'}(P) \not\leq M$, or

(ii) K^* is $Sp_4(2^k)$ or $(S)L_3(2^k)$ and T is nontrivial on the Dynkin diagram of K^*, and we set $K_1 := K$.

In any case, $K_1 \not\leq M$.

Suppose first that $B \leq M$. Then $H_2 := \langle K_1, T \rangle \in \mathcal{H}_*(T, M)$ with $n(H_2) > 1$—unless possibly $K^* \cong {}^3D_4(2)$ with $n = 3$, and K_1 is solvable. In the former case, Hypothesis 6.1.1 is contradicted. In the latter case, our usual argument with the Green Book [**DGS85**] supplies a contradiction: That is, just as in the proofs of 6.1.5 and 6.1.19, $\alpha := (LT, D_L T, D_L H_2)$ satisfies Hypothesis F.1.1, so that α is a weak BN-pair by F.1.9. Also $D_L H_2$ satisfies the hypothesis of F.1.12, so α must be in the list of F.1.12. As $n = 3$ and $k = 1$, the only possibility is the ${}^3D_4(2)$ amalgam of F.1.12.I.4. However, in that case Z is central in the parabolic L_1 with $L_1/O_2(L_1) \cong L_2(8)$, contradicting V the natural module for $L/O_2(L) \cong L_2(8)$.

This contradiction shows that $B \not\leq M$. In particular K^* is not ${}^3D_4(2)$, so $K_1 \in \mathcal{L}(G,T)$. Next as $R^* = [R^*, D_L]$, and $Out(K^*)$ is 2-nilpotent for each K^*, R^* induces inner automorphisms on K^*, so that $R^* \leq O_2(B^*R^*) := C^*$. Then $RU \leq S_0 := S \cap C \in Syl_2(C)$, and as $K_1/O_2(K_1)$ is quasisimple, $S_0 = O_2(C)$. However $N_G(S_0) \leq M$ by 6.1.23, contradicting $B \not\leq M$.

This contradiction shows K^* is not of Lie type and characteristic 2. Thus by our earlier discussion, either $n = 2$ and $K^* \cong L_2(p)$ for $p \equiv \pm 3 \mod 8$ or J_1, or $n = 3$ and $K^* \cong J_1$. In each case as $R^* = [R^*, D_L]$, $R^* \leq O_2(N_{K^*}(S_K^*)R^*) := C^*$; then the argument of the previous paragraph shows $N_{K^*}(S_K^*) \leq M_K^*$.

Suppose $K^* \cong J_1$. Then $N_{K^*}(S_K^*) \cong Frob_{21}/E_8$ is maximal in K^*, so $M_K^* = N_{K^*}(S_K^*)$. Now $D_LT_L \trianglelefteq M_K$, so we conclude D_L is of order 7 rather than 3, and $D_L \leq [D_L, M_K] \leq K \leq C_G(Z_S)$—impossible, as $[Z_S, D_L] = Z_S$.

Therefore $K^* \cong L_2(p)$ with $p \equiv \pm 3 \mod 8$ and $n = 2$. As K^* is not $L_2(4)$ by an earlier reduction, $p \geq 11$. Therefore (1) and (2) are established.

As $n = 2$, D_L is of order 3, so as $m_3(D_LI) \leq 2$, $m_3(I) = 1$ and hence $K = O^{3'}(I)$. As D_L is not inverted in D_LS and D_L is faithful on K^*, S induces inner automorphisms on K^*. As $K = O^{3'}(I)$, if $K_0 \in \mathcal{C}(I)$ with $K_0 \neq K$, then $K_0/O_2(K_0) \cong Sz(2^k)$. As $D_L = O^2(D_L)$, D_L acts on each member of K_0^I by 1.2.1.3, and hence so does $R^* = [R^*, D_L]$. The case $[R^*, K_0^*] \neq 1$ was eliminated in our earlier treatment of groups of Lie type in characteristic 2. Therefore R^* centralizes K_0^{*I}, so R^* centralizes $C_{F^*(I^*)}(K^*)$ in view of 6.1.24.3. Recall S^* induces inner automorphisms on K^*, so as $O_2(I^*) = 1$ by 6.1.24.4, we conclude $R^* \leq K^*$. Thus $R^* \leq S_K^*$, so as $R^* = [R^*, D_L]$, we conclude $R^* = S_K^*$. In particular R^* is of order 4, so by 6.1.24.2, $s = 1$ and hence (3) holds. □

LEMMA 6.1.26. *If there exists $e \in E - V$, then:*

(1) R is transitive on eV.

(2) $|E : V| \leq 4$.

PROOF. Set $L_0 := \langle V^g, V^{gl} \rangle$, where $l \in L$ is as in 6.1.21. Then $\bar{V}^g = \bar{U}$ by 6.1.20.4, and so $\bar{V}^{gl} = \bar{U}^l$. Therefore by 6.1.21.1, $\bar{L} = \bar{L}_1 = \bar{L}_0$ and $L \leq L_1 = L_0R$. By 6.1.20.4, $m(U/C_U(V^g))) = 2$, so $m(E/C_E(V^g)) \leq 2 = m(Z_S)$. Then as $C_{Z_S}(V^{gl}) = 1 = C_{Z_S^l}(V^g)$ and L acts on E by 6.1.21.3,

$$E = Z_SC_E(V^{gl}) = Z_S^lC_E(V^g),$$

so that $E = VC_E(L_0)$.

If $E = V$ then the lemma is trivial, so assume $e \in E - V$. As $E = VC_E(L_0)$ there is $f \in eV \cap C_E(L_0)$. If $[R, f] = 1$, then f is centralized by R and L_0, so $L \leq L_0R \leq C_G(f)$, a contradiction as $C_T(L) = 1$ by 6.1.6.1. This contradiction shows $[R, f] \neq 1$. But by 6.1.21.3, $[R, f] \leq V$, so as L_0 is irreducible on V, $[R, f] = V$. Therefore (1) holds, and we may take $e = f \in C_E(L_0) =: F$. Now $V^gE \leq C_U(F)$ and $\bar{V}^g = \bar{U}$, so $|U : C_U(F)| \leq |U : V^gE| \leq |U \cap R : E|$. But $n = 2$ by 6.1.25.1, and R/E is the natural module for L_1/R by 6.1.25.3, so we conclude $|U : C_U(F)| \leq 4$. We saw R does not centralize f, so as L_0 centralizes F and acts irreducibly on R/E, $[U \cap R, F] \neq 1$. Thus there is $u \in (U \cap R) - C_U(F)$, and for each such u, $[F, u] \leq Z_S$ by 6.1.17.2. Then $|F/C_F(u)| \leq |Z_S| = 4$ by Exercise 4.2.2

in [**Asc86a**]. Therefore to prove (2), it remains to show $F_u := C_F(u) = 1$—since this shows $|F| \leq 4$, and we saw earlier that $E = VF$.

As $\bar{L} = \bar{L}_0$, we may take $D_L \leq L_0 \leq C_G(F)$, so $D_L \leq C_L(F_u)$. Thus as D_L is irreducible on $(U \cap R)/E$ and $u \in (U \cap R) - E$, $U \cap R$ centralizes F_u. Then $R \leq \langle U^{L_0} \rangle \leq C_G(F_u)$, so $L \leq L_0 R \leq C_G(F_u)$, and hence $F_u = 1$ by 6.1.6.1, as desired. □

We now complete this section by eliminating case (1) of 6.1.15—hence reducing Hypothesis 6.1.1 to the case leading to M_{22} in the following chapter:

THEOREM 6.1.27. *Assume Hypothesis 6.1.1 and set $V_L := [V, L]$. Then*

(1) $n = 2$.
(2) V_L is the natural module for $L/O_2(L) \cong L_2(4)$ and $C_T(L) = 1$.
(3) Let $Z_S := C_{V_L}(T_L)$. Then $C_G(Z_S) \leq M$.
(4) Either $N_G(W_0(T, V_L)) \not\leq M$ or $W_1(T, V_L) \not\leq C_T(Z_S)$.

PROOF. By 6.1.6.2, V_L is the natural module for $L/O_2(L) \cong L_2(2^n)$, and $C_T(L) = 1$ by 6.1.6.1. Thus to complete the proof of (2), it suffices to prove (1).

As the statements in Theorem 6.1.27 concerning V are about V_L, we may as well assume $V = V_L$, so that we may apply the results following 6.1.6, which depend upon that assumption.

Suppose first that $C_G(Z_S) \leq M$. Then (3) holds and we are in case (2) of 6.1.15, so (1) and (4) also hold. Therefore as (1) implies (2), Theorem 6.1.27 holds in this case.

Therefore we may assume that $C_G(Z_S) \not\leq M$, so that Hypothesis 6.1.16 is satisfied. Thus we can apply the lemmas in this subsection, which assume Hypothesis 6.1.16. We will derive a contradiction to complete the proof of the Theorem.

First $n = 2$ by 6.1.25.1, so $|U : C_U(V)| = 4$ by 6.1.20.4. Then by 6.1.21.4 and 6.1.25.3, $|C_U(V)/E| = 4$. Finally V is of order 16, and $|E : V| \leq 4$ by 6.1.26.2, so we conclude $|U| \leq 4^5$. Hence $m(\tilde{U}) \leq 8$.

Let W be an noncentral chief factor for K on $\tilde{U}$. By 6.1.25.2, for each extension field F of $\mathbf{F}_2$, the minimal dimension of a faithful FK^*-module is $(p-1)/2$. Hence as $m(\tilde{U}) \leq 8$, $p \leq 17$, so $p = 11$ or 13 by 6.1.25.2. But then $p - 1$ is the minimal dimension of a nontrivial $\mathbf{F}_2\mathbf{Z}_p$-module, so we have a contradiction to $m(\tilde{U}) \leq 8$. This contradiction completes the proof of Theorem 6.1.27. □

6.2. Identifying M_{22} via $L_2(4)$ on the natural module

In this section, we complete the treatment of groups satisfying Hypothesis 6.1.1, by showing in Theorem 6.2.19 that M_{22} is the only group satisfying the conditions established in Theorem 6.1.27. Then applying results in chapter 5, the treatment of those groups containing a T-invariant $L \in \mathcal{L}_f^*(G, T)$ with $L/O_2(L) \cong L_2(2^n)$ is reduced in Theorem 6.2.20 to the case where $n = 2$ and V is the sum of at most two orthogonal modules for $L/O_2(L)$ regarded as $\Omega_4^-(2)$. We treat that final case in Part F2, which is devoted to the groups containing $L \in \mathcal{L}_f^*(G, T)$ with $L/O_2(L)$ a group over $\mathbf{F}_2$.

So in this section, we continue to assume Hypothesis 6.1.1, and as in section 6.1, we let $Z_S := C_V(T \cap L)$, $V_L := [V, L]$, and $S := C_T(Z_S)$. As usual, Z denotes $\Omega_1(Z(T))$. By Theorem 6.1.27, $n = 2$, and by 6.1.6, $C_Z(L) = 1$ and V_L is the natural module for $L/O_2(L) \cong L_2(4)$. Applying these observations to $R_2(LT)$ in the role of V, $Z \leq V_L$. Further replacing V by V_L if necessary, we may assume V *is* the natural module.

By Theorem 6.1.27, $C_G(Z_S) \leq M$, so by 6.1.7.1, $C_G(Z_S) \leq M_V := N_M(V)$; hence by 6.1.9.5:

LEMMA 6.2.1. *$N_G(Z_S) \leq N_G(V) \leq M$.*

Observe that Z_S is the T-invariant 1-dimensional $\mathbf{F}_4$-subspace of V regarded as a 2-dimensional $\mathbf{F}_4$-space. Let $\bar{M}_V := M_V/C_M(V)$.

LEMMA 6.2.2. *(1) $\bar{L}\bar{T} \cong S_5$.*
(2) Z is of order 2.
(3) $C_G(Z) \not\leq M$.

PROOF. Part (3) follows from 6.1.5. Recall $Z \leq V$, so if $\bar{L}\bar{T} \cong A_5$, then $Z_S = C_V(T) = Z$, and (3) contradicts 6.2.1. Hence (1) holds and $Z = C_V(T)$ is of order 2 by (1), establishing (2). □

LEMMA 6.2.3. *If $g \in G$ with $V \leq N_G(V^g)$ and $V^g \leq N_G(V)$, then $[V, V^g] = 1$.*

PROOF. If $[V, V^g] \neq 1$, then 6.1.11 says $V^g \in V^{C_G(Z_S)}$. But $C_G(Z_S) \leq N_G(V)$ by 6.2.1, contradicting our assumption that $1 \neq [V, V^g]$. □

LEMMA 6.2.4. *Assume $U \leq V$ with $m(V/U) = 2$ and $H := C_G(U) \not\leq N_G(V)$. Choose notation so that $T_U := N_T(U) \in Syl_2(N_M(U))$, and let $Q := C_T(V)$, $L_U := O^2(N_L(U))$, $U_H := \langle V^H \rangle$, $\tilde{H} := H/U$, and $H^* := H/C_H(\tilde{U}_H)$. Then*

(1) $U = C_V(t)$ for some $t \in T$ inducing a field automorphism of order 2 on $\bar{L}$.

(2) $F^*(H) = O_2(H)$, $R := Q\langle t \rangle \in Syl_2(H)$, $N_G(R) \leq N_G(J(R)) \leq M$, $T_U \in Syl_2(N_G(U))$, and $|T : T_U| = 2$.

(3) $W_0(R, V) \leq Q$.

(4) U_H is elementary abelian, $\tilde{U}_H \leq Z(O_2(\tilde{H}))$, and $C_H(\tilde{U}_H) = O_2(H)$, so $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$.

(5) $L_U/O_2(L_U) \cong \mathbf{Z}_3$ with $O_2(L_U) = L_U \cap H$.

(6) There is at most one $K \in \mathcal{C}(H)$ of order divisible by 3, and if such a K exists then either

(i) $K = O^{3'}(H)$ and $m_3(K) = 1$, or

(ii) $K/O_2(K) \cong (S)L_3^\epsilon(q)$, and a subgroup of order 3 in L_U induces a diagonal automorphism on $K/O_2(K)$.

PROOF. Observe by 6.1.8 that as $H = C_G(U)$, $H \cap M = N_H(V)$, so that our hypothesis $H \not\leq N_G(V)$ is equivalent to $H \not\leq M$. As $C_G(Z_S) \leq M$, case (3) of 6.1.13 must hold, proving (1). Next by (1), $|T : T_U| = 2$, and the remaining statements of (2)–(4) follow from 6.1.14, except for the inclusion $O_2(H) \geq C_H(\tilde{U}_H)$ in part (4). Part (5) follows from (1), and (6) follows from A.3.18 in view of (5).

Finally $C_H(\tilde{U}_H) \leq N_G(V) \leq M$, and by Coprime Action, $Y := O^2(C_H(\tilde{U}_H)) \leq C_M(V) \leq C_M(L/O_2(L))$. Thus LT normalizes $O^2(YO_2(L)) = Y$. Therefore if $Y \neq 1$ then $H \leq N_G(Y) \leq M = !\mathcal{M}(LT)$, contradicting our initial observation that $H \not\leq M$. Thus $C_H(\tilde{U}_H)$ is a 2-group, completing the proof of (4), and hence also the proof of 6.2.4. □

Define a 4-subgroup F of V^g to be of *central type* if F is centralized by a Sylow 2-subgroup of L^g; of *field type* if F is centralized by an element of M_V^g inducing a field automorphism on $L^g/O_2(L^g)$ and V^g; and of *type 3* if F is of neither of the first two types. By 6.2.2, there exist 4-subgroups of V of field type, and of course Z_S is of central type.

LEMMA 6.2.5. *(1) Let ξ denote the number of orbits of M_V on the 4-subgroups of V of type 3. Then $\xi = 0$ or 1 for $|\bar{M}_V : \bar{L}| = 6$ or 2, respectively. The orbits are of length 0 or 20, respectively.*

(2) M_V is transitive on 4-subgroups of V of each type.

(3) For each 4-subgroup U of V, $U^G \cap V = U^{LT}$.

(4) V is the unique member of V^G containing Z_S.

(5) If $g \in G$ with $V \cap V^g$ noncyclic, then $[V, V^g] = 1$.

(6) V is the unique member of V^G containing any hyperplane of V.

PROOF. As V is the natural module for $\bar{L}$, $\bar{L}$ preserves an $\mathbf{F}_4$-space structure $V_{\mathbf{F_4}}$ on V, in which the central 4-subgroups are the five 1-dimensional subspaces of $V_{\mathbf{F_4}}$, and $\bar{M}_V \leq Aut_{GL(V)}(\bar{L}) = \Gamma L(V_{\mathbf{F_4}})$. In particular, L is transitive on 4-subgroups of central type, and there are 30 4-subgroups not of central type, which form an orbit under $Aut_{GL(V)}(\bar{L})$. This orbit splits into three orbits of length 10 under $\bar{L}$, and $Aut_{GL(V)}(\bar{L})$ induces S_3 on this set of orbits. By 6.2.2, $\bar{M}_V \cong S_5$ or $\Gamma L_2(4)$, so it follows that (1) and (2) hold.

By 6.2.4, we can choose a representative U for each orbit so that $N_T(U) \in Syl_2(N_G(U))$. Now $T = N_T(U)$ iff U is of central type, so groups of central type are not fused to groups of field type or type 3. Similarly if $N_T(U) < T$, then $|T : N_T(U)| = 2$ or 4 for U of field type, or type 3, respectively, so distinct M-orbits are not fused in G. Thus (3) holds.

By (3) and A.1.7.1, $N_G(Z_S)$ is transitive on G-conjugates of V containing Z_S; then as $N_G(Z_S) \leq N_G(V)$ by 6.2.1, (4) holds. As V is a self-dual $\mathbf{F}_2L$-module and L is transitive on $V^\#$, L is transitive on hyperplanes of V, so (4) implies (6).

Assume the hypotheses of (5), and let U be a 4-subgroup of $V \cap V^g$; then by (3) and A.1.7.1, $N_G(U)$ is transitive on G-conjugates of V containing U. Furthermore for U of each type, $Aut_G(U) \cong S_3 \cong Aut_{M_V}(U)$, so that $N_G(U) = C_G(U)N_{M_V}(U)$; we conclude that $C_G(U)$ is transitive on the G-conjugates of V containing U. Thus if $C_G(U) \leq N_G(V)$, then $V = V^g$ and (5) is trivial. If $C_G(U) \not\leq N_G(V)$, then U is of field type by 6.2.4.1, so $\langle V, V^g \rangle$ is abelian by 6.2.4.2, completing the proof of (5). $\square$

LEMMA 6.2.6. *Assume $A := V^g \cap N_G(V)$ and $U := V \cap N_G(V^g)$ are of index 2 in V^g and V, respectively. Then either*

(1) $\bar{A}$ and $U/C_U(V^g)$ are of order 2, $C_A(V)$ and $C_U(V^g)$ are of field type, and $\langle V, V^g \rangle$ is a 2-group, or

(2) $\bar{A} \cong E_4$, $\bar{A} \not\leq \bar{L}$, $Y := \langle V, V^g \rangle \cong S_3/Q_8^2$, $V \cap V^g = [A, U]$ is of order 2, and $O_2(Y) \leq O^2(Y)$.

PROOF. Without loss, we may assume $A \leq T$. First $B := [A, U] \leq A \cap U$, so $B \neq Z_S$ by 6.2.5.4, and hence $\bar{A} \notin Syl_2(\bar{L})$. Also $\bar{A} \neq 1$, as otherwise $V \leq C_G(A) \leq N_G(V^g)$ by 6.2.5.6, contrary to hypothesis.

Suppose first that $\bar{A}$ is of order 2. Then $A_0 := C_A(V)$ is of codimension 2 in V^g, so as $V \leq C_G(A_0)$ but $V \not\leq N_G(V^g)$, we conclude from 6.2.4.1 that A_0 is of field type. Then as U centralizes A_0, $U_0 := C_U(V^g)$ is of index 2 in U since $|C_G(A_0) : C_G(V^g)|_2 = 2$ by 6.2.4.2. Thus we have symmetry between V and V^g, so

U_0 is also of field type. Also $\langle V, V^g \rangle \leq C_G(U_0)$, and by 6.2.4.4, $V \leq O_2(C_G(U_0))$, so $\langle V, V^g \rangle$ is a 2-group and (1) holds.

Thus we may assume that $\bar{A}$ is of order 4, so $\bar{A} \not\leq \bar{L}$ as we saw $\bar{A} \notin Syl_2(\bar{L})$. From (1), our hypotheses are symmetric in V and V^g, so also $Aut_U(V^g)$ is a 4-group not contained in $Aut_{L^g}(V^g)$. Let $Q := UA$ and $\tilde{Q} := Q/B$. From the action of $\bar{A}$ on V, $B = C_V(\bar{A})$ is of order 2 and $C_A(V)$ is the centralizer in A of each hyperplane of V. Also $|V^g : B| = 8$, so as $|V^g| = 16$, it follows that $B = C_A(V) = C_A(U) = V \cap V^g$. Then we conclude $Q \cong Q_8^2$ with $B = Z(Q)$. Further $[[V, A], A] \leq C_V(A) = B \leq A$, so $[V, A] \leq N_V(A) \leq N_V(V^g) = U$ by 6.2.5.6, and thus we conclude $[V, A] = U$ as both groups have rank 3. Thus $[V, A] \leq Q$, so V acts on Q, and then by symmetry, V^g acts on Q. Hence $Y := \langle V, V^g \rangle$ acts on Q. Set $Y^* := Y/Q$, so that Y^* is dihedral, as V^* and V^{g*} are of order 2. We have seen that $[\tilde{A}, V^*] = \tilde{U}$, so we conclude $[\tilde{Q}, V^*] = \tilde{U} = C_{\tilde{Q}}(V^*)$. Therefore V^* is generated by an involution of type a_2 in $Out(Q) \cong O_4^+(2)$, $Y^*/C_{Y^*}(Q) \cong S_3$ with $Q \leq O^2(Y)$, and the images of V^* and V^{*g} are conjugate in this quotient. Thus $\tilde{U} = C_{\tilde{Q}}(V^*)$ is conjugate to $\tilde{A} = C_{\tilde{Q}}(V^{g*})$ in Y, and hence U is conjugate to A in Y. Therefore V^g is conjugate to V in Y by 6.2.5.6. Thus V^* is conjugate to V^{*g} in Y^*, so that $|Y^*| \equiv 2 \mod 4$. Again by 6.2.5.6, $C_Y(Q) \leq N_G(V)$, so as V^* inverts $O(Y^*)$, $C_Y(Q)^* = C_{Y^*}(Q) = 1$. Thus $C_Y(Q) = Z(Q) = B$, so $Y \cong S_3/Q_8^2$, completing the proof of (2). □

LEMMA 6.2.7. *$W_0(T, V) \leq C_T(V) = O_2(LT)$, so that $N_G(W_0(T, V)) \leq M$.*

PROOF. By E.3.34.2, it suffices the prove the first assertion. So assume by way of contradiction that $W_0(T, V) \not\leq C_T(V)$. Then there is $g \in G$ such that $V \leq T^g$ but $[V, V^g] \neq 1$. By 6.2.3, $V^g \not\leq N_G(V)$. Let $U := C_V(V^g)$. Then $m(V/U) = 2$ by 6.1.10.3, and as $V^g \not\leq N_G(V)$, $C_G(U) \not\leq N_G(V)$, so the hypotheses of 6.2.4 are satisfied. Adopt the notation of that lemma (e.g., $H = C_G(U)$, $\tilde{H} = H/U$, $U_H = \langle V^H \rangle$, etc.) and let $A := V^g$, $B := Z_S^g$, and D_U of order 3 in L_U. Then $V = [V, D_U]$. By 6.1.10.2 and E.3.10, $VC_G(A)/C_G(A) \in \mathcal{A}_2(N_G(A)/C_G(A), A)$, so $S^g = VC_{S^g}(V^g)$ and $[A, V] = B$.

We claim next that if K^* is a subgroup of $C_{H^*}(D_U^*)$ with $A^* \leq K^*$, then $[\tilde{U}_H, K^*, D_U^*] \neq 1$: For otherwise using the Three-Subgroup Lemma, $A^* \leq K^* \leq C_{H^*}([\tilde{U}_H, D_U^*] \leq C_{H^*}(\tilde{V})$, contrary to the fact that A does not act on V.

Now $A \leq H := C_G(U)$ and $V \leq U_H$, so $B = [A, V] \leq U_H$, which is abelian by 6.2.4.4. Thus $U_H \leq C_G(B) \leq N_G(A)$ by 6.2.1, so we may take $U_H \leq T^g$. Indeed as U_H centralizes B, we have $V \leq U_H \leq C_{T^g}(Z_S^g) = S^g$. Then $U_H = VC_{U_H}(A)$ by the first paragraph of the proof, so $[U_H, A] = [V, A] = B$, $m(U_H/C_{U_H}(A)) = 2$, and $B = C_A(U_0)$ for $C_{U_H}(A) < U_0 \leq U_H$.

We saw $V \leq C_G(B)$, so $B \leq N_A(V)$. If $B < N_A(V)$, then as $S^g = VC_{S^g}(A)$, $B = [V, N_A(V)] \leq V$; but now $Z_S^g = B \leq V \neq V^g$, contrary to 6.2.5.4. Hence $B = N_A(V)$. We saw $B \leq U_H$, so in particular $B = C_A(\tilde{U}_H)$ as $C_G(\tilde{U}_H) \leq N_G(V)$, and hence $A^* \cong E_4$.

Let $B < A_1 \leq A$. Suppose that $\tilde{U}_1 := C_{\tilde{U}_H}(A_1) > \widetilde{C_{U_H}(A)}$. We saw $B = C_A(U_0)$ for $C_{U_H}(A) < U_0 \leq U_H$. Thus $B = C_A(U_1)$, so as $[U_H, A] = B$, $1 \neq [U_1, A_1] =: B_1 \leq U \cap B$. We will show that $1 \neq U \cap B$ leads to a contradiction. For B is of rank 2, so $m(\tilde{B}) \leq 1$. Then since $[A, U_H] = B$, A^* induces a 4-group of transvections on $\tilde{U}_H$ with center $\tilde{B}$. Thus by G.3.1, there is $K \in \mathcal{C}(H)$

such that $K = [A, K]$, $A^* \leq K^*$, K^* induces $GL(\tilde{U}_1)$ on $\tilde{U}_1 := \langle \tilde{B}^K \rangle$ of rank at least 3, and the kernel of the action lies in $O_2(K^*)$. But $O_2(H^*) = 1$ by 6.2.4.4, so $K^* \cong GL(\tilde{U}_1)$. Then by 6.2.4.6, $K^* \cong L_3(2)$ (so that $m(\tilde{U}_1) = 3$) and $K = O^{3'}(H)$. As $[\tilde{U}_H, A] = \tilde{B} \leq \tilde{U}_1$ and $K = [K, A]$, $\tilde{U}_1 = [\tilde{U}_H, K]$. We saw $m(U_H/C_{U_H}(A)) = 2$, so $\tilde{U}_H = \tilde{U}_1 \oplus C_{\tilde{U}_H}(K)$. (cf. B.4.8.3). Now D_U of order 3 in L_U acts on the subgroup R of 6.2.4.2, and then on $R_K := R \cap K$ in view of 1.2.1.3. But $R_K^* \in Syl_2(K^*)$, so R_K^* is self-normalizing in K^* and hence $[D_U^*, K^*] = 1$. Then D_U centralizes $\tilde{U}_1$ since $K^* = Aut(\tilde{U}_1)$. As $A^* \leq K^*$, this contradicts our claim in paragraph two.

This contradiction shows that $B \cap U = 1$ and that

$$C_{\tilde{U}_H}(A_1) = \widetilde{C_{U_H}(A)} \text{ for each } 1 \neq A_1^* \leq A^*. \qquad (*)$$

Since $A^* \cong E_4$, (*) says

$$A^* \in \mathcal{A}_2(H^*, \tilde{U}_H); \qquad (**)$$

and since $B \cap U = 1$ we have

$$\tilde{B} = [\tilde{U}_H, A^*] \cong E_4. \qquad (!)$$

Further applying (*) when $A_1 = A$ and recalling $m(U_H/C_{U_H}(A)) = 2$, we conclude

$$m(\tilde{U}_H/C_{\tilde{U}_H}(A)) = 2. \qquad (!!)$$

Thus A^* is an offender on the FF-module $\tilde{U}_H$. Recall $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$ by 6.2.4.4, and let $K_A^* := \langle A^{*H} \rangle$. By (**) and E.4.1, A^* centralizes $O(H^*)$, so that $F(K_A^*) \leq Z(K_A^*)$. Next (!!) restricts the possible components K^* of K_A^* in the list of Theorem B.5.6 to alternating groups or groups defined over $\mathbf{F}_2$ or $\mathbf{F}_4$. Now K^* is the image of $K \in \mathcal{C}(H)$, and by 6.2.4.6 and inspection of our restriced list from B.5.6, either

(i) $m_3(K^*) = 1$, so that $K^* \cong L_2(4)$ or $L_3(2)$, or

(ii) $K^* \cong SL_3(4)$ and D_U induces outer automorphisms on K^*.

In particular $K^* = J(K_A^*)^\infty$ is described by Theorem B.5.1. As in the previous paragraph, $R_K := R \cap K \in Syl_2(K)$.

Suppose first that case (ii) holds. By Theorem B.5.1.1, either $\tilde{V}_K := [\tilde{U}_H, K^*] \in Irr_+(K^*, \tilde{U}_K)$, or $\tilde{V}_K$ is the sum of two isomorphic natural modules for K^*. In the former case, $\tilde{V}_K$ is a natural module by B.4.2. In either case, A.3.19 contradicts the fact that $D_U \not\leq K$.

Thus case (i) holds. By Theorem B.5.1.1, either $\tilde{V}_K := [\tilde{U}_H, K^*] \in Irr_+(K^*, \tilde{U}_H)$, or $K^* \cong L_3(2)$ and $\tilde{V}_K$ is the sum of two isomorphic natural modules.

Assume first that $K^* \cong L_3(2)$. If $\tilde{V}_K$ is the sum of two isomorphic natural modules, then by (*), A^* induces the group of transvections with a fixed axis on each of the natural summands, contrary to (!). Thus $\tilde{V}_K \in Irr_+(K^*, \tilde{U}_H)$. Then by B.4.8.4, $\tilde{V}_K = [\tilde{U}_H, K^*]$ is either the natural module or the extension in B.4.8.2. Now as D_U acts on $R_K^* \in Syl_2(K^*)$, D_U centralizes K^* and $\tilde{V}_K/C_{\tilde{V}_K}(K^*)$, and hence D_U centralizes $\tilde{V}_K$ by Coprime Action. As $A^* \leq K^*$, this contradicts our claim in paragraph two.

This contradiction shows $K^* \cong L_2(4)$, so $\tilde{V}_K \in Irr_+(\tilde{U}_H, K)$. Then by B.4.2, either $\tilde{V}_K$ is the A_5-module, or $\tilde{V}_K/C_{\tilde{V}_K}(K)$ is the natural module. The first case is impossible by (*). Thus the second case holds, and $A^* \in Syl_2(K^*)$ by B.4.2.1. Further $C_{\tilde{V}_K}(K) = 1$ by (!), so $\tilde{V}_K$ is the natural module, and $\tilde{U}_H = \tilde{V}_K \oplus C_{\tilde{U}_H}(K)$ by B.5.1.4.

Set $L_K := O^2(N_K(R_K))$, so that $L_K/O_2(L_K) \cong \mathbf{Z}_3$. First suppose $L_K \leq M$. As $K \leq H = C_G(U)$, by 6.1.8 we obtain $L_K \leq K \cap M = N_K(V)$. Then as $[L_K, U] = 1$ and U is of field type, $[L_K, V] = 1$. But $C_{\tilde{U}_H}(L_K) = C_{\tilde{U}_H}(K)$, so $\tilde{V} \leq C_{\tilde{U}_H}(K^*) \leq C_{\tilde{U}_H}(A^*)$, and then $A \leq N_G(V)$, contrary to paragraph one.

Therefore $L_K \not\leq M$. By 6.2.4.2, $N_G(R) \leq M$. If $[D_U^*, K^*] \neq 1$, then

$$R = O_2(D_U R) = O_2(KR)(K \cap R) \trianglelefteq L_K R,$$

so $L_K \leq N_G(R) \leq M$, contradicting the reduction just obtained; hence $[D_U^*, K^*] = 1$. Thus as $A^* \leq K^*$, $[\tilde{V}_K, D_U^*] \neq 1$ by our claim in paragraph two. Thus $D_U^* K^*$ acts on $\tilde{V}_K$ as $GL_2(4)$ with $D_U^* = Z(D_U^* K^*)$. As $N_G(R) \leq M$ but $L_K \not\leq M$, $R^* \neq R_K^*$, so there is $r \in R$ inducing an involutory field automorphism on K^*. This is impossible, as the field automorphism r^* inverts the center D_U^* of $GL_2(4)$, whereas $R \trianglelefteq RD_U$. This contradiction completes the proof of 6.2.7. □

For the remainder of the section, let z denote the generator of Z, set $G_z := C_G(z)$, and $\tilde{G}_z := G_Z/Z$. By 6.2.2.3, $G_z \not\leq M$, so $\mathcal{H}_1 \neq \emptyset$, where

$$\mathcal{H}_1 := \{H \leq \mathcal{H}(T) : H \leq G_z \text{ and } H \not\leq M\}.$$

Consider any $H \in \mathcal{H}_1$, and observe that Hypothesis F.7.6 is satisfied with LT, H in the roles of "G_1, G_2". Form the coset graph Γ as in section F.7, and more generally adopt the notational conventions of section F.7. By 6.2.3 and F.7.11.2, $b := b(\Gamma, V)$ is odd.

LEMMA 6.2.8. *$V \not\leq O_2(G_z)$.*

PROOF. Choose H minimal in $\mathcal{H}_1$; then $H \in \mathcal{H}_*(T, M) \cap G_z$. Thus $n(H) = 1$ by Hypothesis 6.1.1.2. We assume that $V \leq O_2(G_z)$ and derive a contradiction. Then $V \leq O_2(H)$ so $V \leq G_{\gamma_1}^{(1)}$ by F.7.7.2, and hence $b > 1$; thus $b \geq 3$ as we saw b is odd. Let $U_H := \langle V^H \rangle \leq O_2(H)$. As $b \geq 3$, U_H is abelian by F.7.11.4. As usual, let $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$, and γ_i at distance i from γ_0 on a fixed geodesic from γ_0 to γ. By F.7.11.6, $[U_H, U_\gamma] \leq U_H \cap U_\gamma$, where U_γ is the conjugate of U_H defined in section F.7.

As $H \leq G_z$, $H \cap M = N_H(V)$ by 6.1.8. By 3.3.2.4, $H \cap M$ is the unique maximal subgroup of H containing T. Hence we may apply F.7.13 to U_H in the role of "A" to conclude there exists $\alpha \in \Gamma(\gamma)$ such that $m(U_H/N_{U_H}(V_\alpha)) = 1$.

As U_H does not act on V_α, there exists $\beta \in \Gamma(\gamma_1)$ such that V_β does not act on V_α; we consider any β satisfying these two conditions. Notice that as $m(U_H/N_{U_H}(V_\alpha)) = 1$, also $m(V_\beta/N_{V_\beta}(V_\alpha)) = 1$. Let $U_\beta := C_{V_\beta}(V_\alpha)$, so that $U_\beta \leq N_{V_\beta}(V_\alpha) < V_\beta$. Then $V_\alpha \not\leq N_G(V_\beta)$, since otherwise $[V_\alpha, V_\beta] = 1$ by 6.2.7, contradicting $U_\beta < V_\beta$. As $m(V_\beta/N_{V_\beta}(V_\alpha)) = 1$, $C_G(N_{V_\beta}(V_\alpha)) \leq N_G(V_\beta)$ by 6.1.10.1. So as V_α centralizes U_β but does not normalize V_β, $U_\beta < N_{V_\beta}(V_\alpha)$; hence $U_\alpha := [N_{V_\beta}(V_\alpha), V_\alpha]$ is a noncyclic subgroup of V_α. But $U_\alpha \leq [U_H, V_\alpha] \leq U_H$, so as $V_\beta \leq U_H$ which is abelian, $U_\alpha \leq C_{V_\alpha}(V_\beta)$. Now as $V_\beta \not\leq N_G(V_\alpha)$, $C_G(C_{V_\alpha}(V_\beta)) \not\leq N_G(V_\alpha)$, so that $m(C_{V_\alpha}(V_\beta)) \leq 2$ by 6.1.10.1; as U_α is noncyclic, we conclude $U_\alpha = C_{V_\alpha}(V_\beta)$ is a 4-group. Then U_α is of field type by 6.2.4.1. So as V_β centralizes U_α, $m(N_{V_\beta}(V_\alpha)/U_\beta) = 1$ by 6.2.4.2, with $N_{V_\beta}(V_\alpha)$ inducing a field automorphism on V_α. Then $m(V_\beta/N_{V_\beta}(V_\alpha)) = 1 = m(N_{V_\beta}(V_\alpha)/U_\beta)$, so U_β is also a 4-group. Therefore as V_α centralizes U_β but does not normalize V_β, $C_G(U_\beta) \not\leq N_G(V_\beta)$, and then U_β is also of field type by 6.2.4.1.

As $V_\alpha \not\leq N_G(V_\beta)$, $V_\alpha \not\leq G^{(1)}_{\gamma_1}$, so $d(\alpha) = b$ with $\alpha, \gamma, \cdots, \gamma_1$ a geodesic, and we have symmetry between γ and γ_1. By this symmetry (as in the proof of 6.1.19) we can apply F.7.13 to V_α in the role of "A", to conclude that there exists $\beta' \in \Gamma(\gamma_1)$ such that $m(V_\alpha/N_{V_\alpha}(V_{\beta'})) = 1$, and also that there exists $h \in H$ such that V_α^h fixes β' and $I := \langle V_\alpha, V_\alpha^h \rangle$ is not a 2-group.

Observe next that if $\mu, \nu \in \Gamma_0$, and V_μ acts on V_ν, then $[V_\mu, V_\nu] = 1$ by 6.2.7. But as V_α^h fixes β', and $\beta' = \gamma_0^g$ for some $g \in G$, $V_\alpha^h \leq G_1^g \leq N_G(V_{\beta'})$, so V_α^h centralizes $V_{\beta'}$. Similarly as V_α does not centralize $V_{\beta'}$, $V_{\beta'}$ does not act on V_α. Thus β' satisfies the two conditions for "β" in our earlier argument, so we may take $\beta' = \beta$. Then $m(V_\alpha/N_{V_\alpha}(V_\beta)) = 1$, so that we have symmetry between α and β. Thus as we showed that $[N_{V_\beta}(V_\alpha), V_\alpha] = C_{V_\alpha}(V_\beta) = U_\alpha$, by symmetry between α and β, $[N_{V_\alpha}(V_\beta), V_\beta] = C_{V_\beta}(V_\alpha)$. In particular as $U_\beta = C_{V_\beta}(V_\alpha)$, we also have symmetry between U_α and U_β. Further $N_{V_\beta}(V_\alpha)$ and $N_{V_\alpha}(V_\beta)$ are each of rank 3, and induce a field automorphism on V_α and V_β, respectively. Hence

$$1 \neq U_{\alpha,\beta} := [N_{V_\beta}(V_\alpha), N_{V_\alpha}(V_\beta)] \leq U_\alpha \cap U_\beta.$$

Now $U_{\alpha,\beta} \leq U_\beta$ centralizes I as V_α centralizes U_α and V_α^h centralizes $V_{\beta'} = V_\beta$. Thus for $z_0 \in U^\#_{\alpha,\beta}$, $z_0 \in V_\alpha$, but $V_\alpha \not\leq O_2(G_{z_0})$—since $I \leq G_{z_0}$, and $V_\alpha \not\leq O_2(I)$ as $I = \langle V_\alpha, V_\alpha^h \rangle$ is not a 2-group. As the pair (V, z) is conjugate to (V_α, z_0), 6.2.8 is established. □

In the remainder of this section, choose

$$H := G_z,$$

and let $M_z := C_M(z)$, $U := \langle Z_S^H \rangle$, $K := \langle V^H \rangle$, $M_K := K \cap M$, and $H^* := H/C_H(\tilde{U})$. By 6.2.8, $V \not\leq O_2(K)$, so $K \not\leq N_G(V)$. By 6.2.1, $N_G(Z_S) \leq N_G(V) = M_V$, and as V is the natural module for $\bar{L}$, $C_{M_V}(z) \leq N_M(Z_S)$. As $H = G_z$, by 6.1.8 we conclude:

LEMMA 6.2.9. *$H \cap M = N_H(V) = N_H(Z_S)$ and $M_K = N_K(V) = N_K(Z_S)$.*

LEMMA 6.2.10. *(1) $F^*(H) = O_2(H) =: Q_H$ and $\tilde{U} \leq Z(\tilde{Q}_H)$.*
(2) $C_H(\tilde{U}) \leq N_G(V) \leq M$, so $C_V(\tilde{U}) \leq Q_H$.
(3) $O_2(H^*) = 1$.
(4) $V^* \neq 1$.
(5) $[V, U] \leq V \cap U$.

PROOF. The first assertion in (1) holds by 1.1.4.6. Hypothesis G.2.1 is satisfied with Z, Z_S in the roles of "V_1", "V", so G.2.2 completes the proof of (1) and establishes (3). By 6.2.9, $C_H(\tilde{U}) \leq N_H(Z_S) = N_H(V) \leq M$, so $C_V(\tilde{U}) \leq O_2(C_H(\tilde{U})) \leq Q_H$, proving (2). By (1), $\tilde{U}$ is abelian, so by (2), U acts on V. Also $V \leq H \leq N_G(U)$, so (5) holds. As $V \not\leq Q_H$ by 6.2.8, (4) follows from (2). □

LEMMA 6.2.11. *V^* is of order 2.*

PROOF. Assume the lemma fails; then as $V^* \neq 1$ by 6.2.10.4, $m(V^*) \geq 2$. By 6.2.10.1, $Z_S \leq C_V(\tilde{U})$, so that $m(V^*) \leq m(V/Z_S) = 2$. Thus $m(V^*) = 2$, and $Z_S = V \cap Q_H = V \cap U = C_V(\tilde{U})$. Next by (4) and (5) of 6.2.10, $1 \neq [V^*, \tilde{U}] \leq \widetilde{V \cap U} = \tilde{Z}_S$ of order 2. Thus V^* induces a 4-group of transvections on $\tilde{U}$ with center $\tilde{Z}_S$. Also $O_2(H^*) = 1$ by 6.2.10.3. Thus we may apply G.3.1 and the results of section G.6 to H^*. In particular, since $\tilde{U} = \langle Z_S^H \rangle$, we conclude from G.3.1 that

K^* is the direct product of copies of $GL_m(2)$ for some $m \geq 3$. Next as $V \trianglelefteq T$, $1 \neq V^* \cap Z(T^*)$, so by G.6.4.4, $K^* = GL(\tilde{U})$. By 6.2.9,

$$C_{H^*}(\tilde{Z}_S) = M_z^* = N_M(V)^*.$$

Thus as V^* is a 4-group we conclude $m(\tilde{U}) = 3$ and $H^* \cong L_3(2)$.

As $\Phi(Z_S) = 1$ and H^* is transitive on $\tilde{U}^\#$, $\Phi(U) = 1$, so $U \cong E_{16}$. As V^* is the group of transvections with center $\tilde{Z}_S$, $\tilde{Z}_S = C_{\tilde{U}}(V^*)$, so $Z_S = C_U(V)$. Further $U \leq C_T(Z_S) = T_L C_T(V)$, where $T_L := T \cap L$; thus $|\bar{U}| = |U/C_U(V)| = |U : Z_S| = 4 = |\bar{T}_L|$, so $\bar{U} = \bar{T}_L \in Syl_2(\bar{L})$.

Now $[C_H(\tilde{U}), V] \leq C_V(\tilde{U}) \leq V \cap Q_H$ by 6.2.10.2, and we saw that $V \cap Q_H \leq U$. Hence $K = \langle V^H \rangle$ centralizes $C_H(\tilde{U})/C_H(U)$. Next $C_H(\tilde{U})/C_H(U)$ is a subgroup of the group X of all transvections on U with center Z, and $\tilde{U}$ is the dual of X as a module for $C_{GL(U)}(Z)$. Thus as $\tilde{U}$ is the natural module for K^* and K centralizes $C_H(\tilde{U})/C_H(U)$, we conclude $C_H(\tilde{U}) = C_H(U)$.

Next $L = [L, U]$ with $[U, O_2(LT)] \leq C_U(V) = Z_S \leq V$, so L is an $L_2(4)$-block. Also $C_{T^*}(V^*) = V^*$ as V^* is a 4-subgroup of $H^* \cong L_3(2)$; thus $C_T(V) \leq VC_T(\tilde{U})$. Therefore as $C_T(\tilde{U}) = C_T(U)$ by the previous paragraph, we conclude $C_T(V) = VC_T(UV)$. Then as $\bar{U} \in Syl_2(\bar{L})$, it follows from Gaschütz's theorem A.1.39 and C.1.13.a that $LO_2(LT) = LC_T(L)$. On the other hand, $C_T(L) = 1$ by 6.1.6.1. Therefore $V = O_2(LT) = O_2(M)$ using A.1.6. Then $T_L = J(T)$ with $\mathcal{A}(T) = \{A_1, A_2\}$ and $A_1 = V$, so as $m(U) = 4$, $U = A_2$. Thus as $N_L(T_L)$ acts on V, it also acts on U, so that $L_0 := \langle N_L(T_L), H \rangle$ acts on U, and hence $\hat{L}_0 := L_0/C_{L_0}(U) \leq GL(U) \cong A_8$. As $N_L(T_L)$ is transitive on $Z_S^\#$ and H is transitive on $U - Z$, L_0 is transitive on $U^\#$. Further $C_{\hat{L}_0}(z) = \hat{H} \cong L_3(2)$, so we conclude $\hat{L}_0 \cong A_7$. Moreover setting $M_0 := M \cap L_0$, $N_{L_0}(Z_S) \leq M_0 < L_0$ by 6.2.1. The stabilizer of any 4-subgroup of U in $\hat{L}_0$ is the global stabilizer in $\hat{L}_0$ of 3 of the 7 points permuted by $\hat{L}_0$ in its natural representation, which is a maximal subgroup of $\hat{L}_0$. Thus $\hat{M}_0 = N_{L_0}(Z_S)$. Now we can also embed $T \leq Y \leq L_0$ with $\hat{Y} \cong S_5$ and $|Y : Y \cap M_0| = 5$. Thus $Y \in \mathcal{H}_*(T, M)$ with $n(Y) = 2$ by E.2.2, contradicting Hypothesis 6.1.1.2. □

LEMMA 6.2.12. *(1)* $O^2(H \cap M) \leq C_M(V) \leq C_M(L/O_2(L))$.
(2) $O^2(C_H(\tilde{U})) = 1$, *so* $C_H(\tilde{U}) = Q_H$.

PROOF. As V^* has order 2 by 6.2.11, we conclude from 6.2.9 and 6.2.2 that $H \cap M$ acts on the series $V > C_V(\tilde{U}) > Z_S > Z$, and all factors in the series are of rank 1. Therefore $O^2(H \cap M)$ centralizes V by Coprime Action. Then $O^2(H \cap M)$ centralizes $L/O_2(L)$, proving (1).

Next using 6.2.10.2 and (1), $X := O^2(C_H(\tilde{U})) \leq O^2(H \cap M)$. Thus X centralizes $L/O_2(L)$, so that L normalizes $O^2(XO_2(L)) = X$. Now if $X \neq 1$, then $O_2(X) \neq 1$ by 1.1.3.1, since $H \in \mathcal{H}^e$ by 1.1.4.6. But then $H \leq N_G(O_2(X)) \leq M = !\mathcal{M}(LT)$, contradicting $H \not\leq M$. This shows that $C_H(\tilde{U})$ is a 2-group, and then 6.2.10.1 completes the proof of (2). □

We can now isolate the case leading to M_{22}, which we identify via a recent characterization of Chao Ku. Recall that $U = \langle Z_S^H \rangle$, so that $Z_S \leq V \cap U$.

PROPOSITION 6.2.13. *If* $Z_S = V \cap U$, *then* $G \cong M_{22}$.

PROOF. Assume $Z_S = V \cap U$. We begin by arguing much as at the start of the proof of 6.2.11, except this time V^* has order 2 by 6.2.11. By 6.2.10.5, $1 \neq [V^*, \tilde{U}] \leq \widetilde{V \cap U} = \tilde{Z}_S$ of order 2, so that V^* is generated by a transvection on $\tilde{U}$ with center $\tilde{Z}_S$. As $\tilde{U} = \langle \tilde{Z}_S^H \rangle$ and $\tilde{Z}_S = [\tilde{U}, V^*]$, $\tilde{U} = [\tilde{U}, K^*]$ by G.6.2. As $V \trianglelefteq T$, $V^* \leq Z(T^*)$, so G.6.4.4 shows that $K^* \cong L_n(2)$, $2 \leq n \leq 5$, S_6, or S_7; and by G.6.4.2, $\tilde{U}$ is the natural module or the core of the permutation module for S_6. In each case $K^* = N_{GL(\tilde{U})}(K^*)$, so $H^* = K^*$. Next by 6.2.9:

$$C_{H^*}(\tilde{Z}_S) = N_H(Z_S)^* = M_z^* = N_H(V)^* = C_{H^*}(V^*).$$

But if H^* is $L_n(2)$ with $3 \leq n \leq 5$, then V^* is not normal in $C_{H^*}(\tilde{Z}_S)$. Thus $H^* = K^* \cong L_2(2)$, S_6, or S_7. In each case, $V^* \not\leq O^2(H^*)$, so in particular $V \not\leq O^2(X)$, where $X := O^2(M_z)$, and hence $V > V \cap X$. By 6.2.12.1, L acts on $O^2(O^2(H \cap M)O_2(L)) = X$, so L acts on $V \cap X$. Therefore as L is irreducible on V, $V \cap X = 1$.

Suppose first that H^* is S_6 or S_7. Then there are $x, y \in H$ such that $I := \langle V^x, V^y \rangle \leq M_z$ and $I^* \cong S_3$. Then $V^x \not\leq N_G(V^y)$, but $C_{V^x}(\tilde{U}) \leq N_G(Z_S^y) \leq N_G(V^y)$ by 6.2.1; so as V^{*x} has order 2, $N_{V^x}(V^y) = C_{V^x}(\tilde{U})$ is of index 2 in V^x. Similarly $|V^y : N_{V^y}(V^x)| = 2$, so as I is not a 2-group, $O_2(I) \leq O^2(I) \leq X$ and $|Z(O_2(I))| = 2$ by 6.2.6. But as $x, y \in G_z$, $Z \leq V^x \cap V^y = Z(O_2(I))$, so $Z \leq V \cap X$, contrary to the previous paragraph.

This contradiction shows that $H^* \cong S_3$, so $H^* = \langle V^*, V^{g*} \rangle$ for $g \in H - M$ and $|V^H| = |H : M_z| = 3$. Thus $V^H \leq \langle V, V^g \rangle$, so that $K = \langle V, V^g \rangle$. Therefore case (2) of 6.2.6 holds with $K \cong S_3/Q_8^2$, and $Z = V \cap V^g = Z(P)$, where $P := O_2(K)$.

Notice as $Z_S \leq P \trianglelefteq H$ that $U = \langle Z_S^H \rangle \leq P$. Then $R := C_H(\tilde{P}) \leq C_H(\tilde{U}) = Q_H$ by 6.2.12.2. Also as case (2) of 6.2.6 holds, $\overline{N_{V^g}(V)} = \bar{P} \cong E_4$, $C_P(V) = P \cap V$, and $\bar{P} \not\leq \bar{L}$. Therefore $\bar{T} = \bar{P}\langle \bar{t} \rangle$, where $t \in T \cap L$ acts nontrivially on $\bar{P}$. Thus t is nontrivial on $P/(P \cap V)$, so that $t^* \notin V^*$ since $[P, V] \leq P \cap V$. Therefore as $N_{Out(P)}(K^*) \cong S_3 \times S_3$ and $H = KT$, we conclude $H/R \cong S_3 \times \mathbf{Z}_2$ and $C_T(V) = VC_R(V)$. Now $R = PC_R(P)$ as $Inn(P) = C_{Aut(P)}(\tilde{P})$ by A.1.23. But $C_R(P) \leq C_R(V \cap P) = C_R(V)$ by 6.1.10.2, so as $C_R(P) \trianglelefteq H$, $C_R(P)$ centralizes $\langle V^H \rangle = K$. Therefore $C_R(P) = C_R(K)$, so $R = PC_R(K)$ and $C_R(K) \leq C_R(V)$. Thus $C_R(V) = C_R(K)C_P(V) = C_R(K)(P \cap V)$, and hence $C_T(V) = VC_R(V) = VC_R(K) = VC_R(P)$. Then $[P, C_T(V)] = [P, V] \leq V$, so as $L = [L, P]$, $[L, O_2(LT)] \leq V$, and hence L is an $L_2(4)$-block. Now $\Phi(C_T(V)) \leq C_T(L) = 1$ by C.1.13.a and 6.1.6.1. Then since $C_T(V) = VC_R(K)$, $C_R(K)$ is also elementary abelian. Also we chose $t \in T \cap L$ with $\overline{T \cap L} \leq \langle \bar{t} \rangle \bar{P}$; so as $C_T(L) = 1$, by Gaschütz's Theorem A.1.39 $C_T(V) \cap C_G(P\langle t \rangle) = C_V(P\langle t \rangle) = Z$. Thus as $C_R(K)$ centralizes P, $C_R(K) \cap C_G(t) = Z$. But $[t, C_R(K)] \leq C_{[T \cap L, C_T(V)]}(K) = C_V(K) = Z$, so we conclude $m(C_R(K)) \leq 2$, and in case of equality, $[t, C_R(K)] = Z$.

In any case, V is of index at most 2 in $Q := O_2(LT)$. By 1.1.4.6, $F^*(M) = O_2(M)$. Then as Q contains $O_2(M)$ by A.1.6 and Q is abelian, $Q \leq C_M(O_2(M)) \leq O_2(M)$, so $O_2(M) = Q$. Next by 6.2.12.1, $O^2(H \cap M)$ centralizes V, so by Coprime Action, $O^2(H \cap M) \leq C_M(Q) \leq Q$, so $O^2(H \cap M) = 1$. In particular, $C_M(V) = Q$, so that $\bar{M} = M/Q$. An involution in V^g induces a nontrivial inner automorphism on $\bar{L}$, so L/V is not $SL_2(5)$ and hence $V = O_2(L)$.

Now $V = O_2(L) \trianglelefteq M$, so $S_5 \cong \bar{L}\bar{T} \leq \bar{M} \leq N_{GL(V)}(\bar{L}) \cong \Gamma L_2(4)$. Further if $\bar{M} \cong \Gamma L_2(4)$, then an element of order 3 whose image is diagonally embedded in

$\bar{L} \times C_{\bar{M}}(\bar{L})$ centralizes z and hence lies in H, contrary to $O^2(H \cap M) = 1$. Thus $S_5 \cong \bar{M} = \bar{L}\bar{T}$, so that $M = LT$.

Assume first that $C_R(K) = Z$ is of order 2. Thus $M \cong S_5/E_{16}$, with $H = K\langle t\rangle \cong (S_3 \times \mathbf{Z}_2)/Q_8^2$. Then as $C_R(K) = Z$, $C_H(P) \le P$, and $Z = C_P(X)$ for $X \in Syl_3(H)$; thus G satisfies the Hypothesis on page 295 of C. Ku in [**Ku97**]. (Note that the term Z_z there is unnecessary, and also that $\mathbf{Z}_1$ in H/Q should read $\mathbf{Z}_2$). We next verify that G is of *type* M_{22} as defined on p. 295 of that paper—namely we show there exists $z \ne z^d \in P$ with $m(P \cap P^d) = 2$: Let D_L be of order 3 in $N_L(T \cap L)$, and pick $d \in D_L^{\#}$. Then $Z_S = \langle z, z^d\rangle$ and $P \cap P^d = Z_S \cong E_4$, as $\bar{P} \cap \bar{P}^d = 1$ and $Z_S = P \cap P^d \cap V$ from the structure of $\bar{L}$ and its action on V. Thus G is of type M_{22}, so we may apply the Main Theorem of that paper to conclude that $G \cong M_{22}$.

So now we assume that $C_R(K) \cong E_4$, and it remains to derive a contradiction. Then $M \cong S_5/E_{32}$, with $Q \cong E_{32}$. As $C_T(L) = 1$ by 6.1.6.1, Q does not split over V as an L-module. Thus $Q = J(T)$.

Next all involutions in P are fused into V in K, and all involutions in V are fused in L, as are all involutions in $L - V$. Thus all involutions in L are conjugate in G, and are fused to some $j \in P - L$. Next j induces a field automorphism on L/V, so all involutions in jL are conjugate in L. Let $T_0 := P(T \cap L) = \langle j\rangle(T \cap L)$, so that all involutions in T_0 are in z^G. Let $r \in C_R(K) - Z$. Then $r \in Q - V$, and as $Q = J(T)$, $M = N_G(Q)$ controls fusion in Q by Burnside's Fusion Lemma A.1.35. Hence $r \notin z^G$. Therefore $r^G \cap T_0 = \emptyset$, so by Thompson Transfer, $O^2(G) < G$, contradicting simplicity of G. This completes the proof of 6.2.13. □

By 6.2.13, we may assume during the remainder of the section that $Z_S < V \cap U =: V_U$; in Theorem 6.2.19, we will obtain a contradiction under this assumption. Let $Z_U := Z(U)$.

As V^* has order 2 by 6.2.10.4, $m(V_U) \le m(V \cap Q_H) = 3$, so as $Z_S < V_U$:

LEMMA 6.2.14. *$V_U = V \cap Q_H$ is of rank* 3.

LEMMA 6.2.15. *(1) $U = Z_U * U_0$ is a central product, where U_0 is extraspecial of width at least* 2 *and rank at least* 3.

(2) For $v \in V - U$ there exists $g \in H$ with v^*v^{*g} not a 2-element, and for each such g, $|v^*v^{*g}| = 3$ and $\langle V, V^g\rangle \cong S_3/Q_8^2$ with $V_UV_U^g = O_2(\langle V, V^g\rangle) \le U$.

(3) $Z_U \le Z(K)$ and K^* is faithful on U/Z_U.

PROOF. By 6.2.10.3, $O_2(H^*) = 1$, so by the Baer-Suzuki Theorem A.1.2, there is $g \in H$ with v^*v^{*g} not a 2-element. Then $V \not\le N_G(V^g)$, and so $V_U \le N_V(V^g) < V$, so by 6.2.14, $V_U = N_V(V^g)$ is of index 2 in V. Similarly $V_U^g = N_{V^g}(V)$ is of index 2 in V^g, so part (2) follows from 6.2.6. As Z_U centralizes V_U, it centralizes V by 6.1.10.2, so Z_U centralizes $K = \langle V^H\rangle$. Thus $C_{K^*}(U/Z_U) \le O_2(K^*) \le O_2(H^*) = 1$ using 6.2.10.3, so that K^* is faithful on U/Z_U, completing the proof of (3). As $\Phi(U) \le Z$ of order 2 by 6.2.10.1, and U is nonabelian by (2), $\Phi(U) = Z$. We conclude (1) holds, using (2) to see that U_0 is of width at least 2 and rank at least 3. □

Let $\hat{H} := H/Z_U$ and $\dot{H} := H/C_H(\hat{U})$, and identify Z with $\mathbf{F}_2$. Thus by 6.2.15.1, $\hat{U} = \hat{U}_0$ is an $\mathbf{F}_2\dot{H}$-module, and $\dot{H}$ preserves the symplectic form $(\hat{u}_1, \hat{u}_2) := [u_1, u_2]$ on $\hat{U}$, so $\dot{H} \le Sp(\hat{U})$.

LEMMA 6.2.16. *(1)* $V \cap Z_U = Z$, *so* $\dim(\hat{V}_U) = 2$.
(2) $\hat{U} = \langle \hat{Z}_S^H \rangle$ *and* $O_2(\hat{H}) = 1$.
(3) $K^* \cong \dot{K}$.
(4) $\hat{V}_U = [\dot{V}, \hat{U}]$, *and* $\dot{V}$ *is generated by an involution in* $Sp(\hat{U})$ *of type* a_2.
(5) $C_H(\dot{V}) = N_H(\hat{V}_U) = H \cap M$.
(6) $N_{\dot{H}}(\hat{V}_U)$ *is not transitive on* $\hat{V}_U^{\#}$.

PROOF. Part (1) follows from 6.2.15.2, and part (3) from 6.2.15.3. As $U = \langle Z_S^H \rangle$, $\hat{U} = \langle \hat{Z}_S^H \rangle$, so as $\hat{Z}_S \leq Z(\hat{T})$, $\hat{U} \in \mathcal{R}_2(\hat{H})$ by B.2.13, establishing (2). By 6.2.10.2, $[U, V] \leq V_U$, so by 6.2.15.2, $\hat{V}_U = [\hat{U}, \dot{V}]$ is of rank 2 and $U = V^g C_U(V)$ for some $g \in H$. Thus $\dot{V}$ is generated by an involution of type a_2 or c_2 in $Sp(\hat{U})$ in the sense of Definition E.2.6. Indeed for $y \in V^g - Z$ and $v \in V - U$, $[y, v] \in C_{V_U}(y)$ as y induces an involution on V, so $(\hat{y}, \hat{y}^v) = 0$ and hence $\dot{v}$ is of type a_2, establishing (4). As there is a unique involution $i \in Sp(\hat{U})$ of type a_2 with $[\hat{U}, i] = \hat{V}_U$, it follows that $N_{\dot{H}}(\hat{V}_U) = C_{\dot{H}}(\dot{V})$.

Let $h \in C_H(\dot{V})$; then $V^{*h} = V^*$ by (3), so that h acts on $[\tilde{U}, V^*] = \tilde{V}_U$. Thus $C_H(\dot{V}) \leq N_H(V_U)$. But by the previous paragraph, $N_{\dot{H}}(\hat{V}_U) = C_{\dot{H}}(\dot{V})$, so $N_H(V_U) = N_H(\hat{V}_U) = C_H(\dot{V})$. Finally $N_H(V_U) \leq H \cap M$ by 6.2.5.6, while $H \cap M = N_H(V)$ by 6.2.9, and $N_H(V)$ acts on $V \cap U = V_U$, so (5) holds.

By 6.2.9, $H \cap M$ acts on Z_S, so (5) implies (6). □

Let $L_S := O^2(N_L(Z_S))$, $l \in L_S - H$, $E := U \cap U^l$, $W := C_U(Z_S)$, and $X := C_{U^l}(Z_S)$. Observe as $Z_S \leq U$ that $Z_S \leq U^l$, and hence

$$Z \leq Z_S \leq E.$$

LEMMA 6.2.17. *(1)* $Z_U \cap Z_U^l = 1$.
(2) $Z_U \cap U^l = (Z_U \cap Z_U^l)Z$.
(3) $\hat{W} = \hat{Z}_S^{\perp}$ *and* $[\dot{X}, \hat{W}] \leq \hat{E}$.
(4) $\hat{E}$ *is totally singular.*
(5) For $\dot{x} \in \dot{X} - \dot{Z}_U^l$, $C_{\hat{U}}(\dot{x}) \leq \hat{W}$.
(6) $C_X(\hat{U}) = EC_{Z^l}(\hat{U})$.
(7) $\dot{X}$ *induces the full group of transvections on* $\hat{E}$ *with center* $\hat{Z}_S$.
(8) $C_{\hat{E}}(\dot{X}) = \dot{Z}_S$.
(9) $\dot{V} \leq \dot{X}$.
(10) $m(\hat{E}) + m(\dot{X}/\dot{Z}_U^l) = m(\hat{U}) - 1$.

PROOF. Part (1) follows as $V \cap Z_U = Z$ by 6.2.16.1.

Next $\Phi(U^l) = Z^l$ and X acts on Z_U, so $[Z_U \cap U^l, X] \leq Z_U \cap Z^l = 1$ by (1). Thus $Z_U \cap U^l \leq Z(X)$. By 6.2.15.1, $U = U_0 Z_U$ with U_0 extraspecial, so $Z_U^l Z = Z(C_{U^l}(Z)) = Z(X)$. Therefore $Z_U \cap U^l \leq Z_U^l Z$, so as $Z \leq Z_U \cap U^l$, (2) holds.

Observe Hypothesis G.2.1 is satisfied with Z, Z_S, L_S, H in the roles of "V_1, V, L, H", and set $I := \langle U, U^l \rangle$ and $P := O_2(I)$. As U is nonabelian by 6.2.15.1, while $L_S/O_2(L_S) \cong L_2(2)'$, the hypotheses of G.2.3 are also satisfied. So by that lemma, $I = L_S U$, $P = WX$, $1 < Z_S \leq E \leq P$ is an I-series such that $[I, E] \leq Z_S$, and for some nonnegative integer s, and $P/E = W/E \oplus X/E$ is the sum of s natural modules for $I/P \cong L_2(2)$ with $W/E = C_{P/E}(U)$. Now $V = [V, L_S] \leq L_S \leq I$, so $V \leq P$ and hence $\dot{V} \leq \dot{P} = \dot{W}\dot{X} = \dot{X}$, establishing (9).

By definition of the bilinear form on $\hat{U}$, $\hat{Z}_S^{\perp}$ is the image of $C_U(Z_S) = W$ in $\hat{U}$, and the image of a subgroup Y of U in $\hat{U}$ is totally singular iff Y is abelian. As P/E is abelian, $[X, W] \leq E$, completing the proof of (3). As $\Phi(E) \leq \Phi(U) \cap \Phi(U^l) = Z \cap Z^l = 1$, (4) holds.

Pick $u \in U - W$; from the action of I on P/E, the map $\varphi : X \to W/E$ defined by $\varphi(x) := [x, u]E$ is a surjective linear map with kernel E. In particular as $Z \leq E$, $C_{\tilde{U}}(x) \leq \tilde{W}$ for each $x \in X - E$. Further setting $D := \varphi^{-1}(Z_U E/E)$, $D = C_X(U/Z_U E)$. As P/E is a sum of natural modules for I/P, $DZ_U = \langle Z_U^I \rangle E = Z_U Z_U^l E$, so $D = Z_U^l E$. Thus $C_X(\hat{U}) \leq C_X(U/Z_U E) = D = Z_U^l E$. In particular for $u \in U - W$, $C_X(\hat{u}) \leq Z^l E$, and hence (5) follows.

Let $R := C_T(\hat{U})$, and $\tilde{U}_R := C_{\tilde{U}}(R)$ with preimage U_R. By a Frattini Argument, $H = C_H(\hat{U}) N_H(R)$, so as $Z_S \leq U_R$ and $U = \langle Z_S^H \rangle$, $U = U_R Z_U$. Therefore as $Z_U \leq W < U$, R centralizes $\tilde{u} \in \tilde{U} - \tilde{W}$. In particular $C_X(\hat{U}) \leq C_X(\hat{u}) \leq Z^l E$, so (6) holds.

Let $E_0 := EZ_U^l \cap U_0^l$ and $Z_0 := ZZ_U^l \cap U_0^l$. Then $EZ_U^l \leq U^l = U_0^l Z_U^l$, so $EZ_U^l = E_0 Z_U^l$, and similarly $Z_S Z_U^l = ZZ_U^l = Z_0 Z_U^l$. Thus $X = C_{U^l}(Z_S) = C_{U^l}(Z_0)$. As EZ_U^l is abelian, so is E_0. Therefore as U_0 is extraspecial, we conclude that:

$$X \text{ induces the full group of transvections on } E_0 \text{ with center } Z^l \text{ centralizing } Z_0. \quad (!)$$

Let $\hat{e} \in \hat{E} - \hat{Z}_S$. As $EZ_U^l = E_0 Z_U^l$, $eZ_U^l = e_0 Z_U^l$ for some $e_0 \in E_0$. By (2),

$$E \cap Z_S Z_U = Z_S(E \cap Z_U) = Z_S(U^l \cap Z_U) = Z_S(Z_U \cap Z_U^l) = E \cap Z_S Z_U^l.$$

Thus as $\hat{e} \notin \hat{Z}_S$, $e \notin Z_S Z_U^l$, so as $Z_S Z_U^l = Z_0 Z_U^l$, $e_0 \notin Z_0 Z_U^l$. Thus $[e, X] = [e_0, X] = Z^l$ by (!). Hence (7) holds and of course (7) implies (8). Finally

$$m(\hat{U}) = m(\hat{E}) + m(\hat{W}/\hat{E}) + 1 = m(\hat{E}) + m(X/EZ_U^l) + 1 = m(\hat{E}) + m(\dot{X}/\dot{Z}_U^l) + 1,$$

where the last equality follows from (6). Thus (10) holds. □

LEMMA 6.2.18. *(1) $\dot{X}$ and $\dot{Z}_U^l$ are normal in $C_{\dot{H}}(\dot{V})$.*
(2) $\dot{H}$ and its action on $\hat{U}$ satisfy one of the conclusions of Theorem G.11.2.

PROOF. We first verify that $\hat{U}$, $\dot{H}$, $\hat{Z}_S$, $\hat{E}$, $\dot{X}$, $\dot{Z}_U^l$ satisfy Hypothesis G.10.1 in the roles of "V, G, V_1, W, X, X_0". As $\Phi(X) \leq Z^l \leq U$, $\dot{X}$ is elementary abelian, and $\hat{E}$ is totally singular by 6.2.17.4. By construction condition (a) of part (2) of Hypothesis G.10.1 holds. Conditions (b), (c), (d), and (e) are parts (10), (3), (5), and (7) of 6.2.17, respectively. So Hypothesis G.10.1 is indeed satisfied.

Let $M_H := H \cap M$. By 6.2.16.5, $\dot{M}_H = C_{\dot{H}}(\dot{V})$, and by 6.2.9, $M_H = N_H(Z_S)$, so since $[Z_S, U] = Z$, we conclude $M_H = UC_H(Z_S)$. Then as X and Z_U^l are normal in $C_H(Z_S)$, (1) holds.

Next we verify Hypothesis G.11.1. Case (ii) of condition (3) of that Hypothesis holds by 6.2.17.9 and 6.2.16.4. As $\dot{M}_H$ contains the Sylow 2-subgroup $\dot{T}$ of $\dot{H}$, condition (4) of Hypothesis G.11.1 follows from part (1) of this lemma. So Hypothesis G.11.1 is verified. Then part (2) of the lemma follows from Theorem G.11.2. □

We can now complete the elimination of the case remaining after 6.2.13.

THEOREM 6.2.19. *If G satisfies Hypothesis 6.1.1, then $G \cong M_{22}$.*

PROOF. By 6.2.13, we may assume $Z_S < U \cap V$, so the subsequent lemmas in this section are applicable. In particular by 6.2.18.2, $\dot{H}$ and its action on $\hat{U}$ are described in Theorem G.11.2.

By 6.2.16.4, $\dot{V}$ is generated by an involution $\dot{v}$ of type a_2 in $Sp(\hat{U})$ and by 6.2.17.9, $\dot{v} \in \dot{X}$. However in cases (8) and (10)–(13) of G.11.2, $\dot{X}$ contains no involution i with $m([\hat{U}, i]) = 2$, so none of these cases holds. Similarly in case (9), we must have $\dot{H} = \dot{H}_1 \times \dot{H}_2$ with $\dot{H} \cong S_5$, $\dot{H}_2 \cong L_2(2)$, $\hat{U}$ is the tensor product of the natural modules for $\dot{H}_1$ and $\dot{H}_2$, and $\dot{v}$ is a transposition in $\dot{H}_1$. But then $\dot{H}_2$ is transitive on $[\hat{U}, \dot{v}]^\#$, contrary to parts (4) and (6) of 6.2.16. The same argument eliminates case (3) of G.11.2, as there $\dot{v}$ centralizes $Z(O(\dot{H}))$ which is transitive on $[\hat{U}, \dot{v}]^\#$.

Let $d := \dim(\hat{U})$. By 6.2.15.1, $d \geq 4$, so case (1) of Theorem G.11.2 does not hold.

In case (2) of G.11.2, $d = 4$ so $Sp(\hat{U}) \cong S_6$ acts naturally on $\hat{U}$. Thus as $\dot{v}$ is of type a_2, $\dot{v}$ is of cycle type 2^3 in S_6 and $3 \in \pi(\dot{H})$, so 15 or 18 divides $|\dot{H}|$ by G.11.2. Therefore $\dot{H}$ is S_6, S_5 with $\hat{U}$ the $L_2(4)$-module, or a subgroup of $O_4^+(2)$ of order divisible by 9. In each case $N_H(\hat{F})$ is transitive on $\hat{F}^\#$ for each totally singular line $\hat{F}$ in $\hat{U}$, contrary to 6.2.16.6.

As $\dot{v}$ is of type a_2 in $Sp_d(2)$, $|\dot{v}\dot{v}^h| \leq 4$ for each $h \in H$. Thus in case (4) of Theorem G.11.2, $\dot{v}$ is a transposition; in case (5), $\dot{v}$ is a transposition or of type 2^4; in case (6), $\dot{v}$ is a long root involution; and case (7) is eliminated. As $m([\hat{U}, \dot{v}]) = \hat{V}_U$ is of rank 2, while transpositions in cases (4) and (5) act as transvections on $\hat{U}$, we conclude that case (4) does not hold, and in case (5), that $\dot{v}$ is of type 2^4. But now $N_{\dot{H}}(\hat{V}_U)$ is transitive on $\hat{V}_U^\#$, contrary to 6.2.16.6. This contradiction completes the proof of the Theorem. □

We summarize the work of the previous two chapters in:

THEOREM 6.2.20. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_2(2^n)$ and $L \trianglelefteq M \in \mathcal{M}(T)$, and $V \in \mathcal{R}_2(LT)$ with $[V, L] \neq 1$. Then one of the following holds:*

(1) $L/O_2(L) \cong A_5$, and $[V, L]$ is the sum of at most two A_5-modules for $L/O_2(L)$. Further $n(H) = 1$ for all $H \in \mathcal{H}_*(T, M)$.

(2) G is a rank-2 group of Lie type and characteristic 2, but G is $U_5(q)$ only if $q = 4$.

(3) $G \cong M_{22}$ or M_{23}.

PROOF. Suppose first that Hypothesis 5.1.8 holds. Then we may apply Theorem 5.2.3, whose conclusions are among those of (2) and (3) in Theorem 6.2.20. Thus we may suppose that Hypothesis 5.1.8 fails, and hence $n(H) = 1$ for all $H \in \mathcal{H}_*(T, M)$. Then we are done if the first statement in conclusion (1) of 6.2.20 holds; so we may assume it fails, and then we have Hypothesis 6.1.1. Then Theorem 6.2.19 says $G \cong M_{22}$, so that (3) holds. □

In particular, since the groups in conclusions (2) and (3) appear in the list of our Main Theorem, the treatment of QTKE-groups G containing some T-invariant $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_2(2^n)$ is reduced the case where conclusion (1) is satisfied. As mentioned at the outset, we treat this case later in Part F2, which deals with the situation where there exists $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ defined over $\mathbf{F}_2$.

Part 3

Modules which are not FF-modules

In Part 3, we consider most cases where the Fundamental Setup (3.2.1) holds for a pair L, V such that V is not a failure of factorization module for $N_{GL(V)}(Aut_{L_0}(V))$ where $L_0 := \langle L^T \rangle$. The object of Part 3 is to eliminate all but one of the pairs considered here: we will show that $G \cong J_4$ when V is the cocode module for $M/V \cong M_{24}$, and that none of the other pairs lead to examples. However we will also have to deal with a number of shadows whose local subgroups possess the pairs considered in this chapter.

THEOREM Assume the Fundamental Setup (3.2.1). Then one of the following holds:

(1) V is an FF-module for $N_{GL(V)}(Aut_{L_0}(V))$.

(2) V is the cocode module for $M/V \cong M_{24}$ and $G \cong J_4$.

(3) V is the orthogonal module for $Aut_{L_0}(V) \cong L_2(2^{2n}) \cong \Omega_4^-(2^n)$, with $n > 1$.

(4) Conclusion (3) of 3.2.6 is satisfied. In particular $L < L_0$ and $L/O_2(L) \cong L_2(2^n)$, $Sz(2^n)$, or $L_3(2)$.

Note that case (3) and a part of case (1) were handled earlier in Part 2; while case (4) and the remainder of case (1) will be handled later in Part 4 and Part 5.

In the initial chapter of Part 3, we begin to implement the outline for weak closure arguments described in subsection E.3.3. The cases not corresponding to shadows or J_4 will then be quickly eliminated by comparing various parameters associated to the representation of L_0T on V. The remaining two chapters in Part 3 will pursue the deeper analysis required when the configurations do correspond to shadows or J_4.

CHAPTER 7

Eliminating cases corresponding to no shadow

Recall we wish to prove:

THEOREM 7.0.1. *Assume the Fundamental Setup (3.2.1). Then one of the following holds:*

(1) V is an FF-module for $N_{GL(V)}(Aut_{L_0}(V))$.

(2) V is the cocode module for $M/V \cong M_{24}$ and $G \cong J_4$.

(3) V is the $\Omega_4^-(2^n)$-module for $Aut_{L_0}(V) \cong L_2(2^{2n})$.

(4) Conclusion (3) of 3.2.6 is satisfied. In particular $L < L_0$ and $L/O_2(L) \cong L_2(2^n)$, $Sz(2^n)$, or $L_3(2)$.

Recall also that in Part 3, we concentrate on the cases in the FSU not appearing in cases (1), (3), or (4) of Theorem 7.0.1; so we assume the following hypothesis:

HYPOTHESIS 7.0.2. *(1) The Fundamental Setup (3.2.1) holds. In particular $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple, $L_0 := \langle L_0^T \rangle$, and $M := N_G(L_0)$.*

(2) V is not an FF-module for $N_{GL(V)}(Aut_{L_0}(V))$.

(3) Case (3) of 3.2.6 does not hold.

(4) V is not the orthogonal module for $Aut_{L_0}(V) \cong \Omega_4^-(2^n)$.

Part (1) of Hypothesis 7.0.2 has various consequences including the following: As $L \in \mathcal{L}^*(G,T)$, by 1.2.7.3 L_0T is a uniqueness subgroup with $M = !\mathcal{M}(L_0T)$. Furthermore by 3.2.2.8, our module V for L_0T is 2-reduced, and we have various other properties including $Q := O_2(L_0T) = C_T(V)$, $V \trianglelefteq T$, and $M = !\mathcal{M}(N_G(Q))$, so that $C(G,Q) \leq M$, as in 1.4.1.

By part (2) of Hypothesis 7.0.2 and Remark B.2.8, $J(T) \leq C_G(V)$, so Q contains $J(T)$. By 3.2.10, a number of useful properties follow from this fact; for example, $N_G(J(T)) \leq M$, so that $J(T) \leq S \leq T$ implies $N_G(S) \leq M$. Further there are restrictions on the subgroups $H \in \mathcal{H}_*(T,M)$: By 3.1.8.3, H centralizes $Z := \Omega_1(Z(T))$ and $C_V(L_0) = 1$.

Finally by part (3) of Hypothesis 7.0.2 and 3.2.7, V is a TI-set under M. It follows that $H \cap M \leq C_M(Z) \leq N_G(V) = M_V$.

In this chapter we begin the anaysis of groups satisfying Hypothesis 7.0.2. In the first section, we list the cases that can arise. The last of these cases seems difficult to treat using only the methods of this chapter, so in the third section we also add Hypothesis 7.3.1, which excludes that case; the case is treated in the final chapter of part 3. Also the penultimate case and the case where $L_0/O_2(L_0) \cong L_3(2)$ and $m(V) = 6$ cause difficulties, requiring extra analysis; these cases are treated in the last sections of this chapter and the next chapter.

7.1. The cases which must be treated in this part

Recall we are assuming Hypothesis 7.0.2 and the notation established in the discussion following that Hypothesis in the introduction to this chapter.

Section 3.2 determines the list of possibilities for $\bar{L}_0$ and V. We first extract the sublist consisting of those cases where V is not an FF-module for $N_{GL(V)}(Aut_{L_0}(V))$. We begin that deduction, later summarizing the final results in the Table of Proposition 7.1.1.

Recall in the Fundamental Setup that $V = \langle V_\circ^T \rangle$ for some member $V_\circ$ of $Irr_+(L_0, R_2(L_0), T)$, while $V_M := \langle V^M \rangle$, $M_V := N_M(V)$, and $\bar{M}_V := M_V/C_{M_V}(V) = Aut_G(V)$. We wish to determine the cases where V is not an FF-module for $N_{GL(V)}(Aut_{L_0}(V))$.

We first consider the case where $T \not\leq N_G(L)$. Here 3.2.6 applies, and we see that in cases (1) and (2) of 3.2.6, V is not an FF-module and $V_M = V = V_\circ$; these examples appear as the last two cases (below the second horizontal line) in the Table of Proposition 7.1.1. By part (3) of Hypothesis 7.0.2, case (3) of 3.2.6 does not hold. These are the modules where $V \neq V_\circ$; they are treated later in chapter 10 of part 4 in a uniform manner, although some of these examples are FF-modules and some are not.

Therefore we may assume that $T \leq N_G(L)$, so $L_0 = L$ and $\langle L, T \rangle = LT$. We first consider the case where $T \not\leq N_G(V_\circ)$, so that case (3) of 3.2.5 holds. These modules satisfy $V_M = V = V_\circ \oplus V_\circ^t$ for $t \in T - N_T(V_\circ)$; the examples with $\bar{L} \cong L_4(2)$ or $L_5(2)$ are FF-modules, but the others are not, and so the latter appear as the second group in the Table (between the horizontal lines).

Thus we are reduced to the case $T \leq N_G(V_\circ)$, so that $V = V_\circ$. Furthermore $C_V(L) = 1$ as remarked in the introduction to this chapter, so V is an irreducible L-module. These cases are listed in 3.2.9, and form the first group in the Table—except for the first case 3.2.9.1, which is excluded by part (4) of Hypothesis 7.0.2. This case was handled in part 2 in the "Generic Case", since the unitary groups $U_4(2^n)$ arise in that case.

This completes the deduction of Proposition 7.1.1.

We also indicate, in the last two columns of the Table of that result, first the "shadows"(that is, groups having such a local configuration but which are not quasithin or simple), and then the single simple quasithin example given by J_4.

Three of the cases seem to require treatment different from the fairly uniform approach used to treat the remaining cases. In the final case where V is the orthogonal module for $\bar{L}_0 = \Omega_4^+(2^n)$, we have $m = 2$ when $n = 2$—and worse, $a = m = n$ for any n, and as L is not normal in M, we can't appeal to Remark 4.4.2. Because of these difficulties, this case will be treated by more direct methods in the third and final chapter of this part. The penultimate case poses similar difficulties, and is treated in the last section of the second chapter 8 of this part. Finally the case where $\bar{L}_0\bar{T} \cong Aut(L_3(2))$ and V is the sum $3 \oplus \bar{3}$ of the natural and dual module requires special treatment, particularly as $m = 2$ makes it difficult to establish lemma 7.3.2. This case is dealt with at the end of chapter 7.

We have established the list of cases to be treated under Hypothesis 7.0.2:

PROPOSITION 7.1.1. *The cases where V is not an FF-module, and which appear in neither conclusion (3) nor (4) of Theorem 7.0.1, are:*

$\bar{M}_V \geq$	*restr. on n*	$\dim V$	*descr. V*	*shadows*	*example*
$U_3(2^n)$	$n \geq 2$	$6n$	*natural*		
$Sz(2^n)$	*odd* $n \geq 3$	$4n$	*natural*		
$L_3(2^{2n})$	$n \geq 2$	$9n$	$3 \otimes 3^\sigma$	$U_6(2^n), U_7(2^n)$	
$Aut(M_{12})$		10	*irred.perm.*		
$\hat{M}_{22}$		12	*unitary*		
M_{22}		10	*code*	Co_2	
		$\overline{10}$	*cocode*	F_{22}	
M_{23}		11	*code*		
		$\overline{11}$	*cocode*	F_{23}	
M_{24}		11	*code*	Co_1	
		$\overline{11}$	*cocode*	F_{24}	J_4
$SL_3(2^n).2$		$6n$	$3n \oplus \overline{3n}$		
$Sp_4(2^n)'.2$		$8n$	$4n \oplus 4n^{\bar{t}}$		
S_7		8	$4 \oplus \bar{4}$		
$L_3(2) \wr 2$		9	$3 \otimes 3^t$	$L_6(2).2, L_7(2).2$	
$L_2(2^n) \wr 2$	$n \geq 2$	$4n$	$2n \otimes 2n^{\bar{t}}$	$L_4(2^n).2, L_5(2^n).2$	

7.2. Parameters for the representations

Our main task in chapter 7 will be to eliminate the cases not corresponding to a shadow or example. We use the weak closure methods of section E.3. These methods are "numerical", in the sense that they compare parameters—such as a, m, n', α, β determined only by the representation of M on V, and on other parameters r, s, w determined by suitable subspaces U of V with $C_G(U) \leq M$. We will obtain a numerical contradiction from the Fundamental Weak Closure Inequality involving these parameters, established in E.3.29. [1]

Because the initial steps in the weak closure argument involve primarily the parameters m_2 of $\bar{M}_V$ and m, a of the module V, estimates on these values are included in the early columns of the Table in Proposition 7.2.1 below.

Proofs that the parameters are indeed as indicated in the Table appear in corresponding sections of chapter H of Volume I—with the exception of the parameter n', which is determined in 7.3.4. Certain values in the table are given in parentheses; these are values which seem to be well known, but which we do not require in our argument, and hence are not verified in chapter H. The last two columns of the table list parameters α and β primarily relevant to an application of E.6.27 later in this chapter; the derivation of these parameters also appears in chapter H, except in some cases like the last case where they are not used.

We now describe the Table in more detail: Column 1, labeled "case", indicates the pair $\bar{L}_0, V$ discussed in the corresponding row. Column 2, labeled "$a \leq$", gives an upper bound on $a := a(\bar{M}_V, V)$. Column 3, labeled "$m \geq$", gives a lower bound on $m := m(\bar{M}_V, V)$. The definitions of these parameters appear as E.3.9 and E.3.1. Column 4, labeled "$w \geq$", gives the resulting lower bound on the difference $m - a$, which is in turn a lower bound on the parameter w of Definition E.3.23 by 7.3.3. Column 5, labeled "n'", is the parameter $n' := n'(Aut_G(V))$ given in Definition

[1] Of course, local configurations $\bar{L}, V$ that actually exist in shadows are not eliminated numerically. So in the following chapter 8, we instead show that those configurations provide the *unique* solution to the FWCI; and then eliminate the cases by showing those configurations violate our SQTK hypothesis.

E.3.37; by 7.3.4 this column will give an upper bound on w. Column 6, labeled "$m_2 \leq$", gives an upper bound on $m_2 := m_2(\bar{M}_V)$. Columns 7 and 8, labeled "$\beta \geq$" and $\alpha \geq$", give the minimum codimension of a subspace U of V such that $O^2(C_M(U)) \not\leq C_M(V)$, or such that $C_{\bar{M}_V}(U)$ contains an $(F-1)$ offender, respectively. If there are no $(F-1)$-offenders, then $J_1(T)$ centralizes V and column 8 contains ∞. We remark that the minimum of α and β by 7.4.1 gives a lower bound for the parameter r of Definition E.3.3 in the cases where $L \trianglelefteq M$.

PROPOSITION 7.2.1. *The values of various parameters for our modules are:*

case	$a \leq$	$m \geq$	$w \geq$	n'	$m_2 \leq$	$\beta \geq$	$\alpha \geq$
$SU_3(2^n)/6n$	n	$2n$	n	n	$n+1$	$4n$	∞
$Sz(2^n)/4n$	n	$2n$	n	n	n	$\frac{8}{3}n$	∞
$(S)L_3(2^{2n})/9n$	$3n$	$3n$	0	$2n$	$4n$	$4n$	$\infty; 5$ *if* $n=1$
$M_{12}/10$	2	4	2	2	4	6	∞
$3M_{22}/12$	3	4	1	2	5	8	∞
$M_{22}/10$	3	3	0	2	5	6	6
$M_{22}/\overline{10}$	3	3	0	2	5	6	5
$M_{23}/11$	3	4	1	2	4	6	∞
$M_{23}/\overline{11}$	3	4	1	2	4	6	5
$M_{24}/11$	3	4	1	2	6	6	7
$M_{24}/\overline{11}$	3	4	1	2	6	6	5
$SL_3(2^n).2/3n \oplus \overline{3n}$	n	$2n$	n	n	$2n$	$4n$	$\infty; 2$ *if* $n=1$
$Sp_4(2^n)'.2/4n \oplus 4n^{\bar{t}}$	$< 2n$	$3n$	$> n$	n	$3n$	$4n$	∞
$S_7/4 \oplus \bar{4}$	2	4	2	2	3	4	∞
$L_3(2) \wr 2/3 \otimes 3^t$	2	3	1	2	4	6	3
$L_2(2^n) \wr 2/2n \otimes 2n^{\bar{t}}$	(n)	n	0	n	$2n$	$(2n)$	$\infty; 2$ *if* $n=1$

7.3. Bounds on w

We now implement the outline discussed in subsection E.3.3.

As remarked earlier, in chapter 7 and the next chapter 8, we exclude the final case in the Tables of Propositions 7.1.1 and 7.2.1:

HYPOTHESIS 7.3.1. *V is not the orthogonal module for $\bar{L}_0 \cong \Omega_4^+(2^n)$.*

Recall that the case excluded by Hypothesis 7.3.1 will be treated by other methods in the third chapter 9 of this part 3. Thus in this chapter and the next, discussion of "all" cases in the Tables refers to the remaining cases, with the final row of the Tables excluded.

We first discuss the parameters r and s. See Definitions E.3.3, E.3.5, E.3.1, and E.3.9 for the parameters r, s, m, and a.

PROPOSITION 7.3.2. *$r \geq m$, so that $s = m$.*

PROOF. This follows from Theorem E.6.3 when $m > 2$, which we see from Table 7.2.1 holds in all cases except for $L_3(2)$ on $3 \oplus \bar{3}$. In that case we make a direct argument, but as the methods are of a different flavor from the uniform treatment in this chapter, we banish those details to a mini-Appendix at the end of the chapter; see 7.7.1 for the proof. □

In view of 7.3.2, the column headed $m \geq$ in Table 7.2.1 also provides a lower bound for the parameter s. Then comparison with a gives us information on w. Recall from Definition E.3.23 that

$$w := \min\{m(V^g/V^g \cap T) : g \in G \text{ and } [V, V^g \cap T] \neq 1\}.$$

LEMMA 7.3.3. *The column "$w \geq$" of Table 7.2.1 gives a lower bound for w.*

PROOF. Recall from E.3.34.1 that $w \geq s - a$. As $s = m$ by 7.3.2, we subtract the column for a from the column for m in the Table, and obtain the result. □

Having established a lower bound on w, we now apply E.3.35 in order to obtain an upper bound for w.

Let H denote an arbitrary member of $\mathcal{H}_*(T, M)$, although from time to time we may temporarily impose further constraints on H.

PROPOSITION 7.3.4. *$w \leq n(H) \leq n'(\bar{M}_V) = n' < s$, where n' is listed in the column headed "n'" in Table 7.2.1.*

PROOF. Let k denote the value of n' given in Table 7.2.1; we first assume $n' = k$. Recall that $s = m$ by 7.3.2, and observe further that $m > n'$ in all cases in the Table, so that $s > n'$. Next we check that Hypothesis E.3.36 is satisfied: We observed in the introduction to this chapter that $V \trianglelefteq T$, $M = !\mathcal{M}(N_G(Q))$, and V is a TI-set under M, with $H \leq C_G(Z)$, and $H \cap M \leq C_M(Z) \leq N_M(V)$. Further by Hypothesis 7.0.2, V is neither an FF-module nor the orthogonal module for $L_2(2^{2n})$, so whenever $n(H) > 1$ we can apply Theorem 4.4.14 to conclude that a Hall $2'$-subgroup B of $H \cap M$ is faithful on $\bar{L}_0$, and hence also on V. It follows that $C_{H\cap M}(V) \leq O_2(H \cap M)$, completing the verification of Hypothesis E.3.36. Now since $n' < s \leq r$, the lemma holds by E.3.39.1.

Thus it remains to verify that $k = n'$. If $\bar{L}$ is $L_3(2)$ on $3 \oplus \bar{3}$ or $Sp_4(2)' \cong A_6$ on $4 \oplus \bar{4}$, then T is nontrivial on the Dynkin diagram of $\bar{L}$, and hence $\bar{T}$ permutes with no nontrival subgroup of $\bar{M}_V$ of odd order, so that $n' = 1 = k$. In all other cases where $\bar{L}$ is of Lie type, $\bar{T}$ permutes with a Cartan subgroup of $\bar{L}$, which contains a cyclic subgroup of order $2^k - 1$, so that $n' \geq k$ in these cases. Similarly when $\bar{L}$ is sporadic, $\bar{T}$ permutes with a subgroup of order 3 and $k = 2$, so $n' \geq k$. Finally if $n' > k$ then $n' > 2$ and we may apply A.3.15 to some prime $p > 3$ which does not divide $k(2^k - 1)$ and obtain a contradiction which completes the proof. □

We can already see that when $\bar{L}$ is $Sp_4(2^n)$, the value in the column $w \geq$ strictly exceeds the value in the column n', so that 7.3.3 and 7.3.4 provide our first example of a numerical contradiction, eliminating one of our cases from Table 7.1.1:

COROLLARY 7.3.5. *$\bar{L}$ is not $Sp_4(2^n)'$.* [2]

7.4. Improved lower bounds for r

We saw earlier in 7.3.2 that $r \geq m \geq 2$. In many cases, we can improve this bound on r using E.6.28: First $r > 1$, giving hypothesis (1) of E.6.28. As V is not an FF-module, hypothesis (2) of E.6.28 holds. Finally if $L \trianglelefteq M$, and X is an abelian subgroup of $C_M(V)$ of odd order, then $N_G(X) \leq M$ by Theorem 4.4.3.

[2]It would also be possible to eliminate case (iii) of 3.2.6.3.c at this point (adjusting for the fact that V might not be a TI-set under M). However, it seems more natural to treat all cases of 3.2.6.3.c uniformly in chapter 10.

Note that when $L \trianglelefteq M$, Hypothesis 4.4.1 is satisfied by any abelian subgroup X of $C_M(V)$ of odd order, in view of Remark 4.4.2. Thus the hypotheses of E.6.28 are satisfied, so $r \geq \min\{\alpha, \beta\}$ by that result, while column 7 and 8 in Table 7.2.1 give lower bounds on α and β, so:

PROPOSITION 7.4.1. *If $L \trianglelefteq M$ then $r \geq \min\{\alpha, \beta\}$, the bound appearing in the final column of Table 7.2.1.*

7.5. Eliminating most cases other than shadows

We begin with the cases which are simplest to eliminate. Recall the Fundamental Weak Closure Inequality E.3.29:

LEMMA 7.5.1. *(**FWCI**)* $\quad m_2 + w \geq r.$

We add the adjacent columns for $w \leq$ and $m_2 \leq$ in Table 7.2.1, and compare this sum S with the bound R given by the final column $\min\{\alpha, \beta\}$ of the Table. We find in the following cases that we get the contradiction $S < R$ to the FWCI, in view of 7.4.1:

LEMMA 7.5.2. *(1) $\bar{L}$ is not $U_3(2^n)$, $Sz(2^n)$, or $\hat{M}_{22}$.*
(2) If $\bar{L}_0$ is $L_3(2^n)$ on $3 \oplus \bar{3}$, then $n = 1$.

Certain other cases are not immediately ruled out, but require only a slight extension of this argument.

For the rest of the section, adopt the notation of the latter part of section E.3: Let $A := N_{V^g}(V)$, be a "w-offender" on V; that is $m(V^g/A) = w$ with $A \not\leq C_G(V)$, so that $\bar{A} \neq 1$.

LEMMA 7.5.3. *(1) Assume the inequality in 7.5.1 is an equality, and let*

$$\mathcal{B} := \{B \leq A : |B : C_A(V)| = 2\}, \text{ and } W := \langle C_V(B) : B \in \mathcal{B} \rangle.$$

Then $m(\bar{A}) = m_2$, $r = m(V^g/C_A(V))$, and $W \leq N_V(V^g)$. Further $m(V/W) \geq w$, and in case of equality, $W = N_V(V^g)$ is a w-offender on V^g and $m(W/C_V(A)) = m_2$.
(2) $m(\bar{A}) \geq r - w$.
(3) $C_V(A) = C_V(V^g)$.

PROOF. By 7.3.4, $w < s$, so (3) follows from E.3.6. By part (2) of Hypothesis 7.0.2, Hypothesis E.3.24 is satisfied. Thus (1) follows from E.3.31 and (3), and (2) from E.3.28.3. □

In certain cases when the FWCI has a unique solution, the embedding of $\bar{A}$ in $\bar{M}_V$ is determined, which leads to a contradiction:

LEMMA 7.5.4. *$\bar{L}$ is neither M_{12}, nor M_{23} on the code module* 11.

PROOF. Assume otherwise. From Table 7.2.1 and 7.4.1, the FWCI is an equality with $w = 2$. Therefore by 7.5.3.1, $m(\bar{A}) = m_2 = 4$ and $r = 6 = m(V^g/C_A(V))$. Define W as in 7.5.3.1, and observe that $W \leq N_V(V^g)$ and $m(V/W) \geq w = 2$ by that result. But if $\bar{M}_V = M_{23}$, then as $m(\bar{A}) = 4$, H.16.8 says $m(V/W) < 2$, a contradiction.

Therefore $\bar{M}_V = M_{12}$. Here as $m(\bar{A}) = 4$, $U = C_V(A)$ is of dimension at most 3 and $m(W) \geq 8$ by H.11.1.4. But then $m(W/U) \geq 5 > 4 = m_2$, contrary to 7.5.3.1. This contradiction completes the proof. □

In the case of A_7, we can dig a little deeper to increase r:

LEMMA 7.5.5. *$\bar{L}$ is not A_7.*

PROOF. Assume $\bar{L}$ is A_7. First $r \geq 4$ by 7.4.1, and by the FWCI 7.5.1, it suffices to show that $r > 5$. We appeal to E.6.27 with $j = 1$: As ∞ is in the column for α in Table 7.2.1, V is not an $(F-1)$-module for $Aut_{\bar{M}}(\bar{L})$, hence $J_1(M) \leq C_M(V)$. From the proof of 7.4.1, $C_G(X) \leq M$ for any $1 \neq X \leq C_M(V)$ of odd order. Thus for $U \leq V$ with $O^2(C_M(U)) \leq C_M(V)$, E.6.27 says $C_G(U) \leq M$. Let $\mathcal{U}$ consists of those $U_1 \leq V$ with $O^2(C_M(U_1)) \not\leq C_M(V)$; it suffices to show $C_G(U_1) \leq M$, for each $U_1 \in \mathcal{U}$ with $m(V/U_1) < 6$. But if $U_1 \in \mathcal{U}$ with $m(V/U_1) < 6$, then $U_1 < U_s := C_V(\bar{s})$ where $\bar{s}$ is a 3-element of cycle type 3^2 in A_7. Thus it will suffice to show that $C_G(U_1) \leq M$, for each U_1 of codimension at most 1 in U_s. Choose a counterexample U_1, and let $U_1 \leq U_2 \leq V$ be maximal subject to $C_G(U_2) \not\leq M$. Note that $C_{\bar{M}}(U_1) = \langle \bar{s} \rangle$, and in particular $O^{2'}(C_M(U_1)) \leq C_M(V)$: For $V = V_1 \oplus V_2$ where $\{V_1, V_2\} = Irr_+(L, V)$, so that $U_s = (U_s \cap V_1) \oplus (U_s \cap V_2)$ and $U_s \cap V_j$ is a 2-subspace of V_j. If $\bar{i}$ is an involution in $\bar{M}$ centralizing U_1, then i must act on $U_1 \cap V_j \neq 0$ and hence on V_j. Thus i centralizes the projection $U_{1,j}$ of U_1 on V_j, and so for $j = 1$ or 2, $U_{1,j} = U_s \cap V_j$. This is impossible as $C_{\bar{L}}(U_s \cap V_j) = \langle \bar{s} \rangle$. So U_1, and hence also U_2, lies in the set Γ of Definition E.6.4. Then U_2 satisfies the hypotheses of E.6.11, so as $m(V/U_2) < 6$ and $m(V/U_2) \geq r \geq 4$, we conclude from E.6.11 that $Aut_{C_M(U_2)}(V)$ contains an element of order 15 or 31, whereas A_7 has no such element. This contradiction shows that $C_G(U_1) \leq M$, completing the proof of the lemma. □

Finally our weak closure methods provide some numerical information which will be useful in the next chapter in treating two cases arising in certain shadows:

LEMMA 7.5.6. *(1) If $\bar{L}$ is M_{22} on the code module then $w > 0$.*
(2) If $\bar{L}$ is $(S)L_3(2^{2n})$ on $9n$ then $w \geq n$.

PROOF. Assume that the lemma fails. From Table 7.2.1 and 7.4.1, $r \geq 4n$ if $\bar{L}$ is $(S)L_3(2^{2n})$, while $r \geq 6$ if $\bar{L}$ is M_{22} on the code module. From Table 7.2.1, $m_2 \leq 5$ when $\bar{L}$ is M_{22}, so 7.5.1 supplies a contradiction to our assumption that $w = 0$ in that case. Thus $\bar{L}$ is $(S)L_3(2^{2n})$.

By E.3.10, $\bar{A} \in \mathcal{A}_{s-w}(\bar{M}_V, V)$, while by 7.3.2, $s = m$. Thus $s \geq 3n$ by Table 7.2.1, so as $w < n$, $\bar{A} \in \mathcal{A}_{2n+1}(\bar{M}_V, V)$. By 7.5.3.2, $m(\bar{A}) \geq r - w > 3n$. Thus we have verified the hypotheses of lemma H.4.5.

Next if $\bar{B}_1 \leq \bar{A}$ with $m(\bar{A}/\bar{B}_1) \leq 3n$ and B is the preimage in A of $\bar{B}_1$, then $m(V^g/B) \leq 3n + w < 4n \leq r$, so $C_V(\bar{B}_1) = C_V(B) \leq N_G(V^g)$ by E.3.4. Thus

$$W_A = \langle C_V(\bar{B}_1) : m(\bar{A}/\bar{B}_1) \leq 3n \rangle \leq N_V(V^g).$$

Therefore $[W_A, A] \leq W_A \cap V^g \leq C_{W_A}(A)$, so A is quadratic on W_A, contrary to H.4.5.2. This contradiction completes the proof of (2) and establishes the lemma. □

7.6. Final elimination of $\mathbf{L_3(2)}$ on $\mathbf{3 \oplus \bar{3}}$

In this section, we eliminate the case left open in 7.5.2.2. This "small" case of $L_3(2)$ on $3 \oplus \bar{3}$ seems to require special treatment: For example, we've already seen in 7.3.2 that the fact that $m = 2$ requires arguments of a different flavor to prove that $r \geq m$; indeed recall that we are postponing that proof that $r \geq m$ until Theorem 7.7.1 in the final section of the chapter.

A second difficulty is that we cannot improve our lower bound on r using E.6.28: since when V is the $3 \oplus \bar{3}$-module for $L_3(2)$, the elementary groups of rank 1 or 2 in $\bar{L}$ centralize subspaces of codimensions 2 or 3 in V, respectively, and hence are $(F-1)$-offenders. In the next lemma, we use *ad hoc* methods to complete the treatment of the case of $L_3(2)$ on $3 \oplus \bar{3}$.

LEMMA 7.6.1. *$\bar{L}_0$ is not $L_3(2)$ on $3 \oplus \bar{3}$.*

PROOF. From Table 7.2.1, $n' = 1$, so $n(H) = 1$ for each $H \in \mathcal{H}_*(T, M)$ by 7.3.4. Also $w > 0$ by 7.3.3 and Table 7.2.1, while $w \leq n(H) = 1$ by 7.3.4; so in fact $w = 1$.

Next $r \geq 3$ as we will show in 7.7.1 in the final section, so as $m_2 = 2$, 7.5.1 is an equality; hence $m(\bar{A}) = 2$ and $m(V^g/C_A(V)) = 3 = r$ by 7.5.3.1.

Suppose first that $\bar{A} \not\leq \bar{L}$. Then by H.4.3.1, $U := C_V(A)$ has dimension 2, and (for $\mathcal{B}$ as in 7.5.3.1) $A_1 := \langle C_V(B) : B \in \mathcal{B}\rangle$ is of dimension 5, while $A_1 \leq N_V(V^g)$ by 7.5.3.1. Also $U = C_V(V^g)$ by 7.5.3.3, so that $m(Aut_{A_1}(V^g)) = 3$, contradicting $m_2(\bar{M}) = 2$.

Thus $\bar{A} \leq L$. In the notation of 7.7.1 and subsection H.4.1 of chapter H of Volume I, $V = V_1 \oplus V_2$ with $V_i \in Irr_+(L, V)$, $V_2 = V_1^t$ for $t \in T - N_T(V_1)$, and V_1 has basis denoted by $1, 2, 3$. By H.4.3.2, we may take $\bar{A}$ to be the unipotent radical of the centralizer of the vector $1 \in V_1$; then $U := C_V(A) = \langle 1\rangle \oplus \langle 2^t, 3^t\rangle$ is of rank 3, and

$$A_1 = \langle C_V(\bar{a}) : \bar{a} \in \bar{A}^\#\rangle = V_1 \oplus \langle 2^t, 3^t\rangle$$

is of rank 5. So by 7.5.3.1, $A_1 = N_V(V^g)$; thus we have symmetry between V and V^g, in that A_1 is also a w-offender on V^g. Set $(M^g)^* := M^g/C_G(V^g)$. Then $A_1^* \leq L^{g*}$ by the previous paragraph, so $U_1 := C_{V^g}(A_1^*)$ is 3-dimensional and $U_1 = C_A(V)$.

In particular $Z_1 := [A, A_1] \leq V \cap V^g$, and by H.4.3.2, Z_1 is generated by the vector $1 \in V_1$. Thus

$$X := \langle V^g, V\rangle \leq G_1 := N_G(Z_1) = C_G(Z_1).$$

Now A centralizes U and V/U, and by symmetry, A_1 centralizes U_1 and V^g/U_1. It follows that X centralizes the quotients in the series

$$1 < UU_1 < AA_1.$$

Set $\tilde{X} := X/AA_1$. As $\tilde{V}$ and $\tilde{V}^g$ have order 2, $\tilde{X}$ is dihedral; set $\tilde{Y} := O(\tilde{X})$. A Hall $2'$-subgroup Y_0 of the preimage of $\tilde{Y}$ centralizes AA_1 by Coprime Action, and then as $r = 3$ while $m(V/A_1) = 1 = m(V^g/A)$, Y_0 centralizes $\langle V^g, V\rangle = X$. As $\tilde{Y}$ is dihedral, $\tilde{Y}_0 = 1$, so X is a 2-group.

We can now finish the proof of the lemma using later Proposition 7.7.2, which says that $G_1 \in \mathcal{H}^e$; we postpone the statement and proof of Proposition 7.7.2 until the next section, as it is proved in parallel with lemma 7.7.6.

Set $\tilde{G}_1 := G_1/Z_1$; then as $G_1 \in \mathcal{H}^e$, $F^*(\tilde{G}_1) = O_2(\tilde{G}_1)$ by A.1.8. Recall $T_1 := C_T(Z_1)$ is Sylow in G_1 by 3.2.10.4. Now $T_1 \leq LO_2(LT)$, so $C_{\tilde{V}_1}(T_1)$ and $C_{\tilde{V}_1^t}(T_1)$ are nontrivial, and by B.2.14 both lie in $O_2(\tilde{G}_1)$. Then as $C_L(Z_1)$ is irreducible on $\tilde{V}_1 = \widetilde{A_1 \cap V_1}$ and $\langle \tilde{2}^t, \tilde{3}^t\rangle = \widetilde{A_1 \cap V_2}$, it follows that $A_1 \leq O_2(G_1)$.

Since $Z_1 \leq V \cap V^g$ with $[V, V^g] \neq 1$, 3.2.10.6 says that $V \not\leq O_2(G_1)$. So as $|V : A_1| = 2$, $A_1 = V \cap O_2(G_1)$, and hence $m(V/V \cap O_2(G_1)) = 1$. Then for any $h \in G_1$, we have $m(V^h/V^h \cap O_2(G_1)) = 1$, with $V^h \cap O_2(G_1) \leq T_1 \leq N_G(V)$. If V^h centralizes V, then $\langle V, V^h \rangle = VV^h$ is a 2-group, while if V^h does not centralize V then $V^h \cap O_2(G_1)$ is a w-offender on V, so our argument above for V^g applies to V^h to show $\langle V, V^h \rangle$ is again a 2-group. Therefore the Baer-Suzuki Theorem forces $V \leq O_2(G_1)$, which we saw is not the case. This completes the proof. □

7.7. mini-Appendix: $\mathbf{r > 2}$ for $\mathbf{L_3(2).2}$ on $\mathbf{3 \oplus \bar{3}}$

Our goal in this section is to prove the following two results:

THEOREM 7.7.1. *If $\bar{L}_0$ is $L_3(2)$ on $3 \oplus \bar{3}$, then $r > 2$. In particular, $s = m = 2$.*

PROPOSITION 7.7.2. *Assume $\bar{L}_0$ is $L_3(2)$ on $3 \oplus \bar{3}$, and $r > 2$. Then $F^*(C_G(v_1)) = O_2(C_G(v_1))$ for each $V_1 \in Irr_+(L, V)$ and $v_1 \in V_1^\#$.*

So throughout this section, assume we are in the case where $\bar{L}_0$ is $L_3(2)$ on $3 \oplus \bar{3}$. Recall $L \in \mathcal{L}^*(G,T)$, $L \trianglelefteq M \in \mathcal{M}(T)$, $V \in \mathcal{R}_2(LT)$ is normal in M, $\bar{M} := M_V/C_M(V) \cong Aut(L_3(2))$, and $V = V_1 \oplus V_2$, where $V_2 := V_1^t$ for $t \in T - N_T(V_1)$ and V_2 is the dual of the natural module V_1. Recall $Q := O_2(LT)$.

The module V is discussed in subsection H.4.1 of chapter H of Volume I, where we find that we can view $\bar{L}$ as the group of invertible 3×3 matrices over $\mathbf{F}_2$, with respect to some basis of V_1 denoted by $\{1, 2, 3\}$, with $\bar{t}$ the inverse-transpose automorphism.

7.7.1. Reduction to $\mathbf{C_G(V_0) \leq M}$ for $\mathbf{V_0 := \langle 1, 1^t \rangle}$. Our goal in Theorem 7.7.1 is to show that $r(G,V) > 2$, so we need to prove that $C_G(U) \leq M$ for each $U \leq V$ with $m(V/U) \leq 2$. It turns out this can be accomplished by controlling the centralizer of the single subspace $V_0 := \langle 1, 1^t \rangle$, by showing:

PROPOSITION 7.7.3. *$G_0 := C_G(V_0) \leq M$.*

In this short subsection, we prove that Theorem 7.7.1 can be deduced from Proposition 7.7.3.

So assume Proposition 7.7.3, and suppose that for some $U \leq V$ with $m(V/U) \leq 2$, we have $C_G(U) \not\leq M$.

We first consider the case where $m(V/U) = 1$. Since V admits an orthogonal form, $U = v^\perp$ for some $v \in V$. Now replacing the orbit representatives in H.4.2 by conjugates $v = 2, 2 + 3^t, 2 + 2^t$, we see using the form in H.4.1 that $V_0 \leq v^\perp = U$, so that $C_G(U) \leq C_G(V_0) \leq M$ by Proposition 7.7.3.

Thus we have established that $r > 1$, so it remains to treat the case $m(V/U) = 2$.

First assume U is centralized by no involution of $\bar{M}$. Then Q is Sylow in $C_M(U)$, and no nontrivial element of odd order in $\bar{M}$ centralizes a subspace of V of codimension 2, so that $C_M(U) = C_M(V)$. Hence as $r > 1$, we get $C_G(U) \leq M$ from E.6.12.

This leaves the case where U is centralized by some involution $\bar{i} \in \bar{M}$. Since $m(V/U) = 2$, we must have $\bar{i} \in \bar{L}$, and conjugating in $\bar{L}$, we may take $\bar{i}$ to be given by the matrix for the permutation $(2,3)$ (and hence also $(2^t, 3^t)$). So again $V_0 \leq U$, and Proposition 7.7.3 gives $C_G(U) \leq M$.

This completes the proof of Theorem 7.7.1 modulo Proposition 7.7.3. So the remainder of this section is devoted to the proof of Propositions 7.7.3 and 7.7.2.

7.7.2. More detailed properties of $\mathbf{V_0}$ and its centralizer. Observe $C_{\bar{M}}(V_0)$ is the subgroup of $\bar{L}$ fixing 1 and acting on the subspace $\langle 2, 3\rangle$, so $C_{\bar{M}}(V_0) \cong L_2(2) \cong S_3$.

Set $L^0 := O^2(C_L(V_0))$, so that $\bar{L}^0/O_2(\bar{L}^0)$ is of order 3. Let $\theta \in L^0$ be of order 3. Observe

$$[V, L^0] =: V_- = \langle 2, 3\rangle \oplus \langle 2^t, 3^t\rangle = V_0^{\perp}.$$

and

$$V = V_0 \oplus V_-.$$

Set $T_0 := C_T(V_0)$ and $M_0 := C_M(V_0)$. Then $C_{LT}(V_0) = L^0T_0$, $T_0 \in Syl_2(C_M(V_0))$, and $\overline{T_0}$ of order 2 is generated by the involution $\bar{i}$ defined in the previous subsection.

Let $Z_1 := \langle 1\rangle$, $G_1 := C_G(Z_1)$, and $L_1 := O^2(C_L(Z_1))$. Thus $Z_1 \leq V_0$, so $G_0 \leq G_1$ and $L^0 \leq L_1$. Again $L_1/O_2(L_1)$ is of order 3, but $L_1/Q \cong A_4$ while $L^0/Q \cong \mathbf{Z}_3$.

Let V_+ denote either V_0 or Z_1, and define $G_+ := C_G(V_+)$, $L_+ := O^2(C_L(V_+))$, $M_+ := C_M(Z_+)$, and $T_+ := C_T(V_+)$. Then

$$M_+ = C_M(V)L_+T_+,$$

and by 3.2.10.4, T_+ is Sylow in G_+.

We emphasize that

$$Q = O_2(L^0T_0),$$

and that this property is crucial to our proof that $G_0 \leq M$.

LEMMA 7.7.4. *If Y is an abelian subgroup of $C_M(V_+)$ of odd order, then*

(1) $Y_C := C_Y(V)$ is of index at most 3 in Y, and
(2) if $Y_C \neq 1$, then $N_G(Y_C) \leq M$.

PROOF. As Y is of odd order in $O^2(C_M(V_+)) = O^2(C_M(V))L_+$ and $|\bar{L}_+ : O_2(\bar{L}_+)| = 3$, $|Y : Y_C| \leq 3$. By Theorem 4.4.3 and Remark 4.4.2, $N_G(Y_C) \leq M$. □

LEMMA 7.7.5. *If $w \in V^{\#}$ is 2-central in G, and $L_+T_+ \leq H \leq G_+$, then*

$$F^*(C_G(w)) = O_2(C_H(w)).$$

PROOF. We show that the hypotheses of 1.1.4.4 are satisfied with $G_w := C_G(w)$ in the role of "M", and $H \cap G_w$ in the role of "N". First $G_w \in \mathcal{H}^e$ by 1.1.4.3 and our hypothesis that w is 2-central. Set $G_{+,w} := C_G(V_+\langle w\rangle)$, and embed $Q \leq T_w \in Syl_2(G_{+,w})$. Then $J(T) \leq Q \leq T_w$ so $T_w \leq N_G(T_w) \leq M$ by 3.2.10.8. Consequently $T_w \leq M_+$, which we saw above is $C_M(V)L_+T_+$. Then by Sylow's Theorem, $T_w^c \leq L_+T_+$ for some $c \in C_M(V) \leq G_{+,w}$, so without loss $T_w \leq L_+T_+ \leq H$. Hence $V_+ \leq H \cap O_2(G_+) \cap G_w \leq O_2(H \cap G_w)$. So

$$C_{O_2(G_w)}(O_2(H \cap G_w)) \leq C_{O_2(G_w)}(V_+) \leq O_2(G_w) \cap G_{+,w} \leq O_2(G_{+,w})$$

$$\leq T_w \leq H \cap G_w.$$

Thus we finally have the hypothesis for 1.1.4.4, and we conclude from 1.1.4.4 that $H \cap G_w \in \mathcal{H}^e$. □

7.7.3. Proof of Proposition 7.7.2. In the remaining two subsections of the section, we assume that either

(H0) $V_+ = V_0$ and $G_0 \not\leq M$, or

(H1) $V_+ = Z_1$, $r > 2$, and $G_1 \notin \mathcal{H}^e$.

In each case, we work toward a contradiction. In this subsection, we assume (H1) and obtain a contradiction establishing Proposition 7.7.2, and hence also completing the proof of lemma 7.6.1, which depended upon that Proposition. At the same time, we will prove a lemma 7.7.6, necessary for the proof of Proposition 7.7.3. Then in the final subsection we assume (H0) and complete the proof of Proposition 7.7.3, on which various earlier results depended.

Under (H0), choose $H \in \mathcal{H}_*(L^0T_0, M)$ with $H \leq G_0$. Under (H1), choose $H \in \mathcal{H}(L_1T_1, M)$ with $H \leq G_1$, and H minimal subject to $H \notin \mathcal{H}^e$.

In either case set $M_H := H \cap M$. As $H \in \mathcal{H}$, H is an SQTK-group. Set $A := V_+V_-$; and observe that $A = V$ under (H0), while A is a hyperplane of V under (H1). Therefore $C_G(A) \leq M$ under either hypothesis, since $r > 1$ in Hypothesis (H1).

Under Hypothesis (H0) we will prove:

Lemma 7.7.6. *Assume Hypothesis (H0). Then*

(1) $T_0 \in Syl_2(H)$.

(2) $H = J(H)L^0T_0$.

(3) $F^*(H) = O_2(H)$.

We prove lemma 7.7.6 and Proposition 7.7.2 together.

First assume just Hypothesis (H0). Since T_0 is Sylow in G_0, part (1) of 7.7.6 holds. As $O_2(L^0T_0) = Q$, with T_0 Sylow in both L^0T_0 and H, we conclude from A.1.6 that $O_2(H) \leq Q$. By a Frattini Argument, $H = J(H)N_H(R)$, where $R := T_0 \cap J(H) \in Syl_2(J(H))$, and $J(T) = J(R)$. Then $N_H(R) \leq M$ by 3.2.10.8, so as $H \not\leq M$, also $J(H) \not\leq M$—and hence part (2) of 7.7.6 follows from minimality of H.

It now remains to prove part (3) of 7.7.6, as well as Proposition 7.7.2. Thus we assume either (H0) or (H1), and it remains to show that $F^*(H) = O_2(H)$. As a first step, A.1.6 says $O_2(M) \leq Q \leq T_+ \leq H$, so by 1.1.4.5, $F^*(M_H) = O_2(M_H)$.

Next applying A.1.26.1 to L^0 on $V_- = [V_-, L^0]$, V_- centralizes $O(H)$. Therefore

$$O(H) \leq C_H(V_-) = C_H(V_+V_-) = C_H(A).$$

Thus given our earlier observation that $C_G(A) \leq M$, $O(H) \leq O(M_H)$, so $O(H) = 1$ since $M_H \in \mathcal{H}^e$.

It remains to show that $E(H) = 1$. Thus we may assume that there is a component K of H. If $K \leq M$, then $K \leq E(M_H)$, contradicting $M_H \in \mathcal{H}^e$; thus $K \not\leq M$. By 1.2.1.3, $L_+ = O^2(L_+) \leq N_H(K)$, so L_+T_+ acts on $K_0 := \langle K^{T_+} \rangle$. Therefore by minimality of H, $H = K_0L_+T_+$.

Next as L^0 acts on K, so does $V_- = [V_-, L^0]$. We claim V_- acts faithfully on K, so assume otherwise; the proof will require several paragraphs. First $V_+ < W := C_A(K)$, so as L^0 acts on W, W contains at least one of the five nontrivial orbits $\mathcal{O}$ of $\langle \theta \rangle$ on $V_-^\#$. Now $\mathcal{O} = W_-^\#$ for some 2-subspace W_- of W. Observe W contains no involution w 2-central in G: For if w is such an involution, then $K \leq E(H) \cap G_w \leq E(H \cap G_w)$, while $E(H \cap G_w) = 1$ by 7.7.5.

Suppose first that (H0) holds. Then W contains the orthogonal sum of the hyperbolic 2-space V_0 with W_-, and either W_- lies in V_1 or V_2 and hence is totally singular, or W_- is diagonal and definite. Set $w := v_0 + w_-$ for $0 \neq w_- \in W_-$, where we choose v_0 to be singular in $V_0 \cap V_{3-i}$ in case $W_- \leq V_i$, or the non-singular vector in V_0 in case W_- is definite. Then by construction w is singular and diagonal, so by H.4.2, w is 2-central, contrary to the previous paragraph. This establishes the claim when (H0) holds.

So suppose instead that (H1) holds. As W contains no 2-central involution, W is not $C_V(O_2(L_1))$, so $\mathcal{O}$ does not contain 2^t. Therefore W is not centralized by an involution of $\bar{M}$—so that $W \in \Gamma$ in the language of Definition E.6.4. By (H1), $r > 2$, so as $m(W) = 3$, W is maximal subject to $C_G(W) \not\leq M$. But then E.6.11.2 says there is a subgroup of order 7 normal in $N_{\bar{M}}(W)$, which cannot happen—since $N_{\bar{M}}(W)$ is a $7'$-group unless $W = V_1$, where $N_{\bar{M}}(W) \cong L_3(2)$ has no normal subgroup of order 7. This completes the proof of the claim that V_- is faithful on K.

Next observe that V_- induces inner automorphisms on K: We check that the groups listed in Theorem C (A.2.3) have no A_4-group of outer automorphisms, whereas $V_- = [V_-, \theta]$. Thus the projection V_K of V_- on K is faithful of rank 4.

Let $Z_+ := 1$ under (H0) and $Z_+ := Z_1$ under (H1). In either case, set $\tilde{H} := H/Z_+$. Now $O_2(L_+)Q$ is of index 2 in T_+, and centralizes $\tilde{A} = \tilde{V}_+\tilde{V}_-$. Thus $\tilde{A}$ centralizes a subgroup of $\tilde{T}_+$ of index 2. Therefore $\tilde{V}_K$ is centralized by $Q_K := O_2(L_+)Q \cap K$ of index at most 2 in $T_K := T_+ \cap K$, so $\tilde{Z}_K := C_{\tilde{V}_K}(T_K)$ is noncyclic and contained in $Z(\tilde{T}_K)$. Therefore $m_2(K/Z(K)) \geq 4$ and $Z(\tilde{T}_K)$ is noncyclic. We check the groups on the list of Theorem C for groups with these properties: $m_2(K/Z(K)) \geq 4$ eliminates the groups in cases (1) and (2) of Theorem C (other than A_8 which also appears in case (4)), while $Z(\tilde{T}_K)$ noncyclic eliminates those in cases (4) and (5) as well as those in case (3) over the field $\mathbf{F}_2$. Therefore $K/Z(K)$ is of Lie type over $\mathbf{F}_{2^n}$ for some $n > 1$. Now if $\tilde{R}$ is a root group of $\tilde{K}$ in $\tilde{T}_K$, then $1 \neq \tilde{R} \cap \tilde{Q}_K$, so $\tilde{V}_K \leq C_{\tilde{T}_K}(\widetilde{R \cap Q_K}) \leq C_{\tilde{T}_K}(\tilde{R})$, and hence $\tilde{V}_K \leq Z(\tilde{T}_K)$, so $\tilde{A}$ centralizes $\tilde{T}_K$. In particular $m_2(Z(\tilde{T}_K)) \geq 2$, so either $n \geq 4$ or $K/Z(K)$ is $Sp_4(4)$. Thus by I.1.3, the multiplier of $K/Z(K)$ is of odd order, so as $[\tilde{A}, \tilde{T}_K] = 1$, $[A, T_K] \leq K \cap Z_+ \leq O_2(K) = 1$. Therefore $T_K \leq C_T(A) = Q$, so Q is Sylow in QK_0. However $C(G,Q) \leq M$, so $C(K_0, Q) \leq K_0 \cap M < K_0$. Thus we may apply the local $C(G,T)$-theorem C.1.29 to the maximal parabolics of K_0. Now if K is of Lie type G_2, ${}^3D_4$, or ${}^2F_4$, neither of the two maximal parabolics of K are blocks, so by C.1.29, each is contained in M. Thus $K \leq M$ as K is generated by these maximal parabolics, a contradiction. This reduces us to the case where $K/Z(K)$ is a Bender group over F_{2^n}, $L_3(2^n)$, or $Sp_4(2^n)$, and $M \cap K_0$ is either a Borel subgroup of K_0 or a maximal parabolic K_1 of $K \cong L_3(2^n)$ or $Sp_4(2^n)$. In any case $M \cap K_0$ contains a Borel subgroup B of K_0 normalizing T_K. By an earlier remark, either $n \geq 4$ or $K \cong Sp_4(4)$.

Now let Y be a Cartan subgroup of B. By 7.7.4, $Y_C := C_Y(V)$ is of index at most 3 in Y. But when $n \geq 4$, certainly $|Y : Y_C| > 3$, since Y_C centralizes V and hence centralizes $V_K \leq Z(T_K)$, while some subgroup of Y isomorphic to $\mathbf{Z}_{2^n-1}$ is semiregular on $Z(T_K)$. Therefore K_0 is $Sp_4(4)$, with Y_C of order 3—again centralizing V and hence V_K. This is impossible, as the Cartan group of B is faithful on $Z(T_K)$ in $Sp_4(2^n)$.

This completes the proof of Lemma 7.7.6 and Proposition 7.7.2.

7.7.4. Proof of Proposition 7.7.3. Now that Proposition 7.7.2 is established, we work under Hypothesis (H0), and it remains to obtain a contradiction, establishing Proposition 7.7.3.

We are in a position to exploit Thompson factorization: First, lemma B.2.14 tells us that

$$U := \langle \Omega_1(Z(T_0))^H \rangle \in \mathcal{R}_2(H),$$

so setting $H^* := H/C_H(U)$, we have $O_2(H^*) = 1$. Further

$$V = \langle C_V(T_0)^{L^0} \rangle \leq U,$$

so

$$C_H(U) \leq C_H(V) \leq M_H.$$

We saw early in the proof of 7.7.6 that $J(H) \not\leq M$, so $J(H)^* \neq 1$.

Next $J(H)^*$ is described in Theorem B.5.6. In particular as $J(H) \not\leq M$, either $O_3(J(H)^*) \not\leq M_H^*$ or some component K^* of $J(H)^*$ is not contained in M_H^*.

Assume the first case holds. Then

$$X^* := O_3(J(H)^*) = X_1^* \times \cdots \times X_d^*$$

with $X_i^* \cong \mathbf{Z}_3$ and $[U, X] = U_1 \oplus \cdots \oplus U_d$ where $U_i := [U, X_i]$ is of rank 2. Further $d \leq 2$ so that $L^0 = O^2(L^0)$ acts on each U_i. As $J(T) \trianglelefteq L^0T_0$ and L^0 acts on U_i, L^0 acts on $C_{U_i}(J(T)) \cong \mathbf{Z}_2$, so $[U_i, L^0] = 1$. Then $1 = [U, X, L^0]$, and $[X^*, L^{0*}] = 1$ which says $[X, L^0, U] = 1$. So by the Three-Subgroup Lemma we have $[L^0, U, X] = 1$. But recall $V_- = [L^0, V] \leq [L^0, U]$. Thus X centralizes $V_0V_- = V$, contradicting $X \not\leq M$.

Therefore some component K_+^* of $J(H)^*$ is not contained in M_H^*, so taking $K \in \mathcal{C}(H)$ with $K_+^* = K^*$ and setting $K_0 := \langle K^{T_0} \rangle$, $H = K_0L^0T_0$ by minimality of H. Similarly by a Frattini Argument, $H = C_H(U)N_H(C_{T_0}(U))$, so that $K/O_2(K)$ is quasisimple by 1.2.1.4 and minimality of H.

Lemma 7.7.7. *Hypothesis C.2.3 is satisfied with Q in the role of "R".*

Proof. Recall $C(G, Q) \leq M$, so $C(H, Q) \leq M_H < H$. By A.4.2.4, $Q \in Syl_2(C_0)$, where $C_0 := C_{M_H}(L^0/O_2(L^0)) \trianglelefteq M_H$; then $C_0 \geq \langle Q^{M_H} \rangle$, so Q is also Sylow in the latter group. Also $Q \in \mathcal{B}_2(M_H)$ by C.1.2.4, so that $Q \in \mathcal{B}_2(H)$ by C.1.2.3. Thus we have verified Hypothesis C.2.3. □

Lemma 7.7.8. *$Q \leq N_H(K)$.*

Proof. Assume otherwise. Then by C.2.4, $Q \cap K \in Syl_2(K)$, and as $K \not\leq M$, K is a χ_0-block. Further as K^* is quasisimple and $K < K_0$, we conclude from the list in A.3.8.3 that $K^* \cong L_2(2^n)$ with $n \geq 2$. Then by C.2.4, $K_0 \cap M$ is the Borel subgroup B normalizing $Q \cap K_0$. Let Y be a Cartan subgroup of B. By 7.7.4, $|Y : Y_C| \leq 3$ and $N_G(Y_C) \leq M$ because $Y_C \neq 1$ since K_0 is the product of two conjugates of K. On the other hand, $YT_0 = T_0Y$ and T_0 acts on L, so also $Y_CT_0 = T_0Y_C$. Then as $H \not\leq M$, $N_H(Y_C) \not\leq M$ by 4.4.13.1. This contradiction completes the proof. □

Now that $Q \leq N_H(K)$ by 7.7.8 and $K/O_2(K)$ is quasisimple, we may apply C.2.7 to conclude that K is desribed in C.2.7.3.

LEMMA 7.7.9. *(1) If case (a) of C.2.7.3 holds, then K is an A_7-block.*
(2) $[L^{0*}, T_0^* \cap K^*] \not\leq O_2(L^{0*})$.

PROOF. Suppose case (a) of C.2.7.3 holds, where K is a χ-block. Suppose first that K is an $L_2(2^n)$-block, and either $n > 2$ or $K < K_0$. Let Y be a Cartan subgroup of $K_0 \cap M$. An argument in the proof of the previous lemma shows that $Y_C \neq 1$, and supplies a contradiction. Thus $K = K_0$ is a block of type $L_2(4)$, A_5, or A_7.

Suppose next that K is a block of type A_5 or $L_2(4)$, and let $Y \in Syl_3(M \cap KL^0)$, $Y_C := C_Y(V)$, $Y_L := Y \cap L^0$, and $Y_K := Y \cap K$. By 7.7.4.1, $|Y : Y_C| \leq 3$. As $N_K(Y_K) \not\leq M$, 7.7.4.2 says that $Y_K \neq Y_C$, and hence $Y = Y_K Y_C = Y_L Y_C$ as $|Y : Y_C| \leq 3$. Then

$$V_- = [V, L^0] = [V, Y_L] = [V, Y_K] \leq U \cap K \leq O_2(K).$$

But as K is of type A_5 or $L_2(4)$, $m(O_2(K)/Z(K)) = 4 = m(V_-)$, so $V_- Z(K) = O_2(K)$. This is impossible, as $Q \cap K \in Syl_2(K)$, and Q centralizes V but not $O_2(K)$. This establishes (1); in particular K is not a χ_0-block.

Assume that $[L^{0*}, T_0^* \cap K^*] \leq O_2(L^{0*})$. Then $T_0 \cap K \leq C_{T_0}(L^0/O_2(L^0)) = Q$, so $Q \in Syl_2(K_0 Q)$. Therefore K is a χ_0-block by C.2.5, contrary to the previous paragraph. Thus (2) holds. □

LEMMA 7.7.10. *$L^0 \leq K$, and hence $T_0 \leq N_H(K)$, so that $K = K_0$.*

PROOF. We may assume $L^0 \not\leq K$, and it suffices to derive a contradiction. Since $1 \neq [V, L^0] \leq [U, L^0]$, we have $L^{0*} \neq 1$. We will appeal frequently to the fact that L^{0*} is normal in M_H^*, and hence is normalized by $M_K := M \cap K$, with $L^{0*}/O_2(L^{0*})$ of order 3.

Inspecting the groups listed in C.2.7.3 and appealing to 7.7.9.1, either $m_3(K) = 2$ or $K^* \cong SL_3(2^n)$ with n odd. In the former case we apply A.3.18, and A.3.19 when $K^* \cong SL_3(2^n)$ with n even; we conclude that K is the subgroup of H generated by all elements of order 3 so that $L^0 \leq K$, and the lemma holds in this case.

Therefore we are reduced to the case where $K^* \cong SL_3(2^n)$ with n odd, and M_K^* is a maximal parabolic. Assume $L^0 \not\leq K_0$. Then $[L^{0*}, M_K^*] \leq L^{0*} \cap M_K^* \leq O_2(M_K^*)$, so as $C_{Aut(K^*)}(M_K^*/O_2(M_K^*))$ is a $3'$-group, $[L^{0*}, K^*] = 1$, contrary to 7.7.9.2. This contradiction shows $L^0 \leq K_0$. As we are assuming $L^0 \not\leq K$, we must have $K < K_0 = KK^s$ for $s \in T_0 - N_{T_0}(K)$. Hence $K^* \cong L_3(2)$ by A.3.8.3. As $L^0 \not\leq K$ and T_0 acts on L^0, L^{0*} is diagonally embedded in K_0^*. But the Sylow group T_0^* acts on no such diagonally embedded subgroup with Sylow 3-subgroup of order 3, completing the proof of the lemma. □

As $L^0 \leq K$, $L^{0*} \trianglelefteq M_K^*$. Hence as $L^{0*}/O_2(L^{0*})$ is of order 3, K^* is not $L_3(2^n)$ with $n > 1$ odd. Similarly if $K^* \cong SL_3(2^n)$ with n even, then $L^{0*} = Z(K^*)$, so that $[L^{0*}, K^*] = 1$, contrary to 7.7.9.2. Thus $n = 1$ in case (g) of C.2.7.3.

Assume we are in the subcase of case (e) of C.2.7.3 where $K^* \cong Sp_4(4)$ and M_K^* is a maximal parabolic. Then as $L^{0*} \trianglelefteq M_K^*$, $L^{0*} = O_{2,3}(M_K^*))$. But then $[L^{0*}, T_K^*] \leq O_2(L^{0*})$, contrary to 7.7.9.2.

Thus we are left with the subcase of case (a) of C.2.7.3 where K is an A_7-block, or one of cases (b)–(d), case (e) with $K^* \cong A_6$, case (f), case (g) with $n = 1$, or case (h).

We now eliminate the cases (a)–(d), (e) with $K^* \cong A_6$, and (f); in all these cases, K is a block. We have $V_- = [V_-, L^0] \leq K$ using 7.7.10. Recalling that $V \leq U \leq O_2(H)$, we see that $V_- \leq O_2(K)$. Let W be the unique noncentral 2-chief factor of the block K, and W_- the image of V_- in W. As $C_{V_-}(L^0) = 1$, $W_- \cong V_-$. Further Q centralizes W_- and Q is of index 2 in the Sylow group T_0. However in each case, W is of dimension 4 or 6, and no subgroup of index 2 in a Sylow group centralizes a 4-subspace of W.

We are left with case (h), and with the subcase of case (g) where $n = 1$. Thus $K^* \cong L_m(2)$ with $m := 3, 4, 5$. As L^{0*} is normal in the parabolic M_K^* and T_0-invariant, $L^{0*}T_K^*$ is a rank one parabolic determined by a node δ in the Dynkin diagram adjacent to no node in M_K^*. So when m is 4 or 5, unless $K^*T_0^* \cong S_8$ and δ is the middle node, there is an L^0T_0-invariant proper parabolic which does not lie in M, contrary to the minimality of H. When $K^*T_0^* \cong S_8$, Theorems B.5.1 and B.4.2 say $I := [U, K]/C_{[U,K]}(K)$ is either the orthogonal module or the sum of the natural module and its dual. But in either case, $m(C_I(T_0)) = 1$, impossible as V_- is isomorphic to an L^0T_0-submodule of I and $m(C_{V_-}(T_0)) = 2$.

Therefore $K^* \cong L_3(2)$, and C.2.7.3 says that K is described in Theorem C.1.34. As $m(C_{V_-}(T_0)) = 2$, there are at least two composition factors on $U \leq Z(O_2(K))$, ruling out all but case (2) of C.1.34. Hence $O_2(K) = U = U_1 \oplus U_2$ is the sum of two isomorphic natural modules for $K^* = K/U$, with $V_- = W_1 \oplus W_2$ where $W_i = C_{U_i}(Q)$. Then an element θ of L^0 of order 3 has a unique nontrivial composition factor on $O_2(L^{0*})$, (which is realized on Q/U) plus two nontrivial composition factors W_1 and W_2 in U (realized in V). Thus L^0 has just one nontrivial composition factor on Q/V, which is impossible since the outer automorphism $\bar{t}$ of $\bar{L} \cong L_3(2)$ must interchange any natural module and its dual, and these are the only irreducibles with a unique nontrivial L^0-composition factor. This contradiction finally completes the proof of Proposition 7.7.3 and hence also of Theorem 7.7.1.

CHAPTER 8

Eliminating shadows and characterizing the J_4 example

We begin by reviewing the cases remaining after the work of the previous chapter, which eliminated those cases which do not lead to examples or shadows.

We continue to assume Hypotheses 7.0.2 and 7.3.1 from the previous chapter. The latter hypothesis excludes the case where $\bar{L}_0$ is $\Omega_4^+(2^n)$ on its orthogonal module; that case will be treated in chapter 9 of this part, because the methods used to attack that case are different from those in the remaining cases.

The cases $\bar{L}_0/V$ remaining from Table 7.1.1 that were not eliminated in the previous chapter 7, and are not among the cases to be treated in later chapters, are: $L_3(2^{2n})/9n$, $M_{22}/10$ or $M_{22}/\bar{1}0$, $M_{23}/\bar{1}1$, $M_{24}/11$ or $M_{24}/\bar{1}1$, and $(L_3(2) \wr 2)/9$.

In the case of $(L_3(2) \wr 2)/9$, technical complications also arise, primarily because the existence of small $(F-1)$-offenders on V only gives $r \geq 3$. As a result, different methods are required to treat this case; thus we will defer its treatment to 8.3.1 in the final section of this chapter.

As indicated in Table 7.1.1 in the previous chapter, the subgroups M we study in this chapter do arise as maximal 2-locals in various shadows, and in the case of M_{24} on its cocode module $\overline{11}$, in the quasithin example J_4. Thus we should not expect the methods of the previous chapter to eliminate these configurations on simple numerical grounds. Instead we seek to show that our bounds determine a unique solution for the various parameters: namely, the solution corresponding to the shadow or example. Then to eliminate the shadows, we go on to show that this unique solution leads (via study of w-offenders and subgroups $H \in \mathcal{H}_*(T, M)$) to a local subgroup other than M which is not an SQTK-group. In the $M_{24}/\overline{11}$ case, we construct the centralizer of a 2-central involution, which allows us to identify G as J_4.

8.1. Eliminating shadows of the Fischer groups

In this section, we assume $\bar{L}$ is M_{22}, M_{23}, or M_{24} and V is the cocode module for $\bar{L}$. In these cases we take a shortcut bypassing the uniform route we just outlined. This is because the initial bound on r given by the columns in Table 7.2.1 is a little too weak to pin down the structure of appropriate 2-locals, without a much more detailed analysis of elementary subgroups of $\bar{M}$ and their fixed points on V, and we wish to avoid that analysis.

In fact we will be able to eliminate these configurations, which correspond to the shadows of the Fischer groups, not by directly constructing a local subgroup that is not strongly quasithin, but instead by the use of techniques of pushing up from sections C.2, C.3, and C.4. These results implicitly rule out a number of locals which are not SQTK-groups; as a consequence we obtain an improved bound on r, and this slight improvement makes the remaining weak closure analysis much easier. Since this improved bound on r now exceeds the value occurring in the shadows, our calculations will in effect eliminate the Fischer groups—and in the case of M_{24}, will produce the centralizer of a 2-central involution resembling that in J_4.

In brief, we will use methods of pushing up to show for certain $x \in V$ that $C_G(x) \leq M$. Consequently any $U \leq V$ with $C_G(U) \not\leq M$ must contain only elements in conjugacy classes other than that of x. This restriction, added to those from Table 7.2.1, produces the improved bound on r. Then the remaining weak closure analysis proceeds rapidly.

In this section, we will by convention order the cases so that the case $\bar{L} \cong M_{22}$ is first, the case $\bar{L} \cong M_{23}$ is second, and the case $\bar{L} \cong M_{24}$ is third. When we make an argument simultaneously for all cases, we will list values of parameters for the cases in that order, without explicitly writing "respectively". Thus for example, the module V is the cocode module, which we are denoting by $\overline{10}, \overline{11}, \overline{11}$.

We take the standard point of view (cf. section H.13 of Volume I) that the cocode modules are sections of the space spanned by the 24 letters permuted by M_{24}, modulo the 12-dimensional subspace given by the Golay code. For M_{24}, the 11-dimensional cocode module V is the image of the subspace of all subsets of even size. The orbits of M_{24} on V consist of the set $\mathcal{O}_2$ of images of 2-sets and the set $\mathcal{O}_4$ of images of 4-sets, with the latter determined only modulo the code—that is, $\mathcal{O}_4$ is in 1-1 correspondence with the sextets in the terminology of Conway [**Con71**] and Todd [**Tod66**]. For M_{23} and M_{22} we can consider 2-sets containing just one of the letters fixed by this subgroup, and denote the corresponding vector orbit by $\mathcal{O}_2$.

Our subgroup M corresponds to a local subgroup $\dot{M}$ in the shadow group $\dot{G} := F_{22}, F_{23}, F_{24}$. Notice in these shadows that for $\dot{x} \in \dot{\mathcal{O}}_2$, $C_{\dot{G}}(\dot{x}) \not\leq \dot{M}$; in fact $C_{\dot{G}}(\dot{x})$ has a component, which is not strongly quasithin. We will see that the results on pushing up in section C.2 apply, and in fact rule out these components which arise in the shadows, forcing $C_G(x) \leq M$.

PROPOSITION 8.1.1. *$C_G(x) \leq M$ for $x \in \mathcal{O}_2$.*

PROOF. By H.15.1.1,

$$C_{\bar{L}}(x) \cong M_{21},\ M_{22},\ Aut(M_{22}),$$

where $M_{21} \cong L_3(4)$. Let $H := C_G(x)$, $M_H := H \cap M$, and $L_H := C_L(x)^\infty$. Replacing x by a suitable M-conjugate if necessary, we may assume $T_H := C_T(x) \in Syl_2(C_M(x))$. As $F^*(C_{\bar{L}}(x))$ is simple, $O_2(C_L(x)T_H) = Q = O_2(LT)$.

Next we show that Hypothesis C.2.8 is satisfied with Q, L_H in the roles of "R, M_0". Recall first that as part of the general setup in the introduction to chapter 7, $C(G,Q) \leq M$. By A.4.2.7, Q is Sylow in $C_{M_H}(L_H/O_2(L_H))$, so that the second hypothesis of C.2.8 is satisfied. By H.15.1.2, we have $V = [V, L_H]$ for the cocode modules. By construction $Q = O_2(L_HQ)$ centralizes V, with $N_G(V) \leq M$, so that the third hypothesis of C.2.8 is satisfied. Finally $O_2(M) \leq Q \leq H$ using A.1.6, so that $M_H \in \mathcal{H}^e$ by 1.1.4.4, establishing the first hypothesis of C.2.8.

Thus Hypothesis C.2.8 holds, and we may apply Theorem C.4.8. If $C_G(x) \not\leq M$, then $M_H < H$. But L_H is not listed among the possibilities in C.4.8. This contradiction show that $C_G(x) \leq M$. □

COROLLARY 8.1.2. $r \geq 6, 7, 8$.

PROOF. Suppose that $U \leq V$ with $C_G(U) \not\leq M$. We must show that $m(U) \leq 4$, 4, 3. By Proposition 8.1.1, $U \cap \mathcal{O}_2 = \emptyset$. During the proof of 7.4.1, we verified the hypotheses of E.6.28; and hence (as observed in the proof of that result), also hypotheses (1) and (4) of E.6.27 with $j = 1$. So since the conclusion $C_G(U) \leq M$ of that latter result fails, hypothesis (2) or (3) of that result must fail; hence U centralizes either some $(F-1)$-offender on V, or some nontrivial element of $\bar{M}_V$ of odd order.

First we consider the case where $U \leq C_V(\bar{A})$ for some $(F-1)$-offender $\bar{A}$. By H.15.2.3, if $U \leq C_V(\bar{A})$ with $U \cap \mathcal{O}_2 = \emptyset$, then $m(U) \leq 4, 4, 3$, completing the proof in this case.

So it remains to consider the case where $U \leq W := C_V(\bar{y})$ for some nontrivial element $\bar{y}$ of $\bar{L}$ of odd order. In the case of M_{22}, $m(W) \leq 4$ as $\beta = 6$ in Table 7.2.1. When $\bar{L}$ is M_{23} or M_{24}, then as $U \leq W$ with $U \cap \mathcal{O}_2 = \emptyset$, $m(U) \leq 4$ or 2 by H.15.7.3, completing the proof. □

Using this improved bound on r, it is not hard to eliminate the shadows of the Fischer groups, and isolate the configuration leading to J_4:

THEOREM 8.1.3. *If V is the cocode module for $\bar{L} \cong M_{22}$, M_{23}, or M_{24}, then $\bar{L} \cong M_{24}$, and there is a unique solution of the Fundamental Weak Closure Inequality 7.5.1. Indeed that solution satisfies $r = 8$, $m(C_A(V)) = 3$, $w = n(H) = 2$, and $\bar{A} = K_T$ of rank 6, for A a w-offender on V and $H \in \mathcal{H}_*(T, M)$.*

PROOF. Let A be a w-offender, with $A \leq V^g$ for suitable $g \in G$. By 8.1.2, $r \geq 6, 7, 8$, while by Table 7.2.1, $w \leq 2$ and $m_2 \leq 5, 4, 6$. Thus the FWCI is violated when $\bar{L} \cong M_{23}$. When $\bar{L} \cong M_{24}$, the FWCI is an equality, so all inequalities are equalities, and hence $w = 2$ and $r = 8$. Finally when $\bar{L} \cong M_{22}$, $w \geq 1$ by the FWCI. Further $m(\bar{A}) \geq r - w \geq 4$ by 7.5.3.2, and when these inequalities are equalities, we must have $w = 2$ and $r = 6$—since we saw $w \leq 2$ and $r \geq 6$.

In particular, we have eliminated M_{23}. Suppose next that $\bar{L} \cong M_{24}$, where we have shown the FWCI is an equality with $r = 8$ and $w = 2$. Let W be the subspace of V defined in 7.5.3.1, and note $W = \xi_V(\bar{A})$ in the language of H.10.1. As $w = 2 = n'$, $n(H) = 2$ by 7.3.4. By 7.5.3.1, $m(\bar{A}) = m_2 = 6$. Therefore by H.14.1.1, $\bar{A}$ is K_T or K_S. If $\bar{A} = K_S$, then $W = V$ by H.15.3.3, contrary to 7.5.3.1. Thus $\bar{A} = K_T$, so that the Theorem holds in this case.

We have reduced to the case where $\bar{L} \cong M_{22}$. This case is a little harder. Recall $m_2 \leq 5$, $w \leq 2$, and $m(\bar{A}) \geq 4$, with $w = 2$ in case $m(\bar{A}) = 4$. Let $\mathcal{B}$ be the set of $B \leq A$ with $C_A(V) \leq B$ and $m(V^g/B) = 5$. Then for $B \in \mathcal{B}$, $m(V^g/B) < 6 \leq r$, so $C_G(B) \leq N_G(V^g)$ and hence

$$W := \langle C_V(B) : B \in \mathcal{B} \rangle \leq N_V(V^g).$$

Further $m(\bar{B}) = m(\bar{A}) - 5 + w$, so $m(\bar{B}) = 1$ if $m(\bar{A}) = 4$ (since in that case we showed $w = 2$); while if $m(\bar{A}) = 5$, then $m(\bar{B}) = w$ is either 1 or 2. As $W \leq N_V(V^g)$, $m(V/W) \geq w \geq 1$ by definition of w, so in particular $W < V$.

If $m(\bar{A}) = 5$, then by H.14.3.1, $\bar{A} = K_Q$. Then $W = V$ by H.15.4.4, contrary to the previous paragraph. Thus $m(\bar{A}) = 4$, so as $W < V$, H.15.5 says $m(V/W) \leq 1$. But by earlier remarks, $w = 2$ and $m(V/W) \geq w$. This contradiction completes the proof of the Theorem. □

8.2. Determining local subgroups, and identifying J_4

In this section we treat the remaining cases other than $L_3(2)$ wr $\mathbf{Z}_2$, which we consider in the final section of the chapter; thus in addition to Hypotheses 7.0.2 and 7.3.1, we assume:

HYPOTHESIS 8.2.1. *$\bar{L}_0$ is not $L_3(2) \times L_3(2)$ on the tensor module* 9.

As a result of the previous section, we have eliminated M_{22} and M_{23} on their cocode modules, and in the case of M_{24} on its cocode module, we showed there is a unique solution for the weak closure parameters of a w-offender A on V. Indeed in that case we showed that $\bar{A} = K_T$ and $C_V(A) = C_V(K_T)$ is of dimension 3.

Because of Hypothesis 8.2.1, the other cases to be treated in this section are:

$$\bar{L} \cong (S)L_3(2^{2n})/9n, \ M_{22}/10, \ M_{24}/11.$$

As before we will use this ordering in common arguments, and we adjoin $M_{24}/\overline{11}$ as the fourth case on our list. In the first three cases we will show (as we did in case four) that there is a canonical choice for our w-offender A, and for each such canonical A, $C_V(A)$ is determined. Then in all four cases, we construct a sizable part of the local subgroup $N := N_G(C_V(A))$. In some cases N will not be strongly quasithin, so those cases are eliminated. In the surviving cases we study $C := C_G(z)$, where z is a 2-central involution in V; from $C_M(z)$ and $C_N(z)$ we can construct enough of C to see that either C is not strongly quasithin, or that $M \cong M_{24}/\overline{11}$ and C has the structure of the centralizer of an involution in J_4. Then we identify G as J_4 in the final subsection of this section.

8.2.1. Isolating a w-offender. As usual let $H \in \mathcal{H}_*(T, M)$. Recall H is a minimal parabolic by 3.3.2.4, with $H \cap M$ the unique maximal overgroup of T in H. We see in the next lemma that $V \not\leq O_2(H)$, so from lemma E.2.9, the set $\mathcal{I}(H, T, V)$ of Definition E.2.4 is nonempty.

PROPOSITION 8.2.2. *(1) $V \not\leq O_2(H)$.*

(2) There exists $h \in H$ such that $I := \langle V, V^h \rangle$ is in the set $\mathcal{I}(H, T, V)$ and $h \in I$.

(3) $1 \neq Z_I := V \cap V^h \leq Z(I)$.

(4) $T_I := T \cap I \in Syl_2(I)$ and $M_I := M \cap I = N_I(V)$.

(5) $\ker_{M_I}(I) = O_2(I)$, and $I^* := I/O_2(I) \cong D_{2m}$, m odd (in which case we set $k := 1$), $L_2(2^k)$, or $Sz(2^k)$, for some suitable k dividing $n(H)$.

(6) $V^* = Z(T_I^*)$ and $M_I^* = N_{I^*}(T_I^*)$.

(7) $A := V^h \cap O_2(I) = N_{V^h}(V)$, $C_A(V) = Z_I$, A is cubic on V, $r_{Aut_A(V),V} < 2$, $m(\bar{A}) = m(V/Z_I) - k$, and $C_V(A) \leq B := V \cap O_2(I)$.

(8) If $k > 1$, then $C_V(\bar{X}) = Z_I$ for $\bar{X}$ of order $2^k - 1$ in $\bar{M}_I$.

PROOF. From Table B.4.5, either $\bar{M}_V = Aut(M_{22})$ and V is the code module; or $q(\bar{M}_V, V) > 2$, so that $V \not\leq O_2(H)$ by 3.1.8.2, and (1) holds.

Therefore we may assume that $V \leq O_2(H)$ with V the code module for $\bar{M}_V = Aut(M_{22})$, and it remains to derive a contradiction. We first verify that the hypotheses of 3.1.9 hold with LT in the role of "M_0": Recall we saw after Hypothesis 7.0.2 that $H \cap M \leq M_V$; thus case (II) of Hypothesis 3.1.5 holds. Now part (c) in the hypothesis of 3.1.9 holds by hypothesis, part (a) is a consequence of Table B.4.5, and (d) follows as the dual V^* of V satisfies $q(\bar{L}\bar{T}, V^*) > 2$. Finally $M = !\mathcal{M}(LT)$ by 1.2.7.3, so (b) holds.

By A.3.18, $L = O^{3'}(M)$. Then we observe that each element of order 3 in M_{22} has six 3-cycles on 22 points, so it has three noncentral chief factors on each of V and V^*. Set $L_z := O^2(C_L(z))$; then $\bar{L}_z = O^2(C_{\bar{M}}(z))$ and $L_z/O_2(L_z)) \cong A_6$. Thus each $\{2,3\}'$-subgroup of $C_M(Z)$ permuting with T centralizes V. As $q(\bar{M}_V, V) = 2$, but $m(\bar{M}_V, V) = 3$ by H.14.4, each member of $\mathcal{Q}(M_V, V)$ has rank at least 2. Thus 3.1.9.6 says that $H/O_2(H) \cong S_3$ wr $\mathbf{Z}_2$ or $D_8/3^{1+2}$; in particular $X := O^2(H) \in \Xi(G,T)$. Next by 1.2.4, $L_z \leq K_z \in \mathcal{C}(C_G(z))$; and by A.3.12, either $K_z = L_z$ or $K_z/O_2(K_z) \cong A_7$, M_{11}, M_{22}, M_{23}, or $U_3(5)$. By A.3.18, $K_z = O^{3'}(C_G(z))$, so $X \leq K_z$. Thus $K_z/O_2(K_z) \cong M_{11}$ by 1.3.4. But then $H/O_2(H) \cong SD_{16}/E_9$, impossible as $H/O_{2,3}(H) \cong D_8$. This completes the proof of (1).

Then as H is a minimal parabolic, $V \not\leq \ker_{M\cap H}(H)$ by B.6.8.5, so that Hypothesis E.2.8 holds. Then (2) follows from E.2.9. By E.2.11.5, $O_2(I) = \ker_{M_I}(I)$. By 7.3.3 and 7.5.6, $w > 0$, so $W_0(T,V)$ centralizes V. Therefore I^* is not $Sp_4(2^k)$ by E.2.13.5; in particular the remainder of (5) holds by definition of $\mathcal{I}(H,T,V)$. As $q(\bar{M}, V) > 1$, E.2.13.4 says that (3) holds. We recall from the introduction to the previous chapter that V is a TI-set under M, so that with (3), we have the hypotheses of E.2.14. Now (4) follows from the definition of $\mathcal{I}(H,T,V)$ and E.2.14.1, while (6) follows from E.2.14.2. The first few statements in (7) follow from E.2.13.1 and E.2.15. Then we compute $m(\bar{A})$ using (5), (6), and the fact that $C_A(V) = Z_I$. By E.2.10.1, $AB \trianglelefteq I$, while by parts (3), (4), or (10) of E.2.14, $C_I(AB) \leq \ker_{M_I}(I) = O_2(I)$; thus $C_V(A) \leq C_V(AB) \leq O_2(I) \cap V = B$, completing the proof of (7). Finally, (8) follows from (5) using E.2.14.9. □

As in 8.2.2, pick $I = \langle V, V^h \rangle \in \mathcal{I}(H,T,V))$, and adopt the rest of the notation established in the lemma; e.g., $T_I := T \cap I \in Syl_2(I)$, $M_I := M \cap I$, $I^* := I/O_2(I) = I/\ker_{M_I}(I)$, $k := n(I)$, etc.

PROPOSITION 8.2.3. *$k = n(I) = w = n, 1, 1, 2$, and A is a w-offender on V.*

PROOF. By 8.2.2.5, k divides $n(H)$, so $k \leq n(H)$. By definition $w \leq m(V^*)$, while $m(V^*) = k$ using 8.2.2.6. Then we can extend the inequality in 7.3.4 to

$$w \leq m(V^*) = n(I) = k \leq n(H) \leq n' = 2n, 2, 2, 2 \qquad (*)$$

using the values in Table 7.2.1.

In the fourth case $M_{24}/\overline{11}$, $w = 2$ by 8.1.3, so the lemma follows from (*).

Thus we may assume $\bar{L}$ is not M_{24} on $\overline{11}$. If $w = k$, then A is a w-offender. By Table 7.2.1 and 7.5.6, $w \geq n$, 1, 1. Thus if $k \leq n$, 1, 1, then $w = k$ by (*) and the lemma holds. Therefore by (*), we may assume that $k = 2$ if $\bar{L}$ is M_{22} or M_{24}, while $n < k \leq 2n$ if $\bar{L} \cong (S)L_3(2^{2n})$, and it remains to derive a contradiction.

Assume first that $\bar{L} \cong (S)L_3(2^{2n})$. Then $k > n \geq 1$, so $I^* \cong L_2(2^k)$ or $Sz(2^k)$ and hence $Aut_I(V)$ contains a cyclic subgroup $\bar{X}$ of order $2^k - 1 \geq 3$ acting nontrivially on $\bar{A}$. Therefore as $Out(\bar{L})$ is 2-nilpotent, $1 \neq [\bar{A}, \bar{X}] \leq \bar{L}$ is an X-invariant 2-group. Hence $\bar{X}$ acts on some parabolic of $\bar{L}$, and indeed on a maximal parabolic as $\bar{X}$ has odd order. Therefore $2^k - 1$ divides $(2^{4n} - 1)n$, so as $n < k \leq 2n$, it follows that $k = 2n$. Thus $m(\bar{A}) \leq m_2 = 4n = 2k$, so by E.2.14.7,

$m(V/Z_I) = 3k = 6n$ and $m(\bar{A}) = 4n$. Therefore $m(Z_I) = m(V) - 6n = 3n$. By 8.2.2.8, $Z_I = C_V(\bar{X})$. This contradicts H.4.4.4, which says if $m(\bar{A}) = 4n$, no subgroup of $\bar{M}_V$ of order $2^{2n} - 1$ centralizes a subspace of $C_V(\bar{A})$ of rank exactly $3n$.

Therefore $\bar{L}$ is M_{22} or M_{24}, with $k = 2$. This time $m(\bar{A}) \le m_2 \le 6$, so by E.2.14.8, $I^* \cong L_2(4)$, $m(\bar{A}) = 2s$ for $s := 2$ or 3, and $m(V/Z_I) = 2(s+1)$. Again $Z_I = C_V(\bar{X})$, contradicting H.16.7, which says there is no subgroup $\bar{X}$ of order 3 centralizing a subspace of $C_V(\bar{A})$ of corank $2(s+1)$ in V. So the lemma is established. □

We can now eliminate the shadows of the groups $U_6(2^n)$ or $U_7(2^n)$, when $\bar{L} \cong (S)L_3(2^{2n})$ and $n > 1$. Recall that $U_6(2)$ can be regarded as a Fischer group F_{21}.

LEMMA 8.2.4. *If $\bar{L} \cong (S)L_3(2^{2n})$ then $n = 1$, $\bar{L} \cong L_3(4)$, $r = 5$, $k = w = 1$, $m(\bar{A}) = 4$, and $C_A(V) = Z_I$ is of rank* 4.

PROOF. By 8.2.3, $k = w = n$. By 7.4.1 and Table 7.2.1, $r \ge 4n$ with equality only if:

(*) $C_G(U) \not\le M$ for some U of rank $5n$ where U is the centralizer of an element $\bar{y} \ne 1$ of odd order in $\bar{M}_V$.

So by E.3.28.3, $m(\bar{A}) \ge r - w \ge 3n$, and hence by H.4.4.3, $m(V/C_V(\bar{A})) \ge 5n$. But by 8.2.2.7,

$$m(\bar{A}) = m(V/Z_I) - k \ge m(V/C_V(\bar{A})) - n \ge 4n,$$

so as $m(\bar{A}) \le m_2 = 4n$, we conclude that all inequalities are equalities, so that $m(\bar{A}) = 4n$ and $Z_I = C_V(A)$ is of rank $4n$. Then by the FWCI, $r \le m(\bar{A}) + w = 5n$.

Assume $n = 1$. Then from H.4.4.7, $\bar{L} \cong L_3(4)$, and we saw earlier that $k = w = n = 1$, $m(\bar{A}) = 4n = 4$, and $Z_I = C_V(A)$ is of rank $4n = 4$. The lemma holds when $r = 5$, so as $4 \le r \le 5$, we may assume $r = 4$, and it remains to derive a contradiction. Thus (*) holds. By H.4.6.1, $\langle \bar{y} \rangle = C_{\bar{M}_V}(U)$ is of order 3, so U is in the set Γ of Definition E.6.4. But now E.6.11.2 contradicts the fact that U is not centralized by an element of $\bar{M}_V$ of order 15.

Thus we may take $n > 1$, and it remains to derive a contradiction. As $n = k$, there is $\bar{X}$ of order $2^n - 1$ in $\bar{M}_V$ with $C_V(\bar{X}) = Z_I$ by 8.2.2.8. However this contradicts H.4.4.5, completing the proof. □

If $\bar{L} \cong (S)L_3(2^{2n})$ then $\bar{L} \cong L_3(4)$ by 8.2.4 and H.4.4.7. In that event, let U_L denote the unipotent radical of the stabilizer of a line in the natural module for $L_3(4)$. We now obtain the analogue of lemma 8.1.3 in our remaining cases:

PROPOSITION 8.2.5. *Let $U := C_V(A)$. Then $w = k = n(I)$ and:*

$\bar{L}/V$	w	r	$\bar{A}$	$m(\bar{A})$	$m(U)$
$L_3(4)/9$	1	5	U_L	4	4
$M_{22}/10$	1	6	K_Q	5	4
$M_{24}/11$	1	7	K_S	6	4
$M_{24}/\overline{11}$	2	8	K_T	6	3

In each case, $U \trianglelefteq T$, so $N_G(U) \in \mathcal{H}(T)$. Further $U = C_A(V) = Z_I \le Z(I)$ and so $I \le C_G(U)$.

PROOF. Recall $Z_I = V \cap V^h \le Z(I)$ by 8.2.2.3, and $w = k = n(I)$ by 8.2.3. By 8.2.2.7, $Z_I = C_A(V)$, so $m(Z_I) = m(V) - k - m(\bar{A})$. Thus if $m(U)$ and $m(\bar{A})$

are as described in the Table, then $m(U) = m(Z_I)$, so $Z_I = U$. Further the Table says $\bar{A} \trianglelefteq \bar{T}$, so $U = C_V(A) \trianglelefteq T$. Hence it remains to verify the Table.

When $\bar{L}$ is M_{24} on the cocode module, we verified the Table in 8.1.3. If $\bar{L}$ is $L_3(4)$, the Proposition follows from 8.2.4 modulo the following remark: As both $U = C_V(A)$ and $\bar{A}$ have rank 4, H.4.4.2 says that $\bar{A} = U_L$.

Thus we may assume $\bar{L}$ is M_{22} or M_{24} on the code module. By 8.2.3, $k = w = 1$. By 7.4.1 and the values in Table 7.2.1, $r \geq 6$.

Suppose $\bar{L} \cong M_{24}$ and $r = 6$. Then arguing as in the proof of (*) in the previous lemma, $C_G(U_0) \not\leq M$ for some subspace U_0 of V of rank 5 which is the centralizer of an element $\bar{y}$ of order 3 in $\bar{M}_V$. By H.16.6, $\langle \bar{y} \rangle = C_{\bar{M}_V}(U_0)$ so that $U_0 \in \Gamma$. Then by E.6.11.2, there is an element of order 63 centralizing U_0, contradicting H.16.6. Thus $r \geq 7$ when $\bar{L} \cong M_{24}$ on the code module.

Now by E.3.28.3,

$$m(\bar{A}) \geq r - w = r - 1,$$

so as $r - 1 \geq 5, 6 = m_2$, we conclude $m(\bar{A}) = m_2 = r - 1 = 5, 6$. Then by 8.2.2.6,

$$m(\bar{A}) = m(V/Z_I) - k \geq m(V/C_V(\bar{A})) - 1, \qquad (*)$$

so $m(V/C_V(\bar{A})) \leq m(\bar{A}) + 1 = 6, 7$. Since V is not an FF-module, this inequality is an equality, so the inequality in (*) is also an equality. Thus $U = Z_I$ is of rank 4. Further it follows from H.16.5 that $\bar{A} = K_Q, K_S$, so the proof is complete. □

8.2.2. Constructing $\mathbf{N_G(U)}$. We now use the results from the previous subsection to study the subgroup $N := N_G(U)$, where U is defined in 8.2.5. Let $\tilde{N} := N/U$ and $L_U := N_L(U)^\infty$. Recall from 8.2.5 that $T \leq N$, so $N \in \mathcal{H}^e$ by 1.1.4.6.

As $k \leq 2$ by 8.2.5, 8.2.2.5 says $I^* \cong D_{2m}$ or $L_2(4)$. Thus case (i) of E.2.14.2 holds, with $P := O_2(I) = AB$ and $A = B^h$. By 8.2.5, $U = Z_I$; it follows from E.2.14 that $P = [P, O^2(I)]U$.

We first observe:

Lemma 8.2.6. *(1) $L_U \in \mathcal{C}(N_M(U))$.*

(2) L_U acts naturally on U as A_5, A_5, A_6, $L_3(2)$.

(3) Either $O_2(L_U) = C_{L_U}(U)$, or $\bar{L}$ is M_{24} on the code module, $L_U/O_2(L_U) \cong \hat{A}_6$, and $C_{L_U}(U) = O_{2,Z}(L_U)$.

Proof. Part (1) follows from the definitions. Parts (2) and (3) follow from H.4.6.2, H.16.3.2, H.16.1.2, and H.15.6.2. □

As $T \leq N_M(U)$, T acts on L_U, so by 8.2.6.1 and 1.2.4, $L_U \leq K_U \in \mathcal{C}(N)$ with $T \leq N_N(K_U)$.

Lemma 8.2.7. *$K_U/O_2(K_U)$ is quasisimple.*

Proof. Assume not. Then by 1.2.1.4, $K_U/O_{2,F}(K_U) \cong SL_2(q)$ for some odd prime q. Then as $L_U \leq K_U$, A.3.12 says that either $K_U = L_U O_{2,F}(K_U)$ or $L_U/O_2(L_U) \cong L_2(4)$ and $q \equiv \pm 1 \mod 5$. In any case (in the notation of chapter 1) $X := \Xi_p(K_U) \neq 1$ for some prime $p > 3$, and by 1.3.3, $X \in \Xi(G, T)$. By 1.2.1.4 either $p = q$ and $X = O^2(O_{2,F}(K_U))$, or $K_U = L_U O_{2,F}(K_U)$ and $L_U/O_2(L_U) \cong L_2(4)$. In particular V is not the code module for $\bar{L} \cong M_{24}$, since $\hat{A}_6$ is not isomorphic to $L_2(p)$ for any odd prime p.

Now $X = [X, L_U]$, so as $L_U \trianglelefteq N_M(U)$, $X \not\leq M$; hence $XT \in \mathcal{H}_*(T, M)$, so replacing H by XT if necessary, we may take $H = XT \leq N$. Then H and the subgroup I of H are solvable, so that $1 = n(I) = k$ by E.1.13; hence by 8.2.3, V is not the cocode module for M_{24}. Thus $\bar{L}$ is $L_3(4)$ or M_{22}.

Let $Y \in Syl_p(X)$, so that also $Y \not\leq M$. Then $Y \cong E_{p^2}$ or p^{1+2} by definition of $X \in \Xi_p(G,T)$. Suppose $Y \cong p^{1+2}$. Then $\Phi(Y) \leq M$ by B.6.8.2, so as $p > 3$, Y centralizes U from the action of $Aut_M(U)$ on U in 8.2.6. Then as $p > 3$, $[V, \Phi(Y)] = 1$ by H.4.6.3 and H.16.3.4. Thus $Y \leq N_G(\Phi(Y)) \leq M$ by 4.4.3 and Remark 4.4.2, contradicting our observation that $Y \not\leq M$. We conclude $Y \cong E_{p^2}$.

Let $\hat{H} := H/O_2(H)$. As $k = 1$, $H = O_{2,p,2}(H)$ by B.6.8.2. Thus as we saw $O_2(I) = P = [P, O^2(I)]U$ and $H \leq N$, $P \leq O_2(H)$. Then as $U = Z_I \leq Z(I)$ by 8.2.2.3, and $I \leq O^2(H) = X$, there is a chief factor W for H on $O_2(X)U/U$ with $W = [W, Y]$. As $V \not\leq O_2(H)$ by 8.2.2.1, $V \not\leq O_2(X)$; and V/B is of rank $k = 1$, $B = V \cap P = V \cap O_2(H) = V \cap O_2(X)$, so that $\hat{V}$ is of rank 1. Therefore as $\hat{T}$ is irreducible on $\hat{Y}$, $\hat{V}$ inverts $\hat{Y}$, so $m(W) = 2m([W, V])$. But $[O_2(X)U, V] \leq O_2(X) \cap V = B$, so $[W, V] \leq W_B$, where W_B is the image of B in W. Thus

$$m(W) = 2m([W, V]) \leq 2m(W_B) \leq 2m(B/U) \leq 10$$

using 8.2.5. But this is impossible, as $SL_2(q)/E_{p^2}$ for $p > 3$ has no faithful module of dimension less than $5^2 - 1 = 24$. □

PROPOSITION 8.2.8. *(1)* $L_U = K_U \trianglelefteq N$.
(2) $[L_U, C_G(U)] \leq O_2(L_U)$.
(3) Either

(a) $I/P \cong L_2(2^k)$, *or*
(b) $\bar{L}$ *is* $L_3(4)$ *or* M_{24} *on the code module, and* $I/P \cong D_{10}$.

(4) L_U *acts on* I *and* P *with* $O_2(L_U I) = PO_2(L_U) = C_{L_U I}(\tilde{P})$.

(5) Let $\tilde{J} \in Irr_+(I, \tilde{P})$ *and set* $F := \mathbf{F}_2$ *in case (a) of (3), and* $F := \mathbf{F}_4$ *in case (b). Then* $\tilde{P}$, $\tilde{J}$, *and* $\tilde{B}$ *can be regarded as* F*-modules* $\tilde{P}_F$, $\tilde{J}_F$ *and* $\tilde{B}_F$, *for* $L_U I$, I, *and* L_U, *respectively, and* $\tilde{P}_F = \tilde{J}_F \otimes \tilde{B}_F$ *as an* $FL_U I$*-module.*

(6) If V *is the code module for* $\bar{L} \cong M_{24}$, *then case (b) of (3) holds and* T *does not act on* $O^2(I)$.

PROOF. By 8.2.7, $K_U/O_2(K_U)$ is quasisimple, while $L_U \leq K_U$ and $C_{L_U}(U) = O_{2,Z}(L_U)$ by 8.2.6.3. Therefore $C_{K_U}(U) \leq O_{2,Z}(K_U)$. But $[K_U, C_G(U)] \leq C_{K_U}(U)$, so $[K_U, C_G(U)] \leq O_2(K_U)$. Hence (2) follows.

Choose h as in 8.2.2.2. By 8.2.5 and (2), $h \in I \leq C_G(U) \leq N_G(L_U O_2(K_U))$. Therefore as $L_U O_2(K_U)$ acts on V, $L_U O_2(K_U)$ also acts on V^h, and hence on $\langle V, V^h \rangle = I$ and on $O_2(I) = P$.

Set $Y := IL_U$ and $\dot{Y} := Y/C_Y(\tilde{P})$. Since $\tilde{B}$ is an L_U-submodule of rank $m(\tilde{A})$ given in 8.2.5, in the various cases the $L_U/O_2(L_U)$-module $\tilde{B}$ is identified as: the natural module for $L_2(4)$ by H.4.6.2; the 5-dimensional indecomposable (with trivial quotient) for $L_2(4)$ by H.16.3.3; a natural module for $\hat{A}_6$ by H.16.1.3; the sum of two isomorphic natural modules for $L_3(2)$ by H.15.6.3. Furthermore in each case $C_{L_U}(\tilde{B}) = O_2(L_U)$. In particular, the number of $\dot{L}_U$-constituents on $\tilde{B}$ is $1, 1, 1, 2$, and hence is equal to k by 8.2.5.

Now by E.2.10.2, $\tilde{P} = \tilde{B} \oplus \tilde{A}$ is the sum of two I-conjugates of $\tilde{B}$, and $P = C_I(\tilde{P})$ by E.2.14. Therefore as $[L_U, I] \leq O_2(L_U) = C_{L_U}(\tilde{B})$ by (2), $O_2(L_U) = C_{L_U}(\tilde{P})$ and $\dot{L}_U \cong L_U/O_2(L_U)$ is quasisimple and centralized by $\dot{I} \cong I/P$, so $\dot{Y} = \dot{I} \times \dot{L}_U$ and (4) holds.

If $\dot{I}$ is $L_2(2)$, then conclusion (a) of (3) holds for $k = 1$, and $\tilde{P}$ is the sum of copies of the natural module $\tilde{J}$ with $End_{\mathbf{F}_2 I}(\tilde{J}) = \mathbf{F}_2$, so (5) follows from 27.14 in [**Asc86a**] in this case.

Suppose $\bar{L}$ is M_{22}. Then $\tilde{P}/[\tilde{P}, L_U]$ is of rank 2, so as $\tilde{P}$ is the sum of copies of $\tilde{J}$, it follows that $\dot{I}$ is $L_2(2)$, so that (3a) and (5) hold in this case by the previous paragraph. In the remaining cases for $\bar{L}$, $\tilde{B}$ is the sum of k copies of the natural irreducible module Λ for $\dot{L}_U$, so $\tilde{P}$ is the sum of $2k$ copies of Λ. Further $\Delta := End_{\mathbf{F}_2 \dot{L}_U}(\Lambda)$ is $\mathbf{F}_4$, $\mathbf{F}_4$, $\mathbf{F}_2$, respectively; and by 27.14 in [**Asc86a**], $\tilde{P}$ has the structure $\tilde{P}_\Delta$ of a Δ-module for $\dot{L}_U\Sigma$, where $\Sigma := C_{GL(\tilde{P})}(\dot{L}_U) = GL(\Theta)$ for some $2k$-dimensional Δ-module Θ, and $\tilde{P}_\Delta = \Lambda \otimes \Theta$ as a $\dot{L}_U\Sigma$-module. Then $\dot{I} \leq \Sigma$, and among the possibilities for $\dot{I}$ listed in 8.2.2.5, the only ones which are subgroups of $GL_{2k}(\Delta)$ are $\dot{I} \cong L_2(2^k)$, or D_{10} in the case $k = 1$ and $\Delta = \mathbf{F}_4$. Further $\tilde{J}$ is $\mathbf{F}_2\dot{I}$-isomorphic to Θ by parts (3) and (10) of E.2.14. This completes the proof of (3) and (5).

Suppose V is the code module for M_{24}. Then by (3), $\dot{L}_U\dot{I} \cong \hat{A}_6 \times D_{2m}$ for $m := 3$ or 5. Therefore as $m_3(N) \leq 2$ since N is an SQTK-group, $m = 5$. [1] Next $\bar{T}\bar{L}_U/O_2(\bar{T}\bar{L}_U) \cong \hat{S}_6/E_{64}$ with $\bar{A} = O_2(\bar{L}_U)$, where each involution in $\bar{T}$ is fused into $\bar{A}$ under $\bar{M}$, and there is an involution in $\bar{T} - \bar{L}_U$. Therefore there is an involution $t \in T - L_U O_2(L_U T)$. Assume T acts on $O^2(I)$. Then as $I = \langle V, V^h \rangle = O^2(I)(T \cap I)$ since $T \cap I \in Syl_2(I)$, while $V \trianglelefteq T$, $I = O^2(I)V$, so that T acts on I. Extend the earlier "dot notation" to $Y_T := YT$ by defining $\dot{Y}_T := Y_T/O_2(Y_T)$, and let $v \in V - B$. Then $\dot{s} := \dot{t}$ or $\dot{t}\dot{v}$ centralizes $\dot{I}$. Thus $\dot{I}$ acts on $C_{\tilde{P}}(\dot{s})$, whereas by (5), $C_{\tilde{P}}(\dot{s})$ is of 2-rank 6, while all irreducibles for $\dot{I}$ on $\tilde{P}$ are of rank 4. This contradiction completes the proof of (6).

It remains to establish (1). As $K_U \trianglelefteq N$, we must show that $K_U = L_U$. First $Aut_{L_U}(U) \leq Aut_{K_U}(U)$, and by 8.2.7 and 8.2.6.3, either $C_{K_U}(U) = O_2(K_U)$, or V is the code module for M_{24} and $C_{K_U}(U) = O_2(K_U)O^2(O_{2,3}(L_U))$. If $\bar{L}$ is M_{24} on the cocode module then $Aut_{L_U}(U) = GL(U)$, so $K_U = L_U C_{K_U}(U) = L_U O_2(K_U)$, and hence $L_U = K_U$ in this case. Thus we may assume one of the first three cases holds, so $m(U) = 4$ by 8.2.5.

Suppose case (a) of (3) holds. Then by (6), one of the first two cases holds. Now $m_3(I) = 1 = m_3(L_U)$, $L_U \leq K_U$ with $[K_U, I] \leq O_2(K_U)$, and N is an SQTK-group, so $m_3(K_U) = 1$. Also $Aut_{L_U}(U) \cong A_5$, and $Aut_T(U)$ acts on $Aut_{L_U}(U)$. The proper overgroups of $Aut_{TL_U}(U)$ in $GL(U)$ have 3-rank at least 2, so as $m_3(K_U) = 1$, we conclude again that $Aut_{K_U}(U) = Aut_{L_U}(U)$ and $K_U = L_U$.

Finally assume case (b) of (3) holds. As $O_2(K_U)$ acts on I, $X := O^2(I) = O^2(IO_2(K_U))$. Thus as $[K_U, I] \leq O_2(K_U)$, K_U acts on X, and hence also on $O_2(X)U = P$. Now $\mathbf{F}_{16} = End_{\mathbf{F}_2 X}(\tilde{J})$, and $\tilde{P}$ is the sum of $e := m(\tilde{P})/4 = 2$ or 3 copies of $\tilde{J}$, so by 27.14 in [**Asc86a**], $K_U/C_{K_U}(\tilde{P}) \leq GL(\Omega)$, where Ω is an e-dimensional space over $\mathbf{F}_{16}$. Arguing as in the previous paragraph, $m_5(I) = 1 = m_5(L_U)$ so that $m_5(K_U) = 1$. Then inspecting the overgroups of $Aut_{TL_U}(\Omega)$, we conclude as before that $K_U = L_U$. This completes the proof of the lemma. □

LEMMA 8.2.9. *(1) T acts on $O^2(I)$, and $H = IT$.*
(2) T normalizes VA.
(3) V is not the code module for $\bar{L} \cong M_{24}$.

[1] We just eliminated the shadow of Co_1, where $m = 3$ in the 2-local N.

PROOF. We begin with the proof of (1), although we will obtain (3) along the way. Set $H^+ := H/O_2(H)$ and $H_0 := \langle I, T\rangle$. Then $T \leq H_0 \leq H$ but $H_0 \not\leq M$, so $H_0 = H$ by minimality of $H \in \mathcal{H}_*(T, M)$. By 7.3.4 and Table 7.2.1, $n(H) \leq 2$. Next $H = \langle I^T\rangle T$, so $O^2(H) \leq \langle I^T\rangle \leq C_G(U)$ since $I \leq C_G(U)$ and $U \trianglelefteq T$ by 8.2.5. Now I acts on L_U by 8.2.8.2, and hence so does $H = \langle I, T\rangle$; therefore $m_3(HL_U) \leq 2$ as HL_U is an SQTK-group. We conclude from 8.2.8.2 and the description of L_U in 8.2.6 that $O^2(H)$ centralizes $L_U/O_2(L_U)$, and $m_3(H) \leq 1$.

Suppose that V is the code module for $\bar{L} \cong M_{24}$. Then $L_U/O_2(L_U) \cong \hat{A}_6$, so the argument of the previous paragraph shows that H is a $3'$-group. Therefore as $n(H) \leq 2$, and $5 \in \pi(H)$ by 8.2.8.6, we conclude from E.2.2 and B.6.8.2 that H is a $\{2,5\}$-group. Then as $m_5(L_UH) \leq 2$, it follows that $O^2(I) = O^2(H)$, whereas T does not act on $O^2(I)$ by 8.2.8.6. This establishes (3).

Suppose that $\bar{L} \cong M_{24}$, so that V is the cocode module by the previous paragraph. Then $n(H) = 2 = k = n(I)$ by 8.1.3 and 8.2.5, and $I/O_2(I) \cong L_2(4)$ by 8.2.8.3. As $m_3(H) \leq 1$ by the first paragraph, inspecting the possibilities in E.2.2, we conclude that $O^2(H^+) \cong L_2(4)$ or $U_3(4)$. In the former case, $H = IT$ and $O^2(H) = O^2(I)$ so that (1) holds; so we may assume the latter. Then $I^+ \cong L_2(4)$ is generated by the centers of a pair of Sylow 2-groups of $O^2(H^+)$ and hence I^+ is centralized by a subgroup X of $H \cap M$ of order 5. Recall $H \cap M$ acts on V since V is a TI-set under M, so X acts on $\langle V^{O_2(H)I}\rangle = \langle V^I\rangle = I$. Thus X acts on $O_2(I) = P$. As $m(U) = 3$ by 8.2.5, $GL(U)$ is a $5'$-group, as is $C_{GL(\tilde{P})}(Aut_{L_UI}(\tilde{P}))$ by 8.2.8.5. Thus X centralizes P by Coprime Action, and then as $m(V/V \cap P) = k = 2$, X centralizes V. Then as $I = O_2(I)C_I(X)$, X centralizes $\langle V^{C_I(X)}\rangle = I$. Therefore $I \leq N_H(X) \leq H \cap M$ by Remark 4.4.2 and 4.4.3, impossible as we saw that V is normal in $H \cap M$ but not in I.

Thus we may assume that $\bar{L}$ is $L_3(4)$ or M_{22}. Hence $k = n(I) = 1$ by 8.2.5, and by 8.2.8, either

(i) $\bar{L} \cong L_3(4)$, $I/O_2(I) \cong D_{2m}$ for $m := 3$ or 5, and $\tilde{B} = [\tilde{B}, L_U]$, or
(ii) $\bar{L} \cong M_{22}$, $I/O_2(I) \cong L_2(2)$, and $|\tilde{B} : [\tilde{B}, L_U]| = 2$.

Recall H acts on L_U and U, so that $B \leq O_2(L_U)U \leq O_2(H)$ in case (i), and similarly $|B : B \cap O_2(H)| \leq 2$ in case (ii). As $m(V/B) = 1$ and $V \not\leq O_2(H)$, either

(I) $B = V \cap O_2(H)$, so that $V^+ = \langle v^+\rangle$ is of order 2, or
(II) case (ii) holds and $m(V^+) = 2$.

Suppose case (II) holds. As $n(H) \leq 2$, $V \trianglelefteq H \cap M$, and $m_3(H) = 1$, we conclude from E.2.2 that $O^2(H^+) \cong L_2(4)$ or $U_3(4)$ and V^+ is the center of $T^+ \cap O^2(H^+)$. This contradicts $I \in \mathcal{I}(H, T, V)$ with $n(I) = 1$. The argument also shows that $n(H) = 1$.

Therefore case (I) holds and $n(H) = 1$. Thus for any $g \in H$ with $1 \neq |v^+v^{+g}|$ an odd prime power, $I_1 := \langle V, V^g\rangle \in \mathcal{I}(H, T, V)$. Therefore by 8.2.8.4, $|v^+v^{+g}| \in \pi$, where $\pi := \{3, 5\}$ or $\{3\}$, in case (i) or (ii), respectively. Also we saw earlier that $m_3(H) \leq 1$, and as $m_5(L_UH) \leq 2$ while $m_5(L_U) = 1$, $m_5(H) \leq 1$. We conclude by inspection of the list of possibilities for H with $n(H) = 1$ in B.6.8.2 and E.2.2 that either H^+ is $L_2(2)$ or $Aut(L_3(2))$, or case (i) holds and $O^2(H^+)$ is $\mathbf{Z}_5$ or $L_2(31)$. [2]

[2]In particular, we cannot have $H^+ \cong U_3(2)$; thus in the first case we are eliminating the shadow of $U_7(2)$, where N is not an SQTK-group—though the shadow of $U_6(2)$ still survives in that first case.

If $O^2(H^+)$ is of order 3 or 5, then $H = IT$, so that (1) holds. Thus we may assume $O^2(H^+) \cong L_3(2)$ or $L_2(31)$.

Let W be a chief section for $L_U H$ on $O_2(\tilde{L}_U \tilde{H})$ with $[W, O^2(H)] \neq 1$ and set $(L_U H)^! := L_U H / C_{L_U H}(W)$. As $L_U H$ is irreducible on W, $O_2(H) = C_H(W)$ and $O_2(L_U) \leq C_{L_U}(W)$. Then as $O^2(H)$ centralizes $L_U/O_2(L_U)$, $H^+ \cong H^!$ centralizes $L_U^!$, and W is the sum of isomorphic irreducibles for $H^!$ and for $L_U^!$ by Clifford's Theorem. Recall $\tilde{P} = \tilde{A} \oplus \tilde{B}$, with $\tilde{B}$ either natural or a 5-dimensional indecomposable for $\dot{L}_U \cong SL_2(4)$. Thus we may choose W so that W is the sum of $d \geq 2$ copies of the natural module for $L_U^!$, and W is the tensor product of the natural module for $L_U^!$ with a d-dimensional $O^2(H)$-submodule D of W. As case (I) holds, $[O_2(\tilde{L}_U \tilde{H}), V] \leq \widetilde{V \cap O_2(H)} = \tilde{B}$, so $[W, V]$ is the image of $\tilde{B}$ in W. Therefore L_U is irreducible on $[W, v^+]$, so it follows that v^+ induces a transvection on D. Therefore D is a natural module for $O^2(H^!) \cong L_3(2)$, which is impossible as $H^+ \cong Aut(L_3(2))$ and W is a homogeneous $L_U^!$-module. Therefore (1) is established.

Finally V is T-invariant, and by (1) so is $O_2(I) = AB$, establishing (2). □

LEMMA 8.2.10. *(1) L is a block of type $L_3(4)/9$, $M_{22}/10$, or $M_{24}/\overline{11}$.*
(2) $C_T(L) = 1$.
(3) $V = O_2(L)$.
(4) $Z = C_V(T)$ is of order 2.

PROOF. By 8.2.9.3, V is not the code module for $\bar{L} \cong M_{24}$. By 8.2.9.2, T normalizes VA, so $[O_2(L), A] \leq O_2(L) \cap VA \leq VC_A(V) = VU = V$. Then $L = [L, A]$ centralizes $O_2(L)/V$, so that (1) holds. By 3.2.10.9, $C_Z(L) = 1$, so (2) follows. By (1), $[Z, L] \leq V$. Then as the Sylow group T centralizes Z, we conclude from (2) and Gaschütz's Theorem A.1.39 that $VZ = VC_Z(L) = V$. Therefore $Z = C_V(T)$, so Z is of order 2, completing the proof of (4). By (1), L/V is quasisimple, and as $F^*(L) = O_2(L)$, $Z(L/V)$ is a 2-group. Thus as the multiplier of M_{24} is trivial, (3) holds when $\bar{L} \cong M_{24}$; and similarly (3) holds when $Z(L/V) = 1$, so we may assume that $Z(L/V) \neq 1$. If $\bar{L} \cong L_3(4)$, we may consider a quotient of L/V with center of order 2; then from the structure of the covering group in (3b) of I.2.2, $O_2(L_U)V/V$ is an indecomposable extension of a natural $L_2(4)$ module over a nonzero trivial submodule, which is not isomorphic to $\tilde{B}$ as an L_U-module, contrary to 8.2.8.5. Since an extension of M_{22} over a center of order 2 restricts to such an extension of $L_3(4)$, this argument also eliminates extensions of M_{22}. This completes the proof of (3). □

8.2.3. Constructing $\mathbf{C_G(z)}$. At this stage, in view of 8.2.10.1, the cases remaining are

$$L_3(4)/9, \ M_{22}/10, \ \text{and} \ M_{24}/\overline{11}.$$

By 8.2.10.4, $Z = C_V(T)$ is of order 2. In this section we let z denote a generator of Z, and set $C := C_G(z)$.

Using the subgroup of C generated by $C_M(z)$ and H (appearing essentially as $K_z T$ in the proof of 8.2.13), we will show that $O_2(C)$ is extraspecial with center Z. Then using the fact that C is an SQTK-group, we eliminate the $L_3(4)/9$ and $M_{22}/10$ cases, where $C/O_2(C)$ is $U_4(2)$ or $Sp_6(2)$ in the shadows $U_6(2)$ or Co_2. This reduces us to the case where $L/V \cong M_{24}$ and V is the cocode module. There we show C has the structure of the centralizer of a 2-central involution in J_4, which allows us to identify G as J_4.

Let $L_z := C_L(z)^\infty$ and $\tilde{C} := C/Z$.

LEMMA 8.2.11. *(1)* $L_z \in \mathcal{C}(C_M(z))$.

(2) There exists an L_zT-series $1 < Z < V_z < V$ with $V_z := [V, O_2(L_z)]$, and $\tilde{V}_z$ is the natural module for $L_z/O_2(L_z) \cong L_2(4)$, A_6, $\hat{A}_6$.

(3) V/V_z is the A_5-module, the core of the 6-dimensional permutation module, the 4-dimensional irreducible, respectively.

(4) $O_2(L_z)/V$ induces the group of transvections on V_z with center Z, so $O_2(\bar{L}_z\bar{T}) = O_2(\bar{L}_z)$ is L_zT-isomorphic to the dual of $\tilde{V}_z$.

PROOF. Parts (2)–(4) follow from H.4.6.4, H.16.4, and H.15.3. Then (2) implies (1). □

Set $E := \langle V_z^C \rangle$.

LEMMA 8.2.12. *(1) $E \cong D_8^e$ is extraspecial, for $e := 4, 4, 6$.*
(2) $E = O_2(C)$.
(3) $O_2(L_z) = EV$ and $V_z = E \cap V$.

PROOF. By 1.1.4.6, $F^*(C) = O_2(C) =: Q_C$, so $F^*(\tilde{C}) = \tilde{Q}_C$ by A.1.8. Therefore as $V_z \trianglelefteq T$, $1 \neq C_{\tilde{V}_z}(T) \le Z(\tilde{Q}_C)$. Then as L_z is irreducible on $\tilde{V}_z$ by 8.2.11.2, $\tilde{V}_z \le Z(\tilde{Q}_C)$, so $\tilde{E} = \langle \tilde{V}_z^C \rangle \le Z(\tilde{Q}_C)$.

Let $Q_M := O_2(LT)$. By parts (2) and (5) of 8.2.8, $[V, L_U] = B$. Then by H.4.6.5, H.16.4.4, and H.15.8, $V_z \le B$ but $V_z \not\le U$; therefore $V_z^h \le A$ but $V_z^h \not\le U$. Thus as $U = A \cap Q_M$ and $V_z^h \le E$, $E \not\le Q_M$. But by 8.2.11.4, L_z is irreducible on $O_2(\bar{L}_z\bar{T}) = O_2(\bar{L}_z)$, so $\bar{E} = O_2(\bar{L}_z)$. Thus as $V = O_2(L)$ by 8.2.10.3, $EV = O_2(L_z)$, establishing the first statement in (3).

Now $Z \le V = O_2(L)$ with L irreducible on V, so if $\Phi(Q_M) \neq 1$ then $\Phi(Q_M) \ge V$. But $C_{LT}(Q_M) \le Q_M$, so each x of odd order in L is faithful on $Q_M/\Phi(Q_M)$, whereas $[Q_M, x] \le V$ by 8.2.10.1. Thus $\Phi(Q_M) = 1$. Similarly as $Z \le V$, L is indecomposable on Q_M. But by earlier remarks, $\widetilde{Q_M \cap Q_C} \le C_{\tilde{Q}_M}(\tilde{E}) \le C_{\tilde{Q}_M}(O_2(\bar{L}_z))$. Next from the structure of indecomposable extensions of V by a trivial quotient (obtained from the duals of modules described in I.1.6), $C_{\tilde{Q}_M}(O_2(\bar{L}_z)) \le C_{\tilde{V}}(O_2(\bar{L}_z))$, while $C_{\tilde{V}}(O_2(\bar{L}_z)) = \tilde{V}_z$ by H.4.6.6, H.16.4.5, and H.15.3.4. Hence $V_z = Q_M \cap Q_C$. Thus $V_z = V \cap E$, completing the proof of (3). Now using (3) we have

$$|E| = |V_z||E : V_z| = |V_z||EV : V| = |V_z| \cdot |O_2(\bar{L}_z)| = 2^{1+2e}$$

where $e := 4, 4, 6$. By 8.2.11.4, $Z = C_{V_z}(E)$, so (1) holds. (As $e + 1 = m(V_z)$, $E \cong D_8^e$).

As $E \le Q_C \le O_2(C_{LT}(z)) = EQ_M$, and $Q_C \cap Q_M = V_z \le E$, (2) holds. □

PROPOSITION 8.2.13. *(1) V is the cocode module for $L/V \cong M_{24}$.*
(2) $L = M$ and $C/E \cong \hat{M}_{22}.2$

PROOF. By 8.2.11.1, $L_z \in \mathcal{C}(C_M(z))$, and of course L_z is T-invariant. Then by 1.2.4, $L_z \le K_z \in \mathcal{C}(C)$, and the possibilities for $K_z/O_2(K_z)$ are described in A.3.12.

By 8.2.12.2, $E = O_2(C)$. Let $C^* := C/E$. As $K_z \trianglelefteq C$ and $E = O_2(C)$, $O_2(K_z^*) = 1$; in particular $L_z < K_z$ by 8.2.12.3. Indeed using that result, $O_2(L_z^*) \cong V/V_z$ is described in 8.2.11.3. We inspect the lists in A.3.12 and A.3.14 for such subgroups, and conclude that $\bar{L}$ is M_{24} and $K_z^* \cong \hat{M}_{22}$; in particular, notice when

$L_z^*/O_2(L_z^*) \cong A_5$ that the A_5-module V/V_z does not arise in A.3.14. [3] That is, (1) holds.

By 8.2.12.1, $\tilde{E}$ is of rank 12. Therefore by H.12.1, $\tilde{E}$ is irreducible and $End_{K_z^*}(\tilde{E}) \cong \mathbf{F}_4$, so $Z(K_z^*) = C_{C^*}(K_z^*)$. Finally there is $t \in T \cap L$ inducing an outer automorphism on $L_z/O_2(L_z)$ and hence also on K_z^*, so as $|Aut(K_z^*) : K_z^*| = 2$, $C = TK_zC_C(K_z) = TK_z$ with K_z of index 2 in C. Therefore $C_M(z) = L_zT$ with $|T| = 2^{21} = |L \cap T|$. Then as $M = LC_M(z)$, $L = M$, so (2) holds. □

As a corollary we get:

THEOREM 8.2.14. *$G \cong J_4$.*

PROOF. By 8.2.12, $E = F^*(C) \cong D_8^6$, and by 8.2.13, $C/E \cong Aut(\hat{M}_{22})$. Also $z^L \cap V_z \neq \{z\}$, so z is not weakly closed in E with respect to G. These are the hypotheses of Aschbacher-Segev [**AS91**], so we conclude from the main theorem of that paper that $G \cong J_4$. We mention that their work uses the graph-theoretic methods used elsewhere in this work to establish recognition theorems. □

8.3. Eliminating $\mathbf{L_3(2)} \wr \mathbf{2}$ on 9

In this final section of chapter 8, we treat the exceptional case of $L_3(2) \wr 2$ on its tensor product module, which we have been postponing since the previous chapter. We prove:

THEOREM 8.3.1. *The case $\bar{L}_0 \cong L_3(2) \times L_3(2)$ on its 9-dimensional tensor product module cannot arise.*

We begin by defining notation: Let $L_1 := L$, $L_2 := L^t$, $L_0 := L_1L_2$, $V_1 \in Irr_+(L_1, V)$ with V_1 $N_T(L)$-invariant, and $V_2 := V_1^t$, so that V is the tensor product of V_1 and V_2 as an $\bar{L}_0$-module. Thus we can appeal to subsection H.4.4 of chapter H of Volume I.

Let $V_{i,m}$ be the $N_T(L)$-invariant m-dimensional subspace of V_i, and adopt the following notation for the unipotent radicals of the corresponding parabolic subgroups: $R_i := C_{T \cap L_iO_2(L_0T)}(V_{i,2})$, and $S_i := C_{T \cap L_iO_2(L_0T)}(V_i/V_{i,1})$. Let $R := R_1R_2$, $S := S_1S_2$, and as usual set $Q := O_2(L_0T)$. Notice $T_0 := RS$ is Sylow in L_0Q, and of index 2 in T. Let $W_j := W_j(T, V)$ for $j = 0, 1$.

From 3.2.6.2, we have $V = V_M$, so $M = M_V$.

LEMMA 8.3.2. *$s(G, V) = 3$.*

PROOF. This follows from 7.3.2 and Table 7.2.1. □

Recall from 7.3.3 and Table 7.2.1 that $w \geq 1$; indeed we can show:

LEMMA 8.3.3. *Either*
(1) W_1 centralizes V, so that $w > 1$; or
(2) $\bar{W}_1 = \bar{R}$ and $W_1(S, V) = W_1(Q, V)$.

PROOF. Suppose $A \leq V^g \cap T$ with $m(V^g/A) \leq 1$, but $\bar{A} \neq 1$. By 8.3.2, $s = 3$, so that $\bar{A} \in \mathcal{A}_2(\bar{M}, V)$ by E.3.10. Then by H.4.11.2, $\bar{A} \leq \bar{R}$. So if W_1 does not centralize V, $\bar{W}_1 = \bar{R}$ since $N_M(R)$ is irreducible on $\bar{R}$. Similarly $\bar{R} \cap \bar{S}$ contains no members of $\mathcal{A}_2(\bar{M}, V)$, so $W_1(S, V) = W_1(Q, V)$. □

[3] We just eliminated the shadows of $U_6(2)$ and Co_2, where $C/E \cong U_4(2)$, $Sp_6(2)$ are not SQTK-groups.

REMARK 8.3.4. The second case of lemma 8.3.3 in fact arises in the shadows of $G = Aut(L_n(2))$, $n = 6$ and 7. In those shadows, H is the parabolic determined by the node(s) complementary to those determining the maximal T-invariant parabolic M. Further $w = 1$, and $U = C_V(R)$ is the centralizer of a w-offender. In most earlier cases in this chapter, we were able to use elementary weak closure arguments to show that the configuration correponding to a shadow is the unique solution of the Fundamental Weak Closure Inequality FWCI, and then obtain a contradiction to the fact that $N_G(U)$ is an SQTK-group. But here, as in our treatment of the cases corresponding to the Fischer groups, we instead use the fact that G is quasithin to show that $C_G(U) \leq M$ for suitable subgroups U of V, and then use weak closure to obtain a contradiction.

LEMMA 8.3.5. $N_G(W_0(S,V)) \leq M \geq C_G(C_1(S,V))$.

PROOF. By 8.3.3, $W_1(S,V) = W_1(Q,V)$. As $W_0(S,V) \leq W_1(S,V)$ and $M = !\mathcal{M}(N_G(Q))$, the lemma follows from E.3.16. □

LEMMA 8.3.6. *If $H \in \mathcal{H}^e$ with $S \in Syl_2(H)$ and $n(H) = 1$, then $H \leq M$.*

PROOF. Since $s(G,V) = 3$ by 8.3.2, this follows from 8.3.5 using E.3.19 with 0, 1 in the roles of "i, j". □

As usual we wish to show that $C_G(U) \leq M$ for various subspaces U of V. Usually these subspaces will contain a 2-central involution, so it will be useful to establish some restrictions on the centralizers of such involutions.

Let z be a generator for $C_V(T)$; in the notation of subsection H.4.4, we may take z to the involution $x_{1,1}$ generating $V_{1,1} \otimes V_{2,1}$. Set $G_z := C_G(z)$, $M_z := C_M(z)$, $X := O^2(C_{L_0}(z))$, and $K_z := \langle X^{G_z} \rangle$. Note that $O_2(XT) = S$.

LEMMA 8.3.7. $G_z = K_z M_z$, *where either*

(i) $K_z = KK^s$ *for some* $K \in \mathcal{C}(G_z)$ *and* $s \in T - N_T(K)$ *with* $K/O_2(K) \cong L_2(p)$, *p prime, or*

(ii) $K_z/O_2(K_z) \cong L_4(2)$ *or* $L_5(2)$.

PROOF. Let $P \in Syl_3(X)$; then $X \in \Xi(G,T)$, $P \cong E_9$, and $Aut_T(P) \cong D_8$. We apply 1.3.4 to $G_z \in \mathcal{H}(XT)$ in the role of "H". If $X \triangleleft G_z$, define $K_z := X$; otherwise 1.3.4 gives $X \leq K_z := \langle K^T \rangle$, where $K \in \mathcal{C}(G_z)$ is described in one of the cases of 1.3.4. Notice case (3) of 1.3.4 is ruled out, as there $Aut_T(P)$ is cyclic. Similarly case (2) of 1.3.4 and case (4) with $K_z/O_2(K_z) \cong M_{11}$ are eliminated, as in those cases $Aut_T(P)$ contains a quaternion subgroup. We may assume the lemma fails. Thus neither of the remaining possiblities in case (4) of 1.3.4 holds, so case (1) of 1.3.4 holds and we may take $K_z = KK^s$ with $K/O_2(K) \cong L_2(2^n)$ and $n \geq 4$ even, as $p = 3$ and $L_2(4) \cong L_2(5)$.

Note in either case that $K_z \trianglelefteq G_z$. Set $Y_z := C_{G_z}(X/O_2(X))$. As $T \in Syl_2(G)$ acts on X, $T \cap Y_z \in Syl_2(Y_z)$, so by A.4.2.4, $S = T \cap Y_z$. If $K/O_2(K) \cong L_2(2^n)$, X is characteristic in $N_{K_z}(T \cap K_z)$ and $T \cap K_z = O_2(K_z)O_2(X))$, so by Sylow's Theorem $X^{N_G(K_z)} = X^{K_z}$. This holds trivially if $K_z = X$. Hence by a Frattini Argument, $G_z = K_z N_{G_z}(X) = K_z N_{G_z}(Y_z)$. Then as $S \in Syl_2(Y_z)$, $G_z = K_z Y_z N_{G_z}(S)$ by another Frattini Argument. As $J(T) \leq Q \leq S$, $N_G(S) \leq M$ by 3.2.10.8, so $G_z = K_z Y_z M_z$. Next $Y_z = XY$, where $Y := O^3(Y_z)$ is a $3'$-group as $m_3(G_z) = 2$. As $S \in Syl_2(Y_z)$, $S \in Syl_2(YS)$.

We claim $Y \leq M$. If Y is solvable, then $n(Y) = 1$ by E.1.13, so $Y \leq M$ by 8.3.6. So suppose Y is not solvable. Then there is $Y_1 \in \mathcal{C}(Y)$ with $Y_1/O_2(Y_1) \cong Sz(2^k)$. Now a Borel subgroup B of Y_1 is solvable, so as before $B \leq M$ using 8.3.6. Set $H := \langle Y_1, T\rangle$; then $n(H) = k$ is odd and $k \geq 3$. If $H \not\leq M$ then as $B \leq M$, we get $H \in \mathcal{H}_*(T, M)$, contradicting 7.3.4, which says $n(H) \leq 2$. So $H \leq M$, and in particular $Y_1 \leq M$. These arguments apply to each minimal parabolic H of YS over S, so as this set of parabolics generates $O^{2'}(YS)$ by B.6.5, $O^{2'}(YS) \leq M$. Finally as $S \in Syl_2(YS)$, by a Frattini Argument $YS = O^{2'}(YS)N_{YS}(S) \leq M$, since we saw $N_G(S) \leq M$. This completes the proof of the claim.

As $G_z = K_zY_zM_z$, and $Y_z = XY$ with $X \leq K_z$, we conclude $G_z = K_zM_z$, establishing the first assertion of the lemma.

If $K_z = X$, then $G_z = XM_z = M_z \leq M$, contradicting 3.1.8.3.ii, which shows $H \leq G_z$ for each $H \in \mathcal{H}_*(T, M)$. Thus $X < K_z$, so $K/O_2(K) \cong L_2(2^n)$ with $n > 2$. But now we replace Y_1 by K in the argument above, and again obtain a contradiction to $n(H) \leq 2$ in 7.3.4. This completes the proof. $\square$

We can now essentially eliminate the shadows of the linear groups:

LEMMA 8.3.8. *$C_G(C_V(R)) \leq M$.*

PROOF. Set $U := C_V(R)$; our proof relies on the following properties:

(a) $z \in U$.

(b) $N_{L_0}(R) \leq N_{L_0}(U)$, and there is a subgroup $P \cong E_{3^2}$ of $N_{L_0}(R)$ faithful on U.

(c) $T \leq N_G(U)$.

Since $C_G(U) \not\leq M$, using (c) we may choose $H \in \mathcal{H}_*(T, M)$ with $I := O^2(H) \leq C_G(U)$. By (a), $I \leq G_z$, and by (b) and A.1.27, $C_G(U)$ is a $3'$-group.

Next $G_z = K_zM_z$ by 8.3.7. As $I \not\leq M_z$, the projection K_I^* of I on $(K_zT)^* := K_zT/O_2(K_zT)$ is non-trivial. Furthermore $C_G(U)$, and hence also K_I^*, is a T-invariant $3'$-group. In case (ii) of 8.3.7, $K_I^*(T^* \cap K_I^*)$ contains a Sylow 2-subgroup of K_z^* and hence is a parabolic subgroup of K^*; as this parabolic is a $3'$-group, $I \leq TM_z$, contradicting $I \not\leq M$. So instead case (i) of 8.3.7 holds, and $K_z = KK^s$ with $K \cong L_2(p)$. Now $m_2(L_2(p)) = 2$, so if P^* is a $3'$-subgroup of K^*, then $O^2(P^*) = O(P^*)$. Thus as $I = O^2(I)$, the $3'$-group K_I^* is of odd order, so $O_2(I) \leq O_2(K_zT)$, and $O_2(I)$ is Sylow in I. Then since $X \leq K_z$, by A.1.6 we have $O_2(I) \leq O_2(K_zT) \leq O_2(XT) = S$. It follows that $S \in Syl_2(IS)$. But $n(I) = 1$ as I is solvable, so $I \leq M$ by 8.3.6, a contradiction which establishes the lemma. $\square$

Now we achieve our initial goal:

PROPOSITION 8.3.9. *$n(H) = 2$ for each $H \in \mathcal{H}_*(T, M)$.*

PROOF. Recall $n(H) \leq 2$ by 7.3.4. As $w > 0$, $N_G(W_0) \leq M$ by E.3.16.1. Also $s = 3$ by 8.3.2. Thus if $C_G(C_1(T, V)) \leq M$, then $n(H) = 2$ by E.3.19, so the lemma holds. However if $W_1 \leq C_G(V)$, then $C_G(C_1(T, V)) \leq M$ by E.3.16.1.3, so we may assume that $W_1 \not\leq C_G(V)$. Then by 8.3.3, $\overline{W_1} = \bar{R}$. Therefore $C_V(R) \leq C_T(W_1) = C_1(T, V)$, so $C_G(C_1(T, V)) \leq M$ by 8.3.8, completing the proof. $\square$

LEMMA 8.3.10. *(1) $K_z = KK^s$ with $K/O_2(K) \cong L_2(5) \cong L_2(4)$, $K_zT \in \mathcal{H}_*(T, M)$, and $X(T \cap K_z) = K_z \cap M$ is a Borel subgroup of K_z.*

(2) $G_z = K_zT$ and $M = L_0T$.

PROOF. We first prove (1). Assume the first statement in (1) fails. We claim then that $n(H) = 1$ for each $H \in \mathcal{H}_*(T,M)$ with $H \leq K_zT$. We examine the groups listed in 8.3.7. The claim follows in case (i) of 8.3.7 from E.1.14.6 when $p \geq 7$, and in case (ii) from E.1.14.1. Thus the claim is established, and of course it contradicts 8.3.9. Thus the first part of (1) holds, and as the Borel subgroup $X(K_z \cap T)$ of K_z is the unique T-invariant maximal subgroup of K_z, the remaining statements of (1) hold.

Next by 8.3.7, $G_z = K_zM_z$. Assume that $G_z > K_zT$. Then $Y := O^2(C_{G_z}(K_z/O_2(K_z)) \neq 1$, and $Y \leq M_z$. But Y is a $3'$-subgroup of M_z by 1.2.2.a, so as $\bar{M}_z$ is a $\{2,3\}$-group, Y centralizes V. Then $[L_0, Y] \leq C_{L_0}(V) = O_2(L_0)$, so that L_0T normalizes $O^2(YL_0) = Y$, and hence $G_z \leq N_G(Y) \leq M = !\mathcal{M}(L_0T)$, contradicting $K_z \not\leq M$. Thus $G_z = K_zT$, so $M_z = XT$, and hence $C_M(V)$ is a 2-group. Therefore $M = L_0T$, completing the proof of (2). □

LEMMA 8.3.11. $r(G,V) > 3$.

PROOF. Recall $r(G,V) \geq 3$ by 7.3.2. Assume $r(G,V) = 3$. Then there is $U \leq V$ with $m(V/U) = 3$ and $C_G(U) \not\leq M$. By E.6.12, $Q < C_M(U)$, and $C_M(U) = C_{L_0T}(U)$ by 8.3.10.2. Therefore by H.4.12.3 and H.4.10, $U = C_V(\bar{i})$ for some involution $\bar{i} \in \bar{L}_0\bar{T}$. By H.4.12.3, $C_{\bar{M}}(U)$ is a 2-group, so by E.6.27, U is centralized by an $(F-1)$-offender. Thus $\bar{i} \in \bar{L}_0$ by H.4.10.3. Consequently as $m(V/C_V(\bar{i})) = 3$, we may assume $\bar{i} \in \bar{R}_1$, so that $U = C_V(R_1)$. But of course $R_1 \leq R$ and $C_G(C_V(R)) \leq M$ by 8.3.8. This contradiction establishes 8.3.11. □

LEMMA 8.3.12. $V \leq O_2(G_z)$.

PROOF. Let $Q_z := O_2(K_zT)$. If $V \leq Q_z$, then the lemma holds, since $G_z = K_zM_z$ by 8.3.7 and $V \trianglelefteq M$. So we assume $V \not\leq Q_z$. Let $\tilde{G}_z := G_z/\langle z\rangle$ and $K_z^*T^* := K_zT/Q_z$. By H.4.9.2, XT is irreducible on $\tilde{V}_5$ and V/V_5, where V_5 denotes the 5-dimensional space in H.4.9.2.

We claim that $V_5 = V \cap Q_z$: to see this, we apply G.2.2, which is designed for such situations. Note that Hypothesis G.2.1 is satisfied with $\langle z\rangle$, V_5, L_0, X, K_zT in the roles of "V_1, V, L, L_1, H". We conclude from G.2.2 that

$$\tilde{U} := \langle \tilde{V}_5^{K_z}\rangle \leq Z(\tilde{Q}_z),$$

and $\tilde{U}$ is a 2-reduced module for K_z^*. Further as $V \not\leq Q_z$ and XT is irreducible on V/V_5, $V_5 = V \cap Q_z$ as claimed.

Notice as $U \leq Q_z \leq T$ that $[U,V] \leq V \cap Q_z = V_5$; so as $m(\tilde{V}_5) = 4 = m(V/V_5)$, $\tilde{U}$ is a dual FF-module for $K_z^*T^*$, with dual FF^*-offender V^*. Now V^* is a normal E_{16}-subgroup of the Borel subgroup $(M \cap K_zT)^*$ in 8.3.10, so $V^* \in Syl_2(K_z^*)$. In particular there is $h \in K_z$ with $K_z^* = \langle V^*, V^{*h}\rangle$. Observe $\tilde{U} = [\tilde{U}, K_z^*]$, since $\tilde{V}_5 = [\tilde{V}_5, X]$ and $\tilde{U} = \langle \tilde{V}_5^{K_z}\rangle$. Then as $[\tilde{V}_5, V^*] \leq \tilde{V}_5$ and $K_z^* = \langle V^*, V^{*h}\rangle$, we conclude

$$\tilde{U} = [\tilde{U}, K_z^*] = \tilde{V}_5 + \tilde{V}_5^h,$$

so that $\tilde{U}$ has dimension at most 8, and hence is itself an FF-module, with quadratic FF^*-offender V^*. By Theorems B.5.6 and B.5.1, the only possibility is $\tilde{U} = \tilde{U}_K \oplus \tilde{U}_K^s$ for $\tilde{U}_K$ a natural module for K^*. But now P of order 3 in X diagonally embedded in KK^s is fixed-point-free on $\tilde{U}$, and hence on $\tilde{V}_5$ of rank 4. Also X is fixed-point-free on V^*, so $C_V(X) = \langle z\rangle$, contradicting H.4.12.1. This completes the proof of 8.3.12. □

LEMMA 8.3.13. *If $V^g \cap V \cap z^G \neq \emptyset$, then $V^g \leq C_G(V)$.*

PROOF. This is a consequence of 8.3.12 and 3.2.10.6. □

LEMMA 8.3.14. *$W_2(S,V) \leq C_G(V)$.*

PROOF. If not, there is V^g with $m(V^g/A) = 2$ and $A := V^g \cap M$ satisfies $1 \neq \bar{A} \leq \bar{S}$. As $A \not\leq C_G(V)$ we may assume without loss that $A \not\leq C_G(V_1)$—so $\bar{A}$ has non-trivial projection $\bar{A}_1$ on $\overline{S_1}$. If $\bar{A}_1 = \bar{S}_1$, then for any hyperplane $\bar{B}$ of $\bar{A}_1$, $\bar{A}$ is non-trivial on the proper subspace $C_{V_1}(B)$ of V_1. On the other hand, if $\bar{A}_1$ is of rank 1, the same is true for the hyperplane $B := S_2 \cap A$ of A with $C_{V_1}(B) = V_1$. Since $V_1^\# \subseteq z^G$, without loss we may assume $z \in [C_{V_1}(B), A]$. By construction, $m(V^g/B) = 3$, so as $r > 3$ by 8.3.11, $C_{V_1}(B) \leq N_G(V^g)$. Therefore $z \in [C_{V_1}(B), A] \leq V \cap V^g$. But now 8.3.13 says $V^g \leq C(V)$, contrary to our choice of V^g. This establishes the lemma. □

Now we are in a position to complete the proof of Theorem 8.3.1. Recall $S = O_2(XT)$, so from the embedding of X in K_z in 8.3.10, S is Sylow in K_zS and $n(K_z) = 2$. From 8.3.14 and E.3.16.3, $C_G(C_2(S,V)) \leq M$; and from 8.3.5, $N_G(W_0(S,V)) \leq M$. Therefore as $s = 3$ by 8.3.2, E.3.19 says $K_z \leq M$, contradicting 8.3.10.

CHAPTER 9

Eliminating $\Omega_4^+(2^n)$ on its orthogonal module

The results in chapters 7 and 8 almost suffice to establish Theorem 7.0.1, our main result on pairs L, V in the Fundamental Setup (3.2.1) where V is not an FF-module. The only case left to treat is the case where $L_0/O_2(L_0) \cong L_2(2^n) \times L_2(2^n) \cong \Omega_4^+(2^n)$ with $n > 1$, and V is the orthogonal module for $L_0/O_2(L_0)$.

The standard weak closure arguments that handle most of the pairs in chapters 7 and 8 are not so effective in this case. Difficulties are already apparent from the parameters in Table 7.2.1: For example if T contains an orthogonal transvection σ, then $m(\bar{M}, V) = n$, so that if $n = 2$ we cannot immediately apply Theorem E.6.3 to obtain $r(G, V) \geq m(\bar{M}, V)$ as in 7.3.2. We are able to circumvent this difficulty in Lemma 9.2.3 below. There are more serious problems, however: First, $a(\bar{M}, V) = n = s(G, V)$, so 7.3.3 is ineffective. Second, L is not normal in M, so we can't appeal to 7.4.1 to get an effective lower bound on r. Thus we will instead use the fact that G is a QTKE-group to restrict various 2-locals, in order to show that r is large and $n(H)$ is small for each $H \in \mathcal{H}_*(T, M)$. Then weak closure will become effective.

9.1. Preliminaries

We begin by establishing some notation and a few properties of M.

Let $F := \mathbf{F}_{2^n}$ and regard V as a 4-dimensional orthogonal space V_F over F. As usual, let $Q := O_2(L_0T)$. Notice that we are in case (1) of 3.2.6, and in that case $V = V_M \trianglelefteq M$, so $M_V = M$.

Lemma 9.1.1. *$L_0 = O^{p'}(M)$ for each prime divisor p of $2^{2n} - 1$.*

Proof. This follows from 1.2.2.a. □

Lemma 9.1.2. *(1) $\bar{M} := M/C_G(V)$ is a subgroup of $N_{GL(V)}(\bar{L}_0) = N_{\Gamma L(V_F)}(\bar{L}_0)$, which is the product of $\bar{L}_0$ with the F-scalar maps, extended by $\langle f, \sigma\rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_n$, where σ induces an F-transvection on V_F normalizing $\bar{T}$, with $\bar{L}^\sigma = \bar{L}^t$, and f generates the group of field automorphisms (simultaneously) on $\bar{L}$ and $\bar{L}^t$.*

(2) There are elements in $\bar{T} - N_{\bar{T}}(\bar{L})$ of the form σf_0 with $f_0 \in O_2(\langle f\rangle)$.

(3) L_0 has two orbits on F-points of V, consisting of the singular and nonsingular F-points.

(4) $V_N := [V, \sigma]$ is a nonsingular F-point, and setting $L_N := O^2(N_{L_0}(V_N))$, $N_{L_0Q}(V_N) = L_N Q$ with $\bar{L}_N \cong L_2(2^n)$ and $[V, L_N] = C_V(\sigma) = V_N^\perp$ an indecomposable $3n$-dimensional $\bar{L}_N$-module, with $C_V(\sigma)/V_N$ the natural $\bar{L}_N$-module.

(5) Let V_1 denote the singular F-point stabilized by T. Then $N_{L_0T}(V_1)$ is a Borel subgroup of L_0T, and is transitive on $V_1^\#$.

Proof. This is straightforward. □

9.2. Reducing to $n = 2$

Our first goal is to show that $n \leq 2$. We cannot use the uniform approach of chapters 7 and 8, but we can still use some of the underlying techniques. For example we will not be able to bound r as in 7.4.1 using E.6.28 (which relies on E.6.27), but we can instead use extended Thompson factorization to achieve the hypotheses of E.6.26, which we use in place of E.6.27:

LEMMA 9.2.1. *(1)* $[V, J_{n-2}(T)] = 1$.
(2) Either $[V, J_1(T)] = 1$, *or* $n = 2$ *and* $\sigma \in \bar{T}$.

PROOF. This follows from H.1.1.2 and B.2.4.1. □

Recall $Z := \Omega_1(Z(T))$.

LEMMA 9.2.2. *(1)* $V = \Omega_1(Z(Q))$.
(2) If $Q \leq S \leq T$, *then* $\Omega_1(Z(S)) \leq C_V(S)$.
(3) $Z \leq V_1 = C_V(T \cap L)$.

PROOF. By 3.2.10.9, $C_Z(L_0) = 1$. Assume (1) fails. Now $H^1(\bar{L}_0, V) = 0$ (e.g., using Exercise 6.4 in [**Asc86a**]). So we obtain $[\Omega_1(Z(Q)), L_0] \not\leq V$. But $\hat{q}(\bar{L}_0\bar{T}, V) > 1$ since V is not an FF-module, and $\hat{q}(Aut_{L_0T}(W), W) \geq 1$ for any non-trivial L_0T-chief factor W on $\Omega_1(Z(Q))$ by B.6.9.1, so $\hat{q}(Aut_{L_0T}(\Omega_1(Z(Q)), \Omega_1(Z(Q))) > 2$, contrary to 3.1.8.1. Thus (1) is established. Further for $Q \leq S \leq T$, $Z(S) \leq Q$ since $L_0T \in \mathcal{H}^e$, so (1) implies (2) and (3). □

We can now prove the analogue of 7.3.2 in the case $\bar{L}_0 \cong \Omega_4^+(2^n)$, using 9.2.2 as an alternative to E.6.3 when $n = 2$:

LEMMA 9.2.3. $r(G, V) \geq n$.

PROOF. As $m(\bar{M}, V) = n$, this follows from Theorem E.6.3 when $n > 2$. Thus we may assume that $n = 2$ and $r = 1$; that is $C_G(U) \not\leq M$ for some U of corank 1 in V—and without loss, $N_T(U) \in Syl_2(N_M(U))$. Now U contains a unique F-hyperplane U_0, and from 9.1.2.3, there are two M-orbits on F-hyperplanes, each of the form $W^{\perp}$ for an F-point W of V. Next $T_0 := N_{T \cap L}(U_0) \leq N_T(U)$, so that

$$C_V(N_T(U)) \leq C_V(T_0) \leq U_0 \leq U. \qquad (*)$$

But $U \cap Z \neq 1$, so by E.6.10.4, $\Omega_1(Z(N_T(U))) \not\leq U$. On the other hand by 9.2.2.2, $\Omega_1(Z(N_T(U))) \leq C_V(N_T(U))$, so $C_V(N_T(U)) \not\leq U$, contradicting (*). □

From now on, let $H \in \mathcal{H}_*(T, M)$. Recall that H is a minimal parabolic in the sense of Definition B.6.1 by 3.3.2.4. Further by 3.1.8, H centralizes Z. Set $K := O^2(H)$. If $n(H) > 1$, let B be a Cartan subgroup of $H \cap M$.

LEMMA 9.2.4. *(1)* $n(H) \geq n - 1$.
(2) If $n(H) = 1$, *then* $[V, J_1(T)] \neq 1$ *and* $n = 2$.

PROOF. To prove (1), we may assume $n \geq 3$; we will apply E.6.26 with $j := n - 2 \geq 1$. By 9.2.3, $r > j$, and by 9.2.1.1, $J_j(T) \leq C_T(V)$; therefore (1) follows from E.6.26. Similarly (2) follows from E.6.26, this time using $j := n - 1$ and 9.2.1.2. □

LEMMA 9.2.5. *Either* $n(H) = n$, *or* $n = 2$ *and* $n(H) = 1$.

PROOF. Recall that H centralizes Z. By 9.2.4.1, $k := n(H) \geq n - 1$, so either $k > 1$ or $n = 2$. Thus we may assume $k > 1$, and it remains to show that $k = n$. As $k > 1$, $K/O_2(K)$ is of Lie type over $\mathbf{F}_{2^k}$ by E.2.2.

If $k \neq 6$, let p be a Zsigmondy prime divisor of $2^k - 1$; recall by Zsigmondy's Theorem [**Zsi92**] that this means that a suitable element of order p in $GL_k(2)$ acts irreducibly. If $k = 6$, let $p = 3$. Set $B_p := O_p(B)$. By Theorem 4.4.14, B is faithful on $\bar{L}_0$, so as $BT = TB$, either some $b \in B_p^\#$ induces an inner automorphism on $\bar{L}_0$, or $|B_p|$ divides n and B_p induces field automorphisms on $\bar{L}_0$. Assume the former. If p is a Zsigmondy prime divisor of $2^k - 1$, then k divides n; while if $k = 6$, then $p = 3$ so that n is even. Hence as $k \geq n - 1$, either $n = k$ and the lemma holds, or $k = 6$ and $n = 2$ or 4, impossible as then B_3 of order 9 is faithful on $\bar{L}_0$. Therefore we may assume that B_p induces field automorphisms on $\bar{L}$ and $\bar{L}^t$, and $|B_p|$ divides n. Then as $k \geq n - 1$, $k \neq 6$. Thus p is a Zsigmondy prime divisor of $2^k - 1$, so k divides $p - 1$. Hence as p divides n and $k \geq n - 1$, we conclude $p = n = k + 1$. Then n is odd, and so $V_1 = Z \leq Z(H)$ by 9.2.2.3, a contradiction as $[V_1, B_p] \neq 1$ since B_p induces field automorphisms on $\bar{L}_0$. This establishes the lemma. □

LEMMA 9.2.6. *If $n(H) > 1$, then B is contained in a Cartan subgroup D of L_0 acting on $T \cap L_0$.*

PROOF. This is a consequence of 9.1.1 and 9.2.5. □

Lemma 9.2.6 has essentially eliminated the shadows of $Aut(L_m(2^n))$ for $m := 4$ or 5, since in those groups $B \not\leq D$: our argument above that $B \leq D$ assumes G quasithin, whereas in those groups the parabolic $M = N_G(V)$ has 3-rank 3. So the remainder of the proof (or more precisely, the reduction to $n(H) = 1$ in the next section) can be viewed as showing that any embedding of B in D leads to a contradiction. In the previous chapter 8, the road after eliminating the configurations corresponding to shadows was typically fairly short; unfortunately in this case the only route after that we know is fairly long and hard.

We can at least immediately eliminate all cases where $n > 2$:

PROPOSITION 9.2.7. *(1) $n = 2$.*

(2) $n(H) = 1$ or 2.

(3) If $n(H) = 2$, then $K/O_2(K) \cong L_2(4)$, B is cyclic of order 3, and $B = C_D(V_1)$ with $\bar{B} = [\bar{D}, \sigma]$.

PROOF. If $n = 2$ then (2) holds by 9.2.5, so it only remains to prove that (3) holds; thus in this case we may assume $n(H) = 2 = n$. On the other hand if $n > 2$, then $n(H) = n$ by 9.2.5. So in any event we may assume that $n(H) = n > 1$.

By 9.2.6, $B \leq D$, so as $B \leq K \leq C_G(Z)$ and $C_D(Z) = C_D(V_1)$ is cyclic of order $2^n - 1$, B is cyclic of order at most $2^n - 1$. Therefore as $n(H) = n > 1$, E.2.2 says $K/O_2(K) \cong L_2(2^n)$ or $Sz(2^n)$ and $|B| = 2^n - 1$, so $B = C_D(V_1)$. By 9.1.2.2, $\bar{T}$ contains $\bar{t} = \sigma f_0$ with f_0 a field automorphism of order a power of 2. Observe σ inverts $\bar{B} = C_{\bar{D}}(V_1) = [\bar{D}, \sigma]$. Pick a preimage $t \in N_T(D)$. Then either t acts nontrivially on B, or $n = 2$, $f_0 \neq 1$, and t centralizes B. In the latter case the lemma holds, so we may assume the former.

As B is not inverted by an inner automorphism of $K/O_2(K)$ in T, t induces an outer automorphism on $K/O_2(K)$. Therefore n is even, and hence $K/O_2(K) \cong L_2(2^n)$ and t induces a field automorphism of some order 2^i on $K/O_2(K)$. Therefore $n = 2^i m$ and $|C_B(t)| = 2^m - 1$. If $\bar{t} = \sigma$, then t inverts B so $m = 1 = i$, and hence

$n = 2$, so the lemma holds. Finally if $\bar{t} \neq \sigma$, then t induces an automorphism on B of order $|f_0|$, so that $|f_0| = 2^i$. Then since $B = C_D(V_1)$, we calculate in L_0 that $|C_B(t)| = 2^m + 1$. This is impossible as $2^m - 1 \neq 2^m + 1$. Thus the Proposition is established. □

9.3. Reducing to $\mathbf{n(H) = 1}$

In this subsection, we assume $n(H) = 2$, and eventually arrive at a contradiction.

Set $G_1 := N_G(V_1)$. By 9.2.7.3, $K/O_2(K) \cong L_2(4)$, $\bar{B} = [\bar{D}, \sigma]$, and $B = C_D(V_1)$.

PROPOSITION 9.3.1. *D acts on K and $[K, V_1] = 1$.*

PROOF. Define $\bar{D}_\sigma := C_{\bar{D}}(\sigma)$. Then $\bar{D} = [\bar{D}, \sigma]\bar{D}_\sigma$, and hence $D = BD_\sigma$ for a suitable preimage D_σ in D of $\bar{D}_\sigma$. Thus D_σ is of order 3 and faithful on V_1. The proof begins with a series of three reductions:

First, notice if $D_\sigma \leq N_G(K)$, then $D \leq N_G(K)$, and hence $V_1 \leq \langle Z^{D_\sigma} \rangle \leq C_G(K)$, so that we are done. Thus we may assume $D_\sigma \not\leq N(K)$; in particular, K is not normal in G_1.

Second, suppose that $K \leq G_1$. Then $K \in \mathcal{L}(G_1, T)$, so by 1.2.4, $K \leq K_1 \in \mathcal{C}(G_1)$, and indeed $K < K_1$ by the previous paragraph, so K_1 is described in A.3.14. Suppose $m_3(K_1) = 2$. Then $K_1 \trianglelefteq G_1$ by 1.2.2.b. As $D_\sigma \not\leq K$, comparing the list in A.3.18 to that of A.3.14, we conclude D_σ induces diagonal automorphisms on $K_1/O_2(K_1) \cong L_3(4)$ or $U_3(5)$, and so D normalizes K from the embedding described in A.3.14. Thus in this case we are done by our first reduction, so we may assume that $m_3(K_1) = 1$. Then by A.3.14, $K_1/O_2(K_1)$ is J_1, $L_2(25)$, or $L_2(p)$, or $K_1/O_{2,2'}(K_1) \cong SL_2(p)$ for suitable p. We can reduce the fourth case to the third case by noting that $K_0 := N_{K_1}(T \cap O_{2,2',2}(K_1))^\infty$ is D-invariant. But in the first three cases, $D = C_D(K_1/O_2(K_1))B$ acts on K, contrary to the first reduction. Therefore we may assume that $K \not\leq G_1$. In particular, $[K, V_1] \neq 1$.

Third, we recall that K centralizes Z, so $K \leq G_1$ if $\bar{T} \leq \bar{L}_0\langle\sigma\rangle$ by 9.2.2.3, contrary to the second reduction.

In view of our three reductions, we may assume D does not act on K, $K \not\leq G_1$, and $\bar{T} \not\leq \bar{L}_0\langle\sigma\rangle$. To complete the proof, we construct an overgroup X of K, and obtain a contradiction in X.

By the third reduction and 9.1.2.2, there is $t \in T$ with $\bar{t} = \sigma f$, where f is an involution inducing a field automorphism on $\bar{L}_0$. As σ and f invert $\bar{B}$, t centralizes $\bar{B}$, so $T_2 := \langle t \rangle O_2(DT)$ is B-invariant and $D_\sigma T_2/O_2(D_\sigma T_2) \cong S_3$. Set $X := \langle D, H \rangle$. Suppose $O_2(X) = 1$. Then K, $D_\sigma T_2$, T satisfies Hypothesis F.1.1 in the roles of "L_1, L_2, S", so the amalgam $\alpha := (KT, BT, DT)$ is a weak BN-pair of rank 2 by F.1.9. Further T_2 is maximal in $D_\sigma T_2$, so the hypotheses of F.1.12 are satisfied, and hence α is one of the amalgams listed in that lemma. As $D_\sigma T_2/O_2(D_\sigma T_2) \cong L_2(2)$ and $K/O_2(K) \cong L_2(4)$, α is of type $U_4(2)$, J_2, or $Aut(J_2)$, so that $|T| \leq 2^8$. This contradicts $|V| = 2^8$ with $V < T$.

Thus $O_2(X) \neq 1$, so $X \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ by 1.1.4.6. By 1.2.4, $K \leq K_X \in \mathcal{C}(X)$, and $K_X \trianglelefteq X$ by (+) in 1.2.4, so $X = K_X TD$. As $D \not\leq N_G(K)$, $K < K_X$. Next $V_1 \leq V_X := \langle Z^X \rangle \in \mathcal{R}_2(X)$ by B.2.14. As $[K, V_1] \neq 1$, $[K_X, V_X] \neq 1$. Set $X^* := X/C_X(V_X)$. Then $K^* \neq 1$. Also $K = [K, J(T)]$, or else $K \leq N_G(J(T)O_2(K)) \leq M$ using 3.2.10.8. Thus $J(T)^* \neq 1$, so V_X is an FF-module for $K_X^*T^*$. Comparing the list in A.3.14 with the list of FF-modules in B.5.6, we conclude $K_X^* \cong SL_3(4)$, $Sp_4(4)$, $G_2(4)$, or A_7. In the first three cases, D induces inner-diagonal automorphisms on K_X^* in a Cartan group stabilizing the parabolic of K_X^* normalizing K^* and hence K, contrary to an earlier reduction. In the last case as $[Z, K] = 1$ we have a contradiction since $C_{K_X^*}(C_{V_X}(T))$ contains no A_5-subgroup when V_X is either of the FF-modules of dimension 4 and 6 for $K_X^* \cong A_7$ listed in B.4.2. This finally establishes 9.3.1. □

Define $T_K := T \cap K$ and $T_L := T \cap L_0 \in Syl_2(L_0)$.

LEMMA 9.3.2. *$T_LQ = O_2(TD) = T_KO_2(HD)$.*

PROOF. First $T_LQ = O_2(TD)$ and $TD = DT$ from the structure of $\bar{L}_0\bar{T}$. Also $H = KT$ with $D \leq N_G(K)$ by 9.3.1. Then as $K/O_2(K) \cong L_2(4)$, we conclude $HD/O_2(HD)$ is a subgroup of $S_3 \times S_5$, containing $GL_2(4)$. Then from the structure of this group, $O_2(TD) = T_KO_2(HD)$. □

Our strategy for the remainder of the section, much as in the proof of 9.3.1, is to construct an overgroup M_0 of K and L, and use this overgroup to obtain a contradiction.

Set $T_1 := N_T(L)$. Then T_1 is Sylow in $N_M(L)$ of index 2 in M, so $|T : T_1| = 2$. In particular T_1 contains T_LQ, so $T_1 \in Syl_2(LDT_1)$ by 9.3.2. Similarly as $T_LQ = T_KO_2(HD)$, T_1 is Sylow in KDT_1 as well.

Define $M_0 := \langle LDT_1, K\rangle$, and $V_2 := \langle V_1^L\rangle$. Of course $M_0 \not\leq M$ as $K \not\leq M$. Observe V_2 is a natural module for $L/O_2(L) \cong L_2(4)$.

LEMMA 9.3.3. *$O_2(M_0) \neq 1$.*

PROOF. Assume otherwise and let $S := T_LQ$. Then Hypothesis F.1.1 is satisfied by K, L, S in the roles of "L_1, L_2, S", and $S \trianglelefteq DS$ so $\alpha := (KDS, DS, LDS)$ is a weak BN-pair of rank 2 described in F.1.12. As $K/O_2(K) \cong L/O_2(L) \cong L_2(4)$, the amalgam is one of the untwisted types A_2, B_2, G_2 over $\mathbf{F}_4$. As $[K, V_1] = 1$ by 9.3.1, while V_2 is the natural module for $L/O_2(L)$, we conclude α is of type $G_2(4)$. But then $O_2(LS) = [O_2(LS), L] \leq L$, which is not the case since $T_L \cap L^t \not\leq L$. □

LEMMA 9.3.4. *$T_1 \in Syl_2(M_0)$.*

PROOF. Recall $J(T) \leq T_1$, so if $T_1 \leq T_0 \in Syl_2(M_0)$, then $T_0 \leq M$ by 3.2.10.8. If $T_1 < T_0$, then $T_0 \in Syl_2(G)$ as $|T : T_1| = 2$, and then $L_0T_0 \leq M_0 \not\leq M$, contradicting $M = !\mathcal{M}(L_0T_0)$. □

LEMMA 9.3.5. *(1) $[V_2, K] \neq 1$.*
(2) $[L, K] \not\leq O_2(L)$.

PROOF. First $B \leq K$; and B is faithful on V_2 as V_2 is the natural module for $L/O_2(L) \cong L_2(4)$ while $B = C_D(V_1)$. Thus (1) holds. If (2) fails, then as $[V_1, K] = 1$ by 9.3.1, K centralizes $V_2 = \langle V_1^L\rangle$, contrary to (1). □

Now $M_0 \in \mathcal{H}$ by 9.3.3. As $L \in \mathcal{L}(G, T_1)$, and $T_1 \in Syl_2(M_0)$ by 9.3.4, $L \leq K_L \in \mathcal{C}(M_0)$ by 1.2.4. Similarly $K \leq K_K \in \mathcal{C}(M_0)$. If $K_L \neq K_K$, then by 1.2.1.2 $[K, L] \leq [K_K, K_L] \leq O_2(M_0)$, contrary to 9.3.5.2; therefore $K_K = K_L =: K_0 \in \mathcal{C}(M_0)$ and $\langle L, K\rangle \leq K_0$.

LEMMA 9.3.6. *$M_0 = K_0T_1 \in \mathcal{H}^e$, $K_0 = O^2(M_0)$, and $Z(M_0) = 1$.*

PROOF. By 9.2.7, $B = C_D(V_1) \leq K$ is diagonally embedded in LL^t, so $D = B(D \cap L) \leq \langle K, L\rangle \leq K_0$. As $M_0 = \langle LDT_1, K\rangle$, $O^2(M_0) = \langle L, K\rangle \leq K_0$, so $M_0 = K_0T_1$ and $K_0 = O^2(M_0)$. Next using parts (2) and (3) of 9.2.2, $\Omega_1(Z(T_1)) \leq C_V(T_1) \leq V_1$. Hence as $O_2(M_0) \cap \Omega_1(Z(T_1)) \neq 1$, and as D is irreducible on V_1, $V_1 \leq O_2(M_0)$. Therefore $N_G(O_2(M_0)) \in \mathcal{H}^e$ by 1.1.4.1. Next $V_2 = \langle V_1^L\rangle \leq O_2(M_0)$, and then

$$C_T(O_2(M_0)) \leq C_T(V_2) \leq N_T(L) \leq T_1 \leq M_0,$$

so $M_0 \in \mathcal{H}^e$ by 1.1.4.4 with $N_G(O_2(M_0))$ in the role of "M". Also $Z(M_0) = 1$ as $\Omega_1(Z(T_1)) \leq V_1$ and $C_{V_1}(D) = 1$. □

We now proceed as in the last paragraph of the proof of 9.3.1. Let $U := \langle V_1^{M_0}\rangle$. As $V_1 = \langle Z^D\rangle$, $U = \langle Z^{M_0}\rangle$, so by B.2.14, $U \in \mathcal{R}_2(M_0)$. Set $M_0^* := M_0/C_{M_0}(U)$. By 9.3.5.1, $K^* \neq 1$. Now as in the proof of 9.3.1, $K = [K, J(T)]$ and hence $[U, J(T)] \neq 1$, so U is an FF-module for K_0^*. Then we obtain the same four possiblities for K_0^* as in the proof of 9.3.1, and eliminate the fourth case $K_0^* \cong A_7$ as in that proof, to conclude:

LEMMA 9.3.7. *$K_0^* \cong SL_3(4)$, $Sp_4(4)$, or $G_2(4)$, and U is an FF-module for M_0^*.*

LEMMA 9.3.8. *K_0^* is not $SL_3(4)$.*

PROOF. Otherwise $Z(K_0^*) = C_D(L^*) = (D \cap L^t)^*$, as each is of order 3. But then $K/O_2(K) \cong L_2(4)$ is centralized by $\langle (D \cap L^t)^{N_T(D)}\rangle = D$, a contradiction since $B \leq D \cap K$. □

LEMMA 9.3.9. *$K_0^* \cong Sp_4(2^n)$.*

PROOF. If not, by 9.3.7 and 9.3.8, $K_0^* \cong G_2(4)$. Now L and K are normalized by T, so $L = P_1^\infty$ and $K = P_2^\infty$, where P_1^* and P_2^* are the maximal parabolics of K_0^* containing $(T \cap K_0)^*$. By 9.3.7, U is an FF-module for K_0^*, and by 9.3.6, $Z(M_0) = 1$—so U is the natural $G_2(4)$-module by Theorems B.5.1 and B.4.2.4. Therefore by B.4.6.14, $D \cap L$ centralizes $K/O_2(K)$. We again use the action of $N_T(D)$ to obtain the same contradiction obtained at the end of the proof of 9.3.8. □

LEMMA 9.3.10. *K_0 is an $Sp_4(4)$-block.*

PROOF. Recall T_1 is of index 2 in T. If $1 \neq C$ char T_1 with $C \trianglelefteq M_0$, then $N_G(C)$ contains $M_0 \not\leq M$ and $\langle L, T\rangle = L_0T$, contradicting $M = !\mathcal{M}(L_0T)$. Thus no such characteristic subgroup exists, giving the condition (MS3) of Definition C.1.31. We obtain (MS1) and (MS2) using 9.3.9. Then the lemma follows from C.1.32.3. □

We are now in a position to obtain a contradiction, eliminating the case $n(H) = 2$. For by 9.3.6, $Z(M_0) = 1$, so U is the natural module for the $Sp_4(4)$-block K_0 by 9.3.10. Now V/V_2 is the natural module for $L/O_2(L)$. However $L = P^\infty$ for some maximal parabolic P^* of K_0^*, so $O_2(L)/U$ is an indecomposable of F-dimension 3 with no natural submodule. Therefore $V \leq U$, so $V = U$ as both are of order 2^8. Then $K_0 \leq N_G(U) = N_G(V) = M$, a contradiction.

9.4. Eliminating $\mathbf{n(H) = 1}$

As we just showed $n(H) \neq 2$, $n(H) = 1$ for all $H \in \mathcal{H}_*(T, M)$ by 9.2.5. This makes weak closure arguments effective, once we obtain restrictions on the weak closure parameters r and w.

Define V_N and L_N as in 9.1.2.4, and let $U_N := V_N^{\perp}$. By 9.1.2.4, $U_N = [V, L_N]$ and U_N/V_N is the natural module for $L_N/O_2(L_N) \cong L_2(4)$. For $v \in V_N^{\#}$, set $G_v := C_G(v)$.

PROPOSITION 9.4.1. *$L_N \trianglelefteq G_v$.*

PROOF. Assume the lemma fails. Then $G_v \not\leq M$. We can assume $T_v := C_T(v) \in Syl_2(C_M(v))$, and then by 3.2.10.4, $T_v \in Syl_2(G_v)$. By 1.2.4, $L_N \leq L_v \in \mathcal{C}(G_v)$ with L_v described in A.3.14, and $L_v \trianglelefteq G_v$ by (+) in 1.2.4 applied to T_v. We are done if $L_N = L_v$, so assume $L_N < L_v$; thus $L_v \not\leq M$.

We claim that $L_vT_v \in \mathcal{H}^e$. Suppose first that L_v is quasisimple. As $v \in [V, L_N] \leq L_N \leq L_v$, $v \in Z(L_v)$, so the multiplier of $L_v/Z(L_v)$ is of even order. Also $C_V(L_v) \leq C_V(L_N) = V_N$, so $m_2(Aut(L_v)) \geq m(V/V_N) = 6$. Inspecting the lists of A.3.14 and I.1.3 for groups with an automorphism group of 2-rank at least 6, we conclude $L_v/Z(L_v) \cong G_2(4)$. But then by I.1.3, $Z(L_v)$ is of order 2, so $\langle v \rangle = C_{V_N}(L_v)$ and hence $m_2(Aut_V(L_v)) = 7 > m_2(Aut(G_2(4)))$, a contradiction. Thus L_v is not quasisimple. As $z \in U_N = [U_N, L_N]$ and $C_T(O_2(L_vT_v)) \leq C_T(v) = T_v$, we conclude using 1.2.11 that $L_vT_v \in \mathcal{H}^e$.

As $L_vT_v \in \mathcal{H}^e$, it follows from B.2.14 that $U := \langle Z^{L_v} \rangle \in \mathcal{R}_2(L_vT_v)$. Notice using 9.1.2.4 that U contains U_N and V_1. Set $(L_vT_v)^* := L_vT_v/C_{L_vT_v}(U)$.

We next claim that $L_v^* = L_N^*$, so assume otherwise.

Suppose first that $J(T) \not\leq C_G(U)$. Then $[L_v^*, J(T)^*] \neq 1$ and U is an FF-module for $L_v^*T_v^*$. If L_v appears in case (c) or (d) of 1.2.1.4 then $O_\infty(L_v)^*$ is a $3'$-group, so by B.5.6, $[O_\infty(L_v^*), J(T)^*] = 1$. Therefore as $[L_v^*, J(T)^*] \neq 1$ and $L_v^*/O_\infty(L_v^*)$ is quasisimple, $L_v^* = [L_v^*, J(T)^*]$. On the other hand, if $L_v/O_2(L_v)$ is quasisimple, then so is $L_v^* = [L_v^*, J(T)^*]$. Thus in any case, $L_v^* = [L_v^*, J(T)^*]$ is quasisimple. Now L_v^* appears in A.3.14 and B.5.1, and hence as in a previous argument is $SL_3(4)$, $Sp_4(4)$, $G_2(4)$, or A_7. Further by B.5.1 and B.4.2, $[U, L_v]/C_{[U,L_v]}(L_v)$ is either the natural module or the sum of two natural modules for $L_3(4)$. As $v \in [U_N, L_N]$, $v \in C_{[U,L_v]}(L_v)$. Hence the 1-cohomology of the natural module is nontrivial, so that by I.1.6, $L_v^* \cong Sp_4(4)$ or $G_2(4)$, and $[U, L_v]$ is a quotient of a 5-dimensional orthogonal space or the 7-dimensional Cayley algebra over $\mathbf{F}_4$, respectively. Further $L_N^* = P^{*\infty}$ for some maximal parabolic P^* of L_v^*. Then $C_U(O_2(L_N^*)) = C_U(O_2(P^{*\infty}))$ contains U_N, which does not split over V_N, and $v \in C_{V_N}(L_v^*)$. This is impossible, since from the structure of these two modules, $C_U(O_2(P^*)) = C_U(L_v^*) \oplus [C_U(O_2(P^*)), P^*]$.

Therefore $J(T) \leq C_G(U)$. By a Frattini Argument, $L_v^*T_v^* = N_{L_vT_v}(J(T))^*$, so as $N_G(J(T)) \leq M$ by 3.2.10.1, $L_v^* = L_N^*$, completing the proof of our second claim.

In particular as $L_N/O_2(L_N)$ is simple and U is 2-reduced, the second claim says $L_v^* = L_N^* \cong L_2(4)$; hence $O_\infty(L_v) \leq C_{L_v}(U)$. Therefore as $L_v \not\leq M$, $C_{L_v}(U) \not\leq M$, so case (c) or (d) of 1.2.1.4 holds. In the notation of chapter 1, there is at least one prime $p > 3$ with $1 \neq X := \Xi_p(L_v)$. Then X is characteristic in L_v and hence normal in G_v. Further X centralizes U, and hence centralizes V_1V_N. By 1.3.3,

$X \in \Xi(G_v, T_v)$. Now T acts on V_1V_N and there is $g \in N_{L_0}(V_1V_N)$ with $v^g \notin V_N$ and $v^g \in Z(T_v)$. Then $V_1V_N \leq U^g$, so $X^g \leq C_G(v) = G_v$, and hence X^g acts on X. Further $T_v \leq G_v^g \leq N_G(X^g)$, and X centralizes v^g as $v^g \in V_1V_N$, so X acts on X^g. Recall from the definition of $\Xi(G_v, T_v)$ that $X = PO_2(X)$ with $P \cong E_{p^2}$ or p^{1+2}. Set $(XX^gT_v)^+ := XX^gT_v/O_2(XX^gT_v)$. Then T_v is irreducible on $P^+/\Phi(P^+)$ and $P^{g+}/\Phi(P^{g+})$, so $P^+ \cap P^{g+}$ is 1, $\Phi(P^+)$, or P^+. As $m_p(XX^g) \leq 2$, the last case holds, so $X = X^g$. Therefore X is normal in G_v and G_{v^g}, so $L_0 = \langle L_N, L_N^g \rangle$ acts on X. Then as $Aut(X/O_{2,\Phi}(X))^\infty \cong SL_2(p)$, either L or L^t centralizes $X/O_2(X)$, and thus $L_0 = \langle L^{T_v} \rangle$ centralizes $X/O_2(X)$, contradicting $X = [X, L_N]$. This finally establishes 9.4.1. □

LEMMA 9.4.2. *(1) If $v \in V_N^\#$, $g \in L_0 - N_G(V_N)$, and $u \in V_N^{g\#}$, then $C_G(\langle u, v \rangle) \leq M$.*

(2) V is the unique member of V^G containing V_1V_N.

PROOF. Part (1) follows as $C_G(\langle u, v \rangle)$ acts on $\langle L_N, L_N^g \rangle = L_0$ by 9.4.1. As $V_1V_N = V_NV_N^l$ for suitable $l \in L$, $C_G(V_1V_N) \leq M = N_G(V)$ by (1). By 3.2.10.2, M controls fusion in V, so we conclude that $N_G(V_1V_N) \leq M$, and that (2) follows from the proof of A.1.7.2. □

We can finally begin to implement our standard weak closure strategy.

LEMMA 9.4.3. *$r(G, V) > 3$.*

PROOF. Suppose $U \leq V$ with $m(V/U) \leq 3$ and $C_G(U) \not\leq M$. As $m(V/U) \leq 3$, $C_{\bar{M}}(U)$ is a 2-group by 9.1.2. Recall from 9.2.3 that $r > 1$, so by E.6.12, $C_{\bar{M}}(U)$ is a nontrivial 2-group. As $m(V/U) < 4$, we may take $U \leq C_V(t) = U_N$. Now for each $V_N^g \leq U_N$, $1 \neq V_N^g \cap U$ as $m(U_N/U) \leq 1$, so the lemma follows from 9.4.2. □

LEMMA 9.4.4. *$W_0 := W_0(T, V)$ centralizes V, so $w > 0$.*

PROOF. Suppose $A := V^g \leq T$ with $\bar{A} \neq 1$. If $m(C_A(V)) \geq 5$, then $V \leq N_G(V^g)$ by 9.4.3, contrary to E.3.11. Hence $m(\bar{A}) \geq 4$, so as $m_2(\bar{M}) = 4$ and $\bar{T}_L = J(\bar{T})$, $\bar{A} = \bar{T}_L$. Then $C_V(\bar{A}) = V_1$, so if U_1 is the L-irreducible containing V_1, then $C_A(\bar{L})$ centralizes $\langle V_1^L \rangle = U_1$. Now $m(A/C_A(\bar{L})) = 2$, so as $r > 3$, $U_1 \leq M^g$. Similarly $U_1^t \leq M^g$, so $U_1U_1^t = V_1^\perp \leq M^g$, and $[U_1U_1^t, A] = V_1$, so $U_1U_1^t$ induces F-transvections on A with center V_1. This is impossible since M controls fusion in V by 3.2.10.2, while the center of an F-transvection on V is nonsingular by 9.1.2.4, and V_1 is singular by 9.1.2.5. □

LEMMA 9.4.5. *$W_1(T, V)$ centralizes V, so $w > 1$.*

PROOF. If not, then arguing as in the proof of the previous lemma, there is a hyperplane $A := V^g \cap T$ of V^g with $\bar{A} \neq 1$, and this time $m(\bar{A}) \geq 3$. Suppose first $\bar{A} \not\leq \bar{L}_0$. Then $\bar{A}$ has maximal rank (namely 3) subject to $\bar{A} \not\leq \bar{L}_0$, so $\bar{A} \in \mathcal{A}(C_{\bar{M}}(\bar{a}))$ for each $\bar{a} \in \bar{A} - \bar{L}_0$. Observe $m(V^g/C_A(V_1)) \leq 2$, so $V_1 \leq M^g$ since $r > 3$ by 9.4.3. Thus if A does not centralize V_1, then $Z = [V_1, A] \leq M \cap V^g = A$. As $C_A(V_1)$ is of codimension at most 2 in V^g, V_1 induces an F-transvection on V^g with Z contained in the center $[V^g, V_1]$, a contradiction as in the proof of the previous lemma. Therefore $[A, V_1] = 1$, so as $\bar{A} \not\leq L_0$, there is $t \in A$ with $\bar{t} = \sigma$ and $\bar{A} = \langle \bar{t} \rangle (\bar{A} \cap \bar{L}_N)$. But then $m(V^g/C_A(U_N)) \leq 3$, so $U_N \leq M^g$ since $r > 3$. Therefore $V_NV_1 = [A, U_N] \leq A \leq V^g$, contrary to 9.4.2.2.

Thus $\bar{A} \leq \bar{T}_L$, so $\bar{A}$ has rank 3 or 4. We now argue as in the proof of 9.4.4: First $C_V(\bar{A}) = C_V(\bar{T}_L) = V_1$, and $C_A(V^g)$ has rank 4,3, with $C_{\bar{A}}(\bar{L})$ of rank 1,2, respectively. So in any case $m(V^g/C_A(\bar{L})) = 3 < r$, and hence we can continue the argument in the proof of 9.4.4 to get $U_1 U_1^t \leq M^g$, and obtain the same contradiction. □

Observe that by 9.4.4, 9.4.5, and E.3.16, $N_G(W_0) \leq M \geq C_G(C_1(T, V))$. As $m(\bar{M}, V) \geq 2$, $s(G, V) \geq 2$ by 9.4.3. Then as $n(H) = 1$, E.3.19 forces $H \leq M$, a contradiction. This contradiction finally shows that case (1) of 3.2.6 cannot occur, and hence completes the proof of Theorem 7.0.1 begun in chapter 7.

Part 4

Pairs in the FSU over $\mathbf{F}_{2^n}$ for $n > 1$.

In part 4, we prove two theorems about pairs L, V in the Fundamental Setup (3.2.1): In chapter 10, we show that $L = L_0$. Then in chapter 11, we show that L is not of Lie type of Lie rank 2 over $\mathbf{F}_{2^n}$ for $n > 1$.

A counter example in chapter 10 is of the form $L_0 = LL^t$ with $t \in T - N_T(L)$ and $L/O_2(L)$ isomorphic to $L_2(2^n)$ or $Sz(2^n)$ with $n > 1$, or to $L_3(2)$. In the first two cases, we can view $L_0/O_2(L_0)$ as of Lie type of Lie rank 2 over $\mathbf{F}_{2^n}$. Thus the majority of the effort in part 4 is devoted to the elimination of cases in the FSU where $\bar{L}_0$ is of Lie type and Lie rank 2 over $\mathbf{F}_{2^n}$ for some $n > 1$.

One of the main tools for treating such groups is the study of Cartan subgroups, both of L_0 and of $H \in \mathcal{H}_*(T, M)$: a Cartan subgroup of $X := L_0$ or H is defined to be a Hall $2'$-subgroup of $N_X(T \cap X)$.

The most difficult cases are those where the Cartan subgroup is small or trivial: that is, when $n = 2$, or in chapter 10 when $\bar{L} \cong L_3(2)$ is defined over $\mathbf{F}_2$.

CHAPTER 10

The case $L \in \mathcal{L}_f^*(G,T)$ not normal in M.

In this chapter we prove:

THEOREM 10.0.1. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple. Then $T \leq N_G(L)$.*

10.1. Preliminaries

Assume Theorem 10.0.1 is false, and pick a counterexample L. Let $L_0 := \langle L^T \rangle$ and $M := N_G(L_0)$. By 3.2.3, there is $V_\circ \in Irr_+(L_0, R_2(L_0T))$ such that L and $V_T := \langle V_\circ^T \rangle$ are in the Fundamental Setup 3.2.1. Set $V := \langle V_T^M \rangle$, and note that this differs from the notation in the FSU where V_T, V are denoted by "V, V_M". Note in particular that by construction $V \trianglelefteq M$, so that $M = N_G(V)$.

As $L < L_0$, we can appeal to Theorem 3.2.6. In the first two cases of Theorem 3.2.6, V_T is not an FF-module, and those cases were eliminated in Theorem 7.0.1. Thus we are left with case (3) of Theorem 3.2.6. We recall from that result that $V = V_1V_1^t$ for $t \in T - N_T(L)$, with $V_1 := [V,L] \leq C_V(L^t)$.

Recall that in the FSU with $V \trianglelefteq M$, we set $\bar{M} := M/C_M(V)$ and $\tilde{V} = V/C_V(L_0)$. Also set $L_1 := L$, $L_2 := L^t$ for $t \in T - N_T(L)$, and $V_i := [V, L_i]$.

The cases to be treated are listed in the following lemma. Subcases (ii) and (iii) of 3.2.6.3 appear as cases (5) and (6) in 10.1.1. In subcase (i) $V_1 \in Irr_+(L,V)$, and by 3.2.6.3b, $\hat{q}(Aut_{L_0T}(V_1), V_1) \leq 2$, so $\tilde{V}$ appears in B.4.2 or B.4.5. As $L < L_0$, $\bar{L}$ appears in 1.2.1.3. Intersecting those lists leads to the remaining cases in 10.1.1.

LEMMA 10.1.1. *$V = V_1V_2 \in \mathcal{R}_2(M)$ with $V_i := [V, L_i] \leq C_V(L_{3-i})$, $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$, and one of the following holds:*

(1) $\tilde{V}_1$ is the natural module for $\bar{L} \cong L_2(2^n)$, with $n > 1$.

(2) V_1 is the A_5-module for $\bar{L} \cong A_5$.

(3) $\tilde{V}_1$ is the natural module for $\bar{L} \cong L_3(2)$.

(4) V_1 is the orthogonal module for $\bar{L} \cong \Omega_4^-(2^n)$, with $n > 1$.

(5) V_1 is the sum of a natural module for $\bar{L} \cong L_3(2)$ and its dual, with the summands interchanged by an element of $N_T(L)$.

(6) V_1 is the sum of four isomorphic natural modules for $\bar{L} \cong L_3(2)$, and $O^2(C_{\bar{M}}(\bar{L})) \cong \mathbf{Z}_5$ or E_{25}.

(7) V_1 is the natural module for $\bar{L} \cong Sz(2^n)$.

Let $Z := \Omega_1(Z(T))$, $t_0 := T \cap L_0$, $T_1 := N_T(L)$; and $B_0 := O^2(N_{L_0}(T_0))$. Note that $B_0T = TB_0$ and (except when $\bar{L} \cong L_3(2)$ where $B_0 = 1$) $\bar{B}_0$ is a Borel subgroup of $\bar{L}_0$. Set $S := \text{Baum}(T)$.

LEMMA 10.1.2. *(1) Except possibly in the first three cases of 10.1.1, V is not an FF-module for $Aut_{L_0T}(V)$, so $J(T) \leq C_T(V)$.*

(2) $J(T) \leq N_T(L) = T_1$.

(3) $C_T(V) = O_2(L_0T)$ *except in case (6) of 10.1.1, where at least* $C_T(V) \trianglelefteq L_0T$. *In any case,* $M = !\mathcal{M}(N_G(C_T(V))) = !\mathcal{M}(N_G(J(C_T(V))))$.

(4) Except possibly in cases (1) and (3) of 10.1.1, $C_V(L_0) = 1$.

(5) Assume $\bar{L}$ *is not* $L_3(2)$ *and let* D *be a Hall* $2'$*-subgroup of* B_0. *Then either:*

(a) $C_D(Z) = 1$, *or*

(b) V_1 *is the orthogonal module for* $\bar{L} \cong \Omega_4^-(2^n)$ *and* $C_D(Z) \cong \mathbf{Z}_{2^n+1}^2$.

(6) In cases (1) and (2) of 10.1.1, L_0T *is a minimal parabolic in the sense of Definition B.6.1, with* $N_{L_0T}(T_0)$ *the unique maximal overgroup of* T *in* L_0T. *Thus if* $J(T) \not\leq C_T(V)$ *then* L_0T *is described in E.2.3 and* $S \leq N_T(L) = T_1$.

PROOF. Part (2) is clear if $J(T) \leq C_T(V)$, while if $J(T) \not\leq C_T(V)$, it follows from B.1.5.4. Except in cases (5) and and (6) of 10.1.1, $\tilde{V}_1$ is an irreducible for L, and (1) follows from B.4.2. In cases (5) and (6), V is not an FF-module for $Aut_{L_0T}(V)$ by Theorem B.5.6, so (1) is established. Next in all cases of 10.1.1 except case (6), $V = V_T$, so that $C_T(V) = O_2(L_0T)$ by 1.4.1.4. In case (6), $C_{L_0T}(V) \leq O_2(L_0T)$, so as $V \trianglelefteq L_0T$, $C_T(V) = C_{L_0T}(V) \trianglelefteq L_0T$, and hence (3) holds in that case too. Part (4) follows in the final four cases of 10.1.1 from (1) and 3.2.10.9; in the second case it follows from I.1.6. Part (5) follows easily from (4) and the structure of the modules in 10.1.1. Finally the first two remarks in (6) are elementary observations, and then if $J(T) \not\leq C_T(V)$, the remaining remarks are a consequence of E.2.3. □

LEMMA 10.1.3. $L_0 = O^{p'}(M)$ *for each prime divisor* p *of* $|\bar{L}|$.

PROOF. This follows from 1.2.2. □

10.2. Weak closure parameters and control of centralizers

We will make use of weak closure, together with control of centralizers of elements of $V_1^{\#}$. In 10.2.3, we will use the fact that G is a QTKE-group to show $n(H) \leq 2$ for $H \in \mathcal{H}_*(T,M)$; subsequent results provide lower bounds on the weak-closure parameters $r(G,V)$ and $w(G,V)$. In 10.2.13, we will eliminate most cases using the relation $n(H) \geq w(G,V)$ in E.3.39.

LEMMA 10.2.1. *Except possibly in case (3) of 10.1.1,* $N_G(S) \leq M$.

PROOF. We may assume case (3) of 10.1.1 does not hold. If $J(T) \leq C_T(V)$, then as $J(T) \leq S$, $N_G(S) \leq M$ by 3.2.10.8. Thus we may assume $J(T) \not\leq C_T(V)$, so by 10.1.2.1, we are reduced to cases (1) and (2) of 10.1.1. In those cases, L_0T is a minimal parabolic, and is described in E.2.3 by 10.1.2.6.

In case (1) of 10.1.1, E.2.3.2 says $S \in Syl_2(L_0S)$, so we can apply Theorem 3.1.1 with L_0T, $N_G(S)$, S in the roles of "H, M_0, R", to conclude that $O_2(\langle N_G(S), L_0T\rangle) \neq 1$. Thus $N_G(S) \leq M = !\mathcal{M}(L_0T)$, as desired.

Therefore we may assume case (2) of 10.1.1 holds; the proof for this case will be longer. Moreover for each $S_+ \trianglelefteq T$ with $T_0 = T \cap L \leq S_+$, $S_+ \in Syl_2(L_0S_+)$; hence applying 3.1.1 as in the previous paragraph, we conclude that $N_G(S_+) \leq M$. In particular $N_G(T_1) \leq M$. We may also assume that $N_G(S) \not\leq M$, so as $M = !\mathcal{M}(L_0T)$, no nontrivial characteristic subgroup of S is normal in L_0T. Then E.2.3.3 says that L_1 is an A_5-block.

Suppose first that $C_Z(L_0) = 1$. Then $O_2(L_0T) = V$ by C.1.13.c, so that $V = O_2(M)$ using A.1.6. Further using E.2.3, $S = S_1 \times S_2$, where $S_i := C_S(L_{3-i}) =$

$S_{i,1} \times S_{i,2}$ with $S_{i,j} \cong D_8$, and T acts transitively as D_8 on the four members of $\Delta := \{S_{i,j} : i, j\}$. As S is the direct product of the subgroups in Δ, by the Krull-Schmidt Theorem A.1.15, $N_G(S)$ permutes $\Gamma := \{DZ(S) : D \in \Delta\}$. Let K be the kernel of $N_G(S)$ on Γ and $N_G(S)^\Gamma := N_G(S)/K$. Then $D_8 \cong T^\Gamma \leq N_G(S)^\Gamma \leq S_4$. Observe that for $F \in \Gamma$, $\mathcal{A}(F) = \{V_F, A_F\}$ is of order 2, where $V_F := V \cap F$. Thus $O^2(K)$ acts on each V_F. Then as $V \trianglelefteq T$, $K = O^2(K)(K \cap T)$ acts on $\langle V_F : F \in \Gamma\rangle = V$. Hence $K \leq N_G(V) = M$, so as we are assuming $N_G(S) \not\leq M$, there is $x \in N_G(S)$ with x inducing a 3-cycle on Γ. Therefore $N_G(S)^\Gamma \cong S_4$. Let K_R be the preimage in $N_G(S)$ of $O_2(N_G(S)^\Gamma)$ and $R := T \cap K_R$. By a Frattini Argument, $N_G(S) = K(N_G(S) \cap N_G(R))$, so we may take $x \in N_G(R)$. But R normalizes just two members V and $A := \langle A_F : F \in \Gamma\rangle$ of $\mathcal{A}(S) = \mathcal{A}(T)$, so x acts on V and A. Therefore $N_G(S) = KT\langle x\rangle \leq N_G(V) = M$, contrary to our assumption.

Thus in the remainder of the proof, we assume that $C_Z(L_0) \neq 1$. We may choose $H \in \mathcal{H}_*(T, M)$ with $H \leq N_G(S)$. Let $E := \Omega_1(Z(S))$, $V_H := \langle Z^H\rangle$, and $H^* := H/C_H(V_H)$. As usual $V_H \in \mathcal{R}_2(H)$ by B.2.14. Now $Z \leq E$ and hence $V_H \leq E$. As $C_Z(L_0) \neq 1$, $C_H(V_H) \leq C_G(C_Z(L_0)) \leq M = !\mathcal{M}(L_0T)$, so $H^* \neq 1$. Observe applying E.2.3.3 to L_0T that for $t_i \in T \cap L_i - S$, t_i induces a transvection on E with center $v_i \in V_i$. If $t_1 \in C_H(V_H)$, then

$$S_0 := T_0S = \langle t_1, t_2, S\rangle \leq C_T(V_H).$$

But we saw earlier that $N_G(S_+) \leq H$ for each $S_+ \trianglelefteq T$ with $T_0 \leq S_+$, so by a Frattini Argument, $H = N_H(C_T(V_H))C_H(V_H) \leq M$, contrary to our assumption.

Thus $t_i^* \neq 1$, so as $V_H \leq E$, t_i^* induces a transvection on V_H with center v_i. Then comparing the possibilities in E.2.3 to the list of groups in G.6.4 containing $\mathbf{F}_2$-transvections, we conclude that either $H^* \cong O_4^+(2)$ with $m([V_H, H]) = 4$, or H^* is one of S_5 or $S_5 \text{ wr } \mathbf{Z}_2$. The latter cases are out, as then $N_M(S)$ is not a $3'$-group, contrary to 10.1.3 and the fact that $N_{L_0}(S)$ is a $3'$-group. So $[V_H, H]$ is the orthogonal module for $H^* \cong O_4^+(2)$. Let $Y := O^2(C_H(v_2))$; then $Y^* \cong \mathbf{Z}_3$, and $Y \cap M \leq O_2(H)$.

Let $X := C_G(v_2)$. Then $T_1 = C_T(v_2)$, $|T : T_1| = 2$, and $L \leq X$. As $T \not\leq X$ but $N_G(T_1) \leq M$, $T_1 \in Syl_2(X)$. Thus by 1.2.4, $L \leq I \in \mathcal{C}(X)$. Suppose first that $L = I$. Then $L \trianglelefteq X$ by 1.2.1.3, so X acts on $[O_2(L), L] = V_1$. As $Y = [Y, T_1]$ while $Y \cap M \leq O_2(H)$, we conclude from the structure of $Aut(L/O_2(L))$ that $Y \leq O^2(C_X(L/O_2(L)))$. Further $End_{\mathbf{F}_2(L/O_2(L))}(V_1) \cong \mathbf{F}_2$, so that Y must centralize V_1. However, Y does not centralize $v_1 \in V_1$. This contradiction shows that $L < I$.

Suppose that $V_1 \leq O_2(I)$. Then since the A_5-block L has a unique nontrivial 2-chief factor V_1, and V_1 is projective, $W := \langle V_1^I\rangle = V_1 \oplus C_W(L) \leq Z(O_2(I))$ and I has a unique nontrivial 2-chief factor. In particular $W \in \mathcal{R}_2(I)$ and setting $\hat{I} := I/C_I(W)$, $[W, \hat{a}] = [V_1, \hat{a}]$ for each involution $\hat{a} \in \hat{L}$, so $q(\hat{I}, W) \leq 2$. Also we conclude from A.3.14 that $I/O_2(I) \cong A_7$, $\hat{A}_7$, J_1, $L_2(25)$, or $L_2(p)$ with $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$. Then as $q(\hat{I}, W) \leq 2$, we conclude from B.4.2 and B.4.5 that $I/O_2(I) \cong A_7$. Since the unique nontrivial L-chief factor V_1 is the A_5-module, we conclude that W is the A_7-module, so I is an A_7-block. However $I = O^{3'}(X)$ by A.3.18, so $1 \neq O^{3'}(C_{L_2}(v_2)) \leq O^2(C_I(L))$, contradicting $O^2(C_I(L)) = 1$.

Therefore $V_1 \not\leq O_2(I)$, so as L is irreducible on V_1, $V_1 \cap O_2(I) = 1$. Set $\dot{I} := I/O_2(I)$. Then $\dot{L}$ is a T_1-invariant A_5-block in $\dot{I}$, a situation that does not occur in A.3.14. This contradiction completes the proof. □

LEMMA 10.2.2. *Assume $H \in \mathcal{H}_*(T,M)$ with $[Z,H] \neq 1$, and set $W := \langle Z^H \rangle$. Then*

(1) $L_0 = [L_0, J(T)]$ and one of the first three cases of 10.1.1 holds.

If in addition case (1) or (2) of 10.1.1 holds, then:

(2) $O^2(H) = [O^2(H), J(T)]$ and $J(T) \not\leq C_T(W)$, so W is an FF-module for $H/C_H(W)$.

(3) If case (1) of 10.1.1 holds, then $B_0 \leq N_G(S)$ and $S \in Syl_2(L_0S)$.

(4) $O_2(\langle N_G(S), H\rangle) \neq 1$.

PROOF. As $[Z,H] \neq 1$, $[V, J(T)] \neq 1$ by 3.1.8.3. Thus $L_0 = [L_0, J(T)]$, and then 10.1.2.1 completes the proof of (1). In the remaining assertions we may assume case (1) or (2) of 10.1.1 holds. Then by 10.1.2.6, L_0T is a minimal parabolic described in E.2.3, and $S \leq T_1$.

In case (1) of 10.1.1, E.2.3.2 says that $S \in Syl_2(L_0S)$ and $S \leq T_+ := T_0O_2(L_0T)$, so that $S = \text{Baum}(T_+)$. But B_0 normalizes T_+ so $B_0 \leq N_G(S)$, completing the proof of (3).

Assume $[O^2(H), J(T)] < O^2(H)$. Then as $[Z,H] \neq 1$ we conclude from B.6.8.3d that $S = \text{Baum}(O_2(H))$ and hence $H \leq N_G(S)$. However since we are excluding case (3) of 10.1.1, $N_G(S) \leq M$ by 10.2.1. This contradicts $H \not\leq M$, so (2) holds.

If $J(H)^*$ is the product of copies of $L_2(2^m)$ then by E.2.3.2, $S \in Syl_2(O^2(H)S)$. Then using Theorem 3.1.1 as in the proof of 10.2.1, (4) follows. Since we may assume (4) fails, we conclude from E.2.3.1 that $J(H^*)$ is a product of $s \leq 2$ copies of S_5, and that no nontrivial characteristic subgroup of S is normal in H. Therefore by E.2.3.3, $O^2(H) = K_1 \times \cdots \times K_s$ is the product of A_5-blocks K_i.

Next $O^2(H \cap M) \leq L_0$ by 10.1.3, so $O^2(H \cap M) \leq B_0$ and a Sylow 3-subgroup P of $O^2(H \cap M)$ is contained in $P_0 \in Syl_3(B_0)$. As $O^2(H)$ is a product of A_5-blocks, P centralizes Z, so case (2) of 10.1.1 holds since $C_{P_0}(Z) = 1$ in case (1) by 10.1.2.5. Then since $L_0 = [L_0, J(T)]$ by (1), $\bar{L}_0\bar{T} \cong S_5$ wr $\mathbf{Z}_2$ in view of B.4.2.5, so $B_0 \in \Xi(G,T)$. Since $O^2(H \cap M)$ is T-invariant and lies in B_0, while T is irreducible on $B_0/O_2(B_0)$, we conclude $O^2(H \cap M) = B_0$. Therefore $P = P_0$ is of order 9, so $s = 2$ and $O^2(H) = K_1 \times K_2$ is the product of two blocks. Therefore $O^2(H \cap M) = X_1 \times X_2$ with $X_i := O^2(K_i \cap M) \cong \mathbf{Z}_3/Q_8^2$. Now as $O^2(H \cap M) = B_0$, while X_i has just two noncentral 2-chief factors, X_i cannot be diagonally embedded in L_0, so (interchanging L_1 and L_2 if necessary) $X_i = B_0 \cap L_i$. Then X_i is T_1-invariant, and L_i is an A_5-block as X_i has two noncentral 2-chief factors. Now K_i is T_1-invariant, $I := \langle L_1, K_1 \rangle \leq C_G(X_2)$, I is T_1-invariant, and $S = \text{Baum}(T_1)$ since we saw $S \leq T_1$. Hence $N_G(T_1) \leq M$ by 10.2.1. Therefore $N_T(X_2) = T_1 \in Syl_2(N_G(X_2))$, so $T_1 \in Syl_2(IT_1)$. Hence we can apply 1.2.4 to embed $L_1 \leq L_I \in \mathcal{C}(I)$, and then $K_1 = [K_1, X_1] \leq L_I$, so $L_1 < L_I$ since $K_1 \cap M = X_1$. Now $N_G(X_2)$ is an SQTK-group, so $m_3(N_G(X_2)) \leq 2$ and hence $m_3(L_I) = 1$. This rules out the possibility that $O_2(L_1) \leq O_2(L_I)$ and $L_I/O_2(L_I) \cong A_7$ in A.3.14. We now obtain a contradiction via the argument in the last two paragraphs of the proof of 10.2.1. This contradiction completes the proof of (4), and of the lemma. □

PROPOSITION 10.2.3. *If $H \in \mathcal{H}_*(T,M)$ with $n(H) > 1$, then*

(1) $n(H) = 2$.

(2) A Hall $2'$-subgroup of $H \cap M$ is faithful on $\bar{L}_0$.

(3) If $\bar{L} \cong L_3(2)$, then $T_0O^2(H \cap M)$ is a maximal parabolic in L_0 and $H/O_2(H) \cong S_5$ wr $\mathbf{Z}_2$.

(4) Case (1) of 10.1.1 does not hold; that is, $n(H) = 1$ for each H in that case.

PROOF. Let B_H be a Hall $2'$-subgroup of $H \cap M$. Notice B_H permutes with T, so that $B_+ := B_H \cap L_0$ permutes with T_0.

We first establish (2). If V is not an FF-module for $L_0T/C_{L_0T}(V)$, then (2) follows from Theorem 4.4.14; so we may assume that $B := C_{B_H}(\bar{L}_0) \neq 1$ and V is an FF-module for $\bar{L}_0\bar{T}$. We first verify Hypothesis 4.4.1 and then we apply Theorem 4.4.3: By 4.4.13.2 we have $BT = TB$, giving (1) and (2) of Hypothesis 4.4.1. As $BT = TB$, $N_H(B) \not\leq M$ by 4.4.13.1. As $V_i \trianglelefteq O^2(M)$, B acts on V_i. But by 10.1.3, $(|B|, |\bar{L}|) = 1$, so as $|End_{\bar{L}_i}(\tilde{V}_i)|$ divides $|\bar{L}|$, $[V, B] = 1$. Thus we also have 4.4.1.3, with V in the role of "V_B". Since $L < L_0$, case (1) of Theorem 4.4.3 must hold, contradicting our earlier observation that $N_H(B) \not\leq M$. So (2) is established.

Appealing to (2), 10.1.3, and the structure of $Aut(\bar{L}_i)$, we conclude that either

(i) $\bar{L}$ is not $L_3(2)$ and $B_H = B_+F$, with $B_+ \leq B_0$ (since B_+ permutes with T_0), and F induces field automorphisms on $\bar{L}_0$, or

(ii) $\bar{L} \cong L_3(2)$ and $B_H = B_+ \leq L_0$.

Assume first that (ii) holds; this case corresponds to cases (3), (5), and (6) of 10.1.1. Then as B_H permutes with T_0, B_H is a 3-group, and so $n(H) = 2$. Further $B_HO_2(B_HT)$ is T-invariant, so $\bar{B}_H\bar{T}$ contains a Sylow 3-group of $\bar{L}_0$, and hence $B_HT_0 = O^2(H \cap M)T_0$ is a maximal parabolic in L_0. In particular, $(H \cap M)/O_2(H \cap M) \cong S_3$ wr $\mathbf{Z}_2$, and the only case in E.2.2 with $n(H) = 2$ satisfying this condition is $H/O_2(H) \cong S_5$ wr $\mathbf{Z}_2$. For example case (2b) of E.2.2 is ruled out as here $(H \cap M)/O_{2,3}(H \cap M) \cong D_8$. Thus we have established (3), and also proved (1) in this case. So from now on, we may assume that (i) holds.

Suppose next we are in case (2) of 10.1.1, where $\bar{L} \cong A_5$. Then $F = 1$, so that $B_H = B_+ \leq B_0$. Now we may argue much as in the previous paragraph: As B_H permutes with T, it is a 3-group and so $n(H) = 2$, completing the proof of (1) and hence of the lemma in this case.

So at this point, we have reduced to one of cases (1), (4), or (7) of 10.1.1. Since $B_H = B_+F$ by (i), there is a B_H-invariant Hall $2'$-subgroup D of B_0, and $B_+ \leq D$. By 10.1.2.5, $C_D(Z) = 1$ in cases (1) and (7) of 10.1.1, while $C_D(Z) \cong \mathbf{Z}_{2^n+1}^2$ in case (4). Further in any case, $C_F(Z) = 1$.

Suppose first that $[Z, H] = 1$. Then $F = C_F(Z) = 1$, so $B_+ = B_H \leq C_D(Z)$, and hence $C_{B_0}(Z) \neq 1$ so that case (4) of 10.1.1 holds by the previous paragraph. Set $m := n(H) \geq 2$. From E.2.2, B_H has a cyclic subgroup B of order $2^m - 1$. As $B \leq C_D(Z)$, $2^m - 1$ divides $2^n + 1$, so m divides $2n$. If m divides n then $2^m - 1$ divides $2^n - 1$, impossible as $(2^n + 1, 2^n - 1) = 1$. Thus $m = 2d$ is even and d divides n, so as $(2^n + 1, 2^n - 1) = 1$, $2^d - 1 = 1$ and hence $m = 2$. Therefore the lemma holds in this case.

We may now assume that $[Z, H] \neq 1$. Then $L_0 = [L_0, J(T)]$ by 10.2.2.1, eliminating cases (4) and (7) of 10.1.1, leaving only case (1), where it remains to derive a contradiction in order to complete the proof of the lemma. Recall in this case that $C_{DF}(Z) = 1$.

By 10.2.2.2, $O^2(H) = [O^2(H), J(T)]$. By E.2.3.1, $O^2(H) = \langle K^T \rangle$ where $K \in \mathcal{C}(H)$ with $K/O_2(K) \cong L_2(2^m)$ or A_5, and setting $W := \langle Z^H \rangle$ and $V_K := [W, K]$, $V_K/C_{V_K}(K)$ is the natural module for $K/O_2(K)$. Observe V_K is not the A_5-module as $B_H \leq DF$ and $C_{DF}(Z) = 1$, whereas if V_K were the A_5-module then $[Z, B_H] = 1$.

We next claim that $B_0 \leq N_G(K)$: By 10.2.2.3, $B_0 \leq N_G(S)$, so by 10.2.2.4, $\langle B_0, H \rangle \leq M_1 \in \mathcal{M}(T)$. Hence by 1.2.4, $K \leq I \in \mathcal{C}(M_1)$, and as $K = [K, J(T)]$

and $[Z,K] \neq 1$, also $I = [I, J(T)]$ and $1 \neq U := [Z,I] \in \mathcal{R}_2(I)$ using B.2.14. Thus $J(T) \not\leq C_T(U)$ and U is an FF-module. We conclude from intersecting the lists of A.3.12 and B.4.2 that one of the following holds:

(a) $K = I$.
(b) $I/O_2(I) \cong SL_3(2^m)$, $Sp_4(2^m)$, or $G_2(2^m)$.
(c) $m = 2$ and $I/O_2(I) \cong A_7$ or $\hat{A}_7$, with $I/C_I(U) \cong A_7$.
(d) $I/O_2(I)$ is not quasisimple, $I = O_{2,F}(I)K$, and $O_{2,F}(I)$ centralizes U.

By 1.2.1.3, $B_0 = O^2(B_0)$ normalizes I, so we may assume that $K < I$, and so one of (b)–(d) holds. Hence T acts on I by 1.2.1.3, and then also T acts on K by 1.2.8. In case (d), as $C_D(Z) = 1$, $B_0 \cap O_{2,F}(I) \leq T$, so B_0 acts on the unique $(T \cap I)$-invariant supplement K to $O_{2,F}(I)$ in I. Suppose case (c) holds. By A.3.18, $I = O^{3'}(M_1)$, so $D = D_I \times D_C$, where $D_C := O^3(D) = C_D(I/O_2(I))$ and $D_I := O_3(D) = D \cap I$. As D_C acts on K, we may assume $D_I \not\leq K$. Then $D_I \in Syl_3(I)$. But from the structure of the FF-modules for A_7, $C_{D_I}(Z) \neq 1$, contradicting $C_D(Z) = 1$. Suppose case (b) holds. Then $K = P^\infty$ for some T-invariant parabolic P in $I/O_2(I)$, so as $B_0 = O^2(B_0)$ permutes with T, it must also act on K, completing the proof of the claim.

By the claim B_0 acts on K, and by symmetry B_0 also acts on K^t if there is $t \in T - N_T(K)$. Thus B_0 acts on $O^2(H)$. Recall that by construction B_H acts on D, so D acts on $O^2(H) \cap DB_H = B_H$. Therefore $[B_H, D] \leq B_H \cap D \leq C_D(B_H)$ since the Hall subgroup B_H of $O^2(H \cap M)$ is abelian. Now if $F \neq 1$, then F does not centralize $[F, D]$; thus $F = 1$, and hence $B_H = B_+ \leq D$. Since $B_H \cap K$ is cyclic of order $2^m - 1$, while $D \cong \mathbf{Z}^2_{2^n-1}$, m divides n.

In the remainder of the proof, we will show that $B_H = D$, and that $K \neq K^t$ for some $t \in T - T_1$. Then we will see that the embeddings of D in LL^t and KK^t are incompatible.

As $M = !\mathcal{M}(L_0T)$, $C_Z(\langle L_0, H\rangle) = 1$. As $\tilde{V}_K$ is the natural module for $K/O_2(K) \cong L_2(2^m)$, $C_Z(H) = C_Z(b)$ for each $b \in B_H^\#$. Similarly $C_Z(L_0) = C_Z(d)$ for each $d \in D^\#$, so as $1 \neq B_H \leq D$, we conclude $C_Z(L_0) = C_Z(H) = 1$. Thus V_1 and V_K are natural modules, with $C_{V_1}(L) = 1 = C_{V_K}(K)$.

Next C.1.26 says that there are nontrivial characteristic subgroups $C_1(T) \leq Z$ of T and $C_2(T)$ of S, such that one of the following holds: K is a block, $C_1(T) \leq Z(H)$, or $C_2(T) \trianglelefteq H$. As $C_Z(H) = 1$, $C_1(T) \not\leq Z(H)$, so either K is a block or H normalizes $C_2(T)$. Similarly either L is a block or $C_2(T) \trianglelefteq L_0T$. However $C_2(T)$ cannot be normal in both H and L_0T, since $M = !\mathcal{M}(L_0T)$; therefore either K or L is a block.

Next set $E := \Omega_1(Z(J(T)))$ and $E_0 := \langle E^{L_0}\rangle$. By 10.1.2.6 we may apply E.2.3.2 to L_0T, to conclude that $E_0 = C_{E_0}(L_0)V$. Therefore as $C_Z(L_0) = 1$, $E_0 = V = V_1 \times V_2$ is of rank $4n$. In particular, $E \leq V$ and $E = E_1 \times E_2$ with $E_i := E \cap V_i$ of rank n. Also $D = D_1 \times D_2$, where $D_i := D \cap L_i = C_D(E_{3-i})$. Notice that $C_E(d) = 1$ for $d \in D - (D_1 \cup D_2)$. Similarly applying E.2.3.2 to H and using $C_Z(H) = 1$, we conclude that $E = E_K \times C_E(K) = E_K \times C_E(B_K)$, where $E_K := E \cap V_K$ has rank $m = n(H)$, and $B_K := B \cap K$. We saw m divides n, so $m \leq n$, and hence

$$m(C_E(B_K)) = m(E) - m(V_K) = 2n - m \geq n. \qquad (*)$$

So as $C_E(d) = 1$ for each $d \in D - (D_1 \cup D_2)$, we conclude (interchanging the roles of L and L^t if necessary) that $B_K \leq D_1$, so $E_K = [E, B_K] = [E, D_1] = E_1$ and

$C_E(B_K) = C_E(D_1) = E_2$ are of rank n. Thus $m = n$ by (*), and $D_1 = B_K$ as $B_K \leq D_1$ and $|D_1| = 2^n - 1 = 2^m - 1 = |B_K|$.

Next recall from E.2.3.2 that S normalizes L and K. Therefore as $D_1 = B_K$, $S_1 := [S, D_1] \leq L \cap K$. Since either L or K is a block, and $C_Z(L_0) = C_Z(H) = 1$, we conclude S_1 is special of order 2^{3n}; then it follows that both L and K are blocks, with $O_2(L)$ and $O_2(K)$ of rank $2n$, and S_1 is Sylow in both L and K.

Next L_1 and L_2 commute by C.1.9, so $[S_1, D_2] \leq [L_1, D_2] = 1$. So as D_2 centralizes $S_1 \in Syl_2(K)$, D_2 centralizes K from the structure of $Aut(K)$. Similarly $S_2 := [S, D_2] \in Syl_2(L_2)$ and S_2 centralizes K. But $S_2 = S_1^t$ for $t \in T - T_1$, so S_2 is Sylow in the block K^t and $K^t \neq K$. Hence $O^2(H) = KK^t = K \times K^t$ since $C_{V_K}(K) = 1$. Setting $K_1 := K$ and $K_2 := K^t$, S_iD_i is Borel in both L_i and K_i.

Set $M_1 := N_G(S_1D_1)$. As L_2 centralizes L_1, $L_2T_1 \leq M_1$, with $T_1 = T \cap M_1$. Similarly $K_2T_1 \leq M_1$. Embed $T_1 \leq T_+ \in Syl_2(M_1)$. Recall that $S \leq T_1$, so $S = \mathrm{Baum}(T_+)$, and hence $T_+ \leq N_G(S) \leq M$ by 10.2.1. If $T_1 < T_+$ then T_+ is also Sylow in M, so $M = !\mathcal{M}(L_0T_+)$ by 1.2.7.3. However $L_0T_+ = \langle L_2, T_+ \rangle \leq M_1$, so $K_2 \leq M_1 \leq M$, contradicting $K_2 \not\leq M$.

This contradiction shows that $T_1 = T_+$ is Sylow in M_1. Hence $L_2 \leq L_+ \in \mathcal{C}(M_1)$ by 1.2.4. Now $K_2 = O^2(K_2)$ normalizes L_+ by 1.2.1.3, so as $D_2 \leq L_2 \leq L_+$, also $K_2 = [K_2, D_2] \leq L_+$, and hence $L_2 < L_+$. As L_2 and K_2 are distinct members of $\mathcal{L}(L_+, T_1)$ and both are blocks of type $L_2(2^n)$ with trivial centers, we conclude from A.3.12 that $O_2(L_+) = 1$ and $L_+ \cong (S)L_3(2^n)$. Now L_+ normalizes S_1D_1, and so in fact centralizes S_1D_1 since S_1 is special of order 2^{3n}. Therefore for p a prime divisor of $2^n - 1$, $m_p(D_1L_+) > 2$, contradicting M_1 an SQTK-group. This completes the elimination of case (1) of 10.1.1 when $n(H) > 1$, and hence establishes 10.2.3. □

LEMMA 10.2.4. *Assume that $\bar{L} \cong L_3(2)$, but case (5) of 10.1.1 does not hold, so that $\bar{L}\bar{T}_1 \cong L_3(2)$. Let P be one of the two maximal subgroups of L_0T containing T. Set $X := O^2(P)$, assume $H \in \mathcal{H}(XT)$, and set $K := \langle X^H \rangle$. Then one of the following holds:*

(1) $K = X$.

(2) $K = K_1K_1^s$ with $K_1 \in \mathcal{C}(H)$, $K_1/O_2(K_1) \cong L_2(2^m)$ for some even m or $L_2(p)$ for some odd prime p, and $s \in T - N_T(K_1)$.

(3) $K \in \mathcal{C}(H)$ and $KT/O_2(KT) \cong Aut(L_k(2))$, $k = 4$ or 5.

PROOF. As $X \in \Xi(G, T)$, K is described in 1.3.4 with $p = 3$. Further $XT/O_2(XT) \cong S_3 \text{ wr } \mathbf{Z}_2$, which reduces the list to the cases appearing in the lemma. □

LEMMA 10.2.5. *$N_G(T_1) \leq M$.*

PROOF. If $J(T) \leq C_T(V)$ then the lemma follows from 3.2.10.8, so we may assume $J(T) \not\leq C_T(V)$. Then one of the first three cases of 10.1.1 holds by 10.1.2.1. If case (1) or (2) of 10.1.1 holds then $S \leq T_1$ by 10.1.2.6, so $S = \mathrm{Baum}(T_1)$ and then $N_G(T_1) \leq M$ by 10.2.1.

Thus we may assume case (3) of 10.1.1.3 holds, so $\bar{L} = \bar{L}\bar{T} \cong L_3(2)$. Let H_1 and H_2 be the two maximal subgroups of L_0T containing T. Thus $X_i := O^2(H_i) \in \Xi(G, T)$ and $H_i/O_2(H_i) \cong S_3 \text{ wr } \mathbf{Z}_2$. Since $O_2(X_i) \leq T_1$, $T_1 \in Syl_2(X_iT_1)$. Further T is a maximal subgroup of H_i, so applying Theorem 3.1.1 with H_i, $N_G(T_1)$, T_1 in the roles of "H, M_0, R", we conclude $O_2(G_i) \neq 1$, where $G_i := \langle N_G(T_1), H_i \rangle$.

It will suffice to show $N_G(T_1)$ acts on X_i for $i = 1$ and 2, since then $N_G(T_1)$ acts on $\langle X_1, X_2\rangle = L_0$, so $N_G(T_1) \leq N_G(L_0) = M$, as desired. Therefore we may assume $N_G(T_1) \not\leq N_G(X_i)$ for some i, and we now fix that value of i.

Set $K_j := \langle X_j^{G_j}\rangle$ and $K_j^*T^* := K_jT/O_2(K_j)$ for each j. Notice $O_2(K_j) \leq O_2(H_j) \leq T_1$ using A.1.6. Now $N_G(T_1) \leq G_j$ so $N_G(T_1)$ acts on K_j. Thus as $N_G(T_1) \not\leq N_G(X_i)$, $X_i < K_i$, and hence by 10.2.4 either $K_i = K_{i,1}K_{i,1}^s$ with $K_{i,1} \in \mathcal{C}(G_i)$ and $s \in T - N_T(K_{i,1})$, or $K_i^*T^* \cong Aut(L_k(2))$, $k := 4$ or 5. In either case, $N_G(T_1)$ acts on $R := T_1 \cap K_i$ and $O_2(X_i(T \cap K_i)) \leq R$.

Suppose first that either $K_i^* \cong L_k(2)$, or $K_{i,1}^* \cong L_2(2^m)$. As $H_i \cap K_i$ is T-invariant, $H_i \cap K_i \leq J_i$, where J_i^* is a T-invariant parabolic of K_i^* such that X_i is the characteristic subgroup generated by the elements of order 3 in J_i. Notice

$$O_2(J_i) \leq O_2(X_i(T \cap K_i)) \leq R \leq T \cap K_i. \qquad (*)$$

Now when $K_{i,1}^* \cong L_2(2^m)$, $T \cap K_i = O_2(J_i)$, so the inequalities in (*) are equalities, and then $N_G(T_1) \leq N_{G_i}(R) \leq N_{G_i}(J_i) \leq N_G(X_i)$, contrary to our assumption. On the other hand if $K_i^* \cong L_k(2)$, then $O_2(J_i^*)$ is a unipotent radical, and so by I.2.5 is weakly closed in $(T \cap K_i)^*$ with respect to G_j; thus $N_G(T_1) \leq N_{G_i}(O_2(J_i))) \leq N_G(X_i)$, for the same contradiction.

This leaves the case where $K_{i,1}^* \cong L_2(p)$. If $p \equiv \pm 3 \mod 8$, then again $T \cap K_i = O_2(X_i(T \cap K_i)) \leq R$, so $R = T \cap K_i$; and $N_G(T_1)$ normalizes $N_{K_i}(T \cap K_i) = X_i(T \cap K_i)$ and hence also $O^2(X_i(T \cap K_i)) = X_i$, for our usual contradiction. Therefore $p \equiv \pm 1 \mod 8$, and $(T \cap K_{i,1})^*$ is a nonabelian dihedral 2-group. Since $(X_i \cap K_{i,1})^*$ is a $T \cap K_i$-invariant A_4-subgroup of $K_{i,1}^*$, $|(T \cap K_{i,1})^*| = 8$.

Next R is of index $r \leq |T : T_1| = 2$ in $T \cap K_i$. Further if $r = 2$, then $O_2(X_i(T \cap K_i))^* = J(R^*)$, so $N_G(T_1) \leq N_{G_i}(O_2(X_i(T \cap K_i)) \leq N_G(X_i)$, again contrary to assumption. Therefore $R = T \cap K_i$ and there are exactly two subgroups Y of $K_{i,1}$ with $R \cap K_{i,1} \leq Y$ and $Y^* \cong S_4$. So $O^2(N_G(T_1))$ acts on both such subgroups, and in particular on $X_i \cap K_{i,1}$. Similarly $O^2(N_G(T_1))$ acts on $X_i \cap K_{i,1}^s$, and hence on the product X_i of these two subgroups, so $N_G(T_1) = TO^2(N_G(T_1)) \leq N_G(X_i)$, for our final contradiction. □

LEMMA 10.2.6. *(1)* $M = !\mathcal{M}(L_0T_1)$.
(2) $N_G(V_i) \leq M \geq N_G(L)$.

PROOF. Notice (1) implies (2), so it suffices to prove (1). Suppose that there is $H \in \mathcal{M}(L_0T_1) - \{M\}$. Then $|T : T_1| = 2$, and $N_G(T_1) \leq M$ by 10.2.5. By 1.2.7.3, $M = !\mathcal{M}(L_0T_+)$ for each $T_+ \in Syl_2(M)$, so that $T_1 \in Syl_2(H)$. Thus by 1.2.4, $L_i \leq K_i \in \mathcal{C}(H)$, and $K_i \trianglelefteq H$ by (+) in 1.2.4. Now from A.3.12, K_i does not contain $L_0 = L_1L_2$, so $K_1 \neq K_2$. Thus as $m_p(H) \leq 2$ for each prime divisor p of $|\bar{L}|$, while $L_{3-i} \leq C_H(K_i/O_2(K_i))$, we conclude $m_p(K_i) = 1$ for each such prime. As $H \neq M = N_G(L_0)$, L_0 is not normal in H, so $L_i < K_i$ for $i := 1$ or 2; we fix this value of i.

Now if $\bar{L} \cong Sz(2^n)$, A.3.12 says L_i is properly contained in no K_i with $m_p(K_i) = 1$ for each prime p dividing $2^n - 1$, and similarly L_i is proper in no K_i with $m_7(K_i) = 1 = m_3(K_i)$ when $\bar{L} \cong L_3(2)$. Therefore $\bar{L} \cong L_2(2^n)$.

Assume $F^*(K_i) = O_2(K_i)$. Set $H_0 := K_iL_{3-i}T_1$ and $R := O_2(L_0T)$. Then $L_0T_1 \leq M_0 := M \cap H_0$. As $M = !\mathcal{M}(L_0T)$, $C(H_0, R) \leq M_0$, and by A.4.2.7, $R \in \mathcal{B}_2(H_0)$ and $R \in Syl_2(\langle R^{M_0}\rangle)$. Thus Hypothesis C.2.3 is satisfied, so K_i is described in C.2.7.3. Comparing the list of possiblities for K_i appearing there such that $m_p(K_i) \leq 1$ for each $p \in \pi(|\bar{L}|)$ to the list of embeddings of $L_2(2^m)$ in A.3.12, we obtain a contradiction.

Therefore we may assume instead that $F^*(K_i) \neq O_2(K_i)$. By A.1.26, $V = [V, L_0]$ centralizes $O(K_i)$, so $O(K_i) \leq C_G(V) \leq M$. Then as $O_2(M) \leq T_1$, $[O_2(M), O(K_i)] \leq O_2(M) \cap O(K_i) = 1$, so $O(K_i) = 1$ as $M \in \mathcal{H}^e$. Thus as $K_i/O_\infty(K_i)$ is quasisimple, K_i is quasisimple. As L_i does not centralize V_i, $O_2(L_i) \not\leq Z(K_i)$. But now each possibile embedding of L_i in K_i in A.3.12 with $O_2(L_i) \not\leq Z(K_i)$ has $m_p(K_i) > 1$ for some odd prime p dividing $|\bar{L}|$, again contradicting our earlier observation. This completes the proof. □

At this point, we eliminate the sixth case of 10.1.1; this will avoid complications in the proof of 10.2.9.

LEMMA 10.2.7. *Case (6) of 10.1.1 does not hold. In particular,* $C_T(V) = O_2(L_0T)$.

PROOF. The second statement follows from the first by 10.1.2.3. Assume the first statement fails. Then $m := m(\bar{M}, V) = 4$ and $a := a(\bar{M}, V) = 2$. By Theorem E.6.3, $r := r(G, V) \geq m$, so $r \geq 4$ and $s := s(G, V) = 4$.

Indeed we show $r > 4$: For suppose $U \leq V$ with $m(V/U) = 4$ and $C_G(U) \not\leq M$. If $U \leq C_V(\bar{x})$ for some $\bar{x} \in \bar{M}^\#$, then $\bar{x}$ is an involution and $U = C_V(\bar{x}) \geq V_i$ for $i = 1$ or 2. But then $C_G(U) \leq C_G(V_i) \leq M$ by 10.2.6, a contradiction. Therefore $C_M(U) = C_M(V)$, and E.6.12 supplies a contradiction.

We observe next that 10.1.2.3 and 10.2.3.2 establish Hypothesis E.3.36. A maximal cyclic subgroup of odd order in $\bar{M}$ permuting with $\bar{T}$ is of order 15, so $n'(Aut_G(V)) = 4 < r$. Finally by 10.2.3.1, $n(H) \leq 2$ for each $H \in \mathcal{H}_*(T, M)$. Therefore by E.3.39.2,

$$2 = s - a \leq w \leq n(H) \leq 2$$

where $w := w(G, V)$ is the weak closure parameter defined in E.3.23. Thus $w = 2$. Let $A \leq V^g$ be a w-offender in the sense of Definition E.3.27. By E.3.33.4, $\bar{A} \in \mathcal{A}_2(\bar{M}, V)$. Thus $1 \neq C_{V_1}(N_A(V_1)) \leq C_V(A)$, so A acts on V_1. As $\bar{A} \in \mathcal{A}_2(\bar{M}, V)$, $\bar{A}$ centralizes $O(\bar{M})$ by E.3.40, so $m(A/C_A(V_1)) \leq m_2(Aut(\bar{L})) = 2$. Thus $m(V^g/C_A(V_1)) \leq w + 2 = 4 < r$, and hence $V_1 \leq C_G(C_A(V_1)) \leq M^g$. Similarly $V_2 \leq M^g$, so $V \leq M^g = N_G(V^g)$, contrary to E.3.25 since $w > 0$. □

LEMMA 10.2.8. *Assume* $\bar{L} \cong L_3(2)$ *and* $C_{V_1}(L) \neq 1$. *Set* $Q := C_T(V)$. *Then*

(1) $[Z, L] = 1$.

(2) $Z_Q := \Omega_1(Z(Q)) = Z_{T_1}V$, *where* $Z_{T_1} := \Omega_1(Z(T_1)) = C_{Z_Q}(L_0)$.

(3) $L = [L, J(T)]$, *and* $[Z, H] \neq 1$ *for each* $H \in \mathcal{H}_*(T, M)$.

(4) Set $\tilde{U}_i := C_{\tilde{V}_i}(T_1)$, *let* R_i *be the preimage in* T *of* $O_2(C_{\bar{L}_i}(\tilde{U}_i)$, $R := R_1R_2Q$, *and* $v_2 \in U_2 - C_{V_2}(L_0)$. *Then* $C_{\bar{L}_0\bar{T}}(v_2) \cong A_4 \times L_3(2)$, $R = J(T)Q$, $C_T(v_2) = (T \cap L)R_2Q$, *and* $\Omega_1(Z(C_T(v_2))) = Z_{T_1}\langle v_2 \rangle$.

PROOF. As $Z_{T_1} \leq C_T(V) = Q \leq T_1$, $Z_{T_1} = C_{Z_Q}(T_1)$. As $Z_i := C_{V_i}(L_0) \neq 1$ and $\tilde{V}_1$ is a natural module for $\bar{L}$, $Z_i \cong \mathbf{Z}_2$ by B.4.8.1. In particular $C_Z(L) \neq 1$, so (3) follows from 3.1.8.3, since $H \not\leq M = !\mathcal{M}(L_0T) = !\mathcal{M}(C_G(C_Z(L_0)))$.

By 1.4.1.5, $Z_Q = R_2(L_0T)$ with $Q = C_{L_0T}(Z_Q) = C_{L_0T}(V)$ and $V \leq Z_Q$. By (3), Z_Q is an FF-module for L_0T. As $V_1 \in Irr_+(Z_Q, L)$ with $C_{V_1}(L) \neq 1$, by part (1) of Theorem B.5.1, $V = [Z_Q, L_0]$, and that for any $A \in \mathcal{A}(T)$ with $L = [L, A]$ and $\bar{A}$ minimal subject to this constraint, $\bar{A} \leq \bar{L}$ and $Z_Q = V_1C_{Z_Q}(A)$.

By B.4.8.2, $\bar{A} = \bar{R}_1$ and $r_{Z_Q,\bar{A}} = 1$, so by B.4.8.4, $Z_Q = V_1 C_{Z_Q}(L)$. This shows $Z_Q = VC_{Z_Q}(L_0)$, $R = J(T)Q$, and $C_T(v_2) = (T \cap L)R_2Q$, establishing (4) except for its final assertion. Notice it also shows $Z \cap V \leq Z_{T_1} \cap V \leq C_V(L_0)$. But $T_1 = T_0Q$, so $C_{Z_Q}(L_0) \leq Z_{T_1}$. Conversely, $Z_{T_1} \leq Z_Q$ and we saw $V \cap Z_{T_1} \leq C_V(L_0)$, so $Z_{T_1} \leq C_{Z_Q}(L_0)$, and hence (2) holds. Further $Z \leq Z_{T_1}$, so (2) implies (1). Finally $Q \leq C_T(v_2)$ and $Q = F^*(L_0T)$, so $\Omega_1(Z(C_T(v_2))) \leq Z_Q = VZ_{T_1}$; therefore $\Omega_1(Z(C_T(v_2))) = Z_{T_1}C_V(C_T(v_2)) = Z_{T_1}\langle v_2\rangle$, completing the proof of (4), and hence of the lemma. □

We are now in a position to produce a crucial bound on the weak closure parameter r of Definition E.3.3:

PROPOSITION 10.2.9. *(1) $C_G(v) \leq M$ for each $v \in V_i^\#$.*
(2) $r(G,V) \geq m(V_i)$.
(3) If $v \in V_i - C_{V_i}(L_0)$, then $C_G(v) \leq N_M(V_i)$.

PROOF. Part (3) follows from (1) and the fact that M permutes $\{V_1, V_2\}$ and $V_1 \cap V_2 = C_V(L_0)$. Also (1) implies (2), so it remains to prove (1).

Let $v \in V_2^\#$, and suppose by way of contradiction that $H := C_G(v) \not\leq M$. Without loss $T_v := C_T(v) \in Syl_2(C_M(v))$. By 10.2.6.1, $v \notin C_{V_2}(L_0T_1)$.

We claim first that $N_G(T_v) \leq M$. If $J(T) \leq C_T(V)$, this follows from 3.2.10.8; so by 10.1.2.1 we may assume that one of the first three cases of 10.1.1 holds. Suppose first that case (3) of 10.1.1 holds, and also $C_{V_1}(L) \neq 1$. Then by 10.2.8.2, $Z_{T_1} := \Omega_1(Z(T_1)) \geq C_V(L_0)$, so $v \notin C_V(L_0)$ using our observation in the previous paragraph. Therefore as L_2 is transitive on $\tilde{V}_2^\#$, we may assume $\langle\tilde{v}\rangle = C_{\tilde{V}_2}(T_1)$. Hence by 10.2.8.4, $T_v = (T \cap L)R_2Q$, and $Z_v := \Omega_1(Z(T_v)) = Z_{T_1}\langle v\rangle$. By 10.2.8.1, L_0 centralizes Z, so $C_G(Z_v) \leq C_G(Z) \leq M = !\mathcal{M}(L_0T)$, and hence by 10.1.3, L is the unique member of $\mathcal{C}(C_G(Z_v))$ of order divisible by 3. Therefore $N_G(T_v) \leq N_G(Z_v) \leq N_G(L) \leq M$ using 10.2.6.2. We now turn to the remaining subcase of case (3) of 10.1.1, where $C_{V_1}(L) = 1$. Then $T_v = T_1$, so $N_G(T_v) \leq M$ by 10.2.5. Finally in cases (1) and (2) of 10.1.1, $S \leq T_1$ by 10.1.2.6; and in case (2), S centralizes both singular and nonsingular vectors. So in either case, $S \leq T_v$. Therefore $S = \mathrm{Baum}(T_v)$ and $N_G(T_v) \leq N_G(S)) \leq M$ by 10.2.1. This completes the proof of the claim.

As $N_G(T_v) \leq M$ by the claim, while we chose $T_v \in Syl_2(C_M(v))$, $T_v \in Syl_2(H)$. Also $L \leq H$, so by 1.2.4, $L \leq I \in \mathcal{C}(H)$, with $I \trianglelefteq H$ by (+) in 1.2.4. By 10.2.6, $N_G(L) \leq M$, so $L < I$ and hence $I \not\leq M$. Thus I is described in A.3.12.

Suppose first that I is quasisimple. Then $V_1 \cap Z(I) \leq C_{V_1}(L)$, so $\tilde{V}_1 \cong V_1/C_{V_1}(L)$ is a subquotient of $R_2(LZ(I)/Z(I))$. Inspecting the list in A.3.12 for embeddings with such a subquotient appearing in 10.1.1, we conclude that case (1) or (3) of 10.1.1 holds; and keeping in mind that $N_G(V_1) \leq M$ so that $L \trianglelefteq N_I(V_1)$, we conclude that either:

(i) $\bar{L} \cong L_2(2^n)$, and either $I/Z(I)$ is of Lie type and Lie rank 2 over $\mathbf{F}_{2^n}$, or $n = 2$ and $I/Z(I)$ is M_{22}, $\hat{M}_{22}$, or M_{23}; or

(ii) case (3) of 10.1.1 holds with $C_{V_1}(L) \leq Z(I)$, and $I/Z(I)$ is $L_4(2)$, $L_5(2)$, M_{24}, J_4, HS, or Ru.

In particular either $C_T(L) = C_T(I)$, or $I \cong Sp_4(2^n)$ in (i), using I.1.3 to conclude the Schur multiplier of $Sp_4(2^n)$ is trivial when $n > 1$. When $C_T(L) = C_T(I)$, $V_2 \leq C_T(L) = C_T(I)$, so $I \leq C_G(V_2) \leq M$ by 10.2.6, contradicting $I \not\leq M$. On the other hand if $I \cong Sp_4(2^n)$, then L is indecomposable on $O_2(L)$, so $V_1 = O_2(L)$. Then there is $X \leq N_I(L)$ of order $2^n - 1$ centralizing L/V_1 and faithful on V_1. Thus $X \leq N_G(L) \leq M$, so $X \leq L_0$ by 10.1.3, impossible as there is no such subgroup of L_0.

Thus I is not quasisimple. So $E(I) = 1$ by A.3.3.1. We claim $F^*(IT_v) = O_2(IT_v)$: If not, then $O(I) \neq 1$ as $E(I) = 1$. But by A.1.26.1, $V_1 = [V_1, L]$ centralizes $O(I)$, so $O(I) \leq M$ by 10.2.6.2, and hence $O(C_M(v)) \neq 1$, a contradiction as $C_M(v) \in \mathcal{H}^e$ by 1.1.3.2.

We have shown that $F^*(IT_v) = O_2(IT_v)$. So $V_I := \langle C_{V_1}(T_v)^I \rangle \in \mathcal{R}_2(IT_v)$ by B.2.14. Let $(IT_v)^* := IT_v/C_{IT_v}(V_I)$. Now $V_v := \langle C_{V_1}(T_v)^L \rangle \leq V_I$, and from the action of L_0 on V in 10.1.1, either $V_1 = V_v$ or case (3) of 10.1.1 holds with $C_{V_1}(T_v) = C_{V_1}(L_0) \neq 1$ and $V_v = C_{V_1}(L_0)$. Therefore either $C_V(L_0) \neq 1$, or $N_G(V_v) \leq M$ by 10.2.6.2. In the former case, $1 \neq C_Z(L_0) \leq V_I$, so $C_G(V_I) \leq C_G(C_Z(L_0)) \leq M = !\mathcal{M}(L_0T)$; in the latter, $C_G(V_I) \leq C_G(V_v) \leq M$. So in any case, $C_G(V_I) \leq M$, and hence $L^* < I^*$ as $I \not\leq M$, while $L^* \neq 1$ as $I = \langle L^I \rangle$.

Next observe that $J(T) \leq T_v$, so that $J(T) = J(T_v)$ and $S = \mathrm{Baum}(T_v)$: If $J(T) \leq C_T(V)$ this is clear, so by 10.1.2.1 we may assume that one of the first three cases of 10.1.1 holds. But in each of these cases v centralizes some M-conjugate of $J(T)$, so again the remark holds.

We next claim that $I^* = [I^*, J(T_v)^*]$ is quasisimple. Suppose not, so that either $[V_I, J(T_v)] = 1$ or I^* is not quasisimple. Suppose first that $J(T_v)^* \neq 1$. Then I^* is not quasisimple, so I^* is described in case (c) or (d) of 1.2.1.4, and hence $[X^*, J(T_v)^*] \neq 1$ for $X := \Xi_p(I)$ and some prime $p > 3$, contradicting Solvable Thompson Factorization B.2.16. Thus we may take $J(T_v)^* = 1$. However $L^* \neq 1$, so $J(T) \leq O_2(LT_v)$ and hence $J(T) \trianglelefteq L_0T$, so that $N_G(J(T)) \leq M$. Then by a Frattini Argument, $I = C_I(V_I)N_I(J(T)) \leq M$, contradicting $I \not\leq M$. So the claim is established.

By the claim, V_I is an FF-module for $I^*T_v^*$. Now intersecting the list of possibilities for the embedding of L^* in I^* in A.3.12 with the list of B.4.2, we get the following cases:

(a) $\bar{L} \cong L_2(2^n)$, $I^* \cong SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$, and $O_2(L^*) \neq 1$.
(b) $\bar{L} \cong A_5$ or $L_3(2)$, and $I^* \cong A_7$ with $O_2(L^*) = 1$.
(c) $\bar{L} \cong L_3(2)$ and $I^* \cong L_4(2)$ or $L_5(2)$, with $O_2(L^*) \neq 1$.

Observe in particular that I does not appear in case (c) or (d) of 1.2.1.4, so $I/O_2(I)$ is quasisimple.

Assume case (a) holds. Recall we saw earlier that $V_1 = V_v \leq V_I$ and the FF-module V_I is described in Theorem B.5.1. Then $L = N_I(V_1)^\infty$ and $N_{I^*}(V_1)$ is a maximal parabolic of I^*, so $N_I(L)$ contains a subgroup X of order $2^n - 1$ centralizing $L/O_2(L)$ and nontrivial on V_1. We now get a contradiction much as in the earlier case of $Sp_4(2^n)$ where I was quasisimple: for $X \leq N_G(L) \leq M$, and hence $X \leq L_0$ by 10.1.3, whereas there is no such subgroup of L_0.

Thus we have shown that (b) or (c) holds, so $\bar{L} \cong A_5$ or $L_3(2)$. We next show:
In case (b) either

(b1) I is an exceptional A_7-block, $I^*T_v^* \cong A_7$, and V_I is the natural module for $L^* \cong L_2(4)$, or an indecomposable of rank 3 or 4 for $L^* \cong L_3(2)$, or

(b2) I is an A_7-block, $I^*T_v^* \cong S_7$, and $[V_I, L]$ is the A_5-module for $L^* \cong A_5$.

For assume case (b) holds. We saw that $S = \mathrm{Baum}(T_v)$, so applying C.1.24 with I, T_v, T_v in the roles of "L, T, R", either I is an A_7-block or an exceptional

A_7-block, or there is a nontrivial characteristic subgroup C of S normal in IT_v. However in the last case $G_0 := \langle I, T\rangle \leq N_G(C)$, so as $L \leq I$, $L_0T \leq G_0$ and hence $I \leq G_0 \leq M = !\mathcal{M}(L_0T)$. This contradicts $I \not\leq M$, so I is a block. Further if I is an A_7-block, then as $I = [I, J(T_v)]$, $I^*T_v^* \cong S_7$, so $L/O_2(L)$ is not $L_3(2)$ as $L \in \mathcal{L}(IT_v, T_v)$. If I is an A_7-block, then I^* is self-normalizing in $GL(V_I)$, so $I^*T_v^* = I^*$. Thus (b1) or (b2) holds.

In particular in case (b), $O_2(I) = C_I(V_I)$. In case (c) since $I/O_2(I)$ is quasisimple, the list of Schur multipliers in I.1.3 says $I/O_2(I) \cong I^*$, so again $O_2(I) = C_I(V_I)$.

Assume $\bar{L} \cong L_3(2)$; this argument will be fairly lengthy. By 10.2.7, case (3) or (5) of 10.1.1 holds. In case (b), subcase (b1) holds; so L^* is self-normalizing in $I^*T_v^* \cong A_7$, and hence T_v induces inner automorphisms on $\bar{L}$ so that case (3) of 10.1.1 holds. Similarly in case (c): if $I^* \cong L_4(2)$, then $L^* \cong L_3(2)/E_8$, and so T_v induces inner automorphisms on $\bar{L}$ and L^* is self-normalizing in I^*; while if $I^* \cong L_5(2)$, then either T_v induces inner automorphisms on $\bar{L}$, or $I^*T_v^* \cong Aut(L_5(2))$, L^* is the T_v-invariant nonsolvable rank-2 parabolic, and L^* is self-normalizing in I^*. Except in this last case, case (3) of 10.1.1 holds.

Set $Y := O^2(C_{L_2}(v))$. In case (3) of 10.1.1, $Y/O_2(Y) \cong \mathbf{Z}_3$. In case (5) of 10.1.1, either $Y/O_2(Y) \cong \mathbf{Z}_3$, or v is diagonally embedded in the two summands with $Y = 1$, and $T_v = T_1$ with $LT_v/O_2(LT_v) \cong Aut(L_3(2))$.

Suppose $Y \neq 1$. By A.3.18, $I = O^{3'}(H)$ so $Y \leq N_I(L)$. As we saw $C_I(V_I) = O_2(I)$, $1 \neq Y^* \leq N_{I^*}(L^*)$ and $Y^* \not\leq L^*$. Thus $L^* < O^2(N_{I^*}(L^*))$, so by the previous two paragraphs, $I^*T_v^* \cong L_5(2)$, $Y^*L^*T_v^* \cong S_3 \times L_3(2)$, and case (3) of 10.1.1 holds. On the other hand if $Y = 1$, then by the previous two paragraphs, case (5) of 10.1.1 holds, and $I^*T_v^* \cong Aut(L_5(2))$. Therefore in any case for Y, $I^* \cong L_5(2)$.

Suppose that $C_V(L_0) \neq 1$. Then case (3) of 10.1.1 holds by 10.1.2.4, so by the previous paragraph, $LYT_v/O_2(LYT_v) \cong L_2(2) \times L_3(2)$, contrary to 10.2.8.4, which says that $LYT_v/O_2(LYT_v) \cong \mathbf{Z}_3 \times L_3(2)$.

Therefore $C_V(L_0) = 1$. By B.4.2 and Theorem B.5.1 V_I is either an irreducible of rank either 5 or 10, the sum of the 5-dimensional module and its dual, or the sum of isomorphic 5-dimensional modules. If $Y = 1$, we saw that $I^*T_v^* \cong Aut(L_5(2))$ and L^* is the nonsolvable T_v^*-invariant rank 2 parabolic. Thus $V_I = V_{I,1} \oplus V_{I,2}$ with $V_{I,1}$ a natural I^*-submodule and $V_{I,2}$ its dual. But we also saw that case (5) of 10.1.1 holds, and in that case we saw that $V_v = V_1 \leq V_I$. However V_1 is the sum of a natural module for $\bar{L}$ and its dual, whereas the parabolic L^* has no such submodule on V_I.

Thus $Y \neq 1$, $I^*T_v^* \cong L_5(2)$, and $L^*Y^*T_v^* \cong S_3 \times L_3(2)$. In case (5) of 10.1.1, $V_1 \leq V_I$ and V_1 is the sum of a natural module for $\bar{L}$ and its dual. However examining the possibilities for V_I listed above, we see that the parabolic $L^*Y^*T_v^*$ has no such submodule.

Therefore case (3) of 10.1.1 holds. Since $C_V(L_0) = 1$, V_1 is the natural module for L. But from the our list of possibilities for V_I, each natural submodule for L is contained in an I-irreducible. Thus as $V_I = \langle V_1^I\rangle$, V_I is an I-irreducible, and hence $\dim(V_I) = 5$ or 10.

Again since $C_V(L_0) = 1$, $T_v = T_1$, so that T normalizes T_v. Let $t \in T - T_v$, $u := v^t$, and $E := \langle u, v\rangle$. Then $\langle u\rangle = C_{V_1}(T_v)$ and $C_{G_v}(E) = C_{G_v}(u)$. Since V_I is an irreducible of dimension 5 or 10, $C_{I^*T_v^*}(u)$ is a maximal parabolic of $I^*T_v^*$, and so from the structure of such parabolics,

$$C_{IT_v}(E) = O^{3'}(C_G(E))T_v \leq I^tT_v,$$

as $I^t = O^{3'}(C_G(u))$ since $I = O^{3'}(H)$. Then $C_{IT_v}(E) = C_{I^tT_v}(E)$, so that t acts on $C_{IT_v}(E)$.

Let P be the rank one parabolic of IT_v over T_v not contained in M, and let P_c and P_f be the rank one parabolics of L centralizing and not centralizing u, respectively. Observe that as $L^t = L_2$, t interchanges Y and P_c. If $m(V_I) = 10$, then $C_{I^*T_v^*}(u)$ is an $L_3(2)\times L_2(2)$ parabolic and $C_{IT_v}(u) = \langle Y, P\rangle P_c$. Therefore as t interchanges Y and P_c, and t acts on $C_{IT_v}(E) = C_{IT_v}(u)$ by the previous paragraph, $P = P^t$. This is impossible, as $\langle Y, P\rangle$ is of type $L_3(2)$, while PP_c is of type $L_2(2)\times L_2(2)$. Therefore $m(V_I) = 5$, and $C_{IT_v}(u) = \langle Y, P, P_c\rangle$ is of type $L_4(2)$; again $P^t = P$, and as P_f acts on $O^2(P)$, so does P_f^t. This is impossible, as P centralizes E, but $P_fP_f^t$ contains a E_9-subgroup D with $C_E(D) = 1$ so $m_3(DO^2(P)) = 3$, contradicting $DO^2(P)$ an SQTK-group. This concludes the treatment of the case $\bar{L} \cong L_3(2)$.

Therefore $\bar{L} \cong L_2(4)$ and case (b1) or (b2) holds. In (b1), $V_1 = V_I \trianglelefteq I$, so $I \leq N_G(V_1) \leq M$ by 10.2.6.2, contrary to $I \not\leq M$. In (b2), $[V_I, L]$ is the A_5-module, so case (2) of 10.1.1 holds with $V_1 = [V_I, L]$. Then $Y := O^{3'}(C_{L_2}(v)) \neq 1$, and $Y \leq I$ as $O^{3'}(H) = I$ by A.3.18. Hence $1 \neq Y^* \leq N_{I^*}(L^*)$ but $Y^* \not\leq L^*$, contradicting $L^* = O^2(N_{I^*}(L^*)$. This contradiction finally completes the proof of 10.2.9. □

LEMMA 10.2.10. *(1) For $g \in G - M$, $V_2 \cap V_2^g = 1$.*
(2) If $C_V(L_0) = 1$, then V_i is a TI-set in G.

PROOF. As M permutes $\{V_1, V_2\}$ transitively and $V_1 \cap V_2 = C_V(L_0)$, (1) implies (2).

Suppose $g \in G$ with $1 \neq v \in V_2 \cap V_2^g$. By 10.2.9.1, $C_G(v) \leq M \cap M^g$. Let p be an odd prime divisor of $|\bar{L}|$, and for $X \leq G$ let $\theta(X) := O^{p'}(X^\infty)$. By 10.1.3, $L_0 = \theta(M)$, so $L^g \leq L_0$; and $L_0 \leq L_0^g$ if $v \in C_{V_2}(L_0)$. In the latter case $g \in N_G(L_0) = M$, so we may assume $v \notin C_{V_2}(L_0)$. Thus $L = \theta(C_{L_0}(v))$, so $L^g = L$. Then $g \in M$ by 10.2.6.2, establishing (1). □

LEMMA 10.2.11. *Assume case (3) of 10.1.1 holds with $C_V(L_0) = 1$. Let $1 \neq v_i \in C_{V_i}(T_1)$, set $E := \langle v_1, v_2\rangle$, and $z := v_1v_2$. Let $G_z := C_G(z)$, $X := O^2(C_{L_0}(z))$, $K_z := \langle X^{G_z}\rangle$, and $V_z := \langle E^{G_z}\rangle$. Then*
(1) $V_z \leq Z(O_2(G_z))$ and $C_{G_z}(V_z) \leq N_M(V_1)$.
(2) If $X < K_z$ then $V_z \in \mathcal{R}_2(G_z)$.
(3) $V \leq O_2(G_z)$.

PROOF. By construction, $z \in Z(T)$, so $G_z \in \mathcal{H}^e$ by 1.1.4.6. As $XT \leq G_z$, $O_2(G_z) \leq O_2(XT)$ by A.1.6; then as $O_2(XT) \leq T_1 \leq C_{G_z}(E)$, $V_z \leq Z(O_2(G_z))$. Further

$$C_{G_z}(V_z) \leq C_{G_z}(v_1) \leq N_M(V_1)$$

by 10.2.10, since by hypothesis $C_V(L_0) = 1$, so (1) holds.

Set $G_z^* := G_z/C_{G_z}(V_z)$ and let R denote the preimage in T of $O_2(G_z^*)$. By a Frattini Argument, $G_z = C_{G_z}(V_z)N_G(R)$. Thus if $R \leq T_1$, then R centralizes E, and hence also $\langle E^{N_{G_z}(R)}\rangle = V_z$, so that $V_z \in \mathcal{R}_2(G_z)$. Thus to prove (2), we may assume $R \not\leq T_1$. In particular $[X, R] \not\leq O_2(X)$, so as T is irreducible on $X/O_2(X)$ and normalizes R, $X = [X, R]$. Thus $X^* = [X^*, R^*] \leq R^*$, so X^* is a 2-group and hence $X = O^2(X) \leq C_{G_z}(V_z)$. By 10.1.3, $X = O^{3'}(G_z \cap M)$, so by (1), $X = O^{3'}(C_{G_z}(V_z)) \trianglelefteq G_z$ and hence $X = K_z$, establishing (2).

Assume (3) fails. If V centralizes V_z, then as $C_{G_z}(V_z) \leq M$ by (1), $V \leq O_2(C_{G_z}(V_z)) \leq O_2(G_z)$, contrary to assumption. Hence as XT is irreducible on V/E, $E = C_V(V_z)$. If $X \trianglelefteq G_z$, then as X centralizes E, it centralizes V_z; then $V = [V,X]E$ centralizes V_z, a contradiction. Thus $X < K_z$, and hence $K_z \not\leq M$ so $V_z \in \mathcal{R}_2(G_z)$ by (2).

As $E = C_V(V_z)$, V_1^* is a 4-group. By (1), $V_z \leq N_M(V_1)$, so $[V_z, V_1] \leq V_z \cap V_1 = \langle v_1 \rangle$. That is V_1^* is a 4-group inducing transvections on V_z with center v_1. Further $K_z^*T^*$ is described in case (2) or (3) of 10.2.4. Appealing to G.3.1, the only group $K_z^*T^*$ listed there containing a 4-group of $\mathbf{F}_2$-transvections with a fixed center in some representation is $L_3(2)$ wr $\mathbf{Z}_2$ with $[V_z, K_z] = V_{z,1} \oplus V_{z,2}$, where $V_{z,i} := [V_z, K_{z,i}]$ is a natural module. However in that case, $V_i^* \leq K_{z,i}^*$ with $v_i = [V_z, V_i^*] \leq V_{z,i}$, so $z = v_1v_2 \in [V_z, K_z]$, which is impossible as $z \in Z(G_z)$ but $C_{[V_z,K_z]}(K_z^*) = 1$. This contradiction completes the proof. □

We can now prove our major weak closure result, which establishes an effective lower bound on the parameter $w(G,V)$.

PROPOSITION 10.2.12. *One of the following holds:*

(1) $w(G,V) > 2$.

(2) $w(G,V) = 2$, and case (3) of 10.1.1 holds.

(3) $w(G,V) = 2$, and case (1) of 10.1.1 holds with $n = 2$.

PROOF. In case (3) of 10.1.1, and in case (1) when $n = 2$, set $j := 1$. Otherwise set $j := 2$. We must prove $w(G,V) > j$, so we may assume $A := V^g \cap M$ with $k := m(V^g/A) \leq j$ and $[V, V^g] \neq 1$, and it remains to derive a contradiction.

Let $m := m(\tilde{V}_1)$ and $a := a(Aut_M(V_1), V_1)$. Observe $m > j+1$. Recall $a \leq m_2(Aut_M(V_1))$ and in case (2) of 10.1.1, $a = 1$. Thus $k < m - a$ unless case (3) of 10.1.1 holds and $k = 1$.

For $i = 1, 2$, set $A_i := V_i^g \cap A$ and $B_i := N_{A_i}(V_1)$. Suppose A_1A_2 centralizes V_1. Then by 10.2.9.1, $V_1 \leq N_{M^g}(V_i^g)$, so $C_{V_1}(V_i^g) \neq 1$ since $m(V_i) < m_2(Aut_M(V_1))$ in each case. Then $A = V^g$ by another application of 10.2.9.1. But then $V^g = A_1A_2 \leq C_M(V_1) = C_M(V)$, contrary to our choice of V^g. Thus we may assume A_i does not centralize V_1 for some choice of $i := 1$ or 2.

Next $m(V_i^g/A_i) \leq k$ with $m(A_i/B_i) \leq 1$, so $m(V_i^g/B_i) \leq k+1 < m = m(V_i^g/C_{V_i^g}(L_0^g))$ by paragraph two. Thus $B_i \not\leq C_{V^g}(L_0^g)$, so there exists $b \in B_i - C_{V^g}(L_0^g)$. For each such b and each $r = 1, 2$, we may apply 10.2.9.1 to get

$$C_{V_r}(b) \leq N_{V_r}(V_i^g) =: U_r,$$

so $V_0 := [A_i, C_{V_1}(b)] \leq V_i^g \cap V$ and $[B_i, C_{V_1}(b)] \leq V_i^g \cap V_1 = 1$ by 10.2.10.1. Thus if $V_0 \neq 1$ then $A_i > B_i$ and $V \leq C_G(V_0) \leq M^g$ by 10.2.9.1. Thus $[A_i, V] \leq V^g \cap V \leq C_V(b)$, and as $A_i > B_i$, for any $a \in A_i - B_i$, $V = [a, V]V_2$, so b centralizes V/V_2. Thus $b \in C_T(V/V_2) = C_T(V_1)$, so $V_1 = C_{V_1}(b)$ and $V = V_0V_2$. Then by 10.2.9.1,

$$L = \langle C_L(v_0) : v_0 \in V_0^{\#} \rangle \leq M^g,$$

so $L_0 = \langle L^{A_i} \rangle \leq M^g$ and hence $L_0 = L_0^g$ by 10.1.3, contradicting $g \notin M$. Therefore $V_0 = 1$, so

$$C_{V_1}(A_i) = C_{V_1}(b). \qquad (*)$$

As $C_{V_1}(b) \not\leq C_{V_1}(L)$ from the structure of the modules in 10.1.1, A_i acts on V_1 by (*), so $A_i = B_i$. Then as A_i does not centralize V_1, (*) says

$$Aut_{A_i}(V_1) \in \mathcal{A}_{m-k}(Aut_M(V_1), V_1).$$

Thus $k \geq m-a$, so by paragraph two, case (3) of 10.1.1 holds with $w(G,V) = k = 1$. Hence $V \not\leq M^g$ by E.3.25.

Assume first that $C_V(L_0) \neq 1$. Then $m(V_1) = 4$ by I.1.6, so $m(A_i) \geq 3$ as $k = 1$, and hence $C_{A_i}(V_1) \neq 1$ as $m_2(Aut_M(V_1)) = 2$. But then $V_1 \leq M^g$ by 10.2.9.1, and similarly $V_2 \leq M^g$, contradicting $V \not\leq M^g$.

Therefore $C_V(L_0) = 1$, so V_1 is a TI-set in G by 10.2.10.2. As $Aut_{A_i}(V_1) \in \mathcal{A}_2(Aut_M(V_1), V_1)$, $Aut_{A_i}(V_1)$ is a 4-group of transvections with a fixed axis U_1, so $A_i \cong E_4 \cong U_1$.

Set $I := \langle V_i^g, V_1 \rangle$. We've shown that

$$A_i = B_i = N_{V_i^g}(V_1) \neq 1 \neq U_1 = N_{V_1}(V_i^g).$$

By I.6.2.2a, $O_2(I) = A_i \times U_1$ is of rank 4 with $C_I(V_1) = U_1$, and as $|V_1 : U_1| = 2$, $I/O_2(I)$ is dihedral of order $2d$, with d odd. As $D_{2d} \leq GL_4(2)$, $d = 3$ or 5. Now A_i is of index 2 in V_i^g, so as $k = 1$, $A = A_1A_2\langle c \rangle$ with $c = c_1c_2$, where $c_i \in V_i^g - A_i$. Further as $I/O_2(I) \cong D_{2d}$, there is an involution in I interchanging V_1 and V_i^g, and $U := V \cap M^g = U_1U_2\langle w \rangle$, where $w = w_1w_2$ with $w_r \in V_r - U_r$. If w acts on V_i^g then $1 \neq [A_i, w] \leq V_i^g \cap V$, so that $V \leq C_G([w, A_i]) \leq M^g$ by 10.2.9.1, contrary to an earlier reduction. Thus w interchanges L_1^g and L_2^g, so by symmetry, $L^c = L_2$. Now $[c, U_1] \leq V^g$ is diagonally embedded in V, so we may take $z \in Z^\#$ to lie in $[c, U_1] \leq V^g$. Then $V, V^g \leq G_z$, so $I \leq G_z$. Hence as $V_r \not\leq O_2(I)$, $V \not\leq O_2(G_z)$, contradicting 10.2.11.3. This completes the proof. $\square$

COROLLARY 10.2.13. *Case (3) of 10.1.1 holds with $w(G,V) = 2 = n(H)$ for each $H \in \mathcal{H}_*(T,M)$.*

PROOF. Take $H \in \mathcal{H}_*(T,M)$. By 10.1.2.3 and 10.2.3.2, Hypothesis E.3.36 holds. By 10.2.9.2, $r(G,V) \geq m(V_1)$ and it is easy to check in each case of 10.1.1 that $n'(\bar{M}) < m(V_1)$. Thus the hypotheses of lemma E.3.39 are satisfied. By 10.2.3.1, $n(H) \leq 2$, with $n(H) = 1$ in case (1) of 10.1.1. Thus by E.3.39.1, $w(G,V) \leq n(H) \leq 2$, so 10.2.12 completes the proof of the corollary. $\square$

10.3. The final contradiction

LEMMA 10.3.1. *(1) $C_V(L_0) = 1$.*
(2) V_i is a TI-set in G.

PROOF. By 10.2.10.2, (1) implies (2). Thus we may assume $C_V(L_0) \neq 1$, and it remains to derive a contradiction. Let $H \in \mathcal{H}_*(T,M)$ and set $U := \langle Z^H \rangle$ and $H^* := H/C_H(U)$. By 10.2.13, case (3) of 10.1.1 holds, so by 10.2.8.1, $C_H(U) \leq C_G(Z) \leq M = !\mathcal{M}(L_0T)$, and hence $H^* \neq 1$. By 10.2.13, $n(H) = 2$, so by 10.2.3.3, $H^* \cong S_5$ wr $\mathbf{Z}_2$ and $O^2(H \cap M)T_0$ is a maximal parabolic of L_0. In particular by 10.2.8.1,

$$[O^2(H \cap M), Z] \leq [L_0, Z] = 1.$$

Then as 3-elements are fixed-point-free on natural modules for $L_2(4)$, any $I \in Irr_+(H,U)$ satisfies either

(a) $I = I_1 \oplus I_2$, where $I_i := [I, K_i]$ is the A_5-module for $K_i \in \mathcal{C}(H)$, or
(b) $I = I_1 \otimes I_2$ is the tensor product of A_5-modules I_i for K_i.

In either case we compute directly that $a(H^*, I) = 1$. But by 10.2.9, $r(G,V) \geq m(V_1)$ and $m(V_1) = 4$ by I.1.6, so $s(G,V) = m(\bar{M}, V) = 2$ using B.4.8.2. Set $W_0 := W_0(T,V)$. By 10.2.13, $w(G,V) > 0$, so $N_G(W_0) \leq M$ by E.3.16. If $W_0^* = 1$, then

$W_0 \leq O_2(H)$ by B.6.8.3d, so $W_0 = W_0(O_2(H), V)$ and then $H \leq N_G(W_0) \leq M$, contradicting $H \not\leq M$. Thus $W^* \neq 1$; since $s(G,V) = 2$, we must have $a(H^*, I) \geq 2$ by E.3.18, contradicting $a(H^*, I) = 1$. □

LEMMA 10.3.2. *$C_G(z) \not\leq M$ for $z \in Z^\# \cap V$.*

PROOF. Assume that $C_G(z) \leq M$. We first prove that V is a TI-subgroup of G: For as $C_V(L_0) = 1$ by 10.3.1.1, each diagonal involution in V is conjugate in L_0 to z, and hence has centralizer contained in M by hypothesis. By 10.2.9.1, centralizers of nondiagonal involutions are contained in M. Thus these involutions are not 2-central in G, so they are not fused in G to diagonal involutions, and hence M controls fusion of involutions in V. Therefore V is a TI-set in G by I.6.1.1.

As V is a TI-subgroup of G, $r(G,V) = m(V) = 6$. Let A be a w-offender on V. By 10.2.13, $w(G,V) = 2$, so as $m_2(Aut_G(V)) = 4$, $m(\bar{A}) = 4$ by E.3.28.2. But as V is a TI-subgroup of G, I.6.2.2a says that $C_V(a) = V \cap M^g$ for each $a \in A^\#$. This is impossible as no rank-4 subgroup of $\bar{M}$ satisfies $C_V(\bar{a}) = C_V(\bar{A})$ for each $\bar{a} \in \bar{A}^\#$. This contradiction completes the proof. □

By 10.2.13 and 10.3.1.1, the hypotheses of 10.2.11 hold. So for the remainder of the section, we adopt the notation of that lemma; in particular, we study the group $K_z = \langle X^{G_z} \rangle$.

LEMMA 10.3.3. *$K_zT/O_2(K_zT) \cong S_5 \;\; wr \;\; \mathbf{Z}_2$.*

PROOF. We first observe that if $Y \in \mathcal{H}(T)$ is generated by $N_Y(T)$ and a set Δ of minimal parabolics D such that $n(D) = 1$ for each $D \in \Delta$, then $Y \leq M$ by Theorem 3.3.1 and 10.2.13. In particular each solvable member H of $\mathcal{H}(T)$ is contained in M by E.1.13 and B.6.5, since $H = O^{2'}(H)N_H(T)$ by a Frattini Argument.

Let $J := G_z^\infty$. By a Frattini Argument, $G_z = JN_{G_z}(T \cap J)$, and as G_z/J and $N_J(T \cap J)$ are solvable, $N_{G_z}(T \cap J)$ is a solvable member of $\mathcal{H}(T)$. Therefore $N_{G_z}(T \cap J) \leq M$ by the previous paragraph, so $J \not\leq M$ by 10.3.2. Hence by 1.2.1.1 there is $I \in \mathcal{C}(G_z)$ with $I \not\leq M$.

Suppose $I/O_2(I)$ is a Bender group. Then a Borel subgroup of $I_0 := \langle I^T \rangle$ lies in M by the first paragraph, so $I_0T \in \mathcal{H}_*(T,M)$. Hence by 10.2.13, $n(I) = 2$. Then by 10.2.3.3, $I_0T/O_2(I_0T) \cong S_5$ wr $\mathbf{Z}_2$ and $X \leq I_0$, so $I_0 = K_z$, and the lemma holds.

Therefore we may assume $I/O_2(I)$ is not a Bender group. Suppose next that $X = K_z$. As G_z is an SQTK-group, $m_3(IX) \leq 2$, so I is a $3'$-group. Thus $I/O_2(I)$ is a Suzuki group and hence a Bender group, contrary to our assumption.

Thus $X < K_z$, so by 10.2.4, $K_z = \langle I^T \rangle$, for $I \in \mathcal{C}(G_z)$ with $I \not\leq M$, and as $I/O_2(I)$ is not a Bender group, $I/O_2(I) \cong L_2(p)$ for an odd prime $p > 5$, $L_4(2)$, or $L_5(2)$. But then K_zT is generated by $N_{K_zT}(T)$ and minimal parabolics D with $n(D) = 1$, contrary to an earlier remark. □

We are now in a position to obtain our final contradiction.

By 10.3.3, $K_z = \langle K^T \rangle$ with $K \in \mathcal{L}(G,T)$ and $K/O_2(K) \cong A_5$. In particular $X < K_z$, so by 10.2.11.2, $V_z \in \mathcal{R}_2(G_z)$. Thus $K \in \mathcal{L}_f(G,T)$.

Let $M_+ \in \mathcal{M}(K_zT)$ and set $J_z := \langle X^{M_+} \rangle$. As $X < K_z \leq J_z$, by 10.2.4, $J_z = \langle I_z^T \rangle$ for $I_z \in \mathcal{C}(M_+)$. Furthermore arguing as in the proof of 10.3.3, J_z is not generated by minimal parabolics D with $n(D) = 1$, so from 10.2.4, $I_z/O_2(I_z) \cong$

$L_2(2^m)$ with $m \geq 2$. However the embedding $K < I_z$ does not occur in the list of A.3.14, so we conclude that $K_z = J_z$. Therefore $K \in \mathcal{L}_f^*(G,T)$ with $M_+ = N_G(K_z)$. Thus the hypotheses of Theorem 10.0.1 are satisfied with K in the role of "L". As $K/O_2(K)$ is A_5 rather than $L_3(2)$, 10.2.13 applied to K in the role of "L" supplies a contradiction. This contradiction completes the proof of Theorem 10.0.1.

CHAPTER 11

Elimination of $\mathbf{L_3(2^n)}$, $\mathbf{Sp_4(2^n)}$, and $\mathbf{G_2(2^n)}$ for $\mathbf{n > 1}$

In this chapter, we complete the elimination of the groups possessing a pair L, V arising in the Fundamental Setup (3.2.1) such that $L/O_2(L)$ is of Lie type of Lie rank 2 over a field of order 2^n, $n > 1$.

Choose V so that L, V are in the FSU and $L/O_2(L)$ is of Lie type of Lie rank 2 over a field of order $q := 2^n$, $n > 1$. By Theorem 7.0.1, V is an FF-module. The weak closure parameters of FF-modules make it difficult to do weak closure without first doing some extra work. Furthermore corresponding local configurations do actually occur in suitable maximal parabolics in non-quasithin shadows given by certain groups G of Lie type and Lie rank 3: namely for $\bar{L} \cong SL_3(q)$, in $G \cong L_4(q)$, $Sp_6(q)$, $\Omega_8^+(q).2$, and $\Omega_8^-(q)$; and for $\bar{L} \cong Sp_4(q)$, in $G \cong Sp_6(q)$.

We restrict attention at this point to $q = 2^n$ for $n > 1$, largely because for such q, $\bar{L}$ has a Cartan subgroup X of p-rank 2 for primes p dividing $q - 1$. Using our quasithin hypothesis, G contains no member of $\mathcal{H}(X)$ of larger p-rank, whereas the the groups of Lie type in the previous paragraph do contain such subgroups. This leads to a contradiction, which does not arise in the shadows of groups over the small field $\mathbf{F}_2$; the more complicated treatment needed for the subcase of L of rank 2 over $\mathbf{F}_2$ is postponed to part 5.

Thus in this chapter we will prove:

THEOREM 11.0.1. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G,T)$. Then $L/O_2(L)$ is not isomorphic to $(S)L_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ with $n > 1$.*

Throughout this chapter we assume L is a counterexample to Theorem 11.0.1

By 1.2.1.3, L is T-invariant, so by 3.2.3, $M := N_G(L) \in \mathcal{M}(T)$, $M = !\mathcal{M}(LT)$, and we can choose V so that L and V are in the FSU. In particular let $V_M := \langle V^M \rangle$, $\tilde{V}_M := V_M/C_{V_M}(L)$, $M_V := N_M(V)$, and $\bar{M}_V := M_V/C_M(V)$. Let $T_L := T \cap LO_2(LT)$ and let X be a Hall $2'$-subgroup of $N_L(T_L)$; since $n > 1$, $m_p(X) = 2$ for each prime divisor p of $|X|$ (see 11.0.4). As mentioned earlier, the Cartan subgroup X will provide a main focus for our analysis. Set $Z := \Omega_1(Z(T))$ and abbreviate $q := 2^n$.

Lemmas 11.0.2, 11.0.3, and 11.0.4 collect observations from various earlier results, and provide a starting point for the analysis.

LEMMA 11.0.2. *(1) $V \in Irr_+(L, R_2(LT))$ and V is T-invariant. Moreover T is trivial on the Dynkin diagram of $L/O_2(L)$.*

(2) $V/C_V(L)$ is the natural module for $L/O_2(L) \cong \bar{L} \cong SL_3(q)$, $Sp_4(q)$, or $G_2(q)$.

PROOF. By Theorem 7.0.1, V is an FF-module for $Aut_{GL(V)}(\bar{L})$. By construction in the FSU, $V = \langle V_\circ^T \rangle$ for some $V_\circ \in Irr_+(L, R_2(LT), T)$, so V is T-invariant. If $V > V_\circ$, then V is described in case (3) of Theorem 3.2.5. However in that case by Theorem B.5.1, V is not an FF-module for $Aut_{GL(V)}(\bar{L})$. Therefore $V = V_\circ$, so as V is an FF-module, (2) follows since one of cases (2), (3), or (4) of 3.2.8 must hold. Then as V is T-invariant, T is trivial on the Dynkin diagram of $L/O_2(L)$, completing the proof of (1). □

LEMMA 11.0.3. *(1)* $V_M \in \mathcal{R}_2(M)$.

(2) $\tilde{V}_M$ *is a homogeneous* $\mathbf{F}_2L$*-module.*

(3) Either $C_V(L) = C_{V_M}(L) = 1$*; or* $\bar{L} \cong Sp_4(q)$ *or* $G_2(q)$*,* $V = V_M$*,* $m(C_V(L)) \leq n$*, and* $L = [L, J(T)]$*.*

(4) V *is a TI-set under* M*.*

(5) If $\bar{L}$ *is* $Sp_4(q)$ *or* $G_2(q)$ *then* $H \cap M \leq N_M(V)$ *for each* $H \in \mathcal{H}_*(T, M)$*.*

PROOF. Part (1) is 3.2.2.2; part (2) follows from 3.2.2.3; and as $n > 1$, part (4) is a consequence of 3.2.7. By 3.2.2.4, $C_{V_M}(L) = \langle C_V(L)^M \rangle$. If $L \cong SL_3(q)$, then as $n > 1$ we have $H^1(L, V/C_V(L)) = 0$ by I.1.6, so $C_V(L) = 1$. Hence $C_{V_M}(L) = 1$, so that (3) holds in this case. If $C_V(L) \neq 1$, then $L = [L, J(T)]$ by 3.2.2.6, and $V = V_M$ by Theorem 3.2.5, since now neither cases (2) nor (3) of that result hold. Further by I.1.6, $m(C_V(L)) \leq m(H^1(L, V/C_V(L)) = n$, completing the proof of (3).

Finally assume the hypotheses of (5), and suppose $H \in \mathcal{H}_*(T, M)$ with $H \cap M \not\leq N_M(V)$. In particular $V < V_M$ as $V_M \trianglelefteq M$. As V is a TI-set under M by (4), while $Z \cap V \neq 1$, $[Z, H \cap M] \neq 1$ and hence $[Z, H] \neq 1$. Thus $J(T) \not\leq C_T(V)$ by 3.1.8.3, and so $L = [L, J(T)]$. So setting $M^* := M/C_M(V_M)$, by B.2.7 there is $A^* \in \mathcal{P}(M^*, V_M)$ with $L^* = [L^*, A^*]$. Then by Theorem B.5.6, $F^*(J(M^*, V_M)) = L^*$, and then Theorem B.5.1 supplies a contradiction to $V < V_M$. □

LEMMA 11.0.4. $L = O^{p'}(M)$ *for each prime* p *such that*

(1) p *divides* $q^2 - 1$*, if* $\bar{L}$ *is* $Sp_4(q)$ *or* $G_2(q)$*; or*

(2) p *divides* $q - 1$ *and* $p > 3$*, if* $\bar{L}$ *is* $SL_3(q)$*.*

Moreover if $\bar{L} \cong SL_3(q)$ *with* n *even, then* L *contains each element of* M *of order* 3*.*

PROOF. The primes p are chosen so that $m_p(L) = 2$; hence the lemma follows from A.3.18, using A.3.19 for the final assertion. □

11.1. The subgroups $\mathbf{N_G(V_i)}$ for T-invariant subspaces $\mathbf{V_i}$ of V

By 11.0.2.2, $\tilde{V}$ is the natural $\mathbf{F}_q\bar{L}$-module; thus the two classes of maximal parabolics of $\bar{L}$ preserve $\mathbf{F}_q$-subspaces of dimension 1 and 2. We will use our structure theory of QTKE-groups to restrict the normalizers of these subspaces. The results in this section roughly have the effect of forcing these normalizers to resemble those in the shadows mentioned earlier.

For $i = 1, 2, 3$, let $\mathcal{V}_i$ denote the set of $U \leq V$ such that $C_V(L) \leq U$ and $\tilde{U}$ is an i-dimensional $\mathbf{F}_qT'$-subspace of $\tilde{V}$ for some $T' \in Syl_2(M)$. Further for $i = 1, 2$, set $L(U) := N_L(U)^\infty$.

Denote by V_i the unique T-invariant member of $\mathcal{V}_i$. For $i = 1, 2$, let $L_i := L(V_i)$ and $R_i := O_2(L_iT)$. Then $L_i/O_2(L_i) \cong L_2(2^n)$. By construction $T \leq N_G(V_i)$, so that $N_G(V_i) \in \mathcal{H}^e$ by 1.1.4.6. Notice when $\bar{L} \cong SL_3(q)$ that $V_3 = V$, while in the other cases, from the action of $N_{GL(\tilde{V})}(\bar{L})$ on $\tilde{V}$, $N_M(V_3) = N_M(V_1)$.

We begin by considering the embedding of L_i in a $\mathcal{C}$-component K_i of $N_G(V_i)$.
[1]

Lemma 11.1.1. *Assume either*

(i) $i = 1$, $1 \neq V_0 \leq V_1$, $H := N_G(V_0)$, *and* $T_0 := N_T(V_0) \in Syl_2(H)$, *or*

(ii) $i = 2$ *and* $H := N_G(V_2)$.

Then $L_i \leq K \in \mathcal{C}(H)$ *with* $K \trianglelefteq H$, *and one of the following holds:*

(1) $L_i = K$.

(2) $K/O_2(K) \cong (S)L_3(q)$, $Sp_4(q)$, $G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, *or* ${}^3D_4(q^{1/3})$.

(3) $n = 2$ *and* $K/O_2(K)$ *is isomorphic to* A_7, $\hat{A}_7$, $L_2(p)$ *for a prime* p *with* $p \equiv \pm 1 \mod 5$ *and* $p \equiv \pm 3 \mod 8$, $L_2(25)$, $(S)L_3^{\epsilon}(5)$, M_{22}, $\hat{M}_{22}$, M_{23}, J_1, J_2, J_4, HS, Ru, $SL_2(5)/P_0$ *for a suitable nilpotent group* P_0 *of odd order, or* $SL_2(p)/E_{p^2}$ *for a prime* p *satisfying the congruences above.*

Proof. If $i = 1$, V_0 and T_0 are defined in (i); if $i = 2$, set $V_0 := V_2$ and $T_0 := T$. Thus in either case $H = N_G(V_0)$, and $T_0 \in Syl_2(H)$ acts on L_i, so $L_i \in \mathcal{L}(H, T_0)$. Thus by 1.2.4, L_i is contained in a unique $K \in \mathcal{C}(H)$, and the embedding $L_i \leq K$ appears on the list of A.3.12. As T_0 acts on L_i, T_0 also acts on K, so $K \trianglelefteq H$ by 1.2.1.3. The possibilities for K are determined by restricting the list of A.3.12 to $L_i/O_2(L_i) \cong L_2(q)$. The groups in (2) are the groups of Lie type, characteristic 2, and Lie rank 2 in Theorem C (A.2.3). When $n = 2$, we use the list in A.3.14, and get the further examples in (3). □

We next determine the possible embeddings of L_i in $N_G(V_i)$ for $i = 1$ and 2. Recall that X is a Hall $2'$-subgroup of $N_L(T_L)$, so $X \leq N_L(V_i)$.

Proposition 11.1.2. *For* $i = 1, 2$, $L_i \leq K_i \in \mathcal{C}(N_G(V_i))$ *with* $K_i \trianglelefteq N_G(V_i)$ *and* $K_i \in \mathcal{H}^e$. *Furthermore for* $K := K_i$ *either* $L_i = K$, *or* $i = 1$, $q = 4$, *and one of the following holds:*

(1) $K/O_{2,2'}(K) \cong SL_2(p)$ *where* $p = 5$, *or* $p \geq 11$ *is prime.*

(2) $K/O_2(K) \cong L_2(p)$ *for a suitable prime* $p \geq 11$, *and* $L/O_2(L)$ *is not* $SL_3(4)$.

(3) $KX/O_2(KX) \cong PGL_3(4)$. *Further if* K_0 *denotes the member of* $\mathcal{L}(G, T) \cap K$ *distinct from* K *and* L_1, *and* $I := \langle K_0, L_2 \rangle$, *then* $I \in \mathcal{L}_f^*(G, T)$, *and interchanging the roles of* L *and* I *if necessary,* $L/O_2(L) \cong G_2(4)$ *and* $I/O_2(I) \cong Sp_4(4)$.

Proof. By 11.1.1, $L_i \leq K_i \trianglelefteq N_G(V_i)$. Recall $N_G(V_i) \in \mathcal{H}^e$, so $K_i \in \mathcal{H}^e$ by 1.1.3.1. So we may assume $L_i < K_i =: K$, and K appears in case (2) or (3) of 11.1.1, but not among the conclusions of 11.1.2. In particular $K \not\leq M$.

Let $G_i := C_G(V_i)$; observe that $X \leq N_G(V_i)$, and set $(G_i X)^* := G_i X/O_2(K)$. As $N_G(V_i) \in \mathcal{H}^e$, $G_i \in \mathcal{H}^e$ by 1.1.3.1. Set $X_i := C_X(L_i/O_2(L_i))$. By 11.0.2.2, $|X_i| = q - 1$.

Suppose first that $i = 1$. By inspection of the possiblities for K, namely in (2) and (3) of 11.1.1 but not in 11.1.2, $K/O_2(K)$ is quasisimple and either

(i) $m_p(K) = 2$ for some prime p dividing $q - 1$, or

(ii) $q = 4$ and $K/O_2(K) \cong L_2(p)$ for a prime $p \geq 11$, $L_2(25)$, $L_3(5)$, or J_1.

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[1]Notice that in the shadows we expect $L_i = K_i \trianglelefteq N_G(V_i)$.

Next $L_1 = L_1^\infty \leq C_G(V_1) = G_1$, so $K = [K, L_1] \leq G_1$. As X_1 is faithful on V_1, the product KX_1 is semidirect. Thus for each prime p dividing $q-1$, K does not contain all elements of order p centralizing $L_1/O_2(L_1)$, so applying A.3.18 we conclude that in case (i):

(*) $q = 4$, $K^* \cong L_3(4)$, and $K^*X_1^* \cong PGL_3(4)$ with X_1 inducing outer automorphisms on K^*.

We will return to case (*), after treating case (ii). There $q = 4$ so that $|X_1^*| = 3$. If $K^* \cong J_1$ then $K = \langle L_1, N_K(T)\rangle \leq M$ using Theorem 3.3.1, contradicting $K \not\leq M$. Thus $K/O_2(K)$ is $L_2(p)$ or $L_2(25)$ or $L_3(5)$, so $Out(K^*)$ is a $3'$-group, and hence X_1^* centralizes K^*. If $K/O_2(K) \cong L_2(25)$ or $L_3(5)$, then some $t \in T \cap K$ induces an outer automorphism on $L_1/O_2(L_1)$, so t induces a field automorphism on $L/O_2(L)$, impossible as $[t, X_1] \leq O_2(X_1T)$. Thus $K/O_2(K) \cong L_2(p)$, so conclusion (2) will hold in this case, once we show $L/O_2(L)$ is not $SL_3(4)$. But in that case, $X_1O_2(L) = O_{2,Z}(L) \trianglelefteq M$, so $Y := O^2(X_1T) \trianglelefteq LT$, and hence $N_G(Y) \leq M = !\mathcal{M}(LT)$. Then as $[K, X_1] \leq O_2(K) \leq T$, K normalizes $O^2(YO_2(K)) = Y$ so that $K \leq N_G(Y) \leq M$, contrary to $K \not\leq M$.

Thus to complete the treatment of the case $i = 1$, we assume (*) holds; as this is the first requirement of conclusion (3), it remains to establish the remaining assertions of (3). This argument will require several pages.

Our strategy will be to use K and L to construct a third group I, and obtain a triple $L = \langle L_1, L_2\rangle$, $K = \langle L_1, K_0\rangle$, and $I := \langle L_2, K_0\rangle$—where K_0 is essentially the maximal parabolic of K over $T \cap K$ other than $L_1(T \cap K)$. We will be able to exploit some symmetry in this triangle of subgroups.

Let K_0 denote the member of $\mathcal{L}(G,T) \cap K$ distinct from L_1 and K—that is, $K_0/O_2(K)$ is normal in the maximal parabolic of $K/O_2(K)$ stabilized by XT which is distinct from $N_K(L_1)$. In particular, $K_0/O_2(K_0) \cong L_2(4)$, and $K_0 \in \mathcal{H}^e$. Set $S := O_2(XT)$, $H_1 := K_0SX$, $H_2 := L_2SX$, and $H_{1,2} = SX$.

Assume that there is no nontrivial normal subgroup of T normal in $H := \langle H_1, H_2\rangle$. Then Hypothesis F.1.1 is satisfied with K_0, L_2, S in the roles of "L_1, L_2, S", so by F.1.9, $\alpha := (H_1, H_{1,2}, H_2)$ is a weak BN-pair of rank 2. As $S \trianglelefteq H_{1,2}$, α appears on the list of F.1.12. Indeed α must be one of the (untwisted) cases where the nonabelian chief factor of H_1 and H_2 is isomorphic to $L_2(4)$. As L_2 has at least two noncentral 2-chief factors, α is not the $PGL_3(4)$, $SL_3(4)$, or $Sp_4(4)$ amalgam, so α is the $G_2(4)$-amalgam. By construction V_1 is $H_{1,2}$-invariant, and hence plays the role of the long root group of $G_2(4)$ normal in a maximal parabolic. The parabolic H_1 stabilizing this long root group is irreducible on $O_2(H_1)/V_1$, and $H_1^\infty/O_2(H_1)$ has two A_5-modules on this section; but in our construction K_0 has a natural $L_2(4)$-chief factor on $O_2(K_0)$.

This contradiction shows that $O_2(H) \neq 1$, and hence $HT \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ using 1.1.4.6. Now by 1.2.4, $L_2 \leq I \in \mathcal{C}(HT)$, and $I \trianglelefteq HT$ by 1.2.1.3 as L_2 is T-invariant, so also $I \in \mathcal{H}^e$. Similarly $K_0 \leq I_0 \in \mathcal{C}(HT)$. We conclude from (*) that

$$X_K := X \cap K = X \cap L_1 = X \cap K_0.$$

But as $[V_1, L_1] = 1$ while $V_1 = [V_1, X_2]$ from the action of L on V, $X \cap L_1 \neq X_2$; so X_K does not centralize $L_2/O_2(L_2)$, and hence $[I, X_K] \not\leq O_2(I)$. Thus $[I, I_0] \not\leq O_2(I)$, so $K_0 \leq I_0 = I$. Therefore

$$X = (X \cap L_1)(X \cap L_2) = X_K(X \cap L_2) \leq I,$$

so $m_3(I) = 2$. Also $\mathcal{L}(G,T) \cap I$ contains two members L_2, K_0 with $L_2X/O_2(L_2) \cong K_0X/O_2(K_0) \cong GL_2(4)$; inspecting the list of A.3.14, we conclude $I/O_2(I) \cong SL_3(4)$, $Sp_4(4)$, or $G_2(4)$ and $I = \langle K_0, L_2\rangle$. Furthermore $O^2(H) = \langle K_0, L_2, X\rangle \leq I$, so $I = O^2(HT)$ and $HT = IT$.

Suppose first that $\bar{L} \cong SL_3(4)$; we must eliminate this case as part of our proof that (3) holds. This subcase will require approximately a page of argument.

First

$$X_1 = X_2,$$

and X_1 is Sylow in $O_{2,Z}(L)$. Thus $I/O_2(I)$ is not $SL_3(4)$ or else $X_1 = X_2 = C_X(L_2/O_2(L_2)) = C_X(K_0/O_2(K_0))$, which is not the case in $K^*X^* \cong PGL_3(4)$. In the remaining cases the subgroup $X_1 = X_2$ of the Cartan group X of I is inverted by a 2-element projecting on the center of the Weyl group (D_8 or D_{12}) of $I/O_2(I)$, so this element is not in L_2X. Thus $N_I(X_2) \not\leq L_2X = I \cap M$. Therefore as $X_1 = X_2$, $G_{X_1} := N_G(X_1) \not\leq M$.

Let $L_{X_1} := N_L(X_1)^\infty$, so that $L = O_2(L_1)L_{X_1}$ and $X_1 \leq L_{X_1}$. We now show that it suffices to prove $Q_{X_1} := [O_2(L_{X_1}), L_{X_1}] \neq 1$: For then $O_2(L_{X_1}) \neq 1$, so that Theorem 4.2.13 says $M = !\mathcal{M}(L_{X_1})$. Then as $G_{X_1} \not\leq M$ by the previous paragraph, $O_2(G_{X_1}) = 1$. Next as $Q_{X_1} \neq 1$, L_{X_1} has a 2-chief section of rank at least 6, so $m_2(G_{X_1}) \geq m(Q_{X_1}) > 3$. Therefore $m_p(O_p(G_{X_1})) \leq 2$ for each odd p by A.1.28, so $L_{X_1} \leq C_{X_1} := C_{G_1}(O(F(G_{X_1})))$. As $O_2(G_{X_1}) = 1$, $F^*(C_{X_1}) = EZ(C_{X_1})$, where $E := E(G_{X_1})$). Further using (1) of Theorem A (A.2.1), $|J^{G_{X_1}}| \leq 3$ for each component J of G_{X_1}, so $G_{X_1}^\infty$ normalizes J. Hence $E = C_{X_1}^\infty$ as J satisfies the Schreier Conjecture. Then $L_{X_1} \leq E$, so that Q_{X_1} projects nontrivially on some component K_{X_1} of G_{X_1}. As G is quasithin, $m_{2,3}(E) = 2$, so K_{X_1} is the unique component not centralized by L_{X_1}, and hence $L_{X_1} \leq K_{X_1}$, so $X_1 \leq Z(K_{X_1})$. However $K_{X_1}/Z(K_{X_1})$ appears in Theorem B (A.2.2), and inspecting such groups for a 2-local containing a subgroup $\hat{L}$ with $\hat{L}/O_2(\hat{L}) \cong L_3(4)$ and $[O_2(\hat{L}), \hat{L}] \neq 1$, we conclude $K_{X_1}/Z(K_{X_1}) \cong J_4$. This is contradiction as $X_1 \leq Z(K_{X_1})$ but the multiplier of J_4 is trivial. This completes the proof that to eliminate $L/O_2(L) \cong SL_3(q)$, it is sufficient to show $Q_{X_1} \neq 1$.

So we assume $Q_{X_1} = 1$, and it remains to derive a contradiction. We set up the apparatus to apply lemma G.2.5. Set $U := \langle V^{G_1}\rangle$ and $\hat{G}_1 := G_1/V_1$. It is straightforward to check that Hypothesis G.2.1 is satisfied, with $N_G(V_1)$, $O^2(N_L(V_1))$, G_1X_1T, U, V in the roles of "G_1, L_1, H, U, V". Therefore $\hat{U} \leq Z(O_2(\hat{G}_1))$ by G.2.2.

Let P be a Sylow 3-subgroup of KX containing X with $X_K = Z(P)$, so that $P \cong 3^{1+2}$. As $Z(P) = X_K = X \cap L_1$ is nontrivial on V, P is faithful on U; so as $P \cong 3^{1+2}$, $1 \neq [C_U(X_1), X] = [C_U(X_1), X_K]$. If $Y := [C_U(X_1), X] \leq O_2(LT)$, then $L_{X_1} = \langle X_K^{L_{X_1}}\rangle$ is nontrivial on $O_2(L_{X_1})$, which we saw above suffices; so we may assume $Y \not\leq O_2(LT)$. In particular $U \not\leq O_2(LT)$ so the hypotheses of G.2.5 are also satisfied. Thus by G.2.5.1, $R_1 = UO_2(LT)$. Furthermore by G.2.5.2, $L \leq J \leq LT$, where J plays the role of "I" in G.2.5—with the structure of $O_2(J)$ described in detail in the remaining parts of G.2.5. Let $L_+ := N_{L_1}(X_1)^\infty$. As $1 = Q_{X_1}$, $C_{R_1}(X_1)/C_{O_2(LT)}(X_1)$ is the unique noncentral chief factor for L_+ on $C_{R_1}(X_1)$, so $C_{R_1}(X_1)/C_{R_1}(L_+X_1)$ is the natural module for $L_+/O_2(L_+) \cong L_2(4)$. Let W be a KX-chief factor in $O_2(KT)$ with K nontrivial on W. By G.2.5, the nontrivial L-constituents on $O_2(J)$ are natural or dual, so the nontrivial L_+-constituents are all natural. Therefore by B.4.14, W is the adjoint module for $K/O_2(K)$ and $C_W(X_1)$ is indecomposable of $\mathbf{F}_4$-dimension 4 for $L_+/O_2(L_+)$. In particular $C_W(X_1)$ does not split over $[C_W(X_1), L_+]$, a contradiction as $C_{R_1}(X_1)/C_{R_1}(L_+X_1)$ is the natural module for $L_+/O_2(L_+)$. This contradiction finally completes the elimination of the case where (*) holds and $L/O_2(L) \cong \bar{L} \cong SL_3(4)$.

Thus in view of 11.0.2.2, we have shown that if (*) holds then $L/O_2(L) \cong \bar{L} \cong Sp_4(4)$ or $G_2(4)$, and to complete our treatment of the case $i = 1$ it remains to show that (*) implies the remaining statements in (3). Recall $I/O_2(I) \cong SL_3(4)$, $Sp_4(4)$, or $G_2(4)$, so $I \in \mathcal{L}^*(G,T)$ by 1.2.8.4, and then as $[Z, L_2] \neq 1$, even $I \in \mathcal{L}_f^*(G,T)$. This begins to establish some symmetry between L and I; in particular applying 11.0.2 to I, we conclude there is $V_I \in \mathcal{R}_2(IT)$ with $V_I/C_{V_I}(I)$ the natural module.

Assume $[Z, K] \neq 1$. Then $K \leq K_+ \in \mathcal{L}_f^*(G,T)$ by 1.2.9. Now since $K/O_2(K) \cong L_3(4)$, by A.3.12, either $K = K_+$ or $K_+/O_2(K_+) \cong M_{23}$. By B.4.2, neither $L_3(4)$ nor M_{23} has an FF-module, so Theorem 7.0.1 supplies a contradiction.

Therefore $[Z, K] = 1$, so if $C_{V_I}(I) \neq 1$ then $Z_I := C_Z(I) \neq 1$. But then by 1.2.7.3, $N_G(I) = !\mathcal{M}(IT) = !\mathcal{M}(C_G(Z_I))$, so $L_1 \leq K \leq N_G(I) \geq L_2$, and hence $K \leq M = !\mathcal{M}(LT)$, for our usual contradiction. Thus $C_{V_I}(I) = 1$. As $[Z, K] = 1$, K_0 stabilizes the 1-dimensional $\mathbf{F}_4$-subspace $V_{I,1}$ of V_I stabilized by T. Thus K_0 plays the same role in I that L_1 plays in L. As $XT = TX$ and we saw $X \leq I$, X is also a Cartan subgroup of I and $V_{I,1} = [Z \cap V_{I,1}, X_K] \leq C_G(K)$. Therefore K is the member of $\mathcal{C}(N_G(V_{I,1}))$ containing K_0, so K plays the role of "K" for I as well as for L. In particular, (*) is also satisfied by I. Therefore applying our previous reduction to I, $I/O_2(I)$ is not $SL_3(4)$. Notice also that L_2 plays the same role in both L and I: L_2 is the derived group of the stabilizer of a line of V and V_I.

Suppose $L/O_2(L) \cong G_2(4)$. Then $X_1 \leq L_2$ by B.4.6.14. From the previous paragraph, K_0 centralizes $V_{I,1}$, so if $I/O_2(I) \cong G_2(4)$, then by the same argument,

$$C_X(K_0/O_2(K_0)) = X \cap L_2 = X_1.$$

But from (*), $[K_0, X_1] \not\leq O_2(K_0)$, a contradiction. Therefore if $L/O_2(L) \cong G_2(4)$, then $I/O_2(I) \cong Sp_4(4)$ and so (3) holds.

This leaves the case $L/O_2(L) \cong Sp_4(4)$. Interchanging the roles of L and I if necessary, and appealing to the previous paragraph, we may assume that also $I/O_2(I) \cong Sp_4(4)$. As $L/O_2(L) \cong Sp_4(4)$, $X_K = X \cap L_1$ and X_1 are the two diagonally-embedded subgroups of order 3 with respect to the decomposition

$$X = X_2 \times (X \cap L_2).$$

Therefore as K_0 centralizes $V_{I,1}$, and $X_K = X \cap K_0$, from the structure of $I/O_2(I) \cong Sp_4(4)$, the second diagonal subgroup X_1 centralizes $K_0/O_2(K_0)$. But again this does not hold in (*), a contradiction completing the treatment of the case $i = 1$.

Now we turn to the easier case $i = 2$. We assume that $L_2 < K_2 = K$, and it remains to derive a contradiction. Here $V_2/C_{V_2}(L)$ is the natural module for $L_2/O_2(L_2) \cong L_2(q)$, and by 11.0.3.3, either $C_{V_2}(L) = 1$, or $\bar{L}$ is $Sp_4(q)$ or $G_2(q)$ with $m(C_{V_2}(L)) \leq n$. Thus $m(V_2) \leq 3n$. Examining the possibilities in (2) and (3) of 11.1.1 for cases where K possesses a nontrivial module of rank at most $3n$, we conclude that one of the following holds:

(a) $K/O_2(K) \cong SL_3(q)$ and V_2 is the natural module.
(b) $q = 4$, $K/O_2(K) \cong A_7$, and V_2 is the natural module.

(c) $q = 4$, $K/O_\infty(K) \cong L_2(5)$, and $O_2(K) < O_\infty(K)$ centralizes V_2 of rank at least 4.

In case (b), $K = O^{3'}(N_G(V_2))$ by A.3.18, so $X_2 \leq C_K(L_2/O_2(L_2))$, contrary to the structure of A_7. In case (a), $m(V_2) = 3n$ so $m(C_V(L)) = n$, and from the action of $\bar{L} \cong Sp_4(2^n)$ or $G_2(2^n)$ on V in I.2.3.1.ii.a, R_2 centralizes V_2, whereas this is not the case for the parabolic L_2^* in $K^* \cong SL_3(q)$ on V_I. Hence case (c) holds so $K/O_2(K)$ is not quasisimple and there is $Y \in \Xi(G,T)$ contained in $O_{2,F}(K)$ by 1.3.3; in particular Y is normalized by L_2T. Next $YT \leq C_G(V_2)T \leq N_G(V_1)$, so as $Y \in \Xi(G,T)$, we may apply 1.3.4 to conclude that either $Y \trianglelefteq N_G(V_1)$, or $Y \leq K_Y \in \mathcal{C}(N_G(V_1))$ with K_Y described in 1.3.4. Therefore either $K_1 \leq N_G(Y)$ or $K_1 = K_Y$. However comparing the list of possibilities for K_Y in 1.3.4 to the list of possiblities for K_1 in this lemma, we find no overlap. Thus $K_1 \leq N_G(Y)$, so

$$LT = \langle L_1, L_2T\rangle \leq N_G(Y).$$

Then $K \leq N_G(Y) \leq M = !\mathcal{M}(LT)$, for our usual contradiction. This completes the treatment of the case $i = 2$, and hence the proof of 11.1.2. □

In the remainder of the section, we obtain several further technical restrictions on the normalizers of the subspaces V_i.

LEMMA 11.1.3. *L_2 is the unique member of $\mathcal{C}(N_G(V_2))$ which does not centralize V_2.*

PROOF. By 11.1.2, $L_2 \in \mathcal{C}(N_G(V_2))$. If there is $L_2 \neq K \in \mathcal{C}(N_G(V_2))$, then by 1.2.1.2, $[K, L_2] \leq O_2(L_2) \leq C_G(V_2)$, so as $V_2 \in Irr_+(L_2, V_2)$, K centralizes V_2 by A.1.41. □

LEMMA 11.1.4. *$C_G(V_3/V_1) \leq M \geq N_G(V_3) \cap N_G(L_1)$.*

PROOF. If $\bar{L}$ is $SL_3(q)$ then $V_3 = V$, so $N_G(V_3) \leq M = !\mathcal{M}(LT)$. Hence we may take $\bar{L}$ to be $Sp_4(q)$ or $G_2(q)$. Let $\Delta := V_2^{L_1}$ and $H := N_G(V_3)$; note that V_1 is the intersection of the members of Δ, while V_3 is their span. Then by 11.1.3, $N_H(\Delta)$ acts on

$$\langle L(U) : U \in \Delta\rangle = L,$$

where we recall $L(U) = N_L(U)^\infty$. Therefore $N_H(\Delta) \leq N_G(L) = M$. In particular $C_G(V_3/V_1) \leq M$. Further Δ is the set of subspaces $C_{V_3}(S)$ for $S \in Syl_2(L_1)$, so $N_H(L_1) \leq M$. □

LEMMA 11.1.5. *Either*

(1) $N_G(R_1) \leq M$, or

(2) $V = V_M$, L is an $SL_3(q)$-block or $Sp_4(4)$-block, $C_T(L) = 1$ and $V_1 = \Omega_1(Z(R_1))$.

PROOF. Assume $N_G(R_1) \not\leq M$. Then as $M = !\mathcal{M}(LT)$, there is no nontrivial characteristic subgroup of R_1 normal in LT. Therefore L, R_1 is an MS-pair in the language of Definition C.1.31. so L appears on the list of Theorem C.1.32. Therefore L is an $Sp_4(4)$-block or an $SL_3(q)$-block, since the remaining possibilities in C.1.34 explicitly exclude the case where R_1 is the unipotent radical of the point stabilizer. In particular $V = V_M$.

Set $Q := O_2(LT)$ and $Q_1 := VC_T(V)$. If $V = Q$ then $\Omega_1(Z(R_1)) = C_V(R_1) = V_1$ and the lemma holds, so we may assume $V < Q$. By C.1.13, $\Phi(Q) \leq C_T(L)$ and

$m(Q/Q_1) \leq m(H^1(\bar{L}, \tilde{V}))$, so $H^1(\bar{L}, \tilde{V}) \neq 0$. Therefore by I.1.6, L is an $Sp_4(4)$-block and $m(Q/Q_1) \leq 2$, and by I.2.3.1, $Q/C_T(L)$ is a submodule of the dual of the natural 5-dimensional module over $\mathbf{F}_4$ for $\Omega_5(4) \cong Sp_4(4)$. Here we compute (e.g. by restricting to the subgroup $Sp_4(2) \cong S_6$) that $C_Q(R_1) \leq Q_1$ and $C_V(R_1) = V_1$. Therefore $Z_1 := \Omega_1(Z(R_1)) = V_1 C_{Z_1}(L)$.

If $C_T(L) = 1$ then $Z_1 = V_1$ by the previous paragraph, and hence (2) holds. Thus we may assume that $C_T(L) \neq 1$. Now $[O_2(LT), X] \leq [O_2(LT), L] = V$, so

$$Z_1 := \Omega_1(Z(R_1)) = Z_X \times Z_C,$$

where $Z_X := [V_1, X]$, $Z_C := C_{Z_1}(X) = C_{Z_1}(L)$, and $Z_C \neq 1$ as $C_T(L) \neq 1$. For $D \leq G$, let $\theta(D)$ be the subgroup generated by all elements of D whose order lies in Δ, where Δ is the set of divisors of $2^n - 1$ if $\bar{L}$ is $SL_3(2^n)$ with n odd, and $\Delta := \{3\}$ otherwise. Thus $L = \theta(M)$ by 11.0.4. By Theorem 4.2.13, $M = !\mathcal{M}(L)$, so Z_C is a TI-set under the action of $Y := N_G(R_1)$, with $Y_M := Y \cap M = N_Y(Z_C)$: for if $y \in Y$ with $1 \neq Z_C \cap Z_C^y$, then $\langle L, L^y \rangle \leq C_G(Z_C \cap Z_C^y)$, so as $M = !\mathcal{M}(L)$, $L^y \leq \theta(C_G(Z_C \cap Z_C^y)) \leq \theta(M) = L$, and hence $y \in N_Y(L) = Y_M$. Notice $Y_M < Y$ as (1) fails. Set $Y^* := Y/C_Y(Z_1)$. Then X^* is regular on $Z_X^{\#}$, and normal in Y_M^* since $L_1 R_1$ centralizes V_1. Thus we have the hypotheses for a Goldschmidt-O'Nan pair in the sense of Definition 14.1 in [**GLS96**]; so we may apply O'Nan's Lemma 14.2 in [**GLS96**, 14.2], with Y^*, X^*, Z_1 in the roles of "X, Y, V". Observe conclusion (iv) of that result must hold—since in (i), Y normalizes Z_C giving $Y_M = Y$; while in (ii) and (iii), T does not normalize Z_C. In conclusion (iv) of 14.2 of [**GLS96**, 14.2], $q = 4$, $Z_1 \cong E_8$, and Y^* is a Frobenius group of order 21. Next $C_G(Z_1) \leq C_G(Z_C) \leq M = !\mathcal{M}(L)$, so $L_1 \in \mathcal{C}(N_G(Z_1))$ and hence Y acts on L_1. If L is an $SL_3(4)$-block, the noncentral 2-chief factors for L_1 are $VZ(L_1)/Z(L_1)$ and $O_2(L_1)/VZ(L_1)$, and both are natural modules. Therefore the induced action of $N_G(L_1)$ on $Irr_+(L_1, O_2(L_1)/Z(L_1))$ is contained in $\Gamma L_2(4)$, so $O^{7'}(Y)$ acts on $VZ(L_1)$ and then on $[VZ(L_1), L_1] = V$. But then $Y = O^{7'}(Y) Y_M \leq N_G(V) \leq M$, contradicting $Y_M < Y$. Similarly if L is an $Sp_4(4)$-block, then Y acts on $[V, L_1] = V_3$, so $Y \leq M$ by 11.1.4, for the same contradiction. This completes the proof. □

11.2. Weak-closure parameter values, and $\langle \mathbf{V}^{\mathbf{N_G(V_1)}} \rangle$

Since V is an FF-module, we do not have the ideal situation for weak closure described in subsection E.3.3; however, we will be able to establish at least some restrictions on the weak closure parameters $r(G, V)$, $w(G, V)$, and $n(H)$ discussed in Definitions E.3.3, E.3.23, and E.1.6. Recall that the paramter $n'(Aut_M(V))$ is defined in Definition E.3.37, and notice that $n'(Aut_M(V)) = n > 1$: for example this follows from A.3.15.

LEMMA 11.2.1. *For $H \in \mathcal{H}_*(T, M)$, either*

(1) $n(H) \leq n$, or

(2) $\bar{L} \cong SL_3(q)$, V_M is the sum of two isomorphic natural modules for $L/O_2(L)$, $C_V(H) = 1$, $L = [L, J(T)]$, and $n(H) \leq 2n$. [2]

PROOF. Assume (1) fails, so that $n(H) > n > 1$. Then by E.2.2, $O^2(H/O_2(H))$ is of Lie type over $\mathbf{F}_{2^m}$, for $m := n(H) > n$, and $H \cap M$ is a Borel subgroup of

[2]Notice this essentially eliminates the shadow of $\Omega_8^-(2^n)$, in which $n(H) = 2n$ but $V \not\trianglelefteq M$. Our use of the quasithin hypothesis is via reference to the pushing up result Theorem 4.4.3.

H. Let B be a Hall $2'$-subgroup of $H \cap M$. If $A := C_B(V) \neq 1$, then by 4.4.13.1, $N_G(A) \not\leq M$, contrary to Theorem 4.4.3 using Remark 4.4.2. Thus $C_B(V) = 1$.

Suppose that B normalizes V. Then B is faithful on V, giving Hypothesis E.3.36—so that by E.3.38 we have $n(H) \leq n'(Aut_M(V)) = n$, contrary to assumption.

Hence we may assume that B does not normalize V, so in particular $V < V_M$. By 11.0.3.5, $\bar{L} \cong SL_3(q)$, and then by 11.0.3.4, $C_V(B) = 1$, so $C_V(H) = 1$. In particular as $Z \cap V \neq 1$, $[Z, H] \neq 1$, so $L = [L, J(T)]$ by 3.1.8.3. Now the argument in the final paragraph of the proof of 11.0.3 and an appeal to B.5.1.1.ii shows V_M is the sum of two isomorphic natural modules for $\bar{L} \cong SL_3(q)$. Thus $C_{GL(V_M)}(L) \cong GL_2(q)$, so if $m > 2n$ then $C_B(V_M) \neq 1$, contrary to paragraph one. Thus $m \leq 2n$, completing the verification of (2) and hence the proof. □

Recall $M_V = N_M(V) = N_G(V)$ and $T_L = T \cap LO_2(LT) = T \cap LC_T(V)$.

LEMMA 11.2.2. *Set $m := 2n$ if $\bar{L} \cong G_2(q)$ and $m := n$ otherwise. Let $U \leq V$, and set $k := m(V/U)$. Then*

(1) $m(\bar{M}_V, V) = m$.

(2) Assume that $O^{2'}(C_M(U)) \leq C_M(V)$ and $k < 2m$. Then $C_G(U) \leq M$, and so $O^{2'}(C_G(U)) \leq C_M(V)$.

(3) Either $r(G, V) > m$; or $\bar{L} \cong SL_3(q)$, $r(G, V) = m$, and $C_G(V_2) \not\leq M$. In particular, $s(G, V) = m$.

(4) $W_j(T, V) \leq T_L$ for $j < m - 1$, so $V_1 \leq C_V(W_j(T, V))$.

(5) If $\bar{L} \cong G_2(q)$ then $W_0(T, V) \leq C_T(V)$, so $N_G(W_0(T, V)) \leq M$; that is, $w(G, V) > 0$.

(6) If $\bar{L} \cong G_2(q)$ then $C_G(C_1(R_1, V)) \leq M$.

PROOF. Part (1) is a standard fact about the natural module and its nonsplit central extensions in I.2.3.1; cf. B.4.6 when $\bar{L} \cong G_2(q)$.

Next we claim $r(G, V) \geq m$. By (1), $m(\bar{M}_V, V) \geq m$; so if $m > 2$, the claim follows from Theorem E.6.3. Assume $m \leq 2$; then $m = 2 = n$, and $\bar{L}$ is $SL_3(4)$ or $Sp_4(4)$. If $\bar{L}$ is $Sp_4(4)$, assume further that $C_V(L) = 1$. Then L is transitive on non-zero vectors in the dual of V, and hence transitive on $\mathbf{F}_2$-hyperplanes U of V, so in particular each hyperplane is invariant under a Sylow 2-subgroup of M. Hence as $m(\bar{M}_V, V) = m \geq n > 1$ by (1), $r(G, V) > 1$ by E.6.13. Thus we may assume $L/O_2(L) \cong Sp_4(4)$, $C_V(L) \neq 1$, and U is a hyperplane of V with $C_G(U) \not\leq M$. By Theorem 4.2.13, $M = !\mathcal{M}(L)$, so $C_U(L) = 1$; hence U is an $\mathbf{F}_2$-space complement to $C_V(L)$, and so $m(C_V(L)) = 1$. Now V is a quotient of the full covering $\hat{V}$ of the natural module $\tilde{V}$ for $\bar{L}$, which has the structure of a 5-dimensional orthogonal space over $\mathbf{F}_4$. From this structure, L has two orbits on the $\mathbf{F}_4$-complements to $C_{\hat{V}}(L)$ in $\hat{V}$, with representatives $\hat{U}_\epsilon$, $\epsilon = \pm 1$, such that $Aut_{\bar{L}}(\hat{U}_\epsilon) \cong O_4^\epsilon(4)$. Moreover each $\mathbf{F}_2$-hyperplane of $\hat{V}$ supplementing $C_{\hat{V}}(L)$ contains such an $\mathbf{F}_4$-hyperplane, so the images U_ϵ of $\hat{U}_\epsilon$, for $\epsilon = \pm 1$, are representatives for the orbits of L on $\mathbf{F}_2$-complements to $C_V(L)$. In particular $N_{LT}(U)$ is maximal in LT but not of index 2, and there is a subgroup Y of order 3 in $N_L(U)$ faithful on U. Next $Z \cap V \not\leq C_V(L)$, so that $Z \cap U \neq 1$, and hence $N_G(U) \in \mathcal{H}^e$ by E.6.6.4. By E.6.7.1, $C_G(U)$ contains a χ-block invariant under $Y = O^2(Y)$. Then as Y is faithful on U, while $m_3(YC_G(U)) \leq 2$ as M is an SQTK-group, $m_3(C_G(U)) \leq 1$. Hence we have the hypotheses of E.6.14, and that lemma supplies a contradiction, completing the proof of the claim that $r(G,V) \geq m$.

Assume the hypotheses of (2). By 11.0.3.4, $C_M(U) \leq M_V$. Then $C_M(U) = C_M(V)$ since if Y is of odd prime order in $\bar{M}_V$, then $m(V/C_V(Y)) \geq 2m$; notice we use 11.0.4 and A.1.41 to conclude $C_{\bar{M}_V}(\bar{L}) = Z(\bar{L})$, and to exclude diagonal outer automorphisms. Now (2) follows from E.6.12 and the fact that $r(G,V) \geq m > 1$.

We have shown $r(G,V) \geq m$. Further in case of equality, we may pick U with $k = m$, $C_G(U) \not\leq M$, and $O^{2'}(C_{\bar{M}_V}(U)) \neq 1$ by (2). But then $U = C_V(i)$ for a suitable root involution $i \in \bar{L}$, and up to conjugacy either:

(i) $\bar{L} \cong Sp_4(q)$ or $G_2(q)$ and $V_3 \leq U$, or
(ii) $\bar{L} \cong SL_3(q)$ and $U = V_2$.

Case (i) contradicts 11.1.4, so that (3) holds.

Let $A \leq V^g \cap T$ with $m(V^g/A) =: j < m-1$. To prove (4), we must show that $A \leq T_L$, so we may assume $\bar{A} \neq 1$. Then for $B \leq A$ with $m(V^g/B) < m = s(G,V)$, $\bar{A} \in \mathcal{A}_{m-j}(\bar{M}_V, V) \subseteq \mathcal{A}_2(\bar{M}_V, V)$, using E.3.10. Hence (using B.4.6.9 in case $\bar{L}$ is $G_2(q)$) $\bar{A} \leq \bar{L}$, so $W_j(T,V) \leq T_L$. Then as $V_1 = C_V(T_L)$, (4) holds.

Assume next that $\bar{L} \cong G_2(q)$ and $j := m(V^g/A) = 0$ or 1 with $\bar{A} \neq 1$. By B.4.6.3, there is $E_{q^3} \cong A_1 \trianglelefteq N_{\bar{L}}(V_1)$ with $C_V(A_1) \in \mathcal{V}_3$. As $\bar{L}$ is $G_2(q)$, $m = 2n$, so as $n \geq 2$, $\bar{A} \in \mathcal{A}_{m-j}(\bar{T}, V) \subseteq \mathcal{A}_{n+1}(\bar{T}, V)$ by the previous paragraph. Hence by B.4.6.9, $\bar{A} \leq A_1^h$ for some $h \in L$, and if $j = 0$, then $\bar{A} = A_1^h$. Further if $j = 1$ and $A \leq R_1$, then by B.4.6.12, $\bar{A} \leq A_1$. Thus $C_V(W_1(R_1,V)) \geq C_V(A_1) = V_3$, so $C_G(C_1(R_1,V)) \leq C_G(V_3) \leq M$ by 11.1.4. That is, (6) holds.

Now take $j = 0$. Thus $\bar{A} = A_1^h$, so without loss $\bar{A} = A_1$. Let $D \leq A$ with $\bar{D}$ a long root group of $\bar{L}$. Then $m(V^g/D) = m(A/D) = 2n = m < r(G,V)$ by (3), so $C_V(D) \leq N_G(A)$. Set

$$E := \langle C_V(D) : D \leq A,\ \bar{D} \text{ is a long root subgroup of } \bar{L} \rangle.$$

Then by B.4.6.3, $m(V/E) = n$ and $[E, A] = V_3$. We just saw $E_D := C_V(D) \leq N_G(A)$, so E acts on A, and hence $V_3 = [E, A] \leq A$. Thus $V_3 \leq V \cap V^g$, so as V is a TI-set under M by 11.0.3.4, $g \notin M$. Furthermore $D = C_A(E_D)$, so $m(A/C_A(E_D)) = m(A/D) = 2n$. Therefore by B.4.6, the image of E_D in $L^g/O_2(L^g)$ is contained in a long root group, and $[E_D, A] =: A_D \in \mathcal{V}_2^g =: \mathcal{V}_2(A)$. But as $\bar{A} = A_1$, also $[E_D, A] \in \mathcal{V}_2 =: \mathcal{V}_2(V)$. So if we define $\Delta(V_3, V) := \mathcal{V}_2(V) \cap V_3$, we see

$$\Delta(V_3, V) = \Delta(V_3, V)^g := \Delta(V_3, A). \qquad (*)$$

Define

$$L(V_3, V) := \langle L(I) : I \in \Delta(V_3, V) \rangle.$$

By (*) and 11.1.3, $L(V_3, V) = L(V_3, V)^g =: L(V_3, A)$. But we check that $L(V_3, V) = L$, so by Theorem 4.2.13,

$$M = !\mathcal{M}(L(V_3, V)) = !\mathcal{M}(L(V_3, A)) = M^g,$$

contradicting $g \notin M$. Together with E.3.16.1, this establishes (5). □

LEMMA 11.2.3. *(1) If $C_V(L) = 1$, then $N_G(V_1) = C_G(V_1)N_M(V_1)$.*
(2) If $C_G(V_1) \leq M$, then $N_G(V_1) \leq M$.

PROOF. Set $Y := N_G(V_1)$ and $Y^* := Y/C_G(V_1)$. Now $C_T(V_1) = T_L$ and $T = \langle f \rangle T_L$ where f induces a field automorphism on $\bar{L}$, so T^* is cyclic. Hence by Cyclic Sylow 2-Subgroups A.1.38, $Y^* = O(Y^*)T^*$.

Assume $C_V(L) = 1$. Then X^* is regular on $V_1^{\#}$, so by A.1.12, X^* is normal in any overgroup of odd order in $GL(V_1)$. Hence $O(Y^*) \leq N_{GL(V_1)}(X^*)$, where the latter group consists of X^* extended by $\mathbf{Z}_n$, and contains T^*. Then $X^*T^* \trianglelefteq O(Y^*)T^* = Y^*$, so by a Frattini Argument, $Y^* = X^*N_Y(T)^* \leq N_M(V_1)^*$, since $N_G(T) \leq M$ by Theorem 3.3.1. Thus (1) holds.

Assume $G_1 := C_G(V_1) \leq M$. In view of (1), we may also assume that $C_V(L) \neq 1$. Hence $N_G(R_1) \leq M$ by 11.1.5. As $G_1 \leq M$, $L_1 \trianglelefteq G_1$, so as T acts on L_1, $L_1 \trianglelefteq N_G(V_1)$ by 1.2.1.3. Then as $R_1 \in Syl_2(C_{G_1}(L_1/O_2(L_1)))$, by a Frattini Argument $Y = C_{G_1}(L_1/O_2(L_1))N_Y(R_1) \leq M$, so that (2) holds. □

LEMMA 11.2.4. *Assume $\langle V^{N_G(V_1)} \rangle$ is abelian, and $[V, W_0(T, V)] \neq 1$. If $\bar{L} \cong SL_3(q)$, assume further that $C_G(V_2) \leq M$. Then*

(1) $W_0(T, V) \leq R_2$.
(2) If $V^g \leq T$ with $[V, V^g] \neq 1$, then $V \not\leq N_G(V^g)$.
(3) $r(G, V) \leq 2n$.

PROOF. By hypothesis $w(G, V) = 0$, so by 11.2.2.5, $\bar{L}$ is not $G_2(q)$ and $s(G, V) = n$ by 11.2.2.3. Furthermore there is $A := V^g \leq T$ with $\bar{A} \neq 1$. As $s(G, V) = n$, $\bar{A} \in \mathcal{A}_n(\bar{T}, V)$ by E.3.10. Let $\hat{A} := A/C_A(L^g)$ and $\dot{L}^g := L^g/C_{L^g}(A)$.

Our hypothesis that $C_G(V_2) \leq M$ when $\bar{L} \cong SL_3(q)$, together with 11.2.2.3, says that $r(G, V) > n$. Thus if $m(A/B) \leq n$, then $C_G(B) \leq N_G(A)$. Also by hypothesis $\langle V^{N_G(V_1)} \rangle$ is abelian, so $g \notin N_G(V_1)$ as $[V, A] \neq 1$.

We next claim there is no $W \leq V$ with $[\tilde{W}, A] = \tilde{V}_1$ and $m(A/C_A(W)) = n = m(W/C_W(A))$. For if so, $W \leq C_G(C_A(W)) \leq N_G(A)$ by the previous paragraph, and then W induces transvections on the $\mathbf{F}_q$-space $\hat{A}$ with axis $\widehat{C_A(W)}$. If $\bar{L}$ is $SL_3(q)$ then $C_V(L) = 1$ and by hypothesis $V_1 = [A, W]$, so V_1 is a 1-dimensional $\mathbf{F}_q$-subspace of A. If $\bar{L}$ is $Sp_4(q)$ then as $m(W/C_W(A)) = n$, $Aut_W(A)$ is a root subgroup of $\dot{L}^g$ inducing transvections on A, so $[\hat{A}, W]$ is a 1-dimensional $\mathbf{F}_q$-subspace of $\hat{A}$, and $C_A(L^g) \leq [A, W]$ by I.2.3.1.ii.b. Thus as $[\tilde{W}, A] = \tilde{V}_1$, $[W, A] = V_1$. Now in either case L^g is transitive on 1-subspaces of $\hat{A}$ with representative $\hat{V}_1^g$, so conjugating in $N_G(A)$ we may assume $g \in N_G(V_1)$, contrary to the previous paragraph.

Next assume that $A \not\leq R_2$. Then $Aut_A(V_2) \in \mathcal{A}_n(Aut_T(V_2), V_2)$, so as R_2 centralizes V_2 and $\tilde{V}_2$ is the natural module for $L_2/O_2(L_2)$, $Aut_A(\tilde{V}_2) \in Syl_2(Aut_{L_2}(\tilde{V}_2))$. Hence $[\tilde{V}_2, A] = \tilde{V}_1$, and $m(A/C_A(V_2)) = n = m(V_2/C_{V_2}(A))$, contary to the claim applied to V_2 in the role of W. Thus (1) is established.

By (1), $A \leq R_2$, and hence $[V, A] \leq V_2$. Suppose that $[\tilde{V}, A] < \tilde{V}_2$. Then $m([A, \tilde{V}]) = n$ and $\bar{A}$ is contained in the root subgroup of a transvection in $\bar{R}_2$. In particular, $m(\bar{A}) = m(A/C_A(V)) = n$ and conjugating in L_2, we may assume $[\tilde{V}, A] = \tilde{V}_1$, contrary to the claim applied to V in the role of W. Therefore $[\tilde{V}, A] = \tilde{V}_2$, so $C_V(A) = V_2$ and $C_{\tilde{V}}(A) = \tilde{V}_2$ since $A \leq R_2$.

We next reduce (3) to (2). Namely as $A \leq R_2$,

$$V = \langle C_V(B) : m(A/B) \leq 2n \rangle,$$

so if $r(G, V) > 2n$ then $V \leq N_G(A)$, contrary to (2).

Thus it remains to prove (2), so we may assume that $V \leq N_G(A)$.

Assume first that $V_2 = [A, V]$. Then as $V \leq N_G(A)$, $V_2 = [A, V] \leq V \cap A$. By symmetry between A and V, since $\tilde{V}_2 = C_{\tilde{V}}(A)$, $\hat{V}_2 = \widehat{C_A(V)}$ is an L^g-conjugate of

$\hat{V}_2^g$. Thus V_2 is an L_2-conjugate of V_2^g, and hence we may assume $g \in N_G(V_2)$. In particular $V_1^g \leq V_2$. Now by 11.1.3, $g \in N_G(L_2)$, so g permutes the fixed points of Sylow 2-groups of L_2. Of course $V_1^{L_2}$ is the set of these subspaces, so conjugating in L_2, we may assume $g \in N_G(V_1)$, contradicting an earlier observation. Thus $[A, V] < V_2$, so as $\tilde{V}_2 = [A, \tilde{V}]$, $C_V(L) \neq 1$, and hence $\bar{L} \cong Sp_4(q)$.

Now $V_2 = C_V(A)$, so $m(Aut_V(A)) = 2n$, and by symmetry $m(\bar{A}) = 2n$. If some $\bar{a} \in \bar{A}$ induces an $\mathbf{F}_q$-transvection on $\tilde{V}$, then without loss $[V_3, a] = 1$. Hence $\dot{V}_3$ is the root group of a transvection on $\hat{A}$, and then by symmetry $\bar{A}$ contains the root group centralizing V_3 and $[\tilde{V}_3, A] = \tilde{V}_1$. Then $m(A/C_A(V_3)) = n$, contrary to the claim applied to V_3 in the role of W. Therefore $\bar{A}$ contains no transvections.

Let $\bar{R}_l$ and $\bar{R}_k$ be root groups of transvections contained in $\bar{R}_2$, $\bar{S} := \bar{R}_l\bar{R}_k$, and $\bar{A}_S = \bar{A} \cap \bar{S}$. Then $m(\bar{S}/\bar{A}_S) \leq m(\bar{R}_2/\bar{A}) = n$, so as $\bar{R}_l \cap \bar{A} = 1$, we conclude $\bar{S} = \bar{R}_l \times \bar{A}_S$ and $m(\bar{A}_S) = n$. Now $\bar{R}_k \leq \bar{Y} \leq C_{\bar{L}}(\bar{R}_l)$ with $\bar{Y} \cong L_2(q)$, and setting $V_Y := [V, Y]$, $\tilde{V}_Y$ is a nondegenerate 2-dimensional $\mathbf{F}_q$-subspace of $\tilde{V}$ with $V_Y \leq C_V(\bar{R}_l)$. Indeed taking $\bar{R}_k \trianglelefteq \bar{T}$, $V_1 = [V_Y, \bar{R}_k]$, so $V_1 = [V_Y, \bar{S}] = [V_Y, A_S]$; hence as $\tilde{V}_2 = [V, A]$, $V_2 = [V, A]$. This contradicts an earlier reduction, and completes the proof. □

LEMMA 11.2.5. *Let $H \in \mathcal{H}(T)$ with $H \not\leq M$. If $\bar{L} \cong SL_3(q)$, assume further $C_G(V_2) \leq M$. Then either*

(1) $W_0(T, V) \not\leq O_2(H)$, or

(2) $\langle V^{N_G(V_1)}\rangle$ is nonabelian.

PROOF. We observe that Hypothesis F.7.6 is satisfied with LT, H in the roles of "G_1, G_2". Adopt the notation of section F.7, and assume $W_0(T, V) \leq O_2(H)$. Then the parameter b of Definition F.7.8 is even by F.7.9.4. Thus by F.7.11.2, there exists $g \in G$ with $1 \neq [V, V^g] \leq V \cap V^g$, so $\langle V^{N_G(V_1)}\rangle$ is nonabelian in view of 11.2.4.2, and hence (2) holds. □

11.3. Eliminating the shadow of $\mathbf{L_4(q)}$

Notice that when $\bar{L} \cong SL_3(q)$, the condition $C_G(V_1) \leq M$ distinguishes $L_4(q)$ from the other shadows. In this section, we eliminate that troublesome configuration, and also (when we show $C_V(L) = 1$) eliminate the shadow of $Sp_6(q)$ in the case $\bar{L} \cong Sp_4(q)$.

Throughout this section we assume:

HYPOTHESIS 11.3.1. *(1) There exists $H \in \mathcal{H}_*(T, M)$ with $[Z, H] \neq 1$.*

(2) If $\bar{L} \cong SL_3(q)$ then $C_G(V_1) \leq M$.

The object of this section is to prove:

PROPOSITION 11.3.2. *Assume Hypothesis 11.3.1. Then*

(1) $\bar{L} \cong Sp_4(q)$.

(2) $C_G(V_1) \not\leq M$.

(3) $C_V(L) = 1$.

(4) If $W_0(T, V) \leq O_2(H)$, then $\langle V^{C_G(V_1)}\rangle$ is nonabelian.

During this section we assume the pair G, L is a counterexample to Proposition 11.3.2. We begin a series of reductions.

LEMMA 11.3.3. *$W_0(T, V) \not\leq O_2(H)$.*

PROOF. Assume that $W_0(T,V) \leq O_2(H)$. We will show that 11.3.2 holds, contrary to our choice of G as a counterexample. When $\bar{L} \cong SL_3(q)$, we have $C_G(V_2) \leq C_G(V_1) \leq M$ by Hypothesis 11.3.1.2, so we may apply 11.2.5 to conclude that $\langle V^{N_G(V_1)}\rangle$ is nonabelian. As V is a TI-set under M by 11.0.3.4, this forces $N_G(V_1) \not\leq M$, so conclusion (2) of 11.3.2 follows from 11.2.3.2. Next $C_V(L) \leq V_1 \trianglelefteq T$, so if $C_V(L) \neq 1$ then $C_{V_1}(LT) \neq 1$, and hence $C_G(V_1) \leq C_G(C_{V_1}(LT)) \leq M = !\mathcal{M}(LT)$, whereas we just saw 11.3.2.2 holds; this contradiction establishes conclusion (3) of 11.3.2. Hence $N_G(V_1) = C_G(V_1)N_M(V_1)$ by 11.2.3.1, so $\langle V^{C_G(V_1)}\rangle$ is nonabelian as $N_M(V_1) \leq M_V$, establishing conclusion (4) of 11.3.2.

By Hypothesis 11.3.1.2 and as 11.3.2.2 holds, L is not $SL_3(q)$. Since $H \not\leq M = !\mathcal{M}(N_G(O_2(LT))$, while H normalizes $W_0(T,V)$ by E.3.15 since $W_0(T,V) \leq O_2(H)$, $\bar{L}$ is not $G_2(q)$ by 11.2.2.5. Thus $\bar{L} \cong Sp_4(q)$, so that conclusion (1) of 11.3.2 holds. But now the choice of G as a counterexample is contradicted. □

Set $W_0 := W(T,V)$. By 11.3.3, $W_0 \not\leq O_2(H)$, so part (4) of Proposition 11.3.2 is vacously satisfied. Thus we only need to establish parts (1)–(3).

Let $V_H := \langle Z^H\rangle$, $H^* := H/C_H(V_H)$, $m := s(G,V)$, and $k := n(H)$. By B.2.14, $V_H \in \mathcal{R}_2(H)$. As $W_0 \not\leq O_2(H)$ while $C_H(V_H)$ is 2-closed by B.6.8, there exists $A := V^g \leq T$ with $A^* \neq 1$.

LEMMA 11.3.4. *(1) $A^* \in \mathcal{A}_m(H^*, V_H)$.*
(2) $\bar{L}$ is $SL_3(q)$ or $Sp_4(q)$.
(3) Either $k = n$, or $k > n$ and conclusion (2) of 11.2.1 holds.
(4) $N_G(V_1) \leq M_V$.

PROOF. Part (1) follows from E.3.6. By E.3.20, $k \geq m$. By 11.2.2, $m \geq n$, and $m = 2n$ if $\bar{L}$ is $G_2(2^n)$. Finally by 11.2.1, either $k \leq n$, or conclusion (2) of 11.2.1 holds. Thus (2) and (3) hold.

Suppose $C_G(V_1) \not\leq M$. Then conclusion (2) of Proposition 11.3.2 holds, and hence (as we saw during the proof of 11.3.3) also conclusion (3) of 11.3.2 holds. Then by Hypothesis 11.3.1.2, $\bar{L}$ is not $SL_3(q)$, so that conclusion (1) of Proposition 11.3.2 holds, contrary to our choice of G as a counterexample. Thus $C_G(V_1) \leq M$, and hence $N_G(V_1) \leq M_V$ by 11.2.3.2 and 11.0.3.4. This establishes conclusion (4), and completes the proof. □

LEMMA 11.3.5. *(1) $O^2(H) = \langle K^H\rangle$, with $K \in \mathcal{C}(H)$ and $K/O_2(K) \cong L_2(2^k)$.*
(2) If $k > n$ assume $k = 2n$. Then $K = O^2(H)$.

PROOF. By 11.3.4.3, $k \geq n > 1$. Therefore by E.2.2, $O^2(H) = \langle K^H\rangle$, for $K \in \mathcal{C}(H)$ described in E.2.2; in particular $K/O_2(K)$ is of Lie type over $\mathbf{F}_{2^k}$. As $[Z,H] \neq 1$ by Hypothesis 11.3.1.1, $K \in \mathcal{L}_f(G,T)$, so $K \leq K_+ \in \mathcal{L}_f^*(G,T)$ by 1.2.9.2. Now the possibilities for the embedding of K in K_+ are described in the list of A.3.12. In particular if $K/O_2(K)$ is not $L_2(2^k)$, then we conclude by comparing that list with those of Theorems B.5.1 and B.5.6, that K_+T has no FF-module—contrary to Theorem 7.0.1.

Thus (1) is established, so we assume the hypotheses of (2) with $K < O^2(H)$. By 11.3.4.3 and the hypotheses of (2), $k = n$ or $2n$.

Let D be a Hall $2'$-subgroup of $H \cap M$, p a prime divisor of $q-1$, and $D_p := \Omega_1(O_p(D))$. As $k = n$ or $2n$, $D_p \neq 1$. As $K < O^2(H)$, $D = D_1 \times D_1^t$ for $D_1 := D \cap K$ and $t \in N_T(D) - N_T(K)$. Thus $[D_p,t] \neq 1 \neq C_{D_p}(t)$.

By 11.0.4, $D_p \le L$; so as $D_pT = TD_p$, $\bar{D}_p$ is contained in a Cartan subgroup of $\bar{L}$. As $[D_p, t] \ne 1$, t induces a field automorphism on $\bar{L}$. But then either $C_{D_p}(t) = 1$ or t centralizes D_p, contrary to the previous paragraph. This completes the proof of (2). □

LEMMA 11.3.6. $k = n$.

PROOF. Assume $k > n$ and let $\hat{M} := M/C_M(V_M)$ and $C_T(V_M) := Q_M$. By 11.3.4.3, conclusion (2) of 11.2.1 holds, so $\bar{L} \cong SL_3(q)$, V_M is the sum of two conjugates of V, $k \le 2n$, and $m(\hat{M}, V_M) = 2n$. We first observe that the weak closure hypothesis E.6.1 is satisfied with V_M in the role of "V"; in particular LT normalizes $C_T(V) \cap C_M(V_M) = Q_M$, so $M = !\mathcal{M}(N_M(Q_M))$. As $m(\hat{M}, V_M) = 2n > 2$, $m(\hat{M}, V_M) = 2n = s(G, V_M)$ by Theorem E.6.3.

Suppose first that $B := V_M^y \le T$ for some $y \in G$ with $\hat{B} \ne 1$. Then $m(\hat{B}) \le m_2(\hat{M}) = m_2(\hat{L}) = 2n = s(G, V_M)$. On the other hand, $\hat{B} \in \mathcal{A}_{2n}(\hat{T}, V_M)$ by E.3.10, so $m(\hat{B}) = 2n$. Now assume further that $V_M \le N_G(B)$. Then $[V_M, B] \le V_M \cap B$, so by symmetry between B and V_M, $m(V_M/C_{V_M}(B)) = 2n$. Thus from the structure of the natural module V for $\bar{L} \cong SL_3(q)$, $\hat{B} = \hat{R}_2$ and $V_2 = C_V(B)$. Now for $v \in V - V_2$, $[v, B] = [V, B]$, so by the symmetry between V_M and B, v induces a root element of $M^y/C_{M^y}(B)$, and V induces the corresponding root group. Thus $[V, V^y] = V_1$, and by symmetry, $[V, V^y] = V_1^y$; then $y \in N_G(V_1) \le M_V$ by 11.3.4.4, contrary to $[V, V^y] = V_1$. Thus we have shown that if $[V_M, V_M^y] \le V_M \cap V_M^y$, then $[V_M, V_M^y] = 1$.

We next reproduce the argument establishing 11.2.5: Namely we now have Hypothesis F.7.6 with $N_M(Q_M)$, H, V_M in the roles of "G_1, G_2, V"; for example $M = C_M(V_M)N_M(Q_M)$ by a Frattini Argument, so $V_M \in \mathcal{R}_2(N_M(Q_M))$ by 11.0.3.1. If $W_0(V_M, T) \le O_2(H)$, then as in 11.2.5, the parameter b_M for the graph as in Definition F.7.8 is even using F.7.9.4, so $1 \ne [V_M, V_M^g] \le V_M \cap V_M^g$ for some $g \in G$ by F.7.11.2, contrary to the previous paragraph.

Thus there is $B := V_M^g \le T$ with $B \not\le O_2(H)$. By E.3.20, $k \ge s(G, V_M) = 2n$, so as $k \le 2n$, $k = 2n$. Therefore $O^2(H) = K \in \mathcal{C}(H)$ with $K/O_2(K) \cong L_2(2^{2n})$ by 11.3.5.

Next $N_{GL(V_M)}(\bar{L})$ is an extension of $GL_3(q) \times GL_2(q)$ by field automorphisms. Using 11.0.4, we conclude that $N_{\bar{M}}(\bar{L})$ is an extension of $\bar{L}$ by field automorphisms. Therefore for each $U \le V_M$ with $m(V_M/U) = 2n$, $C_{Aut_M(V_M)}(U)$ is a 2-group. Also $m_2(Aut_M(V_M)) \le 3n$.

We claim $r(G, V_M) > 2n$. For assume $U \le V_M$ with $m(V_M/U) = 2n$ and $C_G(U) \not\le M$. Then $1 \ne V \cap U$, so U contains a 2-central involution. By the previous paragraph $O^2(C_M(U)) \le C_M(V_M)$. Thus $O^{2'}(C_M(U)) \not\le C_M(V_M)$ by E.6.12, so conjugating in L if necessary, $V_1 \le U$. But then $C_G(U) \le C_G(V_1) \le M$, so the claim is established.

As $r(G, V_M) > 2n$ while $m(\hat{B}) = 2n$, $V_H \le C_G(C_B(V_H)) \le N_G(B)$. By E.3.6, $B^* \in \mathcal{A}_k(H^*, V_H)$, so as $K^* \cong L_2(2^{2n})$, $B^* \in Syl_2(K^*)$ is of rank $2n$ and $C_{V_H}(B^*) = C_{V_H}(b^*)$ for all $b^* \in B^{*\#}$. Thus by G.1.6, $V_H/C_{V_H}(K)$ is a direct sum of $s \ge 1$ copies of the natural module for K^*. Since V_H normalizes B, $m(Aut_{V_H}(B)) = ks = 2ns$, so as $m_2(Aut_M(V_M)) \le 3n$ by an earlier remark, we conclude that $s = 1$ and $V_H/C_{V_H}(K)$ is the natural module for K^*. Now $m(Aut_{V_H}(B)) = k = m(B/C_B(V_H))$, so as B is the sum of two isomorphic natural modules for $SL_3(q)$, V_H induces R_2^g on B, $m([B, V_H]) = 4n$, and $V_2^g \le [B, V_H]$.

As $m([B, V_H]) = 4n$, V_H is the $6n$-dimensional maximal central extension of the natural module for K^* appearing in I.2.3.1. Then from the structure of that module as orthogonal 3-space over $\mathbf{F}_{2^n}$, $[V_H, a] \cap [V_H, b] = 1$ for $a^* \neq b^*$ in B^*; whereas from the action of R_2^g on V^g, for elements $a, b \in V^g$ in distinct cosets of V_2^g, we $[a, V_H] = [a, R_2^g] = V_2 = [b, R_2^g] = [b, V_H]$. This contradiction completes the proof of the lemma. □

We now obtain successive restrictions forcing various 2-locals to closely resemble those in the shadow $L_4(q)$.

Set $K := O^2(H)$, let D be a Hall $2'$-subgroup of $H \cap M$, and further set $D_0 := O^3(D)\Omega_1(O_3(D))$. Recall X is a Cartan subgroup of L acting on T_L.

LEMMA 11.3.7. *(1)* $K \in \mathcal{C}(H)$ *and* $K^* \cong K/O_2(K) \cong L_2(2^n)$.
(2) $D_0 \leq L$.
(3) $C_Z(L) = C_V(L) = 1 = C_{V_H}(K)$.
(4) $V = A$ *and* $V^* \in Syl_2(K^*)$.
(5) $V_1 = [V_1, D_0]$ *and we may take* $D_0 \leq X$.

PROOF. Part (1) follows from 11.3.6 and 11.3.5.2, and then (2) follows from 11.0.4. Next $\bar{L} \cong SL_3(q)$ or $Sp_4(q)$ by 11.3.4.2, so that $m = s(G, V) = n$ by 11.2.2. Thus $A^* \in \mathcal{A}_n(H^*, V_H)$ by 11.3.4.1, so it follows that $A^* \in Syl_2(K^*)$ is of rank n and $C_{V_H}(A^*) = C_{V_H}(a^*)$ for all $a^* \in A^{*\#}$. Then by G.1.6, $V_H/C_{V_H}(K)$ is a direct sum of s copies of the natural module for K^*.

Let $V_L := [\langle Z^L \rangle, L] = [Z, L]$, so that $V \leq \langle Z^L \rangle = V_L C_Z(L)$ using B.2.14. Similarly $V_H = [V_H, K]C_Z(K)$. As $[Z, H] \neq 1$ by Hypothesis 11.3.1.1, $L = [L, J(T)]$ by Theorem 3.1.8.3; therefore by Theorem B.5.1.1, either $V_L = V$ and $\tilde{V}$ is the natural module for $\bar{L}$, or $\bar{L} \cong SL_3(q)$ and V_L is the sum of two isomorphic natural modules for $\bar{L}$. As $D_0 \leq L$ and $TD_0 = D_0 T$, we may take $D_0 \leq X$, and either $C_Z(D_0) = C_Z(L)$, or conjugating in $N_L(V_2)$ if necessary, we may assume $[V_1, D_0] = 1 = [Z, D_0]$. On the other hand as $V_H/C_{V_H}(K)$ is a sum of natural modules for K^* and $V_H = [V_H, K]C_Z(K)$, $C_Z(D_0) = C_Z(K)$. In particular, $[Z, D_0] \neq 1$, so $C_Z(D_0) = C_Z(L)$. Then as $K \not\leq M = !\mathcal{M}(LT)$, $1 = C_Z(K) = C_Z(D_0) = C_Z(L)$, so (3) follows, [3] $[V_H, K] = V_H$, $V_1 = [V_1 \cap Z, D_0] \leq V_H$, and the proof of (5) is complete. By 11.2.2.4, $V_1 \leq C_{V_H}(W_0) \leq C_{V_H}(A)$.

Next $C_G(V_2) \leq C_G(V_1) \leq M$ using 11.3.4.4. Thus as $m(A/C_A(V_H)) = n$, 11.2.2.3 says $V_H \leq C_G(C_A(V_H)) \leq N_G(A)$. Now V_H centralizes $C_A(V_H)$ of corank n in A, so $V_H/C_{V_H}(A)$ is contained in the group Λ of all $\mathbf{F}_q$-transvections on A with axis $C_A(V_H)$. From the action of A^* on V_H, $C_{V_H}(A)$ is of rank sn, so as $m_2(\Lambda) = 2n$, n for $\bar{L} \cong SL_3(q), Sp_4(q)$, conjugating in L^g if necessary, either:

(i) $s = 1$, $m(V_H/C_{V_H}(A)) = n$, and $[A, V_H] = V_1^g$, or
(ii) $s = 2$, $\bar{L} \cong SL_3(q)$, $Aut_{V_H}(A) = Aut_{R_2^g}(A)$, and $V_2^g = [A, V_H]$.

In case (i), $V_1^g = [A, V_H] = C_{V_H}(A)$, so $V_1^g = V_1$ using our earlier observation that $V_1 \leq C_{V_H}(A)$; hence $A = V$ by 11.3.4.4. Similarly in case (ii), from the action of R_2^g on A, for each $u \in V_H - V_2^g$, $[u, A] = V_1^{gx}$ for some $x \in L_2^g \leq N_G(A)$. Further V_H is the sum of two natural modules for K^* and $V_1 = [V_1, D_0] \leq C_{V_H}(A)$, so V_1 is a 1-dimensional $\mathbf{F}_q$-subspace of $C_{V_H}(A)$. Therefore $V_1 = [A, u]$ for some $u \in V_H - V_2^g$, so again $V_1 = V_1^{gx}$ for some $x \in N_G(A)$, and hence we may assume $V_1 = V_1^g$, so $A = V$ by 11.3.4.4. This completes the proof of (4). □

[3] So we have eliminated the shadow of $Sp_6(q)$ where $\bar{L} \cong Sp_4(q)$ and $C_V(L) \neq 1$.

By 11.3.7.4, $V = A \not\leq O_2(H)$. Therefore we have Hypothesis E.2.8 with H, T, $H \cap M$ in the roles of "H, T, M". Let $h \in H - M$ and set $I := \langle V, V^h \rangle$. By 11.3.7, $K^* \cong L_2(2^n)$ with $V^* \in Syl_2(K^*)$, so $I^* = K^*$. Thus I is in the set $\mathcal{I}(H, T, V)$ defined in Definition E.2.4. Therefore by E.2.9.1, $O_2(H)$ acts on I, so $K = O^2(I)$, and $I \trianglelefteq H$ as T acts on V and $H = KT$.

Next by E.2.11, the hypotheses of E.2.10 are satisfied with I, $M \cap I$, $T \cap I$, V in the roles of "H, M, T, V". Notice also that $B := C_V(V_H) = V \cap O_2(H)$ plays the role of "B" in E.2.10, and by that result $P := BB^h$ is a normal 2-subgroup of I. Since $V \cap V^h \leq Z(I)$ and $K = O^2(I)$, $V \cap V^h = 1$ since $C_{V_H}(K) = 1$ by 11.3.7.3. Therefore $P = B \times B^h$ and $B = C_P(V^*)$ by E.2.10.2. By E.2.10.7, $P = O_2(I)$, and by G.1.7, P is a sum of j natural modules for $K/O_2(K) \cong L_2(q)$, so $P = [P, K]$ and hence $P = O_2(K)$. Therefore B^h acts faithfully on V, with $B = C_V(B^h)$ of corank n in V and $m(B^h) = m(B) = jn$. Thus B^h is a group of transvections with axis B, so $\bar{L} \cong SL_3(q)$, $j = 2$, and B is T-invariant of rank $2n$; hence $B = V_2$ and $\bar{P} = \bar{B}^h = \bar{R}_2$. Thus P is of rank $4n$ with $P \cap V = V_2$. As $V_2 \cap Z \leq V_1$, and $V_1 = [V_1, D_0]$ by 11.3.7.5, $V_K := \langle (Z \cap P)^H \rangle = \langle V_1^H \rangle$. As P is a sum of two natural modules for $K^* \cong L_2(2^n)$, V_K is a natural submodule of P of rank $2n$ and

$$[O_2(LT), V_K] \leq O_2(LT) \cap V_K = V_1.$$

Therefore $L = [L, V_K]$ centralizes $O_2(LT)/V$, so L is an $SL_3(q)$-block. Thus L/V is a covering group of $SL_3(q)$, so from the list of Schur multipliers in I.1.3, either $V = O_2(L) = C_L(V)$, or $q = 4$ and $O_2(L/V) \neq 1$. However in the latter case $(R_2 \cap L)/V$ does not split over $O_2(L)/V$ by I.2.2.3b, whereas $\bar{P} = \bar{R}_2$ and $PV/V \cong P/V_2 \cong E_{q^2}$ is T-invariant. Thus $V = C_L(V) = O_2(L)$, and then as $P = J(PV) \cong E_{q^4}$, $P = O_2(L_2)$.

Recall X is a Cartan subgroup of L acting on T_L and we may take $D_0 \leq X$ by 11.3.7.5. Notice that in the shadow $L_4(q)$, the Cartan group D of $H \cap M$ is not contained in the derived subgroup L of M, and DX is a group of rank 3 for primes dividing $q - 1$. Indeed, $D_0 \not\leq L$, so it remains to show that the unnatural inclusion $D_0 \leq L$ leads to a contradiction. This is accomplished by studying the action of X on P and V.

Assume n is even. Then the subgroup D_3 of D_0 of order 3 is contained in X. However $C_P(D_3) = 1 = C_V(D_3)$ as P is the sum of natural modules for K^* and $V^* = [V^*, D_3]$, whereas $C_X(L/V) \in Syl_3(O_{2,Z}(L))$ is the only subgroup of X of order 3 having no fixed points on V, and it centralizes PV/V.

Thus n is odd, so $D = D_0 \leq X$ and $T = T_L$. Now $C_T(L) = 1$ by 11.3.7, and $H^1(L/V, V) = 0$ by I.1.6, so that $V = O_2(LT)$ by C.1.13.b. Thus $T = T_L \in Syl_2(L)$. Set $G_P := N_G(P)$, $\hat{G}_P := G_P/P$, and $Y := \langle L_2, I \rangle$. We've seen that $P = O_2(L_2) = O_2(K)$ with $\hat{L}_2 \cong \hat{K} \cong L_2(q)$ and $\hat{T} \cong E_{q^2}$. As $K \in \mathcal{L}(G_P, T)$, $K \leq K_+ \in \mathcal{C}(G_P)$ by 1.2.4, and if $K < K_+$, then the embedding of K in K_+ is described in A.3.12. As $\hat{K} \cong L_2(2^n)$ with n odd and $\hat{T}$ is abelian, we conclude $K = K_+ \in \mathcal{C}(G_P)$. Similarly $L_2 \in \mathcal{C}(G_P)$. Next $\hat{L}_2 \neq \hat{K}$ since $K \not\leq M$, so $\hat{Y} = \hat{K} \times \hat{L}_2 \cong \Omega_4^+(q)$. As $|\hat{T}| = q^2 = |\hat{Y}|_2$, $Y = O^{2'}(G_P)$. As P is the sum of two copies of the natural module for $\hat{K}$, and V and P/V are natural modules for $\hat{L}_2$, P is the orthogonal module for $\hat{Y}$. As $X \leq N_G(P)$ and $m_p(N_G(P)) \leq 2$ for each prime divisor p of $q - 1$, X is a Cartan subgroup of Y.

We next show:

LEMMA 11.3.8. *There exist $\mathbf{F}_q$-structures on P and V, preserved by Y and L, respectively, which agree on $P \cap V = V_2$.*

PROOF. Let

$$E_V := End_{\mathbf{F}_q L}(V),\ E_{P\cap V} := End_{\mathbf{F}_q L_2 X}(P \cap V),\ \text{ and } E_P := End_{\mathbf{F}_q Y}(P).$$

Then $E_W \cong \mathbf{F}_q$ for each $W \in \{P, V, P \cap V\}$. In particular we may regard $E_{P\cap V}$ as the restriction of E_W to $P \cap V$ for $W := P, V$, so the lemma holds. $\square$

Let $X_1 := C_X(L_1/O_2(L_1))$ and $X_2 := X \cap L_2$. Let W_1 be an X-complement to $V \cap P$ in V, and W_2 an X-complement to $P \cap R_1$ in P. Finally let $W_3 := [W_1, W_2]$. Then W_3 is X-invariant, and $\langle W_1, W_2 \rangle = W_1 W_2 W_3$ is a special group of order q^3 with center W_3. By 11.3.8, the $\mathbf{F}_q$-structures on P and V restrict to X-invariant $\mathbf{F}_q$-structures on W_i, which agree on W_3. Thus we may regard W_i as an $\mathbf{F}_q X$-module.

LEMMA 11.3.9. *The map $c : W_1 \times W_2 \to W_3$ defined by $c(w, w') := [w, w']$ is X-invariant and $\mathbf{F}_q$-bilinear.*

PROOF. Since X acts on W_i, c is X-invariant and $\mathbf{F}_2$-bilinear. Pick generators w_i for W_i as an $\mathbf{F}_q$-space with $[w_1, w_2] = w_3$. Using the $\mathbf{F}_q$-structure on P, we may write $X_2 = \{x(\lambda) : \lambda \in \mathbf{F}_q^\#\}$ so that $x(\lambda)w_2 = \lambda w_2$. Next $[V, X_2] \le [V, L_2] \le V_2 = P \cap V$ from the action of L on V, so as W_1 is X-invariant, $[W_1, X_2] = 1$. As W_1 centralizes X_2, it acts on the λ-eigenspace of $x(\lambda)$ on P; then as W_2 is contained in that eigenspace, so is $W_3 = [W_1, W_2]$—and hence $x(\lambda)w_3 = \lambda w_3$. Thus

$$\lambda w_3 = x(\lambda)w_3 = [x(\lambda)w_1, x(\lambda)w_2] = [w_1, \lambda w_2],$$

and hence c is linear in its second variable. Similarly X_1 centralizes W_2, since W_2 covers a Sylow 2-group of $L_1/O_2(L_1)$, so $W_1 W_3$ is an eigenspace for each member of $X_1^\#$ on V, and the same argument shows c is linear in its first variable. $\square$

We are now in a position to obtain a contradiction, and hence finally eliminate the shadow of $L_4(q)$. Let y be a generator for $X \cap K = D$. Then y has two eigenspaces on P: $P \cap V = V_2$ and $\langle W_2^{L_2} \rangle$. Let λ be the eigenvalue on the second space; then as y is of determinant 1 on P, y has eigenvalue λ^{-1} on $P \cap V$. Similarly y is of determinant 1 on V, so the eigenvalue for y on W_1 is λ^2. Then by 11.3.9, the eigenvalue for y on W_3 is the product $\lambda^2\lambda = \lambda^3$ of its eigenvalues on W_1 and W_2. This is impossible, as $W_3 = [W_1, W_2] \le [V, P] = P \cap V$ and the eigenvalue for y on $P \cap V$ is λ^{-1}. This contradiction completes the proof of Proposition 11.3.2.

11.4. Eliminating the remaining shadows

Recall from earlier discussion that the shadows other than $Sp_6(q)$ and $\Omega_8^+(q).2$ with $\bar{L} \cong SL_3(q)$, have been eliminated. In these remaining shadows, the centralizer of a 2-central involution is not quasithin, and we essentially eliminate those configurations in 11.4.4 in this section.

LEMMA 11.4.1. *(1) $C_V(L) = 1$. In particular, L is transitive on $V^\#$.*
(2) If $L_1 < K \in \mathcal{C}(C_G(V_1))$, then K is described in case (1) or (2) of 11.1.2.
(3) $R_1 \le O_\infty(KT)$, and $[R_1, X] \le O_2(KT)$.

PROOF. Let $H \in \mathcal{H}_*(T, M)$. If $C_V(L) \neq 1$, then $\bar{L}$ is not $SL_3(q)$ by 11.0.3.3, and $[Z, H] \neq 1$ as $H \not\leq M = !\mathcal{M}(LT)$, contrary to 11.3.2.3. Then 11.0.2.2 completes the proof of (1).

Suppose 11.1.2.3 holds. Observe I satisfies the hypotheses for L, so applying 11.0.2.2 and (1) to $V_I \in Irr_+(I, R_2(IT))$ in the Fundamental Setup (3.2.1), we conclude V_I is the natural module for $I^* := I/C_I(V_I) = I/O_2(I) \cong Sp_4(4)$. From the proof of 11.1.2.3, L_2 stabilizes a line of V_I. In 11.1.2.3, we also have $L/O_2(L) \cong \bar{L} \cong G_2(4)$. By 11.2.2.5, $W_0 := W_0(T, V) \leq C_T(V)$, and hence $W_0 \leq C_{L_2T}(V_2) = O_2(L_2T)$; furthermore $N_G(W_0) \leq M$, so as $I \not\leq M$, $W_0 \not\leq O_2(IT) = C_{IT}(V_I)$. Thus $1 \neq W_0^* \leq O_2(L_2^*T^*) = O_2(L_2^*) = R_2^*$, and R_2^* is of rank 6. Let $A := V^g \leq T$ with $A^* \neq 1$. Let R^* be a root subgroup of $O_2(L_2^*)$, and A_R the preimage in A of $A^* \cap R^*$. Then $m(A^*/A_R^*) \leq m(O_2(L_2^*)/R^*) = 4$. By 11.2.2.3 $r(G, V) > 2n = 4$, so $C_G(A_R) \leq N_G(A)$. Hence from the action of R_2^* on V_I,

$$V_I = \langle C_{V_I}(A_R) : R^* \leq O_2(L_2^*) \rangle \leq N_G(A).$$

On the other hand by 11.2.2.3, $s(G, V) = 4$, so $m(A^*) \geq 4$ by E.3.10. It follows that $[V_I, A] = C_{V_I}(A)$ is of rank 4 so $m(V_I/C_{V_I}(A)) = 4$; thus as A is the natural module for $\bar{L}^g$, $Aut_{V_I}(A)$ is contained in a long root group of $Aut_{L^g}(A) \cong G_2(4)$ of rank 2 (e.g. see (6) and (13) of B.4.6), forcing $m(V_I/C_{V_I}(A)) \leq 2$. This contradiction establishes (2).

If $L_1 = K$, then (3) is immediate. Otherwise by (2), K is described in case (1) or (2) of 11.1.2. In those cases, observe that L_1 has no nontrivial 2-signalizers in $Aut(K/O_\infty(K))$, so that $R_1 \leq O_\infty(KT)$. Since $O_{2,F}(KT)$ is of index 1 or 2 in $O_\infty(KT)$, $[R_1, X] \leq O_2(KT)$, so that (3) holds in these cases also. □

We can now return to our study of the embedding of L_1 in $C_G(V_0)$ for $1 \neq V_0 \leq V_1$, begun in the initial section of the chapter. For $1 \neq V_0 \leq V_1$ with $T \leq N_G(V_0)$, define $K(V, V_0) := \langle L_1^{N_G(V_0)} \rangle$; and for $z \in V_1^\#$, let $K(V, z) = K(V, \langle z \rangle)$.

LEMMA 11.4.2. *Let $z \in C_V(T)^\#$. Then $K(V, z) = K(V, V_1)$.*

PROOF. Let $K := K(V, z)$ and $K_1 := K(V, V_1)$. By 11.1.1, $K \in \mathcal{C}(N_G(\langle z \rangle))$ and $K_1 \in \mathcal{C}(N_G(V_1))$. Then $K_1 \leq C_G(V_1) \leq C_G(z)$, so $K_1 = \langle L_1^{K_1} \rangle \leq K$.

We assume that $K_1 < K$ and derive a contradiction. As $K_1 \trianglelefteq C_G(V_1)$, $K \not\leq C_G(V_1)$. Further $L_1 < K$, so K is described in case (2) or (3) of 11.1.1. As $K \in \mathcal{L}(G, T)$, $K \leq K_+ \in \mathcal{L}^*(G, T)$.

Assume first that $K < K_+$. Then the embedding of K in K_+ is described in A.3.12. The pairs $K/O_2(K)$, $K_+/O_2(K_+)$ appearing there with K described in case (2) or (3) of 11.1.1 (cf. also A.3.13) are: $L_3(4)$, M_{23}; A_7, M_{23}; $L_2(p)$, $(S)L_3^\epsilon(p)$, for a prime $p \geq 11$; $SL_2(p)/E_{p^2}$, $(S)L_3(p)$ for a prime $p \geq 5$; M_{22}, M_{23}; $\hat{M}_{22}$, J_4; and $SL_2(5)/P_0$, $SL_2(5)/P_1$, where P_0 and P_1 are suitable nilpotent groups of odd order. Moreover as $K < K_+$, $[z, K_+] \neq 1$, so in the last case $[z, O_\infty(K_+)] \neq 1$, contrary to 3.2.14. Therefore $K_+/O_2(K_+)$ is quasisimple. Further as $[z, K_+] \neq 1$, $K_+ \in \mathcal{L}_f^*(G, T)$. But this contradicts Theorem 7.0.1, since K_+ has no FF-module by Theorem B.4.2.

Therefore $K = K_+ \in \mathcal{L}^*(G, T)$. As $K_1 < K$, $[V_1, K] \neq 1$. But by A.1.6, $O_2(KT) \leq R_1$, and $V_1 \leq Z(R_1)$, so

$$V_1 \leq \Omega_1(Z(O_2(KT))) =: V_K.$$

In particular $[V_K, K] \neq 1$, so $K \in \mathcal{L}_f^*(G, T)$ by A.4.9.

Suppose first that $K/O_2(K)$ is not quasisimple. Then $O_\infty(K)$ centralizes $R_2(KT)$ by 3.2.14, and we conclude from A.4.11 that $O_{2,F}(K)$ centralizes V_K and hence also V_1. By 11.1.1, either $K = O_{2,F}(K)L_1$; or $K/O_{2,F}(K) \cong SL_2(p)$ for a prime p with $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$. However in the former case, K centralizes V_1, contrary to an earlier remark, so the latter case holds.

By 11.4.1.3, $R_1 \leq O_\infty(KR_1)$. Thus $R_1 \in Syl_2(O_\infty(KR_1))$, so by a Frattini Argument, $K = O_{2,F}(K)K_J$, where $K_J := N_K(R_1)^\infty$. Therefore $K_J/O_2(K_J) \cong K/O_\infty(K) \cong L_2(p)$. As $O_{2,F}(K)$ centralizes V_1 but K does not, $K_J \not\leq C_G(V_1)$.

First $XT \leq N_G(R_1) =: N$. We claim that XT normalizes K_J. By 1.2.4, $K_J \leq K_0 \in \mathcal{C}(N)$. The claim is immediate if $K_J = K_0$, so we may assume that $K_J < K_0$, and hence the embedding is described in A.3.12. As R_1 is Sylow in $O_\infty(KR_1)$ and normal in N, $K_0/O_2(K_0)$ is not $SL_2(p)/E_{p^2}$, so $K_0/O_2(K_0)$ is quasisimple. Hence as $p \geq 11$ since $p \equiv \pm 1 \mod 5$, we conclude that $K_0/O_2(K_0) \cong L_2(p^2)$. But then T does not act on K_1, establishing the claim. Therefore as $K_J \leq C_G(z)$ while $X \leq N_N(K_J)$ by the claim, $V_1 = [z, X] \leq C_G(K_J)$, contrary to the previous paragraph.

Therefore $K/O_2(K)$ is quasisimple, so $V_K \in \mathcal{R}_2(KT)$ and $O_2(KT) = C_T(V_K)$ by 1.4.1.4. Set $K^* := K/C_K(V_K)$. We saw $O_2(KT) \leq R_1$. Thus if $J(R_1) \leq O_2(KT)$ then $J(R_1) = J(O_2(KT))$. Hence $KT \leq N_G(J(R_1))$, so $N_G(J(R_1)) \leq N_G(K)$ by 1.2.7.3. Thus $X \leq N_G(K)$, so as $K \leq C_G(z)$ and $C_V(L) = 1$ by 11.4.1.1, $V_1 = [z, X] \leq C_G(K)$, for our usual contradiction.

Therefore $J(R_1)^* \neq 1$, so we may apply B.2.10.2 with K, T, R_1 in the roles of "L, T, R" to conclude that V_K is an FF-module for K^*T^*, with $K^* = [K^*, J(R_1)^*] = J(K^*T^*, V_K)$. Then Theorems B.5.1 and B.4.2 reduce the list in 11.1.1 to those cases where either K^* is $SL_3(q)$, $Sp_4(q)$, or $G_2(q)$, or $q = 4$ and K^* is A_7.

Assume K^* is not A_7, and let Y be a Hall $2'$-subgroup of $N_K(R_1)$. Then Y centralizes z, and induces a group of order $q - 1$ on $Z_1 := \Omega_1(Z(R_1))$ containing V_1, so as V_1 has order q, we conclude $V_1 < Z_1$. Hence $Y \leq N_G(R_1) \leq M$ by 11.1.5. Then by 11.0.4, $Y_1 := O^3(Y)\Omega_1(O_3(Y)) \leq L$. As Y^* has p-rank 2 for primes p dividing $q - 1$, so does $\bar{Y}$. Therefore as V is the natural module for $L/O_2(L)$. $z \in C_{V_1}(Y_1) = C_V(L)$, contrary to 11.4.1.1.

Thus K^* is A_7. As L_1^* centralizes $z \in V_K - C_{V_K}(K)$, $[V_K, K]$ is not a 4-dimensional irreducible for K^*. Therefore by B.4.2.5, $[V_K, K]$ is the natural 6-dimensional module, $J(R_1)^*$ is generated by a transposition, and $q(K^*T^*, V_K) = 1$, so $m_2(T) = m_2(R_1)$ by B.2.4.3. Thus conjugating in K, there is $A \in \mathcal{A}(T)$ such that A^* induces a field automorphism on L_1^*. However this is impossible since $J(T) \leq LC_T(\bar{L})$ for the natural modules in 11.0.2.2 by (2)–(4) of B.4.2. This contradiction finally shows that $K_1 = K$, completing the proof. □

In the remainder of the section, fix $z \in C_V(T)^\#$.

Set $G_z := C_G(z)$, $M_z := C_M(z)$, and $K := K(V, V_1)$. Then $K = K(V, z) \trianglelefteq G_z$ by 11.4.2. By 11.4.1.2, either $K = L_1$, or K is described in case (1) or (2) of 11.1.2. Furthermore if $G_z \leq M$, then as $z \in V_1$, $C_G(V_1) \leq G_z \leq M$, so for $H \in \mathcal{H}_*(T, M)$, $[Z, H] \geq [z, H] \neq 1$; thus Hypothesis 11.3.1 holds, so $C_G(V_1) \not\leq M$ by 11.3.2.2, a contradiction. Therefore

$$G_z \not\leq M.$$

Define $G_1 := N_G(V_1)$ (as opposed to $C_G(V_1)$ in earlier sections), and $M_1 := N_M(V_1)$. Recall that $X \leq G_1$.

LEMMA 11.4.3. *(1)* $G_z = KC_{G_z}(K/O_2(K))M_z$. *Therefore if* $K = L_1$, *then* $C_{G_z}(K/O_2(K)) \not\leq M$.

(2) $G_1 = K(C_{G_1}(V_1) \cap C_{G_1}(K/O_2(K)))M_1$.

PROOF. Recall $K = K(V, V_1) = K(V, z)$ is normal in G_1 and G_z. Set $Y := C_{G_z}(K/O_2(K))$. From 11.1.2, $Out(K/O_2(K))$ is 2-closed, so $YKT \trianglelefteq G_z$, and hence by a Frattini Argument, $G_z = YKN_{G_z}(T)$. Now by Theorem 3.3.1, $N_G(T) \leq M$, proving the first assertion of (1). So if $K = L_1$, then as $G_z \not\leq M$, $Y \not\leq M$, giving the remaining assertion of (1).

Now instead set $Y := C_{G_1}(V_1) \cap C_{G_1}(K/O_2(K))$. The same proof shows that $C_{G_1}(V_1)T = YTKC_{M_1}(V_1)$. As $C_V(L) = 1$ by 11.4.1.1, $G_1 = C_{G_1}(V_1)M_1$ by 11.2.3.1, proving (2). □

Since $G_z \not\leq M$, *there is* $H \in \mathcal{H}_*(T, M) \cap G_z$; H *has this meaning for the remainder of the chapter.*

Since $n(H) \geq n$ in the shadows, our next result eliminates those groups:

LEMMA 11.4.4. $n(H) = 1$.

PROOF. Assume $n(H) > 1$; then from E.2.2, $O^2(H) = \langle I^T \rangle$ for some $I \in \mathcal{L}(G, T)$ with $I \not\leq M$. By 1.2.4 $I \leq I_1 \in \mathcal{C}(G_z)$, and then by 1.2.1, either $[L_1, I] \leq [K, I_1] \leq O_2(K) \leq O_2(L_1T)$, so $[L_1, I] \leq O_2(L_1)$, or $I \leq I_1 \leq K$. But by 11.4.1.2, either $K = L_1$ or K is described in case (1) or (2) of 11.1.2; in either case L_1 is the unique minimal member of $\mathcal{L}(G, T) \cap K$. Therefore if $I \leq K$ then $I = L_1 \leq M$, contradicting $I \not\leq M$. Thus $[L_1, I] \leq O_2(L_1)$.

Let B be a Hall $2'$-subgroup of $I \cap M$. Then $B \leq C_M(L_1/O_2(L_1))$ and B centralizes z. For each prime divisor p of $q - 1$, L contains each subgroup B_p of B of order p by 11.0.4, so $B_p \leq C_L(z) \cap C_L(L_1/O_2(L_1)) = L \cap R_1$ from the action of L on the natural module V. Therefore $(|B|, q - 1) = 1$.

As B centralizes z, $B \leq M_V$ by 11.0.3.4. Then as $BT = TB$, $[L_1, B] \leq O_2(L_1)$, and $(q - 1, |B|) = 1$, it follows from the action of $N_{Aut(V)}(\bar{L})$ on V that $[V, B] = 1$. But then using Remark 4.4.2, $N_G(B) \leq M$ by Theorem 4.4.3; so $H = \langle H \cap M, N_H(B) \rangle \leq M$, contradicting $H \not\leq M$. □

11.5. The final contradiction

We now work to obtain a contradiction, by analyzing the normal closure $\langle V^{G_1} \rangle$ of V in G_1. The analysis falls into two cases, depending on whether $\langle V^{G_1} \rangle$ is abelian or not. The strong restriction in 11.4.4 will make weak closure methods more effective.

LEMMA 11.5.1. $L/O_2(L)$ *is* $SL_3(q)$ *or* $Sp_4(q)$.

PROOF. In view of 11.0.2, we may assume $\bar{L} \cong G_2(q)$. Hence by parts (5) and (6) of 11.2.2,

$$C_G(C_1(R_1, V)) \leq M \geq N_G(W_0(R_1, V)).$$

Thus it will suffice to find $H_1 \leq H$ with $H_1 \not\leq M$, $n(H_1) = 1$, and $R_1 \in Syl_2(H_1)$: For since $n(H_1) = 1$ and $s(G, V) > 1$ by 11.2.2.3, we may apply E.3.19 to conclude that $H_1 \leq M$, contrary to our choice of H_1.

Suppose first that $K = L_1$. Then by 11.4.3.1, $C_{G_z}(L_1/O_2(L_1)) \not\leq M$, so we may choose H with $[L_1, O^2(H)] \leq O_2(L_1)$ and set $H_1 := R_1O^2(H)$; then $n(H_1) = 1$ using 11.4.4. On the other hand if $L_1 < K$, then by 11.4.1.2, K is described in case

(1) or (2) of 11.1.2. In case (1) of 11.1.2, $O_{2,2'}(K) \not\leq M$, so we may choose $R_1 \leq H_1 \leq R_1 O_{2,2'}(K)$, and then $n(H_1) = 1$ by E.1.13. As $R_1 \in Syl_2(C_G(L_1/O_2(L_1)),$ $R_1 \in Syl_2(H_1)$, completing the proof in these two cases by paragraph one.

In case (2) of 11.1.2, $K/O_2(K) \cong L_2(p)$ for a prime $p \geq 11$, and L_1 has no nontrivial 2-signalizer in $Aut(K/O_2(K))$, so $R_1 = O_2(KT)$. Thus $K \leq N_G(R_1) \leq N_G(W_0(R_1, V)) \leq M$, a contradiction. □

As $G_z \not\leq M$, 11.4.3.1 says $C_{G_z}(K/O_2(K))$ or K is not contained in H. Thus the following choice is possible:

From now on we choose H so that either $[K, O^2(H)] \leq O_2(K)$, or $O^2(H) \leq K$.

In particular notice that if $K = L_1$, then as $H \not\leq M$, $[K, O^2(H)] \leq O_2(K)$. By 11.4.4, $n(H) = 1$. Recall $G_1 = N_G(V_1)$ and set $\tilde{G}_1 := G_1/V_1$.

LEMMA 11.5.2. *(1) If $z \in V \cap V^g$ then $V^g \in V^{G_z}$. That is G_z is transitive on $\{V^g : z \in V^g\}$.*

(2) $K = K(V^g, z)$ for each $g \in G_z$.

PROOF. By 11.4.1.1, L is transitive on $V^\#$, so (1) holds using A.1.7.1.

As $K \trianglelefteq G_z$, for $g \in G_z$

$$K = K^g = K(V, z)^g = K(V^g, z),$$

so (2) holds. □

LEMMA 11.5.3. *Assume $K = L_1$ and set $m := 2n$, $3n$, for $\bar{L} \cong SL_3(q)$, $Sp_4(q)$, respectively. Then*

(1) For each $g \in G - N_G(V)$, $V \cap V^g \leq V_1^y$ for some $y \in L$.

(2) $r(G, V) \geq m$.

(3) $\langle V^{G_1} \rangle$ is nonabelian.

PROOF. Let $g \in G - N_G(V)$. Our first goal is to prove (1), so we may suppose $1 \neq U := V \cap V^g$. By transitivity of L on $V^\#$, we may assume $U \cap V_1 \neq 1$, and we may suppose that $U \not\leq V_1$. For $u \in U^\#$, $V^g = V^{g_u}$ for some $g_u \in C_G(u)$ by 11.5.2.1. Now $u \in V_1^x$ for some $x \in L$, and $K(u) := K(V, u) = K(V^{g_u}, u) = K(V^g, u)$ by 11.5.2.2, while $K(u) = L_1^x$ by our hypothesis that $K = L_1$. Since $U \not\leq V_1$, $L = \langle K(u) : u \in U^\# \rangle$; so as $U^{g^{-1}} \leq V$, $L^{g^{-1}} = L$, and hence $g \in M$; indeed $g \in M_V$ since V is a TI-set under M by 11.0.3.4. This contradicts the choice of g, so (1) is established.

If $U \leq V$ with $m(V/U) < m$, then $U \not\leq V_1^y$ for any $y \in L$, so $C_G(U) \leq M_V$ by (1). Thus (2) holds.

Suppose that $\langle V^{G_1} \rangle$ is abelian. By (2), $C_G(V_2) \leq M$. Hence $W_0 := W_0(T, V) \not\leq O_2(H)$ by 11.2.5. Then since H is a minimal parabolic with $H \cap M$ the unique maximal overgroup of T, $N_H(W_0) \leq H \cap M$. As $m(\bar{M}_V, V) > 1$, $s(G, V) > 1$ by (2). Hence as $n(H) = 1$ by 11.4.4, it will suffice to show $C_G(C_1(T, V)) \leq M$, since then E.3.19 supplies a contradiction. Indeed as $C_G(V_2) \leq M$, it suffices to show $V_2 \leq C_1(T, V)$.

So suppose $A := V^g \cap T$ is of corank at most 1 in V^g, but $[V_2, A] \neq 1$. Let $B := C_A(V_2)$. Then

$$m(V^g/B) \leq m_2(Aut_M(V_2)) + 1 = n + 1 < 2n \leq m,$$

so by (2), $V_2 \leq C_G(B) \leq N_G(V^g)$. Hence $D := C_{V^g}(V_2)$ is of corank at most $n+1$ in V^g, so from the action of M_V^g on V^g either:

(i) D is of corank exactly n in V^g and V_2 induces transvections with axis D on V^g, or

(ii) $n = 2$, $\bar{L} = SL_3(4)$, and V_2 induces a group of field automorphisms on $Aur_{L^g}(V^g)$.

By (1), $[V_2, A] \leq V \cap V^g \leq V_1^y \cap V_1^{gw}$ for some $y \in L$ and $w \in L^g$, so $\bar{A} \leq \bar{T}_L$ and (i) holds; then since $[V_2, \bar{A}] \neq 1$, $[V_2, A] = V_1$. As V_2 induces transvections with axis D on V^g, $V_1 = [A, V_2] \in V_1^{gL^g}$, so we may take $g \in G_1$. But then as $[V, V^g] \neq 1$, we have a contradiction to our assumption that (3) fails. This shows $V_2 \leq C_1(T, V)$, and completes the proof. $\square$

LEMMA 11.5.4. *$C_G(V_1) \not\leq M$, so we may choose $H \leq G_1$ with $O^2(H) \leq C_G(V_1)$.*

PROOF. If $L_1 < K$, then $K \leq C_G(V_1)$ but $K \not\leq M$. On the other hand, if $L_1 = K$, then by 11.5.3.3 $\langle V^{G_1} \rangle$ is nonabelian, so $G_1 \not\leq M$ since $V \leq Z(O_2(M))$. Hence $C_G(V_1) \not\leq M$ by 11.2.3.2, so using 11.4.3.2 and the argument we made just before 11.5.2, we can choose $H \leq G_1$ with $O^2(H) \leq C_G(V_1)$, while aintaining the condition $O^2(H) \leq K$ or $[K, O^2(H)] \leq O_2(K)$. $\square$

Because of 11.5.4, the set $\mathcal{H}_1 := \mathcal{H}(L_1T, M) \cap G_1$ is nonempty. In the remainder of this section we choose $H_1 \in \mathcal{H}_1$ and set $U_H := \langle V_3^{H_1} \rangle$.

LEMMA 11.5.5. *(1) $U_H \leq O_2(H_1)$.*

(2) $\tilde{U}_H \leq Z(O_2(\tilde{H}_1))$, and $\Phi(U_H) \leq V_1$.

(3) If $K = L_1$, then $\tilde{U}_H$ is a direct sum of natural modules for $K/O_2(K)$.

PROOF. Observe that Hypothesis G.2.1 is satisfied with V_3, H_1 in the roles of "V, H"; hence (1) and (2) hold by G.2.2. Further $\tilde{V}_3$ is the natural module for $L_1/O_2(L_1)$, so if $L_1 = K$, then as $K \trianglelefteq G_1$, (3) holds. $\square$

LEMMA 11.5.6. *Let $Y := C_G(V_1) \cap C_G(K/O_2(K))$. Then*

(1) $(|Y|, q-1) = 1$.

(2) $m_3(Y) \leq 1$, and if n is even, then Y is a $3'$-group.

(3) If $I \in \mathcal{C}(Y)$ then $[I, X] \leq O_2(Y)$.

(4) If $P = [P, X] \leq T$ and $\Phi(P) \leq O_2(G_1)$, then $[P, Y] \leq O_2(Y)$.

(5) $[O_2(LT), X] \leq O_2(KT)$.

(6) If $O^2(H_1) \leq K$, then $m(A/A \cap O_2(H_1)) \leq 1$ for each elementary subgroup A of R_1.

PROOF. For p a prime divisor of $q-1$, $m_p(X) = 2$ and $C_X(V_1) = X \cap L_1 = X \cap K$, so $X \cap Y = 1$. Next $O_p(X)$ normalizes Y and hence a Sylow p-group Y_p of Y—so as G_1 is an SQTK-group, $Y_p = 1$, proving (1). Similarly as Y centralizes $L_1/O_2(L_1)$ of order divisible by 3, $m_3(Y) \leq 1$, the first requirement of (2).

Assume n is even. Then Y is a $3'$-group by (1), completing the proof of (2). Further if $I \in \mathcal{C}(Y)$, then $I/O_2(I) \cong Sz(2^k)$. If (3) fails then by (1), $X/C_X(I/O_2(I))$ is a nontrivial group of field automorphisms on $I/O_2(I)$. Let B be an XT-invariant Borel subgroup of $I_0 := \langle I^T \rangle$. Then using 1.2.1.3 as usual, either $B = N_B(T)$, or $I < I_0$ and $N_B(T)T/T \cong \mathbf{Z}_{2^k-1}$. In either case, X acts nontrivially on $N_B(T)T/T$. By 3.3.1, $N_B(T) \leq C_M(V_1) \cap C_G(L_1/O_2(L_1))$; thus a Hall $2'$-subgroup B_0 of $N_B(T)$

acts on R_1X, and hence by a Frattini Argument can be taken to normalize X. Then $[X, B_0] \leq X \cap B_0T = 1$ using (1), contrary to an earlier observation. Thus (3) holds when n is even.

So assume instead n is odd. Then X is a $3'$-group of odd order coprime to $|I|$ by (1). Therefore as $m_3(Y) \leq 1$ by (2), (3) follows from an examination of the list of A.3.15, unless possibly $X/C_X(I/O_2(I))$ induces a nontrivial group of field automorphisms on $I/O_2(I) \cong L_2(2^k)$ or $L_3(2^k)$. In that event, $k = 2^a m$ for some odd m divisible by $|X : C_X(I/O_2(I))|$, and $X/O_2(X)$ induces a faithful group of field automorphisms on the subgroup I_1 of I with $I_1/O_2(I_1) \cong L_2(2^m)$ or $L_3(2^m)$. Further a Borel subgroup of I_1 acts on T, unless possibly some $t \in T$ induces a graph automorphism on $I_1/O_2(I_1) \cong L_3(2^m)$, in which case a subgroup of order $2^m - 1$ acts on T. Then arguing as in the previous paragraph, $[X, B_0] = 1$, contradicting the fact that $X/C_X(I/O_2(I))$ induces nontrivial field automorphisms on $I_1/O_2(I_1)$. This contradiction completes the proof of (3).

Assume the hypotheses of (4). Applying (3) and appealing to 1.2.1.1, we conclude X centralizes $Y^\infty/O_2(Y)$, and hence so does $[P, X] = P$. Thus P centralizes $E(Y/O_2(Y))$. As $\Phi(P) \leq O_2(G_1)$, $P = [P, X]$ centralizes $F(Y/O_2(Y))$ by A.1.26. Thus P centralizes $F^*(Y/O_2(Y))$, establishing (4).

If $L_1 = K$, then $O_2(LT) \leq O_2(KT)$, while if $L_1 < K$, then by 11.4.1.3, $[O_2(LT), X] \leq [O_2(L_1T), X] \leq O_2(KT)$. Thus (5) is established.

Finally if $O^2(H_1) \leq K$, then $L_1 < K$ so K is described case (1) or (2) of 11.1.2. In particular in each case, $m_2(R_1/C_{R_1}(K/O_2(K)) \leq 1$ as $R_1 = O_2(L_1T)$. But $C_{R_1}(K/O_2(K)) \leq O_2(H_1)$, since $O^2(H_1) \leq K$. Thus (6) holds. □

PROPOSITION 11.5.7. *$\langle V^{G_1} \rangle$ is abelian.*

PROOF. Assume that $\langle V^{G_1} \rangle$ is nonabelian. Set $U := \langle V_3^{G_1} \rangle$. By 11.5.5 applied to G_1 in the role of "H_1", $\tilde{U} \leq Z(O_2(\tilde{G}_1))$ and $\Phi(U) \leq V_1$. Let $Y := C_G(V_1) \cap C_G(K/O_2(K))$.

We first treat the case $\bar{L} \cong SL_3(q)$. Then $V_3 = V$ so that U is nonabelian by assumption. Let $x, y \in N_L(X)$ with $V = V_1 \oplus V_1^x \oplus V_1^y$. As $U = \langle V^{G_1} \rangle$ is nonabelian, $V \not\leq Z(U)$, so $\bar{U} \neq 1$. From the proof of 11.5.5.1, Hypothesis G.2.1 is satisfied, so by G.2.5, $L \leq I := \langle U, U^x, U^y \rangle = LU$, $\bar{U} = O_2(\bar{L}_1) = \bar{R}_1$,

$$S := O_2(I) = C_U(V)C_{U^x}(V)C_{U^y}(V),$$

$US/S = O_2(L_1)S/S$, S has an L-series

$$1 =: S_0 \leq S_1 \leq S_2 \leq S_3 \leq S_4 := S$$

such that $S_1 := V$, $S_2 := U \cap U^x \cap U^y$, (and setting $W_i := S_i/S_{i-1}$) L centralizes W_2, W_3 is the direct sum of r copies of the dual V^* of V, and W_4 is the direct sum of s copies of V. As $I = \langle U^L \rangle$, M_1 acts on I and hence on S, as does L since $L \leq I$.

We claim that $S \leq O_2(G_1)$. Set $E := U^x \cap U^y$. By 11.5.5.2, $\Phi(E) \leq V_1^x \cap V_1^y = 1$. From the discussion above, $W_4 = [W_4, L_1]$ and for each irreducible J in W_3, the image of E in J is the X-invariant complement to $[J, L_1]$ in J. Hence $S = [S, L_1]E$. Further X acts on E and $S/S_2 = [S/S_2, X]$, so $E = [E, X]S_2$ and $S = [S, X](U \cap S)$. Now as $\Phi(E) = 1$, $[E, X]$ centralizes $Y/O_2(Y)$ by 11.5.6.4, so as $S_2 \leq U \leq O_2(G_1)$, $E = [E, X]S_2$ centralizes $Y/O_2(Y)$. Then as $[L_1, Y] \leq [K, Y] \leq O_2(Y)$, $S = [S, L_1]E$ centralizes $Y/O_2(Y)$. Also we saw that $S \leq O_2(LT)$, so $[S, X]$ centralizes $K/O_2(K)$ by 11.5.6.5, and hence so does $S = [S, X](S \cap U)$. Thus S centralizes

$KY/O_2(KY)$, so as $G_1 = KYM_1$ by 11.4.3.2 and M_1 acts on S, we conclude that $S \leq O_2(G_1)$, completing the proof of the claim.

As $S \leq O_2(G_1)$, $[\tilde{U}, S] = 1$ by 11.5.52. Consequently $r = 0 = s$, so that $S = S_2 \leq U$ and L is an $SL_3(q)$-block. Since $H^1(\bar{L}, V) = 0$ by I.1.6, C.1.13.b says that $S = C_S(L)V$. As $S_2 \leq U \cap E$ and $\Phi(E) = 1$, $C_S(L) = C_U(L)$ is abelian. As $\bar{U} = \bar{R}_1 = [\bar{R}_1, \bar{L}_1]$, $U = [U, L_1]C_U(L) = (U \cap L)C_U(L)$ and $(U \cap L)/C_{U\cap L}(L)$ is special of order q^5. Further $C_U(L_1) = C_{Z(U)}(L)V_1 = Z(U)$, so that $U/Z(U)$ is of rank $4n$. As L_1 centralizes $Z(U)$, also $[K, L_1] = K$ centralizes $Z(U)$.

Assume that $L_1 < K$. Then by 11.1.2, $n = 2$ and $q = 4$, so that $U/Z(U)$ is of rank $4n = 8$ by an earlier observation. As we are assuming that $\bar{L} \cong SL_3(4)$, case (2) of 11.1.2 does not arise, so by 11.4.1.2, K is described in case (1) of 11.1.2. As $L_1 \trianglelefteq M_1$ and $V \leq U$, $C_K(\tilde{U})$ acts on L_1, so $C_K(\tilde{U}) \leq O_{2,Z}(K)$. Then as K centralizes $Z(U)$, $C_K(U/Z(U)) \leq O_{2,Z}(K)$, impossible as $K/O_{2,Z}(K)$ is not a section of $GL_8(2)$.

Therefore $L_1 = K$. As L is an $SL_3(q)$-block, L_1 has two noncentral 2-chief factors, so 11.5.5.3 says that $\tilde{U}$ is a sum of exactly two copies of the natural module for $K/O_2(K) \cong L_2(q)$. In particular $|U| = q^5$. Therefore as $U = C_{Z(U)}(L)(U \cap L)$ and $|(U \cap L)/C_{U\cap L}(L)| = q^5$, we conclude that $U = U \cap L = T \cap L = O_2(L_1) = O_2(K)$, and $C_{U\cap L}(L) = 1$ so that $V = O_2(L)$.

As $K = L_1 \leq M$, $K_H := O^2(H) \leq Y$ by our choice of H. Let $X_1 := C_X(K/O_2(K)$ and $C := \langle R_1, K_H, X_1\rangle$. Further since $C \leq C_{G_1}(K/O_2(K))$ and $R_1 = C_T(K/O_2(K)$, $R_1 \in Syl_2(C)$. Set $\hat{C} := C/C_C(\tilde{U})$. As $\tilde{U}$ is a sum of two absolutely irreducible modules for $K/O_2(K)$ over $\mathbf{F}_q$, $\hat{C} \leq C_{GL(\tilde{U})}(K/O_2(K)) \cong GL_2(q)$. Since L is an $SL_3(q)$-block with $V = O_2(L)$, as before C.1.13.b says

$$T_L = (T \cap L)C_T(L). \qquad (*)$$

In particular as $U \leq L$ and $\bar{U} = \bar{R}_1$, $R_1 = UC_{R_1}(L)$ by (*) and $C_{R_1}(L)$ centralizes U. Thus $\hat{C}$ is a subgroup of $GL_2(q)$ of odd order. Next $C_H(\tilde{U}) \leq N_H(V) \leq H \cap M$, so $C_H(\tilde{U}) \leq \ker_{H\cap M}(H)$. Therefore by B.6.8, $K_H = O_2(K_H)D$ for some p-group D with $D \cap M = \Phi(D)$. By 11.5.6.1, $(p, q-1) = 1$, so as $\hat{C}$ is a subgroup of $GL_2(q)$ of odd order, $\hat{D}$ is cyclic of order dividing $q+1$ and $[\hat{D}, \hat{X}_1] = 1$. If n is even then $X_0 := C_X(L/V)$ is a subgroup of X_1 of order 3. Then as $\hat{D}$ centralizes $\hat{X}_1$, $H = DT$ acts on $[U, X_0] = V$, contradicting $K_H \not\leq M$. Therefore n is odd so $T = T_L$. Then using (*) and our earlier observations that $O_2(K) = U = T \cap L$,

$$T = (T \cap L)C_T(L) = UC_T(L) = UC_T(U). \qquad (**)$$

Further $[C_T(U), K_H] \leq C_{K_H}(U) \leq O_{2,\Phi}(K_H)$, so H acts on $C_T(U)$ and hence by (**), $H \leq N_G(T) \leq M$ using Theorem 3.3.1, a contradiction. This completes the treatment of the case $\bar{L} \cong SL_3(q)$.

Therefore by 11.5.1 it remains to treat the case $\bar{L} \cong Sp_4(q)$. At several places we use the fact that:

(!) $\bar{L}_1\bar{X}$ is indecomposable on $O_2(\bar{L}_1)$ with chief series $1 < Z(\bar{L}_1) < O_2(\bar{L}_1)$, and $Z(\bar{L}_1) = C_{\bar{M}_V}(V_3)$.

We first observe that $V \leq O_2(G_1)$: For $V = [V, X]$, so by parts (4) and (5) of 11.5.6, V centralizes $KY/O_2(KY)$. Then recalling that $G_1 = KYM_1$ and $V \trianglelefteq M_1$ since V is a TI-set under M by 11.0.3.3.4, the observation is established.

Suppose U is nonabelian. Then as $U = \langle V_3^{G_1} \rangle$, U does not centralize V_3, so $\bar{U} \neq 1$ and $1 \neq [V_3, U] \leq V_1$ by 11.5.5.2. As $U \trianglelefteq L_1T$, $\bar{U} = O_2(\bar{L}_1)$ by (!). Therefore $[V, U] = V_3$, which is impossible since by the previous paragraph, $V \leq O_2(G_1)$, so $[V, U] \leq V_1$ by 11.5.5.2.

Thus U is abelian. If $[V, U] \neq 1$ then as $[V_3, U] = 1$, $\bar{U} = Z(\bar{L}_1)$ by (!). In this case set $W := U = U_H$. On the other hand if $[V, U] = 1$, set $W := W_H$, where $W_H := \langle V^{G_1} \rangle$. As $U \trianglelefteq G_1$, $[U, W] = 1$. As $V \leq O_2(G_1)$, $W \leq O_2(G_1)$, so as we are assuming that W is nonabelian, and as $[V_3, W] = 1$, again $\bar{W} = Z(\bar{L}_1)$. Therefore in either case, $\bar{W} = Z(\bar{L}_1)$, so $[V, W] = [V, Z(\bar{L}_1)] = V_1$, and hence $\Phi(W) = V_1$.

Choose an element $y \in L$ so that $\langle Z(\bar{L}_1), Z(\bar{L}_1)^y \rangle \cong L_2(q)$, and $I := \langle W, W^y \rangle$ contains $X_1 = C_X(L_1/O_2(L_1))$. Observe that $\bar{I} = \langle \bar{W}, \bar{w}^y \rangle$ for each $1 \neq \bar{w} \in \bar{W}$, and $V = V_3 \oplus V_1^y$. Then as $[W, O_2(I)] \leq W \cap O_2(I)$,

$$Q := (W \cap O_2(I))(W^y \cap O_2(I)) \trianglelefteq I,$$

with $[O_2(I), I] \leq Q$. Since $I = \langle W, W^y \rangle$ and $\Phi(W) = V_1$, $[W \cap W^y, I] \leq V_1 V_1^y \leq V$. Also $\Phi(W) = V_1 \leq V$, and

$$Q/(W \cap W^y)V = (WV \cap Q)/(W \cap W^y)V \times (W^y V \cap Q)/(W \cap W^y)V,$$

with $(WV \cap Q)/(W \cap W^y)V = C_{Q/(W \cap W^y)V}(w)$, as $\bar{I} = \langle \bar{W}, \bar{w}^y \rangle$.

We claim that $Q \leq O_2(G_1) =: Q_1$: It follows from G.1.6 that $Q/(W \cap W^y)V$ is a sum of natural modules for $\bar{I}$, so that $C_Q(X_1) \leq (W \cap W^y)V \leq Q_1$. Thus as $Q = [Q, X_1]C_Q(X_1)$, it remains to show $[Q, X_1] \leq Q_1$. Next $\Phi(Q) \leq (W \cap W^y)V \leq Q_1$ and $Q \leq O_2(I) \leq O_2(LT)$, so using parts (4) and (5) of 11.5.6, $[Q, X_1]$ centralizes $KY/O_2(KY)$. Next L_1 is transitive on $V_1^G \cap (V - V_3)$, so by a Frattini Argument, $M_1 = L_1 N_{M_1}(V_1^y)$. Thus as $N_{M_1}(V_1^y)$ acts on $[Q, X_1]$, $G_1 = KYM_1 = KYN_{M_1}([Q, X_1])$, so as $[Q, X_1]$ centralizes $KY/O_2(KY)$, the claim is established.

Next $\bar{L}$ is generated by three conjugates $\bar{I}^{l_i}$, $1 \leq i \leq 3$, of $\bar{I}$ under L_1. As $O_2(L)$ acts on W, it acts on W^{yl_i} for each i, so $[O_2(L), L]$ is the product of $[O_2(L), W] \leq Q \leq Q_1$ and $[O_2(L), W^{yl_i}] \leq Q_1$, so $[O_2(L), L] \leq Q_1$. Then as the Schur multiplier of $Sp_4(q)$ is trivial by I.1.3, $O_2(L) = [O_2(L), L] \leq Q_1$.

Now as $Z(\bar{L}_1) = \bar{W} \leq \bar{Q}_1 \trianglelefteq \bar{L}_1$, $\bar{Q}_1 = \bar{R}_1$ or $\bar{W}$. However in the latter case as $O_2(L) \leq Q_1$, $Q_1 \in Syl_2(IQ_1)$, and then by C.1.29, there is a nontrivial characteristic subgroup Q_0 of Q_1 normal in IQ_1. But then $Q_0 \trianglelefteq \langle I, L_1T \rangle = LT$, so $G_1 \leq N_G(Q_1) \leq N_G(Q_0) \leq M = !\mathcal{M}(LT)$, contradicting 11.5.4. Therefore $\bar{Q}_1 = \bar{R}_1$, so $R_1 \cap LQ_1 = Q_1$. Then by C.1.32, either there is a nontrivial characteristic subgroup Q_0 of Q_1 normal in LQ_1, or L is an $Sp_4(4)$-block. The former case leads to the same contradiction as before.

Therefore L is an $Sp_4(4)$-block. Since $H^1(\bar{L}, V) = 0$ by I.1.6, and we have seen that $O_2(L) = [O_2(L), L]$, we conclude from C.1.13.b that $O_2(L) = V$.

We now treat the case that $[V, U] = 1$. Here we recall that $W = W_H$ and $\bar{W} = Z(\bar{L}_1)$, so $[W, L_1] \leq O_2(L) = V$, and hence $[W, L_1] = [V, L_1] = V_3 \leq U$. Therefore $K = \langle L_1^K \rangle$ centralizes W/U, so $W = UV$. But then as $[V, U] = 1$ and U and V are abelian, $W = W_H$ is abelian, contrary to our assumption.

Therefore $[V, U] \neq 1$. In this case we recall that $W = U$. As $1 \neq [V, U]$, $\bar{W}_H \not\leq C_{\bar{L}\bar{T}}(V_3) = Z(\bar{L}_1)$, so as W_H is L_1X-invariant, $\bar{W}_H = \bar{R}_1$ and there exists $g \in G_1$ with $\bar{V}^g \leq \bar{R}_1$ but $\bar{V}^g \not\leq \bar{Z}(\bar{L}_1)$. Thus conjugating in L_1 if necessary, $V_1 \leq C_V(V^g) \leq V_2 \leq [V, V^g]$. But $W_H \leq Q_1 \leq N_G(V)$, so $V^g \trianglelefteq W_H$, and thus $[V, V^g] \leq V \cap V^g \leq C_V(V^g)$. Hence $V_2 = V \cap V^g$ is conjugate to V_2^g under L^g, so we may choose $g \in N_G(V_2)$. By 11.1.3, g acts on L_2, so as $V = [V, L_2]$, also $V^g = [V^g, L_2]$. But then $\bar{V}^g = [\bar{R}_2, L_2]$, contradicting $\bar{V}^g \leq \bar{R}_1$. This final contradiction completes the proof of 11.5.7. □

With 11.5.7 in hand, we are finally in a position to obtain a contradiction to 11.5.3.3.

PROPOSITION 11.5.8. $K = L_1$.

PROOF. Assume that $L_1 < K$; then we may take $H_1 := KT$ to be our chosen member of $\mathcal{H}_1$. Observe that Hypothesis F.7.6 is satisfied with LT, H_1, L_1T in the roles of "G_1, G_2, $G_{1,2}$". Adopt the notation of section F.7, and in particular let $b := b(\Gamma, V)$. By 11.5.7, $U := \langle V^{H_1} \rangle$ is elementary abelian, so $b > 1$ by F.7.7.2.

Assume first that b is even. Then by F.7.11.2, there is $g \in G$ with $1 \neq [V, V^g] \leq V \cap V^g$, and by F.7.11.5 with the roles of γ_0 and γ reversed, we may choose $V^g \leq O_2(G_{1,2}) = R_1$. Then inspecting the subgroups of $\bar{R}_1$ acting quadratically on V, either

(i) $V_1 = [V, V^g]$ is a 1-dimensional $\mathbf{F}_q$-subspace of V and V^g, or

(ii) $\bar{L} \cong Sp_4(q)$ and (conjugating in L_1 if necessary) $[V, V^g] = V_2$, and $[V_3, V^g] = V_1$ is a 1-dimensional $\mathbf{F}_q$-subspace of V and V^g.

In either case, as L^g is transitive on 1-dimensional $\mathbf{F}_q$-subspaces, we may choose $g \in G_1$. Then 11.5.7 contradicts our choice of $1 \neq [V, V^g]$.

So b is odd. Pick γ at distance b as in F.7.11, choose a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b := \gamma$$

in Γ, and choose g so that $\gamma_1 g = \gamma$. Thus $V \not\leq O_2(H_1^g)$ and as γ_1 is on the geodesic, $[U, U^g] \leq U \cap U^g$ by F.7.11.1. By 11.5.6.6, $U_1 := U \cap O_2(H_1^g)$ and $U_0 := U^g \cap O_2(H_1)$ are of index at most 2 in U and U^g, respectively. Further $V_1 \cap V_1^g = 1$—or else by 11.4.2, $K = K^g$, so $[U, O^2(H_1^g)] \leq [U, K^g] = [U, K] \leq U$, contradicting $V \not\leq O_2(H_1^g)$. Thus $[U_1, U_0] \leq V_1 \cap V_1^g = 1$, so U_0 centralizes $U_1 \cap V$ of corank at most 1 in V. However by 11.2.2.3, $s(G, V) = m(\bar{M}, V) = n > 1$, so U_0 centralizes V by E.3.6. Then as $1 \neq [V, V^g] \leq [V, U^g]$, V induces a group of transvections on U^g with axis U_0. As $V \not\leq G_\gamma^{(1)}$, by F.7.7.2, $V \not\leq O_2(G_\gamma)$.

Since $L_1 < K$, K is described in case (1) or (2) of 11.1.2 by 11.4.1.2. As $C_{H_1}(U) \leq C_{H_1}(V) \leq M_1$ and $L_1 \trianglelefteq M_1$, we conclude from the structure of those groups that $C_K(U) \leq O_{2,Z}(K)$. Thus we may pick an H_1^g-chief section W of U^g such that $F := O_{2,F^*}(K)$ is nontrivial on W. Again from the structure of K, as $V \not\leq O_2(H_1^g)$, V is nontrivial on $Aut_F(W)$, so V induces a transvection on W. But comparing the groups in 11.1.2 to those in G.6.4.2, $Aut_{H_1}(W)$ contains no transvection, completing the proof of the lemma. □

By 11.5.8, $K = L_1$, so $\langle V^{G_1} \rangle$ is nonabelian by 11.5.3.3, contrary to 11.5.7. This contradiction completes the proof of Theorem 11.0.1.

Part 5

Groups over $\mathbf{F}_2$

Results in the previous parts have reduced the choices for L, V in the Fundamental Setup (3.2.1) to the case where $L/O_{2,Z}(L)$ is essentially a group of Lie type defined over $\mathbf{F}_2$, and V is highly restricted. We adopt the convention that A_5 (regarded as $\Omega_4^-(2)$), and A_7 (a subgroup of $A_8 \cong L_4(2) \cong \Omega_6^+(2)$) are considered to be defined over $\mathbf{F}_2$. For a precise description of the pairs L, V which remain to be considered, see conclusion (3) of Theorem 12.2.2 early in this part.

The first chapter 12 of this part contains a number of useful reductions which smooth out the situation. For example, some reductions treat or eliminate certain larger possibilities for L or V. These reductions use special and comparatively elementary techniques, such as the weak-closure methods from section E.3, or control of centralizers of certain elements of V.

The cases that remain after these sections are then treated in chapters 13 and 14, using "generic" techniques for groups over $\mathbf{F}_2$, such as versions of the theory of large extraspecial 2-subgroups in the original classification literature, and variants on the amalgam method from section F.9

CHAPTER 12

Larger groups over $\mathbf{F}_2$ in $\mathcal{L}_f^*(G,T)$

In this chapter we consider the cases remaining in the Fundamental Setup (3.2.1) after the work of the previous parts. Then we reduce that list further, concentrating on cases which can be treated by methods such as weak closure and control of centralizers of certain elements of V.

After an initial reduction in the first section 12.1, the cases that remain are listed in part (3) of Theorem 12.2.2 in the second section. Then in Hypothesis 12.2.3, we add the assumption that G is not one of the groups already treated in earlier analysis; the latter groups are listed in conclusions (1) or (2) of Theorem 12.2.2. In the remaining cases $L/C_L(V)$ is essentially a group defined over $\mathbf{F}_2$. Then the main goal of this chapter is to treat, and in most cases eliminate, the largest of those groups over $\mathbf{F}_2$: namely $\hat{A}_6$, A_7, $L_4(2)$, and $L_5(2)$.

12.1. A preliminary case: Eliminating $\mathbf{L_n(2)}$ on $\mathbf{n \oplus n^*}$

In this section we complete our analysis of case 3.2.5.3 of the Fundamental Setup (3.2.1), where V is a sum of two T-conjugates of $V_\circ \in Irr_+(L, R_2(LT), T)$. Recall that most such cases were eliminated in Theorem 7.0.1. Thus it remains to consider the cases where $L/C_L(V) \cong L_4(2)$ or $L_5(2)$, and $V_\circ$ is a natural module for $L/C_L(V)$. We eliminate these cases using the weak-closure techniques of part 3, together with reductions from chapters E.6 and 11. We must work a little harder however, because $m(M/C_M(V), V) = 2$, so that Theorem E.6.3 is not available to give an initial lower bound on $r(G,V)$.

Once this case is eliminated, we will have completed the treatment of the cases in the FSU where L is T-invariant and L is not irreducible on $V/C_V(L)$; for recall chapter 10 completed the treatment of the case where L is not T-invariant, while Theorems 6.2.20 and 7.0.1 treated the cases where V is not an FF-module.

Thus at the end of this section, the treatment of the FSU will be reduced to the cases described in 3.2.8. The first four subcases of 3.2.8 include all cases where $L/C_L(V)$ is defined over $\mathbf{F}_{2^n}$ with $n > 1$, and those cases were handled in Theorems 6.2.20 and 11.0.1. Hence after this section it remains only to treat the cases where $L/C_L(V)$ is a group defined over $\mathbf{F}_2$; by convention we include $\hat{A}_6$ and A_7 among such groups.

While in this section $L/C_L(V)$ is also a group over $\mathbf{F}_2$, the fact that L is not irreducible on V makes the treatment of this case easier, and different from the treatment of the generic case of groups over $\mathbf{F}_2$.

So in this section we assume G is a simple QTKE-group, $T \in Syl_2(G)$, $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_n(2)$, $n = 4$ or 5, $M := N_G(L)$, $V \in \mathcal{R}_2(M)$, $\bar{M} := M/C_M(V) \cong Aut(L_n(2))$, and $V = V_1 \oplus V_2$, with V_1 the natural module for $\bar{L}$ and $V_2 = V_1^t$ for $t \in T - LQ$, where $Q := O_2(LT) = C_T(V)$. Thus V_2 is the dual of V_1 as an $\mathbf{F}_2L$-module. Let $T_1 := N_T(V_1)$, and for $v \in V$, let $M_v := C_M(v)$, $G_v := C_G(v)$, and $L_v := O^2(C_L(v))$.

By 3.2.5.3, $V \trianglelefteq M$; hence

$$M = N_G(V)$$

as $M \in \mathcal{M}$.

Recall that the module V is described in section H.9 of Volume I. We adopt the notation of that section, including the description of the orbits $\mathcal{O}_i$ $(1 \le i \le 3)$ of M on $V^\#$.

We now proceed to analyze our group G. Eventually we obtain a contradiction, and hence show no such group exists.

LEMMA 12.1.1. *Let $v \in \mathcal{O}_3$. Then $G_v = C_G(v) \le M$.*

PROOF. First from lemma H.9.1.4, $Q = O_2(L_vT_v)$, where $T_v := C_T(v) \in Syl_2(M_v)$. Now $C(G,Q) \le M$ by 1.4.1.1. Thus as $L_v \trianglelefteq M_v$, we conclude from A.4.2.7 that $Q \in \mathcal{B}_2(G_v)$ and Q is Sylow in $\langle Q^{M_v} \rangle$. Therefore Hypothesis C.2.3 is satisfied by G_v, M_v, Q in the roles of "H, M_H, R".

Let $W := [V, L_v]$. By lemma H.9.1.4, $L_v/O_2(L_v) \cong L_{n-1}(2)$ and $W = W_1 \oplus W_2$ with W_1 the natural module for $L_v/O_2(L_v)$ and W_2 its dual. Let z generate $C_W(T_v)$, and observe z is 2-central in G by H.9.1.2.

For each value of n we define a subgroup $K_v \in \mathcal{C}(G_v)$ with $L_v \le K_v \trianglelefteq G_v$: If $n = 4$, then W is not an FF-module for L_v by Theorem B.5.1.1, so $J(T_v) = J(Q)$ by B.2.7. Then as $C(G,Q) \le M$, $N_G(T_v) \le M_v$ and hence $T_v \in Syl_2(G_v)$. So by 1.2.4, $L_v \le K_v \in \mathcal{C}(G_v)$, and as T_v acts on L_v, $K_v \trianglelefteq G_v$ by 1.2.1.3. On the other hand, if $n = 5$, then by 1.2.1.1, L_v projects nontrivially on some $K_v \in \mathcal{C}(G_v)$, so K_v has a section isomorphic to $L_v/O_2(L_v) \cong L_4(2)$. Therefore $K_v = O^{3'}(G_v)$ by A.3.18, so that again $L_v \le K_v \trianglelefteq G_v$.

Suppose that there is a component K of G_v. Then $L_v = O^2(L_v)$ acts on K by 1.2.1.3. By A.1.6, $O_2(M) \le Q \le G_v$, so $M_v \in \mathcal{H}^e$ by 1.1.4.4; thus $K \not\le M_v$. Similarly $G_z \in \mathcal{H}^e$ by 1.1.4.6, so $G_{v,z} := G_v \cap G_z \in \mathcal{H}^e$ by 1.1.3.2; thus $K \not\le G_{v,z}$, so $K \not\le G_z$. But $z \in W = [W, L_v] \le L_v$, so $[K, L_v] \ne 1$. Therefore $[K, K_v] \ne 1$, and hence $K = K_v$ by 1.2.1.2. Then $C_W(K) \le C_W(L_v) = 1$. Set $G_v^* := G_v/C_{G_v}(K)$. Then $W^* \cong W$ as an L_v^*-module, and $L_v^* \trianglelefteq M_v^*$. But no group with such a 2-local M_v^* appears on the list of Theorem C (A.2.3).

This contradiction shows that $E(G_v) = 1$. Next $W = [W, L_v]$, so W centralizes $O(G_v)$ by A.1.26. Therefore $O(G_v) \le G_{v,z}$, and hence $O(G_v) = 1$ as $G_{v,z} \in \mathcal{H}^e$. Thus we have shown $O^2(F^*(G_v)) = 1$, so that $G_v \in \mathcal{H}^e$.

We now assume that $G_v \not\le M$, and derive a contradiction.

Suppose that $L_v \trianglelefteq G_v$ and set $Y := C_{G_v}(L_v/O_2(L_v))$. Then as $Aut(L_v/O_2(L_v))$ is induced in L_vT_v, $G_v = L_vT_vY$, so $Y \not\le M$ as $G_v \not\le M$. Next embed $T_v \le X \in Syl_2(G_v)$; then $N_X(T_v)$ normalizes $C_{T_v}(L_v/O_2(L_v)) = Q$, and so lies in M_v—hence $T_v = X \in Syl_2(G_v)$. Thus $Q = T_v \cap Y$ is Sylow in Y, so we conclude from the $C(G,T)$-Theorem C.1.29 that there is a χ_0-block B of Y with $B \not\le M$. If B is an $L_2(2^n)$-block, then a Cartan subgroup D of B lying in $B \cap M$ centralizes $L_v/O_2(L_v)$; hence D centralizes V, as $\bar{M} = N_{GL(V)}(\bar{L})$ and $C_{\bar{M}}(\bar{L}_v) = 1$. Thus $V \le C_{T_v}(B \cap M) = C_{T_v}(B)$, and hence $B \le C_G(V) \le M$, contrary to the choice of B. If B is an A_5-block, then $O^2(B \cap M) \le O^{3'}(M_v) = L_v$ by A.3.18, whereas $Z(L_v/O_2(L_v)) = 1$. Thus B is an A_3-block. Notice since B centralizes $L_v/O_2(L_v)$ of order divisible by 3, and G_v is an SQTK-group, that $B \trianglelefteq G_v$. Set $H := BT_v$, so that $T_v \in Syl_2(H)$, $B = O^2(H)$, and $Q \in Syl_2(QB)$. As $B \not\leq M = !\mathcal{M}(LT)$, there is no $1 \neq R_0 \leq Q$ with $R_0 \trianglelefteq \langle LT, H\rangle$. Thus Hypotheses C.5.1 and C.5.2 are satisfied with LT, Q in the roles of "M_0, R". Further L_vT_v is maximal in LT, so $L_vT_v = N_{LT}(B)$. Then we have the hypotheses of C.5.7, and as $|LT : L_vT_v| \neq 2$, C.5.7 supplies a contradiction.

This contradiction shows that L_v is not normal in G_v. Thus $K := K_v > L_v$, so as $L_v \trianglelefteq M_v$, $K \not\leq M_v$. As $G_v \in \mathcal{H}^e$, $K \in \mathcal{H}^e$ by 1.1.3.1. By C.2.6.2, $O_{2,F}(K) \leq M_v \leq N_G(L_v)$, so $K/O_2(K)$ is quasisimple by 1.2.1.4. As $L_v \leq K$ and T_v is nontrivial on the Dynkin diagram of $L_v/O_2(L_v)$, K is not a χ_0-block, so Q normalizes K by C.2.4. Thus we have the hypotheses of C.2.7, so K is described in C.2.7.3. Thus as T_v is nontrival on the Dynkin diagram of $L_v/O_2(L_v) \cong L_3(2)$ or $L_4(2)$, and $L_v \trianglelefteq M_v \cap K$, we conclude that case (h) of C.2.7.3 holds with $KT_v/O_2(KT_v) \cong Aut(L_5(2))$ and $L_vT_v \cap K$ is the parabolic subgroup determined by the middle two nodes; in particular $n = 4$. Let $Z_v := \Omega_1(Z(O_2(KT_v)))$, $Y := \langle Z_v^K\rangle$, and $(KT_v)^+ := KT_v/O_2(KT_v)$. By C.2.7.2, Y is an FF-module for $K^+T_V^+ \cong Aut(L_5(2))$, so we conclude from Theorem B.5.1.1 that $[Y, K] = U \oplus U^t$ for $t \in T_v - N_{T_v}(U)$. By B.2.14, $Y = [Y, K] \oplus C_{Z_v}(K)$. Thus the parabolic $L_vT_v \cap K$ determined by the middle nodes of K centralizes Z_v, whereas from the action of LT on V, $C_V(T)$ is not centralized by L_vT_v. This contradiction completes the proof of 12.1.1. □

From Lemma H.9.1, V has the structure of an orthogonal space preserved by $\bar{M}$, and $\mathcal{O}_3$ is the set of nonsingular vectors in that space.

LEMMA 12.1.2. *(1) If $U \leq V$ with $C_G(U) \not\leq M$, then U is totally singular.*
(2) $r(G, V) \geq n$, so that $s(G, V) = m(Aut_M(V), V) = 2$.

PROOF. Part (1) follows from 12.1.1 and the fact that $\mathcal{O}_3$ is the set of nonsingular vectors in V. Then (1) implies (2). □

Using the lower bound on the parameter $r(G, V)$ in 12.1.2.2, we can apply the weak-closure machinery in section E.3 (subsection E.3.3) to establish successively better lower bounds on the parameter $w(G, V)$. Often results are easier to establish in the case $n = 5$; for example, the analogue of 12.1.3 below is not established for $n = 4$ until 12.1.7.

LEMMA 12.1.3. *If $n = 5$ then $W_0 := W_0(T, V)$ centralizes V.*

PROOF. Suppose that $n = 5$ but $W_0 \not\leq C_T(V)$. Then there is $A := V^g \leq T$ with $\bar{A} \neq 1$. Recall $M = N_G(V)$ and $M^g = N_G(A)$.

We begin by showing we may choose A with $m(\bar{A}) \geq 5$. Suppose first that $V \not\leq N_G(A)$; then as $r(G, V) \geq 5$ by 12.1.2.2, $m(\bar{A}) \geq 5$ by E.3.4.2. So suppose instead that $V \leq N_G(A)$. Here, interchanging the roles of A and V if necessary, we may assume that $m(\bar{A}) = m(A/C_A(V)) \geq m(V/C_V(A))$; equivalently $m(C_V(A)) \geq m(C_A(V))$. Suppose that $m(\bar{A}) < 5$. Then by our assumption above, $m(C_V(A)) \geq m(C_A(V)) > 5$. Hence $C_V(A)$ is not totally singular and $1 \neq C_{V_1}(A)$, so $\bar{A} \leq \bar{T}_1 \leq \bar{L}$. Then as A centralizes a nonsingular vector $v \in V$, by lemma H.9.1.4, $\bar{A} \leq \bar{L}_v \cong L_4(2)$ and $V = C_V(L_v) \oplus W$, where W is the sum of a natural module and its dual. Now $m(\bar{A}) \geq m(W/C_W(A))$, so that $\bar{A}$ contains a member $\bar{B}$ of $\mathcal{P}(\bar{L}_v, W)$ by B.1.4.4. Then B.4.9.2iii determines $\bar{B}$ uniquely as $J(C_{\bar{T}}(v))$, so that $\bar{B} = J(C_{\bar{T}_1}(v)) = \bar{A}$. In particular $\bar{A}$ is the unipotent radical of the stabilizer in $\bar{L}_v \cong L_4(2)$ of a 2-subspace of the natural module W. Thus $m(\bar{A}) = m(V/C_V(A))$,

$\bar{A}$ is faithful on each V_i, and for $u \in V_1 - C_{V_1}(A)$, $[u,A] = [V_1,A]$ is of rank 2. Set $A_i := V_i^g$. Now $V \leq N_G(A)$ by our assumption, and we saw that $m(\bar{A}) = m(V/C_V(A)) = 4 < 5$, so we have symmetry between V and A. Reversing the roles of V and A, we conclude from that symmetry that $V/C_V(A)$ is faithful on A_i, so as $[A,V_1]$ is of rank 2, V_1 induces transvections on A_i with center $[A,V_1] \cap A_i$. This is impossible as $m(V_1/C_{V_1}(A)) = 2 > 1$ and A_2 is dual to A_1.

This contradiction establishes the claim that we may take $m(\bar{A}) \geq 5$. Hence by lemma H.9.2.3, we may take $\bar{A} \leq \bar{A}_0$, where $\bar{A}_0$ is the centralizer in $\bar{T}_1$ of a 3-subspace of V_1. Then by H.9.2.5,

$$V = \check{\Gamma}_{4,\bar{A}}(V) = \check{\Gamma}_{4,A}(V);$$

so as $r(G,V) \geq 5$, $V = \check{\Gamma}_{4,A}(V) \leq N_G(A)$ by E.3.32. Then $[V,A] \leq V \cap A \leq C_V(A)$, and applying lemma H.9.2.4 to the action of $\bar{A}$ on V, $C_V(A) = [V,A] = V \cap A$ is of rank 5. Thus $m(V/C_V(A)) = 5$, so we have symmetry between V and A, and by that symmetry $C_A(V) = V \cap A$ is of rank 5. Hence $m(\bar{A}) = 5$. We saw that $\bar{A} \leq \bar{A}_0$, so $\bar{A}$ acts faithfully on each V_i. In particular, $V_2 \not\leq A$, and for $v \in V_2 - A$, $[v,A] \leq A \cap V_2 = C_{V_2}(A)$, with $m(C_{V_2}(A)) = 2$ by H.9.2.4. By symmetry, V_2 normalizes but does not centralize V_i^g, so as $m([V_2,A]) = 2$ and $m(V_2/C_{V_2}(A)) > 1$, we have the same contradiction as in the previous paragraph. This completes the proof of 12.1.3. □

LEMMA 12.1.4. *Assume $n = 4$ and let v generate $C_{V_1}(T_1)$. Then $T_1 \in Syl_2(G_v)$.*

PROOF. Let $T_1 \leq T_0 \in Syl_2(G_v)$. If the lemma fails, then as $|T:T_1| = 2$, $T_0 \in Syl_2(G)$. But $T_1 \in Syl_2(M_v)$, so $T_0 \not\leq M$, and hence $N_G(T_1) \not\leq M$. If C is a nontrivial characteristic subgroup of T_1 normal in L_vT_1, then $C \trianglelefteq \langle T, L_v\rangle = LT$, so $N_G(T_1) \leq N_G(C) \leq M = !\mathcal{M}(LT)$, contrary to the previous sentence. Thus no such C exists, so (L_v, T_1) is an MS-pair in the sense of Definition C.1.31. However L_v has at least three noncentral 2-chief factors, and as $v \in V_1 = [V_1, L_v] \leq L_v$ by H.9.1.3, $v \in Z(L_v)$. Hence L_v must satisfy case (4) of C.1.34. Therefore $Z_1 := \Omega_1(Z(T_1))$ is of rank at least 3, with $m(Z_1/C_{Z_1}(L_v)) = 1$. Now $L = \langle L_v, L_v^t\rangle$ for $t \in T - T_1$, so $1 \neq C_{Z_1}(L_v) \cap C_{Z_1}(L_v^t) \leq C_{Z_1}(L)$, and hence $C_Z(L) \neq 1$.

Next $J(T) \leq T_1$ by B.1.5.4. Thus $J(T_1) = J(T)$, so as $N_G(T_1) \not\leq M$, $N_G(J(T)) \not\leq M$. In particular $J(T) \not\leq Q$, and hence $R_2(LQ) = V \oplus C_{Z_1}(L)$ by B.5.1.4. Now an FF^*-offender in $\bar{T}$ lies in $\mathcal{P}(\bar{T},V)$ by B.2.7, and by B.4.9.2iii the unique member $\overline{J(T)}$ of $\mathcal{P}(\bar{T},V)$ is the unipotent radical of the stabilizer in L of a 2-subspace of V_1. Thus $N_{LT}(J(T)) = XT$, where $X \in \Xi(G,T)$ with $XT/O_2(XT) \cong S_3$ wr $\mathbf{Z}_2$. As $R_2(LQ) = V \oplus C_{Z_1}(L)$ with $C_V(X) = 1$, $C_{Z_1}(X) = C_{Z_1}(L)$. As $J(T) \leq T_1$, $T_0 \leq N_G(T_1) \leq N_G(J(T)) =: G_J$, and of course $TX \leq G_J$. Thus $H := \langle T_0, TX\rangle \leq G_J$, so $H \in \mathcal{H}(XT)$. Suppose $X \trianglelefteq H$. Then T_0 and T act on T_1 and hence on $C_{Z_1}(X) = C_{Z_1}(L)$, so $T_0 \leq N_G(C_{Z_1}(L)) \leq M = !\mathcal{M}(LT)$, contrary to $T_0 \not\leq M$. Therefore X is not normal in H, so by 1.3.4, $X < K_0 := \langle K^T\rangle$ for some $K \in \mathcal{C}(H)$, and K_0 is described in 1.3.4. Now $K \in \mathcal{L}(G,T)$ and $[Z,X] \neq 1$, so $K \in \mathcal{L}_f(G,T)$. By 1.3.9, $K \in \mathcal{L}_f^*(G,T)$, so by 3.2.3 there is $V_K \in \mathcal{R}_2(K_0T)$ such that the pair K, V_K satisfies the Fundamental Setup. Hence by Theorem 3.2.5, this pair is listed in 3.2.5.3, 3.2.8, or 3.2.9. By Theorem 10.0.1, $K = K_0$, so case (1) of 1.3.4 does not hold. Theorem 11.0.1 eliminates case (3) of 1.3.4. Case (2), and case (4) with $K/O_2(K) \cong M_{11}$, do not appear in the indicated lists for the FSU. Thus $KT/O_2(KT) \cong S_8$ or $Aut(L_5(2))$. Let $H^* := H/C_H(K/O_2(K))$, so that $H^* = K^*T^*$ since $K^*T^* = Aut(K^*)$. Then $X^* = O^2(P^*)$, where P^* is the parabolic of K^* determined by the end nodes of the Dynkin diagram. Therefore $R^* := O_2(X^*) \leq T_1^*$, and hence as it is the unipotent radical of P^*, R^* is weakly closed in T^* with respect to H^* by I.2.5. Since $C_T(K^*) \leq C_T(X^*) \leq T_1$, a Frattini Argument shows that $N_H(T_1)^* = N_{H^*}(T_1^*)$. Thus $T_0^* \leq N_H(T_1)^* = N_{H^*}(T_1^*) \leq N_{H^*}(R^*) = N_{H^*}(P^*) = N_{H^*}(X^*)$, so

$$K_0^*T^* = H^* = \langle T_0^*, X^*T^* \rangle \leq N_{H^*}(X^*) = X^*T^*,$$

a contradiction. This completes the proof of 12.1.4. □

LEMMA 12.1.5. *(1) Let v generate $C_{V_1}(T_1)$. Then $T_1 \in Syl_2(G_v)$.*
(2) M controls fusion of involutions in V.

PROOF. If $n = 4$, then (1) follows from 12.1.4. If $n = 5$, then by 12.1.3, $W_0 \leq Q \leq T_1$, so $N_G(T_1) \leq N_G(W_0)$ by E.3.15. As $M = !\mathcal{M}(N_G(Q))$ by 1.4.1, $N_G(W_0) \leq M$ by E.3.34.2. Thus $N_{G_v}(T_1) \leq N_{G_v}(W_0) \leq M_v$, so as T_1 is Sylow in M_v, (1) also holds in this case.

By (1), $|G_v|_2 = |T|/2$ for any $v \in \mathcal{O}_1$, while $|G_z|_2 = |T|$ for $z \in \mathcal{O}_2$. Finally by 12.1.1, $|G_v|_2 < |T|/2$ for $v \in \mathcal{O}_3$. Thus the distinct M-classes of involutions in V are in different G-classes, so (2) holds. □

LEMMA 12.1.6. *(1) For $v \in \mathcal{O}_1$, $\langle V^{G_v} \rangle$ is abelian.*
(2) If $V_1 \cap V^g \neq 1$, then $[V, V^g] = 1$.

PROOF. Let $v \in \mathcal{O}_1$. By 12.1.5.2, $M = N_G(V)$ is transitive on G-conjugates of v in V, so by A.1.7.1, G_v is transitive on G-conjugates of V containing v. Thus (2) follows from (1), so it suffices to establish (1).

We may as well choose v to generate $C_{V_1}(T_1)$. By 12.1.5.1, $T_1 \in Syl_2(G_v)$. By lemma H.9.1, $U := [V, L_v] = V_1 \oplus U_2$, where $U_2 := v^{\perp} \cap V_2$. Let v_2 generate $C_{V_2}(T_1)$. Then $z := vv_2$ generates $C_V(T)$, and $v_2 \in U_2$.

By 1.1.6, the hypotheses of 1.1.5 are satisfied with G_v, G_z in the roles of "H, M". But as $z \in U$, $[O(G_z), z] = 1$ by A.1.26, so $O(G_v) = 1$ as z inverts $O(G_z)$ by 1.1.5.2.

Suppose first that $G_v \notin \mathcal{H}^e$. Then as $O(G_v) = 1$, there is a component K of G_v, and by 1.1.5.3, $K = [K, z] \not\leq M$. As $T_1 \in Syl_2(G_v)$, $L_v \leq K_v \in \mathcal{C}(G_v)$ by 1.2.4. Now $z \in U = [U, L_v] \leq K_v$, so as $K = [K, z]$, we conclude $K = [K, K_v] = K_v$ from 1.2.1.2. Thus $v \in L_v \leq K$. Set $K^* := K/O_2(K)$. Then $U \cap Z(K) = \langle v \rangle$, so $U^* \trianglelefteq L_v^*$, with $U^* = V_1^* \oplus U_2^*$ the sum of the natural module and its dual for $L_v/O_2(L_v) \cong L_{n-1}(2)$. As no group on the list of 1.1.5.3 has such a subgroup invariant under a Sylow 2-group T_1^*, we have a contradiction.

This contradiction shows that $G_v \in \mathcal{H}^e$. Let $Q_v := O_2(G_v)$, and $\tilde{G}_v := G_v/\langle v \rangle$. Now $T_1 \in Syl_2(G_v)$, and L_v is irreducible on $\tilde{V}_1$, so Hypothesis G.2.1 holds with $\langle v \rangle$, V_1, T_1, G_v in the roles of "V_1, V, T, H". Then by G.2.2.1, $\tilde{V}_1 \leq Z(O_2(\tilde{G}_v)) = Z(\tilde{Q}_v)$. Similarly as L_v is irreducible on U_2, $[Q_v, U_2] \leq \langle v \rangle \cap U_2 = 1$, so that $U_2 \leq Z(Q_v)$. In particular $U = V_1U_2 \leq Q_v$, so for any $g \in G_v$, U_2 centralizes U^g. Hence by 12.1.2.2, $U_2 \leq C_G(U^g) = C_{M^g}(V^g)$.

Suppose first that $V \leq Q_v$. Then for all $g \in G_v$, $V^g \leq Q_v \leq M = N_G(V)$. By the previous paragraph, V^g centralizes U_2, so V^g acts on V_1 and V_2; hence by symmetry, V acts on V_1^g and V_2^g. Then as $v \notin V_2$,

$$[V_1, V_2^g] \leq [V_1, Q_v] \cap V_2^g \leq \langle v \rangle \cap V_2^g = 1,$$

so that $V_1 \leq C_{M^g}(V_2^g) = C_{M^g}(V^g)$. Then $V^g \leq C_M(V_1) = C_G(V)$, so (1) holds.

So assume instead that $V \not\leq Q_v$. Then by the Baer-Suzuki Theorem, there is $g \in G_v$ such that $I := \langle V, V^g \rangle$ is not a 2-group. We showed $U_2 \leq C_G(V^g)$, and by symmetry U_2^g centralizes V, so $U_2 U_2^g \langle v \rangle \leq Z(I)$ and $V_1^g \leq C_{T_1}(U_2)$. But $C_{\bar{T}_1}(U_2)$ is the group of transvections on V_2 with axis U_2, so as V_1 is dual to V_2, $C_{\bar{T}_1}(U_2)$ is the group of transvections on V_1 with center $\langle v \rangle$. Hence $[V, V_1^g] \leq U_2\langle v \rangle = U_2 \langle z \rangle$, and then $[V_1 U_2 V_1^g U_2^g, I] \leq U_2 U_2^g \langle z \rangle$. Therefore $O^2(I) \leq C_G(V_1 U_2) = C_G(V)$ using 12.1.2.2, contradicting I not a 2-group. This completes the proof of 12.1.6. □

LEMMA 12.1.7. *$W_0 := W_0(T, V)$ centralizes V, so that $w := w(G, V) > 0$.*

PROOF. Assume that $W_0 \not\leq C_T(V)$. Then $n = 4$ by 12.1.3, and there is $A := V^g \leq T$ with $\bar{A} \neq 1$.

Suppose first that $V \leq N_G(A)$. Then interchanging the roles of A and V if necessary, we may assume $m(A/C_A(V)) \geq m(V/C_V(A))$. Then by B.1.4.4, $\bar{A}$ contains a member of $\mathcal{P}(\bar{T}, V)$, which is $J(\bar{T})$ by B.4.9.2iii. Thus $\bar{A}$ is the unipotent radical of the stabilizer in $\bar{L}$ of a 2-subspace of V_1, so that $[V_1, A]$ is of rank 2. As $\bar{A}$ normalizes V_1, and $V_1 \leq V \leq N_G(A)$, $1 \neq [V_1, A] \leq V_1 \cap A$, contrary to 12.1.6.2.

Therefore we may assume that $V \not\leq N_G(A)$. As $r(G, V) \geq 4$ by 12.1.2.2, $m(\bar{A}) \geq 4$ by E.3.4, so that $m(\bar{A}) = 4 = m_2(\bar{L}\bar{T})$. Then by lemma H.9.3.3, we may take $\bar{A}$ to be one of the groups denoted there by $\bar{A}_i$ for $0 \leq i \leq 2$. As $r(G, V) \geq 4$, we conclude from E.3.32 that

$$\check{\Gamma}_{3,\bar{A}}(V) = \check{\Gamma}_{3,A}(V) \leq U := N_V(A). \qquad (*)$$

As we are assuming $U < V$, $i \neq 0$ by lemma H.9.3.4. Let $B := N_A(V_1)$. Then $\bar{B} = \bar{A} \cap \bar{L}$ has rank 3, as $\bar{A}$ is $\bar{A}_1$ or $\bar{A}_2$. Set $U_i := V_i \cap U$. By 12.1.6.2,

$$1 = V_i \cap A \geq [U_i, B].$$

But for any $\bar{b} \in \bar{B}^\#$, $C_{V_i}(b) \leq U_i$ by (*), so $C_{V_i}(b) = U_i = C_V(B)$. However this is not the case as $\bar{B}$ contains at least one transvection on V_1, but not all elements of $\bar{B}^\#$ induce transvections on V_1. This contradiction completes the proof. □

LEMMA 12.1.8. *$W_1 := W_1(T, V)$ centralizes V, so that $w > 1$.*

PROOF. Assume that $W_1 \not\leq C_T(V)$. As $w > 0$ by 12.1.7, $w = 1$. Thus there is a w-offender $A := N_{V^g}(V) \leq T$ with A a hyperplane of V^g and $\bar{A} \neq 1$. Now $V \not\leq N_G(V^g)$ by E.3.25. As $r(G, V) \geq n$ by 12.1.2.2, $m(\bar{A}) \geq n - 1$ by E.3.28.3. As $r(G, V) \geq n$, E.3.32 says that

$$\check{\Gamma}_{n-2,A}(V) = \check{\Gamma}_{n-2,\bar{A}}(V) \leq U := N_V(V^g). \qquad (*)$$

Let $U_i := V_i \cap U$ and $B := N_A(V_1)$. Then $m(A/B) \leq 1$, so $m(V^g/B) \leq 2$. Also $[U_i, B] \leq V_i \cap V^g = 1$ by 12.1.6.2, so that $U_i \leq C_{V_i}(\bar{B})$.

Suppose $m(\bar{A}) = n - 1$. Then by (*), $C_{V_i}(\bar{b}) \leq U_i$ for each $\bar{b} \in \bar{B}^\#$, so that $U_i = C_{V_i}(\bar{b})$ for each such $\bar{b}$. However, if $\bar{b}$ is a transvection in $\bar{L}$, then U_i is a hyperplane of V_i, so that $\bar{B}$ must be the full group of transvections with axis U_i for $i = 1$ and 2. This is not the case as V_1 is dual to V_2. Thus $\bar{B}$ contains no transvections, and hence $\dim(U_i) = n - 2$ and $\bar{B}$ lies in the unipotent radical $\bar{R}_i$ centralizing U_i. However $m(\bar{R}_1 \cap \bar{R}_2) = 4$ and $\bar{R}_1 \cap \bar{R}_2$ contains a 4-group $\bar{R}$ with each member of $R^\#$ a transvection, so as $\bar{B}$ contains no transvections, $m(\bar{B}) \leq m((\bar{R}_1 \cap \bar{R}_2)/\bar{R}) = 2$. Thus as $m(\bar{B}) \geq n-2$, we conclude $n = 4$ and $\bar{A} > \bar{B}$, so $C_{U_1}(A) = 1$ and hence U_1 is faithful on V^g. But U_1 centralizes the subspace B of codimension 2 in V^g; this forces U_1 to induces a group of transvections on

V_i^g with fixed axis $B \cap V_i^g$ for $i = 1$ and 2. Thus as V_1 is dual to V_2, $m(U_1) \leq 1$, contradicting $m(U_1) = n - 2 = 2$.

This contradiction shows that $m(\bar{A}) \geq n$. Suppose $n = 5$. Then by lemma H.9.2.3, we may take $\bar{A} \leq \bar{A}_0$, where $\bar{A}_0$ is the centralizer in $\bar{T}_1$ of a 3-subspace X of V_1. Let W be a hyperplane of V_1 containing X. Then $m(\bar{A}/C_{\bar{A}}(W)) \leq m(\bar{A}_0/C_{\bar{A}_0}(W)) = 3$, so $W \leq C_V(C_{\bar{A}}(W)) \leq U$ by (*). As this holds for each such hyperplane, we conclude $V_1 \leq U$. But then $1 \neq [V_1, A] \leq V_1 \cap V^g$, contrary to 12.1.6.2.

Therefore $n = 4$. Then by lemma H.9.3.3, we may assume $\bar{A}$ is one of the subgroups there denoted $\bar{A}_i$ for $0 \leq i \leq 2$. Now $U < V$ as $V \not\leq N_G(V^g)$, so $i \neq 0$ in view of (*) and lemma H.9.3.4. Therefore by parts (5) and (6) of lemma H.9.3, $m(U) \geq 6$, and $C_U(A)$ is of rank 1 or 2 for $i = 1$ or 2, respectively. Next as $s(G, V) = 2$ by 12.1.2.2, $C_U(A) = C_U(V^g)$ by E.3.6. Thus $m(U/C_U(V^g)) \geq 5$ or 4 in the respective cases; so as $m_2(\bar{M}) = 4$, we conclude that $\bar{A} = \bar{A}_2$ and $m(U/C_U(V^g)) = 4$. But as $r(G, V) > m(V/U)$, $C_G(U) \leq N_G(V)$; hence $C_A(V) = C_{V^g}(U)$ since $N_{\bar{A}}(U)$ is faithful on U by H.9.3.6. Therefore as $m(\bar{A}) = 4$, $C_A(V) = C_{V^g}(U)$ is of rank 3. This contradicts parts (4)–(6) of lemma H.9.3, which say that $m(C_{V^g}(U)) = 1$, 2, or 4, since $m(U/C_U(V^g)) = 4$. This completes the proof of 12.1.8. □

Having shown that $w > 1$, we turn to the other weak closure parameters of section E.3; as usual we will obtain a contradiction from their interrelations.

Recall by 3.3.2 that we may apply the results of section B.6 to any $H \in \mathcal{H}_*(T, M)$.

LEMMA 12.1.9. *(1) If $1 \neq X$ is of odd order in $C_M(V)$, then $N_G(X) \leq M$.*
(2) If $H \in \mathcal{H}_*(T, M)$, then $n(H) = 2$.
(3) $r(G, V) \geq n + 2$.

PROOF. Assume X is as in (1); replacing X with $Z(F(X))$, we may assume that X is abelian. Then $[X, L] \leq O_2(L)$. By Remark 4.4.2, Hypothesis 4.4.1 is satisfied. As V is not the sum of isomorphic natural modules for $L/O_2(L) \cong L_n(2)$ or $\Omega_6^+(2)$, $N_G(X) \leq M$ by Theorem 4.4.3.

Next suppose $U \leq V$ with $G_U := C_G(U) \not\leq M$. By 12.1.2.1, U is totally singular. Conjugating in L, we may take $T_U := C_T(U)$ Sylow in $C_M(U)$. Let $H \in \mathcal{H}_*(T_U, M) \cap G_U$. By 12.1.8, $W_i = W_i(T, V) \leq Q \leq T_U$ for $i = 0, 1$, so by E.3.15, $W_i = W_i(T_U, V) = W_i(Q, V)$, and $N_G(T_U) \leq N_G(W_i)$. But $M = !\mathcal{M}(N_G(Q))$ by 1.4.1, so by E.3.34.2,

$$N_G(W_0(T_U, V)) \leq M \geq C_G(Z(W_1(T_U, V))) \geq C_G(C_1(T_U, V)).$$

In particular $N_{G_U}(T_U) \leq M_U$, and hence $T_U \in Syl_2(G_U)$. Then as $s(G, V) = 2$ by 12.1.2.2, $n(H) > 1$ by E.3.19, so we may apply E.2.2 to conclude that a Cartan subgroup B of the Borel subgroup $H \cap M$ is nontrivial. For each odd prime p and $1 \neq X \in Syl_p(B)$, $H = \langle H \cap M, N_H(X) \rangle$, so $N_G(X) \not\leq M$ as $H \not\leq M$. If $T = T_U$ and $p > 3$, then as $XT = TX$, $X \leq M = N_G(V)$, while $\bar{M} = \bar{L}\bar{T}$ has no nontrival p-subgroup permuting with $\bar{T}$, we conclude $[X, V] = 1$. This contradicts (1); thus if $H \in \mathcal{H}_*(T, M)$ then $p = 3$ so that $n(H) = 2$, establishing (2).

Indeed this argument shows more generally that $\bar{X} \neq 1$. As X is of odd order, $C_V(X)$ is a nondegenerate subspace of V of codimension at least 4, so as U is a totally singular subspace of $C_V(X)$, $\dim(C_V(X)) \geq 2\dim(U)$. Thus

$$\dim(V/U) \geq \dim(V/C_V(X)) + \dim(C_V(X))/2$$

$$= 2n - \dim(C_V(X))/2 \geq 2n - (2n-4)/2 = n+2,$$

establishing (3). □

LEMMA 12.1.10. *$W_2(T,V)$ centralizes V, so that $w > 2$.*

PROOF. Assume that $W_2(T,V) \not\leq C_T(V)$. Then $w = 2$ by 12.1.8, so there is a w-offender $A := V^g \cap M \leq T$ with $m(V^g/A) = 2$ and $\bar{A} \neq 1$. Let $U := N_V(V^g)$; then $m(V/U) \geq 2$ as $w = 2$. By 12.1.9.3, $m(\bar{A}) \geq n$. Then by E.3.32,

$$\check{\Gamma}_{n-1,A}(V) = \check{\Gamma}_{n-1,\bar{A}}(V) \leq U < V. \qquad (*)$$

Suppose first that $n = 4$. Then $m(\bar{A}) = 4 = m_2(\bar{M})$, so by lemma H.9.3.3, we may take $\bar{A}$ to be one of the subgroups there denoted by $\bar{A}_i$ for $0 \leq i \leq 2$. Set $\bar{B}_i := \bar{A}_i \cap \bar{L}$. By (*) and H.9.3.4, $i \neq 0$. Then we conclude from the last two parts of H.9.3 that $\check{\Gamma}_{2,\bar{A}}(V)$ is of rank 6. As $m(V/U) \geq 2$, $U = \check{\Gamma}_{2,\bar{A}}(V) = \check{\Gamma}_{3,\bar{A}}(V)$, whereas $C_V(\bar{a}) \not\leq \check{\Gamma}_{2,\bar{A}}(V)$ for $\bar{a} \in \bar{A}_1 - \bar{B}_1$, or for $\bar{a}$ a transvection in $\bar{B}_2$.

This contradiction reduces us to the case $n = 5$. Then by lemma H.9.2.3, we may take $\bar{A} \leq \bar{A}_0$ in the notation of that result. Now lemma H.9.2.5 contradicts (*), completing the proof. □

We are now in a position to establish a contradiction. Pick $H \in \mathcal{H}_*(T,M)$. By 12.1.9.2, $n(H) = 2$. However by 12.1.10, $w \geq 3$, while by 12.1.9.3, $r(G,V) \geq 6$. Thus $H \not\leq M$ with $n(H) < \min\{w, r(G,V)\}$, contrary to E.3.35.1.

This contradiction shows:

THEOREM 12.1.11. *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_4(2)$ or $L_5(2)$. Let $M := N_G(L)$. Then there is no $V \in \mathcal{R}_2(M)$ such that $M/C_M(V)) \cong Aut(L/O_2(L))$ and V is the sum of the natural module and its dual for $L/O_2(L)$.*

By Theorems 12.1.11, 3.2.5, 3.2.8, and 3.2.9, the subcase of the Fundamental Setup with $L/O_2(L) \cong L_4(2)$ or $L_5(2)$ is reduced to the cases (i.e. cases (9), (10), and (11) of 3.2.8) with $V/C_V(L)$ a natural module or its exterior square. These cases will be treated along with the other cases where $L/O_2(L)$ is defined over $\mathbf{F}_2$; in particular they are completed in section 12.6, and in the final three sections of this chapter.

12.2. Groups over $\mathbf{F}_2$, and the case V a TI-set in G

We now begin a fairly unified treatment of those simple QTKE-groups G for which there exists $L \in \mathcal{L}_f^*(G,T)$ such that the section $L/O_2(L)$ has not yet been eliminated from the list of cases in section 3.2. Thus in section 12.2, and indeed in many subsequent sections, we assume the following hypothesis:

HYPOTHESIS 12.2.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple.*

We begin with a Theorem which summarizes much of what we have accomplished up to this point:

THEOREM 12.2.2. *Assume Hypothesis 12.2.1. Then one of the following holds:*

(1) G is a group of Lie type of Lie rank 2 over $\mathbf{F}_{2^n}$, $n > 1$, but $G \cong U_5(2^n)$ only for $n = 2$.

(2) $G \cong M_{22}$, M_{23}, or J_4.

(3) $T \leq M := N_G(L)$, and there exists $V \in Irr_+(L, R_2(LT), T)$. For each such V, $V \trianglelefteq T$, $V \in \mathcal{R}_2(LT)$, the pair L, V is in the Fundamental Setup (3.2.1), V is a TI-set under M, and either $V \trianglelefteq M$ or $C_V(L) = 1$. In addition, one of the following holds:

(a) V is the natural module of rank n for $L/O_2(L) \cong L_n(2)$, with $n = 3$, 4, or 5.

(b) $m(V) = 4$ and V is indecomposable under $L/O_2(L) \cong L_3(2)$.

(c) $L/O_2(L) \cong L_5(2)$, and V is an irreducible of rank 10.

(d) $V/C_V(L)$ is the natural module for $L/C_L(V) \cong A_n$, with $5 \leq n \leq 8$.

(e) $m(V) = 4$, and $L/C_L(V) \cong A_7$.

(f) $V/C_V(L)$ is the natural module of rank 6 for $L/O_2(L) \cong G_2(2)' \cong U_3(3)$.

(g) V is a faithful irreducible of rank 6 for $L/O_2(L) \cong \hat{A}_6$.

PROOF. By Theorem 10.0.1, $T \leq N_G(L)$. Hence $\langle L, T\rangle = LT$ and by 3.2.3, there exists $V_\circ \in Irr_+(L, R_2(LT), T)$ and for each such $V_\circ$, L and $V := \langle V_\circ^T\rangle$ satisfy the FSU. Therefore by Theorem 3.2.5, one of the following holds:

(i) $V = V_\circ \trianglelefteq M$.

(ii) $V = V_\circ \trianglelefteq T$, $C_V(L) = 1$, and V is a TI-set under M.

(iii) Case (3) of Theorem 3.2.5 holds.

Case (iii) was eliminated in Theorem 7.0.1 and Theorem 12.1.11. Thus case (i) or (ii) holds, so that $V = V_\circ \in Irr_+(L, R_2(LT))$ and $V \trianglelefteq T$. As $O_2(LT)$ centralizes $R_2(LT)$ and $L/O_2(L)$ is quasisimple, $O_2(LT) \leq C_{LT}(V) \leq O_2(LT)O_{2,Z}(L)$, so that $V \in \mathcal{R}_2(LT)$. In case (i), $V \trianglelefteq M$, and in case (ii), $C_V(L) = 1$, so in either case V is a TI-set under M. Thus it remains only to show either that G is described in (1) or (2), or that L and its action on V are as described in one of the cases (a)–(g) of part (3) of Theorem 12.2.2.

The possibilities for the pair (L, V) when V is not an FF-module under the action of $Aut_{GL(V)}(L/C_L(V))$ are listed in 3.2.9. If the first case of 3.2.9 holds, then V is the $\Omega_4^-(2^n)$-module for $L_2(2^{2n})$ with $n > 1$, so by Theorem 6.2.20, either $G \cong U_4(2^n)$, or $n = 2$ and $G \cong U_5(4)$. Thus conclusion (1) of Theorem 12.2.2 holds in this case. The remaining cases of 3.2.9 were treated in Theorem 7.0.1, where it was shown that G is isomorphic to J_4, so that conclusion (2) of Theorem 12.2.2 holds.

Thus we have reduced to the case where V is an FF-module for $Aut_{GL(V)}(\bar{L})$, where $\bar{L} := L/C_L(V)$. Therefore $\bar{L}$ and its action on $\tilde{V} := V/C_V(L)$ are listed in 3.2.8. In the first case of 3.2.8, $\bar{L} \cong L_2(2^n)$ and V is the natural module. Then by Theorem 6.2.20, the only groups G arising are: the groups of Lie rank 2 and characteristic 2 (arising in our Generic Case), so that conclusion (1) of 12.2.2 holds; and M_{22} and M_{23}, so that conclusion (2) of 12.2.2 holds. Indeed the only case of the FSU with $\bar{L} \cong L_2(2^n)$ left open by Theorem 6.2.20 is the case where $n = 2$ and V is the A_5-module; this case is one of the subcases of 3.2.8.5, and it appears as a subcase of case (d) of conclusion (3) of 12.2.2. The cases with $n > 1$ in (2), (3), and (4) of 3.2.8, were eliminated in Theorem 11.0.1. On the other hand when $n = 1$, one of the conclusions of Theorem 12.2.2 holds—namely (a), (b), the subcase of (d) with $Sp_4(2)' \cong A_6$, or (f). Thus Theorem 12.2.2 holds in the first four cases of 3.2.8. In the remaining cases of 3.2.8, one of the conclusions of part (3) of Theorem 12.2.2 holds; notice that 3.2.8.10 correponds to the subcase of case (d) of conclusion (3) of 12.2.2 where $\bar{L} \cong L_4(2) \cong A_8$. So the proof is complete. □

Thus in the remainder of this section, and in many subsequent sections, we will assume:

HYPOTHESIS 12.2.3. *Hypothesis 12.2.1 holds, and G is not one of the groups in conclusions (1) and (2) of Theorem 12.2.2. Thus conclusion (3) of Theorem 12.2.2 holds. Set $M := N_G(L)$, and let $V \in Irr_+(L, R_2(LT), T)$.*

Since Hypothesis 12.2.3 implies that conclusion (3) of Theorem 12.2.2 holds, the remainder of our treatment of the Fundamental Setup is devoted to the groups and modules listed there.

Observe also that Hypothesis 12.2.3 imposes constraints on all members of $\mathcal{L}_f^*(G,T)$:

REMARK 12.2.4. Assume Hypothesis 12.2.3. Then for any $K \in \mathcal{L}_f^*(G,T)$ with $K/O_2(K)$ quasisimple, Hypothesis 12.2.3 holds for K in the role of "L". Thus K is described in conclusion (3) of Theorem 12.2.2, T normalizes K, there exists $V_K \in Irr_+(K, R_2(KT), T)$, and any such V_K is described in conclusion (3) of Theorem 12.2.2.

Indeed observe that any KT-submodule of $R_2(KT)$ which is irreducible module K-fixed points must contain such a V_K, and hence must itself be of the form V_K. However rather than introducing further notation for KT, we will continue to use the existing notation of $Irr_+(K, R_2(KT), T)$.

Usually when we assume Hypothesis 12.2.3, we adopt the following notational conventions:

NOTATION 12.2.5. (1) $Z := \Omega_1(Z(T))$ and $Q := O_2(LT)$.

(2) $M_V := N_M(V) = N_G(V)$ and $\bar{M}_V := M_V/C_{M_V}(V)$.

(3) For $v \in V^\#$, $G_v := C_G(v)$, $M_v := C_M(v)$, $L_v := O^2(C_L(v))$, and $T_v := C_T(v)$. We have the properties:

(a) $L_v \trianglelefteq M_v$.

(b) Conjugating in L, we may choose v so that $T_v \in Syl_2(M_v)$.

(c) $M_v \in \mathcal{H}^e$.

(d) $C(G_v, Q) \leq M_v$.

(e) $M_v \leq M_V$.

(4) For $z \in V \cap Z^\#$, set $\tilde{G}_z := G_z/\langle z \rangle$.

PROOF. We establish the properties claimed in part (3): Part (a) follows since $L \trianglelefteq M$; (b) follows since $T \in Syl_2(G)$, (c) follows from 1.1.3.2; (d) follows as $C(G,Q) \leq M$ by 1.4.1.1; (e) is a special case of 12.2.6, established in the next subsection. □

12.2.1. Preliminary results under Hypothesis 12.2.3. Recall that we are assuming Hypothesis 12.2.3, so that in particular Theorem 12.2.2.3 holds. Our first result follows from Theorem 12.2.2.3 and 3.1.4.1:

LEMMA 12.2.6. *V is a TI-set in M, so if $1 \neq U \leq V$ and $H \leq N_G(U)$, then $H \cap M = N_H(V)$.*

LEMMA 12.2.7. *Assume $C_G(Z) \leq M$ and $H \in \mathcal{H}_*(T, M)$. Let $K := O^2(H)$ and $V_H := \langle Z^H \rangle$. Then*

(1) $V_H \in \mathcal{R}_2(H)$ and $C_T(V) = O_2(H) \leq C_H(V_H) \leq \ker_{N_H(V)}(H)$.

Assume further that H is not solvable. Then

(2) $K/O_2(K) \cong L_2(4)$.

(3) $K \leq Y \in \mathcal{L}_f^*(G, T)$, and either

(i) $Y = K$ and $[V_H, K]$ is the sum of at most two A_5-modules for $K/O_2(K)$, or

(ii) $Y/O_{2,Z}(Y) \cong A_7$, Hypothesis 12.2.3 is satisfied with Y in the role of "L", and for each $V_Y \in Irr_+(Y, R_2(YT), T)$, V_Y is T-invariant and $m(V_Y) = 4$ or 6.

PROOF. First $V_H \in \mathcal{R}_2(H)$ by B.2.14, so $O_2(H) \leq C_H(V_H)$. Then as $C_G(Z) \leq M$ but $H \not\leq M$, $K \not\leq C_H(V_H)$. The remaining statements in (1) follow from B.6.8.6 and 12.2.6.

Now assume H is not solvable. By E.2.2, $K = \langle X^T \rangle$ for a suitable $X \in \mathcal{C}(H)$ with $X/O_2(X)$ quasisimple and $X \not\leq M$. As $[V_H, X] \neq 1$ by (1), $X \in \mathcal{L}_f(G, T)$. We may embed $X \leq Y \in \mathcal{L}^*(G, T)$, and then by 1.2.9, $Y \in \mathcal{L}_f^*(G, T)$.

Suppose first that $Y/O_2(Y)$ is quasisimple. Then by Remark 12.2.4, Hypothesis 12.2.3 is satisfied with Y in the role of "L". In particular Y is T-invariant; and for each $V_Y \in Irr_+(Y, R_2(YT), T)$, V_Y is T-invariant, and Y, V_Y satisfies one of the conclusions of 12.2.2.3. Therefore T acts on X by 1.2.8.1, so that $X = K$ with KT described in E.2.2.2.

Assume first that $K = Y$. Comparing the lists of 12.2.2.3 and E.2.2.2, we conclude that $K/O_{2,Z}(K) \cong L_2(4)$, $L_3(2)$, or A_6. However if $K/O_{2,Z}(K)$ is $L_3(2)$ or A_6, then by E.2.2.2, T is nontrivial on the Dynkin diagram of $K/O_2(K)$, a contradiction as 12.2.2.3 says $V_Y/C_{V_Y}(K)$ is a natural module. Thus $K/O_2(K) \cong L_2(4)$. Hence by the exclusions in Hypothesis 12.2.3 and Theorem 6.2.20, $[V_H, K]$ is the sum of at most two A_5-modules. Therefore (2) and (3i) hold in this case.

So we may assume that $K < Y$. Therefore by 1.2.4, the embedding of K in Y is described in A.3.12. Searching for pairs K, Y with K appearing in E.2.2.2 and Y appearing in 12.2.2.3, we conclude that either $K/O_{2,Z}(K) \cong L_2(4)$, $L_3(2)$, or A_6, with $Y/O_{2,Z}(Y) \cong A_7$; or $K/O_2(K) \cong L_3(2)$, with $Y/O_2(Y) \cong L_4(2)$ or $L_5(2)$. But again when $K/O_2(K)$ is $L_3(2)$ or A_6, T is nontrivial on the Dynkin diagram of $K/O_2(K)$, whereas there is no such embedding of $KT/O_2(KT)$ in S_7 or $Aut(L_4(2))$, so $KT/O_2(KT) \cong Aut(L_3(2))$ and $YT/O_2(YT) \cong Aut(L_5(2))$. However this is also impossible as V_Y is a T-invariant natural module for $Y/O_2(Y)$ by 12.2.2.3, so $Y/O_2(Y)$ is self-normalizing in $GL(V_Y)$. Thus $K/O_2(K) \cong A_5$ and $Y/O_{2,Z}(Y) \cong A_7$, so that (2) holds. Also as Y, V_Y appears in 12.2.2.3, (3ii) holds.

Finally assume that $Y/O_2(Y)$ is not quasisimple. Then by 1.2.1.4, $Y/O_{2,2'}(Y) \cong SL_2(p)$ for some prime $p > 3$. Now T acts on Y by 1.2.1.3, so again $X = K$ is T-invariant by 1.2.8.1, and hence appears in E.2.2.2; in particular as $K/O_2(K)$ is quasisimple, $K < Y$. Again by 1.2.4, the embedding $X < Y$ is described in A.3.12, so by A.3.12, $K/O_2(K) \cong L_2(p)$ or $L_2(5)$. Now for some odd prime q, $X_0 := \Xi_q(Y) \in \Xi_{rad}(G, T)$, and as $Y \in \mathcal{L}^*(G, T)$, by definition $X_0 \in \Xi^*_{rad}(G, T)$. Then by 1.3.8, $X_0 \in \Xi^*(G, T)$. By 3.2.14 applied to Y, $[Z, X_0] = 1$, so $X_0T \leq C_G(Z) \leq M$. Then

$M = N_G(X_0)$ since $N_G(X_0) = !\mathcal{M}(X_0T)$ by 1.3.7, so $H \leq YT \leq N_G(X_0) \leq M$, contradicting $H \not\leq M$. $\square$

Given a group A, write $\theta(A)$ for the subgroup of A generated by all elements of order 3 in A.

LEMMA 12.2.8. *One of the following holds:*

(1) $O^{3'}(M) = L$.

(2) $L/O_2(L) \cong A_5$ *or* $L_3(2)$.

(3) $L/O_2(L) \cong \hat{A}_6$ *or* $\hat{A}_7$ *and* $L = \theta(M)$ *is the subgroup of* M *generated by all elements of* M *of order* 3.

PROOF. First L is described in 12.2.2.3, so if $m_3(L) = 1$, then (2) holds; thus we may assume $m_3(L) = 2$. Then (1) or (3) holds by 12.2.2 and A.3.18. $\square$

LEMMA 12.2.9. *(1) If* $C_Z(L) \neq 1$, *then* $C_G(Z) \leq M$.

(2) If $C_G(Z) \leq M$, *then* $L = [L, J(T)]$.

PROOF. As $M = !\mathcal{M}(LT)$, (1) holds. Theorem 3.1.8.3 implies (2). $\square$

LEMMA 12.2.10. *(1)* $C_{\bar{M}_V}(\bar{L}) = Z(\bar{L})$.

(2) $\bar{L} = O^2(\bar{M}_V)$ *and* $\bar{M}_V = \bar{L}\bar{T}$.

PROOF. In each case listed in conclusion (3) of Theorem 12.2.2, $Out(L/O_2(L))$ is a 2-group, so $O^2(\bar{M}_V) \leq \bar{L}C_{\bar{M}_V}(\bar{L})$. Further in cases (a)–(f), the irreducible module $I := V/C_V(L)$ satisfies $E := End_{\bar{L}}(I) \cong \mathbf{F}_2$, so that $C_{\bar{M}_V}(\bar{L}) = 1$. Hence (1) and (2) hold in these cases. In case (g), $I = V$ and $E \cong \mathbf{F}_4$, with $Z(\bar{L})$ inducing $E^\#$, so again (1) and (2) follow. $\square$

LEMMA 12.2.11. *Assume* $H \in \mathcal{H}_*(T, M)$ *with* $H \leq N_G(U)$ *for some* $1 \neq U \leq V$. *Assume also that one of the following holds:*

(a) $L/O_2(L) \cong L_5(2)$.

(b) $L/O_2(L) \cong \hat{A}_6$, *and* $V \leq O_2(C_G(v))$ *for* $v \in C_V(T)^\#$.

(c) $L/O_2(L) \cong G_2(2)'$ *and* $C_G(V_3) \leq M$, *where* V_3 *is the* $(T \cap L)$*-invariant subspace of* V *of rank* 3.

Then

(1) $n(H) \leq 2$, *and*

(2) if $n(H) = 2$, *then a Hall* $2'$*-subgroup of* $H \cap M$ *is a nontrivial* 3*-group.*

PROOF. The lemma is vacuously true if $n(H) \leq 1$, so we may assume that $n(H) \geq 2$. Then by E.2.2, $H \cap M$ is the preimage of the normalizer of a Borel subgroup of the group $O^2(H/O_2(H))$ of Lie type and characteristic 2. We take C to be a Hall $2'$-subgroup of $H \cap M$, so that C is abelian and $CT = TC$. We may assume that either:

(I) $n(H) = 2$, but there is a prime $p > 3$ such that $B := O_p(C) \neq 1$, or

(II) $n(H) > 2$, in which case p and B also must exist.

Then also $A := \Omega_1(B) \neq 1$, $BT = TB$, and $AT = TA$. As $n(H) > 1$ and $AT = TA$, $N_H(A) \not\leq M$ by 4.4.13.1.

Next as $B \leq H \cap M \leq N_M(U)$ by hypothesis, B normalizes V by 12.2.6. We claim in fact that B centralizes V: For otherwise $1 \neq \bar{B} \leq O^2(\bar{M}_V) = \bar{L}$ by 12.2.10.2. Thus $\bar{B}$ is an abelian p-subgroup of $\bar{L}$ with $p > 3$, and $\bar{B}\bar{T} = \bar{T}\bar{B}$, so we may apply A.3.15. However, the list of possibilities for $L/O_2(L)$ from A.3.15 does not intersect the list from Theorem 12.2.2.3.

Therefore $B \leq C_{M_V}(V) \leq C_{M_V}(\bar{L})$, and hence $B \leq C_{M_V}(L/O_2(L))$. Visibly Hypothesis 4.2.1 is satisfied, so Hypothesis 4.4.1 is satisfied by Remark 4.4.2. Thus we can apply Theorem 4.4.3 to obtain a contradiction: Namely we showed that $N_G(A) \not\leq M$, so that one of the conclusions of 4.4.3.2 must hold, which is not the case as we are assuming one of hypotheses (a)–(c). □

The statement of the following lemma makes use of Notation 12.2.5.

LEMMA 12.2.12. *Assume $v \in V^\#$ with $O_2(\bar{L}_v\bar{T}_v) = 1$, and choose $T_v \in Syl_2(M_v)$. Then*

(1) $Q = O_2(L_vT_v)$, $Q \in Syl_2(C_{G_v}(L_v/O_2(L_v)))$, and Hypothesis C.2.3 is satisfied with G_v, M_v, Q in the roles of "H, M_H, R".

(2) Assume $\bar{L}_v = \bar{L}_v^\infty$ and $V = [V, \bar{L}_v]$. Then Hypothesis C.2.8 is satisfied with G_v, M_v, L_v^∞, Q in the roles of "H, M_H, L_H, R".

PROOF. Since $V \in \mathcal{R}_2(LT)$, $Q \leq T_v$, so as $O_2(\bar{L}_v\bar{T}_v) = 1$, $Q = O_2(L_vT_v)$ by 1.4.1.4. We chose $T_v \in Syl_2(M_v)$ while $L_v \trianglelefteq M_v$ and $C(G_v, Q) \leq M_v$ by 12.2.5.3. Therefore (1) follows from A.4.2.7.

Assume $\bar{L}_v = \bar{L}_v^\infty$ and $V = [V, L_v^\infty]$. Then $L_v^\infty \in \mathcal{C}(M_v)$, and as $\bar{L}_v = \bar{L}_v^\infty$, the argument in the previous paragraph shows that $Q \in Syl_2(C_{M_v}(L_v^\infty/O_2(L_v^\infty)))$. Also $M_v \in \mathcal{H}^e$ by (3c) of 12.2.5, and the verification of the remainder of Hypothesis C.2.8 is straightforward. □

12.2.2. The treatment of V a TI-set in G. We now come to the main result of this section, in which we treat the case where V is a TI-set in G.

THEOREM 12.2.13. *Assume Hypothesis 12.2.3. Then one of the following holds:*

(1) $C_G(v) \not\leq M$ for some $v \in V^\#$.

(2) L is an $L_n(2)$-block for $n = 3$ or 4, and $G \cong L_{n+1}(2)$.

(3) L is an $L_3(2)$-block, and $G \cong A_9$.

(4) L is an $L_4(2)$-block, and $G \cong M_{24}$.

REMARK 12.2.14. The groups M_{22} and M_{23} contain a pair (L, V) failing 12.2.13.1, with V of rank 4 and $L/O_2(L) \cong A_6$ or A_7, but these groups are explicitly excluded by Hypothesis 12.2.3. Their shadows are eliminated via an appeal to 12.2.7.3, which is violated in M_{22} and M_{23}.

REMARK 12.2.15. As the groups appearing in conclusions (2), (3), and (4) of Theorem 12.2.13 appear as conclusions in our Main Theorem, we will sometimes assume that G is not one of those groups. Then Theorem 12.2.13 tells us that $C_G(v) \not\leq M$ for some $v \in V^\#$.

Until the proof of Theorem 12.2.13 is complete, assume that $C_G(v) \leq M$ for each $v \in V^\#$. We must show that one of (2)–(4) holds. We begin a series of reductions. Recall we have adopted Notation 12.2.5.

LEMMA 12.2.16. *V is a TI-set in G.*

PROOF. By 12.2.6, V is a TI-set in M. Thus if the lemma fails, there is $g \in G - M$ and $v \in V^\#$ with $u := v^g \in V$. As we are assuming that conclusion (1) of 12.2.13 fails, $G_w = M_w$ for $w \in V^\#$, so

(1) $M_v = G_v \cong G_u = M_u$.

Next if $u^x = v$ for some $x \in M$, then $gx \in G_v \leq M$, so $g \in Mx^{-1} = M$, contrary to the choice of g. Hence

(2) $u \notin v^M$.

By (1) and (2) there are $u, v \in V^\#$ with $M_u \cong M_v$ but $v \notin u^M$. Inspecting the list of Theorem 12.2.2.3, we first eliminate the cases where L is irreducible on V. In the remaining cases, let z denote the generator of $C_V(L)$. We also eliminate case (b), since there $Z \cap V = C_V(L)$ by B.4.8.2, so that each $v \in V - C_V(L)$ is T-conjugate to vz, and hence all members of $V - C_V(L)$ are M-conjugate. This leaves the subcases of (d) where V is the core of the permutation module of degree n for $\bar{L} \cong A_n$, $n = 6$ or 8, and the subcase of (f) where V is the Weyl module of dimension 7 for $\bar{L} \cong G_2(2)'$. In the former, we may take v of weight 2, and u of weight $n-2$; in the latter, we may take v singular and u nonsingular in $V - C_V(L)$. Now conjugating in L, we may assume $u = vz$. As $C_V(L) \neq 1$, $V \trianglelefteq M$ by 12.2.2.3, so $z \in Z(M)$ and hence $M = G_z$.

Without loss, $T_v := C_T(v) \in Syl_2(M_v)$, so as $G_v \leq M$, $T_v \in Syl_2(G_v)$. As $u = vz$ with z central in $M_u = G_u$, also $T_v \in Syl_2(G_u)$. Then replacing g by a suitable member of $gC_G(u)$, we may assume $g \in N_G(T_v)$. However if $\bar{L} \cong A_6$ or $G_2(2)'$, then v is 2-central in M so that $T = T_v$, so $g \in N_G(T) \leq M$ by Theorem 3.3.1, contrary to (2). Hence $\bar{L} \cong A_8$.

Let $Z_v := \Omega_1(Z(T_v))$ and $V_v := \langle Z_v^L \rangle$. As $Q \leq T_v$, Q centralizes Z_v and hence V_v. On the other hand, Z_v contains $Z \cap V \not\leq C_V(L)$ by I.2.3.1i; so $V \leq V_v$ as $V \in Irr_+(L, R_2(LT))$, and hence $C_{LT}(V_v) \leq C_{LT}(V) = Q$. Therefore $Q = C_{LT}(V_v)$, and hence $V_v \in \mathcal{R}_2(LT)$. If $[V_v, J(T)] = 1$, then as $V \leq V_v$, also $[V, J(T)] = 1$, and then 3.2.10.2 contradicts (2). Thus $[V_v, J(T)] \neq 1$, so by B.2.7, V_v is an FF-module for $LT/C_{LT}(V_v)$. Then as V is the core of the permutation module for A_8, by Theorem B.5.1.1, $V = [V_v, L]$, and hence $V_v = VZ_L$ by B.2.13, where $Z_L := C_{Z_v}(L)$. Thus $Z_v = \langle v \rangle Z_L$. Now $T = T_v(T \cap L) \leq C_G(Z_L)$, so $Z_L \cap Z_L^g = 1$ using (2), since $M = !\mathcal{M}(LT)$ and $M^g = !\mathcal{M}(L^g T^g)$. Hence as $g \in N_G(T_v) \leq N_G(Z_v)$ and Z_L is a hyperplane of Z_v, we conclude Z_L is of order 2. Therefore $Z_L = \langle z \rangle$ and $Z_v = \langle z, v \rangle$. Then as $v^g = u \notin z^G$ by (1), z is weakly closed in Z_v. Therefore $g \in G_z = M$, contrary to (2), completing the proof of 12.2.16. □

LEMMA 12.2.17. *$W_0(T, V) \leq C_T(V) = Q$, so that $N_G(W_0(T, V)) \leq M$.*

PROOF. Let $V^g \leq T$. By 12.2.16, V is a TI-set in G, so as $V^g = N_{V^g}(V)$, we conclude from I.6.2.1 that $[V, V^g] = 1$. Now the final statement follows from E.3.34.2. □

During the remainder of the proof of Theorem 12.2.13, pick $H \in \mathcal{H}_*(T, M)$, and set $K := O^2(H)$, $V_H := \langle Z^H \rangle = \langle Z^K \rangle$, and $H^* := H/C_H(V_H)$. Observe that $C_G(Z) \leq C_G(Z \cap V) \leq N_G(V) \leq M$ by 12.2.16; thus we may apply 12.2.7 during the course of the proof.

LEMMA 12.2.18. *$V \not\leq O_2(H)$, so that $V^* \neq 1$.*

PROOF. By 12.2.7.1, $O_2(H) = C_T(V_H)$. Thus $V \leq O_2(H)$ iff $V^* = 1$, so we may assume $V \leq O_2(H)$, and it remains to derive a contradiction.

Similarly if $W_0 := W_0(T, V) \leq O_2(H)$, then $H \leq N_G(W_0) \leq M$ by E.3.15 and 12.2.17, contrary to $H \not\leq M$. Thus there is $A := V^g \leq T$, with $A^* \neq 1$, and $K \leq \langle A^H \rangle$ by B.6.8.4.

Suppose that $A \cap O_2(H) = 1$. Then $A^* \cong A$, so $m(A^*) = m(V) \geq 3$ by 12.2.2.3. But if H is solvable, then $m_2(H^*) \leq 2$ as $H = O_{2,p,2}(H)$ for some odd prime p by B.6.8.2, so that $H/O_{2,p}(H)$ is a subgroup of $GL_2(p)$. On the other hand if H is nonsolvable, then by 12.2.7.2, $K^* \cong L_2(4)$, so that again $m_2(H^*) \leq 2$. Thus in either case, we have a contradiction to $m(A^*) \geq 3$.

This contradiction shows that $1 \neq A \cap O_2(H)$. Thus for each $h \in H$, $1 \neq A^h \cap O_2(H) \leq N_{A^h}(V)$ by 12.2.7.1. However as $V \leq O_2(H)$, $\langle V, A^h \rangle$ is a 2-group, so $[V, A^h] = 1$ by I.6.2.1. We saw $K \leq \langle A^H \rangle$, so $K \leq C_G(V) \leq M$, contradicting $H \not\leq M$. This completes the proof of 12.2.18. □

Lemma 12.2.19. *H is solvable.*

Proof. Assume that H is not solvable. Then by 12.2.7, $K/O_2(K) \cong K^* \cong L_2(4)$. By 12.2.18, $V^* \neq 1$. As $V \leq O_2(M)$, $V^* \leq O_2(M \cap H)^* = T^* \cap K^* \in Syl_2(K^*)$. Thus either $V^* = T^* \cap K^* \cong E_4$, or $V^* \leq K^*$ is of order 2. Pick $h \in K - M$, and let $U := V \cap O_2(H)$, $I := \langle V, V^h \rangle$, and $W_I := O_2(I)$. Then either $|V^*| = 4$ and $I^* = K^* \cong L_2(4)$, or V^* is of order 2 and $I^* \cong D_{2m}$, $m = 3$ or 5. As $m(V) \geq 3 > m(V^*)$, $U \neq 1$. Then as $U \leq O_2(H) \leq N_H(V^h)$ by 12.2.7.1, $N_V(V^h) \neq 1$. It follows from (a) and (c) of I.6.2.2 that $W_I := U \times U^h$ is a sum of natural modules for $I/W_I \cong I^*$; in particular if $I^* \cong D_{2m}$, an element of order m is fixed point free on W_I. If V^* is of order 2, pick $x \in H \cap M$ of order 3 and let $K_I := \langle V^x, I \rangle$ and $W := U^x W_I$; if $|V^*| = 4$ let $K_I := I$ and $W := W_I$. Thus in either case $K^* = K_I^*$.

We claim that K_I acts on W, and W is elementary abelian: Suppose first that $|V^*| = 2$. We saw that $U \neq 1$ normalizes V^x, and as $\langle V^*, V^{*x} \rangle$ is a 2-group by our choice of x, $\langle V, V^x \rangle$ is a 2-group. Therefore V centralizes V^x by I.6.2.1. Now by symmetry between $I = \langle V, V^h \rangle$ and $\langle V^x, V^h \rangle$, $\langle V^x, V^h \rangle$ acts on $U^x \times U^h$, so $K_I = \langle V, V^x, V^h \rangle$ acts on $W = UU^xU^h$ and W is elementary abelian. On the other hand if $|V^*| = 4$ then $K_I = I$ acts on $W = U \times U^h$, completing the proof of the claim.

Next by 12.2.7.1, $O_2(H)$ acts on V and V^h, and also on V^x when $|V^*| = 2$, so $O_2(H)$ acts on K_I and W. Thus $KO_2(H) = K_I O_2(H)$ acts on K_I and W. Therefore as $K_I O_2(H)/K_I \cong O_2(H)/(O_2(H) \cap K_I)$ is a 2-group, $K = O^2(H) \leq K_I$.

Now if $|V^*| = 4$, then $K_I = I$, so $W = O_2(I)$ is a sum of natural modules for $K^* \cong K_I/W$. Suppose on the other hand that V^* is of order 2. We saw earlier that V centralizes V^x; hence $W = U^x W_I = C_W(V) W_I$, so that $W = C_W(i) \times W_I$ for i of order m in I, which is fixed point free on W_I; and $C_W(i) = C_W(I) \leq C_W(V) = UU^x = C_W(V^x)$. Thus $C_W(I) = C_W(K_I)$. As $[W, V] = U$ and $[W, V^x] = U^x$ with VV^x abelian, $T^* \cap K^* = V^* V^{*x}$ is quadratic on W. Also i is fixed-point-free on W_I, so by G.1.5 and G.1.7, $W/C_W(I)$ is a sum of natural modules for K^*.

Now $Z \cap V \neq 1$, so $1 \neq V_Z := \langle (Z \cap V)^K \rangle \in \mathcal{R}_2(KT)$ by B.2.14. As V is a TI-set in G by 12.2.16 and $K \not\leq M \geq N_G(V)$, $C_V(K) = 1$. As $Z \cap V \leq O_2(H) \cap V = U \leq W$, $V_Z \leq W$, so by the previous paragraph $1 \neq V_Z/C_{V_Z}(K)$ is a sum of natural modules for K^*.

By 12.2.7.3, $K \leq Y \in \mathcal{L}_f^*(G, T)$, with Y described in case (i) or (ii) of that result. Let $V_Y := \langle (Z \cap V)^Y \rangle$ and $\hat{Y} := Y/C_Y(V_Y)$; then $V_Z \leq V_Y$. As $V_Z/C_{V_Z}(K)$ is a sum of natural $L_2(4)$-modules, case (i) of 12.2.7.3 cannot arise, since there the noncentral chief factors for K on V_Z are A_5-modules. [1] Therefore case (ii) of 12.2.7.3 occurs, so $\hat{Y} \cong A_7$, and each $J \in Irr_+(Y, V_Y, T)$ is T-invariant and of rank 4 or 6. Again using the fact that the noncentral chief factors for K on V_Z are $L_2(4)$-modules, we conclude that J is of rank 4 and the natural module for $\hat{K}$. Therefore $[J, T \cap K] = [J, w] \cong E_4$ for each $w \in V - O_2(K)$. Now $[J, w] \leq V$, and from the action of A_7 on J,

$$Y = \langle C_Y(v) : v \in V \cap J^{\#} \rangle \leq M,$$

whereas $K \leq Y$ with $K \not\leq M$. This contradiction completes the proof of 12.2.19.
□

By 12.2.19, H is solvable, so we may apply B.6.8.2 to conclude that

$$H = PT,$$

where P is a p-group for some odd prime p, $P^* = F^*(H^*)$, and $\Phi(P) = P \cap M$. Thus $[P^*, V^*] \neq 1$ by 12.2.18, and hence $P^* = [P^*, V^*]$, since $V \trianglelefteq T$ and T^* is irreducible on $P^*/\Phi(P^*)$ by B.6.8.2. By Coprime Action, we may pick $h \in P - \Phi(P)$ so that $C_{V^*}(h^*)$ is a hyperplane of V^*. Let $I := \langle V, V^h \rangle$ and $U := O_2(I)$. By I.6.2.2, $U = (V \cap U) \times (V^h \cap U)$ is a sum of natural modules of $I/U \cong D_{2p}$. Thus $V^h \cap U$ is of rank $m(V) - 1 \geq 2$, and induces the full group of transvections in $GL(V)$ with axis $V \cap U$. Therefore we may apply the dual of G.3.1 to the action of LT on V, to restrict the cases in 12.2.2.3 to:

LEMMA 12.2.20. *$\bar{L} = GL(V) \cong L_n(2)$ for $n = 3$, 4, or 5, V is the natural module for $\bar{L}$, $U \cap V$ is a hyperplane of V, and U induces the full group of transvections with axis $U \cap V$ on V. In particular as $T \leq N_G(V)$, $T \leq LC_T(V)$.*

Observe that we are beginnning to show that G has a 2-local structure similar to that of one of the groups in conclusions (2)–(4) of Theorem 12.2.13.

LEMMA 12.2.21. *$U \leq O_2(H)$, so V^* is of order 2 and inverts $P^*/\Phi(P^*)$.*

PROOF. We saw $U = [U, O^2(I)]$, so if $U \not\leq O_2(H)$, then $U^* = [U^*, O^2(I^*)] \neq 1$, impossible as H^* is 2-nilpotent. We also saw that $P^* = [P^*, V^*]$, so as $V \cap U$ is a hyperplane of U by 12.2.20, V^* is of order 2, and V^* inverts $P^*/\Phi(P^*)$. □

Next we obtain some restrictions on the structure of H and its action on $\langle (U \cap V)^H \rangle$.

We observed earlier for each $h \in P - \Phi(P) = P - M$ that $I^* \cong D_{2p}$, so that h has order p. Then by A.1.24, $P \cong \mathbf{Z}_p$, E_{p^2}, or p^{1+2}. Let $v \in V - U$. By the Baer-Suzuki Theorem, v inverts an element h' of order p in K; replacing P by a Sylow group containing h', we may take $h' \in P$. Then as V^* is of order 2 by 12.2.21, we may take $h' = h$ in the definition of I. Let $W := \langle (U \cap V)^H \rangle$. Then $W \leq O_2(H)$ by 12.2.21, and $U = (V \cap U)(V^h \cap U) \leq \langle (U \cap V)^P \rangle \leq W$. Indeed as T acts on $U \cap V = O_2(H) \cap V$, $(U \cap V)^H = (U \cap V)^{TP} = (U \cap V)^P = U^P$, so $W = \langle U^P \rangle$. In particular as $U \leq [W, O^2(I)] \leq [W, P]$, $W = \langle U^P \rangle \leq [W, P]$, and hence $W = [W, P]$. Now $1 \neq U \cap V = O_2(H) \cap V \trianglelefteq O_2(H)$, so $U \cap V$ commutes with its H-conjugates by I.6.2.2, and hence W is elementary abelian. From the action of I on U in I.6.2.2a, $[U, v] = [U \cap V^h, v] = U \cap V$. Also $[W, v] \leq W \cap V = U \cap V$, so $W = (U \cap V^h) \oplus C_W(v)$. Similarly $W = (U \cap V) \oplus C_W(v^h)$, so $W = U \oplus C_W(I)$

[1] This is where Hypothesis 12.2.3 eliminates M_{22} and M_{23} as possible conclusions in Theorem 12.2.13.

with $U = [W, I] = [W, h]$. Further $[W, V] = [U, V] = V \cap U = V \cap W$ is of rank $n - 1$.

We next claim that we may choose P invariant under v; and when P is nonabelian, that $\Phi(P) \leq N_{H \cap M}(V)$. If $P \cong \mathbf{Z}_p$, then v acts on P and we are done. Suppose $P \cong E_{p^2}$. Then $\langle P, v \rangle \leq N := N_H(\langle h \rangle)$. Set $\dot{N} := N/\langle h \rangle$. As v^* inverts P^*, $\dot{v} \notin O_2(\dot{N})$, so we may apply the Baer-Suzuki Theorem again in $\dot{N}$, to conclude that v inverts y of order p in $C_H(h) - \langle h \rangle$. Thus $\langle h, y \rangle \in Syl_p(H)$ is v-invariant, and choosing $P := \langle h, y \rangle$, we are done in this case also. Finally, suppose $P \cong p^{1+2}$. Then $\Phi(P) = Z(P)$ centralizes h and so acts on $[W, h] = U$. Further $O_{2,\Phi}(H) = O_2(H)\Phi(P)$ and V centralizes $O_{2,\Phi}(H)/O_2(H)$, so that $[\Phi(P), V] \leq O_2(H)$. Thus $\Phi(P)$ acts on $1 \neq C_U(O_2(H)V) \leq U \cap V$, and so since V is a TI-set in M by 12.2.6, $\Phi(P)$ acts on V, establishing the final assertion of the claim. Next $\Phi(P)$ centralizes $V/(U \cap V)$ of rank 1, and hence centralizes some $v_0 \in V - U$, which we may take to be v. Set $P_1 := \Phi(P)\langle h \rangle$, $H_1 := N_H(P_1)$, and $\dot{H}_1 := H_1/P_1$. We apply the Baer-Suzuki Theorem one more time to $\dot{H}_1$: As v^* inverts $P^*/\Phi(P^*)$, $\dot{v} \notin O_2(\dot{H}_1)$, so $\dot{v}$ inverts an element $\dot{k}$ of order p in $\dot{H}_1$, and then the preimage of $\langle \dot{k} \rangle$ is a v-invariant Sylow p-group P of H. This completes the proof of the claim.

So in any event we may assume v acts on P. Thus $VPW = \langle v \rangle PW$ is a subgroup of H. Further $[O_2(H), v] \leq V \cap O_2(H) = V \cap U \leq W$, so that v centralizes $O_2(H)/W$. Then as $P = [P, v]$, $[O_2(H), P] \leq W$, and hence $PW \trianglelefteq O_2(H)P$. Thus as $K = O^2(H) \leq PO_2(H)$, $K \leq PW$. We saw earlier that $W = [W, P]$, so $W \leq O^2(PW) \leq O^2(H) = K$, and hence $K = PW$. Summarizing:

LEMMA 12.2.22. *$P \cong \mathbf{Z}_p$, E_{p^2}, or p^{1+2}, and we may choose P so that P is invariant under $v \in V - U$, $W = \langle U^P \rangle = [W, P]$ is elementary abelian, $[W, V] = V \cap W$ is of rank $n - 1$, $K = PW$, and $\Phi(P) \leq N_{H \cap M}(V)$.*

LEMMA 12.2.23. *$P \cong \mathbf{Z}_p$.*

PROOF. Assume P is not $\mathbf{Z}_p$, and let $H_P := KV$, $\Phi := \Phi(P)$, $W_P := C_W(\Phi)$, and $\hat{H}_P := H_P/C_{H_P}(W_P)$.

Suppose $W_P \neq 1$. By 12.2.22, $W = [W, P]$, so as T^* is irreducible on $P^*/\Phi(P^*)$, $\Phi = C_P(W_P)$ and $\hat{P} \cong E_{p^2}$. Then by Generation by Centralizers of Hyperplanes A.1.17, W_P is generated by nontrivial subgroups $W_i := C_{W_P}(P_i)$, where P_i runs over a nonempty collection of subgroups of index p in P generating P. As v inverts P/Φ, v acts on each subgroup P_i and hence on each W_i; further as $W = [W, P]$ so that $C_W(P) = 1$, also $W_i = [W_i, P]$, so that v is nontrivial on W_i. Thus $1 \neq [W_i, v] \leq W_i \cap V$. Therefore as V is a TI-set in G by 12.2.16, $P_i \leq C_G(W_i \cap V) \leq N_G(V) \leq M$, so $H = KT = PT \leq M$, contrary to $H \not\leq M$.

Therefore $W_P = 1$, so $P \cong p^{1+2}$ and $W = [W, \Phi]$. By 12.2.22, $\Phi \leq N_{H \cap M}(V)$, so $W \cap V = [W \cap V, \Phi]$ is of rank $n - 1$ by 12.2.22, and then $m([V, \Phi]) = n - 1$. In particular, Φ is faithful on V. Now $\bar{\Phi} \leq \bar{M}_V = \bar{L} = L/O_2(L) = GL(V) \cong L_n(2)$ by 12.2.20. Therefore as $\Phi T = T\Phi$, $p = 3$ and $\bar{\Phi}\bar{T}$ is a rank one parabolic of $\bar{L}$. However for any X of order 3 in a rank one parabolic, $[V, X]$ is of rank 2; so as $m([V, \Phi]) = n - 1$ we conclude $n = 3$.

As $n = 3$, $U \cap V$ is of rank 2. Now $KV = WPV = \langle V, V^x, V^y \rangle$ for x, y chosen so that $P := \langle x, y \rangle$. Thus

$$W = [W, KV] = [W, V][W, V^x][W, V^y] = (U \cap V)(U \cap V^x)(U \cap V^y).$$

Furthermore $m(W) \leq 3m(U \cap V) = 6$, so $m(W) = 6$, since this is the minimal dimension of a faithful module for $P \cong 3^{1+2}$.

Let $Q_L := O_2(L)$ and $T_L := T \cap L$. Then $T = QT_L$ by 12.2.20, while $Q = C_{LT}(V)$ and $\overline{LT} \cong L_3(2)$. As $\bar{\Phi}$ is inverted in $\bar{T}_L$ and Φ permutes with T, Φ is inverted in $N_{T_L}(X)$ so $\Phi \leq L$.

Now $K = PW$ by 12.2.22, and $W = [W, \Phi]$, so $Y := \Phi W = O^2(\Phi T)$. As ΦW char $PW \trianglelefteq H$, $Y \trianglelefteq H$. Now as $\bar{L} \cong L_3(2)$, $O_2(\bar{Y}) = \bar{W} \cong W/(W \cap Q)$ is of rank 2, as is $W \cap V$; so as W is of rank 6, $(W \cap Q)/(W \cap V) \cong (W \cap Q)V/V$ is of rank 2. Further $(W \cap Q)V/V = [Q/V, \Phi]$ since $W = [W, \Phi] = O_2(Y)$. Therefore L has a unique noncentral chief factor $[Q, L]/Q_0$ (for some suitable Q_0 containing V) on $[Q, L]/V$. Also since the unique noncentral chief factors for Φ on $[Q, L]/Q_0$ and V are in the centralizer of the unipotent radical $\bar{W}$, it follows from the representation theory of $L_3(2)$ that $[Q, L]/Q_0$ is isomorphic to V as an L-module. Further $W[Q, L]/[Q, L] \cong E_4$, so $L/[Q, L]$ is not $SL_2(7)$; and hence as $L = O^2(L)$, $[Q, L] = Q_L$. As W is abelian by 12.2.22, $\bar{W}$ centralizes $(W \cap Q)V/V = [Q/V, X]$, so Q_L/V is not the 4-dimensional indecomposable of B.4.8.2. Thus $V = Q_0$. Then as $V \leq Z(Q)$, while L is transitive on $(Q_L/V)^{\#}$, and $W \cap Q_L$ contains involutions not in V, it follows that $Q_L \cong E_{64}$.

Let $Q_H := O_2(H)$. As H is irreducible on W, $W \leq Z(Q_H)$, so $C_{Q_H}(Y) = C_{Q_H}(\Phi)$. Each involution in $C_T(\Phi)$ is in $Q_H V$, so from the action of L on Q_L, $C_{Q_L}(\Phi) = \langle q, v \rangle$ with $q \in C_{Q_H}(\Phi) = C_{Q_H}(Y)$ and $Q_L = \langle q^T \rangle V$. We saw $Y \trianglelefteq H$, so $C_{Q_H}(Y) \trianglelefteq H$, and hence $\langle q^T \rangle \leq C_T(Y)$, contradicting $Q_L = \langle q^T \rangle V$. This contradiction completes the proof of 12.2.23. □

By 12.2.22 and 12.2.23,

$$K = PW = O^2(I), \text{ and } U = W \trianglelefteq H.$$

In particular as $P \not\leq M$ by construction,

$$I \not\leq M.$$

As $V \trianglelefteq T$, also

$$I = PWV = KV \trianglelefteq KT = H.$$

Furthermore as T acts on $U = O_2(I)$ and $Q = C_T(V)$,

$$[Q, U] \leq C_U(V) = U \cap V \leq V;$$

and hence as $\bar{L} = \langle \bar{U}^{\bar{L}} \rangle$, we have:

LEMMA 12.2.24. *L is an $L_n(2)$-block.*

We remark that 12.2.24 establishes the first statement in each of conclusions (2)–(4) of Theorem 12.2.13, so it only remains to identify G. Our next result shows that $M = L$, and hence determines the structure of $C_G(z)$ as $C_G(z) \leq M$.

By 12.2.20, $U \cap V$ is a hyperplane of V, and U induces on V the full group of transvections with axis $U \cap V$ on V. Let $Y := O^2(N_L(U \cap V))$, so that $Y/O_2(Y) \cong L_{n-1}(2)$. Then $\bar{U} = O_2(\bar{Y}\bar{T}) = \overline{C_T(U \cap V)}$, so that $C_T(U \cap V) = UC_T(V) \trianglelefteq YT$.

LEMMA 12.2.25. *(1) $M = L$ and $V = O_2(L)$.*

(2) $n = 3$ or 4, $p = 3$, $U = C_G(U)$, $N_G(U) = YIT$, $YIT/U \cong L_{n-1}(2) \times L_2(2)$, and U is the tensor product of the natural modules for the factors.

(3) $|Z| = 2$.

PROOF. Let $R := C_T(U)$. By a Frattini Argument, $R = UN_R(P)$, and then $R = U \times C_T(PU)$. Also $[C_T(PU), v] \leq C_V(PU) = 1$, so $C_T(PU) = C_T(I)$ and $R = U \times C_T(I)$. Thus $C_T(UV) = C_R(V) = (U \cap V) \times C_T(I)$. Next from the structure of $Aut_{GL(U)}(I/U)$, $C_T(U \cap V) = VC_T(U)$, so $C_T(U \cap V) = UV \times C_T(I)$.

Next $Q \leq C_T(U \cap V)$, so by the previous paragraph, $Q = C_T(I) \times V = VC_Q(U)$. By 12.2.24, L is an $L_n(2)$-block for $3 \leq n \leq 5$, so by C.1.13, $m(Q/VC_T(L)) \leq m(H^1(L/O_2(L), V))$. If $n \neq 3$, then $H^1(L, V) = 0$, by (6) and (8) of I.1.6 so that $Q = V \times C_T(L)$ If $n = 3$, the same conclusion holds since $Q = VC_Q(U)$ and $\bar{U}$ is of elementary abelian of order 4 in $\bar{L}$, ruling out the indecomposable in B.4.8.2.

Now $[C_T(L), U] \leq C_U(L) = 1$, so $C_T(L) \leq R$. Thus $C_Q(U) = C_T(L) \times C_V(U) = C_T(L) \times (U \cap V)$, so as $C_T(U \cap V) = UQ$, we conclude $R = C_{UQ}(U) = UC_Q(U) = U \times C_T(L)$. We have shown:

$$R = C_T(U) = U \times C_T(I) = U \times C_T(L). \quad (*)$$

Next LT and I act on $T_0 := C_T(L) \cap C_T(I)$, so if $T_0 \neq 1$ then $I \leq N_G(T_0) \leq M = !\mathcal{M}(LT)$, contrary to $I \not\leq M$. Therefore $T_0 = 1$. But by (*), $\Phi(C_T(L)) = \Phi(R) = \Phi(C_T(I))$, so $\Phi(R) \leq T_0 = 1$, and hence R is elementary abelian.

From 12.2.20, $T = (T \cap L)Q$, and we saw $Q = V \times C_T(L)$ with $\Phi(C_T(L)) = 1$, so $T = (T \cap L) \times C_T(L)$ and $Z = C_V(T) \times C_T(L)$. So as $|C_V(T)| = 2$, $C_T(L)$ is a hyperplane of Z. Next as $C_T(L) \leq Z$, from (*) we see that $[C_T(I), T] \leq [U, T] \cap C_T(I) \leq C_U(I) = 1$; thus $C_T(I) \leq Z$ by (*) since R is elementary abelian. Then $C_T(I)$ is also a hyperplane of Z, since $|C_T(L)| = |C_T(I)|$ by (*). Hence as $T_0 = 1$, $|C_T(I)| = |C_T(L)| \leq 2$. Thus $|R : U| \leq 2$ in view of (*). Also we saw $C_T(U \cap V) = UV \times C_T(I)$, with $J(UV) = U$, so $R = J(C_T(U \cap V)) \trianglelefteq YT$, since YT acts on $C_T(U \cap V)$ by an observation just before 12.2.25; in particular,

$$Y \leq N_G(R).$$

We suppose for the moment that $C_T(L) = 1$. Then $Q = V$, so $O_2(M) \leq V$ by A.1.6. Therefore as $L \trianglelefteq M$, $V = O_2(M) = F^*(M)$. Thus $T \leq L$ and $M = LC_M(L/V)$ by 12.2.20. Then as $End_L(V) = \mathbf{F}_2$, $C_M(L/V) = C_M(V) = V$, so $M = L$. Thus (1) will hold once we show that $C_T(L) = 1$. Also $C_T(L) = 1$ implies $U = C_T(U)$ by (*). But $N_G(U) \in \mathcal{H}^e$ by 1.1.4.6, so that $C_G(U) \in \mathcal{H}^e$ by 1.1.3.1, so that $C_G(U) = U$. Thus $U = C_G(U)$ also follows once we establish $C_T(L) = 1$.

Assume first that $n > 3$. Then $Y \in \mathcal{L}_f(G, T)$, so $Y \leq Y_R \in \mathcal{C}(N_G(R))$ by 1.2.4, with $Y_R \in \mathcal{L}(G, T)$. Then $Y_R \leq Y_0 \in \mathcal{L}_f^*(G, T)$ by 1.2.9. If $Y_0/O_2(Y_0)$ is quasisimple, then applying Theorem 12.2.2.3 to restrict the list of A.3.12, we conclude that either $Y_0/O_2(Y_0) \cong L_m(2)$ for some $n - 1 \leq m \leq 5$, or $n = 4$ and $Y_0/O_{2,3}(Y_0) \cong A_7$. If $Y_0/O_2(Y_0)$ is not quasisimple, then from A.3.12, $n = 4$ and $Y_0/O_2(Y_0) \cong SL_2(7)/E_{49}$. As $Y \leq Y_R \leq Y_0$, we conclude that either $Y_R/O_2(Y_R) \cong L_k(2)$ with $n - 1 \leq k \leq m \leq 5$; or $n = 4$ and either $Y_R/O_{2,3}(Y_R) \cong A_7$, or $Y_R/O_2(Y_R) \cong SL_2(7)/E_{49}$. However $Y_R \leq N_G(R)$, so that $YR/R \cong L_{n-1}(2)$ is a T-invariant subgroup of Y_RR/R; so we conclude that either $Y = Y_R \trianglelefteq N_G(R)$, or $n = 4$ and $Y_RR/R \cong A_7$. Assume this last case holds. We showed $|R : U| \leq 2$, so $m(R) \leq 7$. Then as Y has two isomorphic 3-dimensional composition factors on U, we conclude that $U = [R, Y_R]$ is the 6-dimensional permutation module for Y_RR/R. This is impossible, as in the A_7-module, U/V is dual to V as a Y-module.

This contradiction shows that $Y = Y_R \trianglelefteq N_G(R)$. Then using (*), $C_T(L) = C_R(Y) \trianglelefteq N_G(R)$. Now I normalizes R as $R = C_{IT}(U)$, so if $C_T(L) \neq 1$ then

$I \le N_G(R) \le N_G(C_T(L)) \le M = !\mathcal{M}(LT)$, contrary to $I \not\le M$. Thus $C_T(L) = 1$, so that $R = U$ by (*). Recall that this completes the proof of (1), and shows that $U = C_G(U)$.

Set $G_U := N_G(U)$ and $\dot{G}_U := G_U/U = Aut_G(U)$. We showed that $Y \trianglelefteq N_G(R) = G_U$, so as $\dot{Y}$ centralizes $\dot{V}$, we conclude that $\dot{I} = \langle \dot{V}^I \rangle$ centralizes $\dot{Y}$. Now $\dot{Y} \cong L_{n-1}(2)$ has two chief factors on U, both isomorphic to the natural module $U \cap V$, while $\dot{I} \cong D_{2p}$; it follows that IY is irreducible on U. Then as $End_{\dot{Y}}(U \cap V) = \mathbf{F}_2$, $\dot{G}_U = \dot{Y} \times C_{\dot{G}_U}(\dot{Y}) \cong L_{n-1}(2) \times L_2(2)$, so $\dot{I} = C_{\dot{G}_U}(\dot{Y}) \cong L_2(2)$, and U is the tensor product of the natural modules for the factors. In particular $p = 3$. Then as $m_3(YI) \le 2$ as G_U is an SQTK-group, it also follows that $n < 5$. This completes the proof of (2), and hence of 12.2.25, under the assumption that $n > 3$.

We turn to the case $n = 3$. This time let $G_R := N_G(R)$, $L_R := N_L(R)$, $M_R := N_M(R)$, and $\dot{G}_R := G_R/R$. Since $R = C_T(U)$, R is Sylow in $C_G(R)$, while $G_R \in \mathcal{H}^e$ by 1.1.4.4.6; thus $R = C_G(R)$. As $n = 3$, U is of rank 4; so as $|R : U| \le 2$, R is of rank $k := 4$ or 5. Thus $\dot{G}_R \le GL(R) = GL_k(2)$. Further $\dot{I} \cong D_{2p}$ with $U = [R, P]$ of rank 4, so $p = 3$ or 5. As H acts on R and I, as $R = U \times C_R(I)$ by (*), and as $|C_R(I)| \le 2$, H centralizes $C_R(I)$. Thus $\dot{H}$ is faithfully embedded in $GL(U) \cong GL_4(2)$, with $D_{2p} \cong \dot{I} \trianglelefteq \dot{H}$. We conclude that $\dot{H} \cong S_3$, $\mathbf{Z}_2 \times S_3$, D_{10}, or $Sz(2)$. Hence $\dot{T}$ is cyclic or a 4-group. On the other hand, $\dot{L}_R \cong \dot{V} \times S_3$, so that $\dot{T}$ is noncyclic. Hence $\dot{H} \cong \mathbf{Z}_2 \times S_3$, and in particular $p = 3$. Furthermore $L_R \trianglelefteq M_R$, so $\dot{M}_R$ centralizes $C_R(\dot{L}_R)$ as $|C_R(L_R)| \le 2$; hence $\dot{M}_R$ is faithful on the complement $[R, L_R]$ to $C_R(\dot{L}_R)$ in R in (*). Next M_R normalizes $[R \cap O_2(L), L_R] = V \cap U$, and hence normalizes V as V is a TI-set in G by 12.2.16. Therefore $\dot{M}_R$ centralizes $\dot{V}$ as $|\dot{V}| = 2$. Thus $O^2(\dot{L}_R) = O^2(\dot{M}_R)$ from the structure of the normalizer of $O^2(\dot{L}_R)$ in $GL([R, L_R]) \cong GL_4(2)$, so that $\dot{M}_R = \dot{L}_R \dot{T} = \dot{L}_R$. Next $C_{\dot{G}_R}(\dot{V})$ acts on $[R, V] = U \cap V$, so again as V is a TI-set in G, $C_{\dot{G}_R}(\dot{V}) \le \dot{M}_R$, and hence $C_{\dot{G}_R}(\dot{V}) = \dot{L}_R \cong \mathbf{Z}_2 \times S_3$.

Let $\dot{i} \in \dot{T} - \dot{V}$; then $1 \ne [U \cap V, i] \le U \cap V$. But if $\dot{i} = \dot{v}^g$ for some $g \in G_R$, then $[R, \dot{i}] = [R, \dot{v}]^g = (U \cap V)^g$, so as V a TI-set in G, we conclude $V = V^g$, contradicting $\dot{v} \ne \dot{i}$. Therefore $\dot{i}^G \cap \dot{V} = \emptyset$, so by Burnside's Transfer Theorem 37.7 in [**Asc86a**], $\dot{G}$ is 2-nilpotent. As $\dot{Y} = C_{O(\dot{G}_R)}(\dot{V}) \cong \mathbf{Z}_3$ and $\dot{P} \le O(\dot{G}_R)$, we conclude from the structure of $GL_5(2)$ that $\dot{G}_R \cong S_3 \times S_3$ and $C_T(L) = C_R(Y) = C_R(P) = C_T(I)$. We saw earlier that $T_0 = C_T(L) \cap C_T(I) = 1$, so we conclude $C_T(L) = 1$ and $R = U$. As observed earlier, this establishes (1) and shows that $U = C_G(U)$. Along the way we established the other assertions of (2), and (1) implies (3). Thus the proof of 12.2.25 is complete. □

By 12.2.25.2, $N_{YV}(P)$ is a complement to U in $N_L(U)$. Further U is a homogeneous $C_{YT}(P)$-module, so there is a $C_{YT}(P)$-complement U_0 to $V \cap U$ in U, and hence $U_0 C_{YT}(P)$ is a complement to V in $N_L(U)$. As $N_L(U)$ contains the Sylow 2-group T of L, we conclude from Gaschütz's theorem A.1.39:

LEMMA 12.2.26. *L splits over V, and $N_G(U) \cap N_G(P) \cong L_{n-1}(2) \times S_3$ is a complement to U in $N_G(U)$.*

By 12.2.26, the structure of L, and hence also of $C_G(z)$, are determined, so we can move toward the identification G using recognition theorems from our Background References.

PROPOSITION 12.2.27. *(1) If $n = 4$, then $G \cong M_{24}$ or $L_5(2)$.*
(2) If $n = 3$ then $G \cong L_4(2)$ or A_9.

PROOF. Let $z \in V \cap Z^{\#}$. By assumption, $C_G(z) \leq M$, so we conclude from 12.2.25.1 that $C_G(z) = C_M(z) = C_L(z)$. By 12.2.26, L is determined up to isomorphism, so as $L_{n+1}(2)$ satisfies the hypotheses on G, $C_G(z)$ is isomorphic to the centralizer of a transvection in $L_{n+1}(2)$. Hence if $n = 4$ then by Theorem 41.6 in [**Asc94**], $G \cong M_{24}$ or $L_5(2)$. Similarly if $n = 3$ then $G \cong L_4(2)$ or A_9 by I.4.6. □

By 12.2.20, $L/O_2(L) \cong L_n(2)$, and by 12.2.25.2, $n = 3$ or 4. Thus one of the conclusions of Theorem 12.2.13 holds by 12.2.27. Therefore the proof of Theorem 12.2.13 is complete.

12.3. Eliminating $\mathbf{A_7}$

In section 12.3 we eliminate the cases where $L \in \mathcal{L}_f^*(G,T)$ with $L/O_{2,Z}(L) \cong A_7$; namely we prove:

THEOREM 12.3.1. *Assume Hypothesis 12.2.3. Then $L/O_{2,Z}(L)$ is not A_7.*

We adopt the conventions of Notation 12.2.5, including $Z = \Omega_1(Z(T))$.

After Theorem 12.3.1 is established, case (ii) of 12.2.7.3 cannot arise, so we obtain:

COROLLARY 12.3.2. *Assume Hypothesis 12.2.3, and further assume $C_G(Z) \leq M$. Let $H \in \mathcal{H}_*(T,M)$ and set $K := O^2(H)$ and $V_K := \langle Z^K \rangle$. Then either*

(1) H is solvable, or

(2) $K/O_2(K) \cong L_2(4)$, $K \in \mathcal{L}_f^*(G,T)$, and $[V_K, K]$ is the sum of at most two A_5-modules for $K/O_2(K)$.

We mention some shadows which the analysis must at least implicitly handle: As we noted in Remark 12.2.14, in the QTKE-group $G = M_{23}$ there is $L \in \mathcal{L}_f^*(G,T)$ with $L \cong A_7/E_{16}$. The case $G = M_{23}$ is explicitly excluded by Hypothesis 12.2.3, and its shadow is eliminated early in this section by an appeal to Theorem 12.2.13.

The group $G = McL$ is quasithin but not of even characteristic, in view of the involution centralizer isomorphic to $\hat{A}_8$; this group has $L \in \mathcal{L}^*(G,T)$ with $L \cong A_7/E_{16}$. Further $G = \Omega_7(3)$ is not quasithin but has $L \in \mathcal{L}^*(G,T)$ with $L \cong A_7/E_{64}$. The shadows of these two groups are eliminated by control of the centralizer of a 2-central element of V whose centralizer in the shadow is not in $\mathcal{H}^e$.

In the remainder of this section we assume G, L, M afford a counterexample to Theorem 12.3.1. Choose a $V \in Irr_+(L, R_2(LT), T)$; then V is described in 12.2.2.3.

We now begin a series of reductions.

LEMMA 12.3.3. *V is the natural permutation module of rank 6 for $\bar{L} \cong A_7$.*

PROOF. By 12.2.2.3, either $V/C_V(L)$ is the natural module for $\bar{L}$ or $m(V) = 4$. In the first case since $V = [V, L]$ and the 1-cohomology of the natural module is trivial by I.1.6, the lemma holds.

Thus we may assume that V is a 4-dimensional irreducible for A_7, and it remains to derive a contradiction. Then L is transitive on $V^\#$. As V is not invariant under S_7, $\bar{L} = \bar{M}_V \cong A_7$. Since the groups in conclusions (2)–(4) of Theorem 12.2.13 do not have a member $L \in \mathcal{L}_f^*(G,T)$ of this form, conclusion (1) of Theorem 12.2.13 must hold: [2] that is, $G_v \not\leq M$ for each $v \in V^\#$. Now $Z \cap V = \langle z \rangle$ is of order 2, so that $G_z \not\leq M$. Recall from Notation 12.2.5 that $L_z = O^2(C_{M_v}(z))$; set $K_z := L_z^\infty$. From the structure of V as an L-module, $\bar{L}_z = \bar{K}_z \cong L_3(2)$ and $V = [V, \bar{L}_z]$ is the indecomposable L_z-module of B.4.8.2 with $V/\langle z \rangle$ a natural module. Thus $Q = O_2(K_zT) \in Syl_2(C_{G_z}(K_z/O_2(K_z))$ and Hypothesis C.2.8 is satisfied with G_z, M_z, K_z, Q in the roles of "H, M_H, L_H, R" by 12.2.12. Now $K_z \in \mathcal{L}_f(G,T)$, so by 1.2.4, $K_z \leq K \in \mathcal{C}(G_z)$, and then $K \in \mathcal{L}_f(G,T)$ by 1.2.9.1. [3]

We claim that $K_z = K$. Suppose that $K_z < K$. Then $K \not\leq M_z$ by 12.2.5.3a. If $K/O_2(K)$ is not quasisimple, then $K/O_2(K) \cong SL_2(7)/E_{49}$ by A.3.12. On the other hand if $K/O_2(K)$ is quasisimple, then $K/O_2(K) \cong L_4(2)$ or $L_5(2)$ by Theorem C.4.1. In either case $V \leq V_K := [\Omega_1(Z(O_2(KT))), K]$ by 1.2.9.1. But if $K/O_2(K) \cong SL_2(7)/E_{49}$, then by 3.2.14, $\Xi_7(K) \leq C_G(V_K) \leq C_G(V) \leq M$, so $K = \Xi_7(K)L_z \leq M_z$, contrary to $K \not\leq M_z$. Thus $K/O_2(K)$ is $L_n(2)$ for $n = 4$ or 5. As our tuple satisfies Hypothesis C.2.8, it also satisfies Hypothesis C.2.3. Hence by C.2.7.2, $J(Q) \not\leq O_2(KQ)$ and V_K is an FF-module for $KT/O_2(KT)$. Then V_K is described in Theorem B.5.1. As $z \in V \leq V_K$, $C_{V_K}(K) \neq 0$, so by Theorem B.5.1.2, $n = 4$, $V_K \in Irr_+(K, V_K)$, and $V_K/C_{V_K}(K)$ is the 6-dimensional irreducible for $K/O_2(K) \cong A_8$. Indeed as the 1-cohomology of that module is 1-dimensional by I.1.6.1, V_K is the 7-dimensional core of the permutation module for A_8. But then from the structure of that module, $O_2(N_K(V))$ induces the full group of transvections with center $\langle z \rangle$ on V, contrary to $O_2(N_K(V)) \leq O_2(K_zT) = Q \leq C_G(V)$.

Therefore $K_z = K \trianglelefteq G_z$, so $V = [V, K_z] \leq K_z = K$. Let $Y := C_{G_z}(K/O_2(K))$ and recall that $Q \in Syl_2(Y)$. Then by a Frattini Argument, $G_z = YN_{G_z}(Q) = YM_z$, and hence $Y \not\leq M$ as $G_z \not\leq M$. Further $m_3(G_z) \leq 2$ as G_z is an SQTK-group, and L_z contains a subgroup of order 3 intersecting Y trivially, so $m_3(Y) \leq 1$. Notice $Y \in \mathcal{H}^e$ by 1.1.3.1. Then as $Q \in Syl_2(Y)$, while $C(G,Q) \leq M$ by 1.4.1.1, Hypothesis C.2.3 is satisfied now with Y, $Y \cap M$, Q in the roles of "H, M_H, R". Therefore as $m_3(Y) \leq 1$, we conclude from C.2.5 that $Y = (Y \cap M)X$, where $X \not\leq M$ is a block of type A_3, A_5, or $L_2(2^n)$ for some n, and X is normal in G_z. As $[K, X] \leq O_2(K)$, we conclude from C.1.10 that K centralizes X. Hence $X \leq C_G(K) \leq C_G(V) \leq M$, a contradiction which completes the proof of 12.3.3. □

By 12.3.3, V is the 6-dimensional irreducible module for $\bar{L} \cong A_7$, so we now adopt the notation of section B.3 in discussing the action of M_V on V. In particular:

LEMMA 12.3.4. *(1) L has three orbits $\mathcal{O}_m$, $m = 2, 4, 6$, on $V^\#$, where $\mathcal{O}_m$ is the set of vectors in V of weight m.*

(2) $(Z \cap V)^\# = \{e(m) : m = 2, 4, 6\}$, with $e(m) := e_{\theta_m}$ of weight m, where $\theta_2 := \{1, 2\}$, $\theta_4 := \{3, 4, 5, 6\}$, and $\theta_6 := \Omega - \{7\}$.

[2] This application of 12.2.13 eliminates the "shadow" of M_{22} in Theorem 12.3.1.

[3] Notice this eliminates the shadow of $G = McL$, in which $K \cong \hat{A}_8$; thus $O_2(K) = \langle z \rangle$, and hence $K \notin \mathcal{L}_f(G,T)$.

(3) $\bar{M}_V \cong S_7$ *or* A_7*, and for* $m = 2, 4, 6$, $C_{\bar{M}_V}(e(m))$ *is isomorphic to* $\mathbf{Z}_2 \times S_5$ *or* S_5*;* $S_4 \times S_3$ *or a subgroup of index* 2 *in* $S_4 \times S_3$*;* S_6 *or* A_6*; respectively.*

LEMMA 12.3.5. *L controls fusion of involutions in V.*

PROOF. Recall $N_G(T) = N_M(T)$ by Theorem 3.3.1, and this subgroup controls fusion of involutions in Z by Burnside's Fusion Lemma A.1.35. We saw in 12.3.4.3 that the three involutions in $Z \cap V$ are not M_V-conjugate; hence they are not M-conjugate since V is a TI-set in M by 12.2.6. Further by 12.3.4, each member of V is fused into $Z \cap V$ under L, so the lemma holds. □

LEMMA 12.3.6. $G_{e(6)} \leq M$.

PROOF. Let $e := e(6)$. By 12.3.4, $O_2(\bar{L}_e\bar{T}) = 1$ and $\bar{L}_e \cong A_6$, so applying 12.2.12, Hypothesis C.2.3 is satisfied by G_v, M_v, Q. Also $L_e/O_2(L_e) \cong A_6$ or $\hat{A}_6$ for $L/O_2(L) \cong A_7$ or $\hat{A}_7$, respectively, so $L_e \in \mathcal{L}(G,T)$ and hence $L_e \leq K \in \mathcal{C}(G_e) \subseteq \mathcal{L}(G,T)$. [4] As L_e involves A_6, $K/O_2(K)$ is quasisimple by 1.2.1.4; further $G_e \in \mathcal{H}^e$ by 1.1.4.2, so that $K \in \mathcal{H}^e$ by 1.1.3.1. Then if $L_e < K$, K and $K \cap M$ are described in the list of conclusion (3) of Theorem C.2.7; but we find no case where $K \cap M$ contains a T-invariant subgroup L_e with $L_e/O_{2,Z}(L_e) \cong A_6$.

Thus $L_e = K$. Now $\theta(G_e) = L_e$ by A.3.18, so $\theta(G_e) \leq M$. Set $Y := C_{G_e}(K/O_2(K))$; then $Q \in Syl_2(Y)$ by 12.2.12.1. Thus $G_e = YN_{G_e}(Q) = YM_e$ by a Frattini Argument. Further Hypothesis C.2.3 is satisfied with Y, $Y \cap M$, Q in the roles of "H, M_H, R", so by C.2.5, Y is the product of $Y \cap M$ with χ_0-blocks. Hence as each χ_0-block is generated by elements of order 3, $G_e = \theta(Y)M_e \leq M$, completing the proof. □

LEMMA 12.3.7. *(1)* $L = [L, J(T)]$*, so* $\bar{M}_V \cong S_7$ *and* $\Omega_1(Z(O_2(LT))) = V \oplus C_Z(L)$.

(2) Let $K_e := L^{\infty}_{e(2)}$*. Then* $K_e = [K_e, J(T)]$ *and* $\Omega_1(Z(O_2(K_eT))) = [V, K_e] \oplus C_Z(K_e)$.

PROOF. By 12.3.6, $C_G(Z) \leq M$, so $L = [L, J(T)]$ by 12.2.9.2. Hence by B.3.2.4, $\bar{M}_V \cong S_7$, and if $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$, then $\bar{A}$ is generated by transvections and $m(\bar{A}) = m(V/C_V(A))$. In particular $K_e = [K_e, A]$ for some such A. Let $Z_X := \Omega_1(Z(O_2(X))$ for $X := LT$ or K_eT. As $m(\bar{A}) = m(V/C_V(A))$, $Z_{LT} = VC_{Z_{LT}}(A)$, so $V = [Z_{LT}, L]$. Then as the 1-cohomology of V under $L/O_2(L) \cong A_7$ is trivial by I.1.6.1, $Z_{LT} = V \oplus C_{Z_{LT}}(LA)$. Hence as $LT = LAO_2(LT)$, $C_{Z_{LT}}(L) \leq C_{Z_{LT}}(T) \leq Z$, and (1) follows. Similarly $Z_{K_eT} = [V, K_e] \oplus C_Z(K_e)$, so that (2) holds. □

LEMMA 12.3.8. $G_{e(2)} \leq M$.

PROOF. Let $e := e(2)$ and $K_e := L^{\infty}_e$. Then $K_e \leq K \in \mathcal{C}(G_e) \subseteq \mathcal{L}(G,T)$, and $K \leq K_0 \in \mathcal{L}^*(G,T)$. As $K_e \in \mathcal{L}_f(G,T)$, K and K_0 are also in $\mathcal{L}_f(G,T)$ by 1.2.9.1, and $K_0 \in \mathcal{L}^*_f(G,T)$ by 1.2.9.2. Let $V_0 := \Omega_1(Z(O_2(K_0T))$. Then $e, e(6) \in Z \leq V_0$ since $F^*(K_0T) = O_2(K_0T)$ by 1.1.4.6, so $[V, K_e] = [e(6), K_e] \leq V_0$, and hence $[V_0, K_0] \neq 1$. By 12.3.7.2, $K_e = [K_e, J(T)]$, so $K_0 = [K_0, J(T)]$.

Suppose $K_0/O_2(K_0)$ is not quasisimple. Then $K_e < K_0$, so as $K_e/O_2(K_e) \cong A_5$, the embedding of K_e in K_0 is described in cases (13) or (14) of A.3.14. As

[4] Just as for McL in 12.3.3, the shadow of $\Omega_7(3)$ is now eliminated by the application of 1.2.9.1, as in that group K would be $\Omega^-_6(3)$.

$K_0 = [K_0, J(T)]$ does not centralize V_0, but $O_\infty(K_0)$ centralizes V_0 by 3.2.14, we conclude that $K_0/C_{K_0}(V_0) \cong L_2(p)$ for $p = 5$ or $p \geq 11$. But $p \geq 11$ is ruled out by Theorem B.4.2, so $K_0 = C_{K_0}(V_0)K_e$. However as $e, e(6) \in V_0$, $C_{K_0}(V_0) \leq G_{e(6)} \cap G_e \leq M_e$ by 12.3.6, and then K_0 acts on K_e, a contradiction.

Thus $K_0/O_2(K_0)$ is quasisimple, so by Remark 12.2.4, Hypothesis 12.2.3 is satisfied with K_0 in the role of "L". Thus K_0 and its action on any $I \in Irr_+(K_0, V_0, T)$ are described in Theorem 12.2.2.3. Then comparing that list with the possible embeddings in A.3.14, we conclude that either $K_e = K_0$ or $K_0/C_{K_0}(V_0) \cong A_7$.

Suppose first that $K_e < K$. Then as usual $K \not\leq M$ and $K < K_0$ is eliminated, since by A.3.14, there is no $K \in \mathcal{L}(G,T)$ with $K_e < K < K_0$ when $K_0/C_{K_0}(V_0) \cong A_7$. Then $K = K_0 \in \mathcal{L}_f^*(G,T)$, so that K satisfies our hypothesis in this section that $K/O_{2,Z}(K) \cong A_7$. Hence we may apply the results in this section to K. In particular by 12.3.3 and 12.3.7, $I := [V_0, K_0]$ is the A_7-module, $V_0 = I \oplus C_Z(K)$, and

$$[V, K_e] = [\Omega_1(Z(O_2(K_eT))), K_e] = [V_0, K_e].$$

Pick $v \in [V, K_e]$ of weight 4. Then ev is of weight 6 in V, so $C_G(ev) \leq M$ by 12.3.6. But $C_K(v) = C_K(ev)$ as $K \leq G_e$, so $K = \langle K_e, C_K(v)\rangle \leq M$, contrary to $K \not\leq M$.

This contradiction shows that $K_e = K \trianglelefteq G_e$. Then as $Out(K/O_2(K))$ is a 2-group, $G_e = KTY$, where $Y := C_{G_e}(K/O_2(K))$, so it remains to show that $Y \leq M$. Set $U := \langle Z^{G_e}\rangle$ and $G_e^* := G_e/C_{G_e}(U)$. Then $U \in \mathcal{R}_2(G_e)$ by B.2.14. As $K = [K, J(T)]$, Theorems B.5.1 and B.5.6 imply $[U, K] = [V, K]$; so as $End_{K^*}([V,K]) = \mathbf{F}_2$, $Y \leq C_{G_e}([V,K]) \leq C_{G_e}(ev) \leq M$, completing the proof of 12.3.8. □

Recall the weak closure parameters $r := r(G,V)$ and $w := w(G,V)$ from Definitions E.3.3 and E.3.23.

LEMMA 12.3.9. *(1) If $g \in G - N_G(V)$, then $V^\# \cap V^g \subseteq \mathcal{O}_4$.*
(2) $r(G,V) \geq 4$.

PROOF. By 12.2.6, V is a TI-set in M; so by 12.3.5 and A.1.7.3, if $u \in V^\#$ with $G_u \leq M$, then u is in a unique conjugate of V. Thus (1) follows from 12.3.6 and 12.3.8. Up to conjugation, $\langle e_{1,2,3,4}, e(4)\rangle$ is the unique maximal subspace U of V with $U^\# \subseteq \mathcal{O}_4$, so (1) implies (2) since $m(V) = 6$. □

LEMMA 12.3.10. *$W_1(T,V) \leq C_T(V)$, so $w(G,V) > 1$.*

PROOF. Assume the lemma fails. Then we may choose $A := V^g \cap M \leq T \leq N_G(V)$ to be a w-offender in the sense of subsection E.3.3. Thus $\bar{A} \neq 1$ and $w := m(V^g/A) \leq 1$. Now from the action of S_7 on V, for each $\bar{a} \in \bar{A}^\#$, $[V,a]^\# \not\subseteq \mathcal{O}_4$. But if $V \leq N_G(V^g)$, then $[V,a] \leq V \cap V^g$, contrary to 12.3.9.1, so we conclude $V \not\leq N_G(V^g)$. Therefore $m(V^g/C_A(V)) \geq r(G,V) \geq 4$ by 12.3.9.2, so that $m(\bar{A}) \geq 3 = m_2(\bar{L}\bar{T})$. Thus these inequalities must be equalities, so $m(\bar{A}) = 3$, $w = 1$, and $r(G,V) = 4$. Hence $\bar{A}$ is fused under L to

$$\bar{A}_1 := \langle (1,2), (3,4), (5,6)\rangle \text{ or } \bar{A}_2 := \langle (1,2), (3,4)(5,6), (3,5)(4,6)\rangle.$$

Now the Fundamental Weak Closure Inequality of Remark E.3.29 is an equality, so by E.3.31.1:

$$V_A := \langle C_V(\bar{a}) : \bar{a} \in \bar{A}^\#\rangle \leq N_G(V^g).$$

Therefore $[A, V_A] \leq V \cap V^g$, and hence $[A, V_A]^\# \subseteq \mathcal{O}_4$ by 12.3.9.1. We compute that this does not hold if $\bar{A} = \bar{A}_1$. Similarly $[V_A, A] \leq V^g \leq C_G(A)$, so that $[V_A, A, A] =$

1, and we compute that this does not hold if $\bar{A} = \bar{A}_2$. This contradiction completes the proof of 12.3.10. □

LEMMA 12.3.11. *If $H \in \mathcal{H}(T)$ with $n(H) = 1$, then $H \le M$.*

PROOF. By 12.3.9.2 and 12.3.10, $\min\{r, w\} > 1$, so the lemma follows from E.3.35.1. □

Let $e := e(4)$. Since L does not appear in conclusions (2)–(4) of Theorem 12.2.13, conclusion (1) of Theorem 12.2.13 holds: $G_v \not\le M$ for some $v \in V^\#$. By 12.3.4, 12.3.6, and 12.3.8, we may take $v = e$. Thus there is $H \in \mathcal{H}_*(T, M)$ with $H \le G_e$. Set $K := O^2(H)$; as usual, $K \not\le M$.

LEMMA 12.3.12. *$K \trianglelefteq G_e$, $K/O_2(K) \cong A_5$, $K = [K, J(T)]$, and $[Z, K]$ is the A_5-module.*

PROOF. By 12.3.11 and E.1.13, H is not solvable. By 12.3.6, $C_G(Z) \le M$, so we may apply 12.2.7.2 to conclude that $K/O_2(K) \cong A_5$. By 1.2.4, we may embed $K \le K_e \in \mathcal{C}(G_e) \subseteq \mathcal{L}(G, T)$, and $K_e \le K_0 \in \mathcal{L}^*(G, T)$. As $[V_H, K] \neq 1$ by 12.2.7.1, 1.2.9.1 says $K_0 \in \mathcal{L}_f^*(G, T)$. Then by 12.2.7.3, either $K = K_0$ or $K_0/O_{2,Z}(K_0) \cong A_7$.

Assume first that $K < K_0$. Then by 12.2.7.3, Hypothesis 12.2.3 is satisfied with K_0 in the role of "L". Hence as $K_0/O_{2,Z}(K_0) \cong A_7$, the hypotheses of this section hold with K_0 in the role of L, so we may apply the results obtained so far to K_0. Set $V_0 := \Omega_1(Z(O_2(K_0T)))$. By 12.3.7, $V_0 = V_K \oplus C_Z(K_0)$, with $V_K = [Z, K_0]$, $[Z, K]$ is the A_5-module, and $K = [K, J(T)]$. Thus the lemma holds in this case if $K = K_e$. On the other hand if $K < K_e$, then $K_e = K_0$ by A.3.14. Further by 12.2.8, K_0 contains all elements of order 3 in G_e, so in particular $L_e \le K_0$. But K is the unique member of $\mathcal{L}(K_0T, T)$ with $K/O_2(K) \cong A_5$, so $K = L_e^\infty \le M$, contrary to $K \not\le M$.

Thus we may assume that $K = K_0 = K_e \in \mathcal{L}^*(G, T)$. Therefore $G_e \le N_G(K) = !\mathcal{M}(KT)$ by 1.2.7.3. Then there is $H_1 \in \mathcal{H}_*(T, N_G(K))$, and in particular $H_1 \not\le G_e$. Thus $[Z, H_1] \neq 1$, so $K = [K, J(T)]$ and $[Z, K]$ is an FF-module by Theorem 3.1.8.3. By 12.2.7.3, $[Z, K]$ the sum of A_5-modules, and then by Theorem B.5.1.1, $[Z, K]$ is an A_5-module, completing the proof of the lemma. □

Next by 12.3.4 and 12.3.7.1, $C_{\bar{M}_V}(e) = \bar{M}_1 \times \bar{M}_2$, where $\bar{M}_1 \cong S_4$ is the pointwise stabilizer in $\bar{M}_V$ of $\{1, 2, 7\}$, and $\bar{M}_2 \cong S_3$ is the pointwise stabilizer of $\{3, 4, 5, 6\}$. Let $L_i := O^{3'}(M_i)$, so that $L_e = L_1L_2$, and $L_1 = O^{3'}(C_L(Z \cap V)) = O^{3'}(C_L(Z))$ using 12.3.7.1. Let $P \in Syl_3(L_e)$. By 12.3.12, $K \trianglelefteq G_e$, so $P = (P \cap K) \times C_P(K/O_2(K))$, and hence $P \not\cong 3^{1+2}$. Therefore $O_{2,Z}(L) = O_2(L)$, and appealing to 12.2.8:

LEMMA 12.3.13. *$L/O_2(L) \cong A_7$ and $L = O^{3'}(M)$.*

We are now in a position to complete the proof of Theorem 12.3.1. As $L = O^{3'}(M)$ by 12.3.13 and $C_G(Z) \le M$ by 12.3.6, $L_1 = O^{3'}(C_L(Z)) = O^{3'}(C_G(Z))$. By 12.3.12, $[Z, K]$ is the A_5-module, so $O^2(K \cap M)$ centralizes $Z \cap [Z, K]$, and hence $O^2(K \cap M)$ centralizes Z by B.2.14. Thus $L_1 = O^{3'}(K \cap M)$. Then as L_1 and L_2 are the T-invariant subgroups $X = O^2(X)$ of L_e with $|X : O_2(X)| = 3$, it follows that $L_2 = O^2(C_{L_e}(K/O_2(K))$.

Let $Y := KL_2T$, $U := \langle Z^Y\rangle$, and $Y^* := Y/C_Y(U)$. As $L_e = L_1L_2$, $L_1 \le C_K(Z)$, and $Z = C_Z(L)(Z \cap V)$ by 12.3.7.1,

$$[Z, L_e] = [Z \cap V, L_2] = \langle e_{1,7}, e(2)\rangle.$$

Then $C_{L_2}([Z, L_2]) = O_2(L_2)$, and $C_K([Z, K]) = O_2(K)$ by 12.3.12, so $C_Y(U) = O_2(Y)$. Thus $Y^* \cong S_5 \times S_3$ since $M_1M_2/O_2(M_1M_2) \cong S_3 \times S_3$, and $U \in \mathcal{R}_2(Y)$. Also $K = [K, J(T)]$ by 12.3.12, and $L_2 = [L_2, J(T)]$ using 12.3.7.1, so $Y^* = J(Y)^*$. Therefore by Theorem B.5.6,

$$[U, Y] = [U, K] \oplus [U, L_2] = [Z, K] \oplus [Z, L_2],$$

so in particular $K \le C_G([Z, L_2]) \le C_G(e(2)) \le M$ by 12.3.8, contrary to $K \not\le M$. This contradiction completes the proof of Theorem 12.3.1.

12.4. Some further reductions

We begin section 12.4 with a technical lemma 12.4.1, which we use in particular to prove the main result 12.4.2 of the section.

As we will be assuming Hypothesis 12.2.3, as usual we adopt the conventions of Notation 12.2.5, including $Z = \Omega_1(Z(T))$.

Lemma 12.4.1. *Assume Hypothesis 12.2.3. In addition assume:*

(i) $C_G(Z) \le M$, *and*

(ii) $s(G, V) > 1$.

Then there exists $g \in G$ *with* $1 \ne [V, V^g] \le V \cap V^g$.

Proof. Assume the lemma is false. Let $H \in \mathcal{H}_*(T, M)$, $K := O^2(H)$, $V_H := \langle Z^H\rangle$, and $H^* := H/C_H(V_H)$. As $C_G(Z) \le M$ by (i), 12.3.2 says either H is solvable, or $[V_H, K]$ is the sum of at most two A_5-modules for $K^* \cong A_5$. Then $a(H^*, V_H) = 1$, by E.4.1 in the former case, or by an easy direct computation in the latter.

Observe that the triple $G_1 := LT$, $G_2 := H$, V satisfies Hypothesis F.7.6. Form the coset geometry Γ as in that section, with parameter $b := b(\Gamma, V)$. If $W_0(T, V) \le O_2(H)$, then by F.7.14, b is even. Hence by F.7.11.2, there exists $g \in G$ with $1 \ne [V, V^g] \le V \cap V^g$, contrary to our assumption that the lemma fails. Therefore $W_0(T, V) \not\le O_2(H)$. So there is $A := V^g$ with $A^* \ne 1$. Now as $s(G, V) > 1$ by (ii), $A^* \in \mathcal{A}_2(H^*, V_H)$ by E.3.10, contradicting $a(H^*, V_H) = 1$. □

The main result of this section is Theorem 12.4.2. It eliminates two of the four cases in 12.2.2.3 where $C_V(L) \ne 1$ (cases (b) and (f)), leaving only A_6 and A_8 in case (d). In particular when $\bar{L}$ is $L_3(2)$ or $G_2(2)'$, the result reduces V to the natural module. The analogous reduction will be carried out later for $L_4(2)$ and $L_5(2)$ in Theorems 12.6.34 and 12.5.1. Theorem 12.4.2 also moves in the direction (begun in 12.2.13) of showing that $C_G(V \cap Z) \not\le M$.

Theorem 12.4.2. *Assume Hypothesis 12.2.3. Then*

(1) If $L/O_2(L) \cong L_3(2)$ *or* $G_2(2)'$, *then* $C_V(L) = 1$.

(2) If $L/O_2(L) \cong L_5(2)$ *and* $\dim(V) = 10$, *then* $C_G(Z \cap V) \not\le M$.

Until the proof of Theorem 12.4.2 is complete, assume G, L, V affords a counterexample. Let $Z_V := C_V(L)$.

When $L/O_2(L) \cong L_3(2)$ or $G_2(2)'$, $Z_V \ne 1$ as we are in a counterexample to Theorem 12.4.2. Hence by Theorem 12.2.2.3, V is an indecomposable for $L/O_2(L)$, and V/Z_V is a natural module for $L/O_2(L)$. Then as the 1-cohomology of the dual of V/Z_V in I.1.6 is 1-dimensional, $Z_V = \langle z \rangle$ is of order 2. As $M = !\mathcal{M}(LT)$, $G_z \leq M$.

On the other hand, when $L/O_2(L) \cong L_5(2)$, we have $m(V) = 10$ by hypothesis; and as we are in a counterexample to the theorem, $C_G(Z \cap V) \leq M$. As $\dim(V) = 10$, $Z \cap V$ is of order 2, and in this case we take z to be a generator for $Z \cap V$. Thus $G_z \leq M$ in this case also.

We begin a series of reductions.

LEMMA 12.4.3. *(1)* $C_G(Z) \leq M$.
(2) $L = [L, J(T)]$.

PROOF. As $G_z \leq M$ and $z \in Z$, (1) holds; then (2) follows from 12.2.9.2. □

We are already in a position to complete the proof of part (2) of Theorem 12.4.2:

LEMMA 12.4.4. $L/O_2(L)$ *is not* $L_5(2)$.

PROOF. Assume $L/O_2(L)$ is $L_5(2)$. Then L has two orbits on $V^{\#}$, represented by z and some further involution t.

We claim that $t \notin z^G$. First $L = O^{3'}(M)$ by 12.2.8. Next $L_z/O_2(L_z) \cong L_3(2) \times \mathbf{Z}_3$, so as $G_z \leq M$, $m_3(G_z^\infty) = 1$. However $L_t/O_2(L_t) \cong A_6$ is of 3-rank 2, so $t \notin z^G$, establishing the claim.

It follows from the claim that L is transitive on $z^G \cap V$, so as $G_z \leq M$, while V is a TI-set under M by 12.2.6, V is the unique member of V^G containing z by A.1.7.3.

Next $m(\bar{M}_V) = 3$, so $s(G, V) \geq 3$ by Theorem E.6.3. By 12.4.3.1, $C_G(Z) \leq M$, so by 12.4.1, there exists $g \in G$ with $1 \neq [V, V^g] \leq V \cap V^g$. Conjugating in M_V if necessary, we may assume $V^g \leq T$. Let $A := V^g$. Interchanging the roles of A and V if necessary, we may assume $m(\bar{A}) \geq m(V/C_V(A))$. Then by B.1.4.4, $\bar{A}$ contains a member of $\mathcal{P}(\bar{M}_V, V)$. Therefore by B.4.2.11, $C_V(A) = [V, A]$ is a 6-dimensional subspace of A, and $\bar{A}$ of rank 4 is the unipotent radical of the maximal parabolic of $\bar{L}$ over $\bar{T}$ stabilizing $[V, A]$. In particular, $[V, A]$ is T-invariant, so the generator z of $Z \cap V$ is in $[V, A] \leq V \cap V^g$. This contradicts our earlier observation that z is in a unique conjugate of V, completing the proof. □

By 12.4.4, $L/O_2(L) \cong L_3(2)$ or $G_2(2)'$, so as we are in a counterexample to the Theorem, $Z_V \neq 1$. Hence $V \trianglelefteq M$ by Theorem 12.2.2.3. Then M normalizes $C_V(L) = Z_V = \langle z \rangle$, so since $M \in \mathcal{M}$,

$$M = C_G(z) = N_G(V).$$

When $\bar{L} \cong L_3(2)$, let E be the T-invariant 4-subgroup of V, choose $v \in E - Z_V$, and let $L_1 := O^2(C_L(E))$ and $R_1 := C_T(E)$.

LEMMA 12.4.5. *If* $\bar{L} \cong L_3(2)$*, then*
(1) $[Z, L] = 1$.
(2) $Z_Q := \Omega_1(Z(Q)) = ZV$.
(3) $\bar{R}_1 := \bar{A}$ *for each* $A \in \mathcal{A}(R_1)$ *with* $A \not\leq Q$.

PROOF. Observe that (3) holds by B.4.8.2. By 1.4.1.4, $Z_Q = R_2(LT)$, so $V = [Z_Q, L]$ by B.5.1.1. Then $Z_Q = C_Z(L)V$ by B.4.8.4, so (2) holds. Again By B.4.8.2, $Z \cap V = Z_V$, so (2) implies (1). □

LEMMA 12.4.6. *If $\bar{L} \cong L_3(2)$, then $R_1 \in Syl_2(G_v)$ and $|T : R_1| = 2$.*

PROOF. First $R_1 \in Syl_2(M_v)$ and $|T : R_1| = 2$. Thus the result holds if $N_G(R_1) \leq M$. So we assume that $N_G(R_1) \not\leq M$, and it remains to derive a contradiction.

Let $Z_1 := \Omega_1(Z(R_1))$. Then by 12.4.5.2,

$$Z_1 = C_{Z_Q}(R_1) = ZC_V(R_1) = Z\langle v\rangle.$$

Now $[Z, L] = 1$ by 12.4.5.1, so $Z \cap Z^g = 1$ for $g \in N_G(R_1) - M$ by 1.2.7.4. Hence as Z is a hyperplane in Z_1, we conclude Z is of order 2, so that $Z = Z_V$ and $E = Z_1 = \Omega_1(Z(R_1))$. In particular,

$$N_G(R_1) \leq N_G(E).$$

Furthermore $C_G(E) \leq C_G(Z) \leq G_z = M$, and $Aut_G(E) \cong S_3$ with $Aut_M(E)$ of order 2; thus $|N_G(R_1) : N_M(R_1)| = 3$.

Next as $N_G(R_1) \not\leq M = !\mathcal{M}(LT)$, there is no nontrivial characteristic subgroup of R_1 normal in LT. Thus (LR_1, R_1) is an MS-pair as in Definition C.1.31, so that C.1.34 applies. As V is indecomposable, conclusion (5) of C.1.34 holds; hence L is a block with $Q = VC_T(L)$ and $C_{R_1}(L_1) = EC_T(L)$.

Suppose that $N_G(R_1) \leq N_G(L_1)$. Then $N_G(R_1)$ normalizes $\Phi(C_{R_1}(L_1)) = \Phi(EC_T(L)) = \Phi(C_T(L))$, so $\Phi(C_T(L)) = 1$ since $N_G(R_1) \not\leq M = !\mathcal{M}(LT)$. Thus $C_T(L)$ is central in $VC_T(L) = Q$ and in $Q(T \cap L) = T$. We conclude $C_T(L) = Z = Z_V$, so that $Q = VC_T(L) = V$. Since the nontrivial characteristic subgroup $J(R_1)$ of R_1 is not normal in LT, $J(R_1) \not\leq O_2(LT) = C_T(V)$, so there is $A \in \mathcal{A}(R_1)$ with $\bar{A} \neq 1$. By 12.4.5.3, $\bar{A} = \bar{R}_1$. Thus $\mathcal{A}(R_1) = \{A, V\}$ by B.2.21, since V is self-centralizing in G and $C_V(A) = C_V(a)$ for $\bar{a} \in \bar{A}^\#$ by B.4.8.2. Hence $O^2(N_G(R_1))$ acts on V, so $O^2(N_G(R_1)) \leq M$, contradicting $|N_G(R_1) : N_M(R_1)| = 3$.

Thus there is $g \in N_G(R_1) - N_G(L_1)$. We have seen that $N_G(R_1) \leq N_G(E)$ and $C_G(E) \leq M$; so as $L_1 \trianglelefteq C_M(E)$ while $m_3(C_M(E)) \leq 2$, $L_1L_1^g =: X = \theta(C_G(E))$ and $X/O_2(X) \cong E_9$. Then $X_0 := C_X(\bar{L})$ is of order 3, so by C.1.10, $X_1 := O^2(X_0)$ centralizes L and $X_1/O_2(X_1) \cong \mathbf{Z}_3$. Next X_1 is centralized by $t \in T \cap L - R_1$ inverting $L_1/O_2(L_1)$, so L_1 and X_1 are the two T-invariant members of the set $\mathcal{Y}$ of subgroups Y of X such that $Y = O^2(Y)$ and $|Y : O_2(Y)| = 3$. Now $N_G(R_1)$ normalizes X and hence permutes $\mathcal{Y}$. Since $N_G(R_1) \not\leq N_G(L_1)$, while L_1 is stabilized by $N_M(R_1)$ of index 3 in $N_G(R_1)$, the $N_G(R_1)$-orbit of L_1 has length 3, and the fourth member of $\mathcal{Y}$ is fixed by $N_G(R_1)$. Since $T \leq N_G(R_1)$ and X_1 is the only T-invariant member of $\mathcal{Y}$ other than X_1, we conclude $X_1 \trianglelefteq N_G(R_1)$. However $X_1 \trianglelefteq XLT$, so

$$N_G(R_1) \leq N_G(X_1) \leq M = !\mathcal{M}(LT),$$

contrary to our earlier reduction. This completes the proof of 12.4.6. □

LEMMA 12.4.7. *L controls fusion of involutions in V.*

PROOF. Suppose first that $\bar{L} \cong L_3(2)$. By 12.4.6, $v^G \cap Z_V = \emptyset$. Thus as L is transitive on $V - Z_V$, the lemma holds in this case.

Next take $\bar{L} \cong G_2(2)'$. Then $Z \cap V$ is a 4-group containing a representative of each of the three orbits of M on $V^\#$. But $N_G(T) \leq M$ by Theorem 3.3.1, and $N_G(T)$ controls fusion in Z by Burnside's Fusion Lemma A.1.35, so the lemma holds in this case also. □

LEMMA 12.4.8. $r(G, V) > 1$.

PROOF. Assume that $r(G, V) = 1$. Then there is a hyperplane U of V with $C_G(U) \not\leq N_G(V)$. Let $G_U := C_G(U)$, $M_U := C_M(U)$, and $L_U := N_L(U)$. Then $Z_V \not\leq U$ as $C_G(Z_V) = C_G(z) = M$.

Now consider some hyperplane U_0 of U, and set $G_{U_0} := C_G(U_0)$ and $M_{U_0} := C_M(U_0)$; then $G_U \leq G_{U_0}$, so also $G_{U_0} \not\leq M$. As $Z_V \not\leq U$ and $m(V/U) = 1$, also $Z_V \not\leq U_0$ and $m(V/U_0 Z_V) = 1$. For any involution $\bar{t} \in \bar{M}$, $Z_V \leq C_V(\bar{t})$ and $m(V/C_V(\bar{t})) \geq 2$ (cf. B.4.8.2 and B.4.6), so $\bar{t}$ does not centralize U or U_0. Thus U and U_0 lie in the set Γ of Definition E.6.4, and we may apply appropriate results from that section. In particular by E.6.5.1, Q is Sylow in G_{U_0} and G_U. Also M_{U_0} centralizes the quotients of the series $V > U_0 Z_V > U_0 > 1$, so by Coprime Action, $\bar{M}_{U_0}$ is a 2-group. But we just observed that $\bar{M}_{U_0}$ does not contain involutions, so we conclude that $M_{U_0} = C_M(V)$, and hence also $M_U = C_M(V)$.

Now if $\bar{L} \cong L_3(2)$, then T is regular on hyperplanes not containing Z_V, so U is determined up to conjugation under T, and $\bar{L}_U \cong Frob_{21}$. On the other hand, if $\bar{L} \cong G_2(2)'$, then by Theorem 2 in [**Asc87**], L has two orbits on hyperplanes not containing Z_V, exhibited by conclusions (3) and (4) of Theorem 3 in [**Asc87**], and given by representatives U_1 and U_2, where $\bar{L}_{U_1} \cong PSL_2(7)$ and $\bar{L}_{U_2} \cong Q_8/3^{1+2}$. When $\bar{L}\bar{T} = G_2(2)$, the stabilizers in $\bar{L}\bar{T}$ are twice as large. In each case $N_{LT}(U)$ is maximal in LT, but not of index 2; further L_U contains X_U of order 3 faithful on U. Thus if $F^*(G_U) = O_2(G_U)$, then the hypotheses of lemma E.6.14 are satisfied with U and LT in the roles of "W" and "M_0", so by that lemma, $G_U = C_G(U) \leq M$, contrary to our assumption.

This contradiction shows that $G_U \notin \mathcal{H}^e$. Suppose next that there is a component K of G_U. Then K is described in E.6.8, and in particular $K \not\leq M$. Now $K \cap M \leq M_U = C_M(V)$, so that $[V, K \cap M] = 1$. If case (1) of E.6.8 occurs, this forces $n = 1$, so that $K \cong L_3(2)$; we regard this group as $L_2(7)$, and treat it with the groups $L_2(p)$ arising below. In particular K is not a Suzuki group. The existence of X_U of order 3 faithful on U and an appeal to A.3.18 eliminates all cases of 3-rank 2—namely (2)–(4) of E.6.8, and all cases of E.6.8.5 except $K \cong L_2(p)$, p a Fermat or Mersenne prime. Notice now that the case $U = U_2$ for $\bar{L} \cong G_2(2)'$ cannot arise: For in that case there is $Y_U \cong 3^{1+2}$ faithful on U, so as $m_3(N_G(U)) \leq 2$, G_U is a $3'$-group, whereas K is not a Suzuki group. In the remaining two cases choose $l \in N_L(X_U) - L_U$ with $l^2 \in L_U \cap L_U^l$, and choose the hyperlane U_0 of U to be $U_0 := U \cap U^l$. As l acts on X_U and X_U is faithful on U, X_U acts faithfully on U_0. We saw $Q \in Syl_2(G_{U_0})$, so $K \leq K_0 \in \mathcal{C}(G_{U_0})$ by 1.2.4. Since $K \cong L_2(q)$ rather than $SL_2(q)$, K centralizes $O(G_{U_0})$ by A.1.29. Now by I.3.1, K is contained in the product of a U-orbit of 2-components of G_{U_0}, so as K centralizes $O(G_{U_0})$, we conclude those 2-components are ordinary components. Hence $K \leq E(G_{U_0})$, so K_0 is a component of G_{U_0}. As X_U is faithful on U_0, we may argue as before that $K_0 \cong L_2(q)$ for q a Fermat or Mersenne prime. But no proper embedding $K < K_0$ of these groups appears in A.3.12, so we conclude $K_0 = K$ is also a component of $C_G(U_0)$. Indeed $K = O^{3'}(E(G_{U_0}))$ since $m_3(N_G(U_0)) \leq 2$ and X_U is faithful on U_0. But $l^2 \in L_U \cap L_U^l$ so that $U_0 = U_0^l$. Hence $K^l = O^{3'}(E(G_{U_0^l})) = O^{3'}(E(G_{U_0})) = K$. Then as $K \leq G_U$, K centralizes $UU^l = V$, contrary to $K \not\leq M$.

Therefore $F^*(G_U) = F(G_U)$. As $G_U \notin \mathcal{H}^e$, we conclude $O_U := O(G_U) \neq 1$. Again let U_0 denote a hyperplane of U. By 1.1.6, the hypotheses of 1.1.5 are satisfied with G_{U_0}, $M = C_G(z)$ in the roles of "H, M". In particular by 1.1.5.2,

z inverts $O_{U_0} := O(G_{U_0})$. Similarly z inverts O_U, as we may also apply 1.1.6 and 1.1.5 to U. Now O_U is a nontrivial Q-invariant subgroup of $O(C_{G_{U_0}}(U))$.

Suppose first that O_U acts nontrivially on K_0, for some component K_0 of G_{U_0}. Then $1 \neq Aut_{O_U}(K_0) \leq O(C_{Aut(K_0)}(U))$ is Q-invariant. Inspecting the list of 1.1.5.3 for such a centralizer, we conclude $K_0/Z(K_0) \cong A_7$, U induces a group of inner automorphisms of order 2 on K_0, and $Aut_{O_U}(K_0) \cong \mathbf{Z}_3$. But by 1.1.5.3d, z induces an involution of cycle type 2^3, so that $V = Z_V U$ is not normal in $C_{K_0 Z_V}(z)$, contradicting $G_z = N_G(V)$.

Therefore O_U centralizes $E(G_{U_0})$. As z inverts O_U and O_{U_0}, O_U centralizes O_{U_0}. By 31.14.1 in [**Asc86a**], O_U centralizes $O_2(G_{U_0})$. Thus $O_U \leq C_{G_{U_0}}(F^*(G_{U_0})) \leq F(G_{U_0})$, so in particular $O_U \leq O_{U_0}$. Further O_{U_0} abelian since it is inverted by z.

Now given any $l \in M - N_M(U)$, we may choose $U \cap U^l$ as our hyperplane U_0 of U. Then $\langle O_U, O_U^l \rangle$ is contained in the abelian group $O_{U \cap U^l}$, and in particular, O_U and O_U^l commute. Therefore $1 \neq P := \langle O_U^M \rangle$ is abelian of odd order. Thus $LT \leq N_G(P) < G$ as G is simple; and $N_G(P)$ is quasithin. As $m_2(N_G(P)) \geq m(V) \geq 4$, V cannot act faithfully on $O_p(P)$ for any odd prime p by A.1.5, so $m_p(P) \leq 2$ for each odd prime p. Therefore $V = [V, L]$ centralizes P by A.1.26, which is impossible as z inverts O_U and so acts nontrivially on P. This contradiction finally completes the proof of 12.4.8. □

LEMMA 12.4.9. *If $A := V^g \leq N_G(V)$ with $[A, V] \neq 1$, then $V \not\leq N_G(A)$.*

PROOF. Assume otherwise. Interchanging the roles of A and V if necessary, we may assume $m(\bar{A}) \geq m(V/C_V(A))$, so that $\bar{A}$ contains a member of $\mathcal{P}(\bar{M}, V)$ by B.1.4.4. Then by B.4.6.13 or B.4.8.2, $\bar{A}$ is determined up to conjugacy in $\bar{M}$, and $m(\bar{A}) = m(V/C_V(A))$. In particular, we have symmetry between A and V.

Suppose $\bar{L} \cong L_3(2)$. Then $E = [A, V]$, so by symmetry $E = E^g$. Then as Z_V is weakly closed in E by 12.4.7, $g \in C_G(Z_V) = M = N_G(V)$, contradicting $[V, V^g] \neq 1$.

Therefore $\bar{L} \cong G_2(2)'$ and $\bar{A}\bar{L} = \bar{M} \cong G_2(2)$. Thus by B.4.6.3, $[V, A] = C_V(A)$, and again $Z_V \leq C_V(A)$ and by symmetry $[V, A] = [V, A]^g$. Then again Z_V is weakly closed in $[V, A]$ by 12.4.7, and we obtain the same contradiction. This completes the proof of 12.4.9. □

We are now in a position to complete the proof of Theorem 12.4.2. Recall $m(\bar{M}, V) = 2$, so by 12.4.8, $s(G, V) > 1$. Then by 12.4.3.1, we may apply 12.4.1 to conclude that there is $g \in G$ with $1 \neq [V, V^g] \leq V \cap V^g$. In particular, $V^g \leq N_G(V)$ and $V \leq N_G(V^g)$, contrary to 12.4.9. This contradiction completes the proof of Theorem 12.4.2.

12.5. Eliminating $\mathbf{L_5(2)}$ on the 10-dimensional module

In this section we eliminate the exterior-square module in case (3c) of Theorem 12.2.2, hence reducing the treatment of $L_5(2)$ to the natural module in case (3a). This is analogous to the reduction for $L_4(2)$ in Theorem 12.6.34 of the next section. Specifically we prove:

THEOREM 12.5.1. *Assume Hypothesis 12.2.3 with $L/O_2(L) \cong L_5(2)$. Then V is the natural module for $L/O_2(L)$.*

Assume G, L, V afford a counterexample to Theorem 12.5.1. Then case (3c) of Theorem 12.2.2 occurs, so V is one of the 10-dimensional irreducibles for $L/O_2(L)$.

We mention that there is $L \in \mathcal{L}_f^*(G,T)$ with $L \cong L_5(2)/E_{2^{10}}$ in the non-quasithin groups $Sp_{10}(2)$, $\Omega_{10}^+(2)$, $\Omega_{12}^-(2)$, and $O_{12}^+(2)$. These shadows cause little trouble, as they are essentially eliminated immediately in 12.5.3 below.

The proof involves a series of reductions. As usual we adopt the conventions of Notation 12.2.5. Observe that as T acts on V, T induces inner automorphisms on $\bar{L}$, so $\bar{M}_V = \bar{L}$.

We next discuss the parabolic subgroups of $\bar{L}$ over $\bar{T}$, and their action on the module V. Let Γ be the natural 5-dimensional module for $\bar{L}$ with a basis for Γ denoted by $\{1,\ldots,5\}$, and let $\Gamma_k := \langle 1,\ldots,k\rangle$. Choose notation so that T acts on Γ_k for each k. We regard V as the exterior square $\Lambda^2(\Gamma)$, so that V has basis $i \wedge j$ for $1 \leq i < j \leq 5$. Then T acts on the subspaces V_k of dimension k defined by

$$V_1 := \Lambda^2(\Gamma_2) = \langle 1\wedge 2\rangle,\ V_3 := \Lambda^2(\Gamma_3) = \langle 1\wedge 2, 1\wedge 3, 2\wedge 3\rangle,$$
$$V_4 := \Gamma_1 \wedge \Gamma = \langle 1\wedge i : 1 < i \leq 5\rangle,$$
$$V_6 := \Lambda^2(\Gamma_4) = \langle i\wedge j : 1 \leq i < j < 5\rangle,\ V_7 := \Gamma_2\wedge\Gamma = \langle 1\wedge i, 2\wedge j : i > 1, 3 \leq j \leq 5\rangle.$$

For $i = 1,3,4,6$, set $G_i := N_G(V_i)$, $M_i := N_{LT}(V_i)$, $L_i := N_L(V_i)^\infty$, and $R_i := O_2(L_iT)$.

Notice that $L_i \in \mathcal{L}(G,T)$ for each i.

LEMMA 12.5.2. *(1)* $\bar{M}_1 = N_{\bar{L}}(\Gamma_2)$, $\bar{M}_1/O_2(\bar{M}_1) \cong L_3(2) \times L_2(2)$, $\bar{L}_1/\bar{R}_1 \cong L_3(2)$, *and* $0 < V_1 < V_7 < V$ *is a chief series for* M_1.

(2) $\bar{M}_3 = N_{\bar{L}}(\Gamma_3)$, $\bar{M}_3/O_2(\bar{M}_3) \cong L_2(2) \times L_3(2)$, $\bar{L}_3/\bar{R}_3 \cong L_3(2)$, *and* V_3 *is a natural module for* $\bar{L}_3/\bar{R}_3$.

(3) $\bar{M}_4 = \bar{L}_4 = N_{\bar{L}}(\Gamma_1)$, *and* V_4 *is a natural module for* $\bar{L}_4/\bar{R}_4 \cong L_4(2)$.

(4) $\bar{M}_6 = \bar{L}_6 = N_{\bar{L}}(\Gamma_4)$, V_6 *is the orthogonal module for* $\bar{L}_6/\bar{R}_6 \cong L_4(2) \cong \Omega_6^+(2)$, *and* V/V_6 *is a natural module isomorphic to* $\bar{R}_6$.

PROOF. These are easy calculations. □

Observe that from 12.5.2.1, $M_1 = PL_1$, where P is the minimal parabolic of LT over T which is in M_1, but not in L_1T. Further $O^2(P) = O^{3'}(P) \trianglelefteq M_1$ with $[O^2(P), L_1] \leq O_2(M_1)$. Similarly $O^2(P) \leq L_3 \cap L_6$, P is a minimal parabolic of L_iT for $i = 3,6$, and $M_1 \cap M_i$ is the product of P with the minimal parabolic $P_i := M_i \cap L_1T$.

Recall from chapter 1 the definition of $\Xi_p(X)$ for $X \in \mathcal{L}(G,T)$ with $X/O_2(X)$ not quasisimple.

LEMMA 12.5.3. *For each* $i = 1,3,4,6$, $L_i \leq K_i \in \mathcal{C}(N_G(V_i))$ *with* $K_i \trianglelefteq N_G(V_i)$, *and one of the following holds:*

(1) $L_i = K_i$.

(2) $i = 1$, $K_1/O_2(K_1) \cong L_5(2)$, M_{24}, *or* J_4, *and* $O^2(P) \leq K_1$.

(3) $i = 1$, $K_1 = \Xi_7(K_1)L_1$ *and* $K_1/O_2(K_1) \cong SL_2(7)/E_{49}$.

PROOF. The existence and normality of K_i follows from 1.2.4 and the fact that T normalizes L_i. By 12.5.2, $L_i/O_2(L_i) \cong L_3(2)$ if $i = 1,3$ and $L_4(2)$ if $i = 4,6$.

We first treat the case $i = 1$. We may assume that $L_1 < K_1$, so that $K_1/O_2(K_1)$ is described in the sublist of A.3.12 where $B/O_2(B) \cong L_3(2)$. If $K_1/O_2(K_1) \cong L_5(2)$, M_{24}, or J_4, then $K_1 = O^{3'}(G_1)$ by A.3.18, so $O^2(P) \leq K_1$ and hence (2) holds. Thus we may assume that $K_1/O_2(K_1)$ is not one of these groups, nor $SL_2(7)/E_{49}$, so $K_1/O_2(K_1)$ is one of $L_4(2)$, A_7, $\hat{A}_7$, $L_2(49)$, $(S)L_3^\epsilon(7)$, M_{23}, HS,

He, or Ru. To rule out $L_2(49)$ or $(S)L_3^\epsilon(7)$, observe that in those groups, some element of T induces an outer automorphism on $L_1/O_2(L_1) \cong L_3(2)$, while as $\bar{T} \leq \bar{L}$, this is not the case in LT. In the remaining cases by A.3.18, K_1 is the characteristic subgroup $\theta(G_1)$ of G_1 generated by all elements of order 3. Hence K_1 contains $I := O^{3'}(M_1)$, since $I/O_2(I) \cong \mathbf{Z}_3 \times L_3(2)$ by 12.5.2.1. Further $M_1 = IT$ and $M_1/O_2(M_1) \cong S_3 \times L_3(2)$, whereas $Aut(K_1/O_2(K_1))$ contains no such overgroup of a Sylow 2-group. This completes the proof of the lemma when $i = 1$, although we need some more information about K_1 which we develop in the next paragraph.

Set $(K_1T)^* := K_1T/O_2(K_1T)$. When $K_1^* \cong M_{24}$, $L_5(2)$, or J_4, the overgroups of T^* are described by a 2-local diagram, cf. [**RS80**] or [**Asc86b**]; we now describe the embedding of $L_1^*T^*$ and M_1^* in K_1^* in terms of the minimal parabolics in the sense of Definition B.6.1 indexed by the nodes of of this diagram: As $L_1^*T^*/O_2(L_1^*T^*) \cong L_3(2)$ and $M_1^*/O_2(M_1^*) \cong L_2(2) \times L_3(2)$, it follows that if $K_1^* \cong L_5(2)$ then (up to a symmetry of the diagram) $L_1^*T^*$ is generated by the third and fourth minimal parabolics of K_1^*, and the remaining parabolic P^* of M_1^* is the first parabolic of K_1^*. If $K_1^* \cong M_{24}$ or J_4, then $L_1^*T^*$ is generated by the parabolics indexed by the "square node" and by the adjacent node in those diagrams. Further in M_{24}, P^* is the parabolic P_K^* indexed by the node which is adjacent to neither of these nodes, while in J_4, the corresponding parabolic P_K satisfies $P_K^*/O_2(P_K^*) \cong S_5$, and P^* is the Borel subgroup of that parabolic. Thus when $K_1^* \cong M_{24}$, M_1^* is the trio stabilizer in the language used in chapter H of Volume I, while when $K_1^* \cong J_4$, $P_K^*M_1^* \cong S_5 \times L_3(2)/2^{3+12}$ and $M_1^* \cong (S_4 \times L_3(2))/2^{3+12}$.

We next treat the cases $i = 3$ or 4. Here $L_i/C_{L_i}(V_i) = GL(V_i)$ by 12.5.2, so that $K_i = L_iC_{K_i}(V_i)$. Hence if $K_i/O_2(K_i)$ is quasisimple, then $L_i = K_i$, as required. Therefore we may assume that $K_i/O_2(K_i)$ is not quasisimple, and it remains to derive a contradiction. As $K_i/O_2(K_i)$ is not quasisimple and $L_i \leq K_i$, we conclude from A.3.12 that $i = 3$, $K_3/O_2(K_3) \cong SL_2(7)/E_{49}$, and $K_3 = XL_3$ for $X := \Xi_7(K_3)$. Set $Y := O^2(P)$. Recall X char $K_3 \trianglelefteq G_3$. and $O_2(X) \neq 1$ using 1.1.3.1. Also X centralizes $V_3 \geq V_1$, so $X \leq K_{1,3} := O^2(C_{K_3}(V_1)) \leq G_1 \cap G_3$. We saw earlier that $Y = O^2(P) \leq L_3$, so $Y \leq K_{1,3}$. Then $K_{1,3}T/O_2(K_{1,3}T) \cong GL_2(3)/E_{49}$ and $K_{1,3} = YX = \langle Y^{K_{1,3}} \rangle$.

Suppose first that K_1^* is $L_5(2)$, M_{24}, or J_4. We saw earlier that $Y \leq K_1$. Hence $K_{1,3} = \langle Y^{K_{1,3}} \rangle \leq K_1$, so $K_{1,3}^*T^*$ is a subgroup of $K_1^*T^*$ containing T^*. But from the description of overgroups of a Sylow 2-group in terms of the 2-local diagrams for $L_5(2)$, M_{24}, and J_4 mentioned earlier, no such group has a $GL_2(3)/E_{49}$-section.

So we may suppose instead that $K_1 = L_1$ or $K_1^* \cong SL_2(7)/E_{49}$. By an earlier observation $[L_1, Y] \leq O_2(L_1)$. Thus if $K_1 = L_1$, Y centralizes K_1^*, and we claim this also holds when $K_1^* \cong SL_2(7)/E_{49}$: For $K_1 = \Xi_7(K_1)L_1$ and there is $Y < Y_0 \leq P$ with $Y_0/O_2(Y) \cong L_2(2)$ and $[L_1, Y_0] \leq O_2(L_1)$. Then as $L_1^* \cong SL_2(7)$ is centralized in $Aut(\Xi_7(K_1)^*)$ by $Z(GL_2(7)) \cong \mathbf{Z}_6$ which is abelian, we conclude $Y = [Y_0, Y_0]$ centralizes $\Xi_7(K_1^*)$. Hence $Y = O^7(Y)$ centralizes $K_1^* = O^{7'}(K_1^*)$, establishing the claim.

By the claim, $\langle Y^{K_{1,3}} \rangle = K_{1,3}$ centralizes K_1^*, so as $X \leq K_{1,3}$, X centralizes K_1^*. By construction $X \in \Xi(G,T)$, so by 1.3.4, either $X \trianglelefteq G_1$, or $X < K_0 \in \mathcal{C}(G_1)$ with $m_3(K_0) = 2$. In the latter case $K_0 = \langle X^{K_0} \rangle$ centralizes $K_1/O_2(K_1)$, so that $m_3(K_0K_1) > 2$, contradicting G_1 an SQTK-group. In the former case

$LT \leq \langle G_1, G_3\rangle \leq N_G(X)$, so as $M = !\mathcal{M}(LT)$ and $O_2(X) \neq 1$, $G_1 \leq M$, contrary to 12.4.2.2. This completes the proof that $K_i = L_i$ if $i = 3$ or 4.

Finally we treat the case $i = 6$. Then either $K_6 = L_6$ as required, or as $L_6/O_2(L_6) \cong L_4(2)$, we obtain $K_6/O_2(K_6) \cong L_5(2)$, M_{24}, or J_4 from A.3.12. The latter three cases are impossible, since L_6 acts as $\Omega_6^+(2)$ on V_6, and this action does not extend to any 6-dimensional module for $L_5(2)$, while M_{24} and J_4 have no nontrivial modules of dimension 6. This completes the proof of 12.5.3. □

LEMMA 12.5.4. $G_3 \leq M \geq G_6$.

PROOF. Let $i := 3$ or 6. By 12.5.3, $L_i \trianglelefteq G_i$, so as $N_{GL(V_i)}(Aut_{L_i}(V_i)) = Aut_{M_i}(V_i)$, $G_i = M_iC_G(V_i)$. Thus to show $G_i \leq M$, it suffices to show $C_G(V_i) \leq M$. If $C_G(V_i)$ acts on L_1, it acts on $\langle L_1^{L_i}\rangle = L$, so that $C_G(V_i) \leq N_G(L) = M$. Thus we may assume $C_G(V_i) \not\leq N_G(L_1)$.

Set $G_1^* := G_1/O_2(G_1)$. By 12.5.3, $Out(K_1^*)$ is a 2-group, so $G_1 = K_1TC_{G_1}(K_1^*)$. Thus as $C_G(V_i) \leq G_1$, and T and $C_{G_1}(K_1^*))$ act on $O^2(L_1O_2(K_1)) = L_1$, we may assume the preimage Y in K_1 of the projection of $G_i \cap K_1$ with respect to the decomposition $K_1^* \times C_{G_1^*}(K_1^*)$ is not contained in M_1. Therefore $K_1 \neq L_1$. As $T \leq G_i$, $[T \cap K_1, Y] \leq [G_i \cap K_1, Y] \leq Y \cap G_i$.

Suppose that case (3) of 12.5.3 holds, and set $Y_i := O^{3'}(N_{L_1}(V_i))$. Then Y_iT is a minimal parabolic of L_iT, so as $L_1 \not\leq G_i$, $Y_iT = M_i \cap L_1T = P_i$. Then as $[G_i \cap K_1, Y] \leq Y \cap G_i$ and Y_i acts irreducibly as $SL_2(3)$ on $\Xi_7(K_1)^* \cong E_{49}$, $Y \leq G_i$ and $Y = P_i \cap K_1$ or $\Xi_7(K_1)(P_i \cap K_1)$; as $Y \not\leq M_1$, the latter case holds. Then $\Xi_7(K_1) \leq \langle Y_i^Y\rangle \leq L_i$ as Y normalizes L_i by 12.5.3, contrary to $m_7(L_i) = 1$.

Therefore $K_1^* \cong L_5(2)$, M_{24}, or J_4. Recall from the discussion before 12.5.3 that $M_1 \cap M_i = PP_i$ is the product of the two minimal parabolics P and P_i, and $O^2(P) \leq K_1$. Then Y^* is a proper overgroup of $O^2(P^*)O^2(P_i^*)(T^* \cap K^*)$ which does not contain L_1^*. Let $Y_0 := \langle O^2(P)^Y\rangle$ and recall that the discussion during the proof of 12.5.3 determined the embedding of M_1^* in K_1^*. If $K_1^* \cong M_{24}$, the conditions above on Y^* force $Y^*/O_2(Y^*) \cong \hat{S}_6$ and $Y = Y_0$. If $K^* \cong L_5(2)$, then $Y/O_2(Y) \cong L_4(2)$ or $L_3(2) \times L_2(2)$, and $Y_0/O_2(Y_0) \cong L_4(2)$ or $L_3(2)$, respectively. If $K^* \cong J_4$, then $Y/O_2(Y) \cong \hat{M}_{22}$ or $S_3 \times S_5$, and $Y_0/O_2(Y_0) \cong \hat{M}_{22}$ or S_5, respectively. In particular, in each case $Y_0 \not\leq M$, since $M_1 = PL_1$. Further as $[Y, G_i \cap Y] \leq G_i$, $Y_0 \leq G_i$. But as $O^2(P) \leq L_i \trianglelefteq G_i$ by 12.5.3, $Y_0 \leq L_i \leq M$, contrary to the previous remark. This contradiction completes the proof of 12.5.4. □

LEMMA 12.5.5. *(1) $\bar{L}$ has two classes of involutions with representatives j_1 and j_2, where $m([\Gamma, j_i]) = i$, $m([V, j_2]) = 4$, and $C_V(j_1) = V_4 + V_6$ is of codimension* 3 *in V.*

(2) L has two orbits on the points of V with representatives V_1 and

$$V_1' := \langle 1 \wedge 2 + 3 \wedge 4\rangle.$$

(3) $C_{\bar{L}}(V_1')$ is $\bar{R}_6$ extended by S_6.

PROOF. These are straightforward calculations. □

In the remainder of the section, we let V_1' be defined as in 12.5.5.2

LEMMA 12.5.6. $r(G, V) > 3 = m(M_V, V) = s(G, V)$.

PROOF. By 12.5.5.1, $m(\bar{M}_V, V) = 3$, so $r(G, V) \geq 3$ by Theorem E.6.3. Further if $U \leq V$ with $m(V/U) = 3$ and $C_G(U) \not\leq M$, then U is conjugate to $C_V(j_1)$

by E.6.12; then as U is normal in some Sylow 2-subgroup of LT, E.6.13 supplies a contradiction. □

LEMMA 12.5.7. *$W_0 := W_0(T,V)$ centralizes V, so $w := w(G,V) > 0$ and $N_G(W_0) \le M$.*

PROOF. Suppose $A := V^g \le T$ with $[A,V] \ne 1$. By 12.5.6, $s(G,V) = 3$, so that $\bar{A} \in \mathcal{A}_3(\bar{T},V)$ by E.3.10. But $Aut_{\bar{M}}(V_3) \cong L_3(2)$ is of 2-rank 2, so we conclude A centralizes V_3. Next T acts on Γ_1 and Γ_4, and hence acts on $V_3' := \Gamma_1 \wedge \Gamma_4 = \langle 1\wedge 2, 1\wedge 3, 1\wedge 4\rangle$, with $Aut_{\bar{M}}(V_3) \cong L_3(2)$, so the same argument shows A also centralizes V_3'. Similarly A acts on V_6, so that $A \le C := C_{M_6}(V_3 + V_3')$. By 12.5.2.4, V_6 is the orthogonal module for L_6/R_6, so that $m(C/R_6) = 1$; again as $\bar{A} \in \mathcal{A}_3(\bar{T},V)$, we conclude $A \le C_{LT}(V_6) = R_6$. By 12.5.5.1, V_6 is a hyperplane of $C_V(\bar{r})$ for each $\bar{r} \in \bar{R}_6^{\#}$, while by 12.5.2.4, V/V_6 is a natural module for L_6/R_6 isomorphic to $\bar{R}_6$. It follows that $V = W$, where $W := \langle C_V(\bar{r}) : \bar{r} \in \bar{R}_6^{\#}\rangle$, and no hyperplane of $\bar{R}_6$ lies in $\mathcal{A}_3(\bar{T},V)$. We conclude that $m(\bar{A}) > 3$, so that $\bar{A} = \bar{R}_6$, and hence $W = \langle C_V(B) : m(A/B) \le 3\rangle$. Now $r(G,V) > 3$ by 12.5.6, so $W \le N_G(A)$ by E.3.32. Hence as $V = W$, we have symmetry between A and V. As $\bar{A} = \bar{R}_6$, $V_6 = [V,A]$; then by symmetry between A and V, $[A,V]$ is conjugate to V_6^g in L^g. Thus we may take $g \in G_6$, so $g \in N_M(V_6)$ by 12.5.4, and hence $g \in M_V$ by 12.2.6, contrary to $[V,V^g] \ne 1$. This contradiction shows $W_0 \le C_T(V) = O_2(LT)$, and so $N_G(W_0) \le M$ by E.3.34.2. □

Let $U := \langle V^{G_1}\rangle$ and $\tilde{G}_1 := G_1/V_1$.

LEMMA 12.5.8. *(1) $V \le O_2(G_1)$.*
(2) U is elementary abelian.

PROOF. Let $Y := O^2(M_1)$ and $U_1 := \langle V_7^{G_1}\rangle$. By 12.5.2.1, Y has chief series $0 < V_1 < V_7 < V$. Thus Hypothesis G.2.1 is satisfied with Y, G_1, G_1, Y, V_7 in the roles of "L, G, H, L_1, V", so by G.2.2, $\tilde{U}_1 \in \mathcal{R}_2(\tilde{G}_1)$ and $\tilde{U}_1 \le \Omega_1(Z(O_2(\tilde{G}_1)))$. In particular, $V_7 \le O_2(G_1)$.

Next $V = [V,L_1] \le [O_2(L_1),L_1]$, so if $K_1 = L_1$ or $K_1/O_2(K_1) \cong SL_2(7)/E_{49}$, then $V \le O_2(K_1) \le O_2(G_1)$, and hence (1) hold in these cases. We assume that (1) fails, so $K_1^* := K_1/O_2(K_1) \cong L_5(2)$, M_{24}, or J_4 by 12.5.3. Also $V^* \cong V/V_7$ is the natural module for $L_1^*/O_2(L_1^*) \cong L_3(2)$. Then $[V,U_1] \le V \cap U_1 = V \cap O_2(G_1) = V_7$. Further V^* is invariant under M_1^* by 12.2.6, and from the discussion in the proof of 12.5.3, $M_1^*/O_2(M_1^*) \cong L_2(2) \times L_3(2)$, with the embedding of M_1^* in K_1^* determined. When $K_1^* \cong M_{24}$ or $L_5(2)$, this is contrary to the action of Y^* on $O_2(Y^*)$ as the tensor product module of rank 6. Finally suppose $K_1^* \cong J_4$. Our discussion of the embedding of M_1^* showed that $M_1^* < N^*$, with $O_2(N^*)$ special of order 2^{3+12} and $N^*/O_2(N^*) \cong S_5 \times L_3(2)$. It follows that $V^* = Z(O_2(N^*)) \trianglelefteq N^*$. Since $L_1^* \not\le C_{G_1^*}(\tilde{V}_7)$, $K_1^* = \langle L_1^{*K_1}\rangle \not\le C_{G_1^*}(\tilde{U}_1)$, and in particular $V^* \not\le C_{G_1^*}(\tilde{U}_1)$; as we saw $[V,U_1] \le V_7$ and Y^* is irreducible on $\tilde{V}_7$, we conclude $[U_1,V] = V_7$. Therefore N^* normalizes $[\tilde{U}_1,V^*] = \tilde{V}_7$. But this is impossible as $\tilde{V}_7$ is the tensor product module for $Y^*/O_2(Y^*) \cong L_2(2) \times L_3(2)$, and this action does not extend to $N^*/O_2(N^*) \cong S_5 \times L_3(2)$.

Thus (1) is established. By 12.5.7, $V \le \Omega_1(Z(W_0(O_2(G_1),V))))$, so (2) follows from (1). □

LEMMA 12.5.9. *Let* $T_1 := C_T(V_1')$*, and choose notation with* $T_1 \in Syl_2(C_M(V_1'))$*. Then*

(1) $|T : T_1| = 4$*, and* $T_1 \in Syl_2(C_G(V_1'))$*.*
(2) $V_1' \notin V_1^G$*.*
(3) $V_1^G \cap V = V_1^L$*.*
(4) If $g \in G - N_G(V)$ *with* $V_1 \leq V^g$*, then* $V^g \in V^{G_1}$ *and* $[V, V^g] = 1$*.*

PROOF. The first part of (1) follows from 12.5.5.3. By 12.5.7, $W_0 := W_0(T, V) = W_0(T_1, V)$ and $N_G(W_0) \leq M$, so $T_1 \in Syl_2(C_G(V_1'))$. Thus (1) holds, and (1) implies (2). Then (2) and 12.5.5.1 imply (3). Finally under the hypothesis of (4), (3) and A.1.7.1 imply $V^g \in V^{G_1}$, and then 12.5.8.2 implies $[V, V^g] = 1$. □

LEMMA 12.5.10. *(1) Either* $W_1 := W_1(T, V)$ *centralizes* V*, or* $\bar{W}_1 = \bar{R}_6$*.*
(2) $C_G(C_1(T, V)) \leq M$*.*

PROOF. Suppose $A := V^g \cap M \leq T$ with $[A, V] \neq 1$, and $m(V^g/A) \leq 1$. By 12.5.7, $m(V^g/A) = 1$ and $V > I := N_V(V^g)$. We now argue much as in 12.5.7: This time $r(G, V) > 3 = s(G, V)$ by 12.5.6, so $\bar{A} \in \mathcal{A}_2(\bar{T}, V)$ by E.3.10. Therefore either A centralizes V_3, or $Aut_A(V_3) \in \mathcal{A}_2(Aut_T(V_3), V_3)$, so that $m(A/C_A(V_3)) = m_2(L_3/O_2(L_3)) = 2$, and hence $V_3 \leq I$ as $r(G, V) > 3$. But in the latter case, $V_1 \leq [V_3, A] \leq V^g$, contrary to 12.5.9.4. We conclude A centralizes V_3, and similarly that A centralizes the space V_3' of 12.5.7; so again $A \leq C := C_{M_6}(V_3 + V_3')$ with $m(C/R_6) = 1$, and as $\bar{A} \in \mathcal{A}_2(\bar{T}, V)$, $A \leq C_{LT}(V_6) = R_6$. Thus as L_6 is irreducible on $\bar{R}_6$, $\bar{W}_1 = \bar{R}_6$. Hence (1) holds.

By (1), $V_6 \leq C_1(T, V)$, so (2) follows from 12.5.4. □

For the remainder of the section, let $H \in \mathcal{H}_*(T, M)$. Recall from 3.3.2.4 that H is described in B.6.8 and E.2.2.

LEMMA 12.5.11. *(1)* $n(H) > 1$*.*
(2) $K_1 = L_1$*.*

PROOF. By 12.5.7 and 12.5.10, $N_G(W_0) \leq M \geq C_G(C_1(T, V))$; so as $s(G, V) = 3$, (1) follows from E.3.19 with $i, j = 0, 1$. Suppose $L_1 < K_1$, so that in particular $K_1 \not\leq M$. Then using the description of the embedding of M_1^* in K_1^* in 12.5.3 and its proof, there is $H \in \mathcal{H}(T)$ with $H \leq K_1T$, $H \not\leq M$, and either $H/O_2(H) \cong S_3$, or $K_1/O_2(K_1) \cong SL_2(7)/E_{49}$ and $H := \Xi_7(K_1)T$. Thus $H \in \mathcal{H}_*(T, M)$ with $n(H) = 1$, contrary to (1). Thus (2) is also established. □

LEMMA 12.5.12. *If* $H \leq G_1$*, then* $n(H) = 2$*, and a Hall* $2'$*-subgroup of* $H \cap M$ *is a nontrivial* 3*-group.*

PROOF. By 12.5.11.1, $n(H) > 1$. Then applying 12.2.11 to V_1 in the role of "U", the lemma holds. □

We are now in a position to complete the proof of Theorem 12.5.1.

By Theorem 12.4.2.2, $G_1 \not\leq M$, so we may choose $H \in \mathcal{H}_*(T, M) \cap G_1$. Hence by 12.5.12, $n(H) = 2$ and a Hall $2'$-subgroup of $H \cap M$ is a nontrivial 3-group. Set $K := O^2(H)$, so that $K \not\leq M$, and $X := C_{G_1}(L_1/O_2(L_1))$. Then as $L_1 = K_1 \trianglelefteq G_1$ by 12.5.11.2, and $\bar{T} \leq \bar{L}$, $G_1 = L_1X$. In particular, $X \not\leq M$ as $G_1 \not\leq M$, so we may choose $H \leq XT$. As $m_3(L_1) = 1$ and $m_3(G_1) \leq 2$, $m_3(X) \leq 1$. Therefore $m_3(X) =$

$m_3(H) = 1$, so as $n(H) = 2$, and a Hall $2'$-subgroup of $H \cap M$ is a nontrivial 3-group, $K/O_2(K) \cong L_2(4)$. Also $O^{3'}(H \cap M) \leq O^{3'}(X \cap M) = O^{3'}(X \cap L)$ using 12.2.8, so $M_1 = L_1(H \cap M)$ by 12.5.2.1.

Now just as in the proof of 12.5.8, Hypothesis G.2.1 is satisfied with $H_1 := L_1H$, $O^2(M_1)$, V_7 in the roles of "H, L_1, V". Let $\tilde{H}_1 := H_1/V_1$, $U_H := \langle V_7^H \rangle$, and $Q_H := O_2(H_1)$. Then $\tilde{U}_H \leq Z(\tilde{Q}_H)$ by G.2.2.1, and $U_H \leq U$, so that U_H is elementary abelian by 12.5.8.2. If $[U_H, K] = 1$ then as $V_3 \leq V_7$, $K \leq G_3 \leq M$ by 12.5.4, contrary to $K \not\leq M$. Thus as $H_1 = KTL_1$ with $KL_1Q_H/Q_H \cong A_5 \times L_3(2)$, we conclude $Q_H = C_H(\tilde{U}_H)$.

Let $H_1^* := H_1/Q_H$. Now $W_0 := W_0(T,V) \leq C_T(V)$ and $N_G(W_0) \leq M$ by 12.5.7. Hence as $H \not\leq M$, $W_0 \not\leq O_2(H)$ by E.3.15, so there is $A := V^g \leq T$ with $A \not\leq O_2(H)$. As $A \leq W_0(T,V) \leq C_T(V) \leq O_2(M_1) \leq Q_H(T \cap K)$, $A^* \leq K^*$. Let $B := A \cap Q_H = C_A(\tilde{U}_H)$. Then $m(A/B) \leq m_2(K^*) = 2$. Further $[U_H, B] \leq V_1$, so for $u \in U_H$, $m(B/C_B(u)) \leq m_2(V_1) = 1$, and hence $m(A/C_B(u)) \leq 3$. Now $r(G,V) > 3 = s(G,V)$ by 12.5.6, so $u \in N_G(A)$, and hence $U_H \leq N_G(A)$. Thus if $[U_H, B] \neq 1$ then $V_1 = [U_H, B] \leq A$. But then 12.5.9.4 and 12.5.8.1 show $A \in V^{G_1} \subseteq O_2(G_1)$, contrary to $A \not\leq O_2(H)$. Thus U_H centralizes B, so as $m(A/B) < s(G,V)$, A centralizes U_H by E.3.6. But then $[K,A] = K$ centralizes U_H, contrary to our earlier observation that $Q_H = C_H(\tilde{U}_H)$.

This final contradiction completes the proof of Theorem 12.5.1.

12.6. Eliminating A_8 on the permutation module

The main result of this section is Theorem 12.6.34, which eliminates the A_8-subcase in case (d) of Theorem 12.2.2.3, reducing the treatment of $L/O_2(L) \cong L_4(2)$ to case (a) where V is a 4-dimensional natural module. This leaves only one case of Theorem 12.2.2.3 where it is possible that $C_V(L) \neq 1$: case (d) with $\bar{L} \cong A_6$. That case will be treated in section 13.4 of the following chapter.

We mention that $L_4(2)/E_{64}$ arises as $L \in \mathcal{L}_f^*(G,T)$ in the non-quasithin shadows $G \cong \Omega_8^+(2)$, $O_{10}^+(2)$, $Sp_8(2)$, and $P\Omega_8^+(3)$. Also such a 6-dimensional internal module appears in a suitable non-maximal member of $\mathcal{L}_f(G,T)$ in other non-quasithin groups, such as larger orthogonal and symplectic groups, as well as the sporadic groups J_4 and F_{24}'. As a result, the analysis in this case is fairly long and difficult. In particular, these shadows are not eliminated until 12.6.26.

So in section 12.6 we assume Hypothesis 12.2.3, and adopt the conventions of Notation 12.2.5, including $Z = \Omega_1(Z(T))$. In addition set $Z_V := C_V(L)$ and $\hat{V} := V/Z_V$.

Througout this section, we assume that $\bar{L} \cong A_8$ and that $\hat{V}$ is the orthogonal module for $\bar{L} \cong \Omega_6^+(2)$. In particular notice $O_2(L) = O_{2,Z}(L) = C_L(V)$ by 1.2.1.4, since the Schur multiplier of A_8 is of order 2 by I.1.3.1. Further $\bar{M}_V = \bar{L}\bar{T} \cong A_8$ or $S_8 = Aut(A_8)$.

We adopt the notational conventions of section B.3, and assume T preserves the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ of the set Ω of 8 points. In particular by B.3.3, if $Z_V \neq 1$, then V is the core of the permutation module for $\bar{L}$ on Ω, and Z_V is generated by e_Ω. In that case, $V \trianglelefteq M$ by Theorem 12.2.2.3; hence $Z_V = C_V(L) \trianglelefteq M$, and we conclude $M = C_G(Z_V)$ as $M \in \mathcal{M}$. In any case $\hat{V}$ is the quotient of the core of the permutation module, modulo $\langle e_\Omega \rangle$. We can also view $\hat{V}$ as a 6-dimensional orthogonal space for $\bar{L} \cong \Omega_6^+(2)$. Thus we can speak of singular vectors in $\hat{V}$, nondegenerate subspaces of $\hat{V}$, etc. For $i = 1, 2, 5$, let V_i denote the preimage in V of the i-dimensional subspace of $\hat{V}$ fixed by T. Let $G_i := N_G(V_i)$, $M_i := N_M(V_i)$, $L_i := O^2(N_L(V_i))$, and R_i the preimage in T of $O_2(\bar{L}_i\bar{T})$.

12.6.1. Preliminary results.

LEMMA 12.6.1. *(1) L has two orbits on $\hat{V}^{\#}$, consisting of the singular and nonsingular vectors of $\hat{V}$.*

(2) If $Z_V \neq 1$, then $\mathbf{Z}_2 \cong Z_V = Z(T) \cap V$.

(3) Either $J(T) = J(C_T(V))$, or $|R_2(LT) : VC_{R_2(LT)}(L)| \leq 2$.

(4) $J(R_1) = J(C_T(V))$. Hence $N_G(R_1) \leq M$.

(5) $O^{3'}(M) = L$.

(6) If $L = [L, J(T)]$ and $Z_V \neq 1$, then L centralizes Z.

PROOF. Part (5) follows from 12.2.8. Recall that either

(a) $Z_V \neq 1$, and V is the 7-dimensional core of the permutation module for $\bar{L}$, or

(b) $Z_V = 1$, and V is the 6-dimensional quotient of that core, modulo $\langle e_\Omega \rangle$.

Hence (1) and (2) are well known, easy calculations. Also either case (5) or (6) of B.3.2 holds, so if $\bar{A} \in \mathcal{P}(\bar{T}, V)$, then one of the following holds:

(i) $\bar{A} = \langle D \rangle$, for some $D \subseteq \Delta = \{(1,2), (3,4), (5,6), (7,8)\}$.

(ii) $\bar{A} = \langle \Delta \rangle \cap \bar{L}$.

(iii) $\bar{A}$ is conjugate under $\bar{L}$ to $\bar{A}_0 := \langle (1,2)(3,4), (1,3)(2,4), (5,6), (7,8) \rangle$, or to a hyperplane in the group of (ii), given by either

$$\langle (1,2)(3,4), (1,2)(5,6), (7,8) \rangle, \text{ or } \langle (1,2)(3,4), (5,6), (7,8) \rangle.$$

(iv) $Z_V = 1$ and $\bar{A} \cong E_8$ is the unipotent radical of an $L_3(2)/E_8$ parabolic of $\bar{L}$.

Now $\bar{R}_1 \cong E_{16}$ is the unipotent radical of the stabilizer of the partition

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\},$$

so $\bar{R}_1$ contains no transpositions and hence contains no subgroup of type (i) or (iii); nor does it contain a subgroup of type (ii) or (iv). Thus $\bar{R}_1$ contains no FF*-offenders, so that $J(R_1) = J(O_2(LT))$, and hence $N_G(R_1) \leq N_G(J(O_2(LT))) \leq M = !\mathcal{M}(LT)$, so that (4) holds.

Also if $J(T) \not\leq C_T(V)$, then $\hat{V}$ is the unique noncentral chief factor for L on $R_2(LT)$ by Theorem B.5.1.1. Then (3) follows as $H^1(L, \hat{V}) \cong \mathbf{Z}_2$ by I.1.6.1. If in addition $Z_V \neq 1$, then $Z_V = Z \cap V$ by (2), and we've just seen that $V = [R_2(LT), L]$, so (6) follows as $\langle Z^L \rangle = VC_Z(L)$ by B.2.14. □

In the shadows mentioned earlier (such as $\Omega_8^+(2)$), $C_G(v) \leq M$ for each nonsingular $v \in V$, as in the first main reduction Theorem 12.6.2 below. However, this result does eliminate the sporadic configurations in J_4 and F'_{24}, since in those groups $C_G(v) \not\leq M$.

THEOREM 12.6.2. *$C_G(v) \leq M$ for each $v \in V$ with $\hat{v}$ nonsingular.*

Until the proof of Theorem 12.6.2 is complete, let $v \in V$ with $\hat{v}$ nonsingular and set $R_v := O_2(C_{LT}(v))$. Conjugating in L, we may assume $\hat{v} = \hat{e}_{1,2}$. Thus $\bar{L}_v \cong A_6$, so as $O_2(L) = O_{2,Z}(L) = C_L(V)$, $L_v = C_L(v)^\infty$, $L_v/O_2(L_v) \cong A_6$, and either $R_v = Q$ or $\bar{R}_v = \langle(1,2)\rangle$. By choice of v, $T_v := C_T(v) \in Syl_2(M_v)$ and $|T : T_v| = 4$. As $C_{LT}(v) = L_vT_v$, $R_v = O_2(L_vT_v)$.

LEMMA 12.6.3. *There exists $K_v \in \mathcal{C}(G_v)$ with $L_v \leq K_v = O^{3'}(G_v^\infty) \trianglelefteq G_v$, and $K_v/O_2(K_v)$ quasisimple.*

PROOF. By 1.2.1.1, L_v is contained in the product of the $\mathcal{C}$-components of G_v, so L_v projects nontrivally on $K_v/O_\infty(K_v)$ for some $K_v \in \mathcal{C}(G_v)$. As $L_v/O_2(L_v) \cong A_6$, it follows from A.3.18 that $m_3(K_v) = 2$ and $K_v = O^{3'}(G_v^\infty)$. Thus $L_v \leq K_v \trianglelefteq G_v$, and as $L_v/O_2(L_v) \cong A_6$, $K_v/O_2(K_v)$ is quasisimple by 1.2.1.4. □

Let $T_v \leq S \in Syl_2(G_v)$, set $(K_vS)^* := K_vS/O_2(K_vS)$, and choose S so that $N_S(L_v) \in Syl_2(N_{G_v}(L_v))$. Hence $R := C_S(L_v^*/O_2(L_v^*)) \in Syl_2(C_{K_vS}(L_v^*/O_2(L_v^*)))$ and $R_v = R \cap T_v$. Then:

LEMMA 12.6.4. $|S : T_v| \leq |T : T_v| = 4 \geq |R : R_v|$. *Further $O_2(K_vS) \leq R$.*

LEMMA 12.6.5. *R normalizes L_v, and therefore $[L_v, R] \leq O_2(L_v)$ and $R = C_S(L_v/O_2(L_v))$.*

PROOF. Let X be the preimage in K_vS of $O_2(L_v^*)$. As $[R, L_v] \leq X$ while $|R : R_v| \leq 4$, $(L_vX)^\infty = L_v$, so the lemma holds. □

Let $V_v := [V, L_v]$. Then V_v is the 5-dimensional core of the permutation module for $L_v/O_2(L_v) \cong A_6$. In particular V_v is generated by the L_v-conjugates of a vector of weight 4 in that module, which is central in $T_v \in Syl_2(L_vT_v)$, so that $V_v \in \mathcal{R}_2(L_vT_v)$ and $V_v \leq \Omega_1(Z(R_v))$ by B.2.14. Let v_0 denote the generator of $C_{V_v}(L_v)$; thus v_0 has weight 6 in V and V_v, even though v itself may have weight 2 rather than 6 in V.

LEMMA 12.6.6. *If $J(T) \not\leq Q$ then either*

(1) $L_v = [L_v, J(T_v)]$, or

(2) $J(T_v) = J(Q)$.

PROOF. By hypothesis there is some $A \in \mathcal{A}(T)$ not in Q. Assume first that A satisfies one of (i)–(iii) in the proof of 12.6.1. Then some L-conjugate of A centralizes v and is nontrivial on $L_v/O_2(L_v)$, so that $J(T_v) \not\leq C_{T_v}(V_v)$, $L_v = [L_v, J(T_v)]$, and $L_vT_v/O_2(L_vT_v) \cong S_6$, and hence conclusion (1) holds. Thus if $J(T_v) \leq C_{T_v}(V_v)$, then each $\bar{A}$ must satisfy (iv), so in particular $Z_V = 1$ and V_v is a hyperplane of V. But since $\bar{A}$ satisfies (iv), $\bar{A}$ centralizes no vector of weight 2, so $\bar{A} \not\leq \bar{T}_v$ and hence $J(T_v) \leq Q$. As $Q \leq T_v$, we conclude $J(T_v) = J(Q)$ using B.2.3.3, so that conclusion (2) holds. □

LEMMA 12.6.7. *(1) If $J(T) \leq Q$, then $J(T) = J(Q) = J(T_v)$.*

(2) If $L_v = [L_v, J(T_v)]$, then $L = [L, J(T)]$.

PROOF. As $Q \leq T_v \leq T$, (1) holds. Then (1) implies (2). □

LEMMA 12.6.8. *If $J(T_v) \leq Q$, then*

(1) $S = T_v$ and $R = R_v$.

(2) $J(T_v) = J(R_v) = J(Q)$.

(3) $N_G(J(S)) \leq M$.

PROOF. Assume $J(T_v) \leq Q$. Then $J(T_v) = J(Q)$, so $N_G(T_v) \leq N_G(J(T_v)) = N_G(J(Q)) \leq M = !\mathcal{M}(LT)$. Hence as $T_v \in Syl_2(M_v)$, (1) and (3) hold. Then as $Q \leq O_2(L_vT_v) = R_v$, (2) holds. □

LEMMA 12.6.9. *If $J(T_v) \not\leq Q$, then $J(T) \not\leq Q$ and $L_v = [L_v, J(T_v)]$.*

PROOF. Assume $J(T_v) \not\leq Q$. Then by 12.6.7.1, $J(T) \not\leq Q$. So by 12.6.6, $L_v = [L_v, J(T_v)]$. □

LEMMA 12.6.10. *Let $\Delta(v)$ be the set of vectors of weight 2 in V_v. Then*

$$L(v) := \langle L_u : u \in \Delta(v) \rangle = L.$$

PROOF. Straightforward. □

During the remainder of the proof of Theorem 12.6.2, we assume that $G_v \not\leq M$. In addition when $Z_V \neq 1$ and $G_u \not\leq M$ for some u of weight 2 in V, we choose v to be of weight 2 rather than 6.

LEMMA 12.6.11. *$L_v < K_v$, so $K_v \not\leq M$.*

PROOF. Assume $L_v = K_v$. Then $L_v = O^{3'}(G_v)$ by A.3.18. Furthermore $C_{G_v}(V_v)$ permutes $\{L_u : u \in \Delta(v)\}$, and hence $C_{G_v}(V_v) \leq N_G(L(v))$, so $C_{G_v}(V_v) \leq N_G(L) = M$ by 12.6.10. We deduce several consequences of this fact: First, $V_v \leq O_2(L_v) \leq O_2(G_v)$, so $O^2(F^*(G_v)) \leq C_{G_v}(V_v) \leq M$; then $O^2(F^*(G_v)) \leq O^2(F^*(M_v)) = 1$ using 1.1.3.2—that is, $G_v \in \mathcal{H}^e$. Second, suppose that $V_v \trianglelefteq G_v$. Then as $L_v \trianglelefteq G_v$,

$$Aut_{G_v}(V_v) \leq N_{GL(V_v)}(Aut_{L_v}(V_v)) \cong S_6 \cong Aut_{L_vT_v}(V_v),$$

so $G_v = L_vT_vC_{G_v}(V_v) \leq M$, contrary to our choice of v with $G_v \not\leq M$. Therefore V_v is *not* normal in G_v.

Suppose first that $J(T_v) \leq Q$. Let $H_v := C_{G_v}(L_v/O_2(L_v))$. By 12.6.8, $S = T_v$ and $N_G(J(S)) \leq M$. As $Out(A_6)$ is a 2-group, $G_v = L_vSH_v$, so $H_v \not\leq M$; then as $G_v \not\leq N_G(V_v)$, also $H_v \not\leq N_{G_v}(V_v)$. Therefore $V_v < \langle V_v^{H_v} \rangle =: U$. Recall that the core V_v of the permutation module for A_6 is generated by L_v-conjugates of a vector of weight 4 in that module, which is central in $T_v = S \in Syl_2(G_v)$. Then as $G_v \in \mathcal{H}^e, U \leq \Omega_1(Z(O_2(G_v)))$ by B.2.14. As $L_v = O^{3'}(G_v)$ and $Z(L_v/O_2(L_v)) = 1$, H_v is a $3'$-group. Then we conclude from Theorem B.5.6 that U is not a failure of factorization module for $H_v/C_{H_v}(U)$, and hence $J(S) \leq C_{G_v}(U)$ by B.2.7. Now by a Frattini Argument, $H_v = C_{H_v}(U)N_{H_v}(J(S)) \leq C_G(V_v)N_G(J(S)) \leq M$, contrary to our remark that $H_v \not\leq M$.

Therefore $J(T_v) \not\leq Q$. Then by 12.6.9, $L_v = [L_v, J(T_v)]$, so $[R_2(G_v), L_v] = V_v$ by Theorems B.5.6 and B.5.1. Then $V_v \trianglelefteq G_v$, contrary to an earlier reduction. □

LEMMA 12.6.12. *K_v is not quasisimple.*

PROOF. Assume K_v is quasisimple. Then $m_2(K_v) \geq m(V_v) = 5$, so $K_v/Z(K_v)$ is not M_{23}; and if $K_v/Z(K_v) \cong M_{22}$, then $\langle v_0 \rangle = C_{V_v}(L_v) \leq Z(K_v)$ and L_v is an A_6-block. Next as a 2-local of $K_v/Z(K_v)$ contains a quotient of L_v, as $L_v/O_2(L_v) \cong A_6$, and as $[O_2(L_v), L_v] \neq 1$, we eliminate most possibilities for $K_v/Z(K_v)$ in the list of Theorem C (A.2.3), reducing to $K_v/Z(K_v) \cong L_5(2)$, M_{22}, M_{24}, or J_4. As $|S : T_v| \leq 4$ by 12.6.4 with $S \cap K_v \in Syl_2(K_v)$, we conclude that $K_v/Z(K_v) \cong M_{22}$. However $C_V(L_v)$ is of corank 5 in V, so $C_V(K_v) \leq C_V(L_v)$ is of corank at least 5 in V. Hence $V/C_V(K_v)$ is of rank at least 5 in $Aut_{G_v}(K_v)$ and centralizes $V_v/\langle v_0 \rangle$, whereas $\langle v_0\rangle = C_{V_v}(L_v) = C_{V_v}(K_v)$ with $V_v/\langle v_0\rangle$ of rank 4 and self-centralizing in $Aut(K_v)$. This contradiction completes the proof. $\square$

Set $U := \Omega_1(Z(O_2(K_vS)))$. Recall $(K_vS)^* = K_vS/O_2(K_vS)$.

LEMMA 12.6.13. *(1)* $F^*(K_vS) = O_2(K_vS) = C_{K_vS}(U)$.
(2) K_v^* *is simple.*
(3) $V_v \le [U, K_v]$.

PROOF. By 12.6.12, K_v is not quasisimple, while by 12.6.3, K_v^* is quasisimple. Since $L_v/O_2(L_v)$ has trivial center and contains an E_9-subgroup, if $O_2(K_v) < O_{2,3}(K_v)$ then $m_3(L_vO_{2,3}(K_v)) = 3$, contrary to G_v an SQTK-group. Therefore from the list of possibilities in 1.2.1.4b, K_v^* is simple, so $F^*(K_vS) = O_2(K_vS)$ as K_v is not quasisimple. We showed $V_v \in \mathcal{R}_2(L_v)$, so that $L_v \in \mathcal{X}_f$ by A.4.11. By 12.6.4 and 12.6.5, $O_2(K_vS) \le R \le N_S(L_v) \in Syl_2(N_{G_v}(L_v))$, so we may apply A.4.10.3 with L_v, K_v, S in the roles of "X, Y, T" to conclude that $K_v \in \mathcal{X}_f$; then $[R_2(K_vS), K_v] \ne 1$ by A.4.11. As K_v^* is simple, $U = R_2(K_vS)$ and $C_S(U) = O_2(K_vS)$. Similarly by A.4.10.2, $V_v \le [U, K_v]$. $\square$

LEMMA 12.6.14. $J(T_v) \not\le Q$.

PROOF. Assume $J(T_v) \le Q$. By 12.6.8, $S = T_v$, $R = R_v$, and $J(S) = J(R) = J(Q)$, so that $N_G(R) \le N_G(J(R)) \le M = !\mathcal{M}(LT)$. By 12.6.4, $O_2(K_vS) \le R$. Thus $N_{K_v^*}(R^*) = N_{K_v}(R)^* \le M_v^*$. Then as $K_v \not\le M$ by 12.6.11, it follows that $R^* \ne 1$. Since $T_v \in Syl_2(G_v)$, by 1.2.4 the embedding $L_v < K_v$ is described in in A.3.12; so as $R^* \ne 1$, we conclude that $K_v^* \cong M_{22}$ or M_{23}. Then K_v^* has no FF-module by B.4.2, so that $J(S) \le C_S(U)$ by B.2.7. But $C_S(U) = O_2(K_vS)$ by 12.6.13.1, so $K_v \le N_G(J(S)) \le M$, contrary to $K_v \not\le M$. This contradiction completes the proof of 12.6.14. $\square$

LEMMA 12.6.15. U *is an FF-module for* $K_v^*S^*$.

PROOF. By 12.6.14 and 12.6.9, $L_v = [L_v, J(T_v))]$. Thus $K_v = [K_v, J(T_v)]$, so $r_{V,A^*} \le 1$ for some $A \in \mathcal{A}(T_v)$ by B.2.4.1, and hence U is an FF-module for $K_v^*S^*$. $\square$

LEMMA 12.6.16. *Assume* $Z_V \ne 1$ *and* $S \in Syl_2(G)$. *Then*
(1) $N_G(R_v) \not\le M$, *and*
(2) L *is not a block.*

PROOF. By 12.6.14 and 12.6.9, $L = [L, J(T)]$; so as $Z_V \ne 1$ by hypothesis, L centralizes Z by 12.6.1.6. Therefore $C_G(z) \le M = !\mathcal{M}(LT)$ for each $z \in Z^\#$.

As $S \in Syl_2(G)$ by hypothesis, v is 2-central in G, so there is $g \in G$ with $S^g = T$ and hence $v^g \in Z$. Further $L_v^g < K_v^g$ by 12.6.11, so as $G_v^g \le M$, $K_v^g \le O^{3'}(M) = L$ by 12.6.1.5. Also $L \le C_G(Z) \le G_v^g$, and $K_v^g = O^{3'}(G_v^{g\infty})$ by 12.6.3, so $K_v^g = L$. Thus $L_v^g \le L$, and L_v^g is normal in the preimage of $\bar{L}_v^g$ in L by 12.6.4. Hence as $\bar{L}$ is transitive on its subgroups isomorphic to A_6, $L_v^g \in L_v^L$; then as L centralizes Z, without loss $L_v^g = L_v$. Then $R_v^g \in R_v^{N_{LT}(L_v)}$, so we also take $R_v^g = R_v$.

Thus if $N_G(R_v) \le M$, then $g \in N_G(R_v) \le M = N_G(L)$, so $K_v^g = L = L^g$, and hence $K_v = L \le M$, contrary to 12.6.11. Therefore (1) is established.

As $N_G(R_v) \not\le M \ge N_G(Q)$, $Q < R_v$. Then as we saw at the start of the proof of Theorem 12.6.2, $\bar{R}_v = \langle (1,2)\rangle$, so that $LR_v = LT$. As $(K_vS)^g = LT$ and $L_v^g = L_v$, $R = O_2(N_{LT}(L_v)) = R_v$, and hence $R^g = R_v^g = R_v = R$. Since it only remains to establish (2), we may assume that L is a block. Let $a := g^{-1}$ Notice that $V \trianglelefteq R$, so that also $V^a \trianglelefteq R$.

Suppose first that $[V, V^a] = 1$. Then $V^a \leq C_R(V) = Q$, so as L is a block, $[VV^a, L] \leq [Q, L] \leq V \leq VV^a$. Similarly $[VV^a, K_v] \leq VV^a$. Therefore as $LR = LT$, $K_v \leq M = !\mathcal{M}(LT)$, contrary to 12.6.11.

Thus $[V^a, V] \neq 1$, so as $V^a \leq R = R_v$, $\bar{V}^a = \langle(1,2)\rangle$, and hence $[V, V^a] = \langle e_{1,2}\rangle$. Since $Z_V \neq 1$ by hypothesis, v is chosen to have weight 2, so $v = e_{1,2}$. By symmetry, $[V^a, V] = \langle v^a\rangle$, so $v = v^a = v^{g^{-1}}$, and hence $g \in G_v$, impossible as v^g centralizes L. □

LEMMA 12.6.17. *$N_G(R_v) \not\leq M$.*

PROOF. Assume that $N_G(R_v) \leq M$. Then as $R_v \in Syl_2(C_M(L_v/O_2(L_v)))$, $R_v = R$. Hence $N_{K_v^*}(R^*) = N_{K_v}(R_v)^* \leq M_v^*$, so $R^* \neq 1$ as $K_v \not\leq M$ by 12.6.11. In view of 12.6.15, K_v^* appears in the list of Theorem B.4.2, so since $K_v^*S^*$ has a nontrivial 2-subgroup R^* such that $L_v^* \trianglelefteq N_{K_v^*}(R^*) \geq T_v^*$ with $L_v^*/O_2(L_v^*) \cong A_6$ and $|S^* : T_v^*| \leq 4$, we conclude that $K_v^*S^* \cong S_8$ and R^* induces a transposition on K_v^*. Then $|S^* : T_v^*| = 4 = |T : T_v|$, so $S \in Syl_2(G)$. By 12.6.13.3, $V_v \leq [U, K_v] =: U_v$. Then from Theorem B.5.1.1, $U_v/C_{U_v}(K_v)$ is the 6-dimensional quotient of the core of the permutation module for K_v^*. Further $[C_{V_v}(L_v), K_v] \neq 1$, as v_0 is of weight 6 in both U_v and V. But v is central in G_v, so that $v \notin C_{V_v}(L_v) = \langle v_0\rangle$, and hence $Z_V \neq 1$. Now 12.6.16.1 supplies a contradiction, completing the proof of the lemma. □

LEMMA 12.6.18. *(1) L is an A_8-block.*
(2) K_v is an A_7-block.
(3) $Z_V \neq 1$.
(4) $LT = LR_v$.

PROOF. By 12.6.17, $N_G(R_v) \not\leq M$. So as $N_G(Q) \leq M$, $Q < R_v$, and hence $\bar{R}_v = \langle(1,2)\rangle$, so (4) holds. Then $\bar{R}_v \leq \bar{X} \leq \bar{M}$ for some $\bar{X} \cong S_3$—so either there is $1 \neq C$ char R_v with $C \trianglelefteq X$, or we may apply C.1.29 to $R_v \in Syl_2(X)$, to conclude that $O^2(X)$ is an A_3-block. In the former case, $C \trianglelefteq \langle X, C_{LT}(v)\rangle = LT$, so $N_G(R_v) \leq N_G(C) \leq M = !\mathcal{M}(LT)$, contrary to 12.6.17. Thus X an A_3-block, so L that (1) holds and L_v is an A_6-block. Further as L_v is trivial on R/R_v, V_v is the unique non-central chief factor for L_v on R, so V_v is the unique noncentral chief factor for L_v on $O_2(K_vS)$. Thus K_v is also a block. By 12.6.15, U is an FF-module for $K_v^*S^*$, so by Theorem B.4.2, K_v^* is either of Lie type and characteristic 2, or A_7. (The case of $\hat{A}_6$ is ruled out as $L_v/O_2(L_v) \cong A_6$). Indeed as $SL_3(2^n)$ and $G_2(2^n)$ have no subgroup X with $X/O_2(X) \cong A_6$, $K_v^* \cong Sp_4(2^n)$, A_7, A_8, or $L_5(2)$. As T_v^* acts on L_v^* and $|S^* : T_v^*| \leq 4 = |T : T_v|$, K_v^* is A_7, A_8, or $L_5(2)$. Furthermore in the latter two cases $|S : T_v| = 4 = |T : T_v|$, so that $S \in Syl_2(G)$, and we calculate that $R = R_v$, and $L_v^* \cong A_6$ or A_6/E_{16}, respectively. In particular K_v^* is not $L_5(2)$, since in that group $[O_2(L_v^*), R^*] \neq 1$, whereas L_v has a unique noncentral 2-chief factor. Then as $V_v \leq [U, K_v]$, Theorem B.5.1.1 says $U/C_U(K_v)$ is the natural module for $K_v^* \cong A_7$ or A_8. Now we argue as in the proof of 12.6.17: In either case $[v_0, K_v] \neq 1$, so as $v \in Z(G_v)$, $v \neq v_0$ and hence (3) holds. Finally if K_v is an A_8-block, we showed $S \in Syl_2(G)$; then (1) contradicts 12.6.16.2. Thus K_v is an A_7-block, so (2) holds. □

We are now in a position to complete the proof of Theorem 12.6.2. By 12.6.18.2, K_v is an A_7-block. Therefore $S = T_v$, since A_6 is of odd index in A_7. Hence $R_v = R = C_S(L_v/O_2(L_v)) = O_2(K_vS)$. Since the natural module for A_7 has trivial 1-cohomology by I.1.6.1, $O_2(K_vS) = UC_S(K_v)$ by C.1.13.b. Then from the structure of the A_7-module, $C_S(L_v) = C_S(K_v) \times \langle v_0 \rangle$. By 12.6.18.4, $LT = LS$, so $T_0 := C_T(L) \cap C_S(K_v) \trianglelefteq LT = LS$, and hence $T_0 = 1$ as $K_v \not\leq M$ by 12.6.11. Then as $C_T(L) \leq C_S(L_v)$, $|C_T(L)| \leq |C_S(L_v) : C_S(K_v)| = 2$, so $C_T(L) = Z_V$ as $Z_V \neq 1$ by 12.6.18.3. Therefore as the 1-cohomology of the natural module for A_8 is 1-dimensional by I.1.6.1, we conclude from C.1.13.b that either $Q = V$, or Q is isomorphic to the 8-dimensional permutation module P as an L/V-module. In either case $R_v = \langle r \rangle Q$ with $\bar{r} = (1,2)$ and $[R_v, r] = [Q, r] = \langle v \rangle$. Also $|R_v| = 2|Q| = 2^8$ or 2^9. On the other hand as $R_v = C_S(K_v) \times U$ with U the 6-dimensional permutation module for $K_v/O_2(K_v) \cong A_7$, $r \in C_{R_v}(L_v) = \langle v_0 \rangle \times C_S(K_v)$. In particular r centralizes U, so as $[R_v, r] = \langle v \rangle$, $[C_S(K_v), r] = \langle v \rangle$. As R_v is nonabelian, but U is central in in R_v, we conclude $C_S(K_v)$ is nonabelian, so $|C_S(K_v)| \geq 8$ and hence $|R_v| \geq 2^9$. Then using our earlier bounds, $|R_v| = 2^9$, with $Q \cong P \cong E_{2^8}$ and $R_v \cong D_8 \times E_{64}$. As $R_v = O_2(K_vS)$ and $K_v = O^2(K_v)$, K_v acts on both E_{2^8}-subgroups of R_v, so that $K_v \leq N_G(Q) \leq M$, for our usual contradiction to 12.6.11.

This contradiction completes the proof of Theorem 12.6.2.

In the remainder of the subsection, we show $Z_V = 1$.

LEMMA 12.6.19. *(1) L controls G-fusion in V.*

(2) If $Z_V \neq 1$ and v_4 is of weight 4 in V, then $|C_G(v_4) : C_M(v_4)|$ is odd.

PROOF. Suppose $Z_V = 1$. Then (2) is vacuous, and L has two orbits on $V^\#$, consisting of the singular and nonsingular vectors, with the singular vectors 2-central. By Theorem 12.6.2, the nonsingular vectors are not 2-central, so (1) holds in this case.

Thus we may assume $Z_V \neq 1$. In this case, L has four orbits on $V^\#$, with representatives v_2, v_4, v_6, v_8, where v_m is of weight m. By Theorem 12.6.2, $|C_G(v_m)|_2 = |T|/4$ for $m = 2, 6$. Notice v_8 is 2-central, and $|C_M(v_4)|_2 = |T|/2$. Assume that (1) fails. Then it follows that $v_4^g = v_8$ for some $g \in G$. We may choose v_4 so that $V_1 = \langle v_4, v_8 \rangle$; thus $T_4 := C_T(v_4) \in Syl_2(C_M(v_4))$ and $O^2(C_L(v_4)) = O^2(N_L(V_1)) = L_1$. Then $L_1^g \leq C_G(v_8) \leq M$, so $L_1^g \leq L$ by 12.6.1.5, and we may take $T_4^g \leq T$. As $|T : T_4| = 2$ and L_1T_4/R_1 is of index at most 2 in $S_3 \times S_3$, we conclude $L_1^g \in L_1^L$; thus as L centralizes v_8, we may take $L_1^g = L_1$. But then $R_1^g = O_2(L_1T_4)^g = R_1$. Now using 12.6.1.4, $g \in N_G(R_1) \leq M$, contrary to $v_4 \notin v_8^M$. Hence (1) is established.

Suppose finally that (2) fails. Then $v_4^g \in Z$ for some $g \in G$. If $[V, J(T)] = 1$, then $N_G(J(T)) \leq M$ by 3.2.10,1; and by Burnside's Fusion Lemma, $N_G(J(T))$ controls fusion in $Z(J(T)) \geq VZ$, contrary to v_4 not 2-central in M. Thus we may take $L = [L, J(T)]$, so as $Z_V \neq 1$, L centralizes Z by 12.6.1.6. Hence $L_1^g \leq C_G(v_4^g) \leq M = !\mathcal{M}(LT)$. We now repeat the argument of the previous paragraph to obtain the same contradiction, completing the proof of (2). □

LEMMA 12.6.20. *Let $\mathcal{S}$ be the set of vectors in V of weight 4. If $g \in G - N_G(V)$ with $V \cap V^g \neq 1$, then*

(1) $V \cap V^g \subseteq \mathcal{S}$, so $m(V \cap V^g) \leq 3$.

(2) $V^g \in V^{C_G(v)}$ *for each* $v \in V \cap V^g$.

(3) $r(G, V) \geq 3$, *and* $r(G, V) \geq 4$ *if* $Z_V \neq 1$.

(4) If $1 \neq [V, V^g] \leq V \cap V^g$, *then* $Z_V = 1$, $V \cap V^g$ *is a totally singular* 3*-subspace of* V, *and* $\bar{V}^g$ *is the unipotent radical of an* $L_3(2)/E_8$ *parabolic.*

PROOF. Part (2) follows from A.1.7.1 in view of 12.6.19.1. If $v \in V$ is of weight 8 then $G_v = M$ as we saw at the start of the section, while if v has weight 2 or 6, then $G_v \leq M$ by Theorem 12.6.2. So by 12.6.19.1, we may apply A.1.7.2 to see that each element of weight 2, 6, or 8 lies in a unique conjugate of V. Then (1) follows, and (1) implies (3).

Assume the hypotheses of (4). Interchanging V and V^g if necessary, we may assume $m(\bar{V}^g) \geq m(V/C_V(V^g))$. Then by B.1.4.4, $\bar{V}^g$ contains a member of $\mathcal{P}(\bar{T}, V)$, so the possibilities for $\bar{V}^g$ are described in the discussion near the beginning of the proof of 12.6.1. As $[V, V^g] \leq V \cap V^g$, (1) says $[V, V^g]^\# \subseteq \mathcal{S}$, and the only possibility satisfying this restriction is that given in (4). □

LEMMA 12.6.21. $C_G(v) \not\leq M$ *for each* $v \in V^\#$ *of weight* 4.

PROOF. As the groups in conclusions (2)–(4) of Theorem 12.2.13 do not have a member $L \in \mathcal{L}_f^*(G, T)$ with $\bar{L} \cong A_8$ and $V/C_V(L)$ the permutation module, conclusion (1) of 12.2.13 holds: $G_v \not\leq M$ for some $v \in V^\#$. By 12.6.20, $G_v \leq M$ for v of weight 2, 6, or 8, so the lemma holds. □

LEMMA 12.6.22. *If* $Z_V \neq 1$, *then*

(1) $W_0 := W_0(T, V)$ *centralizes* V.

(2) If $m(V^g/V^g \cap M) \leq 1$ *for some* $g \in G$, *then* $V^g \leq N_G(V)$.

(3) $w(G, V) > 1$.

PROOF. Notice (1) and (2) imply (3), so it remains to prove (1) and (2). Assume $Z_V \neq 1$. Then $M = N_G(V)$ by 12.2.2.3. Suppose $A := V^g \cap M$ with $\bar{A} \neq 1$. Assume $k := m(V^g/A) \leq 1$, and if (1) fails, choose $k = 0$. Thus $V \not\leq N_G(V^g)$ if $k = 1$: for otherwise by assumption (1) does not fail, so that $V \leq C_G(V^g)$, contradicting $\bar{A} \neq 1$. Now by 12.6.20.3 and E.3.4, $m(\bar{A}) \geq r(G, V) - k \geq 4 - k$. Similarly using 12.6.20.3 as in E.3.32,

$$U := \langle C_V(B) : B \leq A \text{ and } m(V^g/B) \leq 3 \rangle \ \leq \ N_G(V^g),$$

so $[A, U] \leq V \cap V^g$. Now $\bar{L}\bar{T}$ is A_8 or S_8, and the maximal elementary abelian 2-subgroups of S_8 are

(i) $D_1 \cong E_8$ regular on Ω.

(ii) $D_2 \cong E_{16}$ with two orbits of length 4.

(iii) $D_3 \cong E_{16}$ with one orbit of length 4, and two of length 2.

(iv) $D_4 \cong E_{16}$ with four orbits of length 2.

If $k = 0$, then $m(\bar{A}) = 4$, so $\bar{A} = D_i$ for $i = 2, 3, 4$; while if $k = 1$, then $m(\bar{A}) \geq 3$, so either $\bar{A} = D_i$ for $i = 1, 2, 3, 4$ or $\bar{A}$ is of index 2 in D_j for $j = 2, 3, 4$. In each case we find that $[A, U]$ contains a vector of weight 2 or 8. This contradicts 12.6.20.1, as $[A, U] \leq V \cap V^g$, so the proof is complete. □

LEMMA 12.6.23. $C_V(L) = 1$. *Thus* V *is the* 6*-dimensional orthogonal module for* $\bar{L}$.

PROOF. Assume $Z_V = C_V(L) \neq 1$. Let $H \in \mathcal{H}_*(T,M)$, $K := O^2(H)$, $V_H := \langle Z^H \rangle$, and $H^* := H/C_H(V_H)$. By 12.6.20.3 and 12.6.22.3, $\min\{r(G,V), w(G,V)\} > 1$, so each solvable member of $\mathcal{H}(T)$ is contained in M by E.3.35.1 and E.1.13. In particular H is not solvable. By 12.2.9.1, $C_G(Z) \leq M$, so by Corollary 12.3.2, $1 \neq V_K := [V_H, K]$ is the sum of at most two A_5-modules for $K^* \cong A_5$. As $H = KT$, $O_2(H) = C_H(V_H) = C_T(V_H)$. Let H_M^* be the Borel subgroup of H^* over T^*; then $H_M = H \cap M$ by 3.3.2.4. Now $O^2(H_M) = O^{3'}(H_M) \leq O^{3'}(M) = L$ by 12.6.1.5.

Next if $W_0 := W_0(T,V) \leq C_T(V_H)$, then $H \leq N_G(W_0) \leq M$ by 12.6.22.1 and E.3.34.2, contrary to $H \not\leq M$. Therefore $W_0 \not\leq C_T(V_H)$, so there is $A := V^g \leq T$ with $A^* \neq 1$. Now by 12.6.22.1, $A \leq W_0(T,V) \leq C_T(V) = Q$. Thus as $O^2(H_M) \leq L$, $H_M \leq LT$, so that $A \leq O_2(H_M)$.

Let $B := A \cap O_2(H)$; then $m(A/B) = m(A^*) =: k \leq 2 = m_2(H^*)$. As $k \leq 2$, $V_H \leq C_G(B) \leq N_G(A)$ by 12.6.20.3, so $[A, V_H] \leq A \cap V_H$. However $O_2(H_M^*)$ is not quadratic on V_H, so $A^* < O_2(H_M^*)$. Therefore $k = 1$, so V_H induces a group of transvections with axis B on A. Thus $|V_H : C_{V_H}(A)| = 2$ from the action of $\bar{L}\bar{T} \cong S_8$ on V. This is impossible, since as $A^* \leq O_2(H_M^*)$, $m(V_H/C_{V_H}(A)) > 1$ from the action of involutions in A_5 on its permutation module. This contradiction completes the proof of 12.6.23. □

12.6.2. The amalgam setup, and the elimination of V_H nonabelian. With 12.6.23 in place, we can begin to use some of the techniques from section F.9, which are more representative of the arguments for the $\mathbf{F}_2$-case in the three chapters of this part.

By 12.6.23, $Z \cap V = V_1$ is of order 2. Let z be the generator of V_1. Recall from Notation 12.2.5 that $G_z = C_G(z)$ and $\tilde{G}_z := G_z/V_1$. Now z is of weight 4 in V, so $G_z \not\leq M$ by 12.6.21. Recall that $L_1 = O^2(N_L(V_1)) = O^2(C_L(z)) = L_z$, V_5 is the 5-subspace $V_1^\perp$ of V, $\bar{L}_1 \cong E_9/E_{16}$, and $\bar{L}_1\bar{T}/O_2(\bar{L}_1\bar{T})$ acts on $\tilde{V}_5$ as $\Omega_4^+(2)$ or $O_4^+(2)$ on its natural module.

We now make a definition which, will be repeated in many later sections of the chapters on the $\mathbf{F}_2$ case: let

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \not\leq M \text{ and } H \leq G_z\}.$$

As $G_z \not\leq M$, $G_z \in \mathcal{H}_z$, and hence $\mathcal{H}_z \neq \emptyset$.

For the remainder of the section, H will denote an element of $\mathcal{H}_z$.

Set $Q_H := O_2(H)$, $U_H := \langle V_5^H \rangle$, $V_H := \langle V^H \rangle$, and $H^* := H/Q_H$.

LEMMA 12.6.24. *(1) L_1T is irreducible on $\tilde{V}_5$.*
(2) $U_H \leq O_2(H)$. In particular, $V_5 \leq O_2(G_1)$.
(3) $V \leq O_2(G_2)$.
(4) $G_5 \leq N_G(V) = M_V$.
(5) If $g \in G$ with $V_1 < V \cap V^g$, then $\langle V, V^g \rangle$ is a 2-group.
(6) $C_G(\tilde{V}_5) = C_G(V)R_1 = C_M(V)R_1$ and $C_G(V_5) = C_G(V) = C_M(V)$.
(7) Hypothesis F.9.1 is satisfied for each $H \in \mathcal{H}_z$, with V_5 in the role of "V_+".
(8) $Q_H = C_H(\tilde{U}_H)$, so $H^* = Aut_H(\tilde{U}_H)$.

PROOF. Part (1) is standard and an easy calculation. Since $m(V_5) = 5 > 3$, (4) follows from 12.6.20.1. By (4), $C_G(\tilde{V}_5) \leq M$; so as $\bar{R}_1 = C_{\bar{M}_V}(\tilde{V}_5)$ and $\bar{M}_V$ contains no transvection with axis V_5, (6) holds.

We next prove (7). Most parts of Hypothesis F.9.1 are easy to check: For example, (4) implies hypothesis (c) of F.9.1, (1) implies (b), and (d) holds as $M = !\mathcal{M}(LT)$ but $H \not\leq M$. Assume the hypothesis of (e); as the conclusion of (e) holds trivially if $[V, V^g] = 1$, we may assume $1 \neq [V, V^g]$. Then 12.6.20.4 says $\bar{V}^g$ is the unipotent radical $\bar{R}$ of the stabilizer of some 3-subspace of V, while (6) and the hypothesis of (e) that V^g centralizes $\tilde{V}_5$ forces $\bar{V}^g \leq \bar{R}_1$. This contradicts $\bar{R} \not\leq \bar{R}_1$, and completes the proof of (e). As $H \in \mathcal{H}(T)$, $H \in \mathcal{H}^e$ by 1.1.4.6. If X is a normal subgroup of H centralizing $\tilde{V}_5$, then by (6) and Coprime Action, $O^2(X)$ centralizes V, so $[L, O^2(X)] \leq O_2(L)$. Hence LT normalizes $O^2(XO_2(L)) = O^2(X)$. Then if X is not a 2-group, $H \leq N_G(O^2(X)) \leq M = !\mathcal{M}(LT)$, contrary to $H \not\leq M$. Therefore $X \leq O_2(H) = Q_H$, so that hypothesis (a) of F.9.1 holds. Thus we have established (7).

Next (7) and parts (1) and (3) of F.9.2 imply (2) and (8), respectively. By (2), $V_5 \leq O_2(G_1) \cap C_G(V_2)$, so as $C_G(V_2) \leq G_1$, $V_5 \leq O_2(C_G(V_2)) \leq O_2(G_2)$. Then $V = \langle V_5^{L_2} \rangle \leq O_2(G_2)$. Hence (3) holds. If $g \in G - M_V$ with $V_1 < V \cap V^g$, then by 12.6.20.1, $V \cap V^g$ is totally singular; so without loss $V_2 \leq V^g$, so that $V^g \leq G_2$. But then $\langle V, V^g \rangle$ is a 2-group by (3). Hence (5) holds. □

LEMMA 12.6.25. *The following are equivalent:*

(1) U_H is abelian.
(2) $V \leq Q_H$.
(3) $V \leq C_G(U_H)$.
(4) V_H is abelian.

PROOF. Parts (2) and (3) are equivalent by F.9.3, which applies by 12.6.24.7. Similarly the hypotheses of F.9.4.3 are satisfied by 12.6.24.6, so (1), (2), and (4) are equivalent by F.9.4.3. □

Observe since $H \leq G_z = G_1$, that if V_H is nonabelian, then V_{G_1} is also nonabelian.

In the non-quasithin shadows mentioned earlier (such as $\Omega_8^+(2)$), V_H is nonabelian. Hence these shadows are eliminated in the next result:

LEMMA 12.6.26. *V_H is abelian.*

PROOF. Assume that V_H is nonabelian. Then by 12.6.25, U_H also is nonabelian and $V^* \neq 1$. Notice the hypotheses of F.9.5 are now satisfied, so in particular V^* is of order 2 and $[\tilde{U}_H, V^*] = \tilde{V}_5$ by parts (1) and (2) of F.9.5. By 12.6.24.5, the hypothesis of part (5) of F.9.5 is satisfied. Also $m(V) = 6$, and by 12.6.24.6, $C_H(V_5) = C_H(V)$, so the hypotheses of LL.5.6F.9.5.6ii are satisfied, and hence we can appeal to that result also. For example if $g^* \in H^*$ such that $I^* := \langle V^*, V^{*g} \rangle$ is not a 2-group, then by F.9.5.5, I^* is faithful on $U_I := V_5 V_5^g$; and by F.9.5.6ii, $U_I \cong Q_8^4$ and $I^* \cong D_6$, D_{10}, or D_{12}. Therefore elements of odd order in H^* inverted by V^* are of order 3 or 5.

We show first that $V \leq O_2(C_H(L_1/O_2(L_1)))$: Assume otherwise. Then by the Baer-Suzuki Theorem, for some $g \in C_H(L_1/O_2(L_1))$, $I^* := \langle V^*, V^{*g} \rangle$ is not a 2-group. By the previous paragraph, I^* is faithful on $U_I \cong Q_8^4$ and I^* is dihedral of order $2m$, $m = 3$, 5, or 6. Also as $g \in C_H(L_1/O_2(L_1))$, as V^* centralizes L_1^*, and as $O_2(L_1^*) = C_{L_1^*}(\tilde{V}_5)$, we conclude that I^* centralizes L_1^* and L_1 acts on V_5^g with $O_2(L_1^*) = C_{L_1^*}(\tilde{V}_5^g)$. Let $\widehat{IL_1} := Out_{IL_1}(U_I)$. Then $\hat{L}_1 \cong E_9$ is faithful on $\tilde{V}_5$

and $\tilde{V}_5^g$, so from the structure of $\hat{O} := Out(U_I) \cong O_8^+(2)$, $\hat{I} \le C_{\hat{O}}(\hat{L}_1) \cong S_3 \times S_3$. But now $m_3(H) > 2$, whereas H is an SQTK-group. This contradiction shows that $V \le O_2(C_H(L_1/O_2(L_1)))$.

We next show that V^* centralizes $F(H^*)$. Assume otherwise. Let $P \in Syl_3(L_1)$ and set $C^* := C_{O_3(H^*)}(P^*)$. If $[O_3(H^*), V^*] \ne 1$, then as V^* centralizes P^*, V^* inverts a subgroup X^* of C^* of order 3 by the Thompson $A \times B$ Lemma; but then $m_3(P^*X^*) = 3$, contrary to H an SQTK-group. Therefore V^* centralizes $O_3(H^*)$, so we may assume that $[V^*, O_p(H^*)] \ne 1$ for some prime $p > 3$. We saw that elements of odd order in H^* inverted by V^* are of order 3 or 5, so $p = 5$. Let Y^* be a supercritical subgroup of $O_5(H^*)$. By the previous paragraph $V \le O_2(C_H(L_1/O_2(L_1)))$, so $O_5(H^*) \not\le C_{H^*}(L_1^*)$. Therefore we conclude from A.1.25 that Y^* is E_{25} or 5^{1+2} and P is irreducible on $Y^*/\Phi(Y^*)$. Thus as V^* centralizes P^*, V^* inverts $Y^*/\Phi(Y^*)$. If $Y^* \cong 5^{1+2}$, then a faithful chief section W for Y^*V^* on $\tilde{U}_H$ is of dimension 20 over $\mathbf{F}_2$, and $m([W, V^*]) \ge 8$. If $Y^* \cong E_{25}$, then a faithful Y^*V^*-chief section W for P^*Y^* is of dimension 12 over $\mathbf{F}_2$ and $m([W, V^*]) = 6$. So in any case $m([\tilde{U}_H, V^*]) \ge 6$, whereas we saw $[\tilde{U}_H, V^*] = \tilde{V}_5$ is of rank 4. This contradiction establishes the claim that V^* centralizes $F(H^*)$.

So as $O_2(H^*) = 1$, $[K^*, V^*] \ne 1$ for some $K \in \mathcal{C}(H)$ such that K^* is a component of H^*. Then $K \not\le M$ as $V \le O_2(M)$. As $V^* \le Z(T^*)$, V^* normalizes K^*, so that $K^* = [K^*, V^*]$. Further $L_1 = O^2(L_1)$ normalizes K by 1.2.1.3. As $V \le O_2(C_H(L_1/O_2(L_1)))$, $K^* \not\le C_{H^*}(L_1^*)$, so that $K^* = [K^*, L_1^*]$. Set $X := KL_1V$ and $\hat{X} := X/C_X(K^*)$. By F.9.5.3, $C_{H^*}(V^*) = N_H(V)^*$. Further $N_H(V) \le H \cap M \le N_H(L_1)$ and $C_X(\hat{V}) = C_K(\hat{V})L_1V$. Now $C_K(\hat{V})$ centralizes $V^*Z(K^*)/Z(K^*)$, so as $O_2(K^*) = 1$, $C_{\hat{K}}(\hat{V}) = \widehat{C_K(V^*)} \le \widehat{K \cap M}$, and then

$$\hat{L}_1 \le O_{2,3}(C_{\hat{X}}(\hat{V})) \text{ with } \hat{V} \text{ of order 2 in } Z(\hat{T}) \text{ and } 3 \in \pi(\hat{L}_1). \qquad (*)$$

Since all elements of odd order in $\hat{K}$ inverted by $\hat{V}$ are of order 3 or 5, we conclude $\hat{K}$ is not $Sz(2^n)$. Hence $m_3(K) = 1$ or 2.

Suppose first that $m_3(K) = 2$. Then as $m_3(P) = 2 = m_3(PK)$, either P is faithful on K^*, or $1 \ne P \cap O_{2,Z}(K)$, so that by 1.2.1.4b, $K^* \cong SL_3^\epsilon(q)$, $\hat{A}_6$, $\hat{A}_7$, or $\hat{M}_{22}$. Further in the latter case, $K/O_2(K) \cong \hat{A}_7$ or $\hat{M}_{22}$ by (*).

Suppose that $K^* \cong \hat{A}_7$. Then $\hat{R}_1 = O_2(\hat{L}_1\hat{T}) \cong E_4$, while $N_K(R_1) \le M$ by 12.6.1.4, so $O^{3'}(N_K(R_1)) \le O^{3'}(M) = L$ by 12.6.1.5. Thus $O^{3'}(N_K(R_1)) \le O^2(C_L(z)) = L_1$. This is impossible, as $N_K(R_1)$ has Sylow 3-group 3^{1+2}, while $E_9 \cong P \in Syl_2(L_1)$.

Suppose next that $K^* \cong \hat{M}_{22}$. As $H \le G_z$, $H \cap M = N_H(V)$ by 12.2.6, so $(M \cap K)^* = N_K(V)^* = C_K(V^*)$, and hence $(M \cap K)^* \cong (S_3 \times \mathbf{Z}_2)/(\mathbf{Z}_3 \times Q_8^2)$. We saw $\tilde{V}_5 = [\tilde{U}_H, V^*]$ is of rank 4, so we conclude from H.12.1.3 that $\tilde{U}_K := [\tilde{U}_H, K]$ is the 12-dimensional irreducible for K^*. Choose v_2 of weight 2 in V_5; by parts (5) and (7) of H.12.1, $C_{\hat{K}}(v_2) \cong S_5/E_{16}$ or A_5/E_{16}. In particular $C_K(v_2)$ involves A_5, so $C_K(v_2) \not\le M$, as we saw earlier that $(M \cap K)^*$ is solvable. But this contradicts Theorem 12.6.2.

We have shown that if $m_3(K) = 2$ then P is faithful on K^*, and hence also on $\hat{K}$. Then $m_3(\hat{P}) = 2$, so that $m_3(O_{2,3}(C_{\hat{X}}(\hat{V})) = 2$ by (*). But no simple group $\hat{K}$ of 3-rank 2 in the list of Theorem C satisfies this restriction. This contradiction completes the treatment of the case $m_3(K) = 2$.

Therefore $m_3(K) = 1$. As $m(P) = 2 = m_3(PK)$, we conclude that $P_K := P \cap K$ is of rank 1. Again inspecting the list of Theorem C, and using (*), we conclude that $\hat{K}$ is $L_2(p^e)$ for some odd prime p with $p^e \equiv \pm 1 \mod 12$. Recalling that elements of odd order in $\hat{K}$ inverted by $\hat{V}$ are of order 3 or 5, we reduce to the case $\hat{K} \cong L_2(p^e)$, with $p \neq 23$ or 25, and $\hat{V}$ inverts no element of order p if $p > 5$, so in fact $p^e \equiv -1 \mod 12$.

Set $H_0 := \langle K, L_1T \rangle$, so that $H_0 \in \mathcal{H}_z$ as $K \not\leq M$. As $K^* = [K^*, V^*]$, $V \not\leq O_2(H_0)$, so U_{H_0} is nonabelian by 12.6.25. Thus replacing H by H_0, we may assume $H = \langle K, L_1T \rangle$.

Next as $K^* \cong L_2(p^e)$, K^*V^* is generated by 3 conjugates of V^*. Thus as $\tilde{V}_5 = [\tilde{U}_H, V^*]$ is of rank 4, for each nontrivial chief section W for K^* on $\tilde{U}_H$, $m([W, K^*]) \leq 3m([W, V^*]) \leq 12$. Thus p^e divides $|L_{12}(2)|$, so as $p^e \neq 23$ or 25 and $p^e \equiv -1 \mod 12$, it follows that $p^e = 11$. But as 11 does not divide $|L_9(2)|$, the smallest nontrivial irreducible for K^* is of rank at least 10, so we conclude $\tilde{U}_K := [\tilde{U}_H, K] \in Irr_+(\tilde{U}_H, K)$, $\tilde{V}_5 = [\tilde{U}_K, V^*]$, and $10 \leq m(\tilde{U}_K)$. Thus if $K \neq K^t$ for some $t \in T$, then K^t centralizes $\tilde{U}_K$. However as T acts on V_5 and K does not centralize $\tilde{V}_5$, neither does K^{*t}, a contradiction. Thus T normalizes K, so $K \trianglelefteq H$ by 1.2.1.3, and so $H = \langle K, L_1T \rangle = KL_1T = KPT$.

Next $P = P_C \times P_K$ where $P_C := C_P(K^*)$ and $P_K = P \cap K$ are of order 3. As $H = KPT$ and $O_2(H^*) = 1$, $H^* = K^*P_C^*T^*$ and $P_C^* = O^2(C_{H^*}(K^*) \trianglelefteq H^*$. In particular P_C^* is T-invariant, so $\tilde{V}_5 = [\tilde{V}_5, P_C]$ since L_1T is irreducible on $\tilde{V}_5$. Then as P_C^* and K^* are normal in H^*, $\tilde{V}_5 \leq \tilde{U}_K$, and $U_H = \langle V_5^H \rangle$, $\tilde{U}_K = \tilde{U}_H = [\tilde{U}_H, P_C]$. As $\mathcal{U}_{C_{H^*}(V^*)}(P^*, 2) \subseteq C_{H^*}(P^*)$ from the structure of $Aut(L_2(11))$, $O_2(L_1^*) = 1$, and so $O_2(L_1) \leq Q_H$. Next $[Q_H, V] \leq Q_H \cap V = V_5 \leq U_H$, so as $K = [K, V]$, $[Q_H, K] \leq U_H$. As P_K^* is inverted by a conjugate of V^*, it follows from the first paragraph of the proof that $[U_H, P_K] \cong Q_8^4$. Then as $O_2(L_1) \leq Q_H$ by the previous paragraph, $[O_2(L_1), P_K] \leq [Q_H, K] \leq U_H$, so that $[O_2(L_1), P_K] = [U_H, P_K] \cong Q_8^4$ is of order 2^9. Therefore as $[V, P_K] \cong E_{16} \cong [\bar{R}_1, \bar{P}_K]$, we conclude that L is an A_8-block and $O_2(L_1) = [O_2(L_1), P_K] \cong Q_8^4$. We saw $\tilde{U}_H = [\tilde{U}_H, P_C]$, so $U_H = [U_H, P_C] \leq O_2(L_1)$. Thus $2^{11} \leq |U_H| \leq |O_2(L_1)| = 2^9$. This contradiction completes the proof of 12.6.26. □

12.6.3. Restrictions on H, and the final contradiction. In this section, we use machinery from section F.9 to handle the case V_H abelian. The same approach will be used many times in the remainder of our treatment of groups over $\mathbf{F}_2$.

LEMMA 12.6.27. *If $g \in G$ with $V \cap V^g \neq 1$, then $[V, V^g] = 1$.*

PROOF. By 12.6.20.1, we may assume $z \in V \cap V^g$. Then by 12.6.20.2, we may take $g \in G_z$. Applying 12.6.26 to G_z in the role of "H", we conclude $V_H = \langle V^{G_z} \rangle$ is abelian, so the lemma holds. □

LEMMA 12.6.28. *(1) $\mathcal{A}_3(\bar{M}_V, V) = \emptyset$.*

(2) If $\bar{B} \in \mathcal{A}_2(\bar{M}_V, V)$, then $m(\bar{B}) \geq 3$ and there exists $\bar{D}$ of index 8 in $\bar{B}$ with $|\mathcal{E}(\bar{B}, \bar{D})| > 2$, where

$$\mathcal{E}(\bar{B}, \bar{D}) := \{\bar{E} \leq \bar{B} : m(\bar{E}/\bar{D}) = 1 \text{ and } C_V(\bar{E}) > C_V(\bar{B})\}.$$

(3) $W_0 := W_0(T, V)$ centralizes V, so $N_G(W_0) \leq M$.

(4) $W_1(T, V)$ centralizes V, so $w(G, V) > 1$.

PROOF. Recall $\bar{M}_V$ acts as A_8 or S_8 on the set Ω of eight points. Thus if $\bar{i}$ is an involution in $\bar{M}_V$, the $\bar{M}_V$-conjugacy class of $\bar{i}$ is determined by the number $n(\bar{i})$ cycles of $\bar{i}$ of length 2 on Ω. Thus $n(\bar{i}) = 1, 2, 3, 4$, and we check in the respective cases that $\bar{i}$ is of Suzuki type (cf. Definition E.2.6) b_1, c_2, b_3, a_2 on the orthogonal 6-space V. In particular,

$$C_V(\bar{i}) \neq C_V(\bar{j}) \text{ for involutions } \bar{j} \neq \bar{i}. \qquad (*)$$

We first prove (1) and (2), so we assume that either $\bar{A} \in \mathcal{A}_3(\bar{M}_V, V)$ or $\bar{B} \in \mathcal{A}_2(\bar{M}_V, V)$. Then $m(\bar{A}) > 3$ by (*), so that $m(\bar{A}) = 4 = m_2(\bar{L}\bar{T})$. Similarly $m(\bar{B}) \geq 3$.

Now the possibilities for $\bar{A}$ of rank 4 are described in cases (ii)–(iv) in the proof of 12.6.22. If $\bar{A} \leq \bar{L}$, then $\bar{A}$ is in case (ii); thus $\bar{A}$ is conjugate to $\bar{R}_1 = J(\bar{L} \cap \bar{T})$, whereas $\bar{R}_1 \notin \mathcal{A}_3(\bar{M}_V, V)$. Therefore $\bar{A} \not\leq \bar{L}$, so we are in case (iii) or (iv), and hence we may take $\bar{i} = (1,2) \in \bar{A}$. Let $W := C_V(\bar{i})$ and $\bar{X} := C_{\bar{M}_V}(\bar{i})$. Then $\langle \bar{i} \rangle = C_{\bar{M}_V}(W)$ and $Aut_{\bar{A}}(W) \in \mathcal{A}_3(Aut_{\bar{X}}(W), W)$. However W is the core of the 6-dimensional permutation module for S_6, and we compute directly that $a(S_6, W) \leq 2$. This contradiction completes the proof of (1).

Next $Z(\bar{T})$ is generated by

$$\bar{t} := (1,2)(3,4)(5,6)(7,8);$$

and

$$J := C_V(\bar{t}) = I \oplus \langle v \rangle,$$

where $v := e_{1,3,5,7}$ and $I := \langle e_{1,2}, e_{3,4}, e_{5,6}, e_{7,8} \rangle$. Let $I_0 := [V, \bar{t}]$ and $\bar{Y} := C_{\bar{M}_V}(\bar{t})$. Then I is isomorphic to the 3-dimensional quotient of the permutation module for $Aut_{\bar{Y}}(I) \cong S_4$ or A_4 on $\{e_{1,2}, e_{3,4}, e_{5,6}, e_{7,8}\}$, $Aut_{\bar{Y}}(I) \leq N_{GL(I)}(I_0)$, and $C_{\bar{Y}}(I) = C_{\bar{M}_V}(I)$ is

$$\bar{X} := \langle (1,2), (3,4), (5,6), (7,8) \rangle,$$

when $\bar{T} \not\leq \bar{L}$, or $\bar{X} \cap \bar{L}$, when $\bar{T} \leq \bar{L}$. In either case, $C_{\bar{Y}}(I)/\langle \bar{t} \rangle$ acts faithfully as a group of transvections on J with axis I.

Since $\bar{t}$ is 2-central in $\bar{M}$, we may take $\bar{B} \leq C_{\bar{M}_V}(\bar{t}) = \bar{Y}$, and then either $\bar{B}$ centralizes I or $Aut_{\bar{B}}(I) \in \mathcal{A}_2(Aut_{\bar{Y}}(I), I)$.

Suppose first that $\bar{B}$ centralizes I. Then $\bar{B} \leq C_{\bar{Y}}(I) \leq \bar{X}$, so as $m(\bar{B}) \geq 3$, we check that $C_V(\bar{B}) = I$. If $m(\bar{B}) = 3$ then $m(\bar{B} \cap \bar{L}) \geq 2$, and for each $\bar{b} \in \bar{B}^\# \cap \bar{L}$, $m(C_V(\bar{b})) = 4 > m(I)$, so $\langle \bar{b} \rangle \in \mathcal{E}(\bar{B}, 1)$. Thus $|\mathcal{E}(\bar{B}, 1)| > 2$, and hence (2) holds with $\bar{D} = 1$. On the other hand if $m(\bar{B}) = 4$ then $\bar{B} = \bar{X}$ and $C_V(\bar{B}) = I$. In this case we take $\bar{D} := \langle \bar{d} \rangle$ for $\bar{d} := (1,2)$. Then for each of the three other transpositions $\bar{d}'$ in $\bar{B}$, $m(C_V(\langle d, d' \rangle)) = 4 > m(I)$, so that $\langle d, d' \rangle \in \mathcal{E}(\bar{B}, D)$, and again (2) holds.

So suppose instead that $Aut_{\bar{B}}(I) \in \mathcal{A}_2(Aut_{\bar{Y}}(I), I))$. We saw that $Aut_{\bar{Y}}(I)$ is a subgroup of $P := N_{GL(I)}(I_0) \cong S_4$ containing A_4, so from the action of $GL(I)$ on I, $\mathcal{A}_2(P, I) = \{O_2(P)\}$, and hence $Aut_{\bar{B}}(I) = O_2(P)$ is of rank 2. Hence it is easy to calculate that $\langle \bar{t} \rangle = C_{\bar{X}}(\bar{B})$, so as $m(\bar{B}) \geq 3$ and $C_{\bar{B}}(I) \leq C_{\bar{X}}(\bar{B})$, we conclude that $C_{\bar{B}}(I) = \langle \bar{t} \rangle$ and $m(\bar{B}) = 3$. In particular each member of $\bar{B}^\#$ is regular on Ω, of rank 3, and for each $\bar{b} \in \bar{B}^\#$, $C_V(\bar{b})$ is of rank 4, so that $\langle \bar{b} \rangle \in \mathcal{E}(\bar{B}, 1)$. Hence $|\mathcal{E}(\bar{B}, 1)| = 7$, so that (2) holds with $\bar{D} = 1$.

It remains to prove (3) and (4), so we may assume that for some $y \in G$, $A := V^y \cap T$ with $[A, V] \neq 1$ and $k := m(V^y/A) \leq 1$. Hence $[V^y, V] \neq 1$, so by 12.6.27, $V \cap V^y = 1$. Then $[V \cap N_G(V^y), A] = 1$, so in particular $V \not\leq N_G(V^y)$. On the other hand for each $A_0 \leq A$ with $m(V^y/A_0) \leq 2$, $C_G(A_0) \leq N_G(V^y)$ by 12.6.20.3. Thus for each $A_0 \leq A$ with $m(A/A_0) < 3 - k$, $C_V(A_0) \leq V \cap N_G(V^y) \leq C_V(A)$, so that $C_V(A_0) = V \cap N_G(V^y)$ for all such A_0. That is, $\bar{A} \in \mathcal{A}_{3-k}(\bar{M}_V, V)$. Therefore $k \neq 0$ by (1). The remaining statement in (3) holds in view of E.3.34.2.

Hence we have reduced to the case $k = 1$, so that $m(A) = 5$, and $\bar{A} \in \mathcal{A}_2(\bar{M}_V, V)$. Now by (2), there exists $\bar{D}$ of corank 3 in $\bar{A}$ satisfying $|\mathcal{E}(\bar{A}, \bar{D})| > 2$. Consider any $\bar{A}_1 \in \mathcal{E}(\bar{A}, \bar{D})$; thus the preimage A_1 in V^y has rank 3. Since $C_V(\bar{A}_1) > C_V(\bar{A}) = V \cap N_G(V^y)$ from the previous paragraph, $C_V(A_1) \not\leq N_G(V^y)$. We conclude from Theorem 12.6.2 that the 3-subspace A_1 is totally singular in V^y; in particular, D is a totally singular 2-subspace of V^y. But then D lies in just two totally singular 3-subspaces of V^y, whereas there are at least 3 choices for $\bar{A}_1 \in \mathcal{E}(\bar{A}, \bar{D})$. This contradiction shows that $k \neq 1$, and so completes the proof of (4). □

LEMMA 12.6.29. *If $H_0 \in \mathcal{H}(T)$ with $n(H_0) = 1$ or H_0 solvable, then $H_0 \leq M$.*

PROOF. Assume that $n(H_0) = 1$. We may apply 12.6.20.3 and 12.6.28.4 to see that $\min\{r(G, V), w(G, V)\} > 1$, so $H_0 \leq M$ E.3.35.1. Recall also that if H_0 is solvable, then $n(H_0) = 1$ by E.1.13 □

As $H \not\leq M$, $n(H) > 1$ and H is not solvable by 12.6.29. Thus $H^\infty \neq 1$. Suppose $H^\infty \leq M$. Then as $C_{\bar{L}}(z)$ is solvable, $H^\infty \leq C_M(V) \leq C_M(L/O_2(L))$, so L normalizes $(H^\infty O_2(L))^\infty = H^\infty$. But then $H \leq N_G(H^\infty) \leq M = !\mathcal{M}(LT)$, a contradiction. We conclude $H^\infty \not\leq M$, so that by 1.2.1.1, there exists $K \in \mathcal{C}(H)$ with $K \not\leq M$. As usual by 1.2.1.3, $L_1 = O^2(L_1)$ normalizes K. Let $K_0 := \langle K^T \rangle$, so that $K_0 \trianglelefteq H$ by 1.2.1.3.

Notice that $K_0 L_1 T \in \mathcal{H}_z$.

For the rest of the section, we assume $H = K_0 L_1 T$, where $K \in \mathcal{C}(H_1)$ for some $H_1 \in \mathcal{H}_z$ with $K \not\leq M$.

Let $M_H := M \cap H$. Notice that $L_1^* \leq O_{2,3}(M_H^*)$.

LEMMA 12.6.30. *(1) Hypothesis F.9.8 is satisfied for each $H_2 \in \mathcal{H}_z$ with V_5 in the role of "V_+". In particular it holds for $H = K_0 L_1 T$.*

(2) $q(H^*, \tilde{U}_H) \leq 2$.

(3) $K/O_2(K)$ is quasisimple, and K_0^* and its action on $\tilde{U}_H$ are described in (4) or (5) of F.9.18.

PROOF. By 12.6.24.7, Hypothesis F.9.1 is satisfied, while F.9.8.f holds by 12.6.27, and case (i) of F.9.8.g holds by 12.6.24.6. Thus (1) holds. Then (1) and F.9.16.3 imply (2).

Suppose $K/O_2(K)$ is not quasisimple. Then $K \trianglelefteq H$ by 1.2.1.3, and by 1.2.1.4, $X := \Xi_p(K) \neq 1$ for some prime $p > 3$. By 12.6.29, $X \leq M = N_G(L)$. By 1.3.3, $X \in \Xi(G, T)$; so $X \trianglelefteq LXT$ by 1.3.4 since L cannot play the role of "L" in that result. Thus $H \leq N_G(X) \leq M = !\mathcal{M}(LT)$, a contradiction. Therefore $K/O_2(K)$ is quasisimple, so as $H = K_0 L_1 T$, (3) follows from F.9.18. □

LEMMA 12.6.31. *One of the following holds:*

(1) $H^* \cong Aut(L_3(4))$ or $SL_3(4)$ extended by a 4-group. Further M_H^* is the product of T^* with a Borel subgroup of $O^2(H^*)$.

(2) H^* is of index at most 2 in S_5 wr $\mathbf{Z}_2$, and M_H^* is the product of T^* with a Borel subgroup of K_0^*.

(3) $H^* \cong S_5 \times S_3$, $|H : M_H| = 5$, and $R_1^* = O_2(M_H^*) \cong E_4$.

PROOF. By 12.6.30.3 we may apply F.9.18 to conclude that $K^*/Z(K^*)$ is a Bender group, $L_3(2^n)$, $Sp_4(2^n)'$, $G_2(2^n)'$, $L_4(2)$, $L_5(2)$, or A_7, or $K_0^* = K^* \cong M_{22}$ or $\hat{M}_{22}$. By 12.6.29, $n(H) > 1$ as $H \not\leq M$, so by E.1.14 either

(i) $K/O_{2,Z}(K)$ is a Bender group over $\mathbf{F}_{2^n}$, $L_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$, with $n := n(H) > 1$, or

(ii) $K_0^* \cong M_{22}$ or $\hat{M}_{22}$, and $n(H) = 2$.

Pick $\tilde{I} \in Irr_+(K_0^*, \tilde{U}_H, T^*)$, and adopt the notation of F.9.18; in particular $I_H := \langle I^H \rangle$. Let $L_C := O^2(C_{L_1}(K_0^*))$ and $L_K := O^2(L_1 \cap K_0)$. In case (i), a Borel subgroup of K_0 over $T \cap K_0$ is contained in M by 12.6.29.

Suppose first that $m_3(K_0) = 0$. Then $K^* \cong Sz(2^n)$. In particular, $L_K = 1$ as $L_1 = O^{3'}(L_1)$. By F.9.18.3, $q(Aut_{K_0T}(\tilde{I}), \tilde{I}) \leq 2$, so $\tilde{I}$ is described in B.4.2 or B.4.5. Hence $\tilde{I}$ is the natural module for K^*, and either F.9.18.4i holds with $K = K_0$ and $I = I_H$, or F.9.18.5iiia holds, with $K < K_0$ and $\tilde{I}_H = \tilde{I} \oplus \tilde{I}^t$ for $t \in T - N_T(K)$. Now $Sz(2^n)$ has no FF-modules by B.4.2, so by F.9.18.7, $I_H = [U_H, K_0]$. As $L_1 = [L_1, T]$, while $Out(K^*)$ is cyclic, either $L_1 = L_C$; or $K < K_0$, L_C^* is of order 3, and an element of order 3 in $L_1 - L_C$ acts as a nontrivial field automorphism on each component of K_0^*. As L_C^* is L_1T-invariant and nontrivial, $\tilde{V}_5 = [\tilde{V}_5, L_C]$ since L_1T is irreducible on $\tilde{V}_5$. Then as $L_C \trianglelefteq H$ and $U_H = \langle V_5^H \rangle$, $\tilde{U}_H = [\tilde{U}_H, L_C]$ and L_C^* is a 3-group, so $C_{\tilde{U}_H}(L_C^*) = 1$ by Coprime Action. However L_C stabilizes K and I, and $End_K(\tilde{I}) \cong \mathbf{F}_{2^n}$ with n odd so that 3 does not divide $2^n - 1$. Therefore $[I_H, L_C] = 1$, contradicting $C_{\tilde{U}_H}(L_C^*) = 1$. Therefore $m_3(K_0) > 0$.

Suppose next that $m_3(K_0) > 1$. Then comparing A.3.18 to the list of groups in (i) and (ii), either $K_0 = \theta(K_0)$, so that $L_1 \leq K_0$, or $L_1K_0/O_{2_Z}(K_0) \cong PGL_3^\epsilon(2^n)$ or $L_3^{\epsilon,\circ}(2^n)$.

Suppose first that $K^* \cong M_{22}$ or $\hat{M}_{22}$. Then there is $H_0 \in \mathcal{H}(T) \cap H$ with $O^2(H_0^*/O_{2,Z}(H_0^*)) \cong A_6$. Therefore $n(H_0) = 1$ by E.1.11, E.1.13, and E.1.14.1, so $H_0 \leq M$ by 12.6.29. But then $O^{3'}(H_0) \leq O^{3'}(M) = L$ by 12.6.1.5. impossible as L has no T-invariant A_6-section.

Therefore as $m_3(K_0) = 2$, K_0^* is one of the Lie-type groups $L_2(2^n) \times L_2(2^n)$, $(S)L_3^\epsilon(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ determined earlier. As $L_1^* \leq O_{2,3}(M_H^*)$, and $M_H^*T^*$ is the normalizer of a parabolic subgroup while $n > 1$, it follows that $M_{K_0} = M \cap K_0$ is a Borel subgroup of K_0, with n even; hence K^* is not $(S)U_3(2^n)$ as $m_3(K) = 2$. Recall $L_1T/O_2(L_1T) \cong S_3 \times S_3$ or S_3 wr $\mathbf{Z}_2$, so $T/O_2(L_1T)$ is noncyclic. Thus as $T \cap K_0 \leq O_2(L_1T)$, $T/O_2(L_1T)$ projects on a noncyclic 2-subgroup of $Out(K_0^*)$, so $K_0^* \cong (S)L_3(2^n)$ or $L_2(2^n) \times L_2(2^n)$. We return to these cases in a moment.

We now consider the case $m_3(K_0) = 1$. Here $K = K_0$, and $K^* \cong L_2(2^n)$, $L_3(2^m)$, m odd, or $U_3(2^k)$, k even. As $L_1 = [L_1, T]$, and $Out(K^*)$ is abelian, L_1 induces inner automorphisms on K^*, so that $L_1^* = L_C^* \times L_K^*$. Then $L_C^* \neq 1$ as $m_3(K) = 1$, while $L_C^* < L_1^*$ as L_1^* has 3-rank 2. Now T normalizes K and L_1, and hence normalizes L_K and L_C; hence for $X \in \{K, C\}$, $L_XT/O_2(L_XT) \cong S_3$. As $TL_K = L_KT$, K^* is not $L_3(2^m)$ since m is odd, and if $K^* \cong L_2(2^n)$, then n is even.

We now handle together the remaining cases: $K_0^* \cong (S)L_3(2^n)$, $L_2(2^n) \times L_2(2^n)$, $L_2(2^n)$, and $U_3(2^n)$, with n even. Let $T_L := T \cap L$; then $L_1 = [L_1, T_L]$ and T_L acts on L_K, so $L_K = [L_K, T_L]$. Moreover from the structure of $Aut(K_0^*)$, unless $K^* \cong (S)L_3(4)$ or $L_2(4)$, there exists a prime $p > 3$ and a nontrivial p-subgroup X of the Borel subgroup M_{K_0} with $XT = TX$ and $X = [X, T_L]$, so that

$$X = [X, T_L] \leq [X, L] \leq L.$$

This is a contradiction as T permutes with no nontrivial p-subgroups of L for $p > 3$ as $\bar{L} \cong L_4(2)$.

Therefore $K^* \cong (S)L_3(4)$ or $L_2(4)$. If $K^* \cong SL_3(4)$ and $L_1 \not\leq K$, then as M_H^* contains a Borel subgroup of H^*, a Sylow 3-subgroup of M_H is of order 27, whereas by 12.6.1.5, a Sylow 3-subgroup of M is of order 9. Therefore if K^* is $(S)L_3(4)$, then as $T/O_2(L_1T)$ projects on a noncyclic subgroup of $Out(K^*)$, (1) holds. So suppose $K^* \cong L_2(4)$. If $K < K_0$ then as $T/O_2(L_1T)$ is noncyclic, (2) holds. If $K = K_0$ our earlier analysis shows $L_1^* = L_C^* \times L_K^*$ with $L_1^*T^* \cong S_3 \times S_4$, so that (3) holds. □

LEMMA 12.6.32. $K^* \cong L_2(4)$.

PROOF. Assume otherwise. Then case (1) of 12.6.31.1 holds, so $K_0^* = K^* \cong (S)L_3(4)$, and $T^*K^*/K^* \cong E_4$. We pick $\tilde{I} \in Irr_+(K, \tilde{U}_H, T)$, and by 12.6.30.3, we may adopt the notation of F.9.18.4; in particular $I_H := \langle I^H \rangle$. As T is nontrivial on the Dynkin diagram of K^*, it follows from B.5.1 and B.4.2.2 that K^*T^* has no FF-modules. Thus by F.9.18.7, $[\tilde{U}_H, K] = \tilde{I}_H$. If $I_H = I$, then $q(H^*, \tilde{I}) \leq 2$ by F.9.18.2; so as K^*T^* has no FF-modules, $\tilde{I}/C_{\tilde{I}}(K)$ must appear in B.4.5. As the tensor-product module for $L_3(4)$ in B.4.5 has $q > 2$, we have a contradiction. Thus $I < I_H$, so case (iii) of F.9.18.4 occurs; that is, $K^* \cong SL_3(4)$ and $\tilde{I}_H = \tilde{U}_1 \oplus \tilde{U}_2$, where $\tilde{U}_1 = \tilde{I}$ is a natural K^*-module, and $\tilde{U}_2 = \tilde{U}_1^t$, for $t \in T$ nontrivial on the Dynkin diagram of K^*. As $\tilde{V}_5 = [\tilde{V}_5, L_1] \leq \tilde{I}_H$, $\tilde{U}_H = \langle \tilde{V}_5^H \rangle \leq \tilde{I}_H$, so $\tilde{U}_H = \tilde{I}_H$.

Next as M_H^* is the product of T^* with a Borel subgroup of K^* by 12.6.31.1, $M_H = L_1T$, so $T^* \cap K^* = O_2(M_H^*) = R_1^*$ is Sylow in K^*. As $\tilde{V}_5$ is L_1T-invariant and centralized by R_1^*, we conclude $\tilde{V}_5 = \tilde{V}_{5,1} \oplus \tilde{V}_{5,2}$, with $\tilde{V}_{5,i} = C_{\tilde{U}_i}(T \cap K)$ an $\mathbf{F}_4$-point in $\tilde{U}_i$. In particular $\tilde{V}_{5,i} = C_{\tilde{V}_5}(X_i)$ for some X_i of order 3 in L_1; so $V_{5,i}$ contains a nonsingular vector u_i of V. Now $C_{K^*}(\tilde{u}_i)$ is a maximal parabolic of K^*, so in particular $C_K(u_i)^*$ does not lie in the Borel group $M_{K_0}^*$. This is a contradiction as $C_G(u_i) \leq M$ by Theorem 12.6.2. This contradiction completes the proof of 12.6.32. □

LEMMA 12.6.33. $H^* \cong S_5 \times S_3$.

PROOF. Assume otherwise. As 12.6.32 eliminates case (1) of 12.6.31, we must be in case (2), where H^* of index at most 2 in S_5 wr $\mathbf{Z}_2$. Thus there is $t \in T - N_T(K)$, and we let $K_1 := K$ and $K_2 := K^t$. For $X \in Syl_3(L_1)$, $X = X_1 \times X_2$ with $X_i \in Syl_3(K_i)$ and $V_5 = [V_5, X] \leq [U_H, K_0] = U_1U_2$, where $U_i := [U_H, K_i]$. Thus $U_H = U_1U_2 = [U_H, K_0]$.

Pick $\tilde{I} \in Irr_+(K_0, \tilde{U}_H, T)$; by 12.6.30.3, we may adopt the notation of F.9.18.5. We claim that either

(a) $[U_H/I_H, K_1, K_2] \leq I_H$, and $[I_H, K_1, K_2] = 1$, or
(b) $\tilde{U}_H$ is the $\Omega_4^+(4)$-module for K_0^*.

For notice by Theorems B.5.6 and B.4.2 that H^* has no strong FF-modules. Thus it follows from F.9.18.6 that either $\tilde{U}_H = \tilde{I}_H$, or both $\tilde{I}_H$ and U_H/I_H are FF-modules for H^*. Observe that only cases (i) and (iiia) of F.9.18.5 can arise. In case (i), $\tilde{I}_H$ is not an FF-module for H^*, so $\tilde{U}_H = \tilde{I}_H$ and (b) holds. Suppose case (iiia) holds. Then (a) holds if $\tilde{U}_H = \tilde{I}_H$, so assume otherwise. Thus U_H/I_H is an FF-module for H^*; then as $U_H = [U_H, K_0]$, it follows from B.5.6 that $[U_1, K_2] \leq I_H$, so again (a) holds. This completes the proof of the claim.

Suppose now that $[\tilde{U}_1, K_2] = 1$. Then $\tilde{V}_5 = \tilde{V}_{5,1} \oplus \tilde{V}_{5,2}$, where $\tilde{V}_{5,i} := [\tilde{V}_5, X_i] \cong E_4$. Then, as in the proof of 12.6.32, $V_{5,i}$ contains a nonsingular point u_i, and $K_{3-i} \leq C_G(u_i) \leq M$ by Theorem 12.6.2, contrary to $K_0 \not\leq M$.

This contradiction shows $[\tilde{U}_1, K_2] \neq 1$. Suppose now that case (a) holds. Then $[U_1, K_2] = [U_H, K_1, K_2] \leq [I_H, K_1] \leq C_{U_H}(K_2)$. So $[U_1, K_2] = [U_1, K_2, K_2] = 1$, contrary to $[\tilde{U}_1, K_2] \neq 1$.

Therefore case (b) holds. As in the proof of 12.6.32, $R_1^* = T^* \cap K_0^*$ and $\tilde{V}_5 \leq C_{\tilde{U}_H}(R_1^*)$. But as $\tilde{U}_H^*$ is the $\Omega_4^+(4)$-module, $C_{\tilde{U}_H}(R_1^*)$ is an $\mathbf{F}_4$-point of $\tilde{U}_H$, whereas $E_{16} \cong \tilde{V}_5 \leq C_{\tilde{U}_H}(R_1^*)$. This contradiction completes the proof of 12.6.33. □

We are at last in a position to obtain a contradiction under the hypotheses of this section.

By 12.6.33, $H^* = H_1^* \times H_2^*$ with $H_1^* \cong S_3$ and $H_2^* \cong S_5$. In particular $L_1T/O_2(L_1T) \cong S_3 \times S_3$, so $\bar{T} \leq \bar{L}$ and $\bar{M}_V = \bar{L} \cong A_8$. Also $X \in Syl_3(L_1)$ is of the form $X = X_1 \times X_2$ with $X_i \in Syl_3(H_i)$. As $X_iT = TX_i$, we conclude each X_i moves 6 points of Ω; hence $C_V(X_i)$ is a nondegenerate space of dimension 2 and $\tilde{V}_5 = [\tilde{V}_5, X_i]$. In particular as $X_1^* \trianglelefteq H^*$ and $\tilde{U}_H = \langle \tilde{V}_5^H \rangle$, $\tilde{U}_H = [\tilde{U}_H, X_1^*]$, and then

$$\tilde{U}_H = \tilde{U}_1 \oplus \tilde{U}_2$$

is an H_2-decomposition of $\tilde{U}_H$, where $\tilde{U}_i := [\tilde{U}_H, t_i^*]$ for t_1^* and t_2^* distinct involutions in H_1^*. As the third involution t_3^* in H_1^* commutes with H_2^* while interchanging $\tilde{U}_1$ and $\tilde{U}_2$, and $F^*(H_2^*) = K^*$ is simple, H_2^* is faithful on $\tilde{U}_1$, and $\tilde{U}_2$ is isomorphic to $\tilde{U}_1$ as an H_2-module. Recalling that $H_2^* \cong S_5$ has no strong FF-modules, we must be in case (b) of F.9.18.6, so that U_i^* is an FF-module for H_2^*. Hence either $[\tilde{U}_i, H_2^*]$ is the S_5-module, or $[\hat{U}_i, H_2^*]$ is the $L_2(4)$-module, where $\hat{U}_H := \tilde{U}_H / C_{\tilde{U}_H}(K)$.

In particular, no member of H^* induces a transvection on $\tilde{U}_H$. Thus by F.9.16.1, $D_\gamma < U_\gamma$, in the notation of F.9.16. Hence by F.9.16.4, we can choose γ so that $0 < m := m(U_\gamma^*) \geq m(U_H/D_H)$. Further by F.9.13.2, $U_\gamma \leq O_2(G_{\gamma_1,\gamma_2}) = R_1^h$ for suitable $h \in H$, so $m \leq 2$ as $H^* \cong S_5 \times S_3$. Next for $b \in U_\gamma - D_\gamma$, $[D_H, b] \leq A_1$ by F.9.13.6, where A_1 is the conjugate of V_1 defined in section F.9; thus $m(A_1) = 1$ and

$$m([\tilde{U}_H, b^*]) \leq m(U_H/D_H) + m([\tilde{D}_H, b]) \leq m + 1 \leq 3,$$

impossible as $m([\tilde{U}_H, b]) = m([\tilde{U}_1, b]) + m([\tilde{U}_2, b]) = 4$, since $b^* \in U_\gamma^* \leq R_1^{h*} \leq K^*$ and $\hat{U}_i$ is the natural or A_5-module for K^*.

This contradiction finally eliminates the A_8-subcase of Theorem 12.2.2.3d, and hence establishes:

THEOREM 12.6.34. *If Hypothesis 12.2.3 holds with $\bar{L} \cong L_4(2)$, then V is the natural module for $\bar{L}$.*

12.7. The treatment of $\hat{\mathbf{A}}_6$ on a 6-dimensional module

In this section we prove

THEOREM 12.7.1. *Assume Hypothesis 12.2.3 with $L/C_L(V) \cong \hat{A}_6$. Then G is isomorphic to M_{24} or He.*

We recall that M_{24} has already appeared in Theorem 12.2.13, in the case that V is a TI-set in G. However in this section, our argument does not require Theorem

12.2.13 until after both He and M_{24} have been independently identified; 12.2.13 is only used when we are working toward the final contradiction.

We mention that the groups He and M_{24} will be identified via Theorem 44.4 in [**Asc94**] in our Background References.

12.7.1. Preliminary results. The proof of Theorem 12.7.1 involves a series of reductions.

Assume L, V arise in a counterexample G. Then by Theorem 12.2.2, V is a 6-dimensional module for $L/C_L(V) \cong \hat{A}_6$ and $C_L(V) = O_2(L)$. We adopt the conventions of Notation 12.2.5. Let $T_L := T \cap L$, and P_1 and P_2 the two maximal subgroups of LQ containing T_LQ. Let $R_i := O_2(P_i)$ and $X := O^2(O_{2,Z}(L))$. We can regard V as a 3-dimensional vector space ${}_FV$ over $F := \mathbf{F}_4$, with $\bar{L} \leq SL({}_FV)$ and $\bar{X}$ inducing F-scalars on ${}_FV$.

LEMMA 12.7.2. *(1) Either $\bar{M}_V = \bar{L} \cong \hat{A}_6$ or $\bar{M}_V = \bar{L}\bar{T} \cong \hat{S}_6$.*

(2) P_1T is the stabilizer in LT of an F-point V_1 of ${}_FV$, and $\bar{R}_1 \cong E_4$ is a group of F-transvections on ${}_FV$ with center V_1 and $C_V(R_1) = V_1$.

(3) P_1 is irreducible on V/V_1 and V_1.

(4) P_2T is the stabilizer of an F-line V_2 of ${}_FV$, and $\bar{R}_2 \cong E_4$ is a group of F-transvections on ${}_FV$ with axis V_2 and $[V, R_2] = V_2$.

(5) $O^2(P_i) = L_iX$, where $\{L_i, X\}$ are the unique T-invariant subgroups $I = O^2(I)$ of P_i with $|I : O_2(I)| = 3$.

(6) $L = \theta(M)$ is the characteristic subgroup of M generated by all elelements of order 3 in M.

PROOF. The calculations in (1)–(5) are well-known and easy. Notice in (1) that automorphisms of $A_6 = Sp_4(2)'$ nontrivial on the Dynkin diagram are ruled out, as they do not preserve V. Part (6) follows from 12.2.8. □

In the remainder of the section, we adopt the notation of L_1 and L_2 as in 12.7.2.5.

LEMMA 12.7.3. *$\mathcal{A}_2(\bar{T}, V) = \{\bar{R}_2\}$ and $a(\bar{M}_V, V) = 2$.*

PROOF. Let $\bar{A} \in \mathcal{A}_2(\bar{T}, V)$. Then $C_V(\bar{A}) = C_V(\bar{B})$ for each hyperplane $\bar{B}$ of $\bar{A}$, so as $1 \neq \bar{A}$, $m(\bar{A}) > 1$. If $\bar{A} \not\leq \bar{L}$, then $\bar{B} := \bar{A} \cap \bar{L}$ is a hyperplane of $\bar{A}$, and $C_V(\bar{B})$ is an F-subspace of V, whereas $\bar{a} \in \bar{A} - \bar{L}$ is nontrival on each $\bar{a}$-invariant F-point since $\bar{a}$ inverts $\bar{X}$. We conclude that $\bar{A} \leq \bar{L}$, so as $\bar{R}_i$, $i = 1, 2$, are of rank 2 and are the maximal elementary abelian subgroups of $\bar{T}_L$, $\bar{A} = \bar{R}_i$ for some i. By 12.7.2.2, $i \neq 1$, and 12.7.2.4 shows that $\bar{R}_2 \in \mathcal{A}_2(\bar{T}, V)$, so $\bar{A} = \bar{R}_2$. Since $m(\bar{R}_2) = 2$, $a(\bar{M}_V, V) = 2$. □

LEMMA 12.7.4. *(1) L has two orbits on $V^{\#}$ with representatives $z \in V_1$ and t.*

(2) Let V_t be the F-point of ${}_FV$ containing t. Then $N_{\bar{L}}(V_t) \cong GL_2(4)$, and V is an indecomposable module for $C_L(V_t)$ with V/V_t the natural module.

(3) $t \in V = [V, L_t] \leq L_t$ and $L_t = \theta(M_t)$.

(4) V_2 is partitioned by two conjugates of V_t and three conjugates of V_1.

PROOF. From 12.7.2, $|V_1^L| = |L : P_1| = 15$, leaving a set $\mathcal{O}$ of 6 F-points of ${}_FV$ not in V_1^L. As 6 is the minimal degree of a faithful permutation representation for $\bar{L}/\bar{X} \cong A_6$, it follows that L is transitive on $\mathcal{O}$ (so that (1) holds), and the stabilizer in $\bar{L}$ of $V_t \in \mathcal{O}$ is isomorphic to $GL_2(4)$. As $V = [V, X]$, V/V_t is the natural module for $N_{\bar{L}}(V_t) \cong GL_2(4)$, so a Sylow 2-subgroup $\bar{S}$ of $N_{\bar{L}}(V_t)$ centralizes an

F-hyperplane of ${}_FV$. Hence as $\bar{R}_i$, $i = 1, 2$, are representatives for the conjugacy classes of 4-subgroups of $\bar{L}$, we may take $\bar{S} = \bar{R}_2$ since $\bar{R}_1$ centralizes no hyperplane by 12.7.2.2. Then as $[V, R_2]$ is not a point by 12.7.2.4, V does not split over V_t as an $N_L(V_t)$-module. This completes the proof of (2), and (2) and 12.7.2.6 imply (3). Further V_2/V_t is the 1-dimensional F-subspace centralized by $\bar{S} = \bar{R}_2$, so $V_t \leq V_2$ and P_2 has two orbits on F-points of V_2 of length 2 and 3, and then (4) follows as T acts on V_1. □

For the rest of the section, t and V_t have the meaning given in 12.7.4. Observe that $L_t/O_2(L_t) \cong A_5$ using 12.7.4.2, so that $L_t = L_t^\infty$.

LEMMA 12.7.5. *Either*

(1) $G_t \leq M$, or

(2) $K_t := \langle L_t^{G_t} \rangle$ is a component of G_t with $V_t = Z(K_t)$ and $K_t/V_t \cong L_3(4)$.

PROOF. Assume that (1) fails, and choose t so that $T_t = C_T(t) \in Syl_2(M_t)$. From 12.7.4, $O_2(\bar{L}_t\bar{T}_t) = 1$ and $V = [V, \bar{L}_t]$, and we saw $L_t = L_t^\infty$, so we may apply 12.2.12.2 to conclude that Hypothesis C.2.8 is satisfied with G_t, M_t, L_t, Q in the roles of "H, M_H, L_H, R". By C.2.10.1, $O(G_t) = 1$. By Theorem C.4.8, $L_t \leq K \in \mathcal{C}(G_t)$ with $K/O_2(K)$ quasisimple and K described in one of the conclusions of that result. If conclusion (10) of C.4.8 holds, then for $g \in G_t - M_t$, $L_t \neq L_t^g \leq M_t$, so $L_t^g \leq \theta(M_t) = L_t$ by 12.7.4.2, a contradiction. Thus by C.4.8, $L_t < K$, $K/O_2(K)$ is quasisimple, and K is described in C.3.1 or C.4.1. By 12.7.4.3, $L_t = \theta(M_t)$ and $V = [V, L_t]$, so $L_t = \theta(K \cap M)$ and $t \in V_t \leq V \leq L_t \leq K$, so $t \in Z(K)$.

Suppose first that $F^*(K) = O_2(K)$. Examining the list of C.4.1 for "M_0" with $M_0/O_2(M_0) \cong L_2(4)$ acting naturally on V/V_t, we see conclusion (2) of C.4.1 holds: K is an $Sp_4(4)$-block with $V_t \leq Z(K)$, and $M \cap K$ is the parabolic stabilizing the 2-dimensional F-space V/V_t in $U(K)/V_t$. As $U(K)$ is a quotient of the orthogonal FK-module of dimension 5, V splits over V_t as an L_t-module—contrary to 12.7.4.2.

Thus as $K/O_2(K)$ is quasisimple, K is a component of G_t; and $Z(K)$ is a 2-group since $O(G_t) = 1$. This time examining the list of C.3.1 for "M_0" given by L_t acting as $L_2(4)$ on V/V_t, we see that one of cases (1), (3), or (4) must occur. Then as $V_t \leq Z(L_t)$ and $t \in V_t \cap Z(K)$, we conclude that $V_t \leq Z(K)$. Now by I.1.3, the only case with a multiplier of 2-rank 2 is $K/Z(K) \cong L_3(4)$. As V is L_t-invariant and elementary abelian, $V_t = O_2(K)$ from the structure of the covering group of $L_3(4)$ in I.2.2.3b. Thus as $Z(K)$ is a 2-group, (2) holds in this case, completing the proof of 12.7.5. □

LEMMA 12.7.6. *Assume that L is a $\hat{A}_6$-block. Then*

(1) $Q = O_2(LT) = V \times C_T(L)$.

(2) If $C_T(L) = 1$ then $O_2(L) = V = O_2(M) = C_G(V)$ and $M = LT$.

PROOF. Since the 1-cohomology of V is trivial by I.1.6, (1) follows from C.1.13.b. Assume $C_T(L) = 1$. By (1), $O_2(LT) = V$. Now (2) follows from 3.2.11. □

12.7.2. The identification of He.

In this subsection we prove:

THEOREM 12.7.7. *If $G_t \not\leq M$, then G is isomorphic to He.*

PROOF. Assume $G_t \not\leq M$ and let $K_t := \langle L_t^{G_t} \rangle$. By 12.7.5, K_t is quasisimple with $V_t = Z(K_t)$ and $K_t/V_t \cong L_3(4)$. In particular by A.3.18, K_t is the unique component of G_t of order divisible by 3. Therefore as X normalizes V_t, for each

$x \in X$, $K_{t^x} = K_t^x \leq C_G(V_t^x) = C_G(V_t) \leq G_t$, so that $K_t = K_{t^x}$. Hence X acts on K_t and $XK_t/Z(K_t) \cong PGL_3(4)$.

From the structure of K_t, L_t is an $L_2(4)$-block, so L is an $\hat{A}_6$-block. Let $Y_X \in Syl_3(X)$. As $X = O^2(O_{2,Z}(L))$ and L is an $\hat{A}_6$-block, $X = VY_X$. As $K_t \trianglelefteq G_t$ and $\Omega_1(O_2(K_t)) = V_t$, $G_t \leq N_G(V_t)$. Then as X is transitive on $V_t^{\#}$, V_t is a TI-set in G by I.6.1.1.

Next as $V \leq L_t$ by 12.7.4.2, $C_G(L_t) \leq C_{G_t}(L_t) = C_{G_t}(K_t)$, so as $C_{Aut(K_t)}(L_t) = 1$, $C_G(L_t) = C_{G_t}(K_t)$. Similarly $C_G(L_t) \leq C_G(V) = C_M(V)$, and $[L, C_M(X)] \leq C_L(X) = Z(L)$, so as L is perfect, $C_G(L_tX) = C_G(L)$ is LT-invariant. Further $[C_G(L_t), X] \leq C_X(L_t) = V_t$, so by a Frattini Argument, $C_G(L_t) = V_tC_G(L_tY_X) = C_G(L_tX)$. On the other hand, we saw that $C_G(L_t) = C_{G_t}(K_t)$, so if $C_G(L_tX) \neq 1$, then $K_t \leq C_G(C_G(L_t)) \leq M = !\mathcal{M}(LT)$, contrary to $K_t \not\leq M_t$. Therefore $C_G(L_tX) = C_G(L) = 1$, and $C_{G_t}(K_t) = C_G(L_t) = V_tC_G(L_tX) = V_t$. Thus $V = O_2(M)$ and $M = LT$ by 12.7.6.2. Then by 12.7.2.1, either $M = L$, or $|M : L| = 2$ with $M/V \cong \hat{S}_6$.

Choose t so that $T_t := C_T(t) \in Syl_2(M_t)$. As $K_t \trianglelefteq G_t$, $G_t \notin \mathcal{H}^e$, so t is not 2-central in G by 1.1.4.6. hence $P = C_Y(\tilde{P})$ since $Inn(P)$ induces $C_{Aut(P)}(\tilde{P})$ by A.1.23. Therefore

$$Y/P \leq Aut(\tilde{P}) \cong O_6^+(2),$$

and $D_8 \cong T/P \in Syl_2(Y/P)$. Further $\alpha := (M_z/P, T/P, N_{G_z}(U)/P)$ is a Goldschmidt triple as in Definition Aa.t:defnGldtrpl. As $O_2(M_z/P) \neq O_2(N_{G_z}(U)/P)$, case (i) of F.6.11.2 holds, and so the image in $Y/O_{3'}(Y)$ of α is a Goldschmidt amalgam; therefore as Y is an SQTK-group, $Y/O_{3'}(Y)$ is described in Theorem F.6.18. In view of (*), $Y/O_{3'}(Y)$ appears in case (6) of Theorem F.6.18; that is, $Y/O_{3'}(Y) \cong L_2(q)$ for $q \equiv \pm 7 \mod 16$. Then as $Y/P \leq O_6^+(2)$, we conclude $Y/P \cong L_3(2)$ or A_6.

Next $\tilde{P}^+$ is the sum of the natural module and its dual for $Y^+/P^+ \cong L_3(2)$, so M_z^+ and $N_{G_z^+}(U^+)$ stabilize unique points of $\tilde{P}^+$. Indeed $\tilde{V}_1^+$ is the point stabilized by M_z^+, and we write $\tilde{U}_1^+$ for the point stabilized by $N_{G_z^+}(U^+)$. Applying φ, M_z stabilizes only $\tilde{V}_1$ and $N_{G_z}(U)$ stabilizes only $\tilde{U}_1$. As $\tilde{V}_1^+ \neq \tilde{U}_1^+$, $\tilde{V}_1 \neq \tilde{U}_1$. But if $Y/P \cong A_6$, then Y stabilizes a point of $\tilde{P}$, so $\tilde{V}_1 = C_{\tilde{P}}(Y) = \tilde{U}_1$, contrary to the previous remark. We conclude $Y/P \cong L_3(2)$.

Now $S_4 \cong M_z/P$ is the stabilizer in Y/P of $\tilde{V}_1$, so $\tilde{P}_1 := \langle \tilde{V}_1^Y \rangle$ is a nontrivial quotient of the 7-dimensional permutation module on Y/M_z, and similarly $\tilde{P}_2 := \langle \tilde{U}_1^Y \rangle$ is a nontrivial quotient of the permutation module on $Y/N_{G_z}(U)$. Hence by H.5.3, either $\tilde{P} = \tilde{P}_i$ is the 6-dimensional core of the permutation module for $i = 1$ or 2, or else $\tilde{P} = \tilde{P}_1 \oplus \tilde{P}_2$ with $\dim(\tilde{P}_i) = 3$ for $i = 1$ and 2. Next $\varphi : T^+ \to T$ is an isomorphism, and for each 3-dimensional indecomposable $\tilde{W}$ for a rank one parabolic Y_0^+ of Y_+ containing the fixed point of Y_0^+, $\tilde{P}^+$ splits over $\tilde{W}$ as a T^+-module. However this is not the case when $\tilde{P}$ is the core of the permutation module, and that module is indecomposable. Hence the former case is impossible, so the latter holds.

Now $Q_z \leq P$ is Y-invariant, so $Q_z = \langle z \rangle$, P_i, or P. As $F^*(G_z) = Q_z$, the first case is out. Next suppose $Q_z = P_i$. Now $P_i \cong E_{16}$, and as $T \in Syl_2(G)$ normalizes P_i, $N_G(P_i) \in \mathcal{H}^e$ by 1.1.4.6, so $C_G(P_i) \in \mathcal{H}^e$ by 1.1.3.1. Therefore as $C_T(P_i) = P_i$, we conclude $C_G(P_i) = P_i$. Now $GL(P_i) = Aut(P_i)$ with $Y/P_i = C_{GL(P_i)}(z)$, so $G_z = YC_G(P_i) = Y$ normalizes P, contrary to $O_2(G_z) = Q_z = P_i < P$. Thus

$Q_z = P \trianglelefteq G_z$, and then as $T/P \cong D_8$ is Sylow in $G_z/P \leq O_6^+(2) \cong S_8$, we conclude from the list of maximal subgroups of S_8 that either $Y = G_z$ or $G_z/P \cong A_7$. From the structure of $G_t \cong P\Gamma L_3(4)/E_4$, we see that $G_{t,z}$ is of order $3 \cdot 2^9$, so that Y is transitive on $t^G \cap P$ of order 14. Thus if G_z/P is A_7, $G_{t,z}$ contains an A_6-section, contradicting $G_{z,t}$ a $\{2,3\}$-group.

Therefore $G_z = Y$. We have also seen that z is not weakly closed in P with respect to z, so that Theorem 44.4 of [**Asc94**] applies. Since $M_V/V \cong \hat{S}_6$, $G \not\cong L_5(2)$, and since $G_t \not\leq M$, $G \not\cong M_{24}$. Therefore as G is simple, Theorem 44.4 in [**Asc94**] shows $G \cong He$. □

12.7.3. The case $\mathbf{V \not\leq O_2(G_z)}$, including the identification of $\mathbf{M_{24}}$. Because of Theorem 12.7.7, we can assume in the remainder of this section that

LEMMA 12.7.8. *$G_t \leq M$.*

LEMMA 12.7.9. *(1) M controls fusion of its involutions.*
(2) G_v is transitive on $\{V^g : v \in V^g\}$ for each $v \in V$.
(3) V is the unique conjugate of V containing t.

PROOF. By 12.7.8 and 12.7.4.2, t is not 2-central in G, so $t \notin z^G$. Thus (1) follows from 12.7.4.1. Then (1) and A.1.7.1 imply (2), and (2) and 12.7.8 imply (3) using A.1.7.2. □

LEMMA 12.7.10. *(1) $m(\bar{M}_V, V) = 2$.*
(2) $r(G,V) > 2$. Hence $s(G,V) = 2$.
(3) If $\bar{L} < \bar{M}_V$ then there are two classes $\mathcal{O}_j$, $j = 1,2$, of involutions in $\bar{M}_V$ not in $\bar{L}$. Further $\bar{i}_j \in \mathcal{O}_j$, where $\langle \bar{i}_j \rangle = Z(\bar{L}_j \bar{T})$, and $m(C_V(\bar{i}_j)) = 3$. Finally $\bar{i}_2$ acts on a conjugate of V_t, but $\bar{i}_1$ does not.
(4) If $U \leq V$ with $m(V/U) = 3$, then one of the following holds:
(i) $C_M(U) = C_M(V)$.
(ii) Up to conjugation in L, U is a hyperplane of V_2 and $C_M(U) = C_M(V)R_2$.
(iii) $U = C_V(\bar{i})$ for some involution $\bar{i} \in \bar{M}_V - \bar{L}$, and $C_M(U) = \langle i \rangle C_M(V)$.
(5) If $U \leq V$ with $m(V/U) = 3$ and $C_G(U) \not\leq M$, then $U = C_V(\bar{i})$ for some $\bar{i} \in \mathcal{O}_1$.

PROOF. First L is transitive on the set $\mathcal{O}$ of involutions in $\bar{L}$, and by 12.7.2.4, $V_2 = C_V(\bar{i})$ for $\bar{i} \in \mathcal{O} \cap \bar{R}_2$. Assume $\bar{L} < \bar{M}_V$. Then $\bar{M}_V \cong \hat{S}_6$ by 12.7.2.1, so there are two classes $\mathcal{O}_j$, $j = 1,2$, of involutions in $\bar{M}_V - \bar{L}$, and we can choose notation so that $\bar{i}_j \in \mathcal{O}_j$, where $\bar{i}_j$ is defined in (3). As $\bar{i}_j$ inverts $\bar{X}$, $m([V, \bar{i}_j]) = 3$, completing the proof of (1). If we represent $\bar{M}_V$ on the set Ω of 6 cosets of $\bar{H} := N_{\bar{M}_V}(V_t)$, then each involution $\bar{i} \in \bar{H} - \bar{L}$ induces a transposition on Ω. Consequently the members of the other class $\mathcal{O}_1$ have cycle type 2^3 on Ω. This completes the proof of (3), and part (4) also follows since $C_M(U)$ is a 2-group for each $U \leq V$ with $m(V/U) < 4$.

Next let $U \leq V$. If U is a hyperplane of V, then $1 \neq U \cap V_t$, so $C_G(U) \leq M$ by 12.7.8. Thus $r(G,V) > 1$. Assume $U \leq V$ with $C_G(U) \not\leq M$ and $k := m(V/U) < 4$. By E.6.12, $C_M(U) > C_M(V)$. Hence U is centralized by some involution $\bar{i} \in \bar{M}_V$ by (4). Thus if $k = 2$, we can take $U = V_2$ by the previous paragraph; however $V_2 = V_1 V_t$, so $C_G(U) \leq M$ by 12.7.8. We conclude $k = 3$, so $r(G,V) > 2$, and hence $s(G,V) = 2$ using (1), so (2) holds. Indeed this argument shows $U \not\leq V_2$, as each hyperplane of V_2 intersects V_t nontrivially. Thus $U = C_V(\bar{i})$ and $\bar{i} \notin \bar{L}$

by (4). Finally if $\bar{i} \in \mathcal{O}_2$, we may take $\bar{i}$ to act on V_t by (3), so $1 \neq C_{V_t}(\bar{i}) \leq U$, contradicting $C_G(U) \not\leq M$ in view of 12.7.8. This completes the proof of (5), and hence of 12.7.10. □

LEMMA 12.7.11. *$W_0 := W_0(T, V) \leq C_T(V)$, so that $w(G, V) > 0$ and $N_G(W_0) \leq M$.*

PROOF. Suppose $A := V^g \leq T$ with $\bar{A} \neq 1$. By 12.7.10.2, $s(G, V) > 1$, so by E.3.10, $\bar{A} \in \mathcal{A}_2(\bar{T}, V)$, and hence $\bar{A} = \bar{R}_2$ by 12.7.3. Now $m(A/C_A(V)) = 2$, so by 12.7.10.2, $V \leq C_G(C_A(V)) \leq N_G(A)$. Thus $V_2 = [V, A] \leq A$ by 12.7.2.4. This contradicts 12.7.9.3, as $t \in V_2$. Hence $W_0 \leq C_T(V) = O_2(LT)$, and $N_G(W_0) \leq M$ by E.3.34.2. □

LEMMA 12.7.12. *$C_T(L) = 1$.*

PROOF. If $C_T(L) \neq 1$, also $C_Z(L) \neq 1$; so by 12.2.9.1, $C_G(Z) \leq M$. But by 12.7.10.2, $s(G, V) > 1$, so by 12.4.1 there is $g \in G$ with $V^g \leq T$ and $[V, V^g] \neq 1$, contrary to 12.7.11. □

LEMMA 12.7.13. *(1) Hypothesis G.2.1 is satisfied for each $H \in \mathcal{H}(P_1 T) \cap N_G(V_1)$, with P_1 in the role of "L_1".*
(2) $V \leq O_2(N_G(V_1))$.

PROOF. By 12.7.2.3, P_1 is irreducible on V/V_1, so (1) holds. Then (2) follows from G.2.2.1. □

Much of the rest of the section is devoted to the proof of the following result, which identifies the remaining group in the conclusion of Theorem 12.7.1.

THEOREM 12.7.14. *If $V \not\leq O_2(G_z)$, then as $G_t \leq M$, G is isomorphic to M_{24}.*

Until the proof of Theorem 12.7.14 is complete, assume $V \not\leq O_2(G_z)$. Recall the subgroup L_1 defined in 12.7.2.5. Set $K := \langle V^{G_z} \rangle$, $U := \langle V_1^K \rangle$, $\tilde{G}_z := G_z/\langle z \rangle$, $H := K L_1 T$, $Q_z := O_2(H)$, and $H^* := H/C_H(\tilde{U})$. As V_1 is $L_1 T$-invariant, $U \trianglelefteq H$. As $V \not\leq O_2(G_z)$, $O^2(K) \neq 1$ and $V \not\leq O_2(H)$, so $K \not\leq M$ by 12.2.6.

LEMMA 12.7.15. *(1) $\Phi(U) \leq \langle z \rangle$, and $\tilde{U} \in \mathcal{R}_2(\tilde{H})$, so that $O_2(H^*) = 1$.*
(2) Either
(a) $\bar{U} = \bar{R}_1$, or
(b) $\bar{M}_V \cong \hat{S}_6$ and $\bar{U}$ is either $Z(\bar{L}_1 \bar{T})$ of order 2 or $O_2(\bar{L}_1 \bar{T})$ of order 8.
(3) If U is abelian, then $\bar{U} = \bar{R}_1$.
(4) $m(V^*) = 2$ or 4 and $V^* = [V^*, L_1^*]$, so $L_1^*/O_2(L_1^*) \cong \mathbf{Z}_3$.

PROOF. Observe that Hypothesis G.2.1 is satisfied with $\langle z \rangle$, V_1, G_z, in the roles of "V_1, V, G_1"; hence (1) holds by G.2.2. As

$$C_H(\tilde{U}) \leq C_H(\tilde{V}_1) \leq N_G(V_1),$$

and $V \not\leq O_2(H)$, V does not centralize $\tilde{U}$ by 12.7.13.2. Thus $V^* \neq 1$, so $\bar{U} \neq 1$. However $U \trianglelefteq H$, so $\bar{U} \trianglelefteq \bar{L}_1 \bar{T}$, and hence (2) follows. Further if U is abelian then $\bar{U} \leq C_{\bar{M}}(V_1)$, so (3) holds. As $V/V_1 = [V/V_1, L_1]$, $V^* = [V^*, L_1]$, so as $V^* \neq 1$ and $L_1/O_2(L_1) \cong \mathbf{Z}_3$, (4) holds. □

We now deal with the case leading to the remaining conclusion of Theorem 12.7.1:

LEMMA 12.7.16. *If $\bar{U} = \bar{R}_1$, then $G \cong M_{24}$.*

PROOF. Assume that $\bar{U} = \bar{R}_1$. By 12.7.2.2, $[\tilde{U}, V^*] = \tilde{V}_1$, so V^* induces a group of transvections on $\tilde{U}$ with center $\tilde{V}_1$. Also $m(V^*) = 2$ or 4 by 12.7.15.4. Thus if $\Delta_1, \ldots, \Delta_s$ are the orbits of K on $\tilde{V}_1^{G_z}$, then by G.3.1, $\tilde{U} = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_s$ and $K^* = K_1^* \times \cdots \times K_s^*$, where $\tilde{U}_i := \langle \Delta_i \rangle$ is of dimension $n \geq 3$, K_i^* is generated by the transvections in K^* with centers in Δ_i, $[K_i, \tilde{U}_j] = 0$ for $i \neq j$, and (as $O_2(K_i^*) = 1$ by 12.7.15.1) K_i^* acts faithfully as $GL(\tilde{U}_i)$ on $\tilde{U}_i$. Observe in particular that $L_1^* \leq K^*$. Now each preimage K_i contains a member of $\mathcal{C}(H)$ by 1.2.1.1, so by 1.2.1.3, $s \leq 2$; and in case of equality, $H^* \cong L_3(2)$ wr $\mathbf{Z}_2$. Therefore $s = 1$, as T^* acts on L_1^* and $L_1^*/O_2(L_1^*) \cong \mathbf{Z}_3$ by 12.7.15.4. Thus by Theorem C (A.2.3), $K^* = GL(\tilde{U}) \cong L_n(2)$, $n = 3$, 4, or 5. Then K is transitive on $\tilde{U}^\#$, so $\Phi(U) = 1$. By 12.7.15.4, $L_1^*T^*$ is a rank one parabolic of K^*.

If $n = 5$, then $C_K(V_1)^* \cong L_4(2)/E_{16}$, so as X is faithful on V_1, $m_3(N_G(V_1)) > 2$, contrary to $N_G(V_1)$ an SQTK-group.

Thus $n = 3$ or 4. As we saw $V^* \leq C_{K^*}(U/V_1) \cong E_{2^{n-1}}$ and $m(V^*) = 2$ or 4, we conclude $m(V^*) = 2$. Next

$$m(Q \cap U) = m(U) - m(\bar{U}) = n + 1 - 2 = n - 1.$$

As $V_1 \leq V \cap U \leq C_V(U) = C_V(\bar{R}_1) = V_1$, we conclude $V_1 = C_V(U) = V \cap U$. Thus

$$m((Q \cap U)V/V) = n - 1 - m(V_1) = n - 3 \leq 1.$$

Now $[Q, U] \leq Q \cap U$, so that $m([W, U]) \leq 1$ for W any noncentral chief factor for L on Q/V. However $\bar{U} = \bar{R}_1$ does not induce transvections on any nontrivial irreducible for $\hat{A}_6$; hence $[U, Q] \leq V$, and L is an $\hat{A}_6$-block. In particular, L_1 has exactly three noncentral 2-chief factors.

Suppose $n = 4$. Then as $L_1^* = O^2(P^*)$ for some rank one parabolic P^* of K^*, L_1 has one noncentral chief factor $\tilde{W}$ on the natural module $\tilde{U}$, and two such factors on $O_2(L_1^*)$. We conclude $[Q_z, L_1] \leq U$, so that $[Q_z, K] \leq U$. Thus by the Thompson $A \times B$-Lemma, $O^2(K)/U$ is faithful on $C_U(O_2(KT)/U)$, so as K is irreducible on $\tilde{U}$, $O_2(KT)$ centralizes U. Then as $H^1(K^*, \tilde{U}) = 0$ by I.1.6, $U = [U, K] \oplus \langle z \rangle$. This is impossible as $z \in [V, R_1]$ by 12.7.2.2, while $UC_T(V) = R_1$ by hypothesis, so that $z \in [V, U]$

Therefore $n = 3$, so $U \cong E_{16}$. Assume $\bar{M}_V = \bar{L}$. Then $V_1 \leq Z(T)$, so by B.2.14, $U \in \mathcal{R}_2(G_z)$, and hence $C_{G_z}(U) = C_{G_z}(\tilde{U})$. However we saw that $4 = |V^*| = |V : C_V(\tilde{U})|$ and $C_V(U) = V_1$ is of index 16 in V. Thus $\bar{M}_V \cong \hat{S}_6$ and $U \notin \mathcal{R}_2(G_z)$, so that $C_G(\tilde{U})/C_G(U) \neq 1$. Therefore from the action of $H^* = GL(\tilde{U})$, $C_G(\tilde{U})/C_G(U)$ is the full group of transvections on U with center z, and affords the K^*-module dual to $\tilde{U}$. Recall that L is a $\hat{A}_6$-block, while $C_T(L) = 1$ by 12.7.12. Then by 12.7.6.2, $V = O_2(M)$ and $M = LT$, so that $M/V \cong \hat{S}_6$. Therefore $|T| = 2^{10}$, so as $|T^*| = 8 = |C_T(\tilde{U})/C_T(U)|$ and $|U| = 16$, $C_T(U) = U$. As T normalizes U, $N_G(U) \in \mathcal{H}^e$ by 1.1.4.6, so $U = C_G(U)$. Hence as $Aut_K(U) = C_{Aut(U)}(z)$, $H = N_{G_z}(U) = K$ with $Q_z = O_2(K)$ of order 2^7. In particular, K has 2-chief series

$$1 < \langle z \rangle < U < Q_z.$$

As Q_z/U is dual to $\tilde{U}$, K is transitive on $(Q_z/U)^\#$ and $\tilde{U}^\#$. As $V \cap Q_z \not\leq U$, there are involutions in $Q_z - U$. It follows that $\Phi(\tilde{Q}_z) = 1$, so $Q_z \cong 2^{1+6} \cong D_8^3$. Now G_z normalizes K and hence normalizes $O_2(K) = Q_z$.

Let $G^+ := M_{24}$. Arguing as in the proof of Theorem 12.7.7, M is determined up to isomorphism, so as G^+ satifies the hypotheses of this Theorem, there is an isomorphism $\varphi : M^+ \to M$. As $\tilde{Q}_z^+ = J(\tilde{T}+)$, $\varphi(Q_z^+) = Q_z$. Now either $\tilde{U}$ is the socle of K on $\tilde{Q}_z$, or $\tilde{Q}_z$ is the sum of $\tilde{U}$ and its dual as a K-module. As in the final three paragraphs of the proof of 12.7.7, any $L_1^+T^+$-submodule of $\tilde{Q}_z^+$ isomorphic to $\tilde{U}^+$ splits over $\tilde{U}^+$, so applying φ the same holds in K, and hence again $\tilde{U}$ is a semisimple K-module. Thus $Aut_{GL(\tilde{Q}_z)}(K^*) \cong Aut(L_3(2))$, so as $K \trianglelefteq G_z$ and $T^* \cong D_8$, $K^* = Aut_{G_z}(\tilde{Q}_z)$. Therefore $K = G_z$. As T^+ splits over Q_z^+, applying φ, T splits over Q_z; so K splits over Q_z, and hence $G_z = K$ is determined up to isomorphism. We have seen that z is not weakly closed in Q_z with respect to G, so that we may apply Theorem 44.4 in [**Asc94**]. This time as $G_t \leq M$ by 12.7.8, we conclude that $G \cong M_{24}$. □

By 12.7.16, to complete the proof of Theorem 12.7.14, we may assume that $\bar{U} \neq \bar{R}_1$, and it remains to derive a contradiction. In particular U is not abelian by 12.7.15.3, and $\Phi(U) = \langle z \rangle$ by 12.7.15.1. Let $\bar{U}_1 := Z(\bar{L}_1\bar{T})$. By 12.7.15.2, $\bar{M}_V \cong \hat{S}_6$, so $O_2(\bar{L}_1\bar{T}) = \bar{U}_1 \times \bar{R}_1 \cong E_8$, and either $\bar{U} = \bar{U}_1$, or $\bar{U} = O_2(\bar{L}_1\bar{T})$ contains $\bar{U}_1$. In any case $E_8 \cong [V, U_1] \leq U \cap V$, and L_1 is irreducible on $V/V_1[V, U_1] \cong E_4$, so $V \cap U = V_1[V, U_1]$, and hence:

Lemma 12.7.17. *$V^* \cong E_4$.*

Lemma 12.7.18. *$\bar{U} = O_2(\bar{L}_1\bar{T}) \cong E_8$.*

Proof. If not, by the discussion before 12.7.17, $\bar{U} = \bar{U}_1$ is of order 2. Then V^* induces a group of transvections on $\tilde{U}$ with axis $\widetilde{U \cap Q}$, so using the dual of G.3.1 as in the proof of 12.7.16, $L_1^* \leq K^* \cong L_n(2)$ with $n = 3$, 4, or 5. This time since we are arguing in the dual of $\tilde{U}$, $[\tilde{U}, K^*]$ is the natural module for K^*. Then $\tilde{U} = [\tilde{U}, K^*] \oplus C_{\tilde{U}}(K^*)$ as K^* is genrated by $m([\tilde{U}, K^*])$ transvections. Next as $[V, U_1]$ is of rank 3 and contains z,

$$[\tilde{U}, V^*] = [U_1, V]/\langle z \rangle = [U_1, V, L_1]\langle z \rangle/\langle z \rangle = [\tilde{U}, V^*, L_1^*]$$

is of rank 2. Thus in its action on the natural module $[\tilde{U}, K^*]$, $L_1^*T^*$ is the rank one parabolic stabilizing the line $[\tilde{U}, V^*]$ and centralizing $[\tilde{U}, K^*]/[\tilde{U}, V^*]$. In particular $L_1^*T^*$ fixes no point in the natural module, so $\tilde{V}_1 \leq C_{\tilde{U}}(L_1^*T^*) = C_{\tilde{U}}(K^*)$, contradicting $\tilde{U} = \langle \tilde{V}_1^K \rangle$. □

We may represent LT on $\Omega := \{1, \ldots, 6\}$ so that P_2T is the global stabilize of $\{1, 2\}$. Then by 12.7.18, $\bar{U} = \langle (1, 2), (3, 4), (5, 6) \rangle$. Pick $g \in L$ with $\bar{U}^g = \langle (1, 6), (2, 3), (4, 5) \rangle$. Then

$$\bar{L} = \langle \bar{U}, \bar{U}^g \rangle = \langle \bar{U}, \bar{x} \rangle$$

for each $1 \neq \bar{x} \in \bar{U}^g$ which is not a transposition. Let $I := \langle U, U^g \rangle$. Arguing as in the the proof of G.2.3, $L \leq I$ and

$$[O_2(I), I] =: P = (P \cap U)(P \cap U^g)$$

with $[I, U \cap U^g] \leq V$, and setting $I/(U \cap U^g)V =: I^+$,

$$P^+ = (U \cap P)^+ \oplus (U^g \cap P)^+$$

with $C_{P^+}(U) = (U \cap P)^+$. Indeed if $1 \neq \bar{x} \in \bar{U}^g$ such that $\bar{x}$ is not a transposition, then $I = \langle U, x \rangle$, so

$$C_{(P \cap U)^+}(x) \leq C_{(P \cap U)^+}(I) = (U \cap U^g)^+ = 1,$$

so $(P \cap U^g)^+ = C_{P^+}(x)$.

Recall $X = O^2(O_{2,Z}(L))$, so $X \leq L \leq I \leq N_G(P)$.

LEMMA 12.7.19. $[P^+, X] = 1$.

PROOF. Assume otherwise. Take $\bar{x} \in \bar{L} \cap \bar{U}^g$, and take $\bar{y} \in \bar{U}^g$ to be the product of 3 transpositions. Then $[C_{[P^+,\bar{X}]}(\bar{x}), \bar{y}] \neq 1$ as $\bar{y}$ inverts $\bar{X}$, so $C_{P^+}(\bar{x}) \neq C_{P^+}(\bar{y})$, whereas $C_{P^+}(\bar{x}) = (U^g \cap P)^+ = C_{P^+}(\bar{y})$ since neither $\bar{x}$ nor $\bar{y}$ is a transposition. This contradiction establishes the lemma. □

LEMMA 12.7.20. *L is a $\hat{A}_6$ block, $V = O_2(M)$, and $M = LT$.*

PROOF. As $[P, X] \leq (U \cap U^g)V$ by 12.7.19 and I centralizes $(U \cap U^g)V/V$, it follows by Coprime Action that $[P, X] = V$. Thus $X = VY$ where Y has order 3, so $LT = VN_{LT}(Y)$ by a Frattini Argument.

If L is a $\hat{A}_6$-block then as $C_T(L) = 1$ by 12.7.12, $O_2(M) = V$ and $M = LT$ by 12.7.6.2. Thus we may assume that L is not a $\hat{A}_6$-block. Then $1 \neq Z_Y := \Omega_1(Z(N_T(Y)) \cap O_2(LT)$, and Z_Y is in the center of $VN_T(Y) = T$, so that $Z_Y \leq Z$. Let $V_Y := \langle Z_Y^L \rangle$; then $V_Y \in \mathcal{R}_2(LT)$ by B.2.14 and $V_Y \leq C_G(Y)$. As $C_T(L) = 1$, $C_{V_Y}(L) = 1$. Let $V_0 := V_Y V$, so that also $V_0 \in \mathcal{R}_2(LT)$ by B.2.12. By B.4.2.8, the unique FF*-offender in $\bar{L}\bar{T}$ on V is $\bar{R}_2$ and $m(V/C_V(\bar{R}_2)) = m(\bar{R}_2)$. Then it follows from B.4.2 and B.3.4 that $\hat{q} := \hat{q}(Aut_{LT}(V_0), V_0) \geq 2$, with equality only if $V_Y/C_{V_Y}(L)$ is the A_6-module (so that $m(V_Y) = 4$ since we saw $C_{V_Y}(L) = 1$) and either

(i) $\bar{R}_2$ is an FF*-offender on V_Y and hence L_1 centralizes Z_Y, or

(ii) There is a strong FF*-offender $\bar{A}$ in $\bar{T}$ on V_0 with $m(V/C_V(\bar{A})) = m(\bar{A})+1$, so that $\bar{A} = O_2(\bar{L}_2\bar{T})$ and again L_1 centralizes Z_Y.

However by 3.1.8.1, $\hat{q} \leq 2$, so indeed $\hat{q} = 2$. Therefore V_Y is the 4-dimensional A_6-module in which L_1T centralizes a point, so as $\bar{U} = O_2(\bar{L}_1\bar{T})$ by 12.7.18, $\bar{U}$ is not quadratic on V_Y, impossible as $[V_Y, U, U] \leq [V_Y \cap U, U] \leq V_Y \cap \langle z \rangle = 1$ using 12.7.15.1. □

We are now ready to complete the proof of Theorem 12.7.14.

By 12.7.20, L is a $\hat{A}_6$-block, $V = O_2(M)$, and $M = LT$. By 12.7.18, $\bar{U} \cong E_8$, so $M/V \cong \hat{S}_6$. In particular M, and hence also T, are determined up to isomorphism, so T is isomorphic to a Sylow 2-group of He. Thus $J(\tilde{T}) \cong E_{64}$. But by our remark before 12.7.17, $V \cap U = V_1[V, U_1]$ is of rank 4, so as $\bar{U} \cong E_8$,

$$|U| = |\bar{U}||U \cap V| = 8 \cdot 16 = 2^7,$$

so $\tilde{U} \cong E_{64}$ and hence U is the preimage D_8^3 of $J(\tilde{T})$ in T and $L_1T/U \cong S_4$.

As T is Sylow in He, $C_T(U) \leq U$, so as U induces $C_{Aut(U))}(\tilde{U})$ by A.1.23, $U = C_H(\tilde{U})$. Thus $H^* \leq Out(U) \cong O_6^+(2)$. Recall $O_2(H^*) = 1$ by 12.7.15.1. As $V^* = [V^*, L_1]$, $[O(H^*), V^*] = 1$ by A.1.26. Then since $K = \langle V^{G_z} \rangle$, K^* centralizes $O(H^*)$. Further $H = KL_1T$ and $L_1^*T^* \cong S_4$ with $V^* = O_2(L_1^*T^*)$. Now examining $O_6^+(2)$ for subgroups satisfying these conditions, we conclude H^* is $L_3(2)$, A_6, A_7, S_5, or $\Gamma L_2(4)$. Next $\tilde{U}_T := C_{\tilde{U}}(T) = C_{\widetilde{U \cap V}}(T) \cong E_4$ and $C_{\tilde{U}}(L_1T) = \tilde{V}_1$. Therefore H^* is not A_6 or S_5, since those groups fix a point of of $\tilde{U}$, but K moves $\tilde{V}_1$. If H^* is $\Gamma L_2(4)$, then $[\tilde{U}, K]$ is the A_5-module for K^*, impossible as V^* is quadratic on $\tilde{U}$.

Next the preimage U_T is isomorphic to E_8 and contains V_1, so by 12.7.10.5, $C_H(U_T) \leq M_z = L_1T$. Then $C_H(U_T) = C_{L_1T}(U_T) \leq T$, and hence $C_{H^*}(\tilde{U}_T)$ is a 2-group by Coprime Action, so that H^* is not A_7. Therefore $H^* \cong L_3(2)$. Then arguing as in the proof of 12.7.7, $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$ is the sum of two nonisomorphic natural modules for H^*. Therefore as $\tilde{V}_1 = C_{\tilde{U}}(L_1^*T^*)$, $\tilde{V}_1 \leq \tilde{U}_i$ for some i, so $U = \langle V_1^H \rangle \leq U_i < U$. This contradiction finally completes the proof of Theorem 12.7.14.

12.7.4. The final contradiction. Because of Theorem 12.7.14, we can assume in the remainder of the section that:

LEMMA 12.7.21. *$V \leq O_2(G_z)$.*

LEMMA 12.7.22. *(1) If $g \in G$ with $V \cap V^g \neq 1$, then $[V, V^g] = 1$.*
(2) Either $W_1 := W_1(T, V)$ centralizes V, or $\bar{W}_1 = \bar{R}_2$ and $r(G, V) = 3$.
(3) $C_G(C_1(T, V)) \leq M$.
(4) If $r(G, V) > 3$, then $C_G(C_2(T, V)) \leq M$.
(5) If $C_V(V^g) \neq 1$, then $\langle V, V^g \rangle$ is a 2-group.

PROOF. Under the hypotheses of (1), we may take $z \in V^g$ by 12.7.9.3 and 12.7.4.1. Then by 12.7.9.2, we may take $g \in G_z$. Now by 12.7.21, $V^g \leq O_2(G_z) \leq T$, so by 12.7.11, $[V, V^g] = 1$. That is, (1) holds.

We next prove (2), (3), and (4). Let $A := V^g \cap M \leq T$ be a w-offender. Thus $\bar{A} \neq 1$ and $w := m(V^g/A)$. By 12.7.11, $w > 0$. If $w > 1$, then W_1 centralizes V by definition, so that (2) holds, and then $C_G(C_1(T, V)) \leq M$ by E.3.34.2, so that (3) holds. That result also shows that (4) holds if $w > 2$.

Next as $1 \neq [A, V] \leq [V^g, V]$, $V \cap V^g = 1$ by (1). If $B \leq A$ with $m(V^g/B) < r(G, V) =: r$, then $C_V(B) \leq N_G(V^g)$, so $[C_V(B), A] \leq V \cap V^g = 1$; thus $C_V(B) = C_V(A)$, so that $\bar{A} \in \mathcal{A}_{r-w}(\bar{T}, V)$ Then by 12.7.3, $r - w \leq 2$; and in case of equality, $\bar{A} = \bar{R}_2 = \overline{W_w(T, V)}$. Thus if $r - w = 2$, then $V_2 = C_V(R_2) \leq C_w(T, V)$, so that $C_G(C_w(T, V)) \leq G_t \leq M$ by 12.7.8.

By 12.7.10.2, $r \geq 3$. Assume first that $r > 3$. Then by the previous paragraph: first $w > 1$; and then either $w > 2$—or $w = 2$ and $r = 4$, so that (4) holds. Thus the lemma holds when $r > 3$ by paragraph two, so we may assume that $r = 3$. Then (4) is vacuous, and (2) and (3) hold by paragraph two when $w > 1$, so we may assume that $w = 1$. Then $r - w = 2$, so that (2) and (3) hold by paragraph three. This completes the proof of (2), (3) and (4).

Assume the hypotheses of (5). By 12.7.4.1, we may assume V^g centralizes $v := t$ or z. We observe $V \leq O_2(G_v)$: if $v = t$, this follows from 12.7.8, and if $v = z$ it follows from 12.7.21. Hence $\langle V, V^g \rangle$ is a 2-group, proving (5). □

If $G_z \leq M$, then by 12.7.4.1 and 12.7.8, we may apply Theorem 12.2.13 to conclude that $G \cong M_{24}$; but then $V \not\leq O_2(G_z)$, contrary to 12.7.21. Therefore $G_z \not\leq M$, so we can choose $H \in \mathcal{H}_*(T, M)$ with $H \leq G_z$. By 3.3.2.4, we may apply the results of section B.6 to H.

LEMMA 12.7.23. *(1) $n(H) = 2$.*
(2) $O^2(H/O_2(H)) \cong L_2(4)$ or $L_3(4)$.
(3) $L_1T = H \cap M$.

PROOF. Let $K_H := O^2(H)$. By 12.7.10.2, $s(G, V) > 1$, and by 12.7.11, $N_G(W_0) \leq M$. As $C_G(C_1(T, V)) \leq M$ by 12.7.22.3, E.3.19 says that $n(H) \geq$

2. Then H is not solvable by E.1.13, so H is described in E.2.2; in particular $(K_H \cap M)O_2(H)/O_2(H)$ is a Borel subgroup of $H/O_2(H)$. As $V \leq O_2(G_z)$ by 12.7.21, 12.2.11.1 says $n(H) \leq 2$, proving (1). By 12.2.11.2, $(K_H \cap M)/O_2(H)$ is a nontrivial 3-group. Next by 12.2.8,

$$\theta(H \cap M) \leq \theta(M) = L,$$

where we recall that $\theta(Y)$ is the characteristic subgroup generated by all elements of order 3 in a group Y. Thus $\theta(H \cap M) \leq O^2(C_L(z)) = L_1$, so as $|L_1|_3 = 3$, we conclude that $\theta(H \cap M) = L_1$. Then inspecting the list of groups in E.2.2 with $n(H) = 2$, $H \cap M$ a $\{2,3\}$-group, and $\theta(H \cap M)/O_2(\theta(H \cap M))$ of order 3, we conclude that (2) holds and $O^2(H \cap M) = \theta(H \cap M)$, so that (3) holds. □

LEMMA 12.7.24. $r(G,V) > 3$.

PROOF. Assume otherwise. Then $r(G,V) = 3$ by 12.7.10.2, and then by 12.7.10.5, $C_G(U) \not\leq M$ where $U := C_V(\bar{i}_1)$. Now T acts on U, so we may choose $H \leq C_G(U)T$, so that $H = C_H(U)T$. Hence $L_1 \leq O^2(H) \leq C_H(U)$ by 12.7.23.3, which is impossible as $C_{LT}(U) = Q\langle i_1 \rangle$ by 12.7.10.4. □

We are now in a position to obtain the final contradiction establishing Theorem 12.7.1.

By 12.7.11, $N_G(W_0) \leq M$; hence as $H \not\leq M$, $W_0 \not\leq O_2(H)$ by E.3.16, so that there exists $A := V^g \leq T$ with $A \not\leq O_2(H)$. Let $K_H := O^2(H)$ and $H^+ := H/O_2(H)$. In case $K_H^+ \cong L_2(4)$, set $H_1 := H$, $K_1 := K_H$, and $T_1 := T$. Otherwise by 12.7.23.2, $K_H^+ \cong L_3(4)$. Here as $A \leq Q \leq O_2(L_1T)$ by 12.7.11 and $L_1T = H \cap M$ by 12.7.23.3, $A \leq O_2(H \cap M)$. Therefore we have two subcases: either $A^+ \leq K_H^+$; or $A^+ = \langle a^+ \rangle A_K^+$, where $A_K^+ := A^+ \cap K_H^+$, and a^+ induces a graph automorphism on K_H^+. In the former subcase, replacing A by a suitable conjugate if necessary, $A \not\leq O_2(P)$ for one of the two maximal parabolics P of K_H. In this subcase, we let $H_1 := N_H(P)$, $K_1 := O^2(H_1)$, and $T_1 := T \cap H_1$, and observe that as $A \not\leq O_2(P)$, $C_{T^+}(A^+) \leq T_1^+$. Finally in the latter subcase, $C_{K_H^+}(a^+) \cong L_2(4)$. In this subcase, let $a^+ \neq b^+ \in a^+(T^+ \cap K^+ \cap Z(C_{T^+}(a^+)))$, H_1 the preimage in H of $C_{H^+}(b^+)$, $K_1 := O^2(H_1)$, and $T_1 := T \cap H_1$.

In each case, $K_1/O_2(K_1) \cong L_2(4)$, $T_1 \in Syl_2(H_1)$, $A \not\leq O_2(H_1)$, and $C_{T^+}(A^+) \leq T_1^+$. Also in each case, $K_1 \not\leq M$ as $H \cap M = L_1T$. Let $Q_1 := O_2(H_1)$, $H_1^* := H_1/Q_1$, $B := A \cap Q_1$, and $D := C_2(Q_1, V)$. As $A \leq O_2(H \cap M)$, $A^* \leq O_2((H_1 \cap M)^*) = (T_1 \cap K_1)^* \in Syl_2(K_1^*)$. As $r(G,V) > 3$ by 12.7.24, and $K_1 \not\leq M$, we conclude from 12.7.22.4 that $K_1 \not\leq C_G(C_2(T,V))$. As $n(H) = 2$, $K_1 \in \mathcal{E}_2(H,T,A)$ in the sense of Definition E.1.5 by construction. So we apply E.3.17.1 with 0, 2, 2 in the roles of "i, j, k", to conclude $C_2(T,V) \leq D$, so that $K_1 \not\leq C_G(D)$, and $A \not\leq C_G(D)$ by E.1.4. But $m(A/B) \leq m_2(H_1^*) = 2$, so $D \leq C_G(B) \leq N_G(A)$ as $r(G,V) > 3$. Indeed as D centralizes B with $m(A/B) \leq 2$, but does not centralize A, we conclude from 12.7.10 that $m(A/B) = m(A^*) = 2$, and we may take $B = V_2^g$ and $D \leq R_2^g$. As $A^* \leq (T_1 \cap K_1)^*$ and $m(A^*) = 2$, $A^* = (T_1 \cap K_1)^* \in Syl_2(K_1^*)$. Thus $m(D/C_D(A)) \geq 2$, as $m(W/C_W(A^*)) \geq 2$ for any nontrivial chief section W for K_1^* on D. So as $m(R_2/Q) = 2$, we conclude $R_2^g = DQ^g$ and $|D : C_D(A)| = 4$. Then by 12.7.2.4,

$$B = V_2^g = [R_2^g, A] = [D, A] \leq D.$$

Let $k \in K_1 - M$; then $K_1^* = \langle A^*, A^{*k} \rangle$. Now $[B^k, A] \leq [D, A] = B$, so A acts on BB^k, and by symmetry, so does A^k, so that $I := \langle A, A^k \rangle$ acts on $U := BB^k$, and

$I^* = K_1^* \cong L_2(4)$. Indeed $|D : C_D(A)| = 4$, so

$$|B : C_B(I)| = |B : C_B(A^k)| \leq |D : C_D(A^k)| = 4.$$

So as $m(B) = 4$, $C_B(I) \neq 1$. But this contradicts 12.7.22.5, since I is not a 2-group.

This final contradiction completes the proof of Theorem 12.7.1.

12.8. General techniques for $\mathbf{L_n(2)}$ on the natural module

When Hypothesis 12.2.3 holds and $\bar{L} \cong L_n(2)$ for $n = 3, 4, 5$, Theorems 12.4.2.1, 12.6.34, and 12.5.1 tell us that V is the natural module for $\bar{L}$. We will encounter a similar setup involving $L_2(2)$ after completing our treatment of the Fundamental Setup. Thus in this section we establish some general techniques for treating all four case simultaneously.

The hypotheses below reflect one difference between the treatments for $n = 2$ and $n > 2$: For $n > 2$, we have already analyzed the case where V is a TI-set in G in Theorem 12.2.13, so we simply exclude the groups appearing in conclusions (2)–(4) of 12.2.13 as part of the operating hypothesis 12.8.1 of this section; then by 12.2.13, $C_G(Z \cap V) \not\leq M$ as L is transitive on $V^\#$. However, the treatment of the case where $n = 2$ and V is a TI-set in G does not appear until the end of the analysis of that case, so for the moment we instead assume $Z \leq V$ and $C_G(Z) \not\leq M$ as part of our operating hypothesis when $n = 2$.

Thus in this section, we assume the following hypothesis:

HYPOTHESIS 12.8.1. *Either (1) or (2) holds:*

(1) Hypothesis 12.2.3 holds, with $L/O_2(L) \cong L_n(2)$, $n = 3, 4, 5$, and V the natural module for $L/O_2(L)$. Further G is not $L_{n+1}(2)$, A_9, or M_{24}.

(2) G is a simple QTKE-group, $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, $M \in \mathcal{M}(T)$, $V := \langle Z^M \rangle$ is of rank 2, $L = O^2(L) \trianglelefteq M$ with $M = !\mathcal{M}(LT)$, $C_{LT}(V) = O_2(LT)$, and $LT/O_2(LT) \cong L_2(2) \cong S_3$. Furthermore assume $C_G(Z) \not\leq M$.

We adopt the following notation, which is consistent with that in Notation 12.2.5 when $n > 2$:

NOTATION 12.8.2. (1) $Z := \Omega_1(Z(T))$, $M := N_G(L)$, $M_V := N_M(V)$, and $\bar{M}_V := M_V/C_M(V)$.

(2) $n := m_2(V)$, and for $1 \leq i \leq n$, let V_i denote the i-dimensional subspace of V invariant under T, $G_i := N_G(V_i)$, and $M_i := N_M(V_i)$. Let $L_i := O^2(N_L(V_i))$, unless $n = 5$ and $i = 2$ or 3, where $L_i := N_L(V_i)^\infty$. Set $R_i := O_2(L_iT)$.

(3) Let z be the generator of V_1 and $\tilde{G}_1 := G_1/V_1$. Set

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \leq G_1 \text{ and } H \not\leq M\}.$$

For $H \in \mathcal{H}_z$, set $U_H := \langle V^H \rangle$, $Q_H := O_2(H)$, and $H^* := H/Q_H$.

Note when $n = 2$ that $V \trianglelefteq M$, so that $M_i \leq M_V$, and $L_1 = 1$. When $n > 2$, V is a TI-subgroup in M and $M_i \leq M_V$ by 12.2.6.

12.8.1. General preliminary results.

LEMMA 12.8.3. *(1) $\bar{M}_V = GL(V)$, and either $\bar{M}_V = \bar{L}$, or $n = 2$ and $\bar{M}_V = \bar{L}\bar{T}$.*

(2) L is transitive on i-dimensional subspaces of V, for each i.

(3) G_i is transitive on $\{V^g : V_i \leq V^g\}$.

(4) $G_1 \not\leq M$.

PROOF. Part (1) is an immediate consequence of Hypothesis 12.8.1. Then (1) implies (2), and (2) and A.1.7.1 imply (3).

Assume case (1) of Hypothesis 12.8.1 holds. Then Hypothesis 12.2.3 holds, but conclusions (2)–(4) of Theorem 12.2.13 are excluded by that hypothesis. Thus conclusion (1) of Theorem 12.2.13 holds, so that $C_G(v) \not\leq M$, and then (4) follows from the transitivity of L on nonzero vectors of V in (2). Finally when case (2) of Hypothesis 12.8.1 holds, (4) is a consequence of the assumption in that hypothesis that $C_G(Z) \not\leq M$. $\square$

By 12.8.3.4, $G_1 \in \mathcal{H}_z$, so $\mathcal{H}_z \neq \emptyset$. Observe that $\mathcal{H}_z \subseteq \mathcal{H}^e$ by 1.1.4.6.

LEMMA 12.8.4. *Let $H \in \mathcal{H}_z$. Then*

(1) Hypothesis G.2.1 is satisfied.

(2) $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$ and $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$.

(3) $\Phi(U_H) \leq V_1$.

(4) $Q_H = C_H(\tilde{U}_H)$, so H^* is the image of H in $GL(\tilde{U}_H)$ under the representation of H on $\tilde{U}_H$ by conjugation.

PROOF. As L_1 is irreducible on $\tilde{V}$, (1) holds. Then G.2.2 implies (2) and (3). If (4) fails, then $Y := O^2(C_H(\tilde{U}_H)) \neq 1$. But by Coprime Action, $Y \leq C_G(V) \leq M_V$, so $[Y,L] \leq C_L(V) = O_2(L)$. Hence L normalizes $O^2(YO_2(L)) = Y$, so that $H \leq N_G(Y) \leq M = !\mathcal{M}(LT)$, contrary to the choice of $H \not\leq M$. $\square$

LEMMA 12.8.5. *Assume $n > 2$, so that $L_1 \neq 1$.*

(1) If $H \in \mathcal{H}_z$ with $L_1 \trianglelefteq H$, then $\tilde{U}_H$ is the direct sum of copies of the natural module $\tilde{V}$ for $L_1^* \cong L_{n-1}(2)'$.

(2) If $L_1 \trianglelefteq G_1$, then for $1 < i < n$, $G_i \leq M$ and V is the unique member of V^G containing V_i, so that $m(V \cap V^g) \leq 1$ for $g \in G - M_V$.

PROOF. Observe $\tilde{V}$ is the natural module for $L_1/O_2(L_1) \cong L_{n-1}(2)'$, so (1) holds as $U_H = \langle V^H \rangle$. Now assume $L_1 \trianglelefteq G_1$. Then for $1 < i < n$, $N_L(V_i)$ induces $GL(V_i)$ on V_i by 12.8.3.1, so that $G_i = C_G(V_i)N_L(V_i)$ and $C_G(V_i) \leq G_1 \leq N_G(L_1)$. Hence G_i acts on

$$\langle L_1^g : g \in G_i \rangle = \langle L_1^g : g \in N_L(V_i) \rangle = L.$$

So $G_i \leq N_G(L) = M$. Then $G_i = M_i \leq M_V$, so the remaining assertions of (2) follow from 12.8.3.3. $\square$

The next lemma 12.8.6 shows that the condition "U_H is abelian for all $H \in \mathcal{H}_z$" is equivalent to "$\langle V^{G_1} \rangle$ abelian". Much of our remaining work on the $\mathbf{F}_2$-Case is partitioned via the cases "U_H abelian for all $H \in \mathcal{H}_z$" versus "$\langle V^{G_1} \rangle$ nonabelian". We will discuss this distinction further after 12.8.6.

LEMMA 12.8.6. *The following are equivalent:*

(1) U_H is abelian for each $H \in \mathcal{H}_z$.

(2) $\langle V^{G_1} \rangle$ is abelian.

(3) If $g \in G$ with $V \cap V^g \neq 1$, then $[V, V^g] = 1$.

(4) Hypothesis F.8.1 is satisfied for each $H \in \mathcal{H}_z$.

(5) Hypothesis F.9.8 is satisfied for each $H \in \mathcal{H}_z$, with V in the role of "V_+".

PROOF. First (1) implies (2) trivially as $G_1 \in \mathcal{H}_z$. By 12.8.3.3 and the transitivity of L on $V^\#$, (2) implies (3). If (3) holds, then condition (a) of Hypothesis

F.8.1 is satisfied for each $H \in \mathcal{H}_z$, while the remaining conditions are easily verified; for example, 12.8.4.4 says $\ker_{C_H(\tilde{V})}(H) = Q_H$, giving (c). Thus (3) implies (4). Finally (4) implies (1) by F.8.5.2, and (4) and (5) are equivalent by Remark F.9.9. □

REMARK 12.8.7. Notice that if one of the equivalent conditions in 12.8.6 holds, then from condition (5), $\langle V^H \rangle = \langle V_+^H \rangle$. Thus the subgroups denoted by "U_H" and "V_H" in section F.9 both coincide with the group denoted by U_H in this section.

When U_H is abelian for all $H \in \mathcal{H}_z$, by parts (4) and (5) of 12.8.6, we can apply lemmas from sections F.8 and F.9 to analyze the amalgam defined by LT and H. Notice in particular by F.8.5 and F.9.11 that in this case the amalgam parameter "b" of those sections is odd and at least 3. On the other hand when U_H is nonabelian, we normally specialize to the case $H = G_1$, and apply methods from the theory of large extraspecial 2-subgroups, which are developed further in the following subsection.

12.8.2. $\langle V^{G_1} \rangle$ nonabelian almost extraspecial subgroups. In this subsection, we consider the case where $\langle V^{G_1} \rangle$ is nonabelian. The analysis in the subsection continues to develop the theory of almost extraspecial 2-subgroups U (i.e., U is nonabelian and $|\Phi(U)| = 2$) begun in section G.2 of Volume I. The theory is a variant of the theory of large extraspecial 2-subgroups appearing in the original classification literature.

In the remainder of the section we take $H := G_1$ and assume that $U := U_H = \langle V^H \rangle$ is nonabelian.

As U is nonabelian, 12.8.4.3 says that:

$$\Phi(U) = V_1.$$

Set $\hat{H} := H/Z(U)$ and $\dot{H} := H/C_H(\hat{U})$.

LEMMA 12.8.8. *(1) $U = U_0 Z(U)$, with U_0 an extraspecial 2-group and $\Phi(U_0) = V_1$.*

(2) Regard V_1 as $\mathbf{F}_2$. Then the map

$$(\tilde{u}_1, \tilde{u}_2) := [u_1, u_2]$$

defines a symmetric bilinear form on $\tilde{U}$ with radical $\widetilde{Z(U)}$ preserved by H^*, which induces an H-invariant symplectic form on $\hat{U}$. If $\Phi(Z(U)) = 1$, then

$$q(\tilde{u}) := u^2$$

defines an H^*-invariant quadratic form on $\tilde{U}$ with bilinear form (,), which induces an H^*-invariant orthogonal space structure on $\hat{U}$.

(3) $V \cap Z(U) = V_1$.

(4) Assume $n \leq 3$, let $I := \langle U^L \rangle$, and $S := O_2(I)$. Then $L \leq I$ and S has the I-chief series

$$1 =: S_0 \leq S_1 \leq \cdots \leq S_{n+1} := S$$

described in G.2.3 or G.2.5, for $n = 2$ or 3, respectively.

(5) If $L_1 \trianglelefteq H$ then $n \leq 3$, and when $n = 3$ the chief series in (3) becomes

$$1 =: S_0 < S_1 < S_3 = S$$

with

$$S_1 := V = U \cap U^g \cap U^h, \ S = (U \cap U^g \cap S)(U \cap U^h \cap S)(U^g \cap U^h \cap S),$$

and S/V the sum of copies of the dual of $\tilde{V}$ as an L_1^*-module, for each $g, h \in L$ with $V = V_1 \oplus V_1^g \oplus V_1^h$.

(6) $U = \langle V_2^H \rangle$.

PROOF. Recall $\Phi(U) = V_1$; then (1) and (2) follow from standard arguments (cf. 23.10 in [**Asc86a**]). As L_1 is irreducible on $\tilde{V}$, either (3) holds or $V \leq Z(U)$, and the latter is impossible as $U = \langle V^H \rangle$. By 12.8.4, Hypothesis G.2.1 is satisfied, and we recall as in section G.2 that as U is nonabelian, the hypothesis in G.2.3 and G.2.5 that $U \not\leq C_T(V) = O_2(LT)$ is satisfied, so that (4) follows from those results.

Assume $L_1 \trianglelefteq H$. Then by 12.8.5.1, $\tilde{U}$ is the direct sum of copies of $\tilde{V}$ as a module for $L_1^* \cong L_{n-1}(2)'$. By (2) and (3), the bilinear form (,) induces an L_1-equivariant isomorphism between $U/C_U(V)$ and the dual space of $\tilde{V}$. But if $n > 3$, then $\tilde{V}$ is not isomorphic to its dual as an L_1^*-module; so we conclude $n \leq 3$. Assume $n = 3$. Then $L_1^* \cong \mathbf{Z}_3$, $\tilde{U} = [\tilde{U}, L_1^*]$, and all chief factors for L_1 on $(S \cap U)/V$ are 2-dimensional. Therefore by (4) and G.2.5,

$$V =: S_1 = S_2 = U \cap U^g \cap U^h$$

since $[I, S_2] \leq V$ by G.2.5.5. Similarly $S = S_3$, as if $S/S_3 \neq 1$, then from G.2.5.7, L_1 has a 1-dimensional chief factor on $(U \cap S)S_3/S_3$. This completes the proof of (5).

Next

$$U = \langle V^H \rangle = \langle V_2^{L_1 H} \rangle = \langle V_2^H \rangle,$$

giving (6). This completes the proof of 12.8.8. □

We continue to establish analogues of results in the literature on large extraspecial subgroups. In Hypotheses G.10.1 and G.11.1 in Volume I, we axiomatized some of the properties that are satisfied by $C_{G_0}(z_0)/O_2(G_0)$ acting on $O_2(G_0)/\langle z_0 \rangle$, when $O_2(G_0)$ is a large almost extraspecial 2-subgroup of a group G_0. In 12.8.12, we verify these hypotheses in our setup, and after that we appeal to the results in sections G.10 and G.11, particularly Theorem G.11.2.

Notice for example that G.10.2 is an analogue of 3.8 in Timmesfeld [**Tim78**]. If G is of Lie type, with the involution centralizer G_1 a maximal parabolic, the subgroup I_2 below corresponds to the complementary minimal parabolic. In the theory of large extraspecial 2-subgroups, the inequality in G.10.2 typically produced a lower bound on the 2-rank of $H/C_H(\hat{U})$. But here H is an SQTK-group over which we have some control, so that G.10.2 serves as an upper bound on $m(\hat{U})$, which we then use in 12.8.12 (via an appeal to Theorem G.11.2) in order to strongly restrict the structure of $\dot{H}$ and its action on $\hat{U}$.

Let P be the minimal parabolic of LT acting nontrivially on V_2; notice under part (2) of Hypothesis 12.8.1 that $P = LT$. Set $I_2 := \langle U^P \rangle$, $W := C_U(V_2)$, and let $g \in P - H$. Set $E := W \cap W^g$, $X := W^g$, and $Z_U := Z(U)$. Observe $Z_U \leq W$ as $V_2 \leq U$.

LEMMA 12.8.9. *(1) $\langle U^H \rangle = O^2(P)U = I_2 = \langle U, U^g \rangle$, $C_{I_2}(V_2) = O_2(I_2)$, and $O^2(P)$ and I_2 are normal in G_2.*

(2) $O_2(I_2) = WX$, $[E, I_2] = V_2$, *and* $O_2(I_2)/E = W/E \oplus X/E$ *is the direct sum of natural modules for* $I_2/O_2(I_2) \cong L_2(2) \cong S_3$.

(3) $C_{O_2(I_2)/E}(u) = [O_2(I_2)/E\ ,\ u] = [X/E\ ,\ u] = W/E$ *for* $u \in U - W$.

(4) For $y \in X - W$, $C_{\tilde{U}}(y) \leq \tilde{W}$.

(5) $C_X(\hat{U}) = C_X(\tilde{U}) = E$.

(6) For $u \in U - W$, $C_X(\hat{u}) \leq Z_U^g E$.

(7) $V_1^g \cap Z_U = 1$.

PROOF. Observe (7) holds as $g \in N_L(V_2) - G_1$, and $V \cap Z_U = V_1$ by 12.8.8.3. As V is the natural module for $\bar{L}\bar{T}$ and $O_2(LT) = C_{LT}(V)$, $P = O^2(P)T$ with $C_P(V_2) = O_2(P)$ and $P/O_2(P) = GL(V_2) \cong L_2(2)$. As U is nonabelian, $[V_2, U] \neq 1$ by 12.8.8.6, so $O^2(P) = [O^2(P), U]$ and $P = \langle U, U^g \rangle O_2(P)$. Thus $I_2 = O^2(P)U$. As $Aut_P(V_2) = GL(V_2)$, $G_2 = C_G(V_2)P$, so as $C_G(V_2) \leq G_1 \leq N_G(U)$, we conclude $I_2 = \langle U, U^g \rangle = \langle U^{G_2} \rangle \trianglelefteq G_2$, so $O^2(P) = O^2(I_2) \trianglelefteq G_2$, completing the proof of (1).

By 12.8.8.6, Hypothesis G.2.1 is satisfied with $O^2(P)$, V_2, 1 in the roles of "L, V, L_1"; further $U = \langle V_2^H \rangle$ by 12.8.8.6, so (2) and (3) follow from G.2.3.

Pick $u \in U - W$; by (3), $[X, u] \leq W$, so we can define $\varphi : X \to W/E$ by $\varphi(x) := [x, u]E$. Set $D := \varphi^{-1}(Z_U E/E)$. By (3), $C_X(\hat{U}) \leq D$, and

$$m(X/D) = m(W/Z_U E).$$

As $O_2(I_2)/E$ is the sum of natural modules for $I_2/O_2(I_2) \cong S_3$ by (2),

$$DZ_U = \langle Z_U^{I_2} \rangle E = Z_U Z_U^g E,$$

so $D = Z_U^g E$. Thus if $y \notin Z_U^g E$, then $[y, u] \notin Z_U E$, and in particular $[y, u] \notin Z_U$, so (6) holds. Similarly for $y \in X - W$ and $u \in U - W$, $[y, u] \notin E$ by (2) and in particular $[y, u] \notin V_1$, so $[y, \tilde{u}] \neq 1$. Thus (4) holds.

Of course $E \leq C_X(\tilde{U}) \leq C_X(\hat{U})$. Let $R := C_T(\hat{U})$ and $\tilde{V}_0 := C_{\tilde{U}}(R)$. By a Frattini Argument, $H = C_H(\hat{U})N_H(R)$; so as $\tilde{V}_2 \leq \tilde{V}_0$, as $\tilde{V}_0$ is normalized by $N_H(R)$, and as $\hat{U} = \langle \hat{V}_2^H \rangle$, we conclude that $\tilde{U} = \tilde{V}_0 \tilde{Z}_U$. In particular as $\tilde{Z}_U \leq \tilde{W} < \tilde{U}$, R centralizes some $\tilde{u} \in \tilde{U} - \tilde{W}$, so by (4), $X \cap R \leq X \cap W = E$, completing the proof of (5). □

LEMMA 12.8.10. *(1)* $Z_U \cap U^g = (Z_U \cap Z_U^g)V_1$.

(2) $Z_U \cap Z_U^g = Z(I_2)$.

(3) If $Z_U \cap U^g > V_1$, *then* $Z \cap Z(I_2) \neq 1$.

(4) $[W, Z_U^g] \leq Z_U V_2$, *so* $m([\hat{U}, x]) \leq 2$ *for* $x \in Z_U^g$.

(5) If $Z_U^g \leq U$, *then* $Z_U = Z(I_2) \times V_1$ *and* $[L, Z(I_2)] = 1$.

(6) $C_{Z_U^g}(\hat{U}) = C_{Z_U^g}(\tilde{U}) = Z_U^g \cap Q_H = Z_U^g \cap U = (Z_U^g \cap Z_U)V_1^g = Z(I_2)V_1^g \leq U$.

PROOF. By 12.8.9.7, $V_1^g \cap Z_U = 1$, and by 12.8.8.1, $Z(W) = V_2 Z_U$. Thus by symmetry between U and U^g, $V_1 \cap Z_U^g = 1$ and $Z(X) = V_2 Z_U^g = V_1 Z_U^g$.

By 12.8.4.2, $[Z_U \cap U^g, X] \leq V_1^g \cap Z_U = 1$, so $Z_U \cap U^g \leq Z(X)$. Therefore $Z_U \cap U^g \leq Z_U^g V_1$ by the previous paragraph, so as $V_1 \leq Z_U \cap U^g$, (1) holds by the Dedekind Modular Law.

By 12.8.9, $I_2 = \langle U, U^g \rangle$, so $Z_U \cap Z_U^g \leq Z(I_2)$. To prove the reverse inclusion, observe by 12.8.9.2 that $Z(I_2) \leq W \cap X$, so $Z(I_2) = Z(I_2) \cap U \leq Z(U)$, and similarly $Z(I_2) \leq Z(U^g)$. Thus (2) holds. As T acts on V_2, T acts on I_2 by 12.8.9.1, and hence on $Z(I_2)$. Further if $Z_U \cap U^g > V_1$ then $Z(I_2) \neq 1$ by (1) and (2), so $C_{Z(I_2)}(T) \neq 1$ and hence (3) holds.

Next $[Z_U^g, W] \leq Z_U^g \cap U \leq Z_U V_2$ by (1) and symmetry between U and U^g. Thus any $x \in Z_U^g$ either centralizes the hyperplane $\hat{W}$ of $\hat{U}$, or induces a transvection on $\hat{W}$ with center $\hat{V}_2$, so (4) follows.

To prove (5), assume $Z_U^g \leq U$. Then by (1) and symmetry between U and U^g, $Z_U^g = (Z_U \cap Z_U^g)V_1^g$, so $Z_U^g = Z(I_2) \times V_1^g$ by (2). Then as U is conjugate to U^g in I_2, the first assertion of (5) holds.

Next let $P_1, \ldots, P_{n-1}$ denote the minimal parabolics of L with the usual ordering so that $N_L(V_i) = \langle P_j : j \neq i \rangle$. Define $H_i := \langle O^2(P_j) : j \leq i \rangle$. We argue by induction on j that each H_j centralizes $Z(I_2)$; and then in particular $H_{n-1} = L$ centralizes $Z(I_2)$, which will complete the proof of (5). First $H_1 = O^2(P) = O^2(I_2)$ from 12.8.9.1, and hence H_1 centralizes $Z(I_2)$. Now suppose that $[Z(I_2), H_j] = 1$ for some $1 \leq j < n-1$. Then H_jT is a maximal parabolic subgroup of $H_{j+1}T$, and so there is $k \in P_{j+1} - H_jT$ such that $H_{j+1} = \langle H_j, H_j^k \rangle$ centralizes $F := Z(I_2) \cap Z(I_2)^k$. Now $k \in P_{j+1} \leq N_L(V_1) \leq H \leq N_G(U)$, so that $Z(I_2)$ and $Z(I_2)^k$ are hyperplanes of Z_U using the result of the previous paragraph. Hence FV_1 is of codimension at most 1 in Z_U and is centralized by H_{j+1}, so $H_{j+1} = O^2(H_{j+1})$ centralizes $Z_U \geq Z(I_2)$ by Coprime Action. This completes our inductive proof of the remaining assertion of (5).

Finally by 12.8.9.5, $C_{Z_U^g}(\hat{U}) = C_{Z_U^g}(\tilde{U}) \leq Z_U^g \cap U$, and the reverse inclusion is immediate. Further $C_{Z_U^g}(\tilde{U}) = Z_U^g \cap Q_H$ by 12.8.4.4, and the remaining equalities in (6) follow from (1) and (2). □

LEMMA 12.8.11. *(1)* $[W, X] \leq E$.

(2) $\Phi(E) = 1$*, so* $\hat{E}$ *is totally isotropic in the symplectic space* $\hat{U}$*.*

(3) X *induces the full group of transvections on* $\hat{E}$ *with center* $\hat{V}_2$*.*

(4) $C_{\hat{E}}(X) = \hat{V}_2$*.*

(5) $m(\hat{E}) + m(\dot{X}/\dot{Z}_U^g) = m(\hat{U}) - 1$*.*

(6) If $C_{\hat{U}}(X) > \hat{V}_2$*, then there exists* $1 \neq \dot{x} \in \dot{X}$ *such that* $m([\hat{U}, \dot{x}]) \leq 2$ *and* $\hat{V}_2 \leq [\hat{U}, \dot{x}]$.

PROOF. By 12.8.9.2, (1) holds. As $E = W \cap X$, $[E, X] \leq V_1^g$ by 12.8.4.2. As $\Phi(E) \leq \Phi(U) \cap \Phi(U^g) = V_1 \cap V_1^g = 1$, (2) holds.

By 12.8.8.1, $U = U_0 Z_U$ with U_0 extraspecial. Let $E_0 := EZ_U^g \cap U_0^g$ and $V_0 := V_1 Z_U^g \cap U_0^g$. Then $EZ_U^g = E_0 Z_U^g$ and $V_2 Z_U^g = V_1 Z_U^g = V_0 Z_U^g$. As $V_1 Z_U^g = V_0 Z_U^g$, $X = W^g = C_{U^g}(V_1) = C_{U^g}(V_0)$. As E is abelian and centralizes Z_U^g, E_0 is also abelian. Therefore as U_0 is extraspecial, we conclude from these two remarks that:

(!) X induces the full group of transvections on E_0 which have center V_1^g, and centralize V_0.

Let $\hat{e} \in \hat{E} - \hat{V}_2$. As $EZ_U^g = E_0 Z_U^g$, $eZ_U^g = e_0 Z_U^g$ for some $e_0 \in E_0$. Now by 12.8.10.1, $Z_U \cap E \leq Z_U \cap U^g = (Z_U \cap Z_U^g)V_1 \leq Z_U \cap E$, so that all inequalities are equalities. Hence $E \cap V_2 Z_U = V_2(Z_U \cap E) = V_2(Z_U \cap Z_U^g)$, and so by symmetry between U and U^g, $E \cap V_2 Z_U = E \cap V_2 Z_U^g$. Thus as $\hat{e} \notin \hat{V}_2$, $e \notin V_2 Z_U^g$, so as we saw that $V_2 Z_U^g = V_0 Z_U^g$, $e_0 \notin V_0 Z_U^g$. Thus $[e, X] = [e_0, X] = V_1$ by (!). Hence (3) holds, and of course (3) implies (4).

Next

$$m(\hat{U}) = m(\hat{E}) + m(\hat{W}/\hat{E}) + 1 = m(\hat{E}) + m(X/EZ_U^g) + 1 = m(\hat{E}) + m(\dot{X}/\dot{Z}_U^g) + 1,$$

as $E = C_X(\hat{U})$ by 12.8.9.5. That is, (5) holds.

Let $\hat{F} := C_{\hat{U}}(X)$ and suppose $\hat{F} > \hat{V}_2$. Then by (4), $\hat{F} \not\leq \hat{E}$, while by 12.8.9.5, $C_U(U^g/Z_U^g) = E$, so $F \not\leq C_U(U^g/Z_U^g)$. Now by 12.8.9.2, $F \leq O_2(I_2) \leq N_G(X)$, so $[X, F] \leq Z_U \cap W^g \leq Z_U^g V_2$ by 12.8.10.1. Hence conjugating in I_2, $\dot{X}_0 := C_{\dot{X}}(\hat{W}/\hat{V}_2) \neq 1$. If $1 \neq \dot{x} \in \dot{X}_0$ centralizes $\hat{W}$, then $\dot{x}$ is a transvection on $\hat{U}$ with axis $\hat{W}$ and center $\hat{V}_2$, so (6) holds. If $\dot{x}$ does not centralize $\hat{W}$, then $\hat{V}_2 = [\hat{W}, x] \leq [\hat{U}, x]$ so as $\hat{W}$ is a hyerplane of $\hat{U}$, $m([\hat{U}, x]) = 2$ and again (6) holds. Thus (6) is established. □

We are in a position to appeal to results in Volume I on centralizers with a large almost extraspecial subgroup:

LEMMA 12.8.12. *(1) Hypothesis G.10.1 is satisfied with $\dot{H}$, $\hat{U}$, $\hat{V}_2$, $\hat{E}$, $\dot{X}$, $\dot{Z}_U^g$ in the roles of "G, V, V_1, W, X, X_0".*

(2) Let $H_2 := H \cap G_2$. Then $\dot{X}$ and $\dot{Z}_U^g$ are normal in $\dot{H}_2$, so in particular $\dot{X} \trianglelefteq \dot{T}$.

(3) Hypothesis G.11.1 is satisfied.

(4) $\dot{H}$ and its action on $\hat{U}$ satisfy one of the conclusions of Theorem G.11.2.

PROOF. By 12.8.8.2, $\hat{U}$ is a symplectic space and $\dot{H} \leq Sp(\hat{U})$, so Hypothesis G.10.1.1 holds. By 12.8.11.2, $\hat{E}$ is totally isotropic. By 12.8.8.6, $\hat{U} = \langle \hat{V}_2^H \rangle$. As T acts on V_2, $\dot{T}$ fixes the point $\hat{V}_2$ of $\hat{U}$, so part (a) of Hypothesis G.10.1.2 holds. By 12.8.9.5, E is the kernel of the action of X on $\hat{U}$. Observe also that $\hat{W} = \hat{V}_2^{\perp}$; thus if $\dot{x} \in \dot{X} - \dot{Z}_U^g$, then $x \notin Z_U^g E$, so 12.8.9.6 shows that $C_{\hat{U}}(x) \leq \hat{W} = \hat{V}_2^{\perp}$, establishing hypothesis (d). Hypothesis (b) follows from 12.8.11.5, hypothesis (c) from 12.8.11.1, and hypothesis (e) from 12.8.11.3. This completes the proof of (1).

Next as $[V_2, U] = V_1$, $H_2 = C_G(V_2)U$; so to prove (2), it suffices to show that X and Z_U^g are normal in $C_G(V_2)$. But this follows as $C_G(V_2)$ acts on U^g and V_1.

Observe that part (4) of Hypothesis G.11.1 follows from (2), and hypothesis (3) follows from 12.8.11.6. Thus (3) holds. Finally $\dot{H}$ is a quotient of the SQTK-group H, so (3) and Theorem G.11.2 imply (4). □

Using Hypothesis 12.8.1, we can refine some of the results from sections G.10 and G.11:

LEMMA 12.8.13. *(1) $V \leq E$.*

(2) Z_U^g centralizes V.

(3) If $n = 2$, then $Z_U \cap Z_U^g = Z(I_2) = 1$, so $\dot{Z}_U^g \cong \tilde{Z}_U$ and $[Z_U, Z_U^g] = 1$.

(4) If $Z_U > V_1$ then $\dot{Z}_U^g \neq 1$.

(5) If $\hat{U}$ is the 6-dimensional orthogonal module for $F^*(\dot{H}) \cong A_8$, then $O^{3'}(H) =: K \in \mathcal{C}(H)$ with $K/O_2(K) \cong A_8$, $Z_D := Z \cap Z(I_2) \neq 1$, $V_D := \langle Z_D^K \rangle \leq Z_U$, $V_D \in \mathcal{R}_2(KT)$, $1 \neq [Z_D, K]$, and $K = [K, Z_U^g] \in \mathcal{L}_f(G, T)$.

(6) Conclusion (4) of G.11.2 does not hold; that is, $\hat{U}$ is not the natural module for $F^*(\dot{H}) \cong A_7$.

(7) Conclusion (12) of G.11.2 does not hold.

(8) $m_3(C_H(\tilde{V}_2)) \leq 1$.

PROOF. As $V \leq U$ and $g \in N_G(V)$, $V \leq U^g$, so (1) and (2) hold.

If $n = 2$, then by Hypothesis 12.8.1, $\mathbf{Z}_2 \cong Z \leq V$, so $Z = V_1 \not\leq Z(I_2)$, and hence $Z(I_2) = 1$. Therefore $[Z_U, Z_U^g] \leq Z_U \cap Z_U^g = Z(I_2) = 1$. It follows from

12.8.10.6 that $C_{Z_U^g}(\hat{U}) = V_1^g$, so that $\dot{Z}_U^g \cong Z_U/V_1 = \tilde{Z}_U$, completing the proof of (3).

Suppose (4) fails; then $\tilde{Z}_U \neq 1$ but Z_U^g centralizes $\hat{U}$. First $n > 2$ as there $\dot{Z}_U^g \cong \tilde{Z}_U$ by (3). Next by 12.8.10.6, $Z_U^g = C_{Z_U^g}(\hat{U}) \leq U$; so by 12.8.10.5, $Z_U = V_1 \times Z(I_2)$ with $[L, Z(I_2)] = 1$. Then $N_G(Z(I_2)) \leq M = !\mathcal{M}(LT)$. Let $J := \langle L_1^H \rangle$ and suppose $L_1 < J$. Now $L_1 \leq C_L(V_1) = C_L(V_1 Z(I_2)) = C_L(Z_U)$, so $J \leq C_H(Z_U) \leq N_H(Z(I_2)) \leq M_1$. If $n > 3$ then by 12.2.8, $J \leq O^{3'}(H \cap M) = L_1$ contrary to assumption. Hence $n = 3$, so $L_1/O_2(L_1) \cong \mathbf{Z}_3$. Then since M is an SQTK-group and $J = \langle L_1^H \rangle$, $J/O_2(J) \cong E_9$ and $J = L_1 J_C$, where $J_C := O^2(C_J(\bar{L})$. As $L_1 = [L_1, T \cap L]$, L_1 and J_C are the only T-invariant subgroups Y_1 of J with $|Y_1 : O_2(Y_1)| = 3$. Thus H is not transitive on the four subgroups Y of J with $|Y : O_2(Y)| = 3$, and we conclude $|H : N_H(L_1)| = 3$ and $J_C \trianglelefteq H$. But $J_C \trianglelefteq LT$, so $H \leq N_G(J_C) \leq M = !\mathcal{M}(LT)$, a contradiction. Therefore $L_1 = J \trianglelefteq H$. Thus by 12.8.5.1, $C_{\tilde{U}}(L_1) = 1$. However by hypothesis $\tilde{Z}_U \neq 1$, and we had seen that L_1 centralizes Z_U. This contradiction establishes (4).

By 12.8.9.1, $I_2 \trianglelefteq G_2$, and a Sylow 3-subgroup of I_2 is faithful on V_2. Thus if (8) fails, then $C_H(V_2)$ contains $Y \cong E_9$ and $m_3(G_2) \geq m_3(I_2 Y) > 2$, contradicting G_2 an SQTK-group. So (8) is established.

Define $H_2 := O^{3'}(H \cap G_2)$ and observe that $H_2 = O^{3'}(C_H(\tilde{V}_2))$. We see that if $m_3(C_{\dot{H}}(\hat{V}_2)) > 1$, then by (8), $O^{3'}(C_H(\hat{V}_2)) < H_2$, so H_2 does not centralize Z_U by Coprime Action. Hence $Z_U > V_1$, so $\dot{Z}_U^g \neq 1$ by (4) under this assumption.

Asume the hypotheses of (5). Then by 1.2.1.1, there is $K \in \mathcal{C}(H)$ with $\dot{K} = F^*(\dot{H})$, so by 1.2.1.4, $K/O_2(K) \cong A_8$. Then $K = O^{3'}(H)$ by A.3.18. Next $O^{3'}(C_{\dot{H}}(\hat{V}_2)) \cong E_9/E_{16}$, so by the previous paragraph, $[Z_U, K] \neq 1$ and $\dot{Z}_U^g \neq 1$. As $\dot{Z}_U^g \neq 1$, $K = [K, Z_U^g]$; so as $[Z_U, K] \neq 1$, $Z_U^g \not\leq C_H(Z_U)$. Then $1 \neq [Z_U, Z_U^g] \leq Z_U \cap Z_U^g = Z(I_2)$ by 12.8.10.2. Thus $Z_D := Z \cap Z(I_2) \neq 1$ by 12.8.10.3. Therefore $n > 2$ by (3), so $L_1 \neq 1$ and $L_1 \leq O^{3'}(H) = K$. Thus if $[Z_D, K] = 1$, then $LT = \langle I_2, L_1 T \rangle$ centralizes Z_D, so $K \leq C_G(Z_D) \leq M = !\mathcal{M}(LT)$. This is impossible as $L_1 \trianglelefteq M_1$, but L_1 is not normal in K as $K/O_2(K) \cong A_8$. Thus $[Z_D, K] \neq 1$. As $V_D \in \mathcal{R}_2(KT)$ by B.2.13, $K \in \mathcal{L}_f(G,T)$, so (5) holds.

Assume that (7) fails; thus $F^*(\dot{H}) = \dot{K} \times \dot{K}^x$ for $x \in H - N_H(K)$, $\hat{U} = [\hat{U}, \dot{K}] \oplus [\hat{U}, \dot{K}^x]$ with $[\hat{U}, \dot{K}]$ the A_5-module for $\dot{K} \cong L_2(4)$, and $\dot{X} = \langle \dot{x} \rangle (\dot{X} \cap \dot{K}\dot{K}^x) \cong E_8$. Then $O^2(C_{\dot{H}}(\hat{V}_2))$ is a Borel subgroup of $E(\dot{H})$, and hence of 3-rank 2, so $\dot{Z}_U^g \neq 1$ by an earlier observation. On the other hand, in this case $m(\dot{X}) = 3$ and $m(\hat{U}) = 8$ so that $m(\hat{E}) \leq 4$ as $\hat{E}$ is totally isotropic by 12.8.11.2. Therefore by 12.8.11.5, $m(\hat{E}) = 4$ and $m(\dot{W}^g/\dot{Z}_U^g) = 3$, contradicting $\dot{Z}_U^g \neq 1$. So (7) is established.

Finally assume that (6) fails; that is, $F^*(\dot{H}) \cong A_7$ and $\hat{U}$ the 6-dimensional permutation module. Then $\hat{U}$ is described in section B.3, and we adopt the notation of that section. By 12.8.11.5:

$$m(\hat{E}) = m(\hat{U}) - m(\dot{X}/\dot{Z}_U^g) - 1 \geq 5 - m_2(\dot{H}) \geq 2. \qquad (*)$$

Thus $\hat{E} > \hat{V}_2$ as $m(\hat{V}_2) = 1$, so we conclude from 12.8.11.3 that $\hat{V}_2 = [\hat{E}, X] \leq [\hat{V}_2^{\perp}, T] \leq \hat{V}_2^{\perp}$; it follows that a generator $\hat{v}$ for $\hat{V}_2$ is not of weight 2 or 6, so that $\hat{v}$ is of weight 4. Hence, in the notation of section B.3, we may choose $\hat{v} = e_{1,2,3,4}$, so

$$\hat{V}_2^{\perp} = \{e_J : |J| \text{ and } |J \cap \{1,2,3,4\}| \text{ are even}\}.$$

In particular $\hat{U}_+ := \{0, e_{5,6}, e_{5,7}, e_{6,7}\} \leq \hat{V}_2^\perp$ is T-invariant but does not contain $\hat{V}_2$, so by 12.8.11.4, $\hat{U}_+ \cap \hat{E} = 1$. Then by 12.8.11.1, $[\hat{U}_+, X] \leq \hat{U}_+ \cap \hat{E} = 1$.

Next $O^2(C_{\dot{H}}(\hat{V}_2)) \cong A_4 \times \mathbf{Z}_3$, so by an earlier observation, $\dot{Z}_U^g \neq 1$. Thus (*) implies $m(\hat{E}) \geq 3$, with equality only if $m(\dot{X}) = 3$. By 12.8.11.2, $\hat{E}$ is totally isotropic so that $m(\hat{E}) \leq 3 = m(\dot{X})$. However we showed in the previous paragraph that $\dot{X} \leq C_{\dot{T}}(\hat{U}_+)$, which is a contradiction as $C_{\dot{T}}(\hat{U}_+) \cong D_8$.

This completes the proof of 12.8.13. $\square$

12.9. The final treatment of $\mathbf{L_n(2)}$, $\mathbf{n = 4, 5}$, on the natural module

In this section we prove:

THEOREM 12.9.1. *Assume Hypothesis 12.2.1 with $L/O_2(L) \cong L_n(2)$, $n = 4$ or 5. Then $n = 4$, and G is isomorphic to $L_5(2)$ or M_{24}.*

We recall that the QTKE-groups $G \cong L_5(2)$ and M_{24} appear as conclusions in Theorem 12.2.13. In proving Theorem 12.9.1 we verify that Hypothesis 12.8.1 holds and apply Theorem 12.2.13 to establish 12.8.3.4. Two groups appear as shadows: The sporadic group Co_3 has a 2-local $L \in \mathcal{L}_f^*(G,T)$ with $L \cong L_4(2)/E_{2^4}$; but Co_3 is neither quasithin nor of even characteristic, in view of the involution centralizer $Sp_6(2)/\mathbf{Z}_2$, and is essentially eliminated in 12.9.3 below. Similarly the sporadic Thompson group F_3 contains $L \in \mathcal{L}_f^*(G,T)$ with $L \cong L_5(2)/E_{2^5}$, but F_3 is not quasithin in view of the involution centralizer $A_9/2^{1+8}$, and is eliminated in 12.9.4.

Furthermore in many groups of large rank there is $L \in \mathcal{L}_f(G,T)$ which is not maximal, but satisfies the rest of the hypothesis of Theorem 12.9.1: namely in many groups of Lie type over $\mathbf{F}_2$, as well as in the sporadic groups F_3, the Baby Monster, and the Monster. In addition the Conway group Co_2 has a 2-local L not containing a Sylow group with structure $L_4(2)/(E_{2^4} \times 2^{1+6})$. These groups are of course not quasithin, and the configurations are also eliminated in 12.9.3 and 12.9.4.

The proof of Theorem 12.9.1 involves a series of reductions. Assume G, L afford a counterexample to Theorem 12.9.1, and choose the counterexample so that $n = 5$ if that choice is possible. Neither A_9 nor the groups appearing in conclusions (1) and (2) of Theorem 12.2.2 contain $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L) \cong L_n(2)$ for $n = 4$ or 5. Thus Hypothesis 12.2.3 is satisfied, we can pick V as in Theorem 12.2.2.3, and $N_G(L) =: M \in \mathcal{M}(T)$. Then Theorems 12.5.1 and 12.6.34 eliminate cases (c) and (d) of Theorem 12.2.2.3, so we conclude that V is the natural module for $L/O_2(L)$. As G, L affords a counterexample to Theorem 12.9.1, G is neither $L_5(2)$ nor M_{24}. Also G is not $L_6(2)$ as G is quasithin, and G is not A_9 as we observed earlier. Thus Hypothesis 12.8.1 is satisfied, so we can appeal to the results of section 12.8, and adopt the conventions of Notation 12.8.2 of that section. Recall that $G_1 \not\leq M$ by 12.8.3.4, so that $G_1 \in \mathcal{H}_z$.

LEMMA 12.9.2. *If $n = 4$, then there is no $K \in \mathcal{L}_f^*(G,T)$ with $K/O_2(K) \cong L_5(2)$, M_{24}, or J_4.*

PROOF. Assume such a K exists. By Remark 12.2.4, Hypothesis 12.2.1 is satisfied with K in the role of "L" and conclusion (3) of Theorem 12.2.2 holds, so $K/O_2(K) \cong L_5(2)$. This is a contradiction as $n = 4$ by hypothesis, contrary to our choice of $n = 5$ if such a choice is possible. $\square$

LEMMA 12.9.3. *Let $1 \le i < 5$ when $n = 5$,, and $i = 1$ or 3 when $n = 4$. Then $L_i \le K_i \in \mathcal{C}(N_G(V_i))$ with $K_i \trianglelefteq N_G(V_i)$, and one of the following holds:*

(1) $L_i = K_i$.

(2) $i = 1$, and $K_1/O_2(K_1) \cong L_5(2)$, M_{24}, or J_4.

(3) $i = 1$, $n = 4$, and $K_i/O_2(K_i) \cong L_4(2)$, A_7, $\hat{A}_7$, M_{23}, HS, He, Ru, or $SL_2(7)/E_{49}$.

PROOF. The proof is of course similar to that of 12.5.3: First $L_i \in \mathcal{L}(G,T)$, so the existence and normality of K_i follow from 1.2.4. If $K_i > L_i$, the possibilities for $K_i/O_2(K_i)$ are given by the sublist of A.3.12 for $L_i/O_2(L_i) \cong L_k(2)$ for a suitable choice of $k := 3$ or 4. When $k = 4$ we obtain the groups in conclusion (2). When $k = 3$ we obtain the groups in conclusions (2) and (3), along with $L_2(49)$ and $(S)L_3^\epsilon(7)$—but these last cases are out, since there T acts nontrivially on the Dynkin diagram of $L_1/O_2(L_1)$, which is not the case by 12.8.3.1.

Thus when $i = 1$ the lemma is established, so we may assume $i > 1$ and $L_i < K_i$, and it remains to derive a contradiction. Set $K_1^*T^* := K_1T/O_2(K_1T)$.

Assume first that $i = 3$ or 4. Then $L_i/C_{L_i}(V_i) = GL(V_i)$, so $K_i = L_iC_{K_i}(V_i)$. Hence $K_i = L_i$ if $K_i/O_2(K_i)$ is quasisimple, contrary to our assumption, so that $K_i/O_2(K_i)$ is not quasisimple. Then from the first paragraph, $K_i/O_2(K_i) \cong SL_2(7)/E_{49}$, $K_i = XL_i$, where $X := \Xi_7(K_i)$, and $i = 3$ since $L_4/O_2(L_4) \cong L_4(2)$ is not involved in K_i. We argue much as in the proof of 12.5.3: Set $K_{1,3} := O^2(C_{K_3}(V_1))$. Then $K_{1,3}T/O_2(K_{1,3}T) \cong SL_2(3)/E_{49}$, since $X \le C_{K_3}(V_3) \le C_G(V_1)$. Further $K_{1,3} = YX$ where $Y := O^{3'}(N_{L_1 \cap L_3}(V_1)) \le K_1$, so that $K_{1,3} = \langle Y^X \rangle \le K_1$ since $K_1 \trianglelefteq N_G(V_1)$. Now $K_{1,3}^*T^*$ is a subgroup of $K_1^*T^*$ containing T^*. But from the structure of the overgroups of T^* in the groups listed in (2) and (3), no subgroup of these groups containing a Sylow 2-subgroup has a $GL_2(3)/E_{49}$-section, except when K_1^* is also $SL_2(7)/E_{49}$. In this last case, $X = O^{7'}(K_{1,3}) = \Xi_7(K_1)$ is normal in K_3 and K_1, so that $L = \langle L_1, L_3 \rangle \le N_G(X)$. Hence $X \le N_G(X) \le M = !\mathcal{M}(LT)$, so that $X = [X, L_3] \le L$, contrary to $m_7(L) = 1$. This contradiction completes the proof that $K_i = L_i$ if $i = 3$ or 4.

Finally take $i = 2$. Thus $n = 5$ by our choice of i in the hypothesis, so $L_2/O_2(L_2) \cong L_3(2)$, and $L_2 \le L_1$ with $L_1/O_2(L_1) \cong L_4(2)$. In particular $m_3(L_1) = 2$, and $L_1 = O^{3'}(N_G(L_1))$ by A.3.18. We conclude $G_2 \not\le N_G(L_1)$, since G_2 contains a subgroup X of order 3 faithful on V_2, whereas if $G_2 \le N_G(L_1)$, then $X \le O^{3'}(N_G(L_1)) = L_1 \le G_1$. Similarly when $L_1 < K_1$ we conclude that $G_2 \not\le N_G(K_1)$.

We now claim

$$L_2T < K_2T < K_1T \text{ and } K_2T \neq L_1T.$$

First as $\dim(V_2) = 2$, $K_2 = K_2^\infty \le C_G(V_2) \le C_G(V_1)$. Then as $L_2 < L_1 \le K_1 \trianglelefteq N_G(V_1)$, $K_2 = [K_2, L_2] \le K_1$. As G_2 does not act on L_1 or K_1, $K_2 < K_1$ and $K_2 \neq L_1$. Finally by assumption, $L_2 < K_2$, so the claim holds.

Now if $L_1 = K_1$, then L_2T is maximal in $L_1T = K_1T$, contrary to $L_2T < K_2T < K_1T$. Thus $L_1 < K_1$, so that K_1^* is in the list of (2). Observe in each of those three groups that $L_1^*T^*$ is determined (up to outer automorphism when K_1^* is $L_5(2)$) as the unique overgroup of T^* in K_1^* with $L_1^*T^*/O_2(L_1^*T^*) \cong L_4(2)$. Suppose first that $K_1^* \cong M_{24}$. Then from the list of overgroups of T^*, L_2^* is normal in each overgroup of $L_2^*T^*$ other than $L_1^*T^*$, contradicting $L_2T < K_2T < K_1T$ with L_2 not normal in $K_2T \neq L_1T$. Therefore $K_1^* \cong L_5(2)$ or J_4, and a similar argument shows that $K_2/O_2(K_2)$ is isomorphic to $L_4(2)$ in the former case, and

M_{24} in the latter. Now we can repeat our argument in the first paragraph of the proof for the case $i = 2$: In each case $K_2 = O^{3'}(N_G(K_2))$ by A.3.18. Also we saw $K_2 \leq C_G(V_2)$, while G_2 contains a subgroup X of order 3 fixed-point-free on V_2, so $X \leq O^{3'}(N_G(K_2)) = K_2 \leq G_1$, a contradiction. This completes the proof of 12.9.3. □

For the remainder of the section, let K_1 be defined as in 12.9.3.

LEMMA 12.9.4. *$\langle V^{G_1} \rangle$ is abelian.*

PROOF. Assume otherwise; then we have the hypotheses of the latter part of section 12.8, so we can appeal to the results there. Adopt the notation of the second subsection of section 12.8; in particular take $H := G_1$, $U := U_H = \langle V^H \rangle$, and $H^* := H/C_H(\tilde{U})$.

As $n \geq 4$, we conclude from 12.8.8.5 that L_1 is not normal in H, so that $L_1 < K_1$ and in particular $K_1 \not\leq M$. Hence $K_1/O_2(K_1)$ is described in (2) or (3) of 12.9.3. By 12.8.12.4, $\dot{H}$ and its action on $\hat{U}$ are described in Theorem G.11.2. As $[\hat{V}, L_1] \neq 1$, $\dot{L}_1 \neq 1$, so $\dot{K}_1$ is a nontrivial normal subgroup of $\dot{H}$, and is also a quotient of K_1^*.

If $K_1^* \cong SL_2(7)/E_{49}$ then either $\dot{K}_1 = \dot{L}_1 \cong L_3(2)$ or $\dot{K}_1 \cong K_1^*$. However by inspection of the list in Theorem G.11.2, $\dot{H}$ has no such normal subgroup. Thus one of the remaining cases holds, where K_1^* is quasisimple, and hence $\dot{K}_1/Z(\dot{K}_1) \cong K_1^*/Z(K_1^*)$. Comparing the list in (2) and (3) of 12.9.3 to the normal subgroups of groups listed in Theorem G.11.2, we conclude $n = 4$ and one of conclusions (4), (5), or (8) of Theorem G.11.2 holds. Conclusion (8) does not occur, as there $\dot{H} \cong S_7$, so that there is no $\dot{T}$-invariant subgroup $\dot{L}_1$ with $\dot{L}_1/O_2(\dot{L}_1) \cong L_3(2)$. Conclusion (4) does not hold by 12.8.13.6.

Thus conclusion (5) of Theorem G.11.2 holds; that is $\hat{U}$ is the 6-dimensional natural module for $\dot{K}_1 = F^*(\dot{H}) \cong A_8$. Let $D := Z_U^g$, $Z_D := Z \cap Z(I_2)$, and $V_D = \langle Z_D^{K_1} \rangle$. By 12.8.13.5, $K_1 \in \mathcal{L}_f(G,T)$, K_1 acts nontrivially on the submodule V_D of $Z_U \in \mathcal{R}_2(KT)$, and $K_1 = [K_1, D]$.

As $K_1 \in \mathcal{L}_f(G,T)$, $K_1 \leq K \in \mathcal{L}_f^*(G,T)$ by 1.2.9.2. Then either $K_1 = K$, or $K/O_2(K) \cong L_5(2)$, M_{24}, or J_4 by A.3.12. Thus $K_1 = K \in \mathcal{L}_f^*(G,T)$ by 12.9.2.

As $F^*(\dot{H}) \cong A_8$, $\dot{H} \cong A_8$ or S_8, so as T normalizes L_1 with $L_1/O_2(L_1) \cong L_3(2)$, we conclude $\dot{H} \cong A_8$. Thus $\dot{L}_1$ is a maximal parabolic of $\dot{H}$ corresponding to an end node. Next set $L_0 := O^2(C_L(V_2)) = O^2(C_{L_1}(V_2))$. Then $L_0^*T^*$ is the minimal parabolic of L_1^* centralizing $\tilde{V}_2$. As $\hat{V}_2$ is a singular 1-space in the orthogonal space $\hat{U}$, $\dot{L}_0\dot{T}$ is one of the two permuting minimal parabolics in the maximal parabolic $\dot{P}_0 := C_{\dot{H}}(\hat{V}_2)$ corresponding to the middle node of the Dynkin diagram for $\dot{H}$; in particular P_0 normalizes L_0. Similarly $\dot{L}_1$ is the maximal parabolic of $\dot{H}$ normalizing the totally singular 3-subspace $\hat{V}$ of $\hat{U}$, and so corresponds to an end node of the diagram for $\dot{H}$, with $\dot{L}_0 = \dot{P}_0 \cap \dot{L}_1$. Finally $\bar{L}_0\bar{T}$ is the minimal parabolic of $\bar{L}$ centralizing V_2, with $\bar{I}_2\bar{T}$ the other minimal parabolic in the maximal parabolic $N_{\bar{L}}(V_2)$ for the middle node, so that I_2 normalizes L_0.

By 12.8.13.2, $D \leq C_T(V) = O_2(LT)$, and hence $\dot{D} \leq O_2(\dot{L}_1\dot{T})$. By 12.8.12.2, $\dot{D} \trianglelefteq \dot{H}_2 := H \cap G_2$, so as $L_0T \leq H_2$, $\dot{D} \trianglelefteq \dot{L}_0\dot{T}$. By 12.8.10.4, $[\hat{V}_2^{\perp}, D] = [\hat{W}, D] \leq \hat{V}_2$, so $\dot{D} \leq O_2(\dot{P}_0)$. Therefore

$$1 \neq \dot{D} \leq \dot{D}_0 := O_2(\dot{L}_1\dot{T}) \cap O_2(\dot{P}_0) \cong E_4.$$

Then as $\dot{L}_0$ is irreducible on $\dot{D}_0$, and $1 \neq \dot{D} \trianglelefteq \dot{L}_0\dot{T}$, we conclude $\dot{D} = \dot{D}_0 \cong E_4$.

Next by 12.8.10.6, $C_D(\hat{U}) = (D \cap Z_U)V_1^g$, so using symmetry between U and U^g,

$$2 = m(\dot{D}) = m(D/C_D(\hat{U})) = m(D/(D \cap Z_U)V_1^g) = m(Z_U/(D \cap Z_U)V_1). \qquad (*)$$

Recall $g \in I_2 \leq N_G(L_0)$. Further $\dot{D}$ centralizes $(D \cap Z_U)V_1$; but $\dot{D} \neq 1$ does not centralize Z_U since $K_1 = [K_1, D]$ and K_1 is nontrivial on Z_U. Therefore as $\dot{L}_0$ is irreducible on $\dot{D}$, and hence on its g^{-1}-conjugate $Z_U/(D \cap Z_U)V_1$, while $\dot{L}_0$ normalizes $C_{Z_U}(\dot{D})$, we conclude from (*) that $C_{Z_U}(\dot{D}) = (D \cap Z_U)V_1$ is of index 4 in Z_U. Recall $K_1 \in \mathcal{L}_f^*(G,T)$, and $V_D \in \mathcal{R}_2(K_1T)$. By Theorem 12.6.34, each $I_D \in Irr_+(K_1, V_D)$ is a natural 4-dimensional module for K_1^*. As L_0 is irreducible on $Z_U/C_{Z_U}(\dot{D})$, $m(I_D/C_{I_D}(D)) = 2$, so $Z_U = C_{Z_U}(D)I_D$ and $C_{I_D}(D) = [I_D, D] =: D_I$ is of order 4. As $Z_U \leq C_G(V_1^g) \leq N_G(D)$, $D_I \leq D$. Further as $K_1 = [K_1, D]$, K_1 centralizes Z_U/I_D, so $I_D = [Z_U, K_1]$. Therefore $[V_2, O^2(P_0)] \leq [V_2Z_U, O^2(P_0)] = V_2I_D$, so P_0 acts on $C_{V_2I_D}(L_0) = V_2C_{I_D}(L_0)$. Therefore if $C_{I_D}(L_0) = 1$, then P_0 acts on V_2, contrary to 12.8.13.8.

Thus $C_{I_D}(L_0) \neq 1$, so since $Aut_{L_0}(I_D)$ is a minimal parabolic of $GL(I_D)$ and $\dot{L}_0$ normalizes $\dot{D}$, $C_{I_D}(L_0) = D_I$, and so P_0 acts on V_2D_I and on D_I. Finally I_2 acts on V_2 and centralizes D_I by 12.8.10.2, as $D_I \leq Z_U \cap Z_U^g$, so $Y := \langle P_0, I_2\rangle$ acts on V_2D_I and D_I. Then $Aut_{I_2T}(V_2D_I)$ and $Aut_{P_0}(V_2D_I)$ are the two minimal parabolics of $GL(V_2D_I) \cong L_4(2)$ stabilizing the 2-subspace D_I; in particular, $I_2C_Y(V_2D_I)$ is normal in Y. But now as I_2 centralizes D_I, P_0 normalizes $[V_2D_I, I_2C_Y(V_2D_I)] = V_2$, a case we eliminated in the previous paragraph. This contradiction completes the proof of 12.9.4. □

By 12.9.4, $\langle V^{G_1}\rangle$ is abelian, and hence (cf. 12.8.6) so is $U_H = \langle V^H\rangle$ for each $H \in \mathcal{H}_z$.

LEMMA 12.9.5. *(1)* $C_{G_1}(K_1/O_2(K_1)) \leq M$, *so* $K_1T \in \mathcal{H}_z$.

(2) $K_1/O_2(K_1) \cong A_7$, $L_4(2)$, *or* $L_5(2)$.

(3) If $n = 4$ *and* $K_1/O_2(K_1) \cong L_5(2)$, *then* $L_1O_2(K_1)/O_2(K_1)$ *is the centralizer of a transvection in* $K_1/O_2(K_1)$.

PROOF. Observe first that $Out(K_1/O_2(K_1))$ is a 2-group for each possibility in 12.9.3, including $K_1 = L_1$, so that $G_1 = K_1TC_{G_1}(K_1/O_2(K_1))$.

We will combine the proofs of the three parts of the lemma, but in proving (2) we will assume that (1) has already been proved. Thus when proving (2), $L_1 < K_1$ since $G_1 \not\leq M$, so that K_1 is described in 12.9.3. We consider three cases:

Case I. If (1) fails, pick $H_1 \in \mathcal{H}_*(T, M)$ with $O^2(H_1) \leq C_{G_1}(K_1/O_2(K_1))$, and let $H := H_1L_1$.

Case II. If (2) fails, then $L_1 < K_1$ but $K_1/O_2(K_1)$ is not A_7, $L_4(2)$, or $L_5(2)$, and we let $H := K_1T$.

Case III. If (3) fails, then $L_1O_2(K_1)/O_2(K_1)$ is a parabolic determined by an end node and the adjacent node in the Dynkin diagram for $L_5(2)$, and we pick $H_1 \in \mathcal{H}_*(T, M)$ to be the minimal parabolic of K_1 determined by the remaining end node, and let $H := H_1L_1$.

In each case $H \in \mathcal{H}_z$. As 12.9.4 provides condition (2) of 12.8.6, the latter result says that H satisfies Hypotheses F.8.1 and F.9.8 with V in the role of "V_+". Thus we may apply the results in sections F.8 and F.9. In particular we adopt the notation of sections F.7 and F.8 (or F.9) for the amalgam generated by H and LT.

Suppose first that we are in Case II. Then by the choice made in the first paragraph of the proof, K_1^* is one of the groups listed in 12.9.3 other than A_7, $L_4(2)$ or $L_5(2)$, and $H = K_1T$. Then unless $K_1^* \cong SL_2(7)/E_{49}$, $K_1/O_2(K_1)$ is quasisimple, so the hypotheses of F.9.18 are satisfied. But then F.9.18.4 supplies a contradiction, as none of the groups other than A_7, $L_4(2)$, and $L_5(2)$ appear in both F.9.18.4 and 12.9.3. Thus we have reduced to $K_1^* \cong SL_2(7)/E_{49}$. This case is impossible, since by F.9.16.3, $q(H^*, \tilde{U}_H) \leq 2$, contrary to D.2.17 applied to $K_1^*T^*$ in the role of "G".

Thus Case I or III holds, and in either case, $H = H_1L_1$ with $[L_1, O^2(H_1)] \leq O_2(L_1)$, so $C_H(L_1/O_2(L_1)) \leq H_1$. As $H_1 \in \mathcal{H}_*(T, M)$, 3.3.2 says H_1 is a minimal parabolic in the sense of Definition B.6.1. By F.8.5, the parameter b is odd and $b \geq 3$. Then by F.7.3.2, there is $g \in \langle LT, H\rangle = G_0$ mapping the edge γ_{b-1}, γ to γ_0, γ_1, and $h \in H$ with $\gamma_2 h = \gamma_0$. Set $\beta := \gamma_1 g$, $\delta := \gamma h$, and let $\alpha \in \{\beta, \delta\}$; then $U_\alpha \leq O_2(G_{\gamma_0,\gamma_1})$ by F.8.7.2. Therefore as $L_1 \trianglelefteq H \geq G_{\gamma_0,\gamma_1}$, $[L_1, U_\alpha] \leq O_2(L_1)$, and hence $U_\alpha \leq C_H(L_1/O_2(L_1)) \leq H_1$. Further as H_1 is a minimal parabolic, for each nontrivial H_1-chief factor E_1 on $\tilde{U}$, $m(U_\alpha/C_{U_\alpha}(E_1)) \leq m(E_1/C_{E_1}(U_\alpha))$ by B.6.9.1. However by 12.8.5.1, each H-chief section E on $\tilde{U}$ is the sum of $n-1$ chief sections under H_1, so that $(n-1)m(U_\alpha/C_{U_\alpha}(E)) \leq m(E/C_E(U_\alpha))$. Hence as $n \geq 4$, if $U_\alpha^* \neq 1$ then

$$2m(U_\alpha^*) < (n-1)m(U_\alpha^*) \leq m(\tilde{U}/C_{\tilde{U}}(U_\alpha^*)). \qquad (*)$$

Now take $\alpha = \beta$. Then $U_\alpha = U^g$ and $U = U_\gamma^g$, so (*) shows that U_H does not induce transvections on U_γ. Therefore by F.9.16.1, $D_\gamma < U_\gamma$, so by F.9.16.4, we may choose γ so that $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. Then taking $\alpha = \delta$, we have a contradiction to (*), completing the proof of (1) and (3), and hence of 12.9.5. □

LEMMA 12.9.6. *(1)* $[V_2, O_2(K_1)] \neq 1$.

(2) $I_2 := \langle O_2(G_1)^{G_2}\rangle \trianglelefteq G_2$, $I_2/O_2(I_2) \cong S_3$, $O_2(I_2) = C_{I_2}(V_2)$, *and* I_2T *is a minimal parabolic of* LT.

(3) $m_3(C_G(V_2)) \leq 1$.

PROOF. By 12.9.5.2, $K_1/O_2(K_1) \cong A_7$, $L_4(2)$ or $L_5(2)$; and by 12.9.5.1, $H := K_1T \in \mathcal{H}_z$. Let $Q := O_2(LT)$, $Q_1 := O_2(K_1)$, and $H^* := H/Q_1$.

Assume that Q_1 centralizes V_2. Then Q_1 centralizes $\langle V_2^H\rangle$, and by 12.8.8.6, $U_H = \langle V_2^H\rangle$, so that Q_1 centralizes U_H. Thus as $K_1/O_2(K_1)$ is simple, $U_H \in \mathcal{R}_2(K_1Q)$. Next as Q_1 centralizes V, $Q_1 \leq Q < R_1$, with R_1/Q the natural module for $L_1/R_1 \cong L_{n-1}(2)$. As $H \not\leq M \geq N_G(Q)$, $Q_1 < Q$, so $Q^* \neq 1$. Therefore $1 \neq Q^* < R_1^* = O_2(L_1^*)$. As $O_2(L_1^*) \neq 1$, K_1^* is not A_7, so that $K_1^* \cong L_4(2)$ or $L_5(2)$. As $1 \neq Q^* < O_2(L_1^*)$, the parabolic $L_1^*T^*$ of K_1^* is not irreducible on $O_2(L_1^*)$, so we conclude that $n = 4$ and $K_1^* \cong L_5(2)$. Then using 12.9.5.3, $R_1^* \cong 2^{1+6}$ and $Q^* = O_2(P^*) \cong E_{16}$ for some end-node maximal parabolic P^* of K_1^*. But then $P \leq N_G(Q) \leq M$, contradicting $L_1 \trianglelefteq M_1$. This completes the proof of (1).

Let P_2 be the minimal parabolic of LT nontrivial on V_2, and $R := O_2(G_1)$. Now as $C_G(V_2) \leq G_1$ and P_2 induces $GL(V_2)$,

$$R^{G_2} = R^{C_G(V_2)P_2} = R^{P_2} \subseteq P_2,$$

so $\langle R^{G_2}\rangle = I_2 \leq P_2$. Further by (1), R does not centralize V_2, so $P_2 = I_2T$ and (2) follows. Finally $[I_2, C_G(V_2)] \leq C_{I_2}(V_2) = O_2(V_2)$ by (2), so

$$2 \geq m_3(G_2) = m_3(I_2) + m_3(C_G(V_2)) = 1 + m_3(C_G(V_2)),$$

establishing (3). □

LEMMA 12.9.7. *$G_i \leq M_V$ for $1 < i < n$.*

PROOF. Recall $M_i \leq M_V$ as V is a TI-set in M by 12.2.6, so it suffices to show $G_i \leq M$. Let $1 < i < n$. As $GL(V_i) = Aut_M(V_i)$, $G_i = M_iC_G(V_i)$, so it suffices to show $C_G(V_i) \leq M$. As $C_G(V_i) \leq C_G(V_2)$, it remains to show $C_G(V_2) \leq M$. Set $(K_1T)^* := K_1T/O_2(K_1T)$. As $Out(K_1^*)$ is a 2-group, $G_1 = DK_1T$, where $D := C_{G_1}(K_1^*)$ and $D \leq M_1$ by 12.9.5.1. As $[D, L_1^*] = 1$, $[D, \bar{L}_1] \leq O_2(\bar{L}_1)$, so D centralizes $\tilde{V}_2$. Thus $C_{G_1}(\tilde{V}_2) = DC_{K_1T}(\tilde{V}_2)$, so it suffices to show that $Y := C_{K_1T}(V_2)T = C_{K_1T}(\tilde{V}_2) \leq L_1T$.

Now Y is an overgroup of T in K_1T with $I := O^2(L_1 \cap G_2) \leq Y$, and IT is a parabolic of LT of Lie rank $n-3$ contained in L_1T. By 12.9.5.2, $K_1^* \cong A_7$, $L_4(2)$, or $L_5(2)$.

We assume that $Y \not\leq L_1T$ and derive a contradiction. Then $IT < Y$. If $K_1^* \cong L_4(2)$ or $L_5(2)$, then Y is a parabolic in K_1T of rank at least $n-2 \geq 2$, so $(Y \cap K_1)O_2(Y)/O_2(Y) \cong S_3 \times S_3$, $L_3(2)$, $S_3 \times L_3(2)$, or $L_4(2)$. If $K_1^* \cong A_7$, then examining overgroups of $(T \cap K_1)^*$, we conclude that $(Y \cap K_1)O_2(Y)/O_2(Y)$ is $L_3(2)$, A_6, or a subgroup of index 2 in $S_4 \times S_3$. However by 12.9.6.3, $m_3(Y) \leq 1$, so $(Y \cap K_1)O_2(Y)/O_2(Y) \cong L_3(2)$. Then as Y has Lie rank at least $n-2$, we conclude that $n = 4$, so that $L_1/O_2(L_1) \cong L_3(2)$ and $IT/O_2(IT) \cong S_3$. As T induces inner automorphisms on $L_1/O_2(L_1)$, $T^* \leq K_1^*$, so $Y^* \leq K_1^*$ and $Y/O_2(Y) \cong L_3(2)$.

Set $H := \langle Y, L_1 \rangle$, so that $H \in \mathcal{H}_z$. Now Y and L_1T are of Lie rank 2 and intersect in IT of Lie rank 1, so we conclude from the lattice of overgroups of T in K_1T that $H/O_2(H)$ is A_7 or $L_4(2)$. In either case as L_1^* does not centralize $\tilde{V}_2$, $C_H(\tilde{V}_2) = Y$; so as $Y/O_2(Y) \cong L_3(2)$, we conclude from B.4.12 that $\tilde{U}_H = \langle \tilde{V}_2^H \rangle$ is a 4-dimensional module for $H/O_2(H) = A_7$ or $L_4(2)$. Thus $O^2(Y)$ is irreducible on U_H/V_2, so $U_H = \langle V^{O^2(Y)} \rangle$. Define I_2 as in 12.9.6.2. By that result, I_2 normalizes V, $I_2 \trianglelefteq G_2$, and $I_2/O_2(I_2) \cong S_3$; therefore as $Y \leq G_2$ with $Y/O_2(Y) \cong L_3(2)$, $O^2(Y)$ centralizes $I_2/O_2(I_2)$, and hence I_2 normalizes $O^2(O^2(Y)O_2(I)) = O^2(Y)$. Hence I_2 acts on $\langle V^{O^2(Y)} \rangle = U_H$. But then $LT = \langle I_2, L_1 \rangle \leq N_G(U_H)$, so that $N_G(U_H) \leq M = !\mathcal{M}(LT)$, contrary to $H \not\leq M$. This contradiction completes the proof of 12.9.7. □

LEMMA 12.9.8. *(1) $m(V \cap V^g) \leq 1$ for $g \in G - M$.*
(2) If $V \cap V^g \neq 1$, then $[V, V^g] = 1$.

PROOF. We may assume $V \cap V^g < V$ as $N_G(V) \leq M$. Then if $V \cap V^g \neq 1$, by 12.8.3.2 we may assume $V \cap V^g = V_i$ for some $1 \leq i < n$, and take $g \in G_i$ by 12.8.3.3. If $i > 1$, we have $G_i \leq M$ by 12.9.7, proving (1). Part (2) follows from 12.9.4 and 12.8.6. □

LEMMA 12.9.9. *(1) $W_0 := W_0(T, V)$ centralizes V, so $N_G(W_0) \leq M$.*
(2) If $A := V^g \cap M$ is a hyperplane of V^g contained in T, then $C_A(V) = 1$.

PROOF. Suppose $A := V^g \cap M \leq T$ with $[A, V] \neq 1$ and $m(V^g/A) \leq 1$. Let $I := N_V(A)$. By 12.9.8.2, $A \cap V = 1$, so $[A, I] \leq A \cap V = 1$ and hence $I < V$. By 12.9.7, for each noncyclic subgroup B of A, $C_V(B) \leq N_V(V^g) \leq N_V(A) = I = C_V(A) \leq C_V(B)$, so $C_V(B) = I$. Thus $m(C_A(W)) \leq 1$ for each A-submodule W of V not contained in I; in particular as $I < V$, $m(C_A(V)) \leq 1$.

Assume $C_A(V) \neq 1$. Conjugating in $N_G(V^g)$, we may assume that $C_A(V) = V_1^g$. Thus $V \leq G_1^g \leq N_G(U^g)$, where $U := \langle V^{G_1} \rangle$. Hence $[V, A] \leq U^g \leq C_G(V^g)$ as U is abelian by 12.9.4.

We now prove (2), so we may assume $A < V^g$. Then $[V, A]$ is cyclic: for otherwise $V^g \leq C_G([V, A]) \leq N_G(V) \leq M$ by 12.9.7, contrary to $A < V^g$. As $[V, A]$ is cyclic, A induces a group of transvections on V with center $[V, A]$; so as $C_V(B) = I = C_V(A)$ for each noncyclic subgroup B of A, $|\bar{A}| = 2$. But now $C_A(V)$ is noncyclic, contrary to paragraph one. This completes the proof of (2).

If $W_0 \leq C_T(V)$ then $N_G(W_0) \leq M$ by E.3.34.2. Thus we may assume $A = V^g$, and it remains to derive a contradiction. Suppose first that A acts nontrivially on V_{n-1}. Then $V_{n-1} \not\leq C_V(A) = I$ and hence $m(C_A(V_{n-1})) \leq 1$ by paragraph one. Let $M_{n-1}^* := M_{n-1}/C_M(V_{n-1}))$, and observe $M_{n-1}^* \cong L_{n-1}(2)$. Then

$$m_2(M_{n-1}^*) \geq m(A^*) \geq n - 1,$$

so we conclude $n = 5$ and $A^* = J(T^*)$. But now $C_{V_4}(A) = V_2 < C_{V_4}(B)$ for B a 4-subgroup of A with B^* inducing transvections on V_4 with a fixed axis, contrary to an observation in the first paragraph.

Therefore A centralizes V_{n-1}, so $A \leq R_{n-1}$. Then as $m(C_A(V)) \leq 1$,

$$m(\bar{A}) \geq m(A) - 1 = n - 1 = m(\bar{R}_{n-1}),$$

so that $AC_T(V) = R_{n-1}$. Thus $L_1 = [L_1, A]$ and $A_1 := C_A(V)$ is of order 2, so by paragraph two we may assume $A_1 = V_1^g$, $V \leq N_G(U^g)$, and $V_{n-1} = [A, V] \leq U^g$. Let $Q := O_2(G_1)$. For $y \in Q$, $[U, y] \leq V_1$ by 12.8.4.2, so $m(U/C_U(y)) \leq 1$ and hence as $n \geq 4$, $C_{V_{n-1}}(y^g)$ is noncyclic. Thus $y^g \in C_G(C_{V_{n-1}}(y^g)) \leq N_G(V)$ by 12.9.7, so $[Q^g, V] \leq Q^g \cap V = V_{n-1} \leq U^g$. If $[K_1^g, V] \leq O_2(K_1^g)$, then $V \leq N_G(A)$ by 12.9.5.1, contrary to $I < V$. Thus $K_1^g = [K_1^g, V]$, so K_1^g centralizes Q^g/U^g. Then $[K_1, Q] \leq U \leq C_Q(U)$ as U is abelian by 12.9.4. Therefore $[K_1, O_2(K_1)] \leq C_G(U) \leq C_G(V)$.

Let $P := O_2(K_1T)$ and choose X of order 7 or 5 in L_1 for $n = 4$ or 5, respectively. Recall $K_1T \in \mathcal{H}_z$ by 12.9.5.1, so that $[\tilde{V}, P] = 1$ by 12.8.4.2. Further $V = V_1 \times [V, X]$, and by Coprime Action, $P = C_P(X)[P, X] = C_P(X)[P, K_1] = C_P(X)[O_2(K_1), K_1]$. Now P acts on V, and $[O_2(K_1), K_1]$ centralizes V by the previous paragraph; then $C_P(X)$ acts on $[V, X]$, and hence P acts on $[V, X]$. Therefore as X is irreducible on $[V, X]$ and normalizes P, P centralizes $[V, X]$, so as $P \leq G_1$, P centralizes V. As $O_2(K_1) \leq P$ and $V_2 \leq V$, this is contrary to 12.9.6.1, so the proof of 12.9.9 is complete. □

We are now in a position to complete the proof of Theorem 12.9.1.

By 12.9.5.2, $K_1^* \cong A_7$, $L_4(2)$, or $L_5(2)$. In particular, there is an overgroup H of T in K_1T not contained in M with $H/O_2(H) \cong S_3$. By 12.9.9.1, $N_G(W_0) \leq M$, so by E.3.15, $W_0 \not\leq O_2(H)$. Thus there is $A := V^g \leq T$ with $A \not\leq O_2(H)$. If $V_1 \leq A$, then by 12.8.3.3 and 12.9.4, $A \in V^{G_1} \cap H \subseteq O_2(H)$, contrary to our choice of A. Thus $V_1 \cap A = 1$.

Now $H \notin \mathcal{H}_z$ since H does not contain L_1, but we define some notation similar to that in Notation 12.8.2: Let $U_H := \langle V_2^H \rangle$ and $Q_H := O_2(H)$. Then $U_H \leq \langle V^{G_1} \rangle$, so U_H is abelian by 12.9.4. Indeed Hypothesis G.2.1 is satisfied with H, V_2, 1 in the roles of "G, V, L", so by G.2.2.1, $\tilde{U}_H \in Z(\tilde{Q}_H)$. By 12.9.7, $V_2 < U_H$. As $H/Q_H \cong S_3$ and $A \not\leq Q_H$, $B := A \cap Q_H$ is of index 2 in A. Then $[U_H, B] \leq V_1$, so for $u \in U_H$, $m(B/C_B(u)) \leq m(V_1) = 1$, so $C_B(u)$ is noncyclic, and hence by 12.9.7, $u \in N_G(A)$. Thus $U_H \leq N_G(A)$, and so $[U_H, B] \leq A \cap V_1 = 1$.

As $O^2(H) \leq \langle A^H \rangle$ and $V_2 < U_H$, there is $h \in H$ such that A^h does not act on V_2. But again using 12.9.7,

$$D := B^h \leq C_G(U_H) \leq C_G(V_2) \leq N_G(V).$$

If $[D, V] = 1$, then $V \leq C_G(D) \leq N_G(A^h)$ by 12.9.7, so $A^h \leq C_G(V) \leq C_G(V_2)$ by 12.9.9.1, contrary to our choice of h. Thus $\bar{D} \neq 1$, so $D = V^{gh} \cap M$ by 12.9.9.2 However

$$1 \neq [U_H, A^h] \leq U_H \cap A^h \leq C_D(V),$$

since U_H is abelian. As D is a hyperplane of A^h with $D = V^{gh} \cap M$, 12.9.9.2 supplies a contradiction.

This final contradiction completes the proof of Theorem 12.9.1.

CHAPTER 13

Mid-size groups over $\mathbf{F}_2$

In this chapter we consider the cases remaining in the Fundamental Setup (3.2.1) after the work of the previous chapter. We make more use of the generic methods for the $\mathbf{F}_2$ case, such as results from sections F.7, F.8, and F.9.

In Hypothesis 13.1.1, we essentially extend Hypothesis 12.2.3 which began the previous chapter, by adding the assumption that G is not one of the groups which arose in the course of that chapter. Then after some reductions in the initial sections 13.1 and 13.2, in the remainder of the chapter we assume an additional refinement in Hypothesis 13.3.1.

In particular in 13.1.2.3, we observe that the remaining possibilities for the section $L/O_2(L)$, with $L \in \mathcal{L}_f^*(G,T)$ in the FSU, are A_5, $L_3(2)$, A_6, $\hat{A}_6$, and $U_3(3) \cong G_2(2)'$. The main goal of the chapter is to treat the latter three groups, thus reducing the FSU to the case where $L/O_2(L)$ is $L_3(2)$ or A_5.

In the natural logical sequence, the smallest simple group A_5 is treated last; thus at that point, all other groups are eliminated, so that $K/O_2(K) \cong A_5$ for all $K \in \mathcal{L}_f^*(G,T)$. However, to avoid repeating arguments common to both A_5 and A_6, we prove such results simultaneously for both in sections 13.5 and 13.6. To do so, we *assume* in part (4) of Hypothesis 13.3.1 (and similarly in the hypothesis of 13.2.7) that $K/O_2(K) \cong A_5$ for all $K \in \mathcal{L}_f^*(G,T)$, when the subgroup $L \in \mathcal{L}_f^*(G,T)$ we've chosen satisfies $L/O_2(L) \cong A_5$. That is, we don't make this choice until we are forced to do so, after the treatment of the other groups.

13.1. Eliminating $\mathbf{L} \in \mathcal{L}_\mathbf{f}^*(\mathbf{G},\mathbf{T})$ with $\mathbf{L}/\mathbf{O_2}(\mathbf{L})$ not quasisimple

We now state the initial hypothesis for the chapter, which excludes the groups in the Main Theorem that have arisen so far under the FSU. Namely throughout this section, we assume:

HYPOTHESIS 13.1.1. *(1) G is a simple QTKE-group and $T \in Syl_2(G)$.*
(2) G is not a group of Lie type of Lie rank 2 over $\mathbf{F}_{2^n}$, $n > 1$.
(3) G is not $L_4(2)$, $L_5(2)$, A_9, M_{22}, M_{23}, M_{24}, He, or J_4.

As usual let $Z := \Omega_1(Z(T))$.

As mentioned earlier, Hypothesis 13.1.1 essentially contains Hypothesis 12.2.3, aside from the assumption in Hypothesis 12.2.1 that there is some $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple. In Theorem 13.1.7, we show for each $K \in \mathcal{L}_f^*(G,T)$ that $K/O_2(K)$ is quasisimple.

We record some elementary consequences of Hypothesis 13.1.1.

LEMMA 13.1.2. *Assume there is $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple and set $M := N_G(L)$. Then L is T-invariant, there exists a T-invariant member V of $Irr_+(L, R_2(LT))$, and:*

(1) *Hypothesis 12.2.3 holds.*
(2) $C_G(v) \not\leq M$ *for some* $v \in V^{\#}$.
(3) $L/O_2(L) \cong A_5$, A_6, $\hat{A}_6$, $L_3(2)$, *or* $G_2(2)'$.
(4) *If* $L/O_2(L) \cong \hat{A}_6$, *then* $V/C_V(L)$ *is the natural module for* A_6.
(5) *If* $L_1 \in \mathcal{L}(G,T)$ *and* $L_1 \leq L$, *then* $L_1 = L \in \mathcal{L}^*(G,T)$.

PROOF. As $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple, part (1) of Hypothesis 13.1.1 implies that Hypothesis 12.2.1 holds, allowing us to apply Theorem 12.2.2. Parts (2) and (3) of Hypothesis 13.1.1 exclude the groups in conclusions (1) and (2) of Theorem 12.2.2, so that conclusion (3) of that result holds. Hence T normalizes L and Hypothesis 12.2.3 holds, establishing (1). Part (3) of Hypothesis 13.1.1 excludes the groups in conclusions (2)–(4) of Theorem 12.2.13, as well as the groups in the conclusions of Theorems 12.3.1, 12.7.1, and 12.9.1. Hence those results eliminate the corresponding cases from conclusion (3) in Theorem 12.2.2 and so establish (2)–(4). Finally as the groups in (3) are of Lie type and either of Lie rank 2 over $\mathbf{F}_2$, or A_5 of Lie rank 1, each proper T-invariant subgroup of L is solvable. Then (5) follows from 1.2.4. □

Define
$$\mathcal{L}_+(G,T) := \{L \in \mathcal{L}_f(G,T) : L/O_2(L) \text{ is } \textit{not} \text{ quasisimple } \},$$
and suppose for the moment that $\mathcal{L}_+(G,T)$ is empty. If $K \in \mathcal{L}_f(G,T)$ then by 1.2.9, $K \leq L \in \mathcal{L}_f^*(G,T)$. As $\mathcal{L}_+(G,T) = \emptyset$, $L/O_2(L)$ is quasisimple, so $K = L \in \mathcal{L}_f^*(G,T)$ by 13.1.2.5. That is, once we show that $\mathcal{L}_+(G,T)$ is empty, we will be able to conclude that $\mathcal{L}_f(G,T) = \mathcal{L}_f^*(G,T)$.

REMARK 13.1.3. Recall that non-quasisimple $\mathcal{C}$-components are allowed by the general quasithin hypothesis: they appear as cases (3) and (4) of A.3.6, and cases (c) and (d) of 1.2.1.4. On the other hand, they do not actually arise in $\mathcal{L}_f(G,T)$ in any of the groups in our Main Theorem. Thus after Theorem 13.1.7, we will finally be rid of this nuisance. In particular, if $\mathcal{L}_f^*(G,T)$ is nonempty, then by 3.2.3 there will exist tuples in the Fundamental Setup. Furthermore, as we just observed, we will also have $\mathcal{L}_f(G,T) = \mathcal{L}_f^*(G,T)$.

If $L \in \mathcal{L}_+(G,T)$ then L appears in case (c) or (d) of 1.2.1.4, so $m_p(L) = 2$ for some odd prime p dividing the order of $O_{2,F}(L)$, and $T \leq N_G(L)$ by 1.2.1.3. Also in the notation of chapter 1, $1 \neq \Xi_p(L) \in \Xi(G,T)$ by 1.3.3.

Recall the basic facts about $\Xi(G,T)$ from that chapter. Recall also from Definition 3.2.12 that $\Xi_-(G,T)$ consists of those $X \in \Xi(G,T)$ such that either X is a $\{2,3\}$-group, or $X/O_2(X)$ is a 5-group and $Aut_G(X/O_2(X))$ is a 2-group. Further $\Xi_+(G,T)$ is defined to be $\Xi(G,T) - \Xi_-(G,T)$. Set
$$\Xi_+^*(G,T) := \Xi_+(G,T) \cap \Xi^*(G,T).$$
We will make repeated use of results from section A.4 such as A.4.11.

We next collect some useful properties of the members L of $\mathcal{L}_+(G,T)$. Although the proof of the next lemma contains an appeal to 13.1.2.3, we could in fact have stated and proved 13.1.4 much earlier, after chapter 11. On the other hand, many arguments from now on (eg. the proof of 13.1.9.1) make strong use of 13.1.2.5—which does depend on work done in chapters after chapter 11.

LEMMA 13.1.4. *Assume $L \in \mathcal{L}_+(G,T)$. Then*

(1) Either

(i) $L \in \mathcal{L}^*(G,T)$, or

(ii) $L/O_{2,F}(L) \cong SL_2(5)$, and $L_+ \in \mathcal{L}_+(G,T)$ for each $L_+ \in \mathcal{L}(G,T)$ with $L < L_+$.

(2) There is $M_c \in \mathcal{M}(T)$ with $M_c = !\mathcal{M}(LT)$. If $L \in \mathcal{L}^*(G,T)$, then $M_c = N_G(L)$.

(3) For some prime $p > 3$, $X := \Xi_p(L) \neq 1$, and for each such X, $X \in \Xi(G,T)$ and either

(i) $X/O_2(X) \cong E_{p^2}$ and $L/X \cong SL_2(p)$, or

(ii) $L/O_{2,F}(L) \cong SL_2(5)$.

(4) $X \in \Xi_+^*(G,T)$, and $M_c = !\mathcal{M}(XT) = N_G(X)$.

(5) $C_L(R_2(M_c)) = O_\infty(L)$. In particular, X centralizes $R_2(M_c)$.

(6) $M_c = !\mathcal{M}(C_G(Z))$.

PROOF. Assume $L \leq L_+ \in \mathcal{L}(G,T)$. As $L \in \mathcal{L}_+(G,T)$, $L \in \mathcal{L}_f(G,T)$, so $L_+ \in \mathcal{L}_f(G,T)$ by 1.2.9.1. Recall T acts on L, so T acts on L_+ by 1.2.4.

As $L \in \mathcal{L}_+(G,T)$, $X := \Xi_p(L) \neq 1$ for some prime $p > 3$ by 1.2.1.4, and $X \in \Xi(G,T)$ by 1.3.3. Indeed (3) holds by 1.2.1.4.

Suppose that X is not normal in L_+. Then by 1.3.4, L_+ appears on the list of 1.3.4; in particular $L_+/O_2(L_+)$ is quasisimple in each case. As T acts on L_+, conclusion (1) of 1.3.4 does not hold, and as $p > 3$, conclusion (4) does not hold. Thus $L_+/O_2(L_+) \cong (S)L_3(p)$ or $Sp_4(2^n)$, with n even. Furthermore $L_+ \in \mathcal{L}_f^*(G,T)$ using 1.3.9.1. But this is contrary to the list of possibilities in 13.1.2.3.

This contradiction shows that $X \trianglelefteq L_+$, so $L_+/O_2(L_+)$ is not quasisimple and hence $L_+ \in \mathcal{L}_+(G,T)$ by definition. Further taking L_+ maximal, $L_+ \in \mathcal{L}^*(G,T)$. Therefore $X \in \Xi^*(G,T)$ by 1.3.8. If $L = L_+$, then (1i) holds. Otherwise by 1.2.4, the inclusion $L < L_+$ is described in A.3.12 (see A.3.13 for further detail in this case); so $1 \neq O_\infty(L) \leq O_\infty(L_+)$ and (1ii) holds. This completes the proof of (1).

As $X \in \Xi^*(G,T)$, $M_c := N_G(X) = !\mathcal{M}(XT)$ by 1.3.7. As $p > 3$ and $Aut_L(X)$ is not a 2-group, $X \in \Xi_+(G,T)$; thus $X \in \Xi_+^*(G,T)$, completing the proof of (4). Further as $X \leq L$, it follows that also $M_c = !\mathcal{M}(LT)$. If $L \in \mathcal{L}^*(G,T)$, then $L \in \mathcal{C}(M_c)$ by 1.2.7.1, and then $L \trianglelefteq M_c$ by 1.2.1.3, completing the proof of (2).

Recall $L_+ \in \mathcal{L}^*(G,T)$, so $L_+ \trianglelefteq M_c$ by (2). As $L_+ \in \mathcal{L}_f(G,T)$, $C_{L_+}(R_2(M_c)) < L_+$ by A.4.11. We also saw earlier that $O_\infty(L) \leq O_\infty(L_+)$. Let $Y := O^2(O_{2,F}(L_+))$. Then Y centralizes $R_2(L_+T)$ by 3.2.14, and $Y \trianglelefteq M_c$, so Y centralizes $R_2(M_c)$ by A.4.11. Then as $L_+ \trianglelefteq M_c$ and $R_2(M_c)$ is 2-reduced, $O_{2,F}(L_+) \leq C_{L_+}(R_2(M_c))$, and hence $O_\infty(L_+) = O_{2,F,2}(L_+) = C_{L_+}(R_2(M_c))$. So as $O_\infty(L) = O_\infty(L_+) \cap L$, we conclude that (5) holds.

Finally as $M_c \in \mathcal{H}^e$ by 1.1.4.6, $Z \leq R_2(M_c)$ by B.2.14, so (4) and (5) imply (6). □

Let $\mathcal{L}_+^*(G,T)$ denote the maximal members of $\mathcal{L}_+(G,T)$; thus $\mathcal{L}_+^*(G,T)$ is nonempty whenever $\mathcal{L}_+(G,T)$ is nonempty. By 13.1.4.1,

$$\mathcal{L}_+^*(G,T) \subseteq \mathcal{L}_f^*(G,T).$$

LEMMA 13.1.5. *Assume $\Xi_+^*(G,T) \neq \emptyset$. Then*

(1) There is $M_c \in \mathcal{M}(T)$ with $M_c = !\mathcal{M}(C_G(Z))$.

(2) $\Xi_+^*(G,T) \subseteq M_c$, so $M_c = N_G(X) = !\mathcal{M}(XT)$ for each $X \in \Xi_+^*(G,T)$.

PROOF. Assume $X \in \Xi_+^*(G,T)$. Then $X \in \Xi^*(G,T)$, so $M_c := N_G(X) = !\mathcal{M}(XT)$ by 1.3.7. Also $X \in \Xi_+(G,T)$, so by 3.2.13, $X \notin \Xi_f(G,T)$. Then by A.4.11, X centralizes $R_2(XT)$, so $R_2(XT)$ contains Z by B.2.14; hence $M_c = !\mathcal{M}(XT) = !\mathcal{M}(C_G(Z))$, so that (1) holds. This also establishes (2), as we may vary $X \in \Xi_+^*(G,T)$ independently of Z. □

LEMMA 13.1.6. *Assume $X \in \Xi_+^*(G,T)$, let $M_c \in \mathcal{M}(XT)$, and assume $M \in \mathcal{M}(T) - \{M_c\}$. Then either*

(1) There exists an odd prime d and $Y = O^2(Y) \trianglelefteq M$ such that $Y \not\leq M_c$, $[Z,Y] \neq 1$, and $Y/O_2(Y)$ is a d-group of exponent d and class at most 2, or

(2) There exists $Y \in \mathcal{C}(M)$ with $Y \not\leq M_c$. For each such Y, $Y/O_2(Y)$ is quasisimple, $Y \trianglelefteq M$, $[Z,Y] \neq 1$, and $Y \in \mathcal{L}_f^*(G,T)$.

PROOF. By 13.1.5, $N_G(X) = !\mathcal{M}(XT) = M_c = !\mathcal{M}(C_G(Z))$.

Suppose first that there is $Y \in \mathcal{C}(M)$ with $Y \not\leq M_c$. Then as $M_c = !\mathcal{M}(C_G(Z))$, $[Z,Y] \neq 1$, so that $Y \in \mathcal{L}_f(G,T)$. Let $Y \leq Y_1 \in \mathcal{L}^*(G,T)$; by 1.2.9.2, $Y_1 \in \mathcal{L}_f^*(G,T)$. If $Y_1 \in \mathcal{L}_+(G,T)$, then by (2) and (6) of 13.1.4, $N_G(Y_1) = !\mathcal{M}(Y_1T) = !\mathcal{M}(C_G(Z))$, contrary to our assumption that $Y \not\leq M_c$. Thus $Y_1/O_2(Y_1)$ is quasisimple, so that $Y = Y_1 \trianglelefteq M$ by 13.1.2. Therefore (2) holds in this case.

We may assume that (2) fails, so $\langle \mathcal{C}(M) \rangle \leq M_c$ by the previous paragraph. Let $M^* := M/O_2(M)$, and for d an odd prime, let $\theta_d(M)$ be the preimage of the group $\theta_d(M^*)$ defined in G.8.9; recall that $\theta_d(M^*)$ is of class at most 2 and of exponent d using A.1.24. Let $\theta(M)$ be the product of the groups $\theta_d(M)$, for $d \in \pi(F(M^*))$.

Suppose that $\theta(M) \leq M_c$. Then as $\langle \mathcal{C}(M) \rangle \leq M_c$, $\theta(M)O_{2,E}(M) =: Y \leq M_c$. with M, M_c in the roles of "H, K", $R := O_2(M_c \cap M) = O_2(M)$ and $C(M_c, R) \leq M_c \cap M$. Then M_c, R, $M_c \cap M$ satisfy Hypothesis C.2.3 in the roles of "H, R, M_H". As $X \in \Xi_+(G,T)$, X is a $\{2,p\}$-group for some prime $p > 3$, so X contains no A_3-blocks. Thus by C.2.6.2, $X \leq M_c \cap M$, contrary to $M \neq M_c = !\mathcal{M}(XT)$.

This contradiction shows that $\theta(M) \not\leq M_c$; hence there is some d with $Y := \theta_d(M) \not\leq C_G(Z)$ and $Y = O^2(Y) \trianglelefteq M$; so (1) holds. □

We are now prepared for the main result of the section:

THEOREM 13.1.7. *Assume Hypothesis 13.1.1. Then $\mathcal{L}_+(G,T) = \emptyset$.*

Until the proof of Theorem 13.1.7 is complete, assume G is a counterexample. As $\mathcal{L}_+(G,T)$ is nonempty, we may choose $L \in \mathcal{L}_+^*(G,T)$, so $L \in \mathcal{L}_f^*(G,T)$ by an earlier remark. Set $M_c := N_G(L)$; then $M_c = !\mathcal{M}(LT)$ by 13.1.4.2. By Theorem 2.1.1, $|\mathcal{M}(T)| > 1$, so $\mathcal{H}_*(T, M_c)$ is nonempty.

Let $\mathcal{X}$ consist of the groups $\Xi_p(L)$, $p \in \pi(F(L/O_2(L)))$. By 13.1.4, each $X \in \mathcal{X}$ is in $\Xi_+^*(G,T)$ and

$$M_c = N_G(X) = !\mathcal{M}(XT) = !\mathcal{M}(C_G(Z)). \qquad (+)$$

Set $V_c := R_2(M_c)$, $M_c^* := M_c/C_{M_c}(V_c)$, and

$$U := [V_c, L].$$

Define

$$\mathcal{L}_1 := \{L_1 \in \mathcal{L}(G,T) : L = O_\infty(L)L_1\}.$$

LEMMA 13.1.8. *(1) $L^* \cong L_2(p)$ for some prime $p > 3$.*
(2) $1 \neq [V_c, L] = [V_c, L_1]$ for each $L_1 \in \mathcal{L}_1$.

PROOF. By 13.1.4.5, $O_\infty(L) = C_L(V_c)$. Hence (1) and the statement in (2) that $[V_c, L] \neq 1$ follow from 13.1.4.3. If $L_1 \in \mathcal{L}_1$, then $L^* = L_1^*$ by (1), so (2) holds. □

We next establish an important technical result:

LEMMA 13.1.9. *(1) For each $L_1 \in \mathcal{L}_1$, $M_c = !\mathcal{M}(L_1T)$.*
(2) $L = [L, J(T)]$.

PROOF. We will show in the first few paragraphs that (1) implies (2).

Set $R := C_T(L/O_\infty(L))$ and let $L_1 \in \mathcal{L}_1$. Observe by 13.1.4.3 that R is Sylow in $O_\infty(LT)$, with $R/O_2(LT)$ cyclic and $\Omega_1(R/O_2(LT))$ inverts $O_{2,F}(L)/O_{2,\Phi(F)}(L)$. Also for $X \in \mathcal{X}$, $O_2(X) \leq R$, so that $R \in Syl_2(XR)$. Set $L_R := N_L(R)^\infty$; by a Frattini Argument, $L_1 = O_\infty(L_1)N_{L_1}(R) = O_\infty(L_1)N_{L_1}(R)^\infty$. As $R \trianglelefteq T$, T acts on L_R, so $L_R \in \mathcal{L}(G,T)$. As $\Omega_1(R/O_2(LT))$ inverts $O_{2,F}(L)/O_{2,\Phi(F)}(L)$, $O_\infty(LT) \cap L_RT = R$, so $R = O_2(L_RT)$ and $L_R/O_2(L_R) \cong L^* \cong L_2(p)$ for some $p > 3$ by 13.1.8.1. As $L \in \mathcal{L}_f(G,T)$, $L_R \in \mathcal{L}_f(G,T)$ by A.4.10.3. As $N_{L_1}(R)^\infty$ is an R-invariant subgroup of L_R and $L_R/O_2(L_R)$ is simple, $N_{L_1}(R)^\infty = L_R$.

Now we assume that (1) holds, but (2) fails. We saw at the outset of the proof of Theorem 13.1.7 that we may choose some $H \in \mathcal{H}_*(T, M_c)$. We will appeal to 3.1.7 with $M_0 := L_RT$, so we begin to verify the hypotheses of that result: We've seen that $R = O_2(M_0)$. As we are assuming that (2) fails, $J(T) \leq O_\infty(LT) \cap L_RT = R$. Thus it remains to verify Hypothesis 3.1.5.

Take $V := R_2(M_0)$. As $L_R \in \mathcal{L}_f(G,T)$, $[V, M_0] \neq 1$ by 1.2.10, so as M_0/R is simple, $R = C_T(V)$. Finally we verify condition (I) of Hypothesis 3.1.5: Let $B := O^2(H \cap M_c)$. As $H \not\leq M_c = !\mathcal{M}(XT)$ by (+), $X \not\leq H$, so as T is irreducible on $X/O_{2,\Phi}(X)$, $B \cap X \leq O_{2,\Phi}(X)$. As this holds for each $X \in \mathcal{X}$, $B \cap O_{2,F}(L) \leq O_{2,\Phi(F)}(L)$, so $H \cap O_\infty(L)$ is 2-closed, and hence $H \cap M_c$ acts on $R \cap L$. Thus $H \cap M_c$ acts on $L_R = N_L(R \cap L)$. This completes the verification of Hypothesis 3.1.5.

Applying our assumption that (1) holds to $L_R \in \mathcal{L}_1$, $M_c = !\mathcal{M}(L_RT)$. Then $O_2(\langle M_0, H\rangle) = 1$, which rules out conclusion (2) of 3.1.7. As $M_c = !\mathcal{M}(C_G(Z))$ by (+), $Z \not\leq Z(H)$, which rules out the remaining conclusion (1) of 3.1.7. This contradiction completes the proof that (1) implies (2).

So we may assume that $L_1 \in \mathcal{L}_1$ and $M \in \mathcal{M}(L_1T) - \{M_c\}$, and it remains to derive a contradiction. By paragraph one, $L_1 = O_\infty(L_1)L_R$, so $L_R \in \mathcal{L}(M,T)$. Thus $L_R \leq L_M \in \mathcal{C}(M)$ by 1.2.4, and as T normalizes L_R, $L_M \trianglelefteq M$ by 1.2.1.3.

We apply 13.1.6 to M and choose Y as in case (1) or (2) of that result. In particular $Y \not\leq M_c$, $Y \trianglelefteq M$, and $[Z, Y] \neq 1$. We claim that $[Y, L_M] \leq O_2(Y)$: In case (2) of 13.1.6, $Y \in \mathcal{C}(M)$, so by 1.2.1.2, either $Y = L_M$ or $[Y, L_M] \leq O_2(Y)$. But by 13.1.6, $Y \in \mathcal{L}_f^*(G,T)$ with $Y/O_2(Y)$ quasisimple, so if $Y = L_M$ then $Y = L_R$ by 13.1.2.5, contradicting $Y \not\leq M_c$. Thus the claim holds in this case. Now assume that case (1) of 13.1.6 holds and let $\dot{M} := M/O_2(M)$. In this case $\dot{Y}$ is of class at most 2 and exponent d for an odd prime d, with $m_d(\dot{Y}) \leq 2$. Thus as both Y and L_M are normal in M, using 1.2.1.4, either $[Y, L_M] \leq O_2(Y)$ as claimed, or $1 \neq D := [O^{d'}(Y \cap L_M), L_M] \trianglelefteq L_M$, with $\dot{D} \cong E_{d^2}$ or d^{1+2} and L_M irreducible on $\dot{D}/\Phi(\dot{D})$. In the latter case $Y = D$ by A.1.32.2 applied with D, Y in the roles of "P, R". Then as $[Z, Y] \neq 1$, $L_M \in \mathcal{L}_+(G,T)$, so $Y \leq O_\infty(L_M) \leq C_G(Z)$ by 13.1.4.5, contrary to $Y \not\leq M_c = !\mathcal{M}(C_G(Z))$. This contradiction completes the proof of the claim.

In particular by the claim, L_R centralizes $Y/O_2(Y)$, and hence Y normalizes $(L_R O_2(Y))^\infty = L_R$. As $Y \trianglelefteq M$, $O_2(Y) \leq O_2(L_R T) = R$ by a remark in the first paragraph, so $R \in Syl_2(YR)$. Now $L_R/O_2(L_R) \cong L_2(p)$ has 3-rank 1 and centralizes $Y/O_2(Y)$, so $m_3(Y) \leq 1$ as $L_R Y$ is an SQTK-group. In case (2) of 13.1.6, $Y \in \mathcal{L}_f^*(G,T)$, so by 13.1.2.3, $Y/O_2(Y) \cong A_5$ or $L_3(2)$. In case (2) of 13.1.6, either $d = 3$ and $Y/O_2(Y) \cong \mathbf{Z}_3$, or $Y/O_2(Y)$ is a d-group for $d > 3$.

Recall X is a solvable $3'$-group and $R \in Syl_2(XR)$, so we may apply a standard Thompson factorization theorem 26.18 in [**GLS96**] to conclude that

$$XR = N_{XR}(J(R))N_{XR}(E_R), \text{ where } E_R := \Omega_1(Z(J_1(R))).$$

As $X \in \Xi(G,T)$, T is irreducible on the Frattini quotient of $X/O_2(X)$, so $J(R)$ or E_R is normal in XR; set $J := J_j(R)$ where $j := 0$ or 1 in the respective case, so in either case $J(R) \leq J$ and $N_G(J) \leq M_c$ since $M_c = !\mathcal{M}(XT)$ by (+).

Set $K := [Y,J]$. If $K \leq O_2(Y)$ then Y normalizes $R_1 := JO_2(Y) \leq R$; but $J = J_j(R_1)$ by B.2.3.3, and hence $Y \leq N_G(J) \leq M_c$, contrary to our choice of $Y \not\leq M_c$. This contradiction shows that $K \not\leq O_2(Y)$. Thus $K = Y$ in case (2) of 13.1.6, since there $Y \in \mathcal{C}(M)$ with $Y/O_2(Y)$ quasisimple. In case (1) of 13.1.6, $Y = KN_Y(J)$ by a Frattini Argument applied to KJ, and hence $Y = K(Y \cap M_c)$. Thus in either case $K = [K,J]$ and as $Y \not\leq M_c$,

$$K \not\leq M_c, \text{ and in particular } [Z,K] \neq 1. \qquad (!)$$

As $L_R T$ normalizes Y and J, it also normalizes $[Y,J] = K$ and hence normalizers KR. Further $K \leq Y \leq C_G(L_R/O_2(L_R))$ as we saw after the claim, so $[L_R, KR] \leq O_2(L_R) \cap KR \leq O_2(KR)$.

As K is subnormal in M, $K \in \mathcal{H}^e$ by 1.1.3.1. As $R \in Syl_2(YR)$, $R \in Syl_2(KR)$. Thus if we set $D := \langle \Omega_1(Z(R))^{KR} \rangle$, then $D \in \mathcal{R}_2(KR)$ by B.2.14, and D is $L_R T$-invariant as R and KR are $L_R T$-invariant. Set $H := L_R KR$ and $\hat{H} := H/C_H(D)$. We saw that $L_R \in \mathcal{L}_f(G,T)$ and $R = O_2(L_R T)$, so $[\Omega_1(Z(R)), L_R] \neq 1$ by A.4.8.5 with L_R, $L_R T$, R, T in the roles of "X, M, R, S", and hence $[D, L_R] \neq 1$, so that $\hat{L}_R \neq 1$. As $[Z,K] \neq 1$ by (!), $[D,K] \neq 1$, so that $\hat{K} \neq 1$. We have seen that $[L_R, KR] \leq O_2(L_R) \cap KR \leq O_2(KR)$, while $O_2(\hat{K}\hat{R}) = 1$ as $D \in \mathcal{R}_2(KR)$, so $[\hat{L}_R, \hat{K}\hat{R}] = 1$. Further $O_2(L_R)$ centralizes $\Omega_1(Z(R))$, so $\hat{H} = \hat{L}_R \times \hat{K}\hat{R}$. In particular $F^*(\hat{H}) = \hat{L}_R \times \hat{K}$, so $\hat{R}$ is faithful on $\hat{K}$.

As T acts on D, $1 \neq [D, L_R] \cap Z =: Z_0 \leq Z$. Then as $C_G(Z_0) \leq M_c$ by (+), $K \not\leq C_G(Z_0)$ by (!), and hence $[D, L_R, K] \neq 1$. As $K = [K,J]$ and $\hat{K} \neq 1$, $\hat{J} \neq 1$, so there is $A \in \mathcal{A}_j(R)$ with $\hat{A} \neq 1$. As $\hat{R}$ is faithful on $\hat{K}$, so is $\hat{A}$.

In a moment we will define a subgroup K_B of K, with $\hat{K}_B = [\hat{K}_B, \hat{A}]$ and $\hat{K}_B$ nontrivial on $[D, L_R]$. Set $H_1 := L_R K_B A$. As $\hat{H} = \hat{L}_R \times \hat{K}\hat{A}$, $\hat{H}_1 = \hat{L}_R \times \hat{K}_B \hat{A}$. As $\hat{L}_R$ is simple, we can choose an H_1-chief section W in $[D, L_R]$ with $\hat{L}_R$ faithful on W and $\hat{K}_B$ nontrivial on W. Set $\bar{H}_1 := H_1/C_{H_1}(W)$ and $\epsilon := m(\hat{A}) - m(\bar{A})$; then $\bar{H}_1 = \bar{L}_R \times \bar{K}_B \bar{A}$, and we will see that $\epsilon = 0$ or 1.

Assume $\hat{K}$ is simple. Then as $1 \neq \hat{A}$ is faithful on $\hat{K}$, $\hat{K} = [\hat{K}, \hat{A}]$; and as $[D, L_R, K] \neq 1$, $\hat{K}$ is faithful on $[D, L_R]$. In this case, we set $K_B := K$. As $\hat{A}$ is faithful on $\hat{K}$ and $\hat{K}$ is faithful on W, $\hat{A}$ is faithful on W, so that $m(\bar{A}) = m(\hat{A})$ and $\epsilon = 0$ in this case.

So assume $\hat{K}$ is not simple. Then $\hat{K}$ is a d-group for $d > 3$, and as $\hat{K}$ is of class at most 2, exponent d, and d-rank at most 2, we conclude from A.1.24 that $\hat{K} \cong E_{d^2}$ or d^{1+2}. Therefore $\hat{A} \leq GL_2(d)$, so $m(\hat{A}) \leq 2$. Now as $K = [K,J]$ and

K is nontrivial on $[D_L, R]$, we may choose A so that $K_A := [K, A]$ is nontrivial on $[D_L, R]$. Also by A.1.17, $\hat{K}_A$ is generated by the fixed points of hyperplanes of $\hat{A}$; so we may choose a hyperplane $\hat{B}$ of $\hat{A}$ and a subgroup $\hat{K}_B = [\hat{K}_B, \hat{A}]$ of $C_{\hat{K}_A}(\hat{B})$ of order d, such that $\hat{K}_B$ is nontrivial on $[D_L, R]$. By the Thompson $A \times B$-Lemma, $\hat{K}_B$ is nontrivial on $C_{[D,L_R]}(B)$. In this case as $m(\hat{A}) \leq 2$, $\epsilon = 0$ or 1.

Let I be an irreducible $K_B A$-submodule of W. Then as $\bar{H}_1 = \bar{L}_R \times \bar{K}_B \bar{A}$, Clifford's Theorem says that W is the direct sum of r copies of I for some r, and $C_{GL(W)}(\bar{K}_B \bar{A}) \cong GL_r(F)$, where $F := End_{\bar{K}_B}(I)$. As $\bar{L}_R \leq C_{GL(W)}(\bar{K}_B)$ with $\bar{L}_R$ nonabelian, $r > 1$.

As $A \in \mathcal{A}_j(R)$, by B.2.4.1,

$$m(D/C_D(A)) \leq j + m(\hat{A}).$$

Then

$$r \cdot m(I/C_I(\bar{A})) = m(W/C_W(\bar{A})) \leq m(D/C_D(A)) \leq j + m(\hat{A}) = j + \epsilon + m(\bar{A}). \quad (*)$$

Thus as $r > 1 \geq \epsilon$, it follows from (*) that $m(I/C_I(\bar{A})) \leq m(\bar{A})$, so that I is an FF-module for $\bar{K}_B \bar{A}$. Therefore by Theorem B.5.6, $\bar{K}_B$ is not a d-group for a prime $d > 3$. Therefore $K_B = K$, $\epsilon = 0$, and $\bar{K} \cong \mathbf{Z}_3$, A_5, or $L_3(2)$. Further if I is the natural $L_2(4)$-module, then $m(\bar{A}) \leq m_2(Aut(L_2(4)) = 2 \leq m(I/C_I(\bar{A}))$, contrary to (*). Thus if $\hat{K} \cong A_5$, then I is the A_5-module by B.4.2. It follows from B.4.2 that $F = \mathbf{F}_2$ for each of the possible irreducible FF-modules I for each $\bar{K}$, so $\bar{L}_R \leq GL_r(\mathbf{F}_2)$. Since $\bar{L}_R \cong L_2(p)$ for $p \geq 5$, we conclude $r \geq 3$, with $\bar{L}_R \cong L_2(7) \cong L_3(2)$ and each $I_R \in Irr_+(L_R, W)$ of rank 3 in case of equality. Now $m(\bar{A}) \leq m_2(Aut(\bar{K})) \leq 2$ and $m(W/C_W(\bar{A})) \geq r \geq 3$, so all inequalities in (*) must be an equalities, and in particular $\bar{L}_R \cong L_3(2)$, $m(I_R) = r = 3$, $m(I/C_I(\bar{A})) = 1$, $j = 1$, and $m(\bar{A}) = 2$. As $r = 3$, $\bar{K} \cong L_3(2)$ and $m(I) = 3$, so $N_{GL(W)}(\bar{L}_R) \cong L_3(2) \times L_3(2)$; hence $\bar{A} \leq \bar{K}$, $\bar{L}_R \bar{K} \cong L_3(2) \times L_3(2)$, and W is the tensor product of natural modules for the two factors. Furthermore as $m(\bar{A}) = 2$ and $m(I/C_I(\bar{A})) = 1$, $\bar{A}$ is the group of transvections in $\bar{K}$ with a common axis on I. In particular $\bar{A}$ is the unique such subgroup of $\overline{T \cap K}$, so $\bar{A} = \bar{J} \cong E_4$, and hence $J = J_1(O_2(H_1)A)$. Then $N_K(\bar{A}) \leq N_G(J) \leq M_c$ by an earlier observation. Now $\hat{K}$ is simple, so $\hat{A}$ is faithful on each nontrivial $\hat{K}$-section of D. Since (*) is an equality, we conclude that $[D, \hat{K}] = W \in Irr_+(L_R K T)$. Now $Z \cap W$ is a 1-subspace of W centralized by the parabolic of $\hat{K}$ stabilizing the group of transvections on I with a common center, and $C_G(Z \cap W) \leq M_c = !\mathcal{M}(C_G(Z))$ by (+). Thus $K = \langle C_K(Z \cap W), N_K(\bar{A}) \rangle \leq M_c$, contrary to (!). This contradiction completes the proof of (1), and hence of 13.1.9. □

LEMMA 13.1.10. *One of the following holds:*

(1) $L^* \cong L_2(4)$ and $U/C_U(L)$ is the $L_2(4)$-module.

(2) $L^*T^* \cong S_5$ and U is the S_5-module.

(3) $L^* \cong L_3(2)$ and U is the sum of at most two isomorphic natural modules.

(4) $L^* \cong L_3(2)$, $m(U) = 4$, and $[Z, L] = 1$.

PROOF. By 13.1.9.2, V_c is an FF-module for M_c^* with $L^* \leq J(M_c^*, V_c)$. By 13.1.8.1, $L^* \cong L_2(p)$ for a prime $p \geq 5$. Thus L^* is isomorphic to $L_2(5)$ or $L_2(7)$ by B.5.5.1iv. Then cases (2) and (4) of Theorem B.5.6 do not hold; and in cases (3) and (5) of B.5.6, we see that $U = [V_c, L]$ satisfies one of conclusions (1)–(4). Finally if case (1) of B.5.6 holds, then $L^* = F^*(J(M_c^*, V_c))$ and U is described in Theorem

B.5.1. Now by Theorem B.5.1, either conclusion (3) holds or $U \in Irr_+(L, V_c)$, in which case B.4.2 says U satisfies one of conclusions (1), (2), or (4), using I.1.6.4 in the latter case. Furthermore in any case $Z \leq R_2(M_c) = V_c$, and $\langle Z^L \rangle = UC_Z(L)$ by B.2.14. Thus in conclusion (4), since $Z \cap U = C_U(L)$ by B.4.8.2, we conclude that $[Z, L] = 1$. □

LEMMA 13.1.11. *For $X \in \mathcal{X}$, if $X \leq X_1 \leq M_c$ and X_1 acts irreducibly on $X/O_{2,\Phi}(X)$, then X_1 is contained in a unique conjugate of M_c.*

PROOF. Suppose $X_1 \leq M_c^g$ and let p be the odd prime in $\pi(X)$, $P \in Syl_p(X)$, and $\hat{M}_c^g := M_c^g/O_2(M_c^g)$. We apply A.1.32 with $\hat{M}_c^g$, $\hat{X}^g$, $\hat{P}$, p, p in the roles of "G, R ,P, r, p". The second case of A.1.32.2 does not arise as $\hat{X}^g$ is not of order p. Thus $\hat{X}^g = \hat{X}$, so $X = O^2(X) \leq O^2(X^g) = X^g$, and hence $g \in N_G(X) = M_c$, so $M_c = M_c^g$. □

LEMMA 13.1.12. *Let $u \in U^\#$, $G_u := C_G(u)$, $M_u := C_{M_c}(u)$, and pick u so that $T_u := C_T(u) \in Syl_2(M_u)$. Then*

(1) M_u is irreducible on $X/O_{2,\Phi}(X)$ for each $X \in \mathcal{X}$.
(2) $|T : T_u| \leq 2$.
(3) $G_u \leq M_c$.

PROOF. Let
$$\mathcal{U} := \{u \in U^\# : |T_u| < |T|\}.$$
If $u \in Z$ then $T_u = T$ is irreducible on $X/O_{2,\Phi}(X)$ for each $X \in \mathcal{X}$, while $G_u \leq M_c = !\mathcal{M}(C_G(Z))$ by (+), so that (1)–(3) hold. Thus we may assume that $\mathcal{U}$ is nonempty, and it suffices to establish (1)–(3) for $u \in \mathcal{U}$. In particular M_c is not transitive on $U^\#$, so $C_U(L) \neq 1$ in case (1) of 13.1.10, and U is the sum of two irreducibles for L^* in case (3).

Let $u \in \mathcal{U}$, and recall from 13.1.4.5 that X centralizes $R_2(M_c) = V_c$ and $U = [V_c, L]$, so that $X \leq M_u$. As $X \in \Xi_+(G,T)$, $X \cong E_{p^2}$ or p^{1+2} for some prime $p > 3$. Set $M_u^+ := M_u/C_{M_u}(X/O_{2,\Phi}(X))$; thus $M_u^+ \leq GL_2(p)$. To prove (1), it will suffice to show that M_u^+ is nonabelian; as $C_G(X) \leq O_{2,F}(L) \leq C_G(U)$, it also suffices to show $C_L(u)^*$ is nonabelian. Indeed $L/O_{2,F}(L) \cong SL_2(q)$ for $q = 5$ or 7, so if $C_L(u)^*$ contains a 4-group, then the preimage of this 4-group in $C_L(u)/O_{2,F}(X)$ is the nonabelian group Q_8, which again suffices. To prove (2), it will suffice to show for each orbit $\mathcal{O}$ of LT on $\mathcal{U}$ that $|\mathcal{O}| \equiv 2 \mod 4$.

Assume case (1) of 13.1.10 holds. By I.2.3.1, U is a quotient of the rank-6 extension U_0 of $U/C_U(L)$, so $m(C_U(L)) = 1$ or 2. However if $m(C_U(L)) = 1$ or $T^* \leq L^*$, then all members of $U^\#$ are 2-central in M_c. Therefore $m(C_U(L)) = 2$, so $U = U_0$ and hence U admits an $\mathbf{F}_4$-structure by I.2.3.1. Further $T^* \not\leq L^*$, so $L^*T^* \cong S_5$. Then LT has two orbits on $\mathcal{U}$: one of length 2 in $C_U(L)$, and one of length 30 on $U - C_U(L)$. In either case, $|u^{LT}| \equiv 2 \mod 4$, so that (2) holds by the previous paragraph. Also $(T_u \cap L)^* \in Syl_2(L^*)$, so (1) also holds by the previous paragraph.

Assume case (3) of 13.1.10 holds. We saw U is the sum of two natural modules, so L is transitive on
$$\mathcal{U}_1 := U - \bigcup_{I \in Irr_+(L,U)} I$$

of order 42 with $C_L(u_1)^* \cong E_4$ for $u_1 \in \mathcal{U}_1$. Further either $\mathcal{U} = \mathcal{U}_1$, or $Aut_{M_c}(U) \cong L_3(2) \times \mathbf{Z}_2$ and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where

$$\mathcal{U}_2 := \bigcup_{I \in Irr_+(L,U)} I - U_1 \text{ is of order 14},$$

where U_1 is the unique T-invariant member of $Irr_+(L, U)$, and $C_L(u)^* \cong S_4$ for $u \in \mathcal{U}_2$. In either case, (1) and (2) hold.

In case (2) of 13.1.10, $\mathcal{U}$ is the set of nonsingular vectors in the orthogonal space U, so $|\mathcal{U}| = 10$, and $C_{L^*}(u) \cong S_3$ is nonabelian; thus (1) and (2) hold in this case. Finally in case (4) of 13.1.10, L is transitive on $\mathcal{U} = U - C_U(L)$ of order 14, so $C_{L^*}(u) \cong A_4$ is nonabelian, completing the proof of (1) and (2). Observe we also showed in cases (1), (3), and (4) of 13.1.10 that $(T_u \cap L)^+$ contains a Q_8-subgroup.

It remains to prove (3), so we assume $G_u \not\leq M_c$, and derive a contradiction.

We next claim that $T_u \in Syl_2(G_u)$: For if not, then by (2), $u^g \in Z$ for some $g \in G$. Then $G_u^g = C_G(u^g) \leq M_c = !\mathcal{M}(C_G(Z))$ by (+), and hence $X \leq M_u = M_c \cap G_u \leq M_c^{g^{-1}}$. By (1), we may apply 13.1.11 to conclude that M_u is contained in a unique conjugate of M_c, so that $M_c = M_c^{g^{-1}}$. Then $g \in M_c$ as $M_c \in \mathcal{M}$, so u is centralized by an M_c-conjugate of T, whereas $u \in \mathcal{U}$ so $|M_c : M_u|$ is even. Hence the claim is established.

Set $R_X := O_2(XT)$ and $R := C_{R_X}(u)$. Observe that $[L, R_X] \leq O_\infty(L) = C_L(U)$, so L^* centralizes R_X^*. We claim that $R = C_{R_X}(U)$. Since $C_{R_X}(U) \leq R$, it suffices to show that R centralizes U. If $U \in Irr_+(L, U)$, then as L^* centralizes R_X^*, R_X centralizes U by A.1.41, so that the claim holds. Therefore we may assume that $U \notin Irr_+(L, U)$, so case (3) of 13.1.10 holds. Then $Aut_{M_c}(U) \leq L_3(2) \times S_3$, so we may assume $Aut_R(U) = C_{Aut_{M_c}}(U) \cong \mathbf{Z}_2$. But then $C_U(R)$ is a T-invariant member of $Irr_+(L, U)$, so as $u \in C_U(R)$, u is 2-central in M_c, contrary to $u \in \mathcal{U}$. This completes the proof of the claim.

By the claim, $R \trianglelefteq XT$, so as $M_c = !\mathcal{M}(XT)$ by (+),

$$C(G_u, R) \leq M_c \cap G_u = M_u.$$

Next, the hypotheses of A.4.2.7 are satisfied with G_u, M_u, T_u in the roles of "G, M, T": For $N_{G_u}(R) \leq M_u$ and $X \trianglelefteq M_u$, with T_u Sylow in M_u and G_u, and $R = O_2(XT) = O_2(XT_u)$. Therefore $R \in \mathcal{B}_2(G_u)$ and $R \in Syl_2(\langle R^{M_u} \rangle)$ by A.4.2.7. Thus the pushing up Hypothesis C.2.3 is satisfied with G_u, M_u in the role of "H, M_H". However, before we apply pushing up results from section C.2, we will establish a number of further preliminary results.

We claim next that $O_{2,F}(G_u) \leq M_u$: Set $\hat{G}_u := G_u/O_2(G_u)$ and recall p is the odd prime in $\pi(X)$. Let $\hat{R}_r$ denote a supercritical subgroup of $O_r(\hat{G}_u)$. As M_u is irreducible on $X/O_{2,\Phi}(X)$ by (2), we may apply A.1.32 with $\hat{G}_u$, $\hat{R}_r$, $\hat{X}$ in the roles of "G, R, P". If $r \neq p$, then by part (1) of that result, $\hat{X}$ centralizes $\hat{R}_r$, and hence $O^p(O_{2,F}(G_u))$ normalizes X. If $r = p$, then by part (2) of A.1.32, either $\hat{X} = \hat{R}_p$, or $\mathbf{Z}_p \cong \hat{R}_p = Z(\hat{X})$ and $\hat{X} \cong p^{1+2}$. In the former case, $O_p(\hat{G}_u)$ normalizes the characteristic subgroup $\hat{R}_p = \hat{X}$, so the claim holds. In the latter case, since the supercritical subgroup $\hat{R}_p$ contains all elements of order p in $C_{O_p(\hat{G}_u)}(\hat{R}_p)$, we conclude that $O_p(\hat{G}_u)$ is cyclic. Then as M_u is irreducible on $X/O_{2,\Phi}(X)$, $\hat{X}$ centralizes $O_p(\hat{G}_u)$, completing the proof of the claim. We have also shown that $\hat{X}$

centralizes $O^p(F(\hat{G}_u))$ and either $\hat{X} = \hat{R}_p \leq O_p(\hat{G}_u)$, or $\hat{X}$ centralizes $O_p(\hat{G}_u)$ and hence $F(\hat{G}_u)$.

Let $z \in Z^\#$. By 1.1.6, the hypotheses of 1.1.5 are satisfied with $\langle u \rangle$, G_u, T_u, M_c in the roles of "B, H, S, M". By the claim, $O(F(G_u)) \leq N_{G_u}(X) \leq M_u$, so $O(F(G_u)) \leq C_{G_u}(z)$ for $z \in Z \cap O_2(X)^\#$. But by 1.1.5.2, z inverts $O(G_u)$, so $O(G_u) = 1$.

Assume that $O_{2,F^*}(G_u) \leq N_{G_u}(X)$. Then $\hat{X}$ centralizes $E(\hat{G}_u)$. We saw that $\hat{X}$ centralizes $O^p(F(\hat{G}_u))$ and either $\hat{X} = \hat{R}_p$ or $\hat{X}$ centralizes $F(\hat{G}_u)$. In the latter case $\hat{X}$ centralizes $F^*(\hat{G}_u)$, so that $\hat{X} \leq O_p(\hat{G}_u)$, and then as $m_p(\hat{X}) = 2 = m_p(\Omega_1(O_p(\hat{G}_u)))$, $\hat{X} = \Omega_1(O_p(\hat{G}_u))$. Hence in either case $G_u \leq N_{G_u}(X) = M_c$, contrary to assumption.

Therefore there exists $K \in \mathcal{C}(G_u)$ with $K \not\leq N_G(X)$ and $K/O_2(K)$ quasisimple. Then $X = O^2(X)$ normalizes K by 1.2.1.3, and $K = [K, X]$ by A.3.3.7. Set $K_0 := \langle K^{T_u} \rangle$.

Suppose $N_{M_u}(K)$ is irreducible on $X/O_{2,\Phi}(X)$. Then $C_X(\hat{K}) \leq O_{2,\Phi}(X)$ and as $\hat{K}$ is described in Theorem C (A.2.3), $m_p(Out(\hat{K})) \leq 1$ since $p > 3$. Therefore since $N_{M_u}(\hat{K})$ is irreducible on $X/O_{2,\Phi}(X)$, X induces inner automorphisms on $\hat{K}$. Then $m_p(K) > 1$, so $K = O^{p'}(G_u)$ by A.3.18. Thus $K_0 = K$ and $X \leq K$. Also $T_uX = XT_u$ with $T_u \in Syl_2(G_u)$, so $\hat{K}$ and the embedding of $\hat{X}$ in $\hat{K}$ are described in A.3.15. As $m_p(Aut_X(\hat{K})) > 1$, and $N_{M_u}(K)$ is irreducible on $X/O_{2,\Phi}(X)$, conclusion (3) of A.3.15 is eliminated, so conclusion (2) or (5) of A.3.15 holds.

Let $P \in Syl_p(X)$. During the proof of (1) and (2), we saw that T_u is reducible on $X/O_{2,\Phi}(X)$ in case (2) of 13.1.10, and in the remaining cases T_u is irreducible and $Aut_{T_u \cap L}(P)$ is noncyclic. Suppose T_u is irreducible on $X/O_{2,\Phi}(X)$. Then $X \in \Xi(G_u, T_u)$. We observe that the proof of 1.3.4 does not require the hypotheses that $H \in \mathcal{H}(XT)$, but only that $H \in \mathcal{H}(X)$, and $N_{T\cap H}(X)$ is irreducible on $X/O_{2,\Phi}(X)$. Thus we may apply the analogue of that result with G_u, T_u, K in the roles of "$H, T, \langle L^T \rangle$", to conclude that $\hat{K}$ is described in 1.3.4. Therefore if A.3.15.5 holds, then $\hat{K} \cong Sp_4(2^n)$ with $Aut_{T_u}(P)$ cyclic, contary to a remark earlier in the paragraph. Thus T_u is reducible on $X/O_{2,\Phi}(X)$, so we are in case (2) of 13.1.10, where $C_L(u)^+$ contains an S_3-section. In case (2) of A.3.15, T_u is irreducible on $X/O_{2,\Phi}(X)$, so we are in case (5) of A.3.15. Then as $P \leq K$ with $PT_u = T_uP$ and $p > 3$, $\hat{K}$ is of Lie type over $\mathbf{F}_{2^n}$ with $2^n \equiv 1 \mod p$, and P is contained in the Borel subgroup $N_K(T_u \cap K)$. Hence the S_3-section is induced by outer automorphisms of $\hat{K}$, so from the structure of $Out(K_0/O_2(K_0))$, $\hat{K} \cong (S)L_3(2^n)$ with n even.

Having discussed the case where $N_{M_u}(K)$ is irreducible on $X/O_{2,\Phi}(X)$, we now turn to the remaining case where it is reducible. If T_u normalizes K, then so does M_u by 1.2.1.3, and then (1) contradicts the assumption in this case. Therefore $K < K_0$. However M_u acts on K_0 and is irreducible on $X/O_{2,\Phi}(X)$ by (1). Further by 1.2.1.3, $Out(\hat{K})$ is cyclic. so as $Out(\hat{K}_0)$ is $Out(\hat{K})$ wr $\mathbf{Z}_2$, again X induces inner automorphisms on $\hat{K}_0$. By 1.2.2.a, $K_0 = O^{p'}(G_u)$, so a Sylow p-subgroup P of X is containied in K_0 and $P = P_K \times P_K^t$, where $P_K := P \cap K \cong \mathbf{Z}_p$ and $t \in T_u - N_{T_u}(K)$. As $T_uP = PT_u$, we conclude from 1.2.1.3 and A.3.15 that $\hat{K}$ is isomorphic to $L_2(2^n)$ or $Sz(2^n)$ with $2^n \equiv 1 \mod p$, J_1 with $p = 7$, or $L_2(r)$ for a suitable odd prime r.

Summarizing our list of possibilities, $X \leq K_0$ and either

(i) $K = K_0$, with $\hat{K}$ isomorphic to $L_3(p)$ or $(S)L_3(2^n)$ where $2^n \equiv 1 \mod p$, or

(ii) $K_0 = KK^t$ for $t \in T_u - N_{T_u}(K)$, and $\hat{K}$ is isomorphic to $Sz(2^n)$ or $L_2(2^n)$ with $2^n \equiv 1 \mod p$, $L_2(r)$ for a suitable odd prime r, or J_1 with $p = 7$.

Recall we saw earlier that Hypothesis C.2.3 holds. We are now ready to apply pushing up results from section C.2.

Suppose first that $F^*(K) = O_2(K)$. If $R \not\leq N_{G_u}(K)$ then $K < K_0$, and by C.2.4.2, $R \cap K \in Syl_2(K)$, so K is a χ_0-block of G_u by C.2.4.1. Then from our list of possibilities for $\hat{K}$, $\hat{K} \cong L_2(2^n)$. On the other hand if $R \leq N_{G_u}(K)$, then K is described in C.2.7.3. We compare the list of C.2.7.3 with our list of possibilities for $\hat{K}$ in (i) and (ii): If $\hat{K} \cong (S)L_3(2^n)$, then case (g) of C.2.7.3 occurs, so $K \cap M_c$ is a maximal parabolic of $SL_3(2^n)$, impossible as $X \trianglelefteq K \cap M_c$. The only remaining possibility in both lists is case (a) of C.2.7.3 with K a χ-block, so again $\hat{K} \cong L_2(2^n)$ and $K < K_0$.

Thus in any case, $K_0 = KK^t$ for $t \in T_u - N_{T_u}(K)$, and $[K, K^t] = 1$ by C.1.9. Let $\mathcal{P}$ be the set of subgroups P_0 of P of order p such that $[C_{O_2(X)}(P_0), P] \neq 1$, and set $X_K := X \cap K$ and $P_K := P \cap K$. Then $X = X_K X_K^t$ and $\mathcal{P} = \{P_K, P_K^t\}$. But $M_c = N_G(X)$, so $N_{M_c}(P)$ permutes $\mathcal{P}$, contrary to the fact that $N_L(P)$ induces either $SL_2(p)$ or $SL_2(5)$ on P, and thus has no orbit of order 2 on the set of subgroups of P of order p.

Therefore $F^*(K) \neq O_2(K)$, so as $O(G_u) = 1$ and $K/O_2(K)$ is quasisimple, K is quasisimple. Then as $X \leq K_0$, and $F^*(X) = O_2(X)$, we conclude by comparing the list in 1.1.5.3 with our list of possibilities for $\hat{K}$ in (i) and (ii) that $K_0/Z(K_0) \cong (S)L_3(2^n)$, $L_2(2^n) \times L_2(2^n)$, or $Sz(2^n) \times Sz(2^n)$ for some n, or $L_2(r) \times L_2(r)$ for r a Fermat or Mersenne prime. In the latter three cases the components commute, so as in the previous paragraph we conclude that $N_{M_c}(P)$ permutes the subgroups $P \cap K$ and $P \cap K^t$, for the same contradiction. Furthermore a similar argument works in the first case: Namely X lies in a Borel subgroup of K, so that $O_2(X)$ is the full unipotent group A, which is special of order 2^{3n} with center of order 2^n. Therefore $N_{M_c}(P)$ acts on $C_P(Z(A)) \cong \mathbf{Z}_p$, for the same contradiction. This finally completes the proof of 13.1.12. □

LEMMA 13.1.13. *U is a TI-set in G.*

PROOF. Suppose $1 \neq u \in U \cap U^g$ for some $g \in G$. Then by 13.1.12.3, $X^g \leq C_{M_c^g}(u) \leq M_c$ for $X \in \mathcal{X}$, and by 13.1.12.1, $C_{M_c^g}(u)$ is irreducible on $X^g/O_{2,\Phi}(X^g)$, so 13.1.11 says $M_c = M_c^g$. Thus $g \in N_G(M_c) = M_c$ as $M_c \in \mathcal{M}$, so $U = U^g$, completing the proof. □

Recall the weak-closure parameters $w(G, U)$ and $r(G, U)$ from Defintions E.3.23 and E.3.3.

LEMMA 13.1.14. *(1) $W_i(T, U)$ centralizes U for $i = 0, 1$, so $N_G(W_0(T, U)) \leq M_c$.*

(2) $w(G, U) > 1 < r(G, U)$.

(3) If $H \in \mathcal{H}(T)$ with $n(H) = 1$, then $H \leq M_c$.

PROOF. As U is a TI-set in G by 13.1.13, if $N_{U^g}(U) \neq 1$ and $\langle U, U^g \rangle$ is a 2-group, then $[U, U^g] = 1$ by I.7.6. Therefore as $U \trianglelefteq T$, we conclude that $W_0 := W_0(T, U)$ centralizes U. Hence $W_0 \leq C_T(U) =: R$, so that $W_0 = W_0(R, U)$. Now by a Frattini Argument, $L = O_\infty(L)N_L(W_0)$, so $N_G(W_0) \leq M_c$ by 13.1.9.1.

Next assume $W_1(T,U)$ does not centralize U. Then by the previous paragraph, there is $g \in G$ with $A := U^g \cap T$ a hyperplane of U^g and $A^* \neq 1$. As $A^* \neq 1$, I.6.2.2a says that A^* is the full group of $\mathbf{F}_2$-transvections on U with axis $U \cap M_c^g$. Inspecting the cases listed in 13.1.10, we conclude $L^* \cong L_3(2)$, $m(U) = 3$, and $E_4 \cong A \leq L^*$. Hence A induces a faithful 4-group of inner automorphisms on $L/O_{infy}(L)$. This is impossible as $L/O_2(L) \cong SL_2(7)/E_{49}$, so $Aut(L/O_2(L)) \cong GL_2(7)/E_{49}$. This completes the proof of (1).

By (1), $w(G,U) > 1$, and by 13.1.13, $r(G,U) = m(U) > 1$. Thus (2) holds. Finally the hypotheses of E.3.35 are satisfied with U, R, M_c in the roles of "V, Q, M", so (2) and E.3.35.1 imply (3). □

We are now in a position to complete the proof of Theorem 13.1.7.

We saw at that outset of the proof that $|\mathcal{M}(T)| > 1$, so that there is some $M \in \mathcal{M}(T)$ with $M \neq M_c$. Thus we may choose Y as in 13.1.6. Then $Y \not\leq M_c$, so $n(YT) > 1$ by 13.1.14.3. Thus Y is not solvable by E.1.13, so case (2) of 13.1.6 holds, and in particular $Y \in \mathcal{L}_f^*(G,T)$. Therefore $Y/O_2(Y)$ is described in 13.1.2.3, so as $n(YT) > 1$, $Y/O_2(Y) \cong A_5$ using E.1.14.

Next as $Y \not\leq M_c$, $Y \not\leq N_G(W_0(T,U))$ in view of 13.1.14.1. Thus by E.3.15, there is $A := U^g \leq T$ for some $g \in G$ with $A \not\leq Q := O_2(YT)$. Let $A_Q := A \cap Q$; then $m(A/A_Q) \leq m_2(YT/O_2(YT)) = 2$, so as $m(A) \geq 3$ by 13.1.10, it follows that $A_Q \neq 1$.

As $A \not\leq O_2(YT)$, $O^2(\langle A^Y \rangle) = Y$. As $Y \not\leq M_c$, there is $h \in Y$ with $A^h \not\leq M_c$. But $A^h \leq YT$, so if $U \leq O_2(YT) = Q$, then $\langle A^h, U \rangle$ is a 2-group with $1 \neq A_Q^h = A^h \cap Q \leq N_G(U)$; then by I.7.6, $A^h \leq C_G(U) \leq N_G(U) = M_c$, contrary to our choice of A^h. Thus $U \not\leq Q$, so we may take $A = U$.

Let $I := \langle U, U^h \rangle$. Then as $m_2(YT/Q) \leq 2$ and $Q = \ker_{YT}(N_{YT}(U))$, $V := U \cap M_c^h$ and $B := U^h \cap M_c$ are of codimension at most 2 in U and U^h, respectively. Therefore since $I/O_2(I)$ is a section of $Y/O_2(Y) \cong A_5$, (a) and (c) of I.6.2.2 say that $O_2(I) = V \times B$, and $I/O_2(I) \cong D_6$, D_{10}, or $L_2(4)$, and $O_2(I)$ is a direct sum of natural modules for $I/O_2(I)$. However in the first two cases, B is of index 2 in U^h, so by 13.1.14.1, $[B,U] = 1$, and then $[U^h, U] = 1$, a contradiction. Hence $I/O_2(I) \cong L_2(4)$. Let $D \in Syl_3(N_I(U))$; then $V = [V,D]$ since $O_2(I)$ is the sume of natural modules for $I/O_2(I)$, and so $U = [U,D]$; thus U is the natural module for $L^* \cong L_2(4)$ by 13.1.10. Hence $B \cong E_4$ and $V = C_U(b)$ for each $b \in B^\#$, so B induces a faithful 4-group of inner automorphisms on $L/O_\infty(L)$. As in the proof of 13.1.14, this is a contradiction. This completes the proof of Theorem 13.1.7.

13.2. Some preliminary results on $\mathbf{A_5}$ and $\mathbf{A_6}$

In this section we establish some technical results used in our treatment of the cases $L/O_{2,Z}(L) \cong A_5$ or A_6 in the FSU. Thus in section 13.2, we assume Hypothesis 12.2.3 from the previous chapter. In particular $M = N_G(L)$ for some $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ quasisimple and $V \in Irr_+(L, R_2(LT))$ is T-invariant and satisfies the Fundamental Setup (3.2.1).

As usual we adopt the conventions of Notation 12.2.5; e.g., $Z = \Omega_1(Z(T))$, $M_V = N_G(V)$, and $\bar{M}_V = M_V/C_G(V)$. We also set

$$Z_V := C_V(L) \text{ and } \hat{V} := V/Z_V.$$

Throughout this short section we assume that $\bar{L} \cong A_n$ for $n = 5$ or 6. Then we are in case (d) of 12.2.2.3, with $\hat{V}$ the 4-dimensional chief factor in a rank-n

permutation module for $\bar{L}$. In particular if $L/O_2(L) = \hat{A}_6$, then $O_{2,Z}(L) \leq C_L(V)$. Therefore as $Out(\bar{L})$ is a 2-group and V is T-invariant, $\bar{M}_V = \bar{L}\bar{T} \cong A_n$ or S_n from the structure of $N_{Aut(\bar{L})}(V)$. We also adopt the notational conventions of section B.3; in particular, $\{1, 2, 3, 4\}$ is an orbit under T.

By B.3.3, if $Z_V \neq 1$ then $n = 6$, V is the core of the permutation module for $\bar{L}$ on $\Omega := \{1, \ldots, n\}$, and Z_V is generated by e_Ω. In any event $\hat{V}$ is the irreducible quotient of the core of the permutation module modulo $\langle e_\Omega \rangle$.

When $n = 6$ we can also view $\hat{V}$ as a 4-dimensional symplectic space over $\mathbf{F}_2$ for $\bar{L} \cong Sp_4(2)'$. When $n = 5$, $\hat{V} = V$ since $\hat{V}$ is projective for $\bar{L} \cong A_5$ (cf. I.1.6.1), and we can view $\hat{V}$ as a 4-dimensional orthogonal space for $\bar{L} \cong \Omega_4^-(2)$. Thus we can speak of isotropic or singular vectors in $\hat{V}$, nondegenerate subspaces of $\hat{V}$, etc. We also adopt the following notational conventions:

NOTATION 13.2.1. For $1 \leq i \leq 4$, let V_i be the preimage in V of an i-dimensional subspace of $\hat{V}$ stabilized by T. Set $M_i := N_{LT}(V_i)$, $L_i := O^2(M_i)$, and let R_i be the preimage in T of $O_2(\bar{M}_i)$. Notice for $i < 4$ that $|R_i : O_2(L_iT)| \leq 2$, with equality iff $L/O_2(L) \cong \hat{A}_6$ and $\bar{T} \not\leq \bar{L}$, in which case $O_2(L_iT) = O_2(L_i)O_2(LT)$. When $L/O_2(L) \cong \hat{A}_6$, define $L_0 := O^2(O_{2,Z}(L))$, and for $i = 1, 2$, set $L_{i,+} := O^2([L_i, T \cap L])$; observe $|L_0 : O_2(L_0)| = 3 = |L_{i,+} : O_2(L_{i,+})|$.

13.2.1. Results on $\mathbf{A_6}$. We collect a number of results on A_6 into a single lemma. The first few are easy calculations invoving only L and V, which do not require Hypothesis 12.2.3.

LEMMA 13.2.2. *Assume $n = 6$ and set $Q := O_2(LT)$. Then*

(1) L is transitive on $\hat{V}^\#$.

(2) Each $v \in V^\#$ is in the center of a Sylow 2-subgroup of LT.

(3) If $L/O_2(L) \cong \hat{A}_6$, then $L_i = L_{i,+}L_0$ for $i = 1, 2$.

(4) If $L = [L, J(T)]$, then $\bar{L}_1 = [\bar{L}_1, J(T)]$.

(5) LT controls G-fusion in V.

(6) $m_2(R_1) = m_2(Q)$, so $V \leq J(R_1)$.

(7) Either there is a nontrivial characteristic subgroup of $Baum(R_1)$ normal in LT (and hence $N_G(Baum(R_1)) \leq M$), or L is an A_6-block.

(8) If $L/O_2(L) \cong \hat{A}_6$, then $J(O_2(L_1T)) = J(O_2(LT))$, so every nontrivial characteristic subgroup of $Baum(O_2(L_1T))$ is normal in LT.

(9) If $L/O_2(L) \cong \hat{A}_6$, then $N_G(L_1) \leq M \geq N_G(L_0)$.

(10) One of the following holds:

(I) Some nontrivial characteristic subgroup of $Baum(T)$ is normal in LT.

(II) L is an A_6-block, and $\mathcal{A}(O_2(LT)) \subseteq \mathcal{A}(T)$.

(III) $\bar{L}_2 = [\bar{L}_2, J_1(T)]$.

PROOF. Parts (1) and (3) are easy calculations, and (2) follows from (1) since the elements of V_1 are central in T. If $Z_V = 1$,, then (5) follows from (1). By Burnside's Fusion Lemma A.1.35, $N_G(T)$ controls fusion in Z, while $N_G(T) \leq M$ by Theorem 3.3.1. Therefore if $Z_V \neq 1$, then $M = M_V$ controls fusion in V_1, so as $\bar{M}_V = \bar{L}\bar{T}$, (5) follows from (1) in this case also.

Next we establish (4) and (6), which will follow fairly easily from B.3.4. First $\bar{R}_1$ contains no strong FF*-offenders by parts (1) and (2ii) of B.3.4, so by B.2.4.3, $m_2(R_1) = m_2(Q)$ and $\mathcal{A}(Q) \subseteq \mathcal{A}(R_1)$. Then as $V \leq \Omega_1(Z(Q))$, $V \leq J(Q) \leq J(R_1)$, completing the proof of (6).

If $J(T) \leq C_T(V) = O_2(LT)$, then (4) is vacuously true. Thus we may suppose that there is $A \in \mathcal{A}(T)$ with $\bar{A} \neq 1$; in particular $L = [L, J(T)]$. Now the hypotheses of B.2.10.2 (and hence of B.2.10.1) are satisfied with LT, T in the roles of "G, R", so $\mathcal{P} := \mathcal{P}_{T,LT}$ is a nonempty stable subset of $\mathcal{P}(\bar{L}\bar{T}, V)$ by B.2.10.1. Similarly using R_1 in the role of "R", $J(R_1) \not\leq Q$ iff $\mathcal{Q} := \mathcal{P}_{R_1,LT}$ is a nonempty stable subset of $\mathcal{P}(\bar{L}\bar{T}, V)$. Moreover by definition in B.2.5, $\overline{J(T)} = J_{\mathcal{P}}(\bar{T})$ and $\overline{J(R_1)} = J_{\mathcal{Q}}(\bar{R}_1)$.

If $\bar{M}_V \cong A_6$, then from B.3.4.1, $J_{\mathcal{P}}(\bar{T}) = \bar{R}_2$, so $\bar{L}_1 = [\bar{L}_1, J_{\mathcal{P}}(\bar{T})]$, and hence (4) holds. So assume instead that $\bar{M}_V \cong S_6$. If $J_{\mathcal{Q}}(\bar{R}_1) = 1$, then (4) follows from B.3.4.2iv, while if $J_{\mathcal{Q}}(\bar{R}_1) \neq 1$, then (4) follows from B.3.4.2v. This completes the proof of (4).

For part (7), we observe that the hypotheses of C.1.37 are satisfied with R_1 in the role of "R" and $P := L_1T$, except when $\bar{M}_V \cong \hat{S}_6$, when we take $P := L_{1,+}T$. Thus conclusion (1) or (2) of C.1.37 holds, giving the alternatives of conclusion (7) of the present result. (Recall that L is not a $\hat{A}_6$-block as $O_{2,Z}(L) = C_L(V)$).

Next we will prove (8) and (9), so we may assume $L/O_2(L) \cong \hat{A}_6$. Set $R := O_2(L_1T)$. We claim first that $J(R) = J(Q)$: If $\bar{M}_V = A_6$ then $R = R_1$ by 13.2.1, and by B.3.4.1, $J(R_1) \leq Q \leq R_1$ so that $J(R) = J(R_1) = J(Q)$. If $\bar{M}_V = \hat{S}_6$, then by 13.2.1, $|R_1 : R| = 2$ and $R = O_2(L_1)Q \leq LQ$. But by B.3.4.2v, if $\overline{J(R)} \neq 1$ then $\overline{J(R)} \not\leq \bar{L}$, so again $J(R) \leq Q$ and $J(R) = J(Q)$, completing the proof of the claim. Then by B.2.3.5, $\mathrm{Baum}(R) = \mathrm{Baum}(Q)$, establishing (8).

Recall $L_0 = O^2(O_{2,Z}(L))$. Therefore $L_0 \trianglelefteq LT$, so that $N_G(L_0) \leq M = !\mathcal{M}(LT)$. Finally $N_G(L_1) = C_G(L_1/O_2(L_1))N_G(R)$ by A.4.2 and a Frattini Argument, so as $N_G(R) \leq N_G(J(R)) \leq M$ by (8), and $C_G(L_1/O_2(L_1))$ normalizes L_0 with $N_G(L_0) \leq M$, we conclude $N_G(L_1) \leq M$, completing the proof of (9).

Finally we prove (10). Let $S := \mathrm{Baum}(T)$. If $J(T) \leq C_T(V) = O_2(LT)$, then using B.2.3.3, $J(T)$ is a nontrivial characteristic subgroup of S normal in LT, so (I) holds. Thus we may assume there is $A \in \mathcal{A}(T)$ with $1 \neq \bar{A} \in \mathcal{P}$. If B is a hyperplane of A with $\bar{B}^L \cap \bar{T} \not\subseteq \bar{R}_2$, then as $B \in \mathcal{A}_1(T)$, (III) holds. Thus we may assume no such B exists. Therefore by B.3.4.2vi, $|\bar{A}| = 2$ for each such A. In particular, $\overline{J(T)}$ lies in the subgroup $\bar{R}_2$ of $\bar{T}$ generated by transvections, so $\mathrm{Baum}(T) = \mathrm{Baum}(R_2)$ by B.2.3.5. Observe that we now have the hypothesis of C.1.37 with R_2 in the role of "R" and $P := L_2T$, unless $LT/O_2(LT) \cong \hat{S}_6$, when we take $P := L_{2,+}T$. Further conclusion (5) of C.1.37 does not hold, as there are no FF*-offenders with image of order greater than 2, so only conclusions (1) or (2) of that lemma can hold. In case (1), (I) holds, and in case (2), L is an A_6-block (Again L is not a $\hat{A}_6$-block as $C_L(V) = O_{2,Z}(L)$). Further FF*-offenders of order 2 are not strong by B.3.4.2i, so that $\mathcal{A}(Q) \subseteq \mathcal{A}(T)$ by B.2.4.3, and hence (II) holds. This completes the proof of (10), and of 13.2.2. □

13.2.2. Results on $\mathbf{A_5}$. In this subsection we assume $n = 5$ and establish a series of results culminating in an important reduction: Theorem 13.2.7. Notice that as $n = 5$, we have Hypothesis 5.0.1, of section 5.1, so we can use results from that section and the subsequent sections of chapters 5 and 6.

LEMMA 13.2.3. *If $n = 5$ then*

(1) $O_2(LT) = C_{LT}(V) = C_{LT}(V_3)$.
(2) $N_G(V_3) \leq M_V$.

PROOF. Part (1) follows from the structure of the A_5-module. Then by (1), $R := O_2(LT) \in Syl_2(C_M(V_3))$, so as $C(G, R) \leq M = !\mathcal{M}(LT)$, $N_G(R) \leq M$ and $R \in Syl_2(C_G(V_3))$. Therefore by a Frattini Argument,

$$N_G(V_3) = C_G(V_3)(N_G(R) \cap N_G(V_3)),$$

so it remains to show that $C_G(V_3) \leq M$—since then $N_G(V_3) \leq M_V$ by 12.2.6. So assume $C_G(V_3) \not\leq M$. Then there is $H \in \mathcal{H}_*(T, M)$ with $O^2(H) \leq C_G(V_3)$, and hence $R \in Syl_2(O^2(H)R)$. Then by Theorem 3.1.1 there is $1 \neq R_0 \leq R$ with $R_0 \trianglelefteq \langle LT, H\rangle$, and so $H \leq N_G(R_0) \leq M = !\mathcal{M}(LT)$, contrary to assumption. □

LEMMA 13.2.4. *Assume $n = 5$. Then for any $W \in \mathcal{R}_2(LT)$ with $[W, L] \neq 1$:*

(1) $R_1 = (T \cap L)O_2(LT) = O_2(C_{LT}(Z \cap [W, L]))$. Further $J(R_1) = J(C_T(W))$ and $Baum(R_1) = Baum(C_T(W))$, so that $C(G, Baum(R_1)) \leq M$.

(2) Let $S := Baum(T)$; then either:

(a) $S \leq C_T(W)$ so that $J(T) = J(C_T(W))$, $C(G, S) \leq M$, and $\mathcal{H}_*(T, M) \subseteq C_G(Z)$, or

(b) $\bar{L}\bar{T} \cong S_5$, $\bar{S} = \overline{J(T)} \cong E_4$ is generated by the two transvections in $\bar{T}$, $\langle Z^L\rangle = V \oplus C_Z(L)$, and $C_V(S) = V_2$.

PROOF. Recall that Hypothesis 12.2.3 excludes the groups in conclusions (2) and (3) of Theorem 6.2.20. Thus case (1) of Theorem 6.2.20 holds, so for any $W \in \mathcal{R}_2(LT)$ with $[W, L] \neq 1$, $[W, L]$ is a sum of at most two A_5-modules. Further $O_2(LT)$ is the kernel of the action of L on both W and V. Thus $N_{\bar{L}\bar{T}}(Z \cap [W, L])$ is the Borel subgroup over $\bar{T}$, so the first sentence in (1) holds. Next by B.3.2.4, each member of $\mathcal{P}(\bar{T}, V)$ is generated by transpositions, and hence none lie in $\bar{R}_1$. Thus $J(R_1) \leq C_T(W) = O_2(LT)$, so that $J(R_1) = J(C_T(W))$ and $\mathrm{Baum}(R_1) = \mathrm{Baum}(C_T(W))$ by B.2.3.5; hence $C(G, \mathrm{Baum}(R_1)) \leq M = !\mathcal{M}(LT)$, so (1) holds.

Part (2) is essentially 5.1.2 applied to W in the role of "V". When $J(T) \leq C_T(W)$, the final statement in (a) follows from Theorem 3.1.8.3. When $J(T) \not\leq C_T(W)$, the statements about $\bar{S}$ and V_2 follow from E.2.3. □

LEMMA 13.2.5. *If $n = 5$ then $N_G(Baum(T)) \leq M$.*

PROOF. The lemma follows from 5.1.7. □

LEMMA 13.2.6. *If $n = 5$ then*

(1) $C_T(v) \in Syl_2(C_G(v))$ for $v \in V_2 - V_1$.

(2) Singular vectors of V are not fused in G to nonsingular vectors of V, so that L controls fusion of involutions in V.

PROOF. Let $v \in V_2 - V_1$. By 13.2.4.2, $v \in V_2 \leq C_V(J(T))$, so $S := \mathrm{Baum}(T) \leq T_v := C_T(v)$; then $S = \mathrm{Baum}(T_v)$ by B.2.3.5. Let $T_v \leq T_0 \in Syl_2(C_G(v))$. Then $N_{T_0}(T_v) \leq N_{T_0}(S) \leq M$ by 13.2.5, so as $T_v \in Syl_2(C_M(v))$, $T_v = T_0$ and hence (1) holds. Then (1) implies that $v \notin z^G$, where z is a singular vector in V, so that (2) holds. □

Most of the remainder of the subsection is devoted to Theorem 13.2.7. This result assumes the hypothesis (*) below, which appears later as part (4) of Hypothesis 13.3.1.

THEOREM 13.2.7. *Assume $n = 5$ and*

$$L_+/O_2(L_+) \cong A_5 \text{ for each } L_+ \in \mathcal{L}_f(G,T). \qquad (*)$$

Then $\mathcal{H}_*(T,M) \subseteq C_G(Z)$.

In the remainder of the section, we assume G is a counterexample to Theorem 13.2.7; thus there is $H \in \mathcal{H}_*(T,M)$ with $H \not\leq C_G(Z)$. Hence:

Conclusion (b) of 13.2.4.2 holds.

In particular, $\bar{L}\bar{T} \cong S_5$ rather than A_5. Let $U_H := \langle Z^H \rangle$, $V_H := [U_H, H]$, $L_H := O^2(H)$, and $H^* := H/C_H(U_H)$. As $H \not\leq C_G(Z)$, by 5.1.7.2:

$$L_H = [L_H, J(T)] \text{ and } L = [L, J(T)].$$

As $L_H = [L_H, J(T)]$, we conclude from B.6.8.6d that $[U_H, J(T)] \neq 1$ Therefore $S := \mathrm{Baum}(T)$ does not centralize U_H, and U_H is an FF-module for H^*. Let $Q := O_2(LT)$.

LEMMA 13.2.8. *(1) H is solvable.*

(2) $U_H = V_H \oplus C_Z(H)$.

(3) Either

(i) $H^* \cong S_3$ and $m(V_H) = 2$, or

(ii) $H^* = (H_1^* \times H_2^*)\langle t^* \rangle \cong S_3$ wr $\mathbf{Z}_2$ and $V_H = U_1 \oplus U_2$, where t^* is an involution with $H_1^{*t} = H_2^*$, $H_1^* \cong S_3$, and $U_1 := [U_H, H_1^*] \cong E_4$.

(4) $S^* \in Syl_2(H^*)$ in (3i), and $J(T)^* = S^* \in Syl_2(H_1^* H_2^*)$ in (3ii).

(5) $S \in Syl_2(L_H S)$.

(6) Let $E := \Omega_1(Z(J(T)))$; set $s := 1$ and $U_1 := V_H$ in case (i), and set $s := 2$ in case (ii). Then $E = C_E(L_H) \oplus E_1 \oplus \cdots \oplus E_s$, where

$$E_i := \langle e_i \rangle = C_{U_i}(S) \cong \mathbf{Z}_2.$$

PROOF. Assume H is not solvable. Then L_H is the product of T-conjugates of members of $\mathcal{L}_f(G,T)$, so by hypothesis (*), $L_H^* \cong A_5$; indeed it follows from (*) that $L_H \in \mathcal{L}_f^*(G,T)$. But then $n(H) > 1$, so that the hypothesis of Theorem 5.2.3 is satisfied. Conclusions (2) and (3) of 5.2.3 are ruled out by Hypothesis 12.2.3, while conclusion (1) of 5.2.3 does not hold as $L_H \not\leq C_G(Z)$. This contradiction establishes part (1) of 13.2.8. Then as U_H is an FF-module for H^*, we conclude from Theorem B.5.6 and B.2.14 that (2) holds, and from E.2.3.2 that (3)–(6) hold. □

We now adopt the notation of 13.2.8.6. Two cases appear in 13.2.8.3: $s = 1$ and $s = 2$. When $s = 2$, define H_i^* as in case (ii) of 13.2.8.3, and let H_i be the preimage in H of H_i^*.

LEMMA 13.2.9. *$O_2(H) = C_H(U_H)$. Thus $L_H/O_2(L_H) \cong E_{3^s}$.*

PROOF. Set $\dot{H} := H/O_2(H)$ and $J := \ker_{H\cap M}(H)$. By 13.2.8 and B.6.8.2, $\dot{L}_H$ is a 3-group, $\dot{J} = \Phi(\dot{L}_H)$, and T is irreducible on $\dot{L}_H/\dot{J}$. As $\Phi(L_H^*) = 1$ by 13.2.8.3, $J = C_H(U_H)$. Thus we may assume that $X := O^2(J) \neq 1$, and it remains to derive a contradiction.

First $X \leq M = N_G(L)$. If X centralizes $L/O_2(L)$, then L normalizes $X = O^2(XO_2(L))$, so $H \leq N_G(X) \leq M = !\mathcal{M}(LT)$, contrary to $H \not\leq M$. Therefore $L = [L, X]$. Let $R := O_2(XT)$. As $XT = TX$, X acts on $T \cap L$, so $R \in Syl_2(LR)$. As $L = [L, X]$, R induces inner automorphisms on $L/O_2(L)$, and $J(R) = J(O_2(LR)) \trianglelefteq$

LT by 13.2.4.1, so $N_G(J(R)) \le M$. To complete the proof we show H acts on $J(R)$, contrary to $H \not\le M$.

Assume H does not act on $J(R)$. Then $O_2(H) < R$, so by E.2.1, $L_H \cong \mathbf{Z}_3$, E_9, of 3^{1+2}, and as $\dot{X} \neq 1$, the last case must hold. Then byt 13.2.8.3, $H^* \cong S_3$ wr $\mathbf{Z}_2$, and as $\dot{R} = C_{\dot{T}}(\dot{X})$, $\dot{R} \cong \mathbf{Z}_4$. But then from the action of H^* on U_H, $J(R) = J(O_2(H))$, contrary to assumption. □

LEMMA 13.2.10. *If* $s = 2$, *then* $L_H \in \Xi_f^*(G,T)$.

PROOF. Assume $s = 2$. Then by 13.2.8.3 and 13.2.9, T is irreducible on $L_H/O_2(L_H) \cong E_9$, so that $L_H \in \Xi(G,T)$. As $[Z, L_H] \neq 1$, $L_H \in \Xi_f(G,T)$. Further if $L_H \le L_0$ for some $L_0 := \langle L_+^T \rangle$ with $L_+ \in \mathcal{L}(G,T)$, then $L_+ \in \mathcal{L}_f(G,T)$. Therefore by hypothesis (*), $L_+/O_2(L_+) \cong A_5$ and $L_+ \in \mathcal{L}_f^*(G,T)$, so $L_0 = L_+$ since conclusion (3) of Theorem 12.2.2 holds by Hypothesis 12.2.3; but this contradicts $m_3(L_H) = 2$. Thus no such L_0 exists, so by definition $L_H \in \Xi_f^*(G,T)$. □

LEMMA 13.2.11. *Assume* $Z(H) = 1$. *Then*

(1) $C_T(L) = C_T(L_H) = C_E(L) = C_E(L_H) = 1$.

(2) $\overline{J(T)} = \bar{S} = \langle (1,2),(3,4) \rangle \cong E_4$.

(3) $s = 2$, $E = \langle e_{1,2}, e_{3,4} \rangle = \langle e_1, e_2 \rangle \cong E_4$, *and* $Z = \langle e_1 e_2 \rangle$ *is of order* 2, *and (interchanging* H_1 *and* H_2 *if necessary) we may take* $e_1 = e_{1,2}$ *and* $e_2 = e_{3,4}$.

(4) $T_0 := N_T(H_1) = N_T(H_2) = C_T(e_1) = C_T(e_2) = QS = O_2(H)S$.

(5) L *is not an* A_5*-block.*

(6) $O^2(H_2)$ *is not an* A_3*-block.*

PROOF. As T acts on $C_E(L_H)$ and $Z(H) = 1$, $C_E(L_H) = 1 = C_T(L_H)$, and hence $E \cong E_{2^s}$ by 13.2.8.

Next as we saw that $L = [L, J(T)]$, (2) follows from 13.2.4.2. Thus $V \cap E$ contains $\langle e_{1,2}, e_{3,4} \rangle \cong E_4$, so as $E \cong E_{2^s}$, we conclude $s = 2$ and $E = V \cap E \le V$. As $Z \le E \le V$, $Z = C_V(T)$ has order 2 and is generated by $z := e_{1,2}e_{3,4}$. As $Z \le V$, $C_T(L) = 1$, completing the proof of (1). Further $E = \langle e_{1,2}, e_{3,4} \rangle = E_1E_2$. Then $Z = \langle z \rangle = \langle e_1 e_2 \rangle$, so (3) holds. Most of the equalities in (4) are clear; observe $T_0 = O_2(H)S$ by 13.2.8.4, and $T_0 = QS$ by (2) and (3).

If L is an A_5-block, then by C.1.13.c, $Q = O_2(LT) = V \times C_T(L)$. Thus $Q = V$ by (1). Now $\bar{T}_0 = \langle (1,2),(3,4) \rangle$ by (2) and (4), so as $Q = V$, $T_0 \cong D_8 \times D_8$. Thus $L_H T_0 \cong S_4 \times S_4$, so $O^2(H_2) =: K_2$ is an A_3-block.

Therefore if (5) fails, then so does (6); so to prove both parts of the lemma, we may assume that K_2 is an A_3-block. Thus $K_1 = K_2^t$ is also an A_3-block; and again by C.1.13.c, $K_i \cong A_4$ and $O_2(H) = C_T(L_H) \times V_H$. Thus $O_2(H) = V_H$ by (1), so $H \cong S_4$ wr $\mathbf{Z}_2$.

By 13.2.10, $L_H \in \Xi^*(G,T)$, so $M_1 := N_G(L_H) = !\mathcal{M}(H)$ by 1.3.7. As $O_2(H) = V_H = O_2(L_H)$, $O_2(H) = O_2(M_1)$ using A.1.6. Then as $F^*(M_1) = O_2(M_1)$, $C_{M_1}(V_H) = V_H$ so that $M_1/V_H \le GL(V_H)$. Then as $H/V_H \cong O_4^+(2)$ is a maximal subgroup of $L_4(2)$ with Sylow group $T/V_H \cong D_8$, we conclude that $M_1 = H$. But now Theorem 13.9.1 contradicts the simplicity of G. □

Set $H_0 := \langle H, L_1 \rangle$.

LEMMA 13.2.12. $O_2(H_0) \neq 1$.

PROOF. Assume that $O_2(H_0) = 1$. Since $L_1 = O^2(N_L(T \cap L))$, we conclude from 5.1.7.2iii that $Z(H) = 1$. Thus we can appeal to 13.2.11. In particular, by that lemma $s = 2$ and $E = \langle e_1, e_2\rangle$, where $e_1 = e_{1,2}$ and $e_2 = e_{3,4}$ are nonsingular. Further $T_0 = C_T(V_2) \in Syl_2(C_G(e_2))$ by 13.2.6.1. Set $K_1 := O^2(H_1)$, $K_2 := O^2(C_L(e_2))$, $G_i := K_iT_0$, and $G_0 := \langle G_1, G_2\rangle$. Then $G_0 \leq C_G(e_2)$, so in particular $T_0 \in Syl_2(G_0)$ and G_0 is an SQTK-group. Therefore (G_0, G_1, G_2) is a Goldschmidt triple of Definition F.6.1 in section F.6, so we can appeal to results in that section.

Let $X := O_{3'}(G_0)$, $\dot{G}_0 := G_0/X$, $\alpha := (\dot{G}_1, \dot{T}_0, \dot{G}_2)$, and $Q_i := O_2(G_i)$. Observe that $\bar{K}_2 = \langle (1,2), (1,5)\rangle$ and $\bar{Q}_2 = \langle (3,4)\rangle$. Further X is 2-closed by F.6.11.1.

Suppose first that $Q_1 = Q_2$. By Theorem 4.3.2, $M = !\mathcal{M}(L)$, so as $K_1 \not\leq M$, no nontrivial characteristic subgroup of Q_2 is normal in LQ_2. On the other hand the hypotheses of C.1.24 are satisfied with Q_2 in the role of "R", so L is an A_5-block by C.1.24, contrary to 13.2.11.5.

Therefore $Q_1 \neq Q_2$. In particular $\dot{\alpha}$ is a Goldschmidt amalgam by F.6.11, so as G_0 is an SQTK-group, $\dot{G}_0$ is described in Theorem F.6.18. Further by the previous paragraph, case (1) of F.6.18 does not arise.

Suppose next that $e_1 \in O_2(G_0)$. Then $W := \langle e_1^{G_0}\rangle \leq O_2(G_0)$. As the generator $z := e_1e_2$ of Z lies in $W\langle e_2\rangle$, $N_G(W\langle e_2\rangle) \in \mathcal{H}^e$ by 1.1.4.3, and hence $A := N_G(W) \cap C_G(e_2) \in \mathcal{H}^e$ by 1.1.3.2. Then as $T_0 \in Syl_2(A)$ since $T_0 \in Syl_2(C_G(e_2))$ and $T_0 \leq G_0 \leq A$, we conclude $G_0 \in \mathcal{H}^e$ by 1.1.4.4. As $[K_i, e_1] \neq 1$, $C_{G_i}(W) \leq Q_i$ for $i = 1, 2$, so $C_{G_0}(W)$ is 2-closed and solvable by F.6.8. Further as $T_0 \in Syl_2(G_0)$ and $e_1 \in Z(T_0)$, $W \in \mathcal{R}_2(G_0)$ by B.2.13. As $\bar{K}_2 = \langle (1,2), (1,5)\rangle$, it follows from 13.2.11.2 that $K_2 = [K_2, J(T)]$ and $J(T) = J(T_0)$. By 13.2.8.4, $K_1 = [K_1, J(T)]$. Therefore W is an FF-module for $G_0^* := G_0/C_{G_0}(W)$ with $K_i^* = [K_i^*, J(T_0)^*] \neq 1$.

Assume first that $\dot{G}_0$ satisfies one of conclusions (3)–(13) of F.6.18, and let $L_0 := G_0^\infty$ and $W_0 := [W, L_0]$. Then from F.6.18, $\dot{L}_0$ is quasisimple, so as X is 2-closed, $L_0 \in \mathcal{C}(G_0)$ by A.3.3. As we saw $G_0 \in \mathcal{H}^e$, $L_0 \in \mathcal{H}^e$ by 1.1.3.1, so that $L_0T_0 \in \mathcal{H}^e$. By F.6.18, $\dot{L}_0$ contains $\dot{K}_1$ or $\dot{K}_2$; so as $K_i = [K_i, J(T)]$, $L_0 = [L_0, J(T_0)]$. Hence $L_0^*J(T)^*$ is described in Theorem B.5.1. Comparing that list with the list in F.6.18, we conclude that $\dot{L}_0 \cong L_3(2)$, $Sp_4(2)'$, $G_2(2)'$, or A_7, and $W_0/C_{W_0}(L_0)$ is a natural module for L_0^*, a 4-dimensional module for $L_0^* \cong A_7$, or the sum of two isomorphic natural modules for $L_0^* \cong L_3(2)$. In each case F.6.18 says $L_0 = O^2(G_0)$, so $K_i \leq L_0$. Then the condition that neither K_1 nor K_2 centralizes $e_1 \in C_W(T_0)$ eliminates all cases except the one where W_0 is the natural module for $G_0^* \cong S_7$ and (in the notation of section B.3) for $i := 1$ or 2, G_i^* is the stabilizer of a partition of type $2^2, 3$, while G_{3-i}^* is the stabilizer of a partition of type $2^3, 1$. This is impossible, as in that case $J(T)^* = O_2(G_{3-i}^*)$, contrary to $K_j^* = [K_j^*, J(T)]$ for each j.

This contradiction shows that $\dot{G}_0$ satisfies none of conclusions (3)–(13) of F.6.18; as case (1) of F.6.18 was eliminated earlier, we conclude that case (2) of F.6.18 holds. Therefore $\dot{G}_0 \cong S_3 \times S_3$ or $E_4/3^{1+2}$. As W is an FF-module for G_0^* and $K_i = [K_i, J(T)]$ for $i = 1$ and 2, it follows from Theorem B.5.6 that $K_i^* \trianglelefteq G_0^* \cong L_2(2) \times L_2(2)$, and $W = [W, K_1] \oplus [W, K_2]$, with $[W, K_i] \cong E_4$. Recall that $\bar{K}_2 = \langle (1,2), (1,5)\rangle$, so that as $e_1 = e_{1,2}$, $\langle e_1^{K_2}\rangle = \langle e_{1,2}, e_{1,5}\rangle = [W, K_2]$ is a proper G_0-invariant subgroup of W, whereas by definition $W = \langle e_1^{G_0}\rangle$. This contradiction finally eliminates the subcase $e_1 \in O_2(G_0)$.

So we turn to the remaining subcase $e_1 \notin O_2(G_0)$. First $C_G(z) \in \mathcal{H}^e$ by 1.1.4.3, so that $C := C_G(z) \cap C_G(e_2) \in \mathcal{H}^e$ by 1.1.3.2. Then as $T_0 \in Syl_2(C)$ since $T_0 \in Syl_2(C_G(e_2))$, we conclude from 1.1.4.4 that $C_{G_0}(z) \in \mathcal{H}^e$. Hence $C_{O(G_0)}(z) \leq O(C_{G_0}(z)) = 1$.

Next $z = e_1 e_2 = e_{1,2}e_{3,4}$ generates Z, and as $\bar{K}_2 = \langle (1,2), (1,5) \rangle$, $\langle z^{K_2} \rangle =: F \cong E_8$, with each coset fE_2 of $E_2 = \langle e_2 \rangle = \langle e_{3,4} \rangle$ in F distinct from E_2 containing a K_2-conjugate z_f of z. Therefore $C_{O(G_0)}(f) = C_{O(G_0)}(z_f) = 1$ using the previous paragraph. Thus no hyperplane of F centralizes an element of $O(G_0)$, so by Generation by Centralizers of Hyperplanes A.1.17, $O(G_0) = 1$.

Now $e_1 \in Z(T_0)$, so e_1 centralizes $O_2(G_0)$, but $e_1 \notin O_2(G_0)$ by assumption. Hence as $O(G_0) = 1$, $L_0 = [L_0, e_1]$ for some component L_0 of G_0. Thus $\dot{L}_0 = E(\dot{G}_0)$ is described in one of cases (3)–(13) of Theorem F.6.18. As $[K_i, e_1] \neq 1$ for $i = 1, 2$, and $\dot{e}_1 \in Z(\dot{Q}_i)$, $\dot{K}_i$ does not centralize $Z(\dot{Q}_i)$. Therefore $\dot{G}_0$ must satisfy conclusion (6) or (8) of F.6.18. But then $K_i \cong A_4$, contrary to 13.2.11.6.

This contradiction finally completes the proof of 13.2.12. □

By 13.2.12, $H_0 \in \mathcal{H}(H)$. Let $U := \langle Z^{H_0} \rangle$, so that $\langle Z^H \rangle = U_H \leq U$, and let $H_0^* := H_0/C_{H_0}(U)$.

LEMMA 13.2.13. *$O^2(H_0/O_{3'}(H_0))$ is not a 3-group.*

PROOF. Assume that $O^2(H_0/O_{3'}(H_0))$ is a 3-group. Then

$$O_2(L_H) \leq O_{3',3}(H_0) \cap T \leq C_T(L_1/O_2(L_1)) = O_2(L_1T) = R_1,$$

so $R_1 \in Syl_2(R_1L_H)$. By 13.2.4.1, $B := \mathrm{Baum}(R_1) = \mathrm{Baum}(Q)$. Thus as $L_H \not\leq M$, $J(R_1)$ is not normal in R_1L_H, so as $[Z, L_H] \neq 1$, $B \in Syl_2(BL_H)$ by E.2.3.2. Thus $Q \in Syl_2(QL_H)$, so by Theorem 3.1.1 applied to LT, Q in the roles of "M_0", "R", $O_2(\langle LT, H \rangle) \neq 1$, and hence we obtain our usual contradiction to $H \not\leq M$. □

LEMMA 13.2.14. $s = 1$.

PROOF. Assume that $s = 2$. By 13.2.10, $L_H \in \Xi_f^*(G, T)$, so $L_H \trianglelefteq H_0$ by 1.3.5. Therefore $H_0 = L_H L_1 T = L_1 H$. Recall we are in case (b) of 13.2.4.2, so that $[Z, L_1] = 1$, and hence $U = \langle Z^{H_0} \rangle = \langle Z^{L_1 H} \rangle = \langle Z^H \rangle = U_H$. By 13.2.8.6, $U_H = U_1 \oplus U_2 \oplus C_Z(H)$. Now $L_1 = O^2(L_1)$ fixes the two subgroups $O^2(H_i)$ with image of index 3 in $L_H/O_2(L_H)$ such that $C_{U_H}(L_H) < C_{U_H}(O^2(H_i)) < U_H$. Hence L_1 acts on U_1, U_2, and $C_Z(H)$. Therefore as $[Z, L_1] = 1$ and H_i induces $GL(U_i)$ on U_i, we conclude $[U_H, L_1] = 1$. Therefore $[L_1, L_H] \leq C_{L_H}(U_H) = O_2(L_H)$, so as $H_0 = L_1 H$, $H_0/O_{3'}(H_0)$ is a 3-group, contrary to 13.2.13. □

We are now ready to complete the proof of Theorem 13.2.7.

As $s = 1$ by 13.2.14, $H/O_2(H) \cong S_3$ by 13.2.9. Hence (H_0, L_1T, H) is a Goldschmidt triple. As $O_2(H_0) \neq 1$ by 13.2.12, H_0 is an SQTK-group. Let $\dot{H}_0 := H_0/O_{3'}(H_0)$ and $\alpha := (\dot{L}_1\dot{T}, \dot{T}, \dot{H})$. By 13.2.13 and F.6.11.2, α is a Goldschmidt amalgam; hence as H_0 is an SQTK-group, $\dot{H}_0$ is described in Theorem F.6.18.

Let $L_0 := H_0^\infty$. By 13.2.13, neither conclusion (1) nor (2) of F.6.18 holds, so $\dot{L}_0$ is quasisimple and described in one of cases (3)–(13) of F.6.18. By F.6.11.1, $O_{3'}(H_0)$ is 2-closed, so $L_0 \in \mathcal{C}(H_0)$ by A.3.3. Thus $L_0 \in \mathcal{L}(G, T)$; so if $[Z, L_0] \neq 1$, then $L_0/O_2(L_0) \cong A_5$ by hypothesis (*) of Theorem 13.2.7. As $\dot{L}_0$ is not A_5 in any of the conclusions of F.6.18, we conclude $[Z, L_0] = 1$. Thus $L_H \not\leq L_0$, so case (3) of F.6.18 holds; that is, $O^2(\dot{H}_0) = \dot{D} \times \dot{L}_0$, where $\dot{L}_0 \cong L_2(q)$, $q \equiv 11$ or 13 mod 24, and $\dot{D} \cong \mathbf{Z}_3$. Let D be a Sylow 3-subgroup of the preimage of $\dot{D}$ which permutes with T. Then D does not centralize Z as $O^2(H) = L_H$ does not. Further $\dot{L}_1 \leq C_{O^2(\dot{H})}(Z) = \dot{L}_0$, so $\dot{L}_1\dot{T} \cong D_{24}$ and $L_1/O_2(L_1)$ is inverted in $C_T(D)$. Thus we may choose D to permute with L_1. Then $[D, O_2(DT)] \leq O_{3'}(H_0) \cap T \leq R_1$, so R_1 is Sylow in R_1D.

We argue as in the proof of 13.2.13: Assume that $D \not\leq M$. Then as $R_1 \in Syl_2(R_1D)$, $B := \mathrm{Baum}(R_1) \in Syl_2(BD)$ by E.2.3.2. But $B = \mathrm{Baum}(Q)$ by 13.2.4.1, so $Q \in Syl_2(QD)$. Then by Theorem 3.1.1 applied with Q, LT, DT in the roles of "R, M_0, H", $O_2(\langle LT, DT\rangle) \neq 1$, so that $D \leq M = !\mathcal{M}(LT)$, contrary to our assumption that $D \not\leq M$. Therefore $D \leq M = N_G(L)$. Now as $L_1/O_2(L_1)$ is inverted in $C_T(D)$, D centralizes $L/O_2(L)$, so L acts on $Y := O^2(DO_2(LT)) = \langle D^T \rangle$, and hence $N_G(Y) \leq M = !\mathcal{M}(L)$ by Theorem 4.3.2. Then $L_0 \leq N_{H_0}(Y) \leq H_0 \cap M$, so $H \leq H_0 = DL_0T \leq M$, for our usual contradiction to $H \not\leq M$.

This contradiction completes the proof of Theorem 13.2.7.

13.3. Starting mid-sized groups over $\mathbf{F}_2$, and eliminating $\mathbf{U_3(3)}$

In this section, with the preliminary results from sections 13.1 and 13.2 in hand, we begin to treat those pairs L, V in the Fundamental Setup (3.2.1) which constitute the main topic of the chapter: the pairs such that $L/O_2(L)$ is an intermediate-sized group A_5, A_6, $\hat{A}_6$, or $U_3(3)$ over $\mathbf{F}_2$. As in the previous chapter, we begin by stating our working hypothesis for this chapter, which excludes the groups in the Main Theorem which have arisen in previous sections. In particular, Hypothesis 13.3.1 extends Hypotheses 12.2.3 and 13.1.1. Each section treats one or more pairs L, V in the FSU; the treatment of a given case assumes the existence of $L \in \mathcal{L}_f(G, T)$ with $L/C_L(V)$ of the given type.

We also recall, as mentioned in the introduction to the chapter, that to avoid repetition of arguments, we treat the case $L/O_2(L) \cong A_5$ simultaneously with the other cases. However in the actual logical sequence, that case is the final one in our treatment of the FSU, so we actually consider it only when all other groups have been eliminated. This necessitates the assumption in part (4) of Hypothesis 13.3.1; the effect of this part of Hypothesis 13.3.1 is that we choose $L \in \mathcal{L}_f(G, T)$ with $L/O_2(L) \cong A_5$ only when we are forced to do so, because no other choice is possible. Thus for the purposes of the proof of the Main Theorem, Hypothesis 13.3.1.4 and the results in this chapter which depend on it, are actually invoked only when we reach that final case.

Thus in section 13.3 and indeed for the remainder of the chapter, we assume the following hypothesis:

Hypothesis 13.3.1. *(1) G is a simple QTKE-group, $T \in Syl_2(G)$, and $L \in \mathcal{L}_f(G, T)$.*

(2) G is not a group of Lie type over $\mathbf{F}_{2^n}$, with $n > 1$.

(3) G is not $L_4(2)$, $L_5(2)$, A_9, M_{22}, M_{23}, M_{24}, He, or J_4.

(4) If $L/O_2(L) \cong A_5$, then $K/O_2(K) \cong A_5$ for each $K \in \mathcal{L}_f(G, T)$.

The next result describes the members K of $\mathcal{L}_f(G, T)$ which can arise under Hypothesis 13.3.1; as in Remark 12.2.4 of the previous chapter, we can usually replace our chosen pair L, V in the FSU by K, V_K for some suitable $V_K \in Irr_+(K, R_2(KT))$.

Lemma 13.3.2. *If $K \in \mathcal{L}_f(G, T)$, then*

(1) $K/O_2(K) \cong A_5$, $L_3(2)$, A_6, $\hat{A}_6$, or $U_3(3)$.

(2) $K \trianglelefteq KT$ and $K \in \mathcal{L}_f^*(G, T)$. Hence $N_G(K) = !\mathcal{M}(KT)$.

(3) There is a T-invariant $V_K \in Irr_+(K, R_2(KT))$ and further each member of $Irr_+(K, R_2(KT), T)$ is T-invariant. The pair K, V_K satisfies the FSU and either V_K is the natural module for $K/C_K(V_K) \cong A_5$, A_6, $L_3(2)$, or $U_3(3)$, or V_K is the 5-dimensional core of a 6-dimensional permutation module for $K/C_K(V_K) \cong A_6$.

(4) Hypotheses 13.1.1, 12.2.1, and 12.2.3 are satisfied with K in the role of "L".

(5) Hypothesis 13.3.1 is satisfied with K in the role of "L" unless $K/O_2(K)$ is A_5 but $L/O_2(L)$ is not A_5.

PROOF. The initial argument is similar to that in 13.1.2: First $K \le I \in \mathcal{L}^*(G,T)$, and by 1.2.9, $I \in \mathcal{L}_f^*(G,T)$. By Theorem 13.1.7, $I/O_2(I)$ is quasisimple, so $K = I$ by 13.1.2.5. Therefore Hypothesis 12.2.3 holds with K in the role of "L" by 13.1.2.1. Hence (4) is established. Furthermore parts (1)–(3) of Hypothesis 13.3.1 are satisfied by K in the role of "L', so (5) also follows as part (4) of Hypothesis 13.3.1.4 is satisfied by K unless $K/O_2(K)$ is A_5, but $L/O_2(L)$ is not.

Part (1) follows from 13.1.2.3. Further 13.1.2 says that K is T-invariant and the first sentence of (3) holds. Then $N_G(K) = !\mathcal{M}(KT)$ by 1.2.7.3, completing the proof of (2).

It remains to show V_K is one of the modules described in (3). Theorem 12.2.2.3 supplies an initial list of possibilities for V_K, and by Remark 12.2.4 the list of 12.2.2.3 can be refined using results from the previous chapter. If $C_{V_K}(K) \neq 1$, then Theorem 12.4.2 rules out the indecomposables in cases (b) and (f) of 12.2.2.3, leaving only case (d) with V_K the core of a 6-dimensional permutation module for $K/C_K(V_K) \cong A_6$. Otherwise $C_{V_K}(K) = 1$, so either V_K is one of the natural modules listed in 13.3.2.3, or V_K is the 6-dimensional faithful module for $\hat{A}_6$. The last case is out by Theorem 12.7.1 and the exclusions in Hypothesis 13.3.1.3. □

Of course we may apply 13.3.2 to L in the role of "K", so $V \in Irr_+(L, R_2(LT))$ is T-invariant and V is one of the modules listed in 13.3.2.3. By 13.3.2, L satisfies Hypothesis 12.2.3, so we may appeal to the results from the previous chapter, and when $L/C_L(V) \cong A_5$ or A_6 we may appeal to results from section 13.2 of this chapter. We adopt the conventions in Notation 12.2.5 from the previous chapter.

We will refer to a module V which is the core of a 6-dimensional permutation module for $L/C_L(V) \cong A_6$ as a *5-dimensional module for* A_6. In addition we adopt:

NOTATION 13.3.3. If $\bar{L} \cong L_3(2)$, A_5, or A_6, define the T-invariant subspaces V_i of V for $1 \le i \le \dim(V/C_V(L))$ as in Notations 12.8.2 and 13.2.1. When $\bar{L}$ is $U_3(3)$, V is the 6-dimensional module for $\bar{L}$ regarded as $G_2(2)'$, which is the quotient of the Weyl module discussed in [**Asc87**]; see also B.4.6. In particular, V admits a symplectic form preserved by $\bar{M}_V$, so we can speak of nondegenerate and totally isotropic subspaces of V. In this case, define V_i to be the unique T-invariant subspace of V of dimension i. Notice that if $C_V(L) = 1$, then $m(V_i) = i$ in each case.

In each case define $G_i := N_G(V_i)$, $M_i := N_M(V_i)$, and $L_i := O^2(N_L(V_i))$. When $L/O_2(L)$ is not $\hat{A}_6$, define $R_i := O_2(L_iT)$. When $L/O_2(L) \cong \hat{A}_6$, define R_i L_0, and $L_{i,+}$ as in Notation 13.2.1.

LEMMA 13.3.4. *(1) $V_1 = Z \cap V$.*
(2) $V = \langle (Z \cap V)^L \rangle$.

(3) The proper overgroups of $\bar{T}$ in $\bar{L}\bar{T} = Aut_G(\bar{L})$ are $\bar{L}_1\bar{T}$ and $\bar{L}_2T$—except when $\bar{L} \cong A_5$, when only $\bar{L}_1\bar{T}$ occurs. In particular, all proper overgroups of T in LT are $\{2,3\}$-groups.

(4) Statements analogous to (1)–(3) hold for any $K \in \mathcal{L}_f(G,T)$ and $V_K \in Irr_+(KT, R_2(KT), T)$ in the roles of "L, V".

PROOF. Part (1) follows from an inspection of the modules listed in 13.3.2.3. Then (2) follows since $V \in Irr_+(LT, R_2(LT))$. Part (3) follows from the well-known fact that the overgroups of T in an untwisted group of Lie type over $\mathbf{F}_2$ are parabolics, and as $Out(\bar{L})$ is a 2-group. Finally (4) follows since 13.3.2.3 also applies to each K and V_K. □

As usual in the FSU, by 3.3.2.4, we may apply the results of section B.6 to members $H \in \mathcal{H}_*(T,M)$. Recall that for $v \in V^\#$, $G_v = C_G(v)$ in Notation 12.2.5.3.

LEMMA 13.3.5. *(1) If $\bar{L} \cong L_3(2)$ or $U_3(3)$ then $G_v \not\leq M$ for each $v \in V^\#$.*

(2) If $\bar{L} \cong A_5$ then $\mathcal{H}_*(T,M) \subseteq C_G(Z)$, so $G_z \not\leq M$ for z generating $Z \cap V = V_1$.

(3) If $\bar{L} \cong A_6$, then $G_v \not\leq M$ for some $v \in V_1 - C_V(L)$.

PROOF. As Hypothesis 13.3.1 excludes the groups in conclusions (2)–(4) of Theorem 12.2.13, conclusion (1) of that result holds: namely $G_v \not\leq M$ for some $v \in V^\#$. Next V is described in 13.3.2.3. In particular $C_V(L) = 1$ unless V is a 5-dimensional module for $\bar{L} \cong A_6$, and L is transitive on $(V/C_V(L))^\#$ unless $\bar{L} \cong A_5$. Therefore (1) holds, and if $\bar{L}$ is A_6, then $G_v \not\leq M$ for some $v \in V_1$. Further if $C_V(L) \neq 1$, then $C_V(L) \leq Z(LT)$, so as $M = !\mathcal{M}(LT)$, (3) holds. Finally when $\bar{L} \cong A_5$, $\mathcal{H}_*(T,M) \subseteq C_G(Z)$ by 13.2.7. Then as $Z \cap V = V_1$ by 13.3.4.1, $G_z \not\leq M$ for z generating $Z \cap V$, so (2) holds. □

By 13.3.5:

LEMMA 13.3.6. *Either $G_1 \not\leq M$, or $C_V(L) \neq 1$ so that V is a 5-dimensional module for $\bar{L} \cong A_6$.*

As usual we let $\theta(X)$ denote the subgroup generated by all elements of order 3 in a group X.

LEMMA 13.3.7. *Assume $\bar{L} \cong A_6$. Then either*

(1) $C_G(V)$ is a $3'$-group, or

(2) $L/O_2(L) \cong \hat{A}_6$, $m_3(C_G(V)) = 1$, and $L_0 = \theta(C_G(V))$.

PROOF. Let $D := \theta(C_G(V))$ and $P \in Syl_3(C_G(V))$. Recall that we may apply 12.2.8; then $\theta(M) = L$ so that $D \leq L$, and hence either $D = 1$, or $L/O_2(L) \cong \hat{A}_6$ with $D = \theta(C_L(V)) = L_0$. In the first case, conclusion (1) holds. In the second, as $\Omega_1(P) \leq D = L_0$, $\Omega_1(P)$ is of order 3, so conclusion (2) holds. Thus the lemma is established. □

LEMMA 13.3.8. *Assume $K \in \mathcal{L}_f(G,T)$, let $M_K := N_G(K)$, and assume $H \in \mathcal{H}(T, M_K)$ and $Y = O^2(Y) \trianglelefteq H$ with $Y \leq M_K$. Then*

(1) $K \not\leq YC_{M_K}(K/O_2(K))$.

(2) Y is a $\{2,3\}$-group.

PROOF. As $K \in \mathcal{L}_f(G,T)$, $M_K = !\mathcal{M}(KT)$ by 13.3.2.2,; then as $H \not\leq M_K$,

$$O_2(\langle K, H\rangle) = 1. \qquad (*)$$

Let $M_K^* := M_K/C_{M_K}(K/O_2(K))$. As $K/O_2(K)$ is quasisimple and $T \leq M_K$, $K = [K, T\cap K]$. Suppose (1) fails, so that $K^* \leq Y^*$. Then $K^* = [K^*, (T\cap K)^*] = [Y^*, (T\cap K)^*] = [Y, T\cap K]^*$, and as Y is T-invariant, $[Y, T\cap K] \leq Y$. Thus $K = (K\cap Y)O_2(K)$, so as T acts on Y, $K \leq Y \leq H$, contrary to (*) as $O_2(H) \neq 1$. Thus (1) holds.

Let $Y_0 := O^{\{2,3\}}(Y)$; then $Y_0^* < M_K^*$ by (1). But by 13.3.4, the proper overgroups of T^* in $M_K^* = Aut_G(K^*)$ are $\{2,3\}$-groups, so we conclude that $Y_0^* = 1$. Then $Y_0 \leq C_G(K/O_2(K))$, so K normalizes $O^{\{2,3\}}(Y_0O_2(K)) = Y_0$. However if $Y_0 \neq 1$, then $O_2(Y_0) \neq 1$ by 1.1.3.1, contrary to (*). Thus (2) holds. □

LEMMA 13.3.9. *Assume $\bar{L} \cong A_6$, $H \in \mathcal{H}(T,M)$, and $Y = O^2(Y) \trianglelefteq H$ with $Y \leq C_M(v)$ for some $v \in V_1 - C_V(L)$. Then either*

(1) $Y = 1$, or

(2) $\bar{Y} = \bar{L}_1$. Further if $L/O_2(L) \cong A_6$ then $Y = L_1$, while if $L/O_2(L) \cong \hat{A}_6$ then $Y = L_{1,+}$.

PROOF. As in the proof of the previous lemma, with L in the role of "K",

$$O_2(\langle L, H\rangle) = 1. \qquad (*)$$

By hypothesis $Y = O^2(Y)$, and as Y centralizes v, $Y \leq M_V$ by 12.2.6. Therefore $\bar{Y} \leq O^2(\bar{M}_V) = \bar{L}$ by 12.2.10.2; and by 13.3.8.1, $\bar{Y} < \bar{L}$. By 13.3.8.2, Y is a $\{2,3\}$-group. By 1.1.3.1, $O_2(Y) \neq 1$.

If $\bar{Y} = 1$, then L normalizes $O^2(YO_2(L)) = Y$, and hence (1) holds by (*). Thus we may assume that $\bar{Y} \neq 1$, so that $\bar{Y} = \bar{L}_i$ for $i = 1$ or 2 by 13.3.4.3. Then as Y centralizes v, $i = 1$. Further $Y_1 := \theta(Y) \leq \theta(M) = L$ by 12.2.8, so $Y_1 \leq L_1$.

Suppose first that $C_{Y_1}(V) \not\leq O_2(Y_1)$. Then by 13.3.7, $L/O_2(L) \cong \hat{A}_6$ and $L_0 \leq Y_1$. Now $L_0 \leq Y_1 \leq L_1$, so $Y_1 = L_0$ or L_1, and in either case $H \leq N_G(Y_1) \leq M$ by 13.2.2.9, contrary to (*). Therefore $C_{Y_1}(V) \leq O_2(Y_1)$. So as Y is a $\{2,3\}$-group, $C_Y(V) \leq O_2(Y)$, and hence $\bar{Y} = \bar{Y}_1$ is of order 3. Therefore $Y = Y_1$ and $|Y : O_2(Y)| = 3$, so (2) holds. □

LEMMA 13.3.10. *(1) If $\bar{L} \cong A_5$ then $J(R_1) = J(O_2(LT))$, $B := Baum(R_1) = Baum(O_2(LT))$, and $C(G,B) \leq M$.*

(2) If $\bar{L} \cong A_6$ or $U_3(3)$ then either there is a nontrivial characteristic subgroup of $B := Baum(R_1)$ normal in LT (so that $N_G(B) \leq M$), or L is an A_6-block or a $G_2(2)$-block. Moreover if L is a $G_2(2)$-block, then $N_G(B) \leq M$.

(3) If $\bar{L} \cong A_6$ then either some nontrivial characteristic subgroup of $B := Baum((T\cap L)O_2(LT))$ is normal in LT (so that $N_G(B) \leq M$), or L is an A_6-block.

(4) If $\bar{L} \cong L_3(2)$, then either some nontrivial characteristic subgroup of $B := Baum(R_1)$ is normal in LT (so that $N_G(B) \leq M$), or L is an $L_3(2)$-block.

PROOF. Part (1) follows from 13.2.4.1, and part (3) follows from case (b) of C.1.24; L is not a $\hat{A}_6$-block since $V/C_V(L)$ is the A_6-module by 13.3.2.3. Similarly the first sentence in (2) follows from 13.2.2 when $\bar{L} \cong A_6$, and from C.1.37 when $\bar{L} \cong U_3(3)$. When $\bar{L} \cong L_3(2)$, C.1.37 also establishes (4).

Thus it only remains to establish the final sentence of (2), so we assume that L is a $G_2(2)$-block, but that $N_G(B) \not\leq M$, and it remains to derive a contradiction.

We check that Hypothesis C.6.2 is satisfied, with L, B, T, LT, $N_G(B)$ in the roles of "L, R, T_H, H, Λ": For example C.6.2.3 holds, since $M = !\mathcal{M}(LT)$. The only part of Hypothesis C.6.2 which is not evident is that $Q := O_2(LB) \leq B$, and this was established during the proof of C.1.37 using Baumann's Argument B.2.18. Thus we may apply C.6.3.1 to conclude that there exists $x \in N_G(B)$ with $V^x \not\leq Q$. Now reversing the roles of V and V^x if necessary, we may assume that $m(\bar{V}^x) \geq m(V/C_V(V^x))$. Further since $\bar{T}$ contains no strong FF*-offenders on V by B.4.6.13, B.2.4.3 says $V \leq B$, so that $V^x \leq B \leq R_1$. Also by B.1.4.6, $\bar{V}^x \in \mathcal{P}(\bar{R}_1, V)$; then by parts (13) and (3) of B.4.6, we have the hypotheses of B.2.20, so $\bar{V}^x = \overline{J(R_1)} = \bar{B}$ is the unique member $\bar{B}$ of $\mathcal{P}(\bar{R}_1, V)$, and $m(V/C_V(V^x)) = m(\bar{B})$.

Next since $C_V(L) = 1$ by 13.3.2.3,

$$m(V/C_V(V^x)) = 3 = m(\bar{B}) = m(B/C_B(V)) = m(B/C_B(V^x)) = m(V)/2.$$

Therefore as $\bar{L} \leq \langle \bar{B}, \bar{B}^l \rangle$ for suitable $l \in L$, $L \leq \langle V^x, V^{xl}, V \rangle$, and

$$m(Q/(C_Q(V^x) \cap C_Q(V^{xl})) \leq 2m(B/C_B(V^x)) = m(V).$$

Hence $Q = V \times C_B(\langle V^x, V^{xl} \rangle) = V \times C_B(L)$, and in particular V^x centralizes $C_B(L)$. Also $E_8 \cong [V, V^x] \leq V \cap V^x$, so as $m(\bar{V}^x) = 3$, $C_{V^x}(V) = V \cap V^x$. Then $|V^x C_B(L)| = |V C_B(L)| = |C_B(V)|$ and hence $C_B(V^x) = V^x \times C_B(L)$. Therefore as $x \in N_G(B)$,

$$\Phi(C_B(L)) = \Phi(C_B(V^x)) = \Phi(C_B(V))^x = \Phi(C_B(L))^x.$$

Thus if $\Phi(C_B(L) \neq 1$, then $x \in N_G(\Phi(C_B(L)) \leq M = !\mathcal{M}(LT)$; but then as $M = N_G(L)$, $V^x \leq O_2(L) \leq O_2(LB) = Q$, contrary to the choice of x. Therefore $\Phi(C_B(L)) = 1$, so that $C_B(V) = Q$ is elementary abelian; and then $\mathcal{A}(B) = \{Q, Q^x\}$ is of order 2 by B.2.21 using B.4.6.6. Hence $O^2(N_G(B)) \leq N_G(Q) \leq M$, and then $N_G(B) = O^2(N_G(B))T \leq M$, contrary to our assumption. This contradiction completes the proof of (2), and hence of 13.3.10. □

LEMMA 13.3.11. *Assume $\bar{L} \cong A_5$. Then*

(1) For each $v \in V^\#$, $\{U \in V^G : v \in U\} = V^{G_v}$.
(2) $V_2^L = V_2^G \cap V$ and $V_3^L = V_3^G \cap V$.
(3) V is the unique member of V^G containing V_3.
(4) $V^{G_2} = \{U \in V^G : V_2 \leq U\}$.
(5) If $g \in G$ with $[V_3, V_3^g] = 1$, then $[V, V^g] = 1$.

PROOF. Part (1) follows from 13.2.6.2 and A.1.7.1. As V_k^L, $k = 2, 3$, are the unique classes of subgroups of V of rank k containing a unique singular point, 13.2.6.2 also implies (2). Then (2) and A.1.7.1 imply (4), as well as the analogous statement for V_3 and G_3. Thus as $G_3 \leq M_V$ by 13.2.3.2, (3) holds. If $[V_3, V_3^g] = 1$, then by (3), V_3^g acts on V. Therefore as $C_{\bar{M}_V}(V_3) = 1$, $V \leq C_G(V_3^g) \leq N_G(V^g)$ again using (3), so that $V \leq C_G(V^g)$ again using $C_{\bar{M}_V}(V_3) = 1$. Thus (5) holds. □

LEMMA 13.3.12. *Assume $\bar{L} \cong U_3(3)$. Then*

(1) $s(G, V) > 1$.
(2) If $U \leq V$ with $C_G(U) \not\leq M$, then U is totally isotropic. Hence $r(G, V) \geq 3$.
(3) If $r(G, V) = 3$, then $C_G(V_3) \not\leq M$.
(4) If $g \in G$ with $1 \neq [V, V^g] \leq V \cap V^g$, then $V \cap V^g = [V, V^g] = C_V(V^g) \in V_3^G$, and we may take $g \in C_G(V \cap V^g)$, so that $C_G(V_3) \not\leq M$.

PROOF. Recall V is a TI-set in M by 12.2.6, so Hypothesis E.6.1 is satisfied, and for $1 \neq U \leq V$, $C_M(U) \leq M_V$. By parts (4)–(6) of B.4.6, $m(\bar{M}_V, V) > 1$, so $C_M(W) = C_M(V)$ for each hyperplane W of V. Further the hyperplanes of V are of the form $v^\perp$ for $v \in V^\#$, so as L is transitive on $V^\#$, L is transitive on hyperplanes. Hence each hyperplane is invariant under a Sylow 2-subgroup of LT, so that $r(G,V) > 1$ by E.6.13. Hence (1) is established.

Next we establish some preliminary results, phrased in terms of the usual geometry of points and lines on V: From section 5 in [**Asc87**], we can identify the points and lines of the generalized hexagon of $\bar{G}_0 := N_{GL(V)}(\bar{L}) \cong G_2(2)$ with the points and *doubly singular* lines of V (i.e., totally isotropic as well as singular in the Dickson trilinear form; see p. 194 of [**Asc87**]). By 5.1 in [**Asc87**], $\bar{G}_0$ is transitive on nondegenerate lines of V, and each such line l is generated by a pair u, v of points opposite (i.e., at maximal distance) in the hexagon. Now $N_{\bar{G}_0}(l) = \bar{H}_1 \times \bar{H}_2$, where $\bar{H}_1 := C_{\bar{G}_0}(l) = C_{\bar{G}_0}(u) \cap C_{\bar{G}_0}(v) \cong S_3$ by F.4.5.5, and $\bar{H}_2 := C_{\bar{G}_0}(\bar{H}_1) \cong S_3$ acts faithfully on l. Further $\bar{H}_1\bar{H}_2$ acts faithfully on the 4-space $l^\perp$, with $l^\perp = [l^\perp, O^2(H_i)]$. Now $N_{\bar{M}_V}(l)$ is of index 1 or 2 in $N_{\bar{G}_0}(l)$ in the cases $\bar{M}_V = \bar{G}_0$ or $\bar{L}$, respectively. In particular $Q := O_2(LT)$ is of index at most 2 in $T_H := C_T(l)$, so $J(T_H) = J(Q)$ in view of B.4.6.13. Further $Q = O_2(K_1T_H)$, where $K_1 := O^2(H_1)$, and $K_2 := O^2(H_2)$ induces $\mathbf{Z}_3$ on l. Set $H := C_G(l)$, so that $T_H \in Syl_2(H \cap M)$. As $N_G(Q) \leq M = !\mathcal{M}(LT)$, $C(H,Q) \leq H \cap M =: M_H$. In particular as $J(T_H) = J(Q)$, $N_T(T_H) \leq M_H$, so that $T_H \in Syl_2(H)$. It also follows as $K_1 \leq M_H$ that $Q = O_2(M_H) = O_2(N_H(Q))$. Thus Hypothesis C.2.3 is satisfied with Q in the role of "R".

We are now ready to establish our main preliminary result: we claim that $H = C_G(l) \leq M$. So we assume that $H \not\leq M$, and derive a contradiction. Observe first that as l contains 2-central involutions, $H \in \mathcal{H}^e$ by 1.1.4.3. Next Q is Sylow in $O_{2,F}(H)Q$ by C.2.6.1, and as $M = !\mathcal{M}(LT)$,

$$N_H(W_0(Q,V)) \leq M_H \geq C_H(C_1(Q,V)).$$

Hence as $n(O_{2,F}(H)) = 1$ by E.1.13, $O_{2,F}(H) \leq M$ by (1) and E.3.19. On the other hand, if $O_{2,F^*}(H) \leq M_H$, then $O_2(H) = Q$ by A.4.4.1; thus $H \leq N_G(Q) \leq M$, contrary to our assumption.

This contradiction shows that there is $K \in \mathcal{C}(H)$ with $K/O_2(K)$ quasisimple, and $K \not\leq M$. By 1.1.3.1, $K \in \mathcal{H}^e$. Suppose first that $Q \not\leq N_H(K)$. Then C.2.4.2 shows that $Q \cap K \in Syl_2(K)$, and as $K \not\leq M$, C.2.4.1 then shows that K is a χ_0-block. In particular $m_3(K) = 1$ and hence $m_3(\langle K^Q \rangle) = 2$. But then $m_3(K_2\langle K^Q \rangle) \geq 3$, contrary to $N_G(l)$ an SQTK-group.

Therefore $Q \leq N_G(K)$, so that K is described in C.2.7.3. Notice if case (g) of C.2.7.3 occurs with n even, then we are in one of cases (1)–(4) of C.1.34, in which $Z(O_2(K))$ is the sum of at most two natural modules for $K/O_2(K) \cong SL_3(2^n)$; this case is ruled out by A.3.19 as $K_2 \not\leq K$. The remaining cases of C.2.7.3 where $m_3(K) = 2$ are eliminated by A.3.18 as $K_2 \not\leq K$. Thus $m_3(K) = 1$. Also K is not an A_5-block as M_H contains the Sylow 2-group T_H of H. Thus inspecting C.2.7.3, one of the following holds:

(i) K is an $L_2(2^m)$-block, Q is Sylow in KQ, and $M_K := M_H \cap K$ is a Borel subgroup of K.

(ii) $K/O_2(K) \cong L_3(2^n)$, n odd, M_K is a maximal parabolic of K, and K is described in C.1.34.

Furthermore $O^{3'}(H) = K$, again using the fact that $m_3(K_2O^{3'}(H)) \leq m_3(N_G(l)) = 2$. Thus $K_1 \leq K$, so as $K_1 \trianglelefteq M_H$, we conclude $n = 1$ in case (ii). Next $K_1 = [K_1, t]$ for some $t \in N_{T\cap L}(l)$. Thus if K is an $L_2(2^m)$-block, t induces a field automorphism on $K/O_2(K)$ and $[M_K, t] \leq C_L(l)$, so $[M_K, t]$ is a $\{2, 3\}$-group; we conclude in case (i) that K is an $L_2(4)$-block.

Next

$$E_{16} \cong l^\perp = [l^\perp, K_1] \leq Z(Q) = Z(O_2(K_1T_H)). \qquad (*)$$

Assume $K/O_2(K) \cong L_3(2)$. Then by (*), case (2) of C.1.34 occurs, with $O_2(K)$ the direct sum of two isomorphic natural modules for $K/O_2(K)$. Hence K_1 has exactly three noncentral 2-chief factors, two of which are in $l^\perp \leq V$. Thus as $O_2(\bar{K}_1) = 1$, K_1 has one noncentral chief factor on Q/V, impossible as K_1 has more than one noncentral chief factor on each nontrivial irreducible on Q/V under $\bar{L} \cong U_3(3)$.

Therefore K is an $L_2(4)$-block, and Q is Sylow in QK. Now $Z(Q) \leq C_{KQ}(O_2(KQ)) = Z(O_2(KQ))$ as $KQ \in \mathcal{H}^e$. Hence

$$E_{16} \cong l^\perp = [l^\perp, K_1] \leq [Z(Q), K_1] \leq Z(Q) \cap U(K).$$

However this is impossible as K_1 has at most one noncentral chief factor on $Z(Q) \cap U(K)$, since $Q \in Syl_2(KQ)$ and K is an $L_2(4)$-block. This contradiction finally establishes the claim that $H = C_G(l) \leq M$.

Now (2) follows directly from the claim. Further if $r(G, V) = 3$, then there is a totally isotropic 3-subspace U of V with $C_G(U) \not\leq M$. By 7.3.3 in [**Asc87**], L has two orbits on such subspaces, represented by V_3 and Y where $N_{\bar{M}_V}(Y) \cong L_3(2)$ is faithful on Y. Thus $C_M(Y) = C_M(V)$, so as $r(G, V) > 1$ by (1), we conclude $C_G(Y) \leq M$ by E.6.12. So $U \in V_3^L$, and (3) follows.

Assume the hypotheses of (4). Then interchanging V and V^g if necessary, we may assume $m(\bar{V}^g) \geq m(V/C_V(V^g))$. We apply B.4.6.13 much as in the proof of 13.3.10: First $\bar{T}$ contains no strong FF*-offenders, so that $\bar{V}^g \in \mathcal{P}(\bar{T}, V)$ by B.1.4.6; then there is a unique conjugacy class of FF*-offenders in $\bar{L}\bar{T}$ represented by the subgroup "A_1" of that lemma, so we may assume that $\bar{V}^g = A_1$ and hence $C_V(V^g) = [V, V^g] \in V_3^G$. Thus we may take $V_3 = [V, V^g]$. By hypothesis, $[V, V^g] \leq V \cap V^g$, and $V \cap V^g \leq C_V(V^g) = V_3$, so $V_3 = V \cap V^g$. Also $m(V/(V \cap V^g)) = 3 = m(V^g/(V \cap V^g))$, so we have symmetry between V and V^g. We conclude $V_3 \in V_3^{gL^g}$, and hence we may take $g \in G_3$. Let U_5 be the preimage in V^g of $\bar{V}^g \cap \bar{L}$. By (3) and (4) of B.4.6,

$$U_5 = \{u \in V^g : C_V(u) > V_3\}$$

and similarly

$$V_5 = V_1^\perp = \{v \in V : C_{V^g}(v) > V_3\},$$

so $U_5 \in V_5^{gN_{L^g}(V_3)}$, and hence we may take $V_5^g = U_5$. Then as $V_1 = [U_5, V_5]$, $V_1^g = V_1$, so $g \in G_1 \cap G_3$. But $Aut_{LT}(V_3)$ is the stabilizer in $GL(V_3)$ of V_1, so $G_1 \cap G_3 = N_{LT}(V_3)C_G(V_3)$. Then as LT normalizes V, we may take $g \in C_G(V_3)$, and hence $C_G(V_3) \not\leq M$ as $C_M(V_3) \leq M_V$ by 12.2.6. This completes the proof of (4). □

During the remainder of this subsection, we will assume the following hypothesis:

Hypothesis 13.3.13. *$C_V(L) = 1$, so that V is not a 5-dimensional module when $\hat{L} \cong A_6$. Set $Q_1 := O_2(G_1)$.*

We recall from Notation 13.3.3 that since $C_V(L) = 1$ by Hypothesis 13.3.13, we have $m(V_i) = i$. In particular by 13.3.4.1, $V_1 = Z \cap V$ is of order 2, and from 13.3.2,3, $V_3 = [V_3, L_1] = \langle V_2^{L_1} \rangle$. Also $G_1 = N_G(V_1) = C_G(V_1)$ and $G_1 \in \mathcal{H}^e$ as G is of even characteristic.

LEMMA 13.3.14. *Assume Hypothesis 13.3.13. Then Q_1 does not centralize V_2.*

PROOF. We assume that $[V_2, Q_1] = 1$ and derive a contradiction. The bulk of the proof proceeds by a series of reductions labeled (a)–(g).

Set $U := \langle V_3^{G_1} \rangle$ and $G_1^* := G_1/Q_1$. Set $L_+ := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$, and $L_+ := L_1$ otherwise. Then $O_2(L_+T) = R_1$ is of index 2 in T, and as $V_3 = [V_3, L_1] = \langle V_2^{L_1} \rangle$, $V_3 = [V_3, L_+] = \langle V_2^{L_+} \rangle$.

Observe that Hypothesis G.2.1 is satisfied with V_3, G_1, L_+ in the roles of "V, H, L". Therefore by G.2.2.1, $U \leq Q_1$. As $V_3 = \langle V_2^{L_+} \rangle$, $U = \langle V_2^{G_1} \rangle$. Then as $[V_2, Q_1] = 1$:

(a) $U \leq \Omega_1(Z(Q_1))$.

Observe that as $C_V(L) = 1$ by Hypothesis 13.3.13, $G_1 \not\leq M$ by 13.3.6. Set $Y := O^2(C_G(U))$. Then $Y \leq C_G(V_3) \leq C_G(V_2) \leq G_1$. Also $Y \trianglelefteq G_1$, and $Y \cap M \leq M_V$ by 12.2.6.

(b) Y is solvable with $m_p(Y) \leq 1$ for each odd prime p.

For if $Y = 1$ then certainly (b) holds, so we may assume $Y \neq 1$. Consider any T-invariant subgroup $Y_0 = O^2(Y_0)$ of $Y \cap M$. As $C_{\bar{M}_V}(V_3)$ is a 2-group, Y_0 centralizes V. Then $[L, Y_0] \leq C_L(V) = O_{2,Z}(L)$, so $L = L^\infty$ centralizes $Y_0O_2(L)/O_2(L)$. Hence as Y_0 is T-invariant, L acts on $O^2(Y_0O_2(L)) = Y_0$. Therefore if $Y_0 \neq 1$, then $N_G(Y_0) \leq M = !\mathcal{M}(LT)$. In particular $Y \not\leq M$, as otherwise $G_1 \leq N_G(Y) \leq M$.

Now if $\bar{L} \cong L_3(2)$, then $V = V_3 \leq U$, so $Y \leq C_G(U) \leq C_G(V_3) = C_G(V) \leq M$, contrary to the previous paragraph. Similarly if $\bar{L} \cong A_5$, then $C_G(V_3) \leq M$ by 13.2.3.2, for the same contradiction. Therefore we may assume that $\bar{L}$ is A_6 or $U_3(3)$.

Let $g \in L_2 - G_1$ be of order 3. Then $m(V_3V_3^g) = 4$, so $V_3V_3^g = V$ if $\bar{L} \cong A_6$, and $V_3V_3^g$ is not totally isotropic if $\bar{L}$ is $U_3(3)$. In the latter case $C_G(V_3V_3^g) \leq M$ by 13.3.12.2, while if $\bar{L}$ is A_6, then $C_G(V_3V_3^g) = C_G(V) \leq M$. So in either case, $C_G(V_3V_3^g) \leq M$.

Set $Y_1 := O^2(Y \cap Y^g)$ and $Y_M := O^2(Y \cap M)$. Then $Y_1 \leq C_G(V_3V_3^g) \leq M$, so $Y_1 \leq Y_M$. Further Y centralizes V_2, so $Y \leq G_1^g \leq N_G(Y^g)$, and hence $Y_1 \trianglelefteq Y$; then by symmetry, $Y_1 \trianglelefteq Y^g$. Next $T \leq M_1 \leq N_G(Y_M)$, so using Y_M in the role of "Y_0" above, L acts on Y_M and $N_G(Y_M) \leq M$; hence $Y_M = Y_M^g \leq Y_1$ as $g \in L_2$. We conclude $Y_M = Y_1$, so if $Y_1 \neq 1$, then $Y \leq N_G(Y_1) \leq M$, contrary to the first paragraph. Therefore $Y_1 = 1$, so that $Y \cap Y^g$ is a 2-group.

Set $\hat{G}_2 := G_2/O_2(G_2)$. As $Y \trianglelefteq G_1$ while $C_G(V_2) \leq G_1$, $Y \trianglelefteq C_G(V_2)$, so that $O_2(Y) \leq O_2(G_2)$, and hence $Y_+ := \langle Y, Y^g \rangle \trianglelefteq C_G(V_2)$. Then as Y and Y^g are normal in Y_+, and $Y \cap Y^g$ is a 2-group normal in Y_+, $\hat{Y}_+ = \hat{Y} \times \hat{Y}^g$.

Therefore since G_2 is an SQTK-group, $m_p(Y) = 1$ for each odd prime p. Further if Y is not solvable, then by 1.2.1.1, there is $K \in \mathcal{C}(Y)$, and as $Y \trianglelefteq G_2$, $K \in \mathcal{C}(G_2)$. Then as g is of order 3, g acts on K by 1.2.1.3, contradicting $Y \cap Y^g$ a 2-group. This contradiction completes the proof of (b).

(c) $O_2(L_+^*) \neq 1$.

If $O_2(L_+^*) = 1$, then $O_2(L_+) \leq Q_1 \leq C_G(V_3)$ by (a), impossible as L_+ induces A_4 on V_3/V_1.

(d) $O_2(L_+^*)$ centralizes $F(G_1^*)$.

Assume $O_2(L_+^*)$ is nontrivial on $O_p(G_1^*)$ for some odd prime p. Then as $L_+/O_2(L_+)$ has order 3, $Aut_{O_2(L_+)}(O_p(G_1^*))$ is noncyclic, so by A.1.21 and A.1.25, there is a noncyclic supercritical subgroup P^* of $O_p(G_1^*)$ such that $P^* \cong E_{p^2}$ or p^{1+2} and $Aut(P^*)/O_p(Aut(P^*))$ is a subgroup of $GL_2(p)$. Hence $Aut_{L_+}(P^*) \cong SL_2(3)$. Let $P := O^2(P_+)$, where P_+ is the preimage of P^* in G_1. Then $PL_+T =: H \in \mathcal{H}(T) \cap G_1$. Further as L_+ is normal in M_1 but L_+ is not P-invariant, $P \not\leq M$.

As $Aut_{L_+}(P^*) \cong SL_2(3)$, $P \in \Xi(G,T)$ with $Aut_{T\cap L_+}(P^*) \cong Q_8$. Also $[U,P] \neq 1$ as $m_p(Y) = 1$ by (b). Since $U \leq \Omega_1(Z(Q_1))$ by (a), $P \in \Xi_f(G,T)$ by an application of A.4.9 to P, G_1 in the roles of "X, M". Assume $P \leq \langle K^T \rangle$ for some $K \in \mathcal{C}(G,T)$ with $K/O_2(K)$ quasisimple. Then $\langle K^T \rangle$ is described in 1.3.4. Further $K \in \mathcal{L}_f(G,T)$ by 1.3.9.2, so $K = \langle K^T \rangle$ by 13.3.2.2, and $K/O_2(K)$ is described in 13.3.2.1. As the lists in 1.3.4 and 13.3.2.1 do not intersect, there is no such K, so $P \in \Xi_f^*(G,T)$. Then by 3.2.13, $P \in \Xi_-(G,T)$. Since $Aut_G(P/O_2(P))$ involves $SL_2(3)$ which is not a $\{2,5\}$-group, we conclude from Definition 3.2.12 that P is a $\{2,3\}$-group, so that $p = 3$. As $m_3(PL_+) \leq 2$ with $Aut_{L_+}(P^*) \cong SL_2(3)$, we conclude $P/O_2(P) \cong P^* \cong E_9$ rather than 3^{1+2}. Let $W := R_2(PT)$; as $Aut_T(P^*) \cong Q_8$, we conclude from D.2.17 that $\hat{q}(Aut_{PT}(W), W) > 2$. However $N_G(P) = !\mathcal{M}(PT)$ by Theorem 1.3.7, so that we may apply Theorem 3.1.8.1 to P, W in the roles of "L_0, V" to obtain $\hat{q}(Aut_{PT}(W), W) \leq 2$, contrary to the previous observation. This contradiction completes the proof of (d).

Since $O_2(G_1^*) = 1$, (c) and (d) say there is $K \in \mathcal{C}(G_1)$ with K^* a component of G_1^* and $[K^*, O_2(L_+^*)] \neq 1$. By 1.2.1.3, $L_+ = O^2(L_+)$ normalizes K. In particular, $K/O_2(K)$ is quasisimple and $K = [K, L_+]$.

(e) $K \in \mathcal{L}_f^*(G,T)$, $G_1 \leq N_G(K) = !\mathcal{M}(KT)$, $L \not\leq N_G(K)$, and $K \not\leq M$.

First $[U,K] \neq 1$ by (b), so using (a) and A.4.9 as in the proof of (d), $K \in \mathcal{L}_f(G,T)$. Then by 13.3.2.2, $K \in \mathcal{L}_f^*(G,T)$, $K \trianglelefteq KT$, and $G_1 \leq N_G(K) = !\mathcal{M}(KT)$. As $G_1 \not\leq M$, $N_G(K) \neq M$. So as $M = !\mathcal{M}(LT)$, $L \not\leq N_G(K)$, and as $N_G(K) = !\mathcal{M}(KT)$, $K \not\leq M$, completing the proof of (e).

If $L_2 \leq N_G(K)$, then $L = \langle L_1, L_2 \rangle \leq N_G(K)$, contrary to (e). So:

(f) $L_2 \not\leq N_G(K)$.

As $L_2 \not\leq N_G(K) \geq C_G(Z)$ by (e) and (f), L_2T contains some $H \in \mathcal{H}_*(T, N_G(K))$, and $H \not\leq C_G(Z)$. By (e), $K \in \mathcal{L}_f^*(G,T)$, and we saw $K/O_2(K)$ is quasisimple, so $O_2(KT) = C_T(R_2(KT))$ by 1.4.1.4b. Then applying 3.1.8.3 to K, $R_2(KT)$ in the roles of "L, V", $K = [K, J(T)]$. Set $J := KL_+T$, $W := R_2(J)$, $J^+ := J/C_J(W)$, and $W_K := [W,K]$. Then $R_2(KT) \leq W$ by A.1.11, so that $Irr_+(K, R_2(KT)) \subseteq Irr_+(K, W_K)$. Now $K/O_2(K)$ is described in 13.3.2.1, and the members of the set $Irr_+(KT, R_2(KT), T)$ are described in 13.3.2.3. Thus applying Theorem B.5.6 to the FF-module W_K for J, we conclude that either $W_K \in Irr_+(KT, R_2(KT), T)$ or $K/O_2(K) \cong L_3(2)$ and W_K is the sum of two isomorphic natural modules.

(g) $L_+ \not\leq K$.

Assume that that $L_+ \leq K$. Define $V(K)$ as in Definition A.4.7, and set $\hat{J} := J/C_J(V(K))$. By (a) and A.4.8.4, $V_3 = [V_3, L_+] \leq [\Omega_1(Z(Q_1)), K] \leq V(K)$. Let X be of order 3 in L_+ and set $Q_J := O_2(J)$. By A.4.8.1, $\hat{Q}_J$ centralizes $\hat{X}$. Thus by the Thompson $A \times B$-Lemma, X is faithful on $C_{V_3}(Q_J)$, so Q_J centralizes V_3. Now by A.4.8.4, $V_3 \leq W_K$, so $1 \neq V_1 \leq C_{W_K}(K)$. However we saw that either $W_K \in Irr_+(KT, R_2(KT), T)$ and so is described in 13.3.2.3, or W_K is the sum of natural modules for $K/O_2(K) \cong L_3(2)$. Thus as $C_{W_K}(K) \neq 1$, W_K is a 5-dimensional module for $K^+ \cong A_6$. Therefore $A_4 \cong L_+^+ \leq \hat{K}^+$ and V_3/V_1 is an L_+-invariant line in W_K/V_1, with $[V_3, O_2(L_+)] = V_1$, whereas in the 5-dimensional module W_K, the preimage of such a line is centralized by $O_2(L_+^+)$. This contradiction establishes (g).

Now as $L_+ \not\leq K$ by (g), but $m_3(KL_+) \leq 2$, A.3.18 eliminates the possibilities for $K/O_2(K)$ of 3-rank 2 in 13.3.2.1. Thus $m_3(K) = 1$, so that $K^+ \cong A_5$ or $L_3(2)$. As $L_+ \not\leq K = [K, L_+]$, and $Out(K^+)$ is a 2-group, L_+ is diagonally embedded in $L_K L_C$, where $L_C := C_{KL_+}(K/O_2(K))$ and $L_K = O^2(L_K)$ is the projection of L_+ on K. But if $K/O_2(K) \cong L_3(2)$, then $L_K = [L_K, T \cap K]$, contrary to the fact that L_+ is T-invariant. Thus $K/O_2(K) \cong A_5$, and from earlier discussion W_K is the A_5-module. Then as $L_K^+ \neq 1$, $L_K T = (T \cap K)O_2(KT)$, so as $R_1 = O_2(L_+ T)$ is of index 2 in T, $R_1 = (T \cap K)O_2(KT)$. Since K satisfies Hypothesis 12.2.3 by 13.3.2.4, we may apply 13.2.4.1 with K in the role of "L to conclude that $C(G, \mathrm{Baum}(R_1)) \leq N_G(K)$. It follows as $K \not\leq M$ by (e) that $N_G(\mathrm{Baum}(R_1)) \not\leq M$. Therefore by 13.3.10, L is an $L_3(2)$-block or an A_6-block, and in either case L_+ has exactly two noncentral 2-chief factors.

As W_K is the A_5-module, $End_K(W_K) = \mathbf{F}_2$, so $[W_K, L_C] = 0$. Thus L_+ has at least one noncentral 2-chief factor on W_K, as well as one on $O_2(L_K^+)$; so as L_+ has just two noncentral 2-chief factors, $[O_2(J), L_+] \leq W_K$. Hence as $K = [K, L_+]$, K is an A_5-block. Then as L_C centralizes K^+ and W_K, $[K, L_C] = 1$ by Coprime Action. By C.1.13.c, $O_2(J) = C_{O_2(J)}(K) \times W_K$, so as $J \in \mathcal{H}^e$, L_C has a noncentral 2-chief factor in $C_{O_2(J)}(K)$, contradicting $[O_2(J), L_+] \leq W_K$. This contradiction finally completes the proof of 13.3.14. □

LEMMA 13.3.15. *Assume Hypothesis 13.3.13. and that* $\bar{L} \not\cong A_5$. *Then*

(1) $I_2 := \langle Q_1^{G_2} \rangle \trianglelefteq G_2$.
(2) $I_2 = XQ_1$, *where* $X := L_{2,+}$ *when* $L/O_2(L) \cong \hat{A}_6$ *and* $X := L_2$ *otherwise.*
(3) $C_{I_2}(V_2) = O_2(I_2)$ *and* $I_2/O_2(I_2) \cong S_3$, *with* $O^2(I_2) = X$.
(4) $C_{Q_1}(V_2) \leq O_2(I_2) \leq O_2(G_2)$.
(5) $m_3(C_G(V_2)) \leq 1$.
(6) $C_G(V_3) \leq M_V$. *Hence* $[V, C_G(V_3)] \leq V_1$.

PROOF. Part (1) holds by construction. As $L \not\cong A_5$, $X/O_2(X)$ is of order 3. Recall one consequence of Hypothesis 13.3.13 is that V_2 is of rank 2. Then as $[Q_1, V_2] \neq 1$ by 13.3.14, XQ_1 induces $GL(V_2)$ on V_2, with X transitive on $V_2^{\#}$. Hence $G_2 = C_G(V_2)Q_1 X$, with $C_G(V_2)Q_1 \leq G_1$, so

$$I_2 = \langle Q_1^{G_2} \rangle = \langle Q_1^X \rangle \leq XQ_1,$$

and $X = [X, Q_1] \leq I_2$, so $I_2 = XQ_1$ and (2)–(4) hold. As $X = O^2(I_2) \trianglelefteq G_2$ by (1) and (3),

$$[C_G(V_2), X] \leq C_X(V_2) = O_2(X);$$

so as $m_3(G_2) \leq 2$ and $X/O_2(X)$ is faithful on V_2, (5) holds.

Next $C_G(V_3) \leq G_2 \leq N_G(X)$, so as $N_L(V_3)$ normalizes $C_G(V_3)$, $C_G(V_3)$ acts on $X^{N_L(V_3)}$. Then as $L = \langle X^{N_L(V_3)} \rangle$, we conclude $C_G(V_3) \leq N_G(L) = M$. Hence $C_G(V_3) \leq N_G(V) = M_V$ as V is a TI-set in M by 12.2.6. This establishes (6). □

13.3.1. Eliminating $\mathbf{U_3(3)}$. With the technical results from the earlier part of the section in hand, we are now ready to embark on the main project in this chapter: the treatment of the cases $\bar{L} \cong U_3(3)$, A_6, and A_5.

In this subsection, we handle the easiest of these cases:

THEOREM 13.3.16. *Assume Hypothesis 13.3.1. Then $\bar{L}$ is not $U_3(3)$.*

In the remainder of this section, assume G, L is a counterexample to Theorem 13.3.16. By 13.3.2.3, $C_V(L) = 1$ and $m(V) = 6$. In particular Hypothesis 13.3.13 is satisfied, so we can appeal to 13.3.14 and 13.3.15. Let z be a generator for V_1, so that $G_1 := C_G(z) = G_z$, and $\tilde{G}_1 = G_1/V_1$. As usual define

$$\mathcal{H}_z = \{H \in \mathcal{H}(L_1T) : H \leq G_1 \text{ and } H \not\leq M\}.$$

By 13.3.6, $G_1 \not\leq M$, so $G_1 \in \mathcal{H}_z$ and hence $\mathcal{H}_z \neq \emptyset$.

We first observe:

LEMMA 13.3.17. *(1) $G_1 \cap G_3 \leq M_V \geq C_G(V_3)$.*
(2) $r(G, V) > 3$.
(3) If $[V, V^g] \leq V \cap V^g$, then $[V, V^g] = 1$.
(4) $[O_2(G_1), V_2] \neq 1$.

PROOF. Part (4) holds by 13.3.14, and $C_G(V_3) \leq M_V$ by 13.3.15.6. Further $Aut_{M_1}(V_3)$ is the full stabilizer in $GL(V_3)$ of V_1, so $G_1 \cap G_3 = C_G(V_3)N_{M_1}(V_3)$. As V is a TI-set in M by 12.2.6, this completes the proof of (1). Then (1) together with parts (2) and (3) of 13.3.12 imply (2), while (1) and part (4) of 13.3.12 imply (3). □

LEMMA 13.3.18. *(1) For each $H \in \mathcal{H}_z$, Hypothesis F.9.1 is satisfied with V_3 in the roles of "V_+".*
(2) $\langle V^{G_1} \rangle$ is abelian.

PROOF. We check the various parts of Hypothesis F.9.1:

First hypothesis (c) of F.9.1 follows from 13.3.17.1, and by construction L_1 is irreducible on $\tilde{V}_3$, so hypothesis (b) holds. As $H \in \mathcal{H}(T)$, $H \in \mathcal{H}^e$ by 1.1.4.6. Also by Coprime Action and 13.3.17.1, $Y := O^2(C_H(\tilde{V}_3)) \leq C_{M_V}(V_3))$, so as $O^2(C_{\bar{M}_V}(\tilde{V}_3)) = 1$, $Y \leq C_M(V) \leq C_M(L/O_2(L))$ and therefore L normalizes $Y = O^2(YO_2(L))$. Thus if $Y \neq 1$, then $H \leq N_G(Y) \leq M = !\mathcal{M}(LT)$, contrary to the definition of $H \in \mathcal{H}_z$. Thus $C_H(\tilde{V}_3)$ is a 2-group, so hypothesis (a) follows. As $M = !\mathcal{M}(LT)$ and $H \not\leq M$, hypothesis (d) holds. Finally 13.3.17.3 implies hypothesis (e), completing the proof of (1).

Now let $H := G_1$, and as in Hypothesis F.9.1, define $U_H := \langle V_3^H \rangle$, $V_H := \langle V^H \rangle$, $Q_H = O_2(H)$ and $H^* := H/C_H(\tilde{U}_H)$. It remains to prove (2), so we may assume V_H is nonabelian.

Observe that $O_2(\bar{L}_1) \cong \mathbf{Z}_4^2$ and $\bar{R}_1 = O_2(\bar{L}_1)$ in case $\bar{M}_V = \bar{L}$, while in case $\bar{M}_V \cong G_2(2)$, $\bar{R}_1$ is $O_2(\bar{L}_1)$ extended by an involution $\bar{r}$ inverting $O_2(\bar{L}_1)$ and centralizing a supplement to $O_2(\bar{L}_1)$ in $\bar{L}_1$. In particular, $\bar{A}_0 := \Omega_1(O_2(\bar{L}_1))$ is the unique nontrivial normal elementary abelian subgroup of $\bar{M}_1$ in case $\bar{M}_V = \bar{L}$, while

$\bar{A}_1 := \langle \bar{r} \rangle \bar{A}_0$ and $\bar{A}_0$ are the only such subgroups in case $\bar{M}_V \cong G_2(2)$. Further $C_{\bar{M}_V}(V_3)$ is $\bar{A}_0$ or $\bar{A}_1$ in the respective case.

By F.9.2, $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$, $O_2(H^*) = 1$, $\Phi(U_H) \leq V_1$, and $Q_H = C_H(\tilde{U}_H)$. Thus if V centralizes U_H, then $V \leq Q_H \leq \ker_H(N_H(V))$, and then for $h \in H$, $[V, V^h] \leq V \cap V^h$, so that $[V, V^h] = 1$ by 13.3.17.3. But then V_H is abelian, contrary to our assumption. Therefore $[U_H, V] \neq 1$, so that $V^* \neq 1$, and as $\Phi(U_H) \leq V_1$, $1 \neq \bar{U}_H$ is a normal elementary abelian subgroup of $\bar{M}_1$, so by the previous paragraph, $\bar{U}_H = \bar{A}_0$ or $\bar{A}_1$. In particular U_H centralizes V_3, so $V_3 \leq Z(U_H)$ and thus $U_H = \langle V_3^H \rangle$ is elementary abelian.

Let $Z_H := Z(Q_H)$. By 13.3.17.4, $[Q_H, V_2] \neq 1$, so as L_1 is irreducible on $\tilde{V}_3$, $V_3 \cap Z_H = V_1$. Furthermore as $V_2 \leq U_H$, $Q_C := C_{Q_H}(U_H)$ is properly contained in Q_H.

For $x \in Q_C$, define $\varphi(xQ_C) : U_H/Z_H \to V_1$ by $\varphi(xQ_C) : uZ_H \mapsto [x, u]$ for $u \in U_H$. By F.9.7, φ is an H-equivariant isomorphism between Q_H/Q_C and the $\mathbf{F}_2$-dual space of U_H/Z_H.

As $[V, \bar{A}_0] = [V, \bar{A}_1] = V_3$, $[V^*, \tilde{U}_H] = \tilde{V}_3$. Then as $V_3 \cap Z_H = V_1$, V^* is nontrivial on U_H/Z_H and hence also on Q_H/Q_C as φ is an equivariant isomorphism. But $Q_H \leq T \leq N_G(V)$, so $[Q_H/Q_C, V] \leq Q_C(V \cap Q_H)/Q_C$, and hence $V \cap Q_H \not\leq Q_C$.

Next $V_5 = V_1^{\perp}$ is a hyperplane of V, with $[v, R_1]V_3/V_3 = V_5/V_3$ for each $v \in V - V_5$. Thus as L_1 is irreducible on V_5/V_3 and $V \cap Q_H \not\leq Q_C \geq V_3$, we conclude that $V_5 = V \cap Q_H$ and $V_3 = V \cap Q_C = V \cap U_H$. Also by 13.3.15.4,

$$V_5 \leq C_{Q_H}(V_2) \leq O_2(G_2),$$

so $V = \langle V_5^{L_2} \rangle \leq O_2(G_2)$.

Now V^* is of order 2 and $O_2(H^*) = 1$, so by the Baer-Suzuki Theorem we can pick $h \in H$ so that for $I := \langle V, V^h \rangle$, $I^* \cong D_{2m}$ with $m > 1$ odd. Then V^* is conjugate to V^{*h} in I, so we may assume $h \in I$.

Suppose $[V_3, V_5^h] \neq 1$. Then as $C_{\bar{M}_V}(\tilde{V}_5) \leq C_{\bar{M}_V}(V_3)$, $[V_5, V_5^h]$ contains a hyperplane of V_3 containing V_1. As all such hyperplanes are fused under L_1, we may take $V_2 \leq [V_5, V_5^h]$. Now $V_5 = V \cap Q_H$ is normal in Q_H, and hence $V_5^h = V^h \cap Q_H$ is normal in Q_H, so that $V_2 \leq V_5 \cap V_5^h \leq V \cap V^h \leq Z(I)$. Thus $I \leq G_2$, impossible as I is not a 2-group, while $V \leq O_2(G_2)$ and $h \in I$.

This contradiction shows that V_5^h centralizes V_3, and hence by symmetry, $V_5 V_5^h$ centralizes $V_3 V_3^h$. In particular $\bar{V}_5^h \leq \bar{A}_1$. Therefore $[V, V_5^h] \leq V_3$, and by symmetry I centralizes $V_5 V_5^h / V_3 V_3^h$. Hence as $h \in I$, $V_5 V_3^h = V_5(V_3 V_3^h) = V_5^h(V_3 V_3^h) = V_5^h V_3$. Therefore as $V_1 \leq V_3^h$, $\bar{V}_5^h = \bar{V}_3^h$ is of rank $m(\bar{V}_3^h) \leq m(V_3^h/V_1) \leq 2$. Therefore as $r(G, V) > 3$ by 13.3.17.2, $V \leq C_G(C_{V_5^h}(V)) \leq N_G(V^h)$, once again contradicting I not a 2-group. This completes the proof of 13.3.18. □

LEMMA 13.3.19. *(1) If $g \in G$ with $V \cap V^g \neq 1$, then $[V, V^g] = 1$.*
(2) $W_2(T, V)$ centralizes V.
(3) $H \leq M$ for each $H \in \mathcal{H}(T)$ with $n(H) \leq 2$.

PROOF. Suppose $1 \neq V \cap V^g$. As L is transitive on $V^\#$, we may take $z \in V^g$ and $g \in G_1$ by A.1.7.1. But then $[V, V^g] = 1$ by 13.3.18.2. Thus (1) is established.

Let $A := V^g \cap M \leq T$ with $m(V^g/A) =: k \leq 2$, and suppose $\bar{A} \neq 1$. Let $U := N_V(V^g)$. By (1), $V \cap V^g = 1$, so as $[U, A] \leq V \cap V^g$, $U \leq C_V(A)$. In particular $U < V$ as $\bar{A} \neq 1$. On the other hand, if $B \leq A$ with $m(A/B) < 4 - k$, then as $r(G,V) > 3$ by 13.3.17.2, $C_G(B) \le N_G(V^g)$, so that $C_V(B) = U$. Thus $\bar{A} \in \mathcal{A}_{4-k}(\bar{T},V)$, so by B.4.6.9, $k = 2$ and $\bar{A} \in \bar{A}_1^L$. Then without loss, $\bar{A} = \bar{A}_1$, so from the action of $\bar{A}_1$ on V,

$$V_5 = \langle C_V(\bar{a}) : \bar{a} \in \bar{A}_1^\# \rangle \le U,$$

and hence $V_5 = U$ as $U < V$. As $k = 2$, $W_1(T,V)$ centralizes V. Therefore as $m(V/U) = 1$ and $U = N_V(V^g)$, U centralizes V^g. Then since $r(G,V) > 3$, $V^g \le C_G(U) \le N_G(V)$, so that $V^g = A$, contrary to $k = 2$. This proves (2). Finally by (2) and 13.3.17.2, $\min\{w(G,V), r(G,V)\} \ge 3$, so E.3.35.1 implies (3). □

LEMMA 13.3.20. *$n(H) \le 2$ for each $H \in \mathcal{H}_*(T,M) \cap G_1$.*

PROOF. By 13.3.17.1, $G_1 \cap G_3 \le M$, so hypothesis (c) of 12.2.11 is satisfied. Therefore as $H \le G_1$, we may apply 12.2.11 with V_1 in the role of "U" to conclude that $n(H) \le 2$. □

We can now complete the proof of Theorem 13.3.16: Recall $T \le G_1$, and $G_1 \not\le M$ by 13.3.6, so there exists $H \in \mathcal{H}_*(T,M) \cap G_1$. Then $n(H) \le 2$ by 13.3.20, so that $H \le M$ by 13.3.19.3, for our final contradiction.

13.4. The treatment of the 5-dimensional module for $\mathbf{A_6}$

In section 13.4 we prove:

THEOREM 13.4.1. *Assume Hypothesis 13.3.1 with $C_V(L) \ne 1$. Then $G \cong Sp_6(2)$.*

Set $Z_V := C_V(L)$. By hypothesis, $Z_V \ne 1$, so by 13.3.2.3,

$$V \text{ is a 5-dimensional module for } L/C_L(V) \cong A_6.$$

Recall this means that V is the core of the permutation module for A_6 acting on $\Omega := \{1,\dots,6\}$. Accordingly we adopt the notational conventions of section B.3. We also adopt the conventions of Notations 12.2.5 and 13.2.1.

Of course the parabolic of the target group $Sp_6(2)$ stabilizing a point in the natural module has this structure. Eventually we identify G with $Sp_6(2)$ during the proof of Proposition 13.4.9. We begin that process by setting up some notation to discuss $Sp_6(2)$.

Let $\dot{G} = Sp_6(2)$, $\dot{T} \in Syl_2(\dot{G})$, and $\dot{P}_i$, $1 \le i \le 3$, the maximal parabolics of $\dot{G}$ over $\dot{T}$ stabilizing an i-dimensional subspace of the natural module for $\dot{G}$. The pair $(\dot{G}, \{\dot{P}_1, \dot{P}_2, \dot{P}_3\})$ is a C_3-system in the sense of section I.5. Notice $\bar{L} \cong A_6 \cong \dot{P}_1'/O_2(\dot{P}_1)$. We will produce a corresponding C_3-system for G, and then use Theorem I.5.1 to conclude that $G \cong Sp_6(2)$. To do so, we will need to study the centralizer G_z of a suitable involution $z \in V_1 - Z_V$, and show $G_z/O_2(G_z) \cong S_3 \times S_3 \cong \dot{P}_2/O_2(\dot{P}_2)$. We must also construct a third 2-local H_0 and show $H_0/O_2(H_0) \cong L_3(2) \cong \dot{P}_3/O_2(\dot{P}_3)$. Then it is not difficult to construct our C_3-system.

13.4.1. Preliminary results on the structure of certain 2-local subgroups. As usual $Z = \Omega_1(Z(T))$ from Notation 12.2.5. Notice $Z_V \le Z_L := C_Z(L)$. Recall that Z_V is of order 2 and is of index 2 in $V_1 = Z \cap V$ by 13.3.4.1.

As usual we let $\theta(X)$ denote the subgroup generated by all elements of order 3 in a group X.

LEMMA 13.4.2. *(1)* $M = N_G(L) = N_G(V) = C_G(Z_V)$.

(2) $C_G(Z) \leq C_G(Z_L) \leq C_G(Z_V) = M$, *and* $M = !\mathcal{M}(C_G(Z_L))$.

(3) For each $v \in V^\#$, $C_G(v)$ *is transitive on conjugates of* V *containing* v. *In particular,* V *is the unique member of* V^G *containing* Z_V.

(4) $C_M(V) = C_M(L/O_2(L)) = C_M(V/Z_V)$.

(5) $L = \theta(M)$, *and if* $L/O_2(L) \cong A_6$, *then* $L = O^{3'}(M)$.

PROOF. Theorem 12.2.2.3 shows that $M = N_G(L)$, and (since $Z_V \neq 1$) that $V \trianglelefteq M$. Hence $M \leq C_G(Z_V)$, and (1) follows as $M \in \mathcal{M}$. As $Z_V \leq Z_L \leq Z$ and $M = !\mathcal{M}(LT)$, (2) holds. Since LT controls fusion in V by 13.2.2.5, (3) follows from A.1.7.1. Observe (5) follows from 12.2.8. Finally $C_M(L/O_2(L)) = C_M(V/Z_V) = C_M(V)$ by A.1.41, estabilishing (4). □

As $M = N_G(V)$ by 13.4.2.1, $\bar{M} := M/C_M(V)$ from Notation 12.2.5.2. Recall $V_1 = V \cap Z$, and by 13.3.5.3 there is $z \in V_1 - Z_V$ with $L_1T \leq G_z := C_G(z) \not\leq M$. Fix a choice of z and observe z has weight 2 or 4 in V. Eventually we will see that there is a unique $z \in V_1$ with $G_z \not\leq M$. As usual define

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \leq G_z \text{ and } H \not\leq M\}.$$

In particular $G_z \in \mathcal{H}_z$ so $\mathcal{H}_z \neq \emptyset$. Recall that R_1 is defined in Notation 13.2.1.

LEMMA 13.4.3. *(1)* $L = [L, J(T)]$.

(2) $VZ = VZ_L$ *and* $|Z : Z_L| = 2$.

(3) $Z = V_1 Z_L$, *so* L_1 *centralizes* Z.

PROOF. As $1 \neq Z_V \leq Z_L$, (1) follows from 3.1.8.3. Then by (1) and Theorem B.5.1, $[VZ, L] = V$, so $VZ = VZ_L$ by B.2.14. Then as $|Z \cap V : Z_V| = 2$ by 13.3.4.1, (2) and (3) hold. □

LEMMA 13.4.4. *If* $H \in \mathcal{H}_z$ *and* $V_H \in \mathcal{R}_2(H)$ *with* $Z_L \cap V_H\langle z\rangle \neq 1$, *then*

(1) $C_H(V_H) \leq M$.

(2) Set $L_+ := L_1$ *or* $L_{1,+}$, *for* $L/O_2(L) \cong A_6$ *or* $\hat{A}_6$, *respectively. Then either:*

(i) $O^2(C_H(V_H)) = 1$ *and* $C_H(V_H) = O_2(H) \leq O_2(L_1T) \leq R_1$, *or*

(ii) $L_+ = O^2(C_H(V_H)) \trianglelefteq H$, *and* $H_1 := C_H(L_+/O_2(L_+))$ *is of index* 2 *in* H *with* $R_1 \in Syl_2(H_1)$.

(3) $L_2 \not\leq H$, *and if* $L/O_2(L) \cong \hat{A}_6$, $L_{2,+} \not\leq H$.

PROOF. As neither L_2 nor $L_{2,+}$ centralizes z, (3) holds. Next $C_H(V_H) = C_H(V_H\langle z\rangle) \leq C_G(Z_L \cap V_H\langle z\rangle) \leq M = !\mathcal{M}(LT)$ by 13.4.2.2, so (1) holds.

Set $Y := O^2(C_H(V_H))$. By (1), $Y \leq M$, and as $H \in \mathcal{H}_z$, Y centralizes $z \in V_1 - Z_V$. Thus the hypotheses of 13.3.9 are satisfied, so we can appeal to that lemma. If $Y = 1$, then $C_H(V_H)$ is a 2-group; so as $V_H \in \mathcal{R}_2(H)$, $C_H(V_H) = O_2(H)$. Further $L_1T \leq H$, so $O_2(H) \leq O_2(L_1T)$ by A.1.6, and hence conclusion (i) of (2) holds. Thus we may assume conclusion (2) of 13.3.9 holds. In particular $L_+ = Y \trianglelefteq H$, so conclusion (ii) of (2) holds. □

LEMMA 13.4.5. *Assume* $H \in \mathcal{H}(T, M)$ *is nonsolvable. Then*

(1) There exists $K \in \mathcal{C}(H)$, *and for each such* K, $K \in \mathcal{L}_f^*(G, T)$, $K \trianglelefteq H$, *and* $K/O_2(K) \cong A_5$, $L_3(2)$, A_6, *or* $\hat{A}_6$.

(2) $K \not\leq M$, $L \not\leq N_G(K)$, *and* $[Z_V, K] \neq 1$.

(3) $Irr_+(K, R_2(KT)) \subseteq Irr_+(KT, R_2(H))$, *there is* $V_K \in Irr_+(K, R_2(KT), T)$, *and for each such* V_K, $V_K \trianglelefteq T$, *the pair* K, V_K *satisifies the FSU, and* $V_K = \langle (Z \cap V_K)^K \rangle$ *is either a natural module for* $K/C_K(V_K) \cong A_5$, $L_3(2)$, *or* A_6, *or a* 5*-dimensional module for* $K/C_K(V_K) \cong A_6$.

(4) For V_K *as in (3),* $O_{2,Z}(K) = C_K(V_K) = C_K(R_2(H))$. *In particular,* $R_2(H)$ *contains no faithful* 6*-dimensional modules when* $K/O_2(K) \cong \hat{A}_6$.

PROOF. As H is nonsolvable, there exists $K \in \mathcal{C}(H)$ by 1.2.1.1. If $K \leq M$ then we obtain a contradiction by applying 13.3.8.2 with L, M, $\langle K^T \rangle$ in the roles of "K, M_K, Y". Thus $K \not\leq M$, so $[Z_V, K] \neq 1$ by 13.4.2.1. Thus as $Z_V \leq Z$, $[R_2(H), K] \neq 1$, so $K \in \mathcal{L}_f(G,T)$ by 1.2.10. Therefore $N_G(K) = !\mathcal{M}(KT)$, and (1) holds by parts (1) and (2) of 13.3.2, since Theorem 13.3.16 rules out $K/O_2(K) \cong U_3(3)$. As $K \not\leq M$ and $M = !\mathcal{M}(LT)$, $L \not\leq N_G(K)$ so (2) holds.

By 13.3.2.3, there is $V_K \in Irr_+(K, R_2(KT), T)$ and $V_K \trianglelefteq T$. As $R_2(KT) \leq R_2(H)$ by A.1.11, $V_K \in Irr_+(K, R_2(H))$. Further the action of K on V_K described in 13.3.2.3, and $V_K = \langle (Z \cap V_K)^K \rangle$ by 13.3.4.2, completing the proof of (3).

By (3), either $C_K(V_K) = O_2(K)$, or $K/O_2(K) \cong \hat{A}_6$ with $C_K(V_K) = O_{2,Z}(K)$. Therefore as $O_2(K) \leq C_K(R_2(H)) \leq C_K(V_K) = O_{2,Z}(K)$, either (4) holds, or $K/O_2(K) \cong \hat{A}_6$ with $C_K(R_2(H)) = O_2(K)$. However in the latter case by A.1.42, there is $I \in Irr_+(K, R_2(H), T)$ with $K/C_K(I) \cong \hat{A}_6$, and $I \in Irr_+(K, R_2(KT), T)$ by A.1.41, contrary to (3). Thus (4) holds. □

When G is $Sp_6(2)$, G_z is solvable; thus we must eventually eliminate the case where G_z is nonsolvable. In that case by 13.4.5 there is $K \in \mathcal{C}(G_z)$ with $K \in \mathcal{L}_f^*(G,T)$, so that we can use our knowledge of groups in $\mathcal{L}_f^*(G,T)$ to restrict the structure of G_z. We begin with 13.4.6; notice in particular the very strong restrictions in part (4).

LEMMA 13.4.6. *Assume* $H \in \mathcal{H}_z$ *is nonsolvable. Then*

(1) There exists $K \in \mathcal{C}(H)$, *and for each such* K, $K \in \mathcal{L}_f^*(G,T)$ *and* $K \trianglelefteq H$. *Further* $K \not\leq M$, $L \not\leq N_G(K)$, *and* $[Z_V, K] \neq 1$.

(2) $K \leq G_z \leq N_G(K)$.

(3) $K = [K, J(T)]$.

(4) Either $M = LT$, *or* $L/O_2(L) \cong \hat{A}_6$ *and* $M = LXT$, *where* X *is a cyclic Sylow* 3*-subgroup of* $C_M(L/O_2(L)) = C_M(V)$.

PROOF. Part (1) is a restatement of parts (1) and (2) of 13.4.5. As $KT \leq H \leq G_z$ and $N := N_G(K) = !\mathcal{M}(KT)$ by (1), $G_z \leq N$, so (2) holds.

Let $L_- := L_2$ if $L/O_2(L) \cong A_6$, and $L_- := L_{2,+}$ if $L/O_2(L) \cong \hat{A}_6$. Then $L = \langle L_1, L_- \rangle$, and by (1), $L_1 \leq N$ but $L \not\leq N$; thus $L_-T \in \mathcal{H}_*(T,N)$. Now $[Z, L_-] \neq 1$ as L_- does not centralize $V_1 = Z \cap V$, so (3) follows by applying 3.1.8.3 with $N_G(K)$, $R_2(KT)$ in the roles of "M, V".

Let $Y := O^2(C_M(V))$. As $\bar{M} = \bar{L}\bar{T}$, $M = LTY$ and $Y \trianglelefteq M$. Further $Y \leq G_z \leq N$ by (2), so the hypotheses of 13.3.8 are satisfied with M, N in the roles of "H, M_K". Therefore Y is a $\{2,3\}$-group by 13.3.8.2. In particular if $m_3(C_M(V)) = 0$, then $C_M(V)$ is a 2-group, so that $M = LT$ and (4) holds. So assume $m_3(C_M(V)) \geq 1$. Then by 13.3.7, $L/O_2(L) \cong \hat{A}_6$ and $m_3(C_M(V)) = 1$ with $L_0 = \theta(C_M(V))$. Thus $C_M(V) = C_T(V)X$, where $X \in Syl_3(C_M(V))$ is cyclic, and once again (4) holds as $C_M(V) = C_M(L/O_2(L))$ by 13.4.2.4. □

We will now begin to produce subgroups H_0 of G which are generated by subgroups H_1 and H_2 in $\mathcal{H}(T)$, such that (H_0, H_1, H_2) is a Goldschmidt triple in the sense of Definition F.6.1. Proposition 13.4.7.5 gives fairly strong information about those pairs which also satisfy conditions (a)–(c) of the Proposition. In particular subgroups satisfying (1i) and (1ii) of Proposition 13.4.7 will eventually be identified as the parabolics $\dot{P}_2$ and $\dot{P}_3$ of $Sp_6(2)$.

PROPOSITION 13.4.7. *Assume $H_i \in \mathcal{H}(T)$, $i = 1, 2$, are distinct with $H_i/O_2(H_i) \cong S_3$. Let $K_i := O^2(H_i)$, $H_0 := \langle H_1, H_2\rangle$, and $V_0 := \langle Z^{H_0}\rangle$. Assume:*

(a) Either K_1 or K_2 has at least two noncentral 2-chief factors,

(b) $|Z : C_Z(H_i)| = 2$, for $i = 1$ and 2, and

(c) If $H_0 \in \mathcal{H}(T)$, then $K_i \trianglelefteq C_{H_0}(C_Z(K_i))$ for $i = 1$ and 2, and

$$K_j = O^{3'}(C_{H_0}(C_Z(K_j)) \text{ for } j := 1 \text{ or } 2.$$

Then

(1) $H_0 \in \mathcal{H}(T, M)$, $Z = V_1 = C_Z(H_1) \times C_Z(H_2)$ with $|C_Z(H_i)| = 2$, and one of the following holds:

(i) $H_0 = H_1H_2$, $[K_1, K_2] \leq O_2(K_1) \cap O_2(K_2)$, $H_0/O_2(H_0) \cong S_3 \times S_3$ or $\mathbf{Z}_2/E_9$, and $V_0 = V_1 \oplus V_2$, where $V_i := [V_0, K_i] = C_{V_0}(K_{3-i})$ is of rank 2.

(ii) $H_0 = K_0T$ where $K_0 \in \mathcal{C}(H_0)$ such that $K_0 \in \mathcal{L}^*_f(G, T)$, $K_0/O_2(K_0) \cong L_3(2)$, $J(T) \trianglelefteq H_0$, and V_0 is either the sum of two nonisomorphic natural modules for $K_0/O_2(K_0)$, or the core of the 7-dimensional permutation module.

(iii) $O_2(H_0) = C_{H_0}(V_0)$, $H_0/O_2(H_0) \cong E_4/3^{1+2}$, $m(V_0) = 6$, and $J(T) \trianglelefteq H_0$.

(iv) $H_0 = K_0T$, where $K_0 \in \mathcal{C}(H_0)$ with $K_0/O_2(K_0) \cong A_6$, and $J_i(T) \trianglelefteq H_0$ for $i = 0, 1$.

(2) Assume conclusion (ii) holds, with $K_i = [K_i, J_1(T)]$ for some i, and $X \in \mathcal{H}(H_0)$. Then $V_0 = \langle Z^X\rangle$ and $X = H_0C_X(V_0)$; so if $C_X(Z) = T$, then $X = H_0$.

(3) If $K_2 = L_2$, then conclusion (iii) does not hold.

(4) Assume $K_2 = L_2$. Then conclusion (iv) does not hold, and if conclusion (ii) holds, then $K_2 = [K_2, J_1(T)]$.

PROOF. Let $Q_0 := O_2(H_0)$ and $H_0^* := H_0/C_{H_0}(V_0)$. Observe that the hypotheses say that (H_0, H_1, H_2) is a Goldschmidt triple in the sense of Definition F.6.1, so

$$(H_1/Q_0, T/Q_0, H_2/Q_0)$$

is a Goldschmidt amalgam by F.6.5.1, and hence is listed in F.6.5.2.

Assume that $Q_0 = 1$. By hypothesis (a), some K_i has at least two noncentral 2-chief factors, which eliminates cases (i)–(v) of F.6.5.2, and in case (vi) also eliminates cases (1) and (2) of F.1.12. In cases (3), (8), (12), and (13) of F.1.12, $Z \leq H_i$ for exactly one value of i, contrary to (b). Therefore $Q_0 \neq 1$, and hence $H_0 \in \mathcal{H}(T)$. Then $H_0 \in \mathcal{H}^e$ by 1.1.4.6, so $V_0 \in \mathcal{R}_2(H_0)$ by B.2.14, and hence $O_2(H_0^*) = 1$.

By (b), $[Z, K_i] \neq 1$, so $C_{K_i}(V_0) \leq O_2(K_i)$ and $K_1^* \neq 1 \neq K_2^*$. Therefore $C_T(V_0) \leq Q_0$ by F.6.8, so as $V_0 \in \mathcal{R}_2(H_0)$, $Q_0 = C_T(V_0)$. By (c),

$$C_{H_0}(V_0) \leq C_{H_0}(Z) \leq N_G(K_1) \cap N_G(K_2).$$

Also by (c), there exists an index j such that $K_j = O^{3'}(C_{H_0}(C_Z(K_j))$. Then $O^{3'}(C_{H_0}(V_0)) \leq K_j$, so in fact $O^{3'}(C_{H_0}(V_0)) = 1$ since $C_{K_j}(V_0) \leq O_2(K_j)$. That is, $C_{H_0}(V_0)$ is a $3'$-group.

Let $H_0^+ := H_0/O_{3'}(H_0)$. As $C_{H_0}(V_0)$ is a $3'$-group, H^+ is a quotient of H^*. Observe that H^+ is described in F.6.11. By F.6.6, $O^2(H_0) = \langle K_1, K_2\rangle$, so $O^2(H_0) = \theta(H_0)$.

Suppose $H_0 \leq M$. Then $O^2(H_0) \leq \theta(M) = L$ by 13.4.2.5, so as each K_i is T-invariant, $K_i \leq L_{k(i)}$ for some $k(i) := 1$ or 2 by 13.3.4.3. By 13.4.3.3, L_1 centralizes Z; and if $L/O_2(L) \cong \hat{A}_6$ then $L_0 \leq L_1$, so L_0 centralizes Z. Therefore as $[Z, K_i] \neq 1$ for each i by (b), it follows that $K_1 = K_2 = L_2$ if $L/O_2(L) \cong A_6$, while $K_1 = K_2 = L_{2,+}$ if $L/O_2(L) \cong \hat{A}_6$. But $K_1 \neq K_2$, as otherwise $H_1 = TK_1 = TK_2 = H_2$, contrary to hypothesis. Hence $H_0 \not\leq M$.

As $H_0 \not\leq M$, $H_k \not\leq M$ for some $k := 1$ or 2. Therefore $C_{Z_L}(H_k) = 1$, as otherwise $H_k \leq C_G(C_{Z_L}(H_k) \leq M = !\mathcal{M}(LT)$. But $C_Z(H_k)$ is a hyperplane of Z by (b), while $1 \neq Z_V \leq Z_L$ and Z_L is also a hyperplane of Z by 13.4.3.2. We conclude that $m(Z) = 2$ and $Z_L = Z_V$ and $C_Z(H_k)$ are of rank 1. As $K_j = O^{3'}(C_{H_0}(C_Z(K_j)))$ by (c) and $K_1 \neq K_2$, $C_Z(H_1) = C_Z(K_1) \neq C_Z(K_2) = C_Z(H_2)$. Then we conclude from (b) that $Z = C_Z(H_1) \times C_Z(H_2)$ with $m(C_Z(H_i)) = 1$ for each i. As $V_1 = Z \cap V$ is of rank 2, $Z = V_1 \leq V$. Thus we have established the initial conclusions of (1), so it remains to show that one of conclusions (i)–(iv) of (1) holds.

Suppose first that $[K_1^*, K_2^*] = 1$. Then $O^2(H_0^*) = K_1^*K_2^*$, so $K_iC_{H_0}(V_0)$ is normal in H_0 for $i = 1, 2$. Furthermore $C_{H_0}(V_0)$ normalizes K_i by (c), and we saw $C_{H_0}(V_0)$ is a $3'$-group, so $K_i = O^{3'}(K_iC_{H_0}(V_0)) \trianglelefteq H_0$. It follows that $H_0 = H_1H_2$ and $[K_1, K_2] \leq O_2(K_1) \cap O_2(K_2) \leq O_2(H_0)$; hence $H_0/O_2(H_0) \cong S_3 \times S_3$ or $\mathbf{Z}_2/E_9$. Set $V_i := [V_0, K_i]$. As $Z = C_Z(H_1) \times C_Z(H_2)$ with $|C_Z(H_i)| = 2$, $V_i = \langle C_Z(H_{3-i})^{H_i}\rangle \cong E_4$ is centralized by H_{3-i}, so we conclude that case (i) of (1) holds. Thus we may assume from now on that $[K_1^*, K_2^*] \neq 1$; under this assumption, we will show that one of (ii)–(iv) holds.

We first consider the case where H_0^* is not solvable, which will lead to (ii) or (iv). By 1.2.1.1, there is $K_0 \in \mathcal{C}(H_0)$ with $K_0^* \neq 1$. Then by 13.4.5.1, $K_0 \trianglelefteq H_0$, $K_0 \in \mathcal{L}_f^*(G,T)$, and $K_0/O_2(K_0)$ is listed in 13.4.5.1. In particular K_0 is not a $3'$-group, so that $K_0^+ \neq 1$; hence H_0^+ is described in F.6.18 by F.6.11.2. Also K_0^+ is a quotient of $K_0/O_2(K_0)$, so comparing the possibilities for $K_0/O_2(K_0)$ in 13.4.5 with the possible quotients K_0^+ in cases (3)–(13) of F.6.18, we conclude K_0^+ must be $L_3(2)$, A_6, or $\hat{A}_6$, with $H_0^+ = K_0^+T^+$ appearing in case (6) or (8) of F.6.18. Furthermore if $K_0/O_2(K_0) \cong \hat{A}_6$, then as $C_{H_0}(V_0)$ is a $3'$-group, $K_0^* \cong K_0/O_2(K_0) \cong \hat{A}_6$, contrary to 13.4.5.4. Thus in any case $C_{K_0}(V_0) = O_2(K_0) = O_{3'}(K_0)$, so $K_0/O_2(K_0) \cong K_0^* \cong K_0^+$, and $K_0/O_2(K_0)$ is not $\hat{A}_6$.

As $H_0^+ = K_0^+T^+$, $H_0 = K_0TO_{3'}(H_0)$, so as $K_0 \trianglelefteq H_0$, $K_0 = O^{3'}(H_0)$. Thus $K_i \leq K_0$ for $i = 1, 2$, so $K_0 = O^2(H)$ by F.6.6, and hence $H_0 = K_0T$. Since $K_1 \neq K_2$, H_1^* and H_2^* are the minimal parabolics of H^* over T^*.

By 13.4.5.3, we may choose a T-invariant $I \in Irr_+(K_0, V_0)$ in the FSU. By 13.4.5.4, $C_{K_0}(I) = C_{K_0}(V_0)$, so that $K_0^* = K_0/C_{K_0}(I)$. By 13.4.5.3, I is either a natural module for K_0^* or a 5-dimensional module for $K_0^* \cong A_6$. As H_1^* and H_2^* are the minimal parabolics of H^*, some K_i (say K_1) centralizes $Z \cap I$, so as $[Z, K_1] \neq 1$ by hypothesis (b), $IZ > IC_Z(K_0)$. Thus K_0 is nontrivial on V_0/I. Hence if $J(T) \not\leq O_2(H_0)$, then by Theorem B.5.1, $[V_0, K_0]$ is the sum of two isomorphic natural modules for $K_0^* \cong L_3(2)$; since $m(Z) = 2$, this contradicts $[Z, K_1] \neq 1$. Therefore $J(T) = J(O_2(H_0)) \trianglelefteq H_0$.

As $Z = C_Z(H_1) \times C_Z(H_2)$ with $C_Z(H_i) \cong \mathbf{Z}_2$, there is $v \in C_Z(H_2) - C_Z(H_1)$; set $V_v := \langle v^{K_0} \rangle$.

Assume first that $K_0/O_2(K_0) \cong L_3(2)$; we will show (ii) holds. As v centralizes H_2 but not H_1, V_v is a quotient of the 7-dimensional permutation module for K_0^* on the coset space K_0/H_2, with $m(V_v) > 1$. Thus by H.5.3, $[V_v, K_0]$ is either the 3-dimensional dual of I or the 6-dimensional core of the permutation module. Thus as $m(Z) = 2$, $\dim(V_0) = 6$, and so $V_0 = V_v \oplus I$ or V_v, respectively. This completes the verification of (ii).

So to complete the treatment of the case H_0^* not solvable, we may assume that $K_0^* \cong A_6$; then to establish (iv), it remains to show that $J_1(T) \trianglelefteq H_0$. As above, V_v is a quotient of the 15-dimensional permutation module on H_0/H_2, so by G.5.3, V_v has a K_0^*-irreducible quotient W_2 isomorphic to the conjugate of $W_1 := I/C_I(K_0)$ under a graph automorphism. Therefore K_0 has chief factors isomorphic to W_1 and W_2 on V_0. We may assume that $J_1(T) \not\leq O_2(H_0) = C_{H_0}(V_0)$, so that there is $A \in \mathcal{A}_1(T)$ with $A^* \neq 1$. By B.2.4.1, $m(V_0/C_{V_0}(A)) \leq m(A^*) + 1$, so either

(I) $m(A^*) = 1$ and $m(W_i/C_{W_i}(A)) = 1$ for $i = 1$ and 2, or

(II) $m(A^*) \geq 2$, so that $m(W_i/C_{W_i}(A)) \geq 2$ for $i = 1$ and 2, and hence $m(A^*) = 3$ and A^* is a strong FF*-offender on both W_1 and W_2.

These cases are impossible since by B.3.4, no involution induces a transvection on both W_1 and W_2, nor does there exists a subgroup which is a strong FF*-offender on both W_1 and W_2. This contradiction completes the verification of (iv), and establishes (1) when H_0^* is nonsolvable.

Thus we have reduced to the case where H_0^* is solvable, but $[K_1^*, K_2^*] \neq 1$. This time let $K_0 := O^2(H_0)$ and let $Q_i := O_2(H_i)$.

Suppose first that $Q_1 = Q_2$. Then $Q_0 = Q_1$, $T^* = \langle t^* \rangle$ is of order 2, and $K_i^* \cong S_3$. Thus $m(V_0) \leq 2m(C_{V_0}(t)) = 2m(Z) = 4$. Inspecting the solvable subgroups H_0^* of $GL_4(2)$ with $O_2(H_0^*) = 1$ and generated by a pair of distinct S_3-subgroups with a common Sylow 2-subgroup, we conclude H_0^* is E_9 extended by $\mathbf{Z}_2$. But this contradicts the fact that $[K_1^*, K_2^*] \neq 1$. This contradiction shows that $Q_1 \neq Q_2$.

Suppose next that K_i does not centralize $O_p(H_0/Q_0)$, for some prime $p > 3$ and $i = 1$ or 2, say $i = 1$. Then by A.1.21 there is a supercritical subgroup P of a Sylow p-subgroup of the preimage of $O_p(H_0/Q_0)$. By a Frattini Argument, $H_0 = N_{H_0}(P)Q_0$. Let $\dot{H}_0 := H_0/C_{H_0}(P/\Phi(P))Q_0$. By A.1.25, $\dot{H}_0 = \langle \dot{H}_1, \dot{H}_2 \rangle$ is a subgroup of $GL_2(p)$; of course $\dot{H}_0$ is solvable and $\dot{H}_1/O_2(\dot{H}_1) \cong S_3$. Thus P is noncyclic. Further if $\dot{K}_2 \neq 1$ then we conclude from Dickson's Theorem A.1.3 that $\dot{K}_0 \cong SL_2(3)$ or $\dot{K}_0$ is cyclic, and in either case $\dot{Q}_1 = \dot{Q}_2$, so that $Q_1 = Q_2$, contrary to the previous paragraph. Thus $K_P := \langle K_2^{H_0} \rangle$ centralizes PQ_0/Q_0. Let $1 \neq P_0 \leq P_1 \in Syl_p(K_P)$ with P_1 acting on P; as $m_p(H_0) = 2 = m_p(P)$, P contains all elements of order p in P_0, so $Aut_{K_P}(P_0)$ is a p-group by A.1.21, and hence K_P is p-nilpotent by the Frobenius Normal p-Complement Theorem 39.4 in [**Asc86a**]. As K_P is generated by 3-elements, K_P is a p'-group, so as K_1 is a $\{2, 3\}$ group, we conclude $K_0 = \langle K_1, K_2 \rangle = K_1K_P$ is a p'-group, contrary to $P \leq K_0$.

Therefore K_i centralizes $O^3(F(H_0/Q_0))$ for $i = 1$ and 2, so by F.6.9, H_0 is a $\{2, 3\}$-group. Then as $C_{H_0}(V_0)$ is a $3'$-group, we conclude that $Q_0 = C_{H_0}(V_0)$. Therefore $H_0^* = H_0/Q_0 = H_0^+$ and $H_0 = K_0T$ with K_0 a $\{2, 3\}$-group. Further (H_1^*, T^*, H_2^*) is a Goldschmidt amalgam by F.6.5.1. Since $Q_1 \neq Q_2$ by an earlier reduction, $H_0^* = H_0^+$ is described in Theorem F.6.18 by F.6.11.2. As H is solvable and $Q_1 \neq Q_2$, conclusion (2) of F.6.18 holds, and by earlier reduction $[K_1^*, K_2^*] \neq 1$, so that $H_0^* \cong E_4/3^{1+2}$. Let $X_0^* := Z(K_0^*)$; then $V_0 = U_0 \oplus U_1$, where $U_1 := C_{V_0}(X_0)$ and $U_0 := [V_0, X_0]$ is of rank $6k$ for some $k \geq 1$. Now $Z = Z_0 \oplus Z_1$ is of rank 2 where $Z_i := Z \cap U_i$; so either $m(Z_i) = 1$ for $i = 0$ and 1, or $U_1 = 0$ and $Z = Z_0$. As T^* has order 4, $6 \leq 6k = m(U_0) \leq 4m(Z_0) \leq 8$; hence $k = 1$ and $Z = Z_0$ with $U_1 = 0$, so $V_0 = U_0$ is of rank 6. By Theorem B.5.6, $J(T) \trianglelefteq H_0$. Now conclusion (1iii) of the lemma holds. Hence the proof of (1) is at last complete.

We next prove (2). So assume that the hypotheses of (2) hold. As conclusion (ii) of (1) holds, $K_0 \trianglelefteq X$ by 13.4.5.1, so $O_2(K_0) \leq O_2(X)$. Set $V_X := \langle Z^X \rangle$, so that $V_0 \leq V_X \in \mathcal{R}_2(X)$ by B.2.14. We must show that $V_0 = V_X$ and $X = C_X(V_0)H_0$. Let $\hat{X} := X/C_X(V_X)$; then $C_X(V_X) \leq C_X(V_0)$, so H_0^* is a quotient of $\hat{H}_0$. Furthermore as conclusion (ii) holds, $C_{K_0}(V_0) = O_2(K_0)$, so that $C_{K_0}(V_0) = C_{K_0}(V_X)$ since $O_2(K_0) \leq O_2(X)$, and hence $\hat{K}_0 \cong K_0^*$.

From the structure of V_0 in either case of (ii), $O^2(N_{GL(V_0)}(K_0^*)) = K_0^*$. Suppose we have shown that $V_0 = V_X$. Then $O^2(\hat{X}) = \hat{K}_0$, so that $\hat{X} = \hat{K}_0\hat{T} = \hat{H}_0$. Hence $X = H_0 C_X(V_0)$, giving the remaining conclusion of (2). Thus it suffices to show $V_0 = V_X$, or equivalently that $V_X \leq V_0$.

By the hypotheses of (2), $J_1(T) \not\leq O_2(H_0)$, so as $H_0 = K_0T$, there is $A \in \mathcal{A}_1(T)$ with $K_0 = [K_0, A]$. Thus by B.2.4.1,

$$m(A^*) \geq m(V_0/C_{V_0}(A)) - 1 \text{ and } \ m(\hat{A}) \geq m(V_X/C_{V_X}(A)) - 1.$$

However V_0 is not an FF-module for $K_0^*T^*$ by Theorem B.5.1, so $m(A^*) < m(V_0/C_{V_0}(A))$, and hence $m(A^*) = m(V_0/C_{V_0}(A)) - 1$. Then by B.2.4.2, $B := V_0C_A(V_0) \in \mathcal{A}(T)$; recall $C_A(V_0) = A \cap Q_0$, so as $V_0 \leq C_X(V_X)$,

$$\hat{B} \leq \hat{A} \cap \hat{Q}_0 \leq C_{\hat{X}}(\hat{K}_0).$$

Suppose first that $\hat{B} = 1$, so that $C_A(V_0) = C_A(V_X)$. Then

$$m(V_0/C_{V_0}(A)) - 1 = m(A^*) = m(\hat{A}) \geq m(V_X/C_{V_X}(A)) - 1,$$

so $V_X = V_0C_{V_X}(A)$ and hence $[V_X, A] \leq V_0$. Therefore as $K_0 = [K_0, A]$, $V_0 = [V_0, K_0] = [V_X, K_0]$ is X-invariant; then as $Z \leq V_0$, $V_X = \langle Z^X \rangle = V_0$, so we are done.

Thus we may assume instead that $\hat{B} \neq 1$. Then as $B \in \mathcal{A}(T)$ and $\hat{B}$ centralizes $\hat{K}_0$, $J(C_X(\hat{K}_0)) =: Y \not\leq C_X(V_X)$. Hence as $m_3(X) \leq 2$ with $\hat{K}_0 \cong L_3(2)$, $m_3(\hat{Y}) \leq 1$, so we conclude from Theorem B.5.6 that either $\hat{Y} \cong S_3$ (in which case we set $Y_0 := Y$), or $\hat{Y} = \hat{Y}_0$ for some $Y_0 \in \mathcal{C}(X)$ with $m_3(Y_0) = 1$. In either case $\hat{Y}_0$ is normal in $\hat{X}$. In the latter case since $O_2(Y_0) \leq O_2(X) \leq C_X(V_X)$, we obtain $\hat{Y}_0 \cong L_3(2)$ or A_5 from 13.4.5.1; then by Theorem B.5.1 and 13.4.5.1, $[V_X, Y_0]$ is either the natural module for $\hat{Y}_0$ or the sum of two natural modules for $\hat{Y}_0 \cong L_3(2)$. Then $End_{\hat{Y}_0}([V_X, Y_0])$ is either a field or the ring of 2×2 matrices over $\mathbf{F}_2$, so that $[V_X, Y_0, K_0] = 1$. Hence $[Z, Y_0] \leq [K_0, V_X, Y_0] = 1$ using the Three-Subgroup Lemma. So as $\hat{Y}_0 \trianglelefteq \hat{X}$, Y_0 centralizes $V_X = \langle Z^X \rangle$, contrary to $\hat{Y}_0 \neq 1$. Thus the proof of (2) is complete.

We next prove (4), so we assume that $K_2 = L_2$, and that either conclusion (ii) or (iv) of (1) holds. Now $J(T) \trianglelefteq H_0$ in either of those cases, so that $S := \mathrm{Baum}(T) = \mathrm{Baum}(O_2(H_0))$ by B.2.3.4. Hence as $H_0 \not\leq M = !\mathcal{M}(LT)$, no nontrivial characteristic subgroup of S is normal in LT. Thus conclusion (I) of 13.2.2.10 does not hold. If conclusion (III) of 13.2.2.10 holds, then $K_2 = [K_2, J_1(T)]$, so we are not in case (iv), as there $J_1(T) \trianglelefteq H_0$. Hence we are in case (ii), so that (4) holds. Thus we may assume that conclusion (II) of 13.2.2 holds, so that L is an A_6-block with $\mathcal{A}(O_2(LT)) \subseteq \mathcal{A}(T)$. As L is an A_6-block, L_2 has exactly three noncentral 2-chief factors. Let $k := 2$ in case (ii), and $k := 3$ in case (iv). As $L_2 = K_2$, L_2 has at least k chief factors on V_0 and one on $O_2(L_2)^*$, so (ii) holds and $[O_2(K_0T), K_0] \leq V_0$. Thus as $J(T) \trianglelefteq H_0$, each $A \in \mathcal{A}(T)$ contains V_0, so $[A, K_0] \leq [O_2(K_0T), K_0] = V_0 \leq A$, and hence $A \trianglelefteq K_0A$. Further as $\mathcal{A}(O_2(LT)) \subseteq \mathcal{A}(T)$, $J(O_2(LT)) = \langle A \in \mathcal{A}(T) : A \leq O_2(LT)\rangle$, so $K_0 \leq N_G(J(O_2(LT))) \leq M$, contrary to $H_0 \not\leq M$ by (1).

It remains to prove (3), so we may assume that $K_2 = L_2$ and conclusion (iii) holds, and we must produce a contradiction. As $m_3(L_2) = m_3(K_2) = 1$ by hypothesis, $L/O_2(L)$ is A_6 rather than $\hat{A}_6$. Let Y_Z be the preimage in H_0 of $Z(O_3(H_0^*))$, and set $Y_2 := O^2(Y_Z)$. Notice that as Y_2^* is fixed point free on V_0 of rank 6, while $Z = V_1$ is of rank 2, $[Z, Y_2]$ is of rank 4. In particular $Y_2 \not\leq C_G(Z_V) = M$ by 13.4.2.1.

Set $Y := \langle L_1T, Y_2T\rangle$, $Q_Y := O_2(Y)$, and $V_Y := \langle Z^Y\rangle$. Observe (Y, L_1T, Y_2T) is a Goldschmidt triple, so $(L_1T/Q_Y, T/Q_Y, K_2T/Q_Y)$ is a Goldschmidt amalgam by F.6.5.1, and hence is listed in F.6.5.2. Now $K_2 = L_2$ has at least three noncentral 2-chief factors in L; so as this does not hold in any case in F.6.5.2, we conclude $Q_Y \neq 1$, so that $Y \in \mathcal{H}(T)$. Hence $V_Y \in \mathcal{R}_2(Y)$ by B.2.14.

We saw $Y_2 \not\leq M$, so $Y \not\leq M$. On the other hand, for $z \in V_1 - Z_V$, $C_Y(V_Y) \leq C_Y(Z_V) \cap C_Y(z) \leq C_M(z)$, so applying 13.3.9 with Y, $O^2(C_Y(V_Y))$ in the roles of "H, Y", and recalling that $L/O_2(L) \cong A_6$, we conclude that $O^2(C_Y(V_Y)) = 1$ or L_1.

In the latter case, Y_2 acts on L_1, and hence centralizes $L_1/O_2(L_1)$ so that L_1 normalizes $O^2(Y_2O_2(L_1)) = Y_2$. Then as $L_2 = K_2$, $L = \langle L_1, L_2\rangle \leq N_G(Y_2)$, contrary to $Y_2 \not\leq M = !\mathcal{M}(LT)$. Thus $O^2(C_Y(V_Y)) = 1$ so that $C_Y(V_Y) = Q_Y \leq O_{3'}(Y)$. In addition this argument shows that $[L_1, Y_2] \not\leq O_2(L_1)$.

Let $Y^* := Y/C_Y(V_Y)$ and $Y^+ := Y/O_{3'}(Y)$, so that Y^+ is a quotient of Y^*, and is described in F.6.11.2. Now $L = [L, J(T)]$ by 13.4.3.1, so that $L_1 = [L_1, J(T)]$ by 13.2.2.4; so as $J(T)^*$ centralizes $O^3(F^*(Y^*))$ by Theorem B.5.6, so does L_1. In particular L_1 centralizes $F^*(O_{3'}(Y^*))$, so as L_1 is generated by conjugates of an element of order 3, we conclude from A.1.9 that $L_1 \leq C_Y(O_{3'}(Y^*))$. Thus $L_1 = O^{3'}(L_1O_{3'}(Y))$, so if $[Y_2^+, L_1^+] \leq O_2(L_1^+)$, then $[Y_2, L_1] \leq O_2(L_1)$, which we showed earlier is not the case. We conclude $[Y_2^+, L_1^+] \not\leq O_2(L_1^+)$. Now as $J(T) \trianglelefteq Y_2T$ since we are in case (iii), but $L_1 = [L_1, J(T)]$, $O_2(Y_2T) \neq O_2(L_1T)$. Thus case (i) of F.6.11.2 holds, so Y^+ is described in F.6.18, where F.6.18.1 is similarly ruled out. As $L_1 = [L_1, J(T)]$, Y^+ is not $E_4/3^{1+2}$ by Theorem B.5.6, while the condition $[Y_2^+, L_1^+] \not\leq O_2(L_1^+)$ rules out the other possibility in F.6.18.2. In the remaining cases in Theorem F.6.18, Y is not solvable, so there is $K_Y \in \mathcal{C}(Y)$, and by 13.4.5, $K_Y/O_2(K_Y) \cong A_5$, $L_3(2)$, A_6, or $\hat{A}_6$. The A_5 case is ruled out, as A_5 does not appear as a composition factor in the groups listed in Theorem F.6.18. Similarly conclusion (3) of F.6.18 does not hold, so $L_1 \leq K_Y$.

As $L_1 = [L_1, J(T)]$, $K_Y = [K_Y, J(T)]$, so by Theorem B.5.1 and 13.4.5.3, $[V_Y, K_Y]$ is a natural module for $K_Y^* \cong L_3(2)$ or A_6, a 5-dimensional module for $K_Y^* \cong A_6$, or the sum of two natural modules for $K_Y^* \cong L_3(2)$. As $Z = V_1$ is of rank 2, with $\langle Z^{Y_2}\rangle \cong E_{16}$, V_Y is the sum of two natural modules for $K_Y^* \cong L_3(2)$. As $L_1^*T^*$ is the parabolic of K_Y^* centralizing Z, $J(R_1) \leq C_Y(V_Y) = Q_Y$, and hence $\mathrm{Baum}(R_1) = \mathrm{Baum}(Q_Y)$ by B.2.3.5. Then each nontrivial characteristic subgroup of $\mathrm{Baum}(R_1)$ is normal in Y, and hence not normal in LT as $Y \not\leq M = !\mathcal{M}(LT)$. Therefore L is an A_6-block by 13.2.2.7, and in particular L_1 has two noncentral chief factors. This is impossible, as L_1 has two noncentral chief factors on V_Y and one on $O_2(L_1^*)$. So the proof of (3), and hence of Proposition 13.4.7, is finally complete. □

13.4.2. The case $\mathbf{G_z}$ solvable, leading to $\mathbf{Sp_6(2)}$. Recall the definitions of z and $\mathcal{H}_z$ given before 13.4.3, and recall that $G_z := C_G(z)$. In the next lemma, we begin to identify G_z and a suitable 2-local H_0 with the parabolics $\dot{P}_2$ and $\dot{P}_3$ of $\dot{G} = Sp_6(2)$.

LEMMA 13.4.8. *Assume $H \in \mathcal{H}_z$ is solvable, choose $V_H \in \mathcal{R}_2(H)$ with $Z_V \leq V_H$, and let $I := \langle J(R_1)^H \rangle$. Then*

(1) $L/O_2(L) \cong A_6$.

(2) $I/O_2(I) \cong S_3$ with $m([V_H, I]) = 2$.

(3) $H = L_1IT$ and $H/O_2(H) \cong S_3 \times S_3$. In particular IT is the unique member of $\mathcal{H}_*(T, M)$ in H.

PROOF. Let $H^* := H/C_H(V_H)$. As usual, $O_2(H^*) = 1$ by B.2.14. As $Z_V \leq V_H$, we may apply 13.4.4.2, to conclude that

$$C_{R_1}(V_H) = O_2(H).$$

For in case (i), $C_H(V_H) = O_2(H) \leq R_1$; and in case (ii), $R_1 \in Syl_2(C_H(L_+/O_2(L_+))$, where $L_+ = O^2(C_H(V_H))$.

We claim that $[V_H, J(R_1)] \neq 1$, so we assume that $[V_H, J(R_1)] = 1$ and derive a contradiction. Then

$$B := \mathrm{Baum}(R_1) = \mathrm{Baum}(C_{R_1}(V_H)) = \mathrm{Baum}(O_2(H)), \qquad (*)$$

by B.2.3.5 and the previous paragraph. Hence as $H \not\leq M = !\mathcal{M}(LT)$, no nontrivial characteristic subgroup of B is normal in LT, so by 13.2.2.7, L is an A_6-block. In particular, $L/O_2(L) \cong A_6$ rather than $\hat{A}_6$.

Calculating in the core V of the permutation module:

$$V_3 = [V, L_1] = [V, T \cap L_1] = [V, O_2(L_1)] = \{e_J : J \subseteq \{1, 2, 3, 4\} \text{ and } |J| \text{ is even}\},$$

and $[V_3, O_2(L_1)] = \langle e_{1,2,3,4} \rangle$. Further if $\bar{M} \cong S_6$, then also $Z_V = \langle e_\Omega \rangle \leq [V, R_1]$.

By 13.2.2.6, $V \leq J(R_1)$, so by (*), $V \leq O_2(H) \leq N_H(V)$ and V centralizes V_H. Hence $U := \langle V^H \rangle \leq O_2(H)$ and $[V, V^h] \leq V \cap V^h$ for each $h \in H$. If $U \leq C_T(V) =: Q$, then L normalizes U because $[Q, L] = V$ since L is a block. But then $H \leq N_G(U) \leq M = !\mathcal{M}(LT)$, contrary to $H \not\leq M$. So we conclude instead that $[V, V^h] \neq 1$ for some $h \in H$.

Suppose that $L_1 \trianglelefteq H$. Then as $V_3 = [V_3, L_1]$, $V_3^h = [V_3^h, L_1] \leq O_2(L_1)$. Thus either $V_3^h = [C_{O_2(L_1)}(V), L_1] = V_3$, or $V_3^h \cap Q = V_1$, $O_2(L_1) = V_3^h C_{O_2(L_1)}(V)$ and $\bar{V}^h = \bar{R}_1 \not\leq \bar{L}$, since V_3/V_1 is the unique minimal L_1-invariant subgroup of V/V_1. Assume the former case holds. Then V^h centralizes V_3, so $\bar{V}^h = \langle (5, 6) \rangle$ and hence $[V, V^h] = \langle e_{5,6} \rangle$. But then $Z_V \leq V_3 \langle e_{5,6} \rangle \leq V \cap V^h$, contrary to 13.4.2.3. In the latter case, $Z_V \leq [V, R_1] = [V, V^h]$, for the same contradiction.

This contradiction shows that L_1 is not normal in H. Hence $[V_H, L_1] \neq 1$ by 13.4.4.2. We saw $V_H \leq Q$, so $1 \neq [V_H, L_1] \leq [Q, L] = V$, and hence $[V_H, L_1] = [V, L_1] = V_3$. By C.1.13.d, $O_2(L) \leq C_T(Q) \leq C_H(V_H)$, so L_1^* is a quotient of $\bar{L}_1 \cong A_4$. Then by A.1.26, $O_2(L_1^*)$ centralizes $F^*(H^*)$ of odd order, so $O_2(L_1^*) = 1$, and hence $O_2(L_1) \leq C_H(V_H) \leq C_H(V_3)$, whereas we saw $[V_3, O_2(L_1)] \neq 1$.

This establishes the claim that $[V_H, J(R_1)] \neq 1$. By the first paragraph of the proof, $C_{R_1}(V_H) = O_2(H)$, so we may apply B.2.10.1 to conclude that

$$\mathcal{P}_{R_1,H} = \{A^{*h} \neq 1 : A \in \mathcal{A}(R_1), h \in H\}$$

is a nonempty stable subset of $\mathcal{P}(H^*, V_H)$. Hence by B.1.8.5, $I^* = \langle J(R_1)^{*H} \rangle = I_1^* \times \cdots \times I_s^*$ with $I_i^* \cong S_3$, and $[V_H, I]$ is the direct sum of the subgroups $U_i := [V_H, I_i] \cong E_4$. Further $s \leq 2$ by Theorem B.5.6.

Recall that $L = \theta(M)$, $L_1 = O^{3'}(C_L(z))$, and $H \leq G_z$. Thus $L_1 = \theta(C_M(z)) = \theta(H \cap M)$. Similarly if $L/O_2(L) \cong A_6$, then $L_1 = O^{3'}(H \cap M)$ using 13.4.2.5.

Next by B.1.8.5, $J(R_1)^* \in Syl_2(I^*)$; thus $J(R_1)^*$ is self-normalizing in I^*. We claim that $O^2(I^*) \cap L_1^* = 1$: If $L/O_2(L) \cong A_6$, then L_1 normalizes R_1, so this follows from the previous observation. So suppose $L/O_2(L) \cong \hat{A}_6$. Then $L_{1,+}$ normalizes R_1, so $O^2(I^*) \cap L_1^*$ is trivial or L_0^*. Assume the latter case holds. Then as L_0 is T-invariant, $L_0^* = O^2(I_i^*)$ for some i, and then $L_0^* \trianglelefteq H^*$ since $s \leq 2$. In case (i) of 13.4.4.2, $C_H(V_H) = O_2(H)$ acts on L_0, so $L_0 = O^2(L_0 C_H(V_H)) \trianglelefteq H$. In case (ii) of 13.4.4.2, $L_{1,+} = O^2(C_H(V_H))$, so $L_1 = L_{1,+}L_0 = O^2(L_0 C_H(V_H)) \trianglelefteq H$. In either case $H \leq M$ by 13.2.2.9, contrary to $H \not\leq M$. This contradiction completes the proof of the claim that $O^2(I^*) \cap L_1^* = 1$.

Since $L_1 = \theta(H \cap M)$, it follows from the claim that $I \not\leq M$. Furthermore $O^2(I^*) = O^{3'}(N_{GL([V_H,I])}(I^*))$, so the claim says $I^* L_1^* = I^* \times L_1^*$. Thus when $L_1^* \neq 1$, it follows from A.1.31.1 applied in the quotient $I^* L_1^*/O_2(L_1^*)$ that $s = 1$.

We first treat case (i) of 13.4.4.2, where $C_H(V_H) = O_2(H)$. Then $m_3(L_1) = m_3(L_1^*) = 1$, so $s = 1$ by the previous paragraph and $L/O_2(L) \cong A_6$. Thus (1) and (2) hold. By 13.4.3.2, $|Z : Z_L| = 2$, so as $z \in Z(H)$ does not lie in U_1,

$$1 \neq Z_L \cap \langle z \rangle (Z \cap U_1) =: Z_1$$

and $H = IC_H(U_1) = IC_H(Z_1) = I(H \cap M)$, where the final equality holds as $C_G(Z_1) \leq M = !\mathcal{M}(LT)$. As $C_H(V_H) \leq M$, $|H : H \cap M| = |I : O_2(I)| = 3$, so $O^{\{2,3\}}(H) \leq C_M(z)$. Then applying 13.3.9 to $O^{\{2,3\}}(H)$ in the role of "Y", we conclude that H is a $\{2,3\}$-group. So as $L_1 = O^{3'}(H \cap M)$, $H = I(H \cap M) = IL_1T$, with $H/O_2(H) \cong S_3 \times S_3$, since $R_1 \leq C_H(L_1/O_2(L_1))$. Thus (3) holds.

We must treat case (ii) of 13.4.4.2, where $O_2(H) < C_H(V_H)$ with $O^2(C_H(V_H)) = L_+ = L_1$ or $L_{1,+}$, when $L/O_2(L) \cong A_6$ or $\hat{A}_6$, respectively, and R_1 is Sylow in the normal subgroup $H_1 := C_H(L_+/O_2(L_+))$ of H. Thus $I = \langle J(R_1)^H \rangle \leq H_1$, and hence $R_1 \in Syl_2(IR_1)$.

Assume that $L/O_2(L) \cong \hat{A}_6$. As $L_{1,+} = O^2(C_H(V_H))$, $L_1^* = L_0^*$ is of order 3, and hence $s = 1$ and $L_1/O_2(L_1) \cong E_9 \cong O^2(I^*) \times L_1^*$ by an earlier remark. Therefore as $m_3(H) \leq 2$, $O^2(I)L_1/O_2(O^2(I)L_1) \cong 3^{1+2}$. Then $O^2(I)$ normalizes $O^2(L_1 O_2(O^2(I)L_1)) = L_1$, so that $I \leq N_G(L_1) \leq M$ by 13.2.2.9, contrary to $I \not\leq M$.

Therefore $L/O_2(L) \cong A_6$, so (1) holds. If $s = 1$, then (2) holds, and an argument above shows that (3) holds. Thus we may assume that $s = 2$. Then as $L_1 = L_+ = O^2(C_H(V_H))$ and $m_3(H) \leq 2$, $I/O_2(I) \cong E_4/3^{1+2}$ with $L_1 = O^2(O_{2,\Phi}(I))$. This is impossible, since $R_1 \in Syl_2(IR_1)$, and $J(R_1)$ centralizes $L_1/O_2(L_1)$. This completes the proof of 13.4.8. □

PROPOSITION 13.4.9. *If G_z is solvable then $G \cong Sp_6(2)$.*

PROOF. Assume G_z is solvable. Then using B.2.14 as usual, the pair $H := G_z$, $V_H := \langle Z^{G_z} \rangle$ satisfy the hypotheses of 13.4.8. Therefore by 13.4.8.3, $H = IL_1T$, where $I := \langle J(R_1)^H \rangle$ and $H/O_2(H) \cong S_3 \times S_3$. Also $M = LTC_M(V) = LC_M(z) = L(H \cap M)$ and $H \cap M = L_1T$, so $M = LT$.

We next check next that the hypotheses of Proposition 13.4.7 are satisfied with IT, L_2T in the roles of "H_1, H_2": For example, L_2 has at least three noncentral 2-chief factors, two on V and one on $O_2(\bar{L}_2)$, giving (a). Further $Z_L = C_Z(L_2T)$ is of index 2 in Z by 13.4.3.2; while $C_Z(IT)$ is of index 2 in Z as $m([V_H, I]) = 2$ by 13.4.8.2, and $V_H = [V_H, I]C_Z(I)$ by B.2.14, so that (b) holds. Suppose $H_0 := \langle IT, L_2T \rangle \in \mathcal{H}(T)$. As $H = IL_1T \not\leq M$, $I \not\leq M$, so $H_0 \not\leq M = !\mathcal{M}(LT)$ and hence $L \not\leq H_0$. But $L_2 \leq H_0$ and L_2T is maximal in $LT = M$, so $L_2 = O^{3'}(H_0 \cap L) = O^{3'}(H_0 \cap M)$ since $L/O_2(L) \cong A_6$ by 13.4.8.1. Hence $L_1 \cap H_0 = O_2(L_1)$. Further $Z_L = C_Z(L_2)$ and $C_G(Z_L) \leq M$, so $L_2 = O^{3'}(C_{H_0}(Z_L))) \trianglelefteq C_{H_0}(Z_L)$. As $G_z = H = IL_1T$ with $L_1 \cap H_0 = O_2(L_1)$, $O^2(I) = O^{3'}(C_{H_0}(z)) \trianglelefteq C_{H_0}(z)$, and $C_{H_0}(C_Z(I)) \leq C_{H_0}(z)$. Hence (c) holds. This completes the verification of the hypotheses of Proposition 13.4.7.

Now by 13.4.7.1, $H_0 \in \mathcal{H}(T)$ and $m(Z) = 2$. Therefore $m(V_H) = 3$ as $z \notin [V_H, I] \cong E_4$. Furthermore one of the cases (i)–(iv) holds. As $L_2 = O^2(H_2)$, conclusion (iii) is ruled out by 13.4.7.3, and conclusion (iv) is ruled out by 13.4.7.4. If $[O^2(I), L_2] \leq O_2(O^2(I))$, then $LT = \langle L_1T, L_2T \rangle \leq N_G(O^2(I))$, contrary to $I \not\leq M = !\mathcal{M}(LT)$; this rules out conclusion (i). Thus H_0 satisfies conclusion (ii), and so $H_0/O_2(H_0) \cong L_3(2)$.

Let $E_0 := M$, $E_1 := H$, $E_2 := H_0$, $\mathcal{F} := \{E_0, E_1, E_2\}$, and $E := \langle \mathcal{F} \rangle$. We show that $(E, \mathcal{F})$ is a C_3-system as defined in section I.5. First hypothesis (D5) holds as $Z_V \leq Z(E_0)$. By 13.4.8.1, $E_0/O_2(E_0) \cong A_6$ or S_6, verifying hypothesis (D1). We have already observed that hypothesis (D2) holds, and hypothesis (D3) holds by construction. Finally as $M \in \mathcal{M}$ and $H \not\leq M$, $\ker_T(E) = 1$, so hypothesis (D4) is satisfied.

As $(E, \mathcal{F})$ is a C_3-system, $E \cong Sp_6(2)$ by Theorem I.5.1. Thus it remains to show that $E = G$. To do so we appeal to a fairly deep result on groups disconnected at the prime 2, which we used earlier in our appeal to Goldschmidt's Theorem in chapter 2. Let $W := O_2(E_2)$; as $E \cong Sp_6(2)$, W is the core of the permutation module for E_2/W and $W = J(T)$. Thus H.5.3.4 tells us that E_2 has four orbits β_1, α_2, γ_2, β_3 on $W^\#$, consisting of vectors of weights 6, 4, 2, 4, and the orbits have length 7, 7, 21, 28, respectively. As $W = J(T)$, E_2 controls G-fusion in W by Burnside's Fusion Lemma A.1.35. As $E \cong Sp_6(2)$, it follows from [**AS76a**] that E has four classes of involutions, determined by the Suzuki type of each on the natural module—so these orbits contain representatives for the classes, namely the Suzuki types b_1, a_2, c_2, b_3 suggested by the notation above. Hence E controls G-fusion of its involutions. As $M = C_G(Z_V) \leq E$, it follows that E is the unique fixed point on G/E of a generator d of Z_V. For $j \notin \beta_3$, we may choose $T \in Syl_2(C_{E_2}(j))$, so that $F^*(C_G(j)) = O_2(C_G(j))$ by 1.1.4.6; hence $d \in O_2(C_G(j))$, so E is the unique fixed point of $O_2(C_G(j))$ on G/E, and hence $C_G(j) \leq E$.

Set $D := d^G$. We claim that D is product-disconnected in G with respect to E, in the sense of Definition ZD on page 20 of [**GLS99**]; cf. the proof of I.8.2. Condition (a) of that definition is trivial. Since $E \cong Sp_6(2)$ we check that $d^E \cap T = b_1 \cap T = \beta_1$. Since E controls G-fusion of its involutions, $D \cap E = d^E$, while by the previous paragraph, $C_G(d) \leq E$. Thus condition (b) of the definition holds by A.1.7.2. Finally consider any $e \in C_D(d) - \{d\}$. Then $e \in C_G(d) \le E$, so since E controls fusion of its involutions, by conjugating in E we may assume that $e \in \beta_1$. Then $de \in \gamma_2$, so that $C_G(de) \le E$ by the previous paragraph, verifying condition (c) of the definition and establishing the claim.

Therefore as G is simple, we may apply Corollary ZD on page 22 of [**GLS99**], to conclude that G is a simple Bender group, and E is a Borel subgroup, which is strongly embedded in G. This is impossible by 7.6 in [**Asc94**], as E has more than one class of involutions. □

13.4.3. Eliminating the case $\mathbf{G_z}$ nonsolvable. If G_z is solvable then Theorem 13.4.1 holds by Proposition 13.4.9. Thus we may assume for the remainder of the proof of the Theorem that G_z is not solvable, and we will work to a contradiction.

In particular there exist nonsolvable members of $\mathcal{H}_z$. Our first result is a refinement of the information produced earlier in 13.4.6.

LEMMA 13.4.10. *Let $H \in \mathcal{H}_z$ be nonsolvable. Then*

(1) There exists $K \in \mathcal{C}(H)$, $K \not\le M$, $K \in \mathcal{L}_f^*(G,T)$, and $K \trianglelefteq H$. Set $V_H := \langle Z^K \rangle$ and $(KT)^* := KT/C_{KT}(V_H)$; then $K^* \cong L_3(2)$ or A_6.

(2) $L_1 \le K$.

(3) $L/O_2(L) \cong A_6$ and if $K^* \cong A_6$, then $K/O_2(K) \cong A_6$.

(4) Let $V_K := [V_H, K]$. Then $V_K = [R_2(KT), K]$ and either V_K is the natural module for K^*, or V_K is a 5-dimensional module for $K^* \cong A_6$ with $\langle z \rangle = C_{V_K}(K)$.

(5) $|Z| = 4$, so $Z = Z \cap V = V_1$, $|Z_L| = 2$, and $C_Z(K) = \langle z \rangle$.

(6) $Z_V = Z_L$.

(7) $M = LT$ and $H = KT = G_z$. Thus $\mathcal{H}_z = \{G_z\}$ and $V_H = \langle Z^H \rangle$. Furthermore G_z contains a unique member of $\mathcal{H}_*(T,M)$: the minimal parabolic of H over T distinct from L_1T.

(8) Let $H_2 \in \mathcal{H}(T)$ be the minimal parabolic of H distinct from L_1T, and set $H_0 := \langle H_2, L_2T \rangle$. Then $H_0 \in \mathcal{H}(T)$, H_2 is the unique member of $\mathcal{H}_*(T,M)$ in H_0, and either:

(i) Conclusion (i) of 13.4.7.1 holds, z is of weight 4 in V, and $Z_V \le V_K$. Further if V_K is a 5-dimensional module for $K^* \cong A_6$, then Z_V is of weight 4 in V_K.

(ii) Conclusion (ii) of 13.4.7.1 holds and $H_0 = N_G(J(T)) \in \mathcal{M}(T)$.

PROOF. First by 13.4.6.1, there exists $K \in \mathcal{C}(H)$, $K \in \mathcal{L}_f^*(G,T)$, $K \trianglelefteq H$, and $K \not\le M$. By 13.4.5.1, $K/O_2(K)$ is A_5, $L_3(2)$, A_6, or $\hat{A}_6$.

Set $U := [R_2(KT), K]$. By 13.4.5.3 with KT in the role of "H", there is $W_K \in Irr_+(K, R_2(KT), T)$, and for each such W_K, $W_K = \langle (Z \cap W_K)^K \rangle \le U$ and W_K is either a natural module for $K/O_{2,Z}(K)$ or a 5-dimensional module for $K/O_{2,Z}(K) \cong A_6$.

Now $K = [K, J(T)]$ by 13.4.6.3, so Theorem B.5.1 shows that either $U \in Irr_+(K, R_2(KT))$, or U is the sum of two isomorphic natural modules for $K^* \cong L_3(2)$, which are T-invariant since then $T^* \le K^*$. In particular U is the A_5-module if $K^* \cong A_5$, and $U = \langle (Z \cap U)^K \rangle \le V_H$, so $U = V_K$. By B.2.14, $V_H = UC_Z(K)$, so $C_{KT}(U) = C_{KT}(V_H)$.

As $V_H = UC_Z(K)$, $C_{KT}(Z) = C_K(Z \cap U)T$, so that $C_K(Z)^*T^*$ is a maximal parabolic of K^*T^* containing T^*. Now $C_K(Z)T = L_KT$, where $L_K :=$

$O^2(C_K(Z)) \leq M$ by 13.4.2.2. Then $L_K \leq \theta(M) = L$ by 13.4.2.5, so that $L_K \leq O^2(C_L(Z)) \leq O^2(C_L(z)) = L_1$. Let $L_C := O^2(C_{L_1}(K/O_2(K))$; as $L_K \leq L_1 \leq H \leq N_G(K)$ and $Out(K^*)$ is a 2-group, it follows that $L_1 = L_K L_C$. In each case $L_K/O_2(L_K)$ is an elementary abelian 3-group of rank 1 or 2; similarly $L_1/O_2(L_1)$ is of rank 1 or 2 for $L/O_2(L)$ isomorphic to A_6 or $\hat{A}_6$, respectively. In particular if $L/O_2(L) \cong A_6$, then $3 \leq |L_K : O_2(L_K)| \leq |L_1 : O_2(L_1)| = 3$, so equality holds and $L_K = L_1$.

Now set $R := O_2(L_1T)$, $S := \text{Baum}(R)$, and $T_K := R(T \cap K)$. Then $T_K \in Syl_2(KT_K)$. Further $[T \cap K, L_C] \leq O_2(K) \leq R$, so as $L_1 = L_K L_C$, $O_2(L_K T_K) = O_2(L_1 T_K) = R$. Also $C_{KT_K}(Z) = L_K T_K$, so $R = O_2(L_K T_K) = O_2(C_{KT_K}(Z))$.

We are now in a position to complete the proof of (1). We showed that $K^* \cong A_5$, $L_3(2)$, or A_6; thus it remains to assume $K^* \cong A_5$, and derive a contradiction. In this case we saw that U is the A_5-module, and we also saw that $R^* = O_2(L_K^* T^*)$, so $R^* = T_K^*$. Then R^* contains no FF*-offenders on U by B.3.2.4, so by B.2.10.1,

$$S = \text{Baum}(R) = \text{Baum}(O_2(KT_K)) \trianglelefteq KT_K.$$

If C is a nontrivial characteristic subgroup of S normal in LT, then $K \leq N_G(C) \leq M = !\mathcal{M}(LT)$, contrary to $K \not\leq M$; hence no such C exists. This eliminates the case $L/O_2(L) \cong \hat{A}_6$, since there 13.2.2.8 shows that each C is indeed normal in LT. Thus $L/O_2(L) \cong A_6$, so by an earlier remark $L_K = L_1$, and hence $R = R_1$. Now 13.2.2.7 shows that L is an A_6-block. Therefore L_1 has exactly two noncentral 2-chief factors; so also K is an A_5-block since $L_1 = L_K$. As $S = \text{Baum}(O_2(KT_K))$, S centralizes $O_2(K)$ by C.1.13.c; so by B.2.3.7, each $A \in \mathcal{A}(S)$ contains $O_2(K)$. Then $[A, K] \leq [O_2(KS), K] = O_2(K) \leq A$, so $A \trianglelefteq KA$. However $m_2(O_2(LT)) = m_2(S)$ by 13.2.2.6, so that $\mathcal{A}(O_2(LT)) \subseteq \mathcal{A}(S)$; hence $J(O_2(LT)) \trianglelefteq KT$, so that $K \leq M$ for our usual contradiction. Therefore K^* is not A_5, completing the proof of (1).

We next prove (2), so we suppose that $L_1 \not\leq K$, and derive a contradiction. If K^* is A_6, then $K = \theta(H)$ by 12.2.8, and hence $L_1 \leq K$, contrary to our assumption. Thus K^* is $L_3(2)$ by (1). If $L/O_2(L) \cong A_6$, then we saw earlier that $L_1 = L_K \leq K$, contrary to our assumption. Thus $L/O_2(L) \cong \hat{A}_6$. As L_0 and $L_{1,+}$ are the T-invariant subgroups with images of order 3 in $L_1/O_2(L_1)$, we conclude that $\{L_C, L_K\} = \{L_0, L_{1,+}\}$. Indeed as $K \not\leq M$, while K acts on $O^2(L_C O_2(K)) = L_C$ and $N_G(L_0) \leq M$ by 13.2.2.9, we conclude that $L_K = L_0$ and $L_{1,+} = L_C$.

As $K^* \cong L_3(2)$, $R^* = O_2(L_K^* T_K^*)$ is the unipotent radical of the maximal parabolic $L_K^* T_K^*$ of K^* stabilizing $Z \cap U$. As $L/O_2(L) \cong \hat{A}_6$, $S \trianglelefteq LT$ by 13.2.2.8, so no nontrivial characteristic subgroup of S is normal in KT, since $K \not\leq M$. Therefore we may apply C.1.37 to conclude that K is an $L_3(2)$-block. But then L_K has just two noncentral 2-chief factors, whereas we saw earlier that $L_K = L_0$, and L_0 has at least three noncentral chief factors on an L-chief section of $O_2(L)$ not centralized by L_0. This contradiction shows that $L_1 \leq K$, completing the proof of (2).

Recall that $L_K \leq L_1$, while by (2), $L_1 \leq L_K$, so $L_1 = L_K$. Thus $L/O_2(L) \cong \hat{A}_6$ iff $m_3(L_1) = 2$ iff $m_3(L_K) = 2$ iff $K/O_2(K) \cong \hat{A}_6$. But then by 13.2.2.8 applied to both LT and KT,

$$J(O_2(LT)) = J(O_2(L_1T)) = J(O_2(L_KT)) = J(O_2(KT)),$$

so that $K \leq M$ for usual contradiction, establishing (3).

As $L/O_2(L) \cong A_6$ by (3), $R = R_1$. Also either $U \in Irr_+(K, R_2(KT))$, or $K^* \cong L_3(2)$ and U is a sum of two isomorphic natural modules. Suppose that the latter case holds. Again R^* is the unipotent radical of the parabolic $C_{K^*}(Z \cap U)T_K^*$ fixing a point in each summand of U, so we can finish much as in the proof of (1): R^* contains no FF*-offenders on U, so $S \trianglelefteq KT_K$ by B.2.10.1. Then no nontrivial characteristic subgroup of S is normal in LT, so L is an A_6-block by 13.2.2.7, and L_1 has exactly two noncentral 2-chief factors. This is a contradiction since $L_1 \le K$ by (2), so that L_1 has a noncentral chief factor on each summand of U, plus one more on $O_2(L_1^*)$.

This contradiction shows that $U \in Irr_+(K, R_2(KT))$. Thus from earlier remarks, $V_H = UC_Z(K)$ and U is the natural module for K^* or a 5-dimensional module for $K^* \cong A_6$. In particular, $Z = (Z \cap U)C_Z(K)$. Next $C_{Z_L}(K) = 1$, as otherwise $K \le C_G(C_{Z_L}(K) \le M$ by 13.4.2.2. By 13.4.3.2, $|Z : Z_L| = 2$, so $|C_Z(K)| \le 2$, and hence $C_Z(K) = \langle z \rangle$. In particular if $K^* \cong A_6$ and $m(U) = 5$, then $C_U(K) = \langle z \rangle$, establishing (4). Also $|Z \cap U : C_{Z \cap U}(K)| = 2$, so as $Z = (Z \cap U)C_Z(K)$ and $C_Z(K) = \langle z \rangle$, (5) and (6) hold.

Using (3) and 13.4.6.5, $M = LT$ so $C_G(Z) = L_1T$. Let $W_H := \langle Z^H \rangle$ and $U_H := [W_H, K]$. By 13.4.5.4, $O_{2,Z}(K) = C_K(W_K) = C_K(U_H) = C_K(W_H)$. As $K = [K, J(T)]$, Theorems B.5.1 and B.5.6 say that either $U_H \in Irr_+(K, W_H)$, so that $U_H = U$, or $K^* \cong L_3(2)$ and U_H is the sum of two isomorphic natural modules for $K^* \cong L_3(2)$. Assume the latter holds. Then as K is irreducible on U and $O_2(H/C_H(V_H)) = 1$ by B.2.14, $Aut_H(U_H) = L_3(2) \times L_2(2)$ and U_H is the tensor product module. Then $Aut_R(U_H)$ contains no FF*-offenders, so as in earlier arguments we obtain a nontrivial characteristic subgroup of R normal in KT and M, a contradiction. Thus $U_H = U$.

By A.1.41, $C_H(K/O_2(K)) \le C_H(U)$, so as $Z = (Z \cap U)\langle z \rangle$ and $Out(K/O_2(K))$ is a 2-group, $H = KTC_H(K/O_2(K)) = KC_H(U) = KC_H(Z) = KL_1T = KT$ since $L_1 \le K$ by (2). Since G_z satisfies the hypotheses for H, we conclude $G_z = KT = H$. Thus (7) holds since $K \not\le M$.

Define H_2 and H_0 as in (8), and let $H_1 := L_2T$. Observe that the hypotheses of Proposition 13.4.7 are satisfied: For example (5) establishes part (b), with $Z_V = C_Z(L_2)$ and $\langle z \rangle = C_Z(H_2)$. Also if $X \in \mathcal{H}(H_0)$, then $L_1 \not\le X$: as otherwise $M = LT = \langle L_1, L_2T \rangle \le X$ whereas $H_2 \le X$ but $H_2 \not\le M$ by (7). Therefore as L_2T is maximal in $M = C_G(Z_V)$ and H_2 is maximal in $H = G_z$, we conclude that $L_2T = C_X(Z_V)$, so $L_2 = O^{3'}(C_{H_0}(Z_V))$; and $H_2 = C_X(z)$, so $O^2(H_2) = O^{3'}(C_{H_0}(z))$. Thus part (c) holds. Finally L_2 has at least three noncentral 2-chief factors, two on V and one on $O_2(\bar{L}_2)$, giving part (a). We conclude from 13.4.7.1 that $H_0 \in \mathcal{H}(T)$ and one of conclusions (i)—(iv) of that result holds. In applying 13.4.7, we interchange the roles of "H_1" and "H_2", so the hypothesis "$K_2 = L_2$" in parts (3) and (4) of 13.4.7 also holds; hence conclusions (iii) and (iv) do not hold here.

Suppose conclusion (ii) holds. Then $J(T) \trianglelefteq H_0$. Further for any $X \in \mathcal{H}(H_0)$,

$$C_X(Z) = C_X(Z_V) \cap C_X(z) = L_2T \cap H_2 = T,$$

so we conclude from 13.4.7.2 that $X = H_0$. Thus $H_0 \in \mathcal{M}(T)$, and in particular $H_0 = N_G(J(T))$. That is, conclusion (ii) of (8) holds.

Finally suppose that conclusion (i) of 13.4.7.1 holds. Then $V_0 := \langle Z^{H_0} \rangle$ is of rank 4. Set $K_2 := O^2(H_2)$; then $[L_2, K_2] \le O_2(L_2) \cap O_2(K_2)$, so L_2 and K_2

normalize each other. Thus L_2 centralizes $U_2 := \langle Z_V^{K_2} \rangle$, and K_2 centralizes $U_1 := \langle z^{L_2} \rangle$. But as conclusion (i) of 13.4.7.1 holds, $C_{V_0}(L_2) =: U_2' \cong E_4$, so that $U_2 = U_2' \cong E_4$; in particular, $Z_V \leq U_2 \leq V_K$. Similarly $U_1 \cong E_4$, so it follows that z is of weight 4 rather than 2 in V. Similarly Z_V is of weight 4 in V_K when V_K is the 5-dimensional module for $K^* \cong A_6$. Thus conclusion (i) of (8) holds, and the proof of 13.4.10 is complete. □

LEMMA 13.4.11. *(1)* $L/O_2(L) \cong A_6$.
(2) $M = LT$.
(3) $\mathcal{H}_z = \{G_z\}$.

PROOF. Part (1) follows from 13.4.10.3, and (2) and (3) follow from 13.4.10.7. □

LEMMA 13.4.12. *(1)* $Z = V_1$ *has rank* 2*, and there exists a unique* $z \in Z^\#$ *such that* $C_G(z) \not\leq M$.
(2) There is a unique member H_2 *of* $\mathcal{H}_*(T, M)$ *contained in* G_z.

PROOF. By 13.4.10.5, $Z = V_1$ is of rank 2. Recall $z \in V_1^\#$ with $G_z \not\leq M$, and z has weight 2 or 4 in V while a generator of Z_V is of weight 6. Let z_k denote the element of V_1 of weight k and choose m with $G_{z_m} \not\leq M$. In this subsection G_{z_m} is not solvable, so by parts (1) and (7) of 13.4.10, $G_{z_m} = K_{z_m}T$ for $K_{z_m} \in \mathcal{C}(G_{z_m})$, and there is a unique $H_{m,2} \in \mathcal{H}_*(T, M)$ contained in G_{z_m}. Set $K_{m,2} := O^2(H_{m,2})$.

As $H_{m,2}$ is the unique member of $\mathcal{H}_*(T, M)$ contained in G_{z_m}, (2) holds. Moreover by 13.4.10.5, $C_Z(H_{m,2}) = \langle z_m \rangle$. As $z_2 \neq z_4$, $H_{2,2} \neq H_{4,2}$.

It remains to prove the final statement in (1), so we assume that $G_{z_m} \not\leq M$ for both $m = 2$ and 4. Set $H_{m,0} := \langle L_2T, H_{m,2} \rangle$. Then by 13.4.10.8, $H_{m,0} \in \mathcal{H}(T)$, and $H_{m,0}$ satisfies conclusion (i) or (ii) of both 13.4.7.1 and 13.4.10.8. As z_2 has weight 2 in V, $H_{2,0}$ satisfies conclusion (ii) rather than (i) of 13.4.10.8, and hence

$$H_{2,0} = N_G(J(T)) \in \mathcal{M}(T).$$

Suppose that $H_{4,0}$ also satisfies conclusion (ii) of both results. Then by 13.4.10.8, $H_{2,0} = N_G(J(T)) = H_{4,0}$ and $H_{m,0}$ contains a unique member $H_{m,2}$ of $\mathcal{H}_*(T, M)$. Therefore $H_{2,2} = H_{4,2}$, contrary to an earlier observation. Hence $H_{4,0}$ satisfies conclusion (i) of both results.

Next let $H_0 := \langle H_{2,2}, H_{4,2} \rangle$. We check that the hypotheses of Proposition 13.4.7 are satisfied: We already observed that $m(Z) = 2$ and $\langle z_k \rangle = C_Z(H_{k,2})$, establishing (b). We saw that $H_{k,2}$ does not centralize z_{6-k}, so $H_0 \not\leq G_{z_k}$ and hence $O^{3'}(C_{H_0}(z_k))T < G_{z_k}$. Now $G_{z_k} = K_{z_k}T$ for $k = 2, 4$, and in each case $K_{k,2}T$ is a maximal subgroup of G_{z_k}, so we conclude $K_{k,2} = O^{3'}(C_{H_0}(z_k))$, giving (c). Finally by 13.4.10.4, K_{z_k} has at least two noncentral 2-chief factors, one in $V_{G_{z_k}}$ and one in $K_{z_k}/C_{K_{z_k}}(V_{G_{z_k}})$, giving (a).

So we may apply 13.4.7. Assume first that H_0 satisfies one of conclusions (ii)–(iv) of 13.4.7.1. Then $H_0 \leq N_G(J(T)) = H_{2,0}$. Recall however that H_0 is generated by distinct members $H_{k,2}$ of $\mathcal{H}_*(T, M)$, whereas $H_{2,2}$ is the unique member of $\mathcal{H}_*(T, M)$ contained in $H_{2,0}$.

Therefore H_0 satisfies conclusion (i) of 13.4.7.1. Thus $K_{2,2} \leq N_G(K_{4,2})$ and hence $H_{2,0} = \langle K_{2,2}, L_2T \rangle \leq N_G(K_{4,2})$, since $H_{4,0}$ also satisfies conclusion (i) of 13.4.7.1. Indeed we saw $H_{2,0} \in \mathcal{M}(T)$, so $H_{2,0} = N_G(K_{4,2})$, and hence $K_{4,2} \leq O_{2,3}(H_{2,0})$. However, we also saw that $H_{2,0}$ satisfies conclusion (ii) of 13.4.7.1, so that $H_{2,0}/O_2(H_{2,0}) \cong L_3(2)$, and hence $O_{2,3}(H_{2,0}) = O_2(H_{2,0})$ is a 2-group. This final contradiction completes the proof of 13.4.12. □

By 13.4.12, there is a unique $z \in V_1^{\#} = Z^{\#}$ with $G_z \not\leq M$. For the remainder of the section, set $H := G_z$ and $V_H := [\langle Z^H \rangle, O^2(H)]$. By 13.4.10.7 there is a unique $H_2 \in \mathcal{H}_*(T, M)$ contained in H. Let $K_2 := O^2(H_2)$, $H_0 := \langle L_2T, H_2 \rangle$, and $V_0 := \langle Z^{H_0} \rangle$.

LEMMA 13.4.13. *(1) $H = KT$ with $K \in \mathcal{L}_f^*(G,T)$, $K/O_2(K) \cong L_3(2)$ or A_6, and V_H is the natural module for $K/O_2(K)$ or a 5-dimensional module for A_6.*

(2) $\langle Z^H \rangle = V_H \langle z \rangle$, so $V_H \in \mathcal{R}_2(H)$.

PROOF. By 13.4.10, (1) holds and $Z = Z_V \langle z \rangle$ where $\langle z \rangle = C_Z(K)$. So by B.2.14, $\langle Z^H \rangle = [Z,K]C_Z(K) = V_H\langle z \rangle$ and $V_H \in \mathcal{R}_2(H)$. □

LEMMA 13.4.14. *H_0 satisfies conclusion (i) or (ii) of 13.4.7.1, and:*

(1) If V_0 is semisimple then z is of weight 4 in V and $Z_V \leq [Z, K_2] \leq V_H$. If further V_H is the 5-dimensional module for A_6, then Z_V is of weight 4 in V_H.

(2) If $H_0/O_2(H_0) \cong L_3(2)$ and V_0 is the core of the permutation module, then either:

(i) $Z_V \not\leq Soc(V_0)$, z is of weight 4 in V, and either

(a) $Z_V \not\leq V_H$, or

(b) V_H is a 5-dimensional module for A_6 and Z_V is of weight 2 in V_H;

or else

(ii) $Z_V \leq Soc(V_0)$, z is of weight 2 in V, $Z_V \leq V_H$; and if V_H is a 5-dimensional module for A_6 then Z_V is of weight 4 in V_H.

PROOF. The initial statement follows from 13.4.10.8. In the remainder of the proof, we extend arguments used in the last few lines of the proof of that result: First z is of weight 4 in V iff $z \in [Z, L_2]$. Further $Z_V \leq [Z, K_2]$ iff $Z_V \leq V_H$ with Z_V of weight 4 in V_H when V_H is of dimension 5. Thus the subcase of conclusion (1) where H_0 is solvable can be treated exactly like the subcase in the earlier proof corresponding to 13.4.10.8i.

So assume $H_0/O_2(H_0) \cong L_3(2)$. Then $Z_V = C_{V_0}(L_2T)$ and $\langle z \rangle = C_{V_0}(K_2T)$. In case (1), where V_0 is semisimple, $C_{V_0}(L_2T) \leq [Z, K_2]$ and $C_{V_0}(K_2T) \leq [Z, L_2]$, completing the proof of (1) in view of the equivalences in the previous paragraph.

It remains to prove (2), so we assume V_0 is the core of the permutation module. Suppose first that $Z_V \not\leq Soc(V_0)$. Then L_2 centralizes the generator for Z_V, which lies in $V_0 - Soc(V_0)$. Thus we may apply section H.5 with L_2, K_2 in the roles of "L_p, L_l": Then $\langle z \rangle = C_{V_0}(K_2T) \leq Soc(V_0)$ by H.5.2.5 and H.5.3.3, and $z \in [Z, L_2]$ by H.5.4.2, so that z is of weight 4 in V. Further $Z_V = C_{V_0}(L_2T) \not\leq [Z, K_2]$ by H.5.4.1. Therefore if $Z_V \leq V_H$, then $z \in Z \leq Z_V[Z, K_2] \leq V_H$, so V_H is a 5-dimensional module for A_6, and hence Z_V is of weight 2 in V_H by our earlier equivalences. Thus either (a) or (b) of (2i) holds in this case.

On the other hand if $Z_V \leq Soc(V_0)$, then the roles of L_2 and K_2 are reversed in the application of section H.5. Thus $Z_V = C_{V_0}(L_2T) \leq [Z, K_2]$ and $z \notin [Z, L_2]$, so that (2ii) holds. □

Recall since $L/O_2(L) \cong A_6$ by 13.4.11 that $R_2 = O_2(L_2T)$.

LEMMA 13.4.15. *Assume for some $g \in G$ that $V_0^g \leq R_2$ and $V \leq R_2^g$. Then $1 = [V, V_0^g]$.*

PROOF. Assume $[V, V_0^g] \neq 1$. By hypothesis $V_0^g \leq R_2$ and $V \leq R_2^g$, so V_0^g and V normalize each other. Let $H_0^{g*} := H_0^g/O_2(H_0^g)$. By 13.4.14, case (i) or (ii) of 13.4.7.1 holds. Now $1 \neq V^* \leq R_2^{g*}$. But in case (i), as R_2^{g*} centralizes L_2^{g*}, $H_0^{g*} \cong S_3 \times S_3$, V^* is of order 2, and $[V_0^g, V] \leq [V_0^g, K_2^g] = [Z, K_2]^g$. Similarly in case (ii), $V^* \leq R_2^{g*} \cong E_4$, so $V^* = R^{*g}$ if $|V^*| > 2$; further by 13.4.7.1, H_0^{g*} contains no transvections on V_0^g.

Suppose first that $|V^*| = 2$. Then $\bar{V}_0^g$ induces a nontrivial group of transvections on V with axis $C_V(V_0^g)$, so as V is a 5-dimensional module for $\bar{L} \cong A_6$, it follows that $[V, V_0^g] = \langle v \rangle$ with v of weight 2 in V. Conjugating in L, we may assume $v \in V_1$. Further $2 = |\bar{V}_0^g| = |V_0^g/C_{V_0^g}(V)|$. Hence V^* is a group of transvections on V_0^g with center $\langle v \rangle$, so $H_0^{g*} \cong S_3 \times S_3$ and $v \in [V_0^g, V] \leq [Z, K_2]^g$ by paragraph one. But by 13.4.14.1, $Z_V \leq [Z, K_2]$, so $\langle v \rangle$ is conjugate in G to Z_V of weight 6, contradicting 13.2.2.5 since v has weight 2.

Therefore $|V^*| > 2$, so by paragraph one, $H_0^{g*} \cong L_3(2)$ and $V^* = R^{g*}$ is of order 4. From the action of H_0 on V_0, $[V, V_0^g] = C_{V_0^g}(V)$ and $V_0^g/C_{V_0^g}(V)$ are of rank 3: this is clear if V_0 is semisimple, and it follows from H.5.2 if V_0 is the core of the permutation module. Hence $\bar{V}_0^g = \bar{R}_2$ and $m(\bar{R}_2) = 3$. As $m(\bar{R}_2) = 3$, $Z_V \leq [V, R_2] = [V, V_0^g] = C_V(V_0^g)$. Then Z_V is weakly closed in $[V, V_0^g]$ by 13.2.2.5. Also $(L_2T)^g$ acts on $[R_2^g, V_0^g] = [V, V_0^g]$, and then also on the subgroup Z_V weakly closed in $[V, V_0^g]$, so $(L_2T)^g \leq M$. Then T^g is conjugate to T in M, so as $N_G(T) \leq M$ by Theorem 3.3.1, $g \in M$. Now as $\bar{R}_2 = \bar{V}_0^g \leq \bar{R}_2^g = O_2(\bar{L}^g\bar{T}^g)$, $L_2T = (L_2T)^g$. As $M = LT$ by 13.4.11, L_2T is maximal in M, so $g \in L_2T \leq H_0$, so $H_0 = H_0^g$. Thus $V_0^g = V_0 \trianglelefteq T$, so as $C_{V_0^g}(V) = [V, V_0^g] \leq V \cap V_0^g$,

$$[O_2(LT), V_0] \leq C_{V_0}(V) \leq V.$$

Therefore $[O_2(LT), L] = V$ and L is an A_6-block. Set $K_0 := O^2(H_0)$; similarly $[O_2(H_0), V] \leq V_0$ and then $[O_2(H_0), K_0] = V_0$.

If $V_0 = U_1 \oplus U_2$ is the sum of non-isomorphic 3-dimensional modules for K_0, we saw that $Z_V \leq U := U_i$ for $i := 1$ or 2 during the proof of 13.4.14.1. If instead V_0 is the core of the permutation module and $Z_V \leq Soc(V_0)$, set $U := Soc(V_0)$. In either of these two cases, since $V^* = R_2^*$ and L_2 centralizes Z_V, $[U, V] = C_U(L_2) = Z_V = C_U(V)$, so U induces a 4-group of transvections on V with center Z_V, impossible as $C_M(V/Z_V) = C_M(V)$ by 13.4.2.4. Therefore we are in case (i) of 13.4.14.2, where V_0 is the core of the permutation module and $Z_V \not\leq Soc(V_0)$; so by that result, z is of weight 4 in V.

As L is an A_6-block, L_1 has two noncentral 2-chief factors, so K is an $L_3(2)$-block or an A_6-block using 13.4.13.1. Further as z is of weight 4 in V, $\langle z \rangle = [V \cap O_2(L_1), O_2(L_1)]$, so that $z \in V_H$. Therefore since $z \in Z(H)$, V_H is the 5-dimensional module for the A_6-block K. By 13.4.12.1, $Z = Z_V\langle z \rangle$ is of order 4, and by symmetry between L and K, $Z \leq V_H$ and $Z \cap O_2(L_1) \not\leq Z(K) \cap V_H = \langle z \rangle$; so as $z \in L_1$, $Z \leq L_1$. Calculating in the A_6-block K, $|Z(K)| \leq 4$ and $O_2(L_1/Z(K)) \cong Q_8^2$, so $|Z(O_2(L_1)/\langle z \rangle)| \leq 4$. Therefore as $[V \cap O_2(L_1), O_2(L_1)] = \langle z \rangle$ and $|V \cap O_2(L_1)| = 8|Z_V \cap O_2(L_1)| = 16$, we have a contradiction. □

LEMMA 13.4.16. *If $g \in G$ with $V_0^g \leq R_2$ and $V_0 \leq R_2^g$, then $[V_0, V_0^g] = 1$.*

PROOF. Assume V_0^g is a counterexample, and let $H_0^* := H_0/C_{H_0}(V_0)$. Interchanging V_0 and V^g if necessary, we may assume that $m(V_0^{g*}) \geq m(V_0/C_{V_0}(V_0^g))$, so V_0 is an FF-module for H_0^*. The modules V_0 in case (ii) of 13.4.7.1 are not

FF-modules by Theorem B.5.1, so by 13.4.14, we are in case (i) of 13.4.7.1. Arguing as in the proof of the previous lemma, $m(R_2^*) = 1$ and then $m(V_0^{g*}) = 1 = m(V_0/C_{V_0}(V_0^g))$ and $[V_0, V_0^g] = Z_V$. By symmetry between V_0 and V_0^g, $Z_V^g = [V_0, V_0^g] = Z_V$, so $g \in N_G(Z_V) = M = N_G(V)$ by 13.4.2.1. Then $V = V^g \leq R_2^g$, so by 13.4.15, $[V, V_0^g] = 1$. Thus as $V = V^g$, also $[V, V_0] = 1$.

Let $U_0 := [Z, K_2]$, so that U_0 is a 4-group as case (i) of 13.4.7.1 holds. As H_0 is solvable, z is of weight 4 in V and $Z_V \leq U_0 \leq V_H$ by 13.4.14.1. Therefore $U_0 = \langle Z_V^{K_2} \rangle$ and $C_{K_2T}(U_0) = O_2(K_2T)$. By 13.4.12.1, $C_G(v) \leq M = N_G(V)$ for each $v \in V^\#$ not of weight 4, so by 13.4.2.3, V is the unique member of V^G containing v. But up to conjugation under L, $\langle e_{1,2,3,4}, e_{1,2,5,6} \rangle$ is the unique maximal subspace U of V all of whose nontrivial vectors are of weight 4, so $r(G, V) \geq 3$.

Let $W_0 := W_0(T, V)$. We claim $[V, W_0] = 1$, so that $N_G(W_0) \leq M$ by E.3.34.2: For suppose $A := V^y \cap M$ with $\bar{A} \neq 1$. Assume for the moment that also $V \leq M^y$. Then $1 \neq [V, A] \leq V \cap V^y$, so by the previous paragraph, $[V, A]^\#$ contains only vectors of weight 4. We conclude that all involutions of $\bar{A}$ are of cycle type 2^3, and hence $|\bar{A}| = 2$ and $|V : C_V(V^y)| \geq |V : C_V(A)| = 4$. Therefore $A < V^y$ when $V \leq M^y$—since if $A = V^y$, then we have symmetry between V and V^y, so that $2 = |\bar{A}| = |V : C_V(V^y)| = 4$.

Now assume $A = V^y$; then $V \not\leq M^y$ by the previous paragraph. Therefore $m(\bar{A}) \geq r(G, V) \geq 3$, and hence $m(\bar{A}) = 3$ as $m_2(\bar{M}) = 3$. Further

$$U = \langle C_V(\bar{B}) : 1 \neq \bar{B} \leq \bar{A} \rangle = \langle C_V(D) : m(A/D) < 3 \rangle \leq M^y.$$

Thus $U < V$, which is not the case if $\bar{A}$ is conjugate to $\bar{R}_2$. Therefore $\bar{A}$ is conjugate to $\bar{R}_1$, and then $m(V/U) = 1$ so that $U = V \cap M^y$ and $m(U/C_U(A)) = 2$. But applying the previous paragraph with U, A in the roles of "A, V", we conclude that $m(U/C_U(A)) = 1$. This contradiction establishes the claim that $W_0 \leq C_T(V)$ and $N_G(W_0) \leq M$.

Thus W_0 is not normal in K_2T, as $K_2 \not\leq M$, and hence $W_0 \not\leq O_2(K_2T) = C_{K_2T}(U_0)$ by E.3.15. Therefore there is $D := V^x \leq T$ for some $x \in G$ with $[U_0, D] \neq 1$. But $|D : C_D(U_0)| = 2$ as $|T : O_2(K_2T)| = 2$, so $U_0 \leq C_G(C_D(U_0)) \leq N_G(V^x)$ as $r(G, V) \geq 3$. Then as D does not centralize U_0, $Z_V = [U_0, D] \leq D$. By 13.4.2.3, V is the unique member of V^G containing Z_V, so $D = V$. But now $[D, U_0] \neq 1$, whereas $U_0 \leq V_0$ and $[V, V_0] = 1$ by the first paragraph. This contradiction completes the proof of 13.4.16. □

Our final lemma shows that the 2-locals M and H_0 resemble the parabolics $\dot{P}_1$ and $\dot{P}_3$ of $Sp_6(2)$, except that z is of weight 2 in V and $Z_V \leq Soc(V_0)$. Still with this information we will be able to obtain a contradiction to our assumption that G_z is not solvable, completing the proof of Theorem 13.4.1.

LEMMA 13.4.17. *(1) z is of weight 2 in V.*

(2) There exists $g \in H$ such that $[V, V^g] = \langle z \rangle \leq V^g$.

(3) $H_0/O_2(H_0) \cong L_3(2)$, V_0 is the core of the permutation module, $Z_V \leq Soc(V_0)$, $Z_V \leq V_H$, and V_H is not a 5-dimensional module if $H/O_2(H) \cong A_6$.

(4) H is transitive on $V_H^\#$, and each subgroup of V_H of order 2 is in Z_V^G.

(5) $r(G, V) \geq 3$.

(6) For each $g \in G - M$, $V^\# \cap V^g$ consists of elements of weight 2.

PROOF. Let $G_1 := LT = M$ and $G_2 := H_0$, and form the coset graph Γ with respect to these groups as in section F.7. Adopt the notational conventions that section including Definition F.7.2, where in particular γ_0, γ_1 are the cosets G_1, G_2. For $\gamma = \gamma_0 g$ set $V_\gamma := V^g$, while for $\gamma = \gamma_1 g$ set $V_\gamma := V_0^g$. Let

$$\alpha := \alpha_0, \ldots, \alpha_n =: \beta$$

be a geodesic in Γ, of minimal length n subject to $V_\alpha \not\leq G_\beta^{(1)}$; such an n exists by F.7.3.8. As $G_\beta^{(1)} = O_2(G_\beta) = C_{G_\beta}(V_\beta)$ for each $\beta \in \Gamma$, $[V_\alpha, V_\beta] \neq 1$, and so we have symmetry between α and β. This symmetry is fairly unusual among our applications of section F.7, as we almost always consider only geodesics whose orgin is conjugate to γ_0; however the approach in this lemma is the one most commonly used in the amalgam method in the literature. By minimality of n,

$$V_\alpha \leq G_{\alpha_{n-1}}^{(1)} \leq G_\beta,$$

so V_α acts on V_β, and by symmetry, V_β acts on V_α. By F.7.9.1, $V_\alpha \leq O_2(G_{\alpha,\alpha_{n-1}}^{(1)})$, and

$$O_2(G_{\alpha,\alpha_{n-1}}^{(1)}) = O_2(H_0 \cap M) = O_2(L_2T)^g = R_2^g,$$
$$\text{for } g \in G \text{ with } \{\gamma_0 g, \gamma_1 g\} = \{\alpha_{n-1}, \beta\} \qquad (*)$$

using transitivity of $\langle G_1, G_2\rangle$ on the edges of the graph in F.7.3.2. Thus $V_\alpha \leq R_2^g$.

Suppose first that $\beta = \gamma_1 g$; then $V_\beta = V_0^g$. If α is conjugate to γ_1 then we may take $\alpha = \gamma_0$ and $\alpha_1 = \gamma_1$, so $V_0 \leq R_2^g$, and by (*) and symmetry between α and β, $V_0^g = V_\beta \leq R_2$. As $1 \neq [V_\alpha, V_\beta] = [V_0, V_0^g]$, we have a contradiction to 13.4.16. Thus at most one of α and β is conjugate to γ_1. Therefore if $\beta \notin \gamma_0 G$, then $\alpha \in \gamma_0 G$, so reversing the roles of α and β, we may assume $\beta \in \gamma_0 G$. Then conjugating in G, we may take $\beta := \gamma_0$ and $\alpha_{n-1} := \gamma_1$. Thus $V_\alpha \leq R_2$ by (*). Similarly $\alpha = \gamma_i g$ for $i = 0$ or 1, and by (*) we may take $V \leq R_2^g$.

If $\alpha = \gamma_1 g$ then $V_\alpha = V_0^g$, contrary to 13.4.15. Hence $\alpha = \gamma_0 g$ and $V_\alpha = V^g$. In particular $1 \neq [V, V^g] \leq V \cap V^g$.

Let $v \in [V, V^g]^\#$. If $C_G(v) \leq M$, then by 13.4.2.3, V is the unique member of V^G containing v; hence $C_G(v) \not\leq M$. However for any $t \in T - C_T(V)$, $[V, t]$ contains a vector of weight 2, so z is of weight 2 by the uniqueness of z in 13.4.12.1; thus (1) holds. Indeed by (1) and that uniqueness, $C_G(w) \leq M$ for $w \in V$ of weight 4, and hence V is the unique member of V^G containing w by 13.4.2.3. This establishes (6).

By (6), all vectors in $[V, V^g]^\#$ are of weight 2, so $[V, V^g]$ is of rank 1—since up to conjugacy, $E := \langle e_{5,6}, e_{4,6}\rangle$ is the unique maximal subspace of V with all nonzero vectors of weight 2, and $E \neq [V, A]$ for any elementary 2-subgroup of $\bar{M}$. Then conjugating in $L_2 \leq G_\beta$, we may assume $[V, V^g] = \langle z\rangle$. Now by 13.4.2.3, we may take $g \in H$, so (2) is established.

As z is of weight 2 in V, we are in case (ii) of 13.4.14.2. Hence either (3) holds, or else V_H is a 5-dimensional module for $H/O_2(H) \cong A_6$ and Z_V of weight 4 in V_H. But in the latter case we have symmetry between L, V and $K := O^2(H)$, V_H, so as Z_V is weight 4 in V_H, we have a contradiction to (1) applied to K, V_H. Hence (3) is established. By (3), V_H is not a 5-dimensional module for $K/C_K(V_H) \cong A_6$, and in the remaining two cases in 13.4.13, V_H is the natural module for $K/O_2(K)$, so H is transitive on the points of V_H; thus (4) is established as $Z_V \leq V_H$ by (3).

If $U \leq V$ with $C_G(U) \not\leq M$, then all vectors in $U^\#$ are of weight 2 by (6). But we saw that up to conjugation, the unique maximal subspace with this property is $\langle e_{5,6}, e_{4,6}\rangle$ of rank 2, so (5) holds. □

We will now obtain a contradiction to our assumption that H is not solvable. This contradiction will complete the proof of Theorem 13.4.1.

Pick $g \in H$ as in 13.4.17.2. Then $\bar{V}^g = \langle (5,6) \rangle$, so there is $l \in L$ with $\bar{V}^{gl} = \langle (3,4) \rangle$. Let $y := gl$. Then $A := V^y \leq T$ with $L_1 = [L_1, A]$. Let $K := O^2(H)$, so that $K \in \mathcal{C}(H)$ by 13.4.10 and our assumption that H is not solvable. As $L_1 = [L_1, A]$, $K = [K, A]$, and hence $[V_H, A] \neq 1$ as $[V_H, K] \neq 1$. Let $U := V_H \cap M^y$ so that $[U, A] \leq U \cap A$, and set $(KT)^* := KT/C_{KT}(V_H)$.

Suppose first that $[A, U] \neq 1$. By 13.4.17.4, H is transitive on $V_H^\#$ and $Z_V \leq V_H$, so $Z_V^h \leq [A, U] \leq A \cap U$ for some $h \in H$. Then as V^h is the unique member of V^G containing Z_V^h by 13.4.2.3, $V^h = A = V^y$, and hence $Z_V^h = Z_V^y$ as $N_G(V) = M = C_G(Z_V)$. Indeed this argument shows Z_V^y is the unique point of $V_H \cap A$, and hence of $[A, U]$; thus $[A, U] = Z_V^y$, and hence U induces transvections on A with center Z_V^y, whereas $\bar{M}$ contains no such transvection, since $C_M(V) = C_M(V/Z_V)$ by 13.4.2.4.

This contradiction shows that $[A, U] = 1$. In particular $V_H \not\leq M^y$, as $[V_H, A] \neq 1$; hence as $r(G, V) \geq 3$ by 13.4.17.5, $m_2(K^*T^*) \geq m(A^*) > 2$, and then examining the cases listed in 13.4.13, we conclude that $H^* \cong S_6$ and $m(A^*) = 3$. Hence for $1 \neq b^* \in A^*$, $\langle b^* \rangle = B^*$ for some $B \leq A$ with $m(A/B) \leq 2$, so $C_{V_H}(b^*) = C_{V_H}(B) \leq M^y$ as $r(G, V) \geq 3$, and therefore

$$\langle C_{V_H}(b^*) : 1 \neq b^* \in A^* \rangle \leq U \leq C_{V_H}(A^*),$$

so that $A^* \in \mathcal{A}_3(T^*, V_H)$. However H^* has no such rank-3 subgroup, since each such subgroup is the radical of some minimal parabolic and hence contains a transvection whose axis is centralized only by that transvection.

This contradiction establishes Theorem 13.4.1.

13.5. The treatment of $\mathbf{A_5}$ and $\mathbf{A_6}$ when $\langle \mathbf{V_3^{G_1}} \rangle$ is nonabelian

In this section, we continue our treatment of the remaining alternating groups A_5 and A_6, postponing treatment of the final group $L_3(2)$ of $\mathbf{F}_2$-type until the following chapter. More specifically, this section begins the treatment of the case where $\langle V^{G_1} \rangle$ is nonabelian, by handling in Theorem 13.5.12 the subcase $\langle V_3^{G_1} \rangle$ nonabelian. In fact if $L/O_2(L)$ is A_5 and $\langle V_3^{G_1} \rangle$ is abelian, we will see that $\langle V^{G_1} \rangle$ is also abelian; thus in this section we also deal with the case where $L/O_2(L) \cong A_5$ and $\langle V^{G_1} \rangle$ is nonabelian.

In this section, with Theorem 13.4.1 now established, we assume the following hypothesis:

HYPOTHESIS 13.5.1. *Hypothesis 13.3.1 holds and G is not $Sp_6(2)$.*

In addition we continue the notation established earlier in the chapter, and the notational conventions of section B.3. In particular we adopt Notations 12.2.5 and 13.2.1.

LEMMA 13.5.2. *Assume Hypothesis 13.5.1. If $K \in \mathcal{L}_f(G, T)$, then*

(1) $K/O_2(K) \cong A_5$, $L_3(2)$, A_6, or $\hat{A}_6$.

(2) $K \trianglelefteq KT$ and $K \in \mathcal{L}^*(G, T)$.

(3) There is $V_K \in Irr_+(K, R_2(KT), T)$, and for each such V_K, $V_K \leq R_2(KT)$, $V_K \trianglelefteq T$, the pair K, V_K satisfies the FSU, $C_{V_K}(K) = 1$, and V_K is the natural module for $K/C_K(V_K) \cong A_5$, A_6, or $L_3(2)$.

PROOF. As in the proof of 13.4.5, this follows from 13.3.2, once we observe that by 13.3.2, we may apply various results to K in the role of "L": Theorem 13.3.16 says $K/O_2(K)$ is not $U_3(3)$. Then since Hypothesis 13.5.1 excludes $G \cong Sp_6(2)$, we conclude from Theorem 13.4.1 that $C_{V_K}(K) = 1$ when $K/O_{2,Z}(K) \cong A_6$. □

13.5.1. Setting up the case division on $\langle \mathbf{V}_3^{\mathbf{G}_1} \rangle$ for $\mathbf{A}_5$ and $\mathbf{A}_6$.

REMARK 13.5.3. In the remainder of this section (and indeed in the remainder of the chapter), in addition to assuming Hypothesis 13.5.1, we also assume $L/O_2(L)$ is not $L_3(2)$; that is, we restrict attention to the cases where $L/O_2(L) \cong A_5$, A_6, or $\hat{A}_6$.

Then by 13.5.2.3, $C_V(L) = 1$ and V is the natural module for $L/C_L(V) \cong A_n$, $n = 5$ or 6.

As usual we adopt the notational conventions of section B.3 and Notation 13.2.1. We view V as the quotient of the core of the permutation module for $L/C_L(V)$ on $\Omega := \{1, \ldots, n\}$, modulo $\langle e_\Omega \rangle$. Recall from Notation 12.2.5.2 that $M_V := N_M(V)$ and $\bar{M}_V := M_V/C_M(V)$. So there is an $\bar{M}_V$-invariant symplectic form on V, and when $n = 5$, an invariant quadratic form. Thus we use terminology (e.g., of isotropic or singular vectors) associated to those forms.

As in Notation 13.2.1, V_i is the T-invariant subspace of V of dimension i and $G_i := N_G(V_i)$.

LEMMA 13.5.4. *When $L/O_2(L) \cong A_6$ or $\hat{A}_6$, set $I_2 := O_2(G_1)L_2$ or $O_2(G_1)L_{2,+}$, respectively. Then:*

(1) $I_2 = \langle O_2(G_1)^{G_2} \rangle \trianglelefteq G_2$.

(2) $C_{I_2}(V_2) = O_2(I_2)$ and $I_2/O_2(I_2) \cong S_3$.

(3) $m_3(C_G(V_2)) \leq 1$.

(4) $C_G(V_3) \leq M_V$. Hence $[V, C_G(V_3)] \leq V_1$.

(5) $[O_2(G_1), V_2] \neq 1$.

(6) $O^2(I_2) = L_2$ or $O^2(L_{2,+})$, respectively, $O^2(\bar{I}_2) = \bar{L}_2$, and $O_2(O^2(I_2))$ is nonabelian.

PROOF. The equalities in (6) follow from the definition of I_2 and the fact that L_2 and $L_{2,+}$ are T-invariant. Then as $V = [V, \bar{L}_2]$ and $O_2(\bar{L}_2) \neq 1$, the remaining statement in (6) follows. Hypothesis 13.3.13 is satisfied by 13.5.2.3, so (5) follows from 13.3.14. Then parts (1)–(4) of the lemma follow from 13.3.15. □

LEMMA 13.5.5. *$G_1 \cap G_3 \leq M_V$.*

PROOF. When $n = 5$, $G_3 \leq M_V$ by 13.2.3.2. When $n = 6$, $Aut_{L_1T}(V_3) = C_{GL(V_3)}(V_1)$, so as $C_G(V_3) \leq M_V$ by 13.5.4.4, and as $L_1T \leq M_V$, $G_1 \cap G_3 \leq M_V$. □

As $C_V(L) = 1$:

$$V_1 = Z \cap V \text{ is of order } 2.$$

Let z be a generator for V_1. By 13.3.6,

$$G_1 = C_G(z) \not\leq M,$$

so $G_1 \in \mathcal{H}_z \neq \emptyset$, where as usual

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \leq G_1 \text{ and } H \not\leq M\}$$

and $\tilde{G}_1 := G_1/V_1$.

LEMMA 13.5.6. *Assume $n = 6$; then*

(1) $V \leq O_2(G_2)$.
(2) $V_2^G \cap V = V_2^L$.
(3) If $V_2 \leq V \cap V^g$ then either $[V, V^g] = 1$ or $[V, V^g] = V_2$, and in the latter case $V^g \leq M_V$ and $\bar{V}^g = O_2(\bar{L}_2)$.

PROOF. As L has two orbits on 4-subgroups of V, either (2) holds or for some $g \in G$, V_2^g is a nondegenerate 2-subspace of V.

Assume the latter holds. Then $Q_0 := C_T(V_2^g) \in Syl_2(C_M(V_2^g))$, and either $\bar{T} \leq \bar{L}$ and $Q_0 = Q := O_2(LT) = C_T(V)$, or Q_0 is the preimage in LT of the subgroup generated by a transposition. Now if $N_G(Q_0) \leq M$, then $Q_0 \in Syl_2(C_G(V_2^g))$, contradicting $|C_G(V_2^g)|_2 = |T|/2 > |Q_0|$.

Thus $N_G(Q_0) \not\leq M$, so $Q < Q_0$, and as $M = !\mathcal{M}(LT)$, no nontrivial characteristic subgroup of Q_0 is normal in LT. Therefore L is an A_6-block by case (c) of C.1.24. Now $Q = O_2(C_M(V_2^g))$, and $O^{3'}(N_M(V_2^g)) = X_1 \times X_2$ is the product of A_3-blocks, with $V = O^{2'}(X_1X_2)$. Let $X := O^2(I_2)$, in the notation of 13.5.4. Now X_1X_2 acts on X^g by 13.5.4.1, so $[X_1X_2, X^g] \leq O_2(X^g)$ by 13.5.4.2. Therefore as $m_3(G_2) \leq 2$, $X^g \leq X_1X_2$, impossible as $O^{2'}(X)$ is nonabelian by 13.5.4.6, while $V = O^{2'}(X_1X_2)$ is abelian.

This contradiction establishes (2); thus it remains to prove (1) and (3). Observe that Hypothesis G.2.1 is satisfied with V_2, V in the roles of "V_1, V" as L_2 is irreducible on V/V_2. Let $U := \langle V^{G_2} \rangle$. By G.2.2, (1) holds and $V_2 \geq \Phi(U)$.

Finally suppose $V_2 \leq V \cap V^g$. By (2) and A.1.7.1 we may take $g \in G_2$, so $V^g \leq U \leq M_V$ and $[V, V^g] \leq V \cap V^g$ as $V \trianglelefteq U$. Further if U is abelian, then $[V, V^g] = 1$ and (3) holds. Thus we may take U nonabelian. As $X \trianglelefteq G_2$ by 13.5.4.1 while $V = [V, X]$, $U = [U, X] \leq O_2(X)$. Therefore using 13.5.4.6, $\bar{U} = O_2(\bar{L}_2)$ is of rank 2. Thus $\bar{V}^g \leq O_2(\bar{L}_2)$, so $[V, V^g] = V_2$ and $m(V/C_V(V^g)) = 2$, so by symmetry, $m(V^g/C_{V^g}(V)) = 2$ and hence $\bar{V}^g = O_2(\bar{L}_2)$, completing the proof of (3). □

LEMMA 13.5.7. *For each $H \in \mathcal{H}_z$, Hypothesis F.9.1 is satisfied with V_3 in the role of "V_+".*

PROOF. Most of this proof is exactly parallel to that of 13.3.18.1: namely part (c) of F.9.1 follows, this time using 13.5.5 rather than 13.3.17.1 to obtain $G_1 \cap G_3 \leq M_V$; parts (b) and (d) follow just as before; and part (a) is proved as before. Thus it remains to verify F.9.1.e.

Assume $1 \neq [V, V^g] \leq V \cap V^g$; then $\bar{V}^g$ is quadratic on V. To verify hypothesis F.9.1.3, we may assume that $g \in G_1$ with $[V^g, \tilde{V}_3] = 1 = [V, \tilde{V}_3^g]$. Then $\bar{V}^g \leq O_2(\bar{L}_1\bar{T}) = \bar{R}_1$, so as $\bar{V}^g$ is quadratic on V, either $m(\bar{V}^g) = 1$; or $n = 6$, $m(\bar{V}^g) = 2$, and conjugating in L_1, we may assume that $V_2 = [V, V^g]$. But in the latter case as $[V, V^g] \leq V \cap V^g$, $V_2 \leq V \cap V^g$; then by 13.5.6.3, $\bar{V}^g = O_2(\bar{L}_2)$, contradicting $\bar{V}^g \leq \bar{R}_1$.

Therefore $m(V^g/C_{V^g}(V)) = 1$, and hence also $1 = m(V/C_V(V^g))$ by symmetry. Suppose $[V_3, V^g] = 1$. Then as $[V, V^g] \neq 1$, $n \neq 5$, since in that case $C_M(V_3) = C_M(V)$. Thus $n = 6$ and $\bar{V}^g$ is generated by a transvection with center V_1, so $[V, V^g] = V_1$. Thus V induces a transvection on V^g with center V_1, so $C_{V^g}(V) = V_1^\perp = V_3^g$; hence $[V_3^g, V] = 1$, and F.9.1.e holds.

It remains to treat the case where $[V_3, V^g] = V_1 = [V_3^g, V]$. Here $m(V^g/C_{V^g}(V)) = 1$, so V induces a transvection with center V_1 on V^g, and so again $[V_3^g, V] = 1$, contrary to assumption. This contradiction completes the proof of 13.5.7. $\square$

NOTATION 13.5.8. Recall $\tilde{G}_1 = G_1/V_1$. By 13.5.7, we can appeal to the results of section F.9 with V_3 in the role of "V_+" in F.9.1. Recall from Hypothesis F.9.1 that for $H \in \mathcal{H}_z$, $U_H := \langle V_3^H \rangle$, $V_H := \langle V^H \rangle$, and $Q_H := O_2(H)$. By F.9.2.1, $U_H \leq Q_H$, and by F.9.2.2, $\Phi(U_H) \leq V_1$. By F.9.2.3, $Q_H = C_H(\tilde{U}_H)$; set $H^* := H/Q_H$.

Notice $G_1 \cap G_3 \leq M$ by 13.5.5, and $H \not\leq M$, so that:

LEMMA 13.5.9. $V_3 < U_H$

We begin our treatment of the case $\langle V^{G_1} \rangle$ nonabelian by considering the subcase where $\langle V_3^{G_1} \rangle$ is nonabelian; the next observation shows that if $n = 5$ and $\langle V^{G_1} \rangle$ is nonabelian, then $\langle V_3^{G_1} \rangle$ is also nonabelian:

LEMMA 13.5.10. *If $n = 5$ and $H \in \mathcal{H}_z$, then the following are equivalent:*
(1) U_H is abelian.
(2) V_H is abelian.
(3) $V \leq Q_H$.

PROOF. When $n = 5$, $C_M(V_3) = C_M(V)$, so the lemma follows from F.9.4.3. $\square$

LEMMA 13.5.11. *If $n = 5$ and $V_2 \leq V \cap V^g$, then $[V, V^g] = 1$ or V_2; in either case, $V^g \leq M_V$.*

PROOF. We may assume $[V, V^g] \neq 1$. By hypothesis $V_1 \leq V \cap V^g$, so by 13.3.11.1 we may take $g \in G_1$. By 13.3.11.5, $[V_3, V_3^g] \neq 1$, so by F.9.2.2, $[V_3, V_3^g] = V_1$. Thus $X := V_3 V_3^g \cong D_8 \times \mathbf{Z}_2$, and $\mathcal{A}(X) = \{V_3, V_3^g\}$. Now V_3^g acts on V_3, and also $V_3^g \leq U_{G_1} \leq M_V$, so $[V, V_3^g] \leq V_3 \leq X$, and hence V acts on X. Then as $\mathcal{A}(X) = \{V_3, V_3^g\}$, $V \leq N_{G_1}(V_3^g) \leq N_G(V^g)$ by 13.2.3.2. By symmetry V^g acts on V, so $[V, V^g] \leq V \cap V^g \leq C_V(V^g)$. As $[V_3, V_3^g] = V_1$ is singular, $\bar{V}^g$ does not induce a transvection on V, so $m(C_V(V^g)) \leq 2 \leq m([V, V^g])$, and hence $[V, V^g] = V \cap V^g$ is of rank 2. Then as $V_2 \leq V \cap V^g$ by hypothesis, we conclude $[V, V^g] = V_2$. This completes the proof. $\square$

13.5.2. The treatment of the subcase $\langle \mathbf{V}_3^{\mathbf{G}_1} \rangle$ nonabelian. We come to the main result of the section, which determines the groups where $\langle V_3^{G_1} \rangle$ is nonabelian:

THEOREM 13.5.12. *Assume Hypothesis 13.3.1 with $L/C_L(V) \cong A_n$ for $n = 5$ or 6, $G \not\cong Sp_6(2)$, and $\langle V_3^{G_1} \rangle$ nonabelian. Then either*
(1) $n = 5$ and $G \cong U_4(2)$ or $L_4(3)$.
(2) $n = 6$ and $G \cong U_4(3)$.

The remainder of this section is devoted to the proof of Theorem 13.5.12.

Observe since $G \not\cong Sp_6(2)$ that Hypothesis 13.5.1 holds. Thus we may apply results from earlier in the section; in particular by 13.5.7, we may apply results from section F.9, and continue to use the conventions of Notation 13.5.8.

In the remainder of the section we assume $\langle V_3^{G_1} \rangle$ is nonabelian. Thus as $G_1 \in \mathcal{H}_z$, there exists $H \in \mathcal{H}_z$ such that U_H is nonabelian.

In the remainder of the section, H will denote any member of $\mathcal{H}_z$ with U_H nonabelian.

Then $\Phi(U_H) = V_1$ by F.9.2.2. By F.9.4.1, $V \not\leq Q_H$, while by F.9.2.1, $V_3 \leq Q_H$. Thus as $|V : V_3| = 2$, $V_3 = V \cap Q_H$ and V^* is of order 2. By F.9.2.1, $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$.

LEMMA 13.5.13. *(1) If $g \in H$ with $V_1 < V \cap V^g$, then $\langle V, V^g \rangle$ is a 2-group.*
(2) The hypotheses of F.9.5.5 and F.9.5.6 hold.

PROOF. As $C_H(\tilde{U}_H) = Q_H$ is a 2-group, we may assume $V^* \neq V^{g*}$; so as V^* is of order 2, $V \cap V^g \leq V \cap Q_H = V_3$. Then as L_1 is transitive on $\tilde{V}_3^{\#}$, we may take $V_2 \leq V \cap V^g$. Now 13.5.11 and 13.5.6.3 show that V^g normalizes V, and so (1) follows.

By (1), we have the hypothesis of F.9.5.5. Further by 13.5.5, $\overline{C_H(V_3)} \leq C_M(V_3)$. Now if $n = 5$ then $C_M(V_3) = C_M(V)$, while if $n = 6$ then $\overline{C_M(V_3)}$ is trivial or induces transvections with center V_1 on V. Thus we also have the hypotheses for F.9.5.6. $\square$

LEMMA 13.5.14. *If $n = 6$, assume $\tilde{U}_H = [\tilde{U}_H, L_1]$. Let $l \in L - L_1T$, and if $n = 6$, choose $\bar{l}$ to fix a point $\omega \in \Omega$ fixed by $\bar{L}_1$. Set $K := \langle U_H, U_H^l \rangle$ and $L_- := O^2(O_2(LT)K)$. Then*

(1) If $\tilde{U}_H = [\tilde{U}_H, L_1]$ then $U_H = [U_H, L_1] \leq L_1 \leq L$.

(2) $\bar{U}_H = O_2(\bar{L}_1) \cong E_4$.

(3) If $n = 5$ then $\bar{K} = \bar{L}$ and $L = L_-$, while if $n = 6$, then $\bar{K} \cong A_5$ is the stabilizer in $\bar{L}$ of ω. Thus in any case $L_1 \leq K$.

(4) The hypotheses of G.2.4 are satisfied with V_1, V_3, V, L_-, U_H, K in the roles of "V_1, V, V_L, L, U, I", so $K = L_-$ and K is described in that lemma.

PROOF. Suppose first that $\tilde{U}_H = [\tilde{U}_H, L_1]$. Then as $V_1 \leq [V_3, L_1]$, $U_H = [U_H, L_1]$. Thus (1) holds. Moreover if $n = 6$, then $\tilde{U}_H = [\tilde{U}_H, L_1]$ by hypothesis, so $U_H \leq L$ by (1).

As $U_H = \langle V_3^H \rangle$ is nonabelian, $\bar{U}_H \neq 1$, and as $L_1T \leq H \leq N_H(U_H)$, $\bar{U}_H \trianglelefteq \bar{L}_1\bar{T}$. Thus (2) holds if $n = 5$. Similarly if $n = 6$, then $\bar{U}_H \leq \bar{L}$ by the first paragraph, so as $\bar{U}_H \trianglelefteq \bar{L}_1\bar{T}$, (2) holds again. Part (3) is immediate from (2) and the choice of l. Then (3) implies the first statement in (4). Finally $L_1 \leq O^2(K) = L_-$ by (3) and $U_H \leq L_1$ by (1), so that $K = L_-U_H = L_-$ by G.2.4. $\square$

13.5.2.1. *Identifying the groups.* In the branch of the argument that will lead to the groups in Theorem 13.5.12, $L_1 \trianglelefteq G_1$ and G_1 is the unique member of $\mathcal{H}_z$. We begin by deriving some elementary consequences of the hypothesis that L_1 is normal in some member H of $\mathcal{H}_z$ with U_H nonabelian.

LEMMA 13.5.15. *Assume $L_1 \trianglelefteq H$, U_H is nonabelian, and $L/O_2(L)$ is not $\hat{A}_6$. Then*

(1) $U_H = [U_H, L_1] \leq L$ and $L_1^* \cong \mathbf{Z}_3$.

(2) $\bar{U}_H = O_2(\bar{L}_1) \cong E_4$.

Choose l and $K := \langle U_H, U_H^l \rangle$ as in 13.5.14. Then

(3) K is an A_5-block contained in L.

(4) If $n = 5$ then $L = K$, so that L is an A_5-block; if $n = 6$, then L is an A_6-block.

(5) $O_2(L_1) = U_H \cong Q_8^2$.

(6) $H = G_1$ and G_1 is the unique member of $\mathcal{H}_z$.
(7) $M = LT$ and $V = O_2(M)$.
(8) If $n = 5$ then $Q_H = U_H$, $H^* \leq \Omega_4^+(2)$, and either

(i) $M = L$ with $H^* \cong S_3 \times \mathbf{Z}_3$, or
(ii) $M/V \cong S_5$ with $H^* = \Omega_4^+(2) \cong S_3 \times S_3$.

(9) If $n = 6$ then $H^* = \Omega_4^+(2) \cong S_3 \times S_3$, and either

(i) $M = L$ with $Q_H = U_H$, or
(ii) $M/V \cong S_6$ with $Q_H = U_H C_T(L_1)$ and $|C_T(L_1)| = 4$.

PROOF. We saw $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$, so as $L_1 \trianglelefteq H$, $O_2(L_1^*) = 1$. Hence $L_1^* \cong \mathbf{Z}_3$. Also $\tilde{V}_3 = [\tilde{V}_3, L_1]$, so as $U_H = \langle V_3^H \rangle$, $\tilde{U}_H = [\tilde{U}_H, L_1]$. Then (1) follows from 13.5.14.1, and (2) from 13.5.14.2.

Choose l and K as in 13.5.14, and let $Q := O_2(LT)$ and $L_- := O^2(KQ)$. By 13.5.14.4, the hypotheses of G.2.4 are satisfied and $K = L_-$. By (1), the hypotheses of G.2.4.8 are satisfied. Therefore by G.2.4.8, K is an A_5-block with $V = O_2(K)$ and $U_H = O_2(L_1) \cong Q_8^2$. Now if $n = 5$, then $L = L_-$ by construction. If $n = 6$, then as Q acts on L_- and $L_- = K$, $[K, Q] \leq O_2(K) = V$, so L is an A_6-block. Thus (3), (4), and (5) are established.

We saw V^* is of order 2 and $O_2(H^*) = 1$, so by the Baer-Suzuki Theorem, there is $g \in H$ with I^* not a 2-group, where $I := \langle V, V^g \rangle$. Now by 13.5.13.2, we may apply F.9.5.6 to conclude that $O_2(I) = U_I := V_3 V_3^g \cong Q_8^2$ and $I/O_2(I) \cong I^* \cong S_3$. Therefore since $U_I \leq U_H$ and $U_H \cong Q_8^2$, we conclude $O_2(I) = U_H$, and hence $I^* = IQ_H/Q_H \cong I/U_H \cong S_3$.

Next as $C_H(\tilde{U}_H) = Q_H$, $H^* = H/Q_H \leq Out(U_H) \cong O_4^+(2)$. As $L_1^* \cong \mathbf{Z}_3$ is normal in H^*, and L_1^* centralizes V^*, L_1^* centralizes I^*. But the centralizer in $Out(U_H) \cong O_4^+(2)$ of L_1^* is isomorphic to S_3, so we conclude $I^* = C_{H^*}(L_1^*)$. Therefore either $H^* \cong S_3 \times S_3$, or $H^* = I^* \times L_1^* \cong S_3 \times \mathbf{Z}_3$ with $T = O_2(L_1 T)$, and the latter case can only occur when $n = 5$ and $\bar{M}_V = \bar{L} \cong A_5$. In either case $H = Q_H L_1 I T = L_1 I T$. Further if we establish (7), then $Q_H \cap Q = Q_H \cap V = V_3$, so $|Q_H| = 8|\bar{Q}_H|$, and then (8) and (9) follow using (5). So it remains to prove (6) and (7).

As Q_H acts on V and V^g, Q_H acts on I. Then as $I^* \trianglelefteq H^*$, H acts on $O^2(IQ_H) = O^2(I)$, and then $L_1 T$ acts on $O^2(I)V = I$. Thus $I \trianglelefteq L_1 I T = H$.

As K is an A_5-block by (3), $Q = V \times C_T(K)$ by C.1.13.c. Further $L_1 \leq K$ by 13.5.14.3, and $C_T(L_1) = V_1 C_T(K)$. Thus as $U_H = [U_H, L_1] \leq K$, $C_T(L_1) \leq C_T(U_H)$, so $[I, C_T(L_1)] \leq C_I(U_H) = V_1$, and therefore $[O^2(I), C_T(L_1)] = 1$ by Coprime Action. In particular $O^2(I)$ centralizes $C_T(L)$. As $O^2(I) = [O^2(I), V]$ and $V \leq O_2(M)$, $O^2(I) \not\leq M$. Therefore as $M = !\mathcal{M}(LT)$, we conclude $C_T(L) = 1$.

We will show next that $D := C_T(K) = 1$. If $n = 5$, then $K = L$, so $D = C_T(L) = 1$; thus we may assume $n = 6$. Then as L is an A_6-block, we conclude from C.1.13.b and I.1.6.5 that either $D = C_T(L) = 1$ or $|D| = 2 = |Q : V|$. Assume that the latter case holds and set $G_D := C_G(D)$. We saw that $O^2(I)$ centralizes $C_T(L_1) \geq D$, so $\langle O^2(I), K \rangle \leq G_D$. Let $G_D^+ := G_D/D$ and $T_0 := C_T(D) \in Syl_2(C_M(D))$. Let $T_0 \leq T_D \in Syl_2(G_D)$; then $|T_D : T_0| \leq |T : T_0| = 2$, so as $K \in \mathcal{L}(G_D, T_0)$, there exists a unique $K_D \in \mathcal{C}(G_D)$ containing K by 1.2.5, and $O^2(I)$ normalizes K_D by 1.2.1.3. As $O^2(I) = [O^2(I), V]$ and K is irreducible on V, $V \cap O_2(K_D D) = 1$. Let $T_1 := T_0 \cap K_D D$; thus $T_1 \in Syl_2(M \cap K_D D)$. Then as $N_G(Q) \leq M = !\mathcal{M}(LT)$, $T_1 \in Syl_2(N_{K_D D}(Q))$. Therefore as $Q = O_2(KT_1) =$

$V \times D$, we conclude $Q \in \mathcal{B}_2(K_D D)$, and in particular Q contains $O_2(K_D D)$ by C.2.1.2. Then as $V \cap O_2(K_D D) = 1$, $D = O_2(K_D D)$. As $C_{C_M(D)}(Q) = Q$, $KT_0 = C_M(D)$. Hence $K^+ T_1^+ \cong KT_1/D$ is the 2-local $N_{K_D^+}(Q^+) = N_{K_D^+}(V^+)$ in the quasisimple group K_D^+. But inspecting the groups in Theorem C (A.2.3), we find no such 2-local. This contradiction establishes the claim that $C_T(K) = 1$.

As $C_T(K) = 1$, $V = Q = O_2(LT)$, so $O_2(M) = V$ and $M = LT$ by 3.2.11. Thus (7) holds, and hence also (8) and (9) by an earlier observation. Thus it remains to establish (6).

By A.1.6, $Q_1 := O_2(G_1) \leq Q_H$. Also $Q_1 \leq O_2(L_1 T)$, and by (8) and (9), either $O_2(L_1 T) = O_2(L_1)V = U_H V$, or $L/V \cong S_6$ and $O_2(L_1 T) = U_H V C_T(L_1)$, with $C_T(L_1)$ of order 4. Next $U_H V \cap Q_H = U_H$ and as $H \leq G_1$, $U_H \leq U_{G_1} \leq Q_1$. We conclude that $Q_1 = U_H$ or $U_H C_T(L_1)$. In either case, $Q_1 = U_H Z_1$ where $Z_1 := Z(Q_1)$ is of order at most 4, and $\Phi(Q_1) = V_1$. Thus G_1 preserves the usual symplectic form on $\hat{Q}_1 := Q_1/Z_1$. Now $m(\hat{Q}_1) = 4$ as $U_H \cong Q_8^2$, So $G_1/Q_1 \leq Sp(\hat{Q}_1) \cong S_6$. Then as $T \leq H$ and $H/Q_1 \cong S_3 \times S_3$ or $S_3 \times \mathbf{Z}_3$, it follows that $G_1 = H$. Thus $L_1 \trianglelefteq G_1$. Finally for any $H_1 \in \mathcal{H}_z$, as $L_1 \leq H_1 \leq G_1$ and $L_1 \trianglelefteq G_1$, $L_1 \trianglelefteq H_1$; so by symmetry between H and H_1, $H_1 = G_1$. This completes the proof of (6), and hence of the lemma. □

We can now proceed to the identification of the groups in Theorem 13.5.12, under the assumption that L_1 is normal in H.

PROPOSITION 13.5.16. *If $L/O_2(L) \cong A_6$ and $L_1 \trianglelefteq H$, then $G \cong U_4(3)$.*

PROOF. By 13.5.15.6, $H := G_1$ is the unique member of $\mathcal{H}_z$. Let $U := U_H$ and $y \in L_2 - T$, so that $U \cong Q_8^2$ by 13.5.15.5. We consider the two cases of 13.5.15.9.

Suppose first that $M = L$. Then $O_2(G_1) = U$ by 13.5.15.9, and $U \cap U^y = V_2$. Hence G is of type $U_4(3)$ in the sense of section 45 (page 244) of [**Asc94**], so by 45.11 in [**Asc94**], $G \cong U_4(3)$.

Thus we may assume that $M/V \cong S_6$; in this case we will obtain a contradiction using transfer, eliminating shadows of extensions of $U_4(3)$. By 13.5.15.9, $Z(Q_H) = C_T(L_1)$ is of order 4.

Let $T_L := T \cap L \in Syl_2(L)$, and define I as in the proof of 13.5.15. From the proof of 13.5.15, $U = O_2(L_1) = O_2(I)$, $VU \in Syl_2(I)$, and $[L_1, I] \leq U$. Then $VU = VO_2(L_1) \leq T_L \in Syl_2(L)$, so $T_L \in Syl_2(T_L I L_1)$, Now L is transitive on $V^\#$, while IL_1 is transitive on the involutions in $U - V_1$, and all involutions in L are fused into U under L, so we conclude all involutions in T_L are in z^G.

Suppose that Q_H is not weakly closed in H with respect to G. Observe that $V_1 = \Phi(Q_H)$, so $N_G(Q_H) = H$. Then by A.1.13 there is $x \in G$ with $Q_H \neq Q_H^x$ and $[Q_H, Q_H^x] \leq Q_H \cap Q_H^x$. In particular $Q_H \leq N_G(Q_H^x) = C_G(z^x)$, so that $z^x \in C_H(Q_H) = Z(Q_H)$. As $Q_H \neq Q_H^x$, $x \notin H$, so $z^x \neq z$; thus $E_4 \cong \langle z, z^x \rangle = Z(Q_H)$, and then by symmetry between Q_H and Q_H^x, also $\langle z, z^x \rangle = Z(Q_H^x)$. Now 13.5.15 shows that H^* acts as $\Omega_4^+(2)$ on U/V_1, so that $Q_H/Z(Q_H) = J(T/Z(Q_H))$; hence $Q_H = Q_H^x$, contrary to the choice of Q_H^x.

This contradiction shows that Q_H is weakly closed in H. Hence H controls fusion in $Z(Q_H)$ by Burnside's Fusion Lemma A.1.35, so that z is weakly closed in $Z(Q_H)$ with respect to G. Now $Z(Q_H) = \langle z, j \rangle$ with $j \in T - T_L$. Therefore if j is an involution then $j \notin z^G$, so as all involutions in T_L are in z^G, Thompson Transfer gives $j \notin O^2(G)$, contrary to the simplicity of G. Hence $Z(Q_H) = \langle j \rangle \cong \mathbf{Z}_4$, so $Z(Q_H) \cap Z(Q_H)^y = 1$ as $z \neq z^y$.

Next as $y \in L_2 - H$, $z \neq z^y \in V_2 \leq Q_H \leq C_G(j)$ so that $j \in C_G(z^y) = H^y$, and similarly $j^y \in H$. Therefore $[j, j^y] \leq Z(Q_H) \cap Z(Q_H^y) = 1$. Thus $\langle j^{L_2} \rangle =: B$ is abelian with $\Phi(B) = V_2$; so as $\bar{B}$ is the E_8-subgroup generated by the transvections in $\bar{T}$, $B \cong \mathbf{Z}_4^2 \times \mathbf{Z}_2$. Let $A := \Omega_1(B)$. Then $\bar{A}$ is of order 2 and normal in $\bar{L}_2\bar{T}$, so $\bar{A} = \langle (1,2)(3,4)(5,6) \rangle$. Further for $a \in A - V_2$, $[a, V] = V_2$, so V is transitive on $A - V_2$, and hence $\overline{C_M(a)} = C_{\bar{M}}(\bar{a}) \cong \mathbf{Z}_2 \times S_4$ for each such a. Therefore $B = O_2(C_M(a))$ and $X := O_2(O^2(C_M(a))) \cong \mathbf{Z}_4^2$ with $V_2 = \Phi(X)$. As before by Thompson Transfer, there is $r \in G$ with $a^r = z$. Then $O^2(C_M(a))^r \leq O^2(H)$, so as U_H is Sylow in $O^2(H)$ by 13.5.15, $X^r \leq U_H$. Then $V_2^r = \Phi(X)^r \leq \Phi(U_H) = V_1$ of rank 1. This contradiction completes the proof. □

PROPOSITION 13.5.17. *If $n = 5$ and $L_1 \trianglelefteq H$, then $G \cong U_4(2)$ or $L_4(3)$.*

PROOF. By 13.5.15.6, $H := G_1$ is the unique member of $\mathcal{H}_z$. By 13.5.15.7, $V = O_2(M)$. By 13.5.15.8, $Q_H = U_H =: U \cong Q_8^2$, and either $M = L$ with $H^* \cong S_3 \times \mathbf{Z}_3$, or $M/V \cong S_5$ and $H^* \cong S_3 \times S_3$. Let $T_L := T \cap L \in Syl_2(L)$, so that $|T : T_L| = 1$ or 2. Define I as in the proof of 13.5.15. Observe that Hypothesis F.1.1 is satisfied with I, L, T in the roles of "L_1, L_2, S": In particular, recall that during the proof of 13.5.15 we showed that $I \trianglelefteq H$ and $H = L_1IT$, so that $O_2(\langle I, L, T \rangle) = 1$ as $H \not\leq M$ and $M = !\mathcal{M}(LT)$.

Therefore $\gamma := (H, L_1T, M)$ is a weak BN-pair by F.1.9. As $T \cap I$ is self-normalizing in I, the hypotheses of F.1.12 are satisfied; so as $I/O_2(I) \cong S_3$, while $L/O_2(L) \cong A_5$ does not centralize Z, we conclude from F.1.12 that γ is of type $U_4(2)$ when $M = L$, and γ is of type $O_6^-(2)$ when $M/V \cong S_5$.

Next we verify the hypotheses of Theorem F.4.31: Let $G_0 := \langle M, H \rangle$. Then the inclusion $\gamma \to G_0$ is a faithful completion of γ. As $M \in \mathcal{M}$, $M = N_G(V)$. We saw $H = G_1 = C_G(z)$. Thus hypotheses (a) and (b) of F.4.31 hold. Hypotheses (c) and (d) are vacuously satisfied, and hypothesis (e) holds as G is simple.

We now appeal to Theorem F.4.31, and conclude as G is simple that either $M = L$ and $G \cong U_4(2)$ or $M/V \cong S_5$ and $G \cong L_4(3)$. □

We mention that the shadows of extensions of $U_4(2)$ and $L_4(3)$ were essentially eliminated during the proof of Theorem F.4.31.

13.5.2.2. *Obtaining a contradiction in the remaining cases.* During the remainder of the section we assume that G is a counterexample to Theorem 13.5.12. Thus appealing to 13.5.16 and 13.5.17, it follows that:

LEMMA 13.5.18. *Assume $L/O_2(L)$ is not $\hat{A}_6$. Then L_1 is* not *normal in any $H \in \mathcal{H}_z$ such that U_H is nonabelian.*

LEMMA 13.5.19. *Let $Y := L_0$ if $L/O_2(L) \cong \hat{A}_6$, and $Y := L_1$ otherwise. Let $H \in \mathcal{H}_z$ and $H_1 := N_H(Y)$. Then*

(1) $H_1^* = N_{H^*}(Y^*)$, and

(2) $V \leq O_2(H_1)$.

PROOF. Recall that $Q_H = C_H(\tilde{U}_H)$; hence as $Y = O^2(YQ_H)$, (1) holds. Notice (2) holds when $L/O_2(L) \cong \hat{A}_6$, since $N_G(L_0) \leq M$ by 13.2.2.9. Therefore we may assume $L/O_2(L)$ is A_5 or A_6, and $V \not\leq O_2(H_1)$. Hence $H_1 \not\leq M$, so $H_1 \in \mathcal{H}_z$.

As $V \not\leq O_2(H_1)$, by the Baer-Suzuki Theorem there is $h \in H_1$ such that I^* is not a 2-group, where $I := \langle V, V^h \rangle$. By 13.5.13.2, we may apply F.9.5.6 to conclude that $\langle V_3^I \rangle$ is nonabelian. Thus U_{H_1} is nonabelian, contrary to 13.5.18. □

LEMMA 13.5.20. *V^* centralizes $F(H^*)$.*

PROOF. If $[O_p(H^*), V^*] \neq 1$ for some prime p, then by 13.5.13.2, we may apply F.9.5.6 to conclude that $p = 3$. Let P^* be a supercritical subgroup of $O_3(H^*)$. Then $[P^*, V^*] \neq 1$, and $m(P^*) \leq 2$ since $H^* = H/Q_H$ is an SQTK-group. Define Y and H_1 as in 13.5.19. By definition Y normalizes V, so as V^* is of order 2, Y^* centralizes V^*. Suppose $O_2(Y^*)$ centralizes P^*. Then as $Y^*/O_2(Y^*)$ is of order 3, the Thompson $A \times B$ Lemma shows that $[C_{P^*}(Y^*), V^*] \neq 1$. This is a contradiction as $C_{P^*}(Y^*) \leq H_1^*$ by 13.5.19.1, and $V \leq O_2(H_1)$ by 13.5.19.2.

Therefore $O_2(Y^*)$ is nontrivial on P^*; then as $Y = O^{3'}(Y)$, P^* is not cyclic, so using A.1.25, $P^* \cong E_9$ or 3^{1+2} and $Y/C_Y(P^*/\Phi(P^*)) \cong SL_2(3)$. In particular Y is irreducible on $P^*/\Phi(P^*)$, so as $[Y^*, V^*] = 1$, $V^* = Z(Y^*)$ inverts $P^*/\Phi(P^*)$. However by F.9.5.2, $m([\tilde{U}_H, V^*]) = 2$; since a faithful irreducible for $SL_2(3)/E_9$ is of rank 8 and the commutator space of $Z(SL_2(3))$ on such a module is of rank 4, we conclude $P^* \cong 3^{1+2}$. But now X of order 3 in Y centralizes an E_9-subgroup of P, contradicting $m_3(H) \leq 2$. □

By 13.5.20, $[K^*, V^*] \neq 1$ for some $K \in \mathcal{C}(H)$ with $K^* \cong K/O_2(K)$ quasisimple. *Let K have this meaning for the remainder of the section.*

LEMMA 13.5.21. *(1) $K^* = [K^*, V^*]$ and $L_1 \leq K$.*
(2) $K^*V^* \cong S_6$, A_8, or $G_2(2)'$.
(3) $\tilde{U}_H = [\tilde{U}_H, K]$, and $\tilde{U}_H/C_{\tilde{U}_H}(K^*)$ is the natural module for K^*.
(4) If $n = 6$, then $L/O_2(L) \cong A_6$ rather than $\hat{A}_6$.
(5) $K \trianglelefteq H$, $L_1^*T^*$ is the stabilizer in K^*T^* of the 2-subspace $\tilde{V}_3 = [\tilde{U}_H, V^*]$ of $\tilde{U}_H$, and $U_H = [Q_H, K]$.
(6) $\tilde{U}_H = [\tilde{U}_H, L_1]$.

PROOF. As V^* has order 2 and $V \trianglelefteq T$, $V^* \leq Z(T^*)$. Therefore V^* centralizes $(T \cap K)^* \in Syl_2(K^*)$, and hence normalizes K^* by 1.2.1.3, as does $L_1 = O^2(L_1)$ by that result. Then as $[K^*, V^*] \neq 1$ by choice of K, $K^* = [K^*, V^*]$, establishing the first part of (1).

Define $Y \leq L_1$ as in 13.5.19. As $K^* = [K^*, V^*]$, $K^* \not\leq H_1^*$ by 13.5.19.2, so Y^* does not centralize K^* by 13.5.19.1. Therefore $K^* = [K^*, Y^*]$ as K^* is quasisimple. Then since $Y \leq L_1$, $K^* = [K^*, L_1^*]$.

Let $T_X := N_T(K)$, $X := KL_1T_X$, and $\hat{X} := X/C_X(K^*)$. As $T_X \in Syl_2(N_H(K))$ by 1.2.1.3, $T_X \in Syl_2(X)$. As K^* is quasisimple, $F^*(\hat{X}) = \hat{K}$ is simple. We claim:

(i) $\hat{V}$ is generated by an involution in the center of the Sylow 2-subgroup $\hat{T}_X$ of $\hat{X}$.

(ii) $1 \neq \hat{Y} \leq \hat{L}_1 \leq O_{2,3}(C_{\hat{X}}(\hat{V}))$.

(iii) $[\tilde{U}_H, V] = \tilde{V}_3 = [\tilde{V}_3, L_1]$ is of rank 2.

(iv) If $\langle \hat{V}, \hat{V}^x \rangle$ is not a 2-group, then $\langle \hat{V}, \hat{V}^x \rangle \cong S_3$.

Part (i) follows as $V^* \leq Z(T^*)$ and V^* is of order 2 and faithful on K^*. Part (iii) follows from F.9.5.2, and (iv) is a consequence of F.9.5.6.2. As K^* is quasisimple with $O_2(K^*) = 1$, $C_{\hat{K}}(\hat{V}) = \widehat{C_{K^*}(V^*)}$. Further by F.9.5.3, $C_{K^*}(V^*) = N_K(V)^*$. Then as $L_1 \trianglelefteq H \cap M_V = N_H(V)$, $C_{\hat{K}}(\hat{V})$ acts on $\hat{L}_1$. Thus as L_1T acts on L_1 and $C_{\hat{X}}(\hat{V}) = C_{\hat{K}}(\hat{V})\hat{L}_1\hat{T}_X$, and as we saw earlier that $K^* = [K^*, Y^*]$, we conclude that (ii) holds.

By (iii), $m([\tilde{U}_H, V]) = 2$, so as $m(V^*) = 1$, $q(K^*V^*, \tilde{U}_H) \leq 2$. Therefore B.4.2 and B.4.5 describe K^* and the possible noncentral 2-chief factors W for KV on $\tilde{U}_H$. As $m(K^*V^*, \tilde{U}_H) = 2$, $\hat{K}$ is not one of the sporadic groups in B.4.5; cf. chapter H of Volume I, and recall that the 12-dimensional module for J_2 is the restriction of the natural module for $G_2(4)$. If $\hat{K} \cong A_7$, then by (iv), $\hat{V}$ induces a transposition on $\hat{K}$, contradicting (ii).

In the remaining cases in B.4.2 and B.4.5, $\hat{K}$ is of Lie type over $\mathbf{F}_{2^m}$ for some m. By (i), either $\hat{V}$ is generated by a long-root involution, or $\hat{K} \cong Sp_4(2^m)'$. Then by (iv), $m = 1$, and if $\hat{K}$ is A_6, then $\hat{V} \not\leq \hat{K}$. Furthermore if $K^* \cong \hat{A}_6$, then as $H \in \mathcal{H}^e$, for some choice of W, W is the faithful 6-dimensional module for K^*. But then as $\hat{V} \not\leq \hat{K}$, $m([\tilde{U}_H, V]) \geq 3$, contrary to (iii). Thus K^* is not $\hat{A}_6$, so in particular $Z(K^*) = 1$, and hence $K^* \cong \hat{K}$ is simple. Similarly by (iii), W is never the 10-dimensional module for $K^* \cong L_5(2)$.

The cases remaining appear in B.4.2. Further $\hat{K}\hat{V} \cong L_l(2)$, $3 \leq l \leq 5$, S_6, or $G_2(2)'$, (keeping in mind that $\hat{V} \not\leq \hat{K}$ iff $\hat{K} \cong A_6$) and W is either the natural module for K^*, or the 6-dimensional orthogonal module for $K^* \cong L_4(2)$. As $\hat{V}$ centralizes $\hat{Y} \neq 1$ by (ii), $\hat{K}$ is not $L_3(2)$. Therefore $m_3(K) = 2$, so by A.3.18, $L_1 \leq O^{3'}(H) = K$, completing the proof of (1). Further in each case $m_3(C_{\hat{K}}(\hat{V})) = 1$, so as $\hat{L}_1 \leq C_{\hat{K}}(\hat{V})$, we conclude that $m_3(L_1) = 1$, and hence (4) holds. Next $W = U_1/U_2$ for suitable submodules U_i of U_H, and by (iii),

$$[W, V] \leq V_W := (V_3 \cap U_1)U_2/U_2,$$

and L_1 is irreducible on $\tilde{V}_3$, so $V_W = [V_W, L_1]$ is of rank 2 and $V_W = [W, V]$. This eliminates the possibility that W is a natural module for $L_4(2)$ or $L_5(2)$, since there $\hat{V}$ is a long-root involution, so V induces a transvection on W. Hence W is the natural module for $K^*V^* \cong S_6$, A_8, or $G_2(2)'$, establishing (2). Furthermore by (iii), W is the unique noncentral chief factor for K on U_H. As $\tilde{V}_3 = [\tilde{V}_3, L_1] \leq [\tilde{U}_H, K]$, $\tilde{U}_H = \langle \tilde{V}_3^H \rangle = [\tilde{U}_H, K]$. This completes the proof of (3).

Finally we verify (5) and (6). As $L_1 \leq K$ and T acts on L_1, T acts on K, so $K \trianglelefteq H$ by 1.2.1.3. By (iii), $[\tilde{U}_H, V] = \tilde{V}_3$ is of rank 2 and is $L_1^*T^*$-invariant, and in each of the cases in (2), $P^* := N_{K^*T^*}([\tilde{U}_H, V])$ is a minimal parabolic of K^*T^*, so $|P^* : T^*| = 3$. Thus $P^* = L_1^*T^*$. Further $[V, Q_H] \leq V \cap Q_H = V_3$, so that $U_H = [Q_H, K]$. This completes the proof of (5). From the action of P^* on $\tilde{U}$, we determine that (6) holds in each case. Thus the proof of the lemma is complete. □

By 13.5.21.6, the hypotheses of 13.5.14 are satisfied. Choose l as in 13.5.14, and set $L_- := \langle U_H, U_H^l \rangle$. By 13.5.14, $U_H \leq L_1$, and $L_- = O^2(L_- O_2(LT))$ is described in G.2.4. Further if $n = 5$ then $L_- = L$ by 13.5.14.3. As in G.2.4, let $S := O_2(L_-)$, $S_2 = V(U_H \cap U_H^l)$ and let s denote the number of chief factors for L_- on S/S_2, as in G.2.4.6. We maintain this notation throughout the remainder of the section.

LEMMA 13.5.22. *(1)* $|S| = 2^{4(s+1)}|S_2 : V|$.
(2) L_1 *has* $2s + 2$ *noncentral* 2*-chief factors.*
(3) $|U_H| = 2^{2s+5}|S_2 : V|$.
(4) $U_H \leq L_1$.

PROOF. By G.2.4, L_- has s natural chief factors on S/S_2 and one A_5-factor on V, so (1) and (2) hold. We already observed that (4) holds, and (3) follows from G.2.4.7. □

LEMMA 13.5.23. *K^* is not A_8.*

PROOF. Assume $K^* \cong A_8$. Then by 13.5.21.3 and I.1.6.1, $\tilde{U}_H$ is either the 7-dimensional core of the permutation module for K^*, or its 6-dimensional irreducible quotient, which we regard as an orthogonal space for $K^* \cong \Omega_6^+(2)$. By 13.5.21.5, $[\tilde{U}_H, V^*] = \tilde{V}_3$ is of rank 2, while if $\tilde{U}_H$ were 7-dimensional, then $[\tilde{U}_H, V^*]$ would be of rank 3. Therefore $\tilde{U}_H$ is 6-dimensional orthogonal space. Moreover $C_K(\tilde{V}_2)^*$ is of 3-rank 2, so $n = 5$ by 13.5.4.3.

By 13.5.21.5, $U_H = [Q_H, K]$, and $L_1^*T^*$ is the minimal parabolic of K^*T^* stabilizing the 2-space $\tilde{V}_2$. Thus L_1 has exactly four noncentral 2-chief factors, two on $\tilde{Q}_H$ and two on $O_2(L_1)^* = O_2(L_1^*) \cong Q_8^2$. Therefore by 13.5.22.2, the parameter s of 13.5.22 is equal to 1. Thus by 13.5.22.3,

$$|U_H| = 2^{2s+5}|S_2 : V| = 2^7 \cdot |S_2 : V|. \qquad (*)$$

Next as $\tilde{U}_H$ is the orthogonal module, $U_H \cong D_8^3$ is of order 2^7. Thus $S_2 = V$ by (*), and $|S| = 2^8$ by 13.5.22.1, so

$$|O_2(L_1)| \leq |O_2(\bar{L}_1)| \cdot |S| = 2^2 \cdot 2^8 = 2^{10}. \qquad (**)$$

Further $|O_2(L_1)^*| = 2^5$ using 13.5.21.5, and $U_H \leq O_2(L_1)$ by 13.5.22.4, so that $|O_2(L_1)| \geq 2^{12}$ by (*), contrary to (**). This contradiction completes the proof of 13.5.23. □

LEMMA 13.5.24. *K^* is not A_6.*

PROOF. Assume $K^* \cong A_6$. Then by 13.5.21.3 and I.1.6.1, $\tilde{U}_H$ is either the 5-dimensional core of the permutation module for K^*, or its 4-dimensional irreducible quotient. In either case by 13.5.21.5, $U_H = [Q_H, K]$ and $L_1^*T^*$ is the maximal parabolic stabilizing the line $\tilde{V}_3$. Thus L_1 has two noncentral chief factors on $\tilde{U}_H$, and one on $O_2(L_1^*)$, so L_1 has exactly three noncentral 2-chief factors. This is a contradiction, since by 13.5.22.2, the number of noncentral 2-chief factors of L_1 is even. This completes the proof of 13.5.24. □

LEMMA 13.5.25. *K^* is not $G_2(2)'$.*

PROOF. Assume $K^* \cong G_2(2)'$. Then by 13.5.21.3 and I.1.6.5, $\tilde{U}_H$ is either the 7-dimensional Weyl module for K^* or its 6-dimensional irreducible quotient. However $U_H = Z(U_H)U_0$ where U_0 is extraspecial, and if $\Phi(Z(U_H)) = 1$, then H preserves a quadratic from on $U_H/Z(U)$. Therefore as $G_2(2)'$ does not preserve a quadratic form on its 6-dimensional module, we conclude $m(\tilde{U}) = 7$ and $Z(U_H) = \langle j \rangle \cong \mathbf{Z}_4$.

Let $X \in Syl_3(L_1)$; then $\langle j \rangle = C_{U_H}(X)$. But by 13.5.22.4, $U_H \leq O_2(L_1)$, and by G.2.4.6, S/S_2 is the sum of natural modules for L_-/S. So $j \in C_{O_2(L_1)}(X) \leq S_2$. However $S_2 = V(U_H \cap U_H^l)$ with $\Phi(U_H \cap U_H^l) \leq \Phi(U_H) \cap \Phi(U_H^l) = V_1 \cap V_1^l = 1$, so

$$V_1 = \Phi(\langle j \rangle) \leq \Phi(S_2) = \Phi(V)\Phi(U_H \cap U_H^l) = 1,$$

a contradiction. □

By 13.5.21.2, K^* is A_6, A_8, or $G_2(2)'$. But this contradicts 13.5.24, 13.5.23, and 13.5.25. This contradiction completes the proof of Theorem 13.5.12.

13.6. Finishing the treatment of $\mathbf{A_5}$

In this section, we complete the treatment of the case in the Fundamental Setup where $L/O_2(L) \cong A_5$, using assumption (4) in Hypothesis 13.3.1 as discussed earlier. To do so, we treat the only case remaining after the reduction in the previous section 13.5. We adopt the notational conventions of section B.3 and Notations 12.2.5 and 13.2.1.

We will prove a result summarizing the work of both this section and the previous section:

THEOREM 13.6.1. *Assume Hypothesis 13.3.1 with $L/O_2(L) \cong A_5$. Then G is isomorphic to $U_4(2)$ or $L_4(3)$.*

The groups in Theorem 13.6.1 have already appeared as conclusions in Theorem 13.5.12.1, under the hypothesis that $\langle V_3^{G_1} \rangle$ is nonabelian; we will prove that there are no examples in the remaining case. (Indeed, as far as we can tell, there are not even any shadows.) We assume throughout this section that G is a counterexample to Theorem 13.6.1, and work toward a contradiction. The contribution from the previous section 13.5 is:

LEMMA 13.6.2. *$\langle V^{G_1} \rangle$ is abelian, so V_H is abelian for each $H \in \mathcal{H}_z$.*

PROOF. By Theorem 13.5.12.1, $\langle V_3^{G_1} \rangle$ is abelian; hence $\langle V^{G_1} \rangle$ is abelian by 13.5.10. $\square$

LEMMA 13.6.3. *(1) $C_G(z) \not\leq M$.*
(2) $C_Z(L) = 1$.

PROOF. This follows from 13.3.5.2 and the fact that $M = !\mathcal{M}(LT)$. $\square$

Set $Q := O_2(LT)$, $S := \mathrm{Baum}(T)$, let $v \in V_2 - V_1$, and let $G_v := C_G(v)$ and $M_v := C_M(v)$. In the notation of section B.3, the generator z of V_1 is $e_{1,2,3,4}$, and we may take $v = e_{1,2}$. Set $Q_v := O_2(G_v)$, $\check{G}_v := G_v/\langle v \rangle$, $L_v := O^2(C_L(v))$ and $V_v := \langle z^{L_v} \rangle$. Then $L_v/O_2(L_v)$ is of order 3, $V_v = \langle v \rangle \times [V, L_v]$, and $\check{V}_v = [\check{V}, L_v]$ is a natural module for $L_v/O_2(L_v)$, of rank 2. By 13.2.6.1,

$$T_v := C_T(v) \in Syl_2(G_v).$$

By 13.2.4.2, $S \leq T_v$, so it follows from B.2.3 that:

LEMMA 13.6.4. *$S = Baum(T_v)$ and $J(T) = J(T_v)$.*

Observe also that there is $a \in z^G \cap C_V(L_v)$ (e.g., $a = e_{1,3,4,5}$) and $\check{a} \in Z(\check{T}_v)$.

LEMMA 13.6.5. *$N_{G_v}(S) \leq M_v = C_{M_V}(v)$.*

PROOF. First $N_G(S) \leq M$ by 13.2.5, and then $M_v \leq M_V$ by 12.2.6. $\square$

LEMMA 13.6.6. *$F^*(G_v) = Q_v$.*

PROOF. Assume that $Q_v < F^*(G_v)$. Then by 1.1.4.3, $z \notin Q_v$. By 1.1.6 we can appeal to lemma 1.1.5 with G_v, T_v, G_1, z in the roles of "H, S, M, z". In particular $F^*(C_{G_v}(z)) = O_2(C_{G_v}(z))$ and z inverts $O(G_v)$. On the other hand, $z \in V_v = \langle v \rangle \times [V, L_v]$, and $[V, L_v]$ centralizes $O(G_v)$ by A.1.26.1, so z centralizes $O(G_v)$, and hence $O(G_v) = 1$. Thus there is a component K of G_v. By 1.1.5.3, $K = [K, z]$ and K is described in 1.1.5.3. Also $L_v = O^2(L_v)$ normalizes K by 1.2.1.3, so as $z \in \langle v \rangle L_v$, also $K = [K, L_v]$.

Notice cases (c) and (d) of 1.1.5.3 cannot arise: for in those cases z induces an outer automorphism on K, whereas L_v induces inner automorphisms on K by A.3.18, so that $z \in \langle v \rangle L_v$ does too. In the remaining cases of 1.1.5.3, z induces an inner automorphism of K.

Recall there is $a \in z^G \cap C_V(L_v)$ with $\check{a} \in Z(\check{T}_v)$. As $a \in z^G$, $F^*(C_{G_v}(a)) = O_2(C_{G_v}(a))$ by 1.1.4.3 and 1.1.3.2. Therefore $[K, a] \neq 1$, and then as $\check{a} \in Z(\check{T}_v)$, a normalizes K and $K = [K, a]$. Set $X := N_{G_v}(K)$, $T_X := T_v \cap X$, and $X^* := X/C_X(K)$. Since L_v^* centralizes a^* but not z^*, $a^* \neq z^*$. Then

$$E_4 \cong E^* := \langle a^*, z^* \rangle \leq Z(T_X^*),$$

so neither case (e) or (f) of 1.1.5.3 holds. Thus we have reduced to cases (a) or (b) of 1.1.5.3, with K^* of Lie type over $\mathbf{F}_{2^m}$ for some m. Again as $E^* \leq Z(T_X^*)$, either $m > 1$ and $E^* \leq K^*$, or $X^* \cong S_6$. Suppose the latter case holds. Then $L_v \leq O^{3'}(G_v) = K$ by A.3.18, so $L_v \cong A_4$, and hence L is an A_5-block. By 13.6.3.2, $C_T(L) = 1$. Therefore $V = O_2(LT)$ by C.1.13.c, so $V = O_2(M) = C_G(V)$ by 3.2.11. Thus as $\langle V^{G_1} \rangle$ is abelian by 13.6.2, $G_1 \leq N_G(V) \leq M$, contrary to 13.6.3.1. This contradiction shows that $m > 1$ with $E^* \leq K^*$.

Next we conclude from A.3.18 that one of the following holds:

(i) $L_v \leq O^{3'}(G_v) = K$.

(ii) $m_3(K) \leq 1$.

(iii) $K^* L_v^* \cong PGL_3^{\epsilon}(2^m)$ or $L_3^{\epsilon,\circ}(2^m)$ and $K = O^{3'}(E(G_v))$.

We claim that T_v normalizes K; assume otherwise and let $K_0 := \langle K^{T_v} \rangle$ and $T_0 := T_v \cap K_0$. In (iii), $K \trianglelefteq G_v$; and in (i), T_v acts on K since T_v acts on L_v. Therefore we may assume that (ii) holds. Comparing the groups in (a) or (b) of 1.1.5.3 to those in 1.2.1.3, we conclude that $K^* \cong L_2(2^m)$ or $Sz(2^m)$. If $K^* \cong L_2(2^m)$ then by 1.2.2.a, $L_v \leq O^{3'}(G_v) = K_0$, so $L_v^* \leq K_0^* = K^*$ and L_v^* centralizes a^*, impossible as involutions of K^* are centralized only by a Sylow 2-subgroup. Therefore $K^* \cong Sz(2^m)$, so $K_0/Z(K_0) \cong Sz(2^m) \times Sz(2^m)$. Let B be the Borel subgroup of K_0 containing T_0, and set $V_0 := \Omega_1(T_0)$. Then (using I.2.2.4 when $Z(K_0) \neq 1$ and in particular $m = 3$) $J(T_v)$ centralizes V_0, so by B.2.3.5, $\mathrm{Baum}(T_v) \trianglelefteq B$. Then $B \leq C_{M_V}(v)$ by 13.6.4 and 13.6.5. Now $O^2(B) \leq O^{2,3}(C_{M_V}(v)) \leq C_M(V)$. Hence $[L_v, O^2(B)] \leq O_2(L_v)$, impossible as as subgroup of order 3 in L_v induces field automorphisms on K. This contradiction completes the proof of the claim that T_v normalizes K.

Hence $T_v = T_X$. Also $L_v^* \leq K^*$ in cases (i) and (ii); this is clear in (i), and it holds in (ii) as $L_v = [L_v, T_v]$ while $Out(K^*)$ is abelian when $m_3(K^*) \leq 1$.

Next as T_v acts on L_v, by inspection of the groups in (a) or (b) of 1.1.5.3, L_v^* is contained in a Borel subgroup B^* of $K^* L_v^*$. Hence as

$$[L_v^*, z^*] \neq 1 = [L_v^*, a^*]$$

and $E^* \leq Z(T_X^*)$, we conclude $K^* \cong Sp_4(2^m)$, as otherwise $Z(T_X^* \cap K^*) =: R^*$ is a root subgroup of K^*, so $C_{B^*}(R^*) = C_{B^*}(z^*)$.

As $m > 1$, K is simple by I.1.3, so $J(T_v) = J_K \times J(T_C)$, where $T_K := T_v \cap K$ and $T_C := C_{T_v}(K)$. Thus $Z_J := \Omega_1(Z(J(T_v))) = Z_K \times Z_C$, where $Z_K := Z(T_K)$ and $Z_C := \Omega_1(Z(J(T_C)))$. Then using 13.6.4,

$$S = \mathrm{Baum}(T) = \mathrm{Baum}(T_v) = T_K \times S_C,$$

where $S_C := \mathrm{Baum}(T_C)$. Therefore $B \leq N_G(S)$, so as before $B \leq C_{M_V}(v)$ by 13.6.5. Let $B_C := O^2(C_B(V))$. Since $|B : O_2(B)| > 3$ and $\bar{B} = \bar{L}_v$ with $|\bar{L}_v :$

$O_2(\bar{L}_v)| = 3$, we conclude $|B : B_C O_2(B)| = 3$ and $B_C \neq 1$. Then $L_v/O_2(L_v)$ is inverted in $T_v \cap L \leq C_{T_v}(B_C/O_2(B_C))$. This is impossible since each element in $Aut(Sp_4(2^m))$ acting on B and inverting an element of order 3 in B^* induces a field automorphism on K^* inverting $\Omega_1(O_3(B^*/O_2(B^*))) \cong E_9$. This contradiction completes the proof of 13.6.6. □

We come to the main technical result of the section, requiring the bulk of the argument; afterwards the remainder of the proof of Theorem 13.6.1 is fairly brief.

THEOREM 13.6.7. *(1) $G_v \leq M_V$. Hence $G_v = C_{M_V}(v)$.*
(2) $N_G(V_v) \leq M$.

Until the proof of Theorem 13.6.7 is complete, assume G is a counterexample. We begin a series of reductions.

Recall $V_v = \langle v\rangle \times [V, L_v]$ with $\{v\} \cup [V, L_v]^\#$ the set of nonsingular vectors in V_v. Therefore by 13.2.6.2, $N_G(V_v)$ acts on the three singular vectors of V_v, and hence preserves their product v—so that $N_G(V_v) \leq G_v$, and hence (1) implies (2). On the other hand, if $V_v \trianglelefteq G_v$, then G_v permutes

$$\mathcal{V} := \{V_u : u \in V_v \text{ and } u \text{ is nonsingular}\},$$

so G_v acts on $V = \langle \mathcal{V}\rangle$, and hence $G_v \leq N_G(V) = M_V$, so (1) holds. Thus we may assume:

LEMMA 13.6.8. *$G_v > M_v$ and V_v is not normal in G_v.*

Set $U_v := \langle z^{G_v}\rangle$ and $G_v^* := G_v/C_{G_v}(U_v)$. Regard G_v^* as a subgroup of $GL(U_v)$ and write u^{x^*} for the image of $u \in U_v$ under $x^* \in G_v^*$.

As $L_v \leq G_v$, $V_v \leq U_v$. As $z \in Z(T_v)$, by 13.6.6 we may apply B.2.14 to conclude $U_v \in \mathcal{R}_2(G_v)$. In particular $O_2(G_v^*) = 1$ and $U_v \leq Z(Q_v)$.

If $[U_v, J(T_v)] = 1$, then $S = \mathrm{Baum}(C_{T_v}(U_v))$ by B.2.3.5 and 13.6.4, so by a Frattini Argument and 13.6.5,

$$G_v = C_{G_v}(U_v)N_{G_v}(S) \leq C_G(U_v)C_{M_V}(v).$$

But then $V_v^{G_v} = V_v^{M_v} = \{V_v\}$, contrary to 13.6.8. Hence

LEMMA 13.6.9. *$J(G_v)^* \neq 1$.*

Thus U_v is an FF-module for G_v^*.

LEMMA 13.6.10. *If $L_v^* = [L_v^*, J(T_v)^*]$, then L_v^* is not subnormal in G_v^*.*

PROOF. Suppose otherwise. Then $O_2(L_v^*) \leq O_2(G_v^*) = 1$, so that L_v^* has order 3. Further $m([U_v, L_v^*]) = 2$ by Theorem B.5.6, so $[U_v, L_v] = [V, L_v]$, and hence $V_v = \langle v\rangle \times [V, L_v] = \langle v\rangle [U_v, L_v]$. Now Theorem B.5.6 also shows that $|L_v^{*G_v^*}| \leq 2$, so as L_v is T_v-invariant, L_v^* is normal in G_v^*, so that $\langle v\rangle[U_v, L_v] = V_v$ is G_v-invariant, contrary to 13.6.8. □

Let $\mathcal{X}_0$ be the set of $L_v^* T_v^*$-invariant subgroups $X^* = O^2(X^*)$ of G_v^* such that $1 \neq X^* = [X^*, J(T_v)^*]$. Let $\mathcal{X}$ denote the set of all members of $\mathcal{X}_0$ normal in G_v^*, and $\mathcal{X}_z$ the set of those X^* in $\mathcal{X}_0$ with $[z, X^*] \neq 1$. For $X^* \in \mathcal{X}_0$, set $U_X := [\langle z^{X^* L_v^*}\rangle, X^*]$.

LEMMA 13.6.11. *For each $X^* \in \mathcal{X}_z$, $[z, X^*, L_v^*] \neq 1$, so $[U_X, L_v^*] \neq 1$.*

PROOF. Let $U := [z, X^*] \neq 1$ and suppose that $[U, L_v^*] = 1$. Then $[X^*, L_v^*] \leq C_{X^*}(U) \leq C_{X^*}(z)$ by Coprime Action, so

$$1 = [X^*, L_v^*, z] = [z, X^*, L_v^*],$$

and hence by the Three-Subgroup Lemma, $[L_v^*, z, X^*] = 1$. Then X^* centralizes $\langle v\rangle[z, L_v^*]$ which contains z, contrary to our choice of X^*. □

LEMMA 13.6.12. $\mathcal{X} \subseteq \mathcal{X}_z$.

PROOF. If $X^* \in \mathcal{X}$ with $[z, X^*] = 1$, then as $X^* \trianglelefteq G_v^*$, X^* centralizes $\langle z^{G_v^*}\rangle = U_v$, contrary to $X^* \neq 1$. □

LEMMA 13.6.13. *No member of $\mathcal{X}$ is solvable.*

PROOF. Suppose $X^* \in \mathcal{X}$ is solvable, and choose X^* minimal subject to this constraint. By Theorem B.5.6, $X^* = O^2(H^*)$ for some normal subgroup H^* of G_v^* with $H^* = J(H)^* = H_1^* \times \cdots \times H_s^*$ where $H_i^* \cong S_3$ and $s \leq 2$. Further $U_H := [U_v, H] = U_1 \oplus \cdots \oplus U_s$, where $U_i := [U_H, H_i]$ is of rank 2. By minimality of X^*, T_v^* is transitive on the H_i^*, so $X^*T_v^*$ is irreducible on U_H. Now $[z, X] \neq 1$ by 13.6.12, so $U_X = U_H$ as $X^*T_v^*$ is irreducible on U_H. By 13.6.11 $[U_H, L_v^*] \neq 1$, so the projection of L_v^* on H^* with respect to the decomposition $H^* \times C_{G_v^*}(U_X)$ is nontrivial. Then as L_v is T_v-invariant, it follows that $L_v^* = [L_v^*, J(T)^*] = O^2(H_i^*)$ for some i, contrary to 13.6.10. □

We conclude from 13.6.13 that $F(J(G_v)^*) = Z(J(G_v)^*)$. Then by 13.6.9 and Theorem B.5.6, $J(G_v)^*$ is a product of components of G_v^*. By C.1.16, $J(T_v)^*$ normalizes the components of G_v^*. Thus there exists $K_+ \in \mathcal{C}(G_v)$ such that K_+^* is a component of G_v^* and $K_+^* = [K_+^*, J(T_v)]$. Thus $\langle K_+^{*T_v}\rangle$ is normal in G_v^* by 1.2.1.3, and so lies in $\mathcal{X}$. Hence $\langle K_+^{*T_v}\rangle \in \mathcal{X}_z$ by 13.6.12.

Let $\mathcal{Y}_z$ consist of those $K \in \mathcal{L}(G_v, T_v)$ such that $K^*/O_2(K^*)$ is quasisimple and $\langle K_v^{*T_v^*}\rangle \in \mathcal{X}_z$. By the previous paragraph, $K_+ \in \mathcal{Y}_z$, so $\mathcal{Y}_z$ is nonempty. Observe that if $K \in \mathcal{Y}_z$ and $K_0 \in \mathcal{L}(G_v, T_v)$ with $K_0^* = K^*$, then $K_0 \in \mathcal{Y}_z$.

For $K \in \mathcal{Y}_z$, let $K_- := \langle K^{T_v}\rangle$, $W_K := \langle z^{K_-L_v}\rangle$, and set $(K_-L_vT_v)^+ := K_-L_vT_v/C_{K_-L_vT_v}(W_K)$. Since $F^*(G_v) = O_2(G_v)$, $W_K \in \mathcal{R}_2(K_-L_vT_v)$ by B.2.14.

In the remainder of the proof of Theorem 13.6.7, let $K \in \mathcal{Y}_z$.

Then K^+ is a quotient of $K^*/O_2(K^*)$, so K^+ is also quasisimple, and W_K is an FF-module for $K_-^+T_v^+$. Also the action of $K_-^+L_v^+T_v^+$ on W_K is described in Theorem B.5.6.

LEMMA 13.6.14. *K is T_v-invariant, $K^* \in \mathcal{X}_z$, and $U_K = [W_K, K]$.*

PROOF. Assume 13.6.14 fails. By 1.2.1.3, $K_- = KK^t$ for some $t \in T_v - N_{T_v}(K)$, and comparing the list of groups in 1.2.1.3 to that in Theorem B.5.6, $K^+ \cong L_2(2^m)$ or $L_3(2)$. Then by 1.2.2, $L_v \leq K_-$. Since L_v is T_v-invariant with $L_v/O_2(L_v)$ of order 3, L_v^+ is diagonally embedded in K_-^+, K^+ is not $L_3(2)$, and $K_-^+T_v^+$ is not S_5 wr $\mathbf{Z}_2$. Therefore by Theorem B.5.6, $U_{K_-} = U_KU_K^t$, where $U_K := [W_K, K]$ and $U_K/C_{U_K}(K)$ is the natural module for $K^+ \cong L_2(2^m)$. Thus by E.2.3.2, $\mathrm{Baum}(T_v)$ is normal in the preimage B of the Borel subgroup B^+ of K_-^+ normalizing $(T_v \cap K_-)^+$. But $S = \mathrm{Baum}(T_v)$ by 13.6.4 and $N_B(S) \leq M_V$ by 13.6.5, so $B \leq M_V$. As $z \in Z(T_v)$, the projection of z on U_{K_-} is diagonally embedded in $U_KU_K^t$, so that $C_B(V) \leq C_B(\langle z^B\rangle) = O_2(B)$. This is a contradiction as $B/O_2(B)$ is noncyclic of odd order, while $O^2(\bar{M}_V) \cong A_5$ had cyclic Sylow groups for odd primes. This contradiction completes the proof. $\square$

LEMMA 13.6.15. *If $K^+ \cong A_5$, then no $I \in Irr_+(K, \langle z^K \rangle)$ is the A_5-module.*

PROOF. Assume $K^+/O_2(K^+) \cong A_5$ and some $I \in Irr_+(K, \langle z^K \rangle)$ is the A_5-module. Then as $K^+ = [K^+, J(T_v)^+]$, $U_K = I$ by Theorem B.5.6.

Further as there are no strong FF*-offenders on I by B.4.2.5, by that result and B.2.9.1, there is $A \in \mathcal{A}(T)$ with A^+ of order 2 inducing a transposition on K^+ and $K^+ = [K^+ L_v^+, A^+] = K^+$. By 13.6.11, $[U_K, L_v] \neq 1$. Then as T_v acts on L_v, the projection L_K^+ of L_v^+ on K^+ is a Borel subgroup of K^+ with $L_K^+ = [L_K^+, A]$. Therefore $L_v^+ = [L_v^+, A^+] = L_K^+$ as L_v is T_v-invariant and $K^+ = [K^+ L_v^+, A^+]$. As $L_v^+ = L_K^+$ and U_K is the A_5-module, the L_v-module $[W_K, L_v] = [U_K, L_K]$ is an indecomposable extension of the trivial module $C_{U_K}(L_v^+)$ by a natural module for $L_v^+/O_2(L_v^+) \cong \mathbf{Z}_3$. This is a contradiction, as $[V, L_v]$ is an L_v-submodule of $[W_K, L_v]$ of rank 2. $\square$

LEMMA 13.6.16. *Assume $K_0 \in \mathcal{L}(G_v, T_v)$ is T_v-invariant with $[z, K_0] \neq 1$. Then*

(1) $O_2(\langle K_0, T \rangle) = 1$.

(2) If C is a nontrivial characteristic subgroup of T_v, then $O_\infty(K_0) N_{K_0}(C) < K_0$.

(3) $K_0 = [K_0, J(T_v)]$.

PROOF. Let $H := \langle K_0, T \rangle$ and assume $O_2(H) \neq 1$; then $H \in \mathcal{H}(T)$. As $|T : T_v| = 2$, by 1.2.5 there is $K_2 \in \mathcal{C}(N_G(O_2(H)))$ containing K_0. As $[z, K_0] \neq 1$, $K_2 \in \mathcal{L}_f(G, T)$. By 13.3.2.2, $K_2 \in \mathcal{L}_f^*(G, T)$. We now make a particularly fundamental use of the special assumption in part (4) of Hypothesis 13.3.1 that we have chosen L with $L/O_2(L) \cong A_5$ only in the final case of the FSU when no other choice was possible: namely by Hypothesis 13.3.1.4, $K_2/O_2(K_2) \cong A_5$. Thus as A_5 is a minimal nonsolvable group, $K_2 = O_2(K_2)K_0$, and then as $|T : T_v| = 2$ and K_2 is perfect, $K_2 = K_0$. As v centralizes K_0, $1 \neq C_T(K_2)$, contrary to 13.6.3.2, since by 13.3.2 K_2 satisfies Hypothesis 13.3.1 in the role of "L". This completes the proof of (1).

Next assume thht C is a nontrivial characteristic subgroup of T_v with $K_0 = O_\infty(K_0) N_{K_0}(C)$. Then there is $K_1 \in \mathcal{L}(N_{K_0}(C), T_v)$ with $K_0 = O_\infty(K_0) K_1$. Replacing K_0 by K_1, we may assume K_0 acts on C. Since T_v is of index 2 in T, C is normal in T, and hence $1 \neq C \leq O_2(\langle K_0, T \rangle)$, contrary to (1). This establishes (2).

Finally if $J(T_v) \leq O_\infty(K_0)$, then $K_0 = O_\infty(K_0) N_{K_0}(J(T_v))$ by a Frattini Argument, contrary to (2). Thus (3) holds. $\square$

LEMMA 13.6.17. *Assume $a := z^g$ with $\tilde{a} \in Z(\tilde{T}_v)$. Then $[a, K] \neq 1$.*

PROOF. Assume $K \leq G_a := C_G(a)$. As a centralizes a subgroup of T_v of index 2, $|O_2(G_a) : (O_2(G_a) \cap N_G(K))| \leq 4$. Thus as $K = K^\infty$, K centralizes $O_2(G_a)/(O_2(G_a) \cap N_G(K))$, and hence $K \trianglelefteq KO_2(G_a)$. Since $[z, K] \neq 1$ with $z \in U_v \in \mathcal{R}_2(G_v)$, $V(K) \neq 1$ in the language of Definition A.4.7. Then $[Z(O_2(KO_2(G_a))), K] \neq 1$ by A.4.9 with K, $KO_2(G_a)$ in the roles of "X, M". Then as $G_a \in \mathcal{H}^e$, K does not centralize $Z_a := Z(O_2(G_a))$. By 1.2.1.1, $\langle K^{T^g} \rangle =: K_0 \leq \langle \mathcal{C}(G_a) \rangle$, so as $[Z_a, K] \neq 1$, some $K_1 \in \mathcal{C}(K_0)$ is in $\mathcal{L}_f(G, T^g)$ by A.4.9 with

K_1, G_a in the roles of "X, M". Thus as $a \in Z(T^g)$ centralizes K_1, 13.6.3.2 applied to K_1 in the role of "L" supplies a contradiction. □

We now begin to eliminate various cases for K^+ from the list of possible quasisimple groups in Theorem B.4.2.

LEMMA 13.6.18. *K^+ is not $L_2(2^m)$.*

PROOF. Assume otherwise. Then by Theorem B.5.6 and 13.6.15, $U_K/C_{U_K}(K)$ is the natural module for $K^+ \cong L_2(2^m)$. Let K_0 be a minimal member of $\mathcal{L}(KT_v, T_v)$. Then $K_0^* = K^*$, so $K_0 \in \mathcal{Y}_z$, and by minimality of K_0, K_0 is a minimal parabolic in the sense of Definition B.6.1. Then by 13.6.16.2 and C.1.26, K_0 is an $L_2(2^m)$-block. Hence replacing K by K_0, we may assume K is a block. Then by E.2.3.2, $J(T_v)$ is normal in the Borel subgroup B of KT_v over T_v, and $S = \mathrm{Baum}(T_v)$ by 13.6.4, so B acts on S in view of B.2.3.4. Hence $B \leq C_{M_V}(v)$ by 13.6.5. Thus $O^2(B) \leq O^2(C_{M_V}(v)) \leq L_v C_M(V)$, so that $[V, O^{2,3}(B)] = 1$. But if $n > 2$, then $z \in C_{W_K}(O^{2,3}(B)) \leq C_{W_K}(K)$ as $U_K/C_{U_K}(K)$ is the natural module for $L_2(2^m)$, contradicting $[z, K] \neq 1$. Thus $n = 2$ and $BC_M(V) = L_v C_M(V)$. Then B centralizes the element $a := z^g \in C_V(L_v)$ with $\check{a} \in Z(\check{T}_v)$ described before 13.6.5. Therefore as B contains a Borel subgroup of K, and K is an $L_2(4)$-block, $K \leq C_G(a)$. This contradicts 13.6.17, completing the proof of the lemma. □

LEMMA 13.6.19. *K^+ is not $SL_3(2^m)$, $Sp_4(2^m)$, or $G_2(2^m)$ with $m > 1$.*

PROOF. If the lemma fails, then for some maximal parabolic P of K containing $T_v \cap K$, $K_1 := P^\infty$ does not centralize z. Then $K_1 \in \mathcal{L}(G_v, T_v)$ with $K_1^+/O_2(K_1^+) \cong L_2(2^m)$. As K_1 is not a block, this contradicts 13.6.16.2 in view of C.1.26. □

By Theorem B.5.6, K^+ is either a Chevalley group over a field of characteristic 2 in Theorem C (A.2.3), or $\hat{A}_6$ or A_7. Lemmas 13.6.18 and 13.6.19 say in the former case that K^+ is a group over $\mathbf{F}_2$. Therefore the list of B.5.6 is reduced to $K^+ \cong L_3(2)$, $Sp_4(2)'$, $G_2(2)'$, $\hat{A}_6$, A_7, $L_4(2)$, or $L_5(2)$. We next show:

LEMMA 13.6.20. *$L_v \leq K$.*

PROOF. If $m_3(K) = 2$, then by A.3.18, $L_v \leq \theta(KL_v) = K$. Thus we may assume $m_3(K) = 1$, so $K^+ \cong L_3(2)$. By Theorems B.5.1 and B.5.6, either $U_K/C_{U_K}(K)$ is a natural module for K^+, or U_K is the sum of two isomorphic natural modules for K^+. By 13.6.11, $[U_K, L_v] \neq 1$. So either $K^+ = [K^+, L_v^+]$, or $[K^+, L_v^+] = 1$ and U_K is the sum of two isomorphic natural modules for K^+, with $L_v^+ \leq Aut_{K^+}(U_K) \cong L_2(2)$.

Assume first that $[K^+, L_v^+] = 1$. Then $J(T_v)^+$ is the unipotent radical $O_2(P^+)$ of the maximal parabolic P^+ of K^+ stabilizing a line in each summand of U_K, and $S^+ = J(T_v)^+$ by B.2.20. Therefore since $S = \mathrm{Baum}(T_v)$ by 13.6.4, $P^+ = N_K(S)^+$ by a Frattini Argument, while $N_K(S) \leq C_{M_V}(v)$ by 13.6.5. But as $[K^+, L_v^+] = 1$, $O^2(N_K(S))$ acts on $O^2(O_2(K)L_v) = L_v$, and hence as $N_K(S) \leq M_v$, $O^2(N_K(S))$ also acts on $[V, L_v] = [z, L_v] \cong E_4$. But $[z, L_v]$ contains a point of each summand of U_K, and so is generated by those two points; whereas we saw that P is the stabilizer of a line in each summand, so that $P^+ = N_P(S)^+$ acts irreducibly on each such line.

Therefore $[K^+, L_v^+] \neq 1$. Hence the projection L_K^+ of L_v^+ on K^+ is nontrivial. So as L_K^+ is T_v-invariant, it is a maximal parabolic of $K^+ \cong L_3(2)$, and hence $L_K = [L_K, T_v \cap K]$. Then as L_v is T_v-invariant, $L_v = L_K \leq K$, as desired. □

LEMMA 13.6.21. *$K \not\leq M$.*

PROOF. By 13.6.20, K^* does not act on L_v^*, so as $L_v \trianglelefteq M_v$, 13.6.21 holds. $\square$

LEMMA 13.6.22. *K^+ is not $L_3(2)$, $Sp_4(2)'$, $G_2(2)'$, $\hat{A}_6$, or A_7.*

PROOF. Assume otherwise. Let K_0 be minimal subject to $K_0 \in \mathcal{L}(KT_v, T_v)$ and $K_0^* = K^*$. Then $K_0/O_2(K_0)$ is quasisimple by minimality of K_0, and $K_0 \in \mathcal{Y}_z$ as $K_0^* = K^*$, so replacing K by K_0, we may assume $K/O_2(K)$ is quasisimple.

If K^+ is not A_7, then using 13.6.6 and 13.6.16.2, (KT_v, T_v) is an MS-pair in the sense of Definition C.1.31. So by C.1.32, either K is a block of type A_6, $\hat{A}_6$, or $G_2(2)$, or K^+ is $L_3(2)$ and by C.1.32.5, K is described in C.1.34. Similarly if K^+ is A_7, C.1.24 says K is an A_7-block or exceptional A_7-block. Set $U := [Z(O_2(K)), K]$. When K is a block, $U_K = U \in Irr_+(K, W_K)$. If K is not a block, then $K/O_2(K) \cong L_3(2)$ and K is described in C.1.34, so $U = U_K$ is a sum of at most two isomorphic natural modules for $L_3(2)$.

As $L_v \leq K$ by 13.6.20, L_v^+ is a T_v^+-invariant $\{2,3\}$-subgroup of K^+ with Sylow 3-group of order 3. Set $P := L_v(T_v \cap K)$. When K^+ is $L_3(2)$, $Sp_4(2)'$, or $G_2(2)'$, P^+ is a minimal parabolic.

Suppose first that K is an A_7-block. Then by B.3.2.4 and B.2.9.1, $J(T_v)^+$ is the subgroup of T_v^+ generated by its three transpositions, and $S^+ = J(T_v)^+$ by B.2.20. Further $N_{K^+}(S^+) = N_K(S)^+$ by a Frattini Argument, and $N_K(S) \leq M_v$ by 13.6.5. From the structure of S_7, $N_{KT_v}(S)$ is maximal in KT_v subject to containing a normal subgroup $\{2,3\}$-subgroup which is not a 2-group, so it follows that $N_{KT_v}(S) = (T_v \cap K)L_v$ and $L_v = O^2(N_K(S))$. Now $L_vT_v \leq K_1T_v \leq KT_v$ with $K_1/O_2(K_1) \cong A_6$, and as $[z, L_v] \neq 1$, $[z, K_1] \neq 1$. Further $K_1^+ = [K_1^+, J(T_v)^+]$, so $K_1 \in \mathcal{Y}_z$; thus replacing K by K_1, we may assume K is not an ordinary A_7-block.

Similarly if K is an $\hat{A}_6$-block, then U has the structure of a 3-dimensional $\mathbf{F}_4K$-module and $J(T_v)^+$ is the 4-subgroup of $T_v^+ \cap K^+$ centralizing an $\mathbf{F}_4$-line U_2 of U, so $S^+ = J(T_v)^+$, $N_K(S)^+ = N_K(U_2)^+$, and hence $N_K(U_2) \leq M_V$ by 13.6.5.

Let $l := [V, L_v]$. Then l is an L_vT_v-invariant line in $[z, L_v] \leq [z, K] \leq U$ with $l = [l, L_v]$. It follows that if $K/O_2(K) \cong L_3(2)$, then $m(U) \neq 4$, since in that case no minimal parabolic of K^+ acts on such a line (cf. B.4.8.2). If K is an $\hat{A}_6$-block, then from the previous paragraph, $N_K(U_2) \leq M_V$, so $N_K(U_2)$ acts on l, a contradiction as $N_K(U_2)$ acts on no E_4-subgroup of U.

Let $\hat{U} := U/C_U(K)$. In the remaining cases, if K is irreducible on $\hat{U}$, then there is a unique T_v-invariant line in U, so $\hat{l}$ is that line. Then if K is not an exceptional A_7-block, P^+ is the parabolic stabilizing $\hat{l}$, while if K is an exceptional A_7-block, then L_v^+ is one of the three T_v-invariant subgroups $L_0 = O^2(L_0)$ of $N_K(\hat{l})$ with $L_0/O_2(L_0) \cong \mathbf{Z}_3$ and $\hat{l} = [\hat{l}, L_0]$. If K is not irreducible on $\hat{U}$, then U is the sum of two isomorphic modules for $K^+ \cong L_3(2)$, $\hat{l}$ is a T_v-invariant line in one of those irreducibles, and $P = N_K(l)$. For our purposes the important fact is that in each case $C_{\hat{U}}(L_v) = 0$, so $C_{\check{U}}(L_v) = C_{\check{U}}(K)$.

Let $Z_K := Z(O_2(K))$ and Z_0 the preimage in K of $Z(O_2(\check{K}))$. If K is a block, then by definition $U = [Z_K, K] = [O_2(K), K]$, so $[Z_0, K] = U$. If K is not a block, then from the description of K in C.1.34, again $[Z_0, K] = U$. So in any event $[Z_0, K] = U$.

Now recall there is $a \in z^G \cap C_V(L_v)$ with $\check{a} \in Z(\check{T}_v) \leq Z(O_2(\check{K}\check{T}_v))$ since $F^*(\check{K}\check{T}_v) = O_2(\check{K}\check{T}_v)$. Then

$$[\check{a}, K] \leq [Z(O_2(\check{K})), K] = \check{U},$$

by the previous paragraph, so K acts on $\check{F} := \langle \check{a} \rangle \check{U}$. Further by B.2.14, $\check{F} = \check{U}C_{\check{F}}(K)$. Then as we saw earlier that $C_{\check{U}}(L_v) = C_{\check{U}}(K)$, it follows that K centralizes $\check{a}$, and hence K centralizes a by Coprime Action, contrary to 13.6.17. □

LEMMA 13.6.23. *(1)* $K = O^{3'}(G_v)$.
(2) $K^+ \cong L_4(2)$ *and* T_v^+ *is nontrivial on the Dynkin diagram of* K^+.
(3) $L_v^+ T_K^+$ *is the middle-node minimal parabolic of* $K^+T_K^+$, *where* $T_K := T_v \cap K$.

PROOF. By 13.6.22 and the remarks before 13.6.20, we have reduced to the cases where $K^+ \cong L_m(2)$ for $m = 4$ or 5. Since $L_v \leq K$ by 13.6.20 we conclude as in the proof of 13.6.22 that $L_v^+ T_K^+$ is a minimal parabolic of K^+.

If T_v is trivial on the Dynkin diagram of K^+, then T_v acts on a parabolic K_1^+ of K^+ containing L_v^+ with $K_1^+/O_2(K_1^+) \cong L_3(2)$. However as $[z, L_v] \neq 1$, also $[z, K_1] \neq 1$. By 13.6.16.3, $K_1^+ = [K_1^+, J(T_v)]^+$ so that $K_1^\infty \in \mathcal{Y}_z$ and 13.6.22 supplies a contradiction.

Hence T_v is nontrivial on the Dynkin diagram of K^+. So as T_v acts on the minimal parabolic L_vT_K , $m = 4$ and L_vT_K is the middle-node minimal parabolic of K. Thus (2) and (3) hold.

By 1.2.4, $K \leq K_+ \in \mathcal{C}(G_v)$; then $K_+ \in \mathcal{Y}_z$, so by symmetry between K_+ and K, $K/O_2(K) \cong L_4(2) \cong K_+/O_2(K_+)$, and hence $K = K_+$. Then by A.3.18, $K = O^{3'}(G_v)$, so (1) is established. □

By 13.6.23, $K^+T_v^+ \cong S_8$ with $L_v^+T_K^+$ the middle-node minimal parabolic of K^+. As $[z, L_v] \neq 1$ and U_K is an FF-module for K^+, we conclude from Theorem B.5.1 that $U_K/C_{U_K}(K)$ is the 6-dimensional orthogonal module. Thus $C_K(z)$ is the maximal parabolic determined by the end nodes, so using 13.6.23.1 we conclude that

$$X := O^{3'}(C_K(z)) = O^{3'}(C_{G_v}(z)) = O^{3'}(C_G(V_2)),$$

and hence that X is T-invariant and $XT_v/R_v \cong S_3 \text{ wr } \mathbf{Z}_2$, where $R_v := O_2(XT_v)$. As $U_K/C_{U_K}(K)$ is the orthogonal module, $J(R_v) = J(O_2(KT_v))$ by B.3.2.4. But as T acts on X and T_v, T acts on R_v, so that $J(R_v) \trianglelefteq \langle K, T \rangle$, contrary to 13.6.16.1. This contradiction finally completes the proof of Theorem 13.6.7.

With Theorem 13.6.7 now in hand, we can now use elementary techniques such as weak closure in a fairly short argument to complete the proof of Theorem 13.6.1.

LEMMA 13.6.24. *If* $g \in G - N_G(V)$ *with* $V \cap V^g \neq 1$, *then*
(1) $V \cap V^g$ *is a singular point of* V, *and*
(2) $[V, V^g] = 1$.

PROOF. By 13.3.11.1, G_v is transitive on $\{V^x : v \in V^x\}$, so as $G_v \leq M_V$ by Theorem 13.6.7.1, V is the unique member of V^G containing v. Hence $V \cap V^g$ is totally singular, so that (1) holds. In particular conjugating in L we may assume $V \cap V^g = V_1$, and then take $g \in C_G(z) = G_1$ by 13.3.11.1. Hence (2) follows from 13.6.2. □

LEMMA 13.6.25. *(1)* $W_i(T,V) \trianglelefteq LT$ *for* $i = 0,1$.
(2) $n(H_1) > 1$ *for each* $H_1 \in \mathcal{H}(T,M)$.
(3) Each solvable member of $\mathcal{H}(T)$ *is contained in* M.
(4) $r(G,V) = 3$ *and* $w(G,V) > 1$.

PROOF. We first observe that by 13.6.3.1 and 13.6.24.1, $r(G,V) = 3$. Let $g \in G - M$ with $A := V^g \cap M \leq T$ and B a hyperplane of A. Suppose $m(V^g/A) \leq 1$ but $[V,A] \neq 1$. Then $m(V^g/B) \leq 2 < r(G,V)$, so $C_V(B) \leq N_G(V^g)$ and hence $[C_V(B),A] \leq V \cap V^g$; therefore $[C_V(B),A] = 1$ by 13.6.24.2. Thus $\bar{A} \in \mathcal{A}_2(\bar{M}_V,V)$, whereas we compute directly that $a(\bar{M}_V,V) = 1$. This contradiction shows that $W_i(T,V) \leq C_T(V) = O_2(LT)$ for $i = 0,1$, establishing (1).

By (1), $w(G,V) > 1$, where $w(G,V)$ appears in Definition E.3.23; this completes the proof of (4). By (4), $\min\{r(G,V), w(G,V)\} > 1$, so (2) and (3) follow from E.3.35.1. □

LEMMA 13.6.26. *For* $H \in \mathcal{H}_z$*:*
(1) No member of $\mathcal{C}(H)$ *is contained in* M.
(2) $O_{2,p}(H) = Q_H$ *for each prime* $p > 3$.

PROOF. Assume $K \in \mathcal{C}(H)$. Part (1) follows from 13.3.8.2 with L, M, $\langle K^T \rangle$ in the roles of "K, M_K, Y". Let $p > 3$ be prime and set $X := O^{p'}(O_{2,p}(H))$. By 13.6.25.3, $X \leq M$, so as $p > 3$, $X = 1$ by 13.3.8.2. Hence (2) holds. □

LEMMA 13.6.27. *There exists* $K \in \mathcal{C}(G_1)$ *such that one of the following holds:*
(1) $G_1 = KT$ *and* $K/O_2(K) \cong J_2$ *or* M_{23}.
(2) $K/O_2(K) \cong L_3(4)$*,* T *is nontrivial on the Dynkin diagram of* $K/O_2(K)$*,* $G_1 = O^{3'}(G_1)T$*, and either* $K = O^{3'}(G_1)$ *or* $O^{3'}(G_1/O_2(G_1)) \cong PGL_3(4)$.

PROOF. By 3.3.2 there exists $H_1 \in \mathcal{H}_*(T,M)$. By 13.3.5.2, $H_1 \leq G_1$, and by 13.6.25.2, $n(H_1) > 1$. Now we apply Theorem 5.2.3: Hypothesis 13.3.1 rules out conclusions (2) and (3) of that Theorem, so we are left with conclusion (1) of 5.2.3. In particular $K_1 := O^2(H_1)$ lies in some $K \in \mathcal{C}(C_G(Z))$. As T normalizes K_1, it normalizes K, and as $K_1 \not\leq M$, $KT \in \mathcal{H}(T,M)$. Thus $n(KT) > 1$ by 13.6.25.2, so in particular $K/O_2(K) \not\cong A_7$. So by Theorem 5.2.3.1, either

(a) $K_1/O_2(K_1) \cong L_2(4)$ and $K/O_2(K) \cong J_2$ or M_{23}, or

(b) $K_1 = K$ with $K/O_2(K) \cong L_3(4)$, and T nontrivial on the Dynkin diagram of $K/O_2(K)$ by E.2.2.

Next as $C_G(Z) \leq G_1$, $K \in \mathcal{L}(G_1,T)$, so $K \leq K_+ \in \mathcal{C}(G_1)$ by 1.2.4. If $K/O_2(K) \cong J_2$ or M_{23}, we conclude $K = K_+$ from 1.2.8.4. If $K/O_2(K) \cong L_3(4)$, then either $K = K_+$ or $K_+/O_2(K_+) \cong M_{23}$ by A.3.12, and the latter case is impossible as T is nontrivial on the Dynkin diagram of $K/O_2(K)$. Thus $K = K_+ \in \mathcal{C}(G_1)$.

By A.3.18, either $O^{3'}(G_1) = K$ or $O^{3'}(G_1/O_2(G_1)) \cong PGL_3(4)$. In particular K is the unique member of $\mathcal{C}(G_1)$ which is not a $3'$-group, and $O_{2,3}(G_1) = 1$ so that $O_{2,F}(G_1) = O_2(G_1)$ using 13.6.26.2.

Suppose $K_0 \in \mathcal{C}(G_1) - \{K\}$. By an earlier observation, K_0 is a $3'$-group, so $K_0/O_2(K_0)$ is $Sz(2^m)$. By 13.6.26.1, $K_0 \not\leq M$, while by 13.6.25.3, a Borel subgroup of K_0 is contained in M. Therefore $\langle K_0, T\rangle \in \mathcal{H}_*(T,M)$, which is contrary to Theorem 5.2.3 as we saw above.

Let $\dot{G}_1 := G_1/O_2(G_1)$; we have shown that $\dot{K} = F^*(\dot{G}_1)$. But $Out(\dot{K})$ is a 2-group if $\dot{K}$ is J_2 or M_{23}, while $Out(L_3(4)) \cong \mathbf{Z}_2 \times S_3$. It follows that either

$G_1 = KT$, or $G_1 = O^{3'}(G_1)T$ with $O^{3'}(\dot{G}_1) \cong PGL_3(4)$. Hence the proof of the lemma is complete. □

Let $H := G_1$ and $K := G_1^\infty$. Now $H \in \mathcal{H}_z$, so by 13.5.7, Hypothesis F.9.1 is satisfied with V_3 in the role of "V_+". Then by 13.6.24.2, Hypothesis F.9.8.f is satisfied, while case (i) of Hypothesis F.9.8.g holds in view of 13.2.3.2. We now adopt the standard conventions from section F.9 given in Notation 13.5.8, including $H^* := H/Q_H$, $U_H := \langle V_3^H \rangle$, and $\tilde{H} := H/V_1$. By 13.6.27, $F^*(H^*) = K^* \cong L_3(4)$, M_{23}, or J_2, and in the first case T^* is nontrivial on the Dynkin diagram of K^*. Therefore $q(H^*, \tilde{U}_H) > 2$ by B.4.2 and B.4.5, contrary to F.9.16.3. This contradiction completes the proof of Theorem 13.6.1.

13.7. Finishing the treatment of $\mathbf{A_6}$ when $\langle \mathbf{V^{G_1}} \rangle$ is nonabelian

In this section, and also in the final section 13.8 of the chapter, we adopt a hypothesis excluding the groups identified in previous sections:

HYPOTHESIS 13.7.1. *Hypothesis 13.3.1 holds, $L/C_L(V) \cong A_6$, and G is not $Sp_6(2)$ or $U_4(3)$.*

Thus since Hypothesis 13.7.1 includes Hypothesis 13.3.1 and Hypothesis 13.5.1, we may appeal to results in sections 13.4 and 13.5.

Set $Q := O_2(LT)$. We continue with the notation established in section 13.5: Namely we adopt the notational conventions of section B.3 and Notations 12.2.5 and 13.2.1.

By 13.5.2.3, $C_V(L) = 1$, so that V is the core of permutation module for $\bar{L} \cong A_6$, given by the vectors e_S for subsets S of even order in $\Omega := \{1, 2, 3, 4, 5, 6\}$, modulo e_Ω. In particular $V_1 = Z \cap V$ is generated by $z := e_{1,2,3,4} \equiv e_{5,6}$.

By 13.5.7, Hypothesis F.9.1 is satisfied with V_3 in the role of "V_+", so we may use results from section F.9. We also adopt the conventions from that section given in Notation 13.5.8, including $\tilde{G}_1 := G_1/V_1$. As usual define

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \leq G_1 \text{ and } H \not\leq M\}.$$

By 13.3.6, $G_1 \in \mathcal{H}_z$, and so $\mathcal{H}_z$ is nonempty.

In the remainder of the section, H denotes a member of $\mathcal{H}_z$.

From Notation 13.5.8 $U_H := \langle V_3^H \rangle$, $V_H := \langle V^H \rangle$, $Q_H := O_2(H)$, and $H^* := H/Q_H$ so that $O_2(H^*) = 1$. By F.9.2.3, $Q_H = C_H(\tilde{U}_H)$. Set $H_C := C_H(U_H)$; then $H_C \leq Q_H$.

By Theorem 13.5.12:

LEMMA 13.7.2. $\langle V_3^{G_1} \rangle$ *is abelian, so U_H is abelian.*

There are no quasithin examples satisfying 13.7.2, so in the remainder of this section we will be working toward a contradiction. As far as we can tell, there are not even any shadows.

13.7.1. Preliminary results. We begin with several consequences of Hypothesis 13.7.1 and 13.7.2, which we can apply both in the next subsection where $\langle V^{G_1} \rangle$ is nonabelian, and in the final section 13.8 where $\langle V^{G_1} \rangle$ is abelian.

LEMMA 13.7.3. *(1)* $V_H \le Q_H$.
(2) $U_H \le Z(V_H)$*, so that* $V_H \le H_C$.
(3) $\langle U_H^L \rangle \le O_2(LT) = Q$.
(4) For $h \in H$*, either* $[V, V^h] = 1$*, or* $[V, V^h] = [V, V_H] = V_1$ *with* $\bar{V}_H = \bar{V}^h = \langle (5,6) \rangle$.
(5) Either V_H *is abelian, or* $\Phi(V_H) = V_1$.
(6) $O_2(\bar{L}_1) \le \bar{Q}_H \le \bar{R}_1$, $V_3 = [V, Q_H]$, $V_1 = [V_3, Q_H] = [U_H, Q_H] = C_{V_3}(Q_H)$*, and* $[V_H, Q_H] = U_H$.
(7) Either

(i) $H_C \le Q$*, so* $H_C = C_H(V_H)$*, or*

(ii) $|H_C : Q \cap H_C| = 2$*, so* $\bar{H}_C = \langle (5,6) \rangle$, $[V_H, H_C] = V_1$*, and* $H_C \le C_H(\tilde{V}_H)$.

(8) If $L/O_2(L) \cong \hat{A}_6$*, then* V_H *is abelian.*
(9) $H \cap M = N_H(V)$ *and* $L_1 = \theta(H \cap M)$.

PROOF. As U_H is abelian,

$$\bar{U}_H \le C_{\bar{T}}(V_3) = C_{\bar{T}}(\tilde{V}),$$

so $V \le C_H(\tilde{U}_H) = Q_H$ and hence (1) holds. By (1) and F.9.3, $V \le C_{Q_H}(U_H)$, so (2) and (3) hold.

Let $h \in H$. Then by (2),

$$V^h \le C_{Q_H}(V_3) = C_{Q_H}(\tilde{V}),$$

so (4) holds, since $(5,6)$ is the transvection in $\bar{T}$ with center V_1. Then (4) implies (5).

If $[L_1, Q_H] \le Q = C_T(V)$, then $[L_1, Q_H] \le C_{L_1}(V_3)$; so as $L_1/C_{L_1}(V_3) \cong A_4$ has trivial centralizer in $GL(V_3)$, $[V_3, Q_H] = 1$, contrary to 13.5.4.5 since $O_2(G_1) \le Q_H$. Thus $[L_1, Q_H] \not\le Q$, so $O_2(\bar{L}_1) = \overline{[Q_H, L_1]} \le \bar{Q}_H \le \bar{R}_1$, and hence $V_3 = [V, O_2(\bar{L}_1)] = [V, Q_H]$. Then as $U_H = \langle V_3^H \rangle$, (6) holds.

Observe $\bar{H}_C \le C_{\bar{R}_1}(V_3) = 1$ or $\langle (5,6) \rangle$. If $\bar{H}_C = 1$, then (7i) holds. Otherwise $\bar{H}_C = \langle (5,6) \rangle$, and then as $[V, (5,6)] = V_1$, $[V, H_C] = V_1$, so that (7ii) holds.

If $L/O_2(L) \cong \hat{A}_6$, then each $t \in T$ inducing a transposition on $\bar{L}$ inverts $L_0/O_2(L_0)$ (see Notation 13.2.1), and hence $t \notin Q_H$ as $L_0 \le L_1 \le H$. We conclude $[V, V^h] = 1$ for all $h \in H$—since if not, some $t \in V^h$ induces a transposition on $\bar{L}$ by (4), whereas $V^h \le Q_H$ by (1), contrary to $t \notin Q_H$. Thus (8) is established.

Finally as $H \le G_z$, $H \cap M = N_H(V)$ by 12.2.6, so the remaining statement of (9) follows using 13.3.7. □

By 13.7.2, $U_H \le H_C$.

LEMMA 13.7.4. *(1) If* $L/O_2(L) \cong A_6$*, then* H^* *is faithful on* $U_H/C_{U_H}(Q_H)$.
(2) There is an H*-isomorphism* φ *from* Q_H/H_C *to the dual of* $U_H/C_{U_H}(Q_H)$*, defined by* $\varphi(xH_C) : uC_{U_H}(Q_H) \mapsto [x, u]$.

PROOF. Part (2) holds by F.9.7, so it remains to establish (1). Set $U_0 := C_{U_H}(Q_H)$ and observe $Q_H = C_H(\tilde{U}_H) \le C := C_H(U_H/U_0)$. On the other hand $\tilde{V}_3 = [\tilde{V}_3, L_1]$ with $V_1 = V \cap U_0$ by 13.7.3.6, so that $L_1 \not\le C$.

Assume that $L/O_2(L) \cong A_6$, but that (1) fails. Then $C^* \ne 1$, so as $O_2(H^*) = 1$, either $E(C^*) \ne 1$, or $O_p(C^*) \ne 1$ for some odd prime p. In the former case there is $K \in \mathcal{C}(C)$ with $K^* \cong K/O_2(K)$ quasisimple. In the latter case we take

$K := O^2(K_1)$, where K_1^* is a minimal normal subgroup of H^* contained in $O_p(C^*)$; thus $K^* \cong K/O_2(K) \cong E_{p^n}$, $n := 1$ or 2 since H is an SQTK-group. The subgroup K satisfies one of these two hypotheses throughout the rest of the proof.

In either case, $K = O^2(K)$ is subnormal in H, so $O_2(K) \leq Q_H$ and Q_H normalizes K. As $K \leq C$, $[U_H, K] \leq U_0$, so

$$1 \neq [U_H, K] = [U_0, K] \leq O_2(K) \leq Q_H \leq C_H(U_0). \qquad (!)$$

Then $1 \neq [U_0, K] \leq [\Omega_1(Z(O_2(K))), K]$, so that $K \in \mathcal{X}_f$.

Consider for the moment the case where $K \in \mathcal{C}(H)$. Then $K \in \mathcal{L}_f(G,T)$, so that $K^* \cong K/O_2(K)$ is described in the list of 13.5.2.1, and $K \trianglelefteq H$ by 13.3.2.2. We saw $L_1 \not\leq C$, so $L_1 \not\leq K$. Thus $K/O_2(K)$ is not A_6 or $\hat{A}_6$ by A.3.18, so $K^* \cong L_3(2)$ or A_5 by 13.5.2.1. Then as $L_1 = [L_1, T]$, either $[K, L_1] \leq O_2(K)$, or $K^* \cong A_5$ and $K = [K, L_1]$. Further if K^* is A_5, then by 13.3.2.3, each $I \in Irr_+(K, R_2(KT), T)$ is a T-invariant A_5-module.

We now return to consideration of both cases. By the previous paragraph we have $K \trianglelefteq H$. Since K^* is either simple or a p-group for p odd, and $C_K(\tilde{U}_H) = O_2(K) \leq C_K(U_0)$ by (!), we conclude from Coprime Action that

$$C_K(U_0) = C_K(\tilde{U}_H) = O_2(K). \qquad (*)$$

Observe that if $K \leq M$, then K normalizes V by 13.7.3.9, so $[V_3, K] \leq V \cap U_0 = V_1$, and hence $\bar{K} \neq \bar{L}_1$, contrary to 13.3.9 applied to K in the role of "Y". Thus $K \not\leq M$.

Suppose next that L is an A_6-block. Then L_1 has just two noncentral 2-chief factors, while L_1 is nontrivial on V_3U_0/U_0 and hence also nontrivial on Q_H/H_C by (2). Therefore L_1 centralizes U_0, and hence $[K, L_1] \leq C_K(U_0) = O_2(K)$ using (*), so K acts on $O^2(L_1O_2(K)) = L_1$. Then as $V_3 \leq L_1$ and $K \leq C$, $[V_3, K] \leq O_2(L_1) \cap U_0 \leq Z(L)V_1$, so K centralizes V_3 by Coprime Action as $|Z(L)| \leq 2$ by C.1.13.b. Thus $K \leq G_1 \cap G_3 \leq M_V$ by 13.5.5, whereas we saw $K \not\leq M$.

Therefore L is not an A_6-block. Then by 13.2.2.7, $N_G(B) \leq M$, where $B := \text{Baum}(R_1)$. In particular as $K \not\leq M$, B is not normalized by K.

Assume next that $K^* \cong A_5$ and $K = [K, L_1]$. Then $R_1 = (K \cap T)O_2(KR_1)$, and we saw earlier that each $I \in Irr_+(K, R_2(KT), T)$ is an A_5-module, so $J(R_1)$ centralizes I by B.4.2. Then $B \trianglelefteq KT$ by B.2.3.5, contrary to the previous paragraph. This contradiction shows that $[K, L_1] \leq O_2(K)$ in the case that $K \in \mathcal{C}(H)$.

Next consider the case where K^* is a p-group. As L_1 acts on $O_2(K)$, $O_2(K) \leq R_1$, so $R_1 \in Syl_2(KR_1)$. As $C_K(U_0) = O_2(K)$ by (*), $C_{KR_1}(R_2(KR_1)) = O_2(KR_1)$ by A.1.19. Thus if $J(R_1)$ centralizes $R_2(KR_1)$, then $B \trianglelefteq KR_1$ using B.2.3.5, whereas we saw B is not normalized by K. Thus KR_1 satisfies case (2) of Solvable Thompson Factorization B.2.16; so in particular $p = 3$. Since K^* is a minimal normal subgroup of H^*, T is irreducible on K^*, so by B.2.16.2, $J(KR_1)/O_2(J(KR_1)) \cong S_3$ or $S_3 \times S_3$. As $L_1 \not\leq C$ acts on $J(KR_1)$, the latter case is impossible as $m_3(H) \leq 2$. Thus if K^* is a p-group, we conclude $K^* \cong \mathbf{Z}_3$.

We have now shown that $K^* \cong \mathbf{Z}_3$, A_5, or $L_3(2)$, and that L_1 centralizes K^*. In particular, K acts on $O^2(O_2(K)L_1) = L_1$. Let $U \leq U_0$ be minimal subject to $U \trianglelefteq X := KL_1T$ and $[U, K] \neq 1$. Set $X^+ := X/C_X(U)$; we claim that $O_2(X^+) = 1$: For $O_2(K^+) = 1$ as $O_2(K)$ centralizes U_0, so K^+ centralizes $O_2(X^+)$, and hence $O_2(X^+) = 1$ by the Thompson $A \times B$ Lemma and mininality of U, as claimed. As K^* is simple, $K^* \cong K^+$ and $C_{KR_1}(U) = O_2(KR_1)$.

As L_1 centralizes K^*, L_1 acts on $T \cap K$, so $R_1 \in Syl_2(KL_1R_1)$. Thus if $J(R_1) \leq C_{R_1}(U)$, then $B = \mathrm{Baum}(O_2(KR_1))$ by B.2.3.5, whereas we saw K does not normalize B. Hence $K^+ = [K^+, J(R_1)^+]$. Also K acts on $[C_{\tilde{U}_H}(O_2(L_1)), L_1] =: \tilde{U}_1$. If $[U_1, K] = 1$ then $K \leq C_G(V_3) \leq M_V$ using 13.5.4.4, again contrary to $K \not\leq M$. Thus $[U_1, K] \neq 1$, so as $[U_1, K] \leq U_0$, $U_2 := [U_0, L_1, K] \neq 1$. Thus we may take $U \leq U_2$, so $U = [U, L_1]$. Then as L_1 centralizes K^*, $L_1^+ \trianglelefteq X^+$, so that $L_1^+ \cong \mathbf{Z}_3$ as $O_2(X^+) = 1$. Then since $K^+ = [K^+, J(R_1)^+]$, it follows from Theorem B.5.6 that U is the sum of two isomorphic natural modules for $K^+ \cong L_3(2)$. Therefore by B.2.20, $B^+ = J(R_1)^+$ is the unipotent radical of a minimal parabolic $K_0^+ R_1^+$ of $K^+ R_1^+$. Then by B.2.3.4, $B = \mathrm{Baum}(O_2(K_0R_1))$, so that $K_0 \leq N_G(B) \leq M$. Then $O^2(K_0) \leq O^2(H \cap M) = L_1$ by 13.7.3. But $|L_1|_3 = 3$ since we are assuming that $L/O_2(L)$ is A_6 rather than $\hat{A}_6$, so $O^2(K_0) = L_1$, contradicting $L_1 \not\leq K$. Thus the proof of (1) and hence of the lemma is at last complete. □

LEMMA 13.7.5. *Let $X := L_1$ if $L/O_2(L) \cong A_6$, and $X := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$. Assume $K \in \mathcal{C}(H)$ and $X \leq K$. Then*

(1) $K \trianglelefteq H$.

(2) $U_H = [U_H, K]$.

(3) $N_H(\tilde{V}_3) \leq M$.

(4) If $m_3(N_K(\tilde{V}_3)) = 2$, then $L/O_2(L) \cong \hat{A}_6$ and $L_1 = \theta(N_K(\tilde{V}_3))$.

(5) If $L_1 \leq K$ and $m_3(N_K(\tilde{V}_3)) = 1$, then $L/O_2(L) \cong A_6$.

(6) $O^{3'}(N_K(\tilde{V}_3))$ is solvable.

(7) If $L_1 \not\leq K$, then $Aut_{L_1}(K/O_2(K)) \neq Aut_X(K/O_2(K))$.

PROOF. Since T normalizes $X \leq K$, $K \trianglelefteq H$ by 1.2.1.3, proving (1). Further $V_3 = [V_3, X] \leq [U_H, K]$, so that $U_H = \langle V_3^H \rangle = [U_H, K]$, establishing (2). Part (3) follows from 13.5.5. Then by (3) and 13.7.3, either $\theta(N_K(V_3)) = L_1$ or $L/O_2(L) \cong \hat{A}_6$ with $\theta(N_K(V_3)) = X$. Now $m_3(X) = 1$, while $m_3(L_1) = 1$ when $L/O_2(L) \cong A_6$, and $m_3(L_1) = 2$ when $L/O_2(L) \cong \hat{A}_6$; so it follows that (4) and (5) hold. As $\theta(N_K(V_3)) \leq L_1$ which is solvable, (6) holds.

Finally suppose that $L_1 \not\leq K$, but the conclusion of (7) fails. Since $X \leq K$, $X < L_1$, so $L/O_2(L) \cong \hat{A}_6$; then as (7) fails, $L_1 = XL_C$, where $L_C = O^2(C_{L_1}(K/O_2(K))$. As X and L_0 are the only proper nontrivial T-invariant subgroups Y of L_1 with $Y = O^2(Y)$, it follows that $L_C = L_0$. But then K normalizes $O^2(O_2(K)L_0) = L_0$ and so lies in M by 13.2.2.9, contrary to 13.3.9. □

The next result eliminates various possibilities for H^* and its action on $\tilde{U}_H$. As usual Theorem C (A.2.3) determines the possibilities for n in (1) and (3). The lemma considers all cases where $[\tilde{U}_H, K] \in Irr_+(\tilde{U}_H, K)$ is an FF-module, except the cases where the noncentral chief factor for K on $\tilde{U}_H$ is the natural module for $K/O_2(K) \cong L_2(2^n)$ or $\hat{A}_6$.

LEMMA 13.7.6. *Assume $K \in \mathcal{C}(H)$ and let $U_K := [U_H, K]$. Then*

(1) If $K/O_2(K) \cong L_n(2)$ and $\tilde{U}_K/C_{\tilde{U}_K}(K)$ is the natural $K/O_2(K)$-module, then $n = 4$, $\tilde{U}_H$ is the natural module for $H^* \cong L_4(2)$, and $L/O_2(L) \cong \hat{A}_6$.

(2) If $K/O_2(K) \cong L_5(2)$, then $\tilde{U}_K/C_{\tilde{U}_K}(K)$ is not a 10-dimensional irreducible for $K/O_2(K)$.

(3) If $K/O_2(K) \cong A_n$ and $\tilde{U}_K/C_{\tilde{U}_K}(K)$ is the natural module, then $U_H = U_K$, $L_1 \leq K$, $H = KT$, and applying the notation of section B.3 to $\tilde{U}_H$, either

(a) $n = 6$, $L/O_2(L) \cong A_6$, $\tilde{V}_2 = \langle e_{1,2,3,4} \rangle$, *and* L_1 *has two noncentral chief factors on* U_H, *or*

(b) $n = 7$, $L/O_2(L) \cong \hat{A}_6$, $\tilde{V}_2 = \langle e_{5,6} \rangle$, *and* $\tilde{V}_3 = \langle e_{5,6}, e_{5,7} \rangle$.

(4) If $K/O_2(K) \cong A_7$ *then* $\tilde{U}_K$ *is not a 4-dimensional* A_7*-module.*

(5) If $K/O_2(K) \cong (S)L_3(2^n)$, $Sp_4(2^n)'$, *or* $G_2(2^n)'$ *and* $\tilde{U}_K/C_{\tilde{U}_K}(K)$ *is a natural module for* K^*, *then* $n = 1$.

PROOF. Assume $K/O_2(K)$, $\tilde{U}_K$ is one of the pairs considered in the lemma. We obtain a contradiction in (2) and (4), and in (5) under the assumption that $n > 1$. In (1) and (3), we establish the indicated restrictions. Observe that, except possibly in (5) when $K/O_2(K) \cong SL_3(2^n)$, $K/O_2(K)$ is simple so that $K^* \cong K/O_2(K)$. In that exceptional case $\tilde{U}_K$ is a natural module by hypothesis, so $C_K(\tilde{U}_K) = O_2(K)$ and thus again $K^* \cong K/O_2(K)$.

The first part of the proof treats the case where $L_1 \leq K$. Here $K \trianglelefteq H$ by 13.7.5.1, and $U_H = U_K$ by 13.7.5.2.

Next $\tilde{V}_3 = \langle \tilde{V}_2^{L_1} \rangle$ is a T-invariant line in $\tilde{U}_K$, so:

(i) If $K^* \cong L_3(2)$ then $C_{\tilde{U}_H}(K) = 0$ (cf. B.4.8.2).

(ii) Under the hypotheses of (3), $n > 5$.

Further

(iii) $\tilde{V}_2$ is a T-invariant $\mathbf{F}_2$-point of $\tilde{U}_H$. Set $K_0 := O^2(C_K(V_2))$, so that also $K_0 = O^2(C_K(\tilde{V}_2))$.

By 13.5.4.3, $m_3(K_0) \leq 1$, so we conclude from (iii) and the structure of $C_{K^*}(\tilde{V}_2)$ that: (2) holds; $n < 5$ in (1); in (3), $n \leq 7$ and in case of equality $\tilde{V}_2 = \langle e_{5,6} \rangle$, when $\tilde{U}_H$ is described in the notation of section B.3.

Assume the hypothesis of (3) with $n = 7$. As $\tilde{V}_2 = \langle e_{5,6} \rangle$ and $\tilde{V}_3 = \langle \tilde{V}_2^{L_1} \rangle$ is a T-invariant line, $\tilde{V}_3 = \langle e_{5,6}, e_{5,7} \rangle$. Hence $N_K(V_3)$ has 3-rank 2, so $L/O_2(L) \cong \hat{A}_6$ by 13.7.5.4. Since $N_{GL(\tilde{U}_K)}(K^*) \cong S_7$, $H = KT$. Hence conclusion (b) of (3) holds.

Thus under the hypotheses of (3), we have reduced to the case $n = 6$. Then as $\tilde{V}_3 = \langle \tilde{V}_2^{L_1} \rangle$ is a T-invariant line, $\tilde{V}_2 = \langle e_{1,2,3,4} \rangle$, L_1 has two noncentral chief factors on $\tilde{U}_H$, and $m_3(N_K(\tilde{V}_3)) = 1$, so that $L/O_2(L) \cong A_6$ by 13.7.5.5. Since $End_K(\tilde{U}_K)$ is of order 2, we conclude that $K^* = F^*(H^*)$. As T normalizes U_K, it is trivial on the Dynkin diagram of K^*, so as $Out(K^*)$ is a 2-group, we conclude that $H = KT$. This gives conclusion (a) of (3), and so completes the proof of (3).

Similarly when $n = 4$ in case (1), or in case (4), $N_K(V_3)$ has 3-rank 2, so that $L/O_2(L) \cong \hat{A}_6$ and $L_1 = O^2(N_K(V_3))$ by 13.7.5.4. Thus $L_1T/O_2(L_1T) \cong S_3 \times \mathbf{Z}_3$ or $S_3 \times S_3$ from the structure of L. Further as $U_H = U_K$ and $K^* \trianglelefteq H^*$, $H^* = N_{GL(\tilde{U}_H)}(K^*) = K^*$. Thus in case (4) where $K^* \cong A_7$, $L_1T/O_2(L_1T)$ is E_9 extended by an involution inverting the E_9, so this case is eliminated. When $K^* \cong L_4(2)$, the conclusions of (1) hold using I.1.6.6.

Assume the hypotheses of (5) with $n > 1$, and let $U_H^+ := U_H/C_{U_H}(K)$. By (iii), V_2^+ is contained in a T-invariant $\mathbf{F}_{2^n}$-point W of U_H^+. As $L_1 \leq K$ and L_1 is T-invariant, L_1 is contained in the Borel subgroup of K containing $T \cap K$. In particular, n is even. Thus L_1 acts on W, so $V_3^+ = [V_2^+, L_1] \leq W$. But now as $O^{3'}(C_K(W))$ is not solvable, 13.7.5.6 supplies a contradiction, establishing that $n = 1$ under the hypotheses of (5).

So to complete the treatment of the case $L_1 \leq K$, it remains only to eliminate case (1) with $n = 3$. So suppose $K^* \cong L_3(2)$. Since $L_1 \leq K$ which has 3-rank 1, $L/O_2(L) \cong A_6$ by 13.7.5.5, and $L_1^* T^*$ is a maximal parabolic of $K^* T^*$. Set $\mathcal{F} := (LT, K_0 L_2 T, KT)$ and $G_0 := \langle \mathcal{F} \rangle$. Here $K_0 T/O_2(K_0 T) \cong S_3$, and by 13.5.4.1, $[K_0, L_2] \leq O_2(L_2)$. As $M = !\mathcal{M}(LT)$, $O_2(G_0) = 1$. Thus if $|Q| > 2^5$, then $(G_0, \mathcal{F})$ is a C_3-system as defined in section I.5, so by Theorem I.5.1, $G_0 \cong Sp_6(2)$. Therefore $Z(KT) = 1$, whereas $z \in Z(KT)$ as $H \in \mathcal{H}_z$. Thus $|Q| \leq 2^5$, so L is an A_6-block. But then L_1 has just two noncentral 2-chief factors, whereas L_1 has noncentral chief factors on each of $O_2(L_1^*)$, $\tilde{U}_H$, and Q_H/H_C by 13.7.4.2. This contradiction completes the proof of (1) and of the lemma in the case $L_1 \leq K$.

It remains to treat the case $L_1 \not\leq K$. When $m_3(K) = 2$, $K = O^{3'}(H)$ by A.3.18 and A.3.19, so $K/O_2(K) \cong L_3(2^n))$, n odd, or A_5. Let $X := L_1$ if $L/O_2(L) \cong A_6$, and $X := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$. Suppose $[K^*, X^*] = 1$. Then as $End_K(\tilde{U}_K/C_{\tilde{U}_K}(K)) = \mathbf{F}_{2^n}$ with n odd, X centralizes $\tilde{U}_K = [K, \tilde{U}_H]$. But then by the Three-Subgroup Lemma, $[\tilde{U}_H, X, K] = 1$; so as $\tilde{V}_3 = [\tilde{V}_3, X]$, $K = O^2(K) \leq C_G(V_3) \leq M_V$ by 13.5.4.4, and hence $\langle K^T \rangle \leq M$, contrary to 13.3.9. Therefore $K = [K, X]$. So as $X = [X, T]$, we conclude that X induces inner automorphisms on $K/O_2(K)$. Then again as $X = [X, T]$, $K/O_2(K)$ is not $L_3(2^m)$ for $m > 1$, and either:

(a) $X \leq K$, and hence $X < L_1$ as $L_1 \not\leq K$, so that $L/O_2(L) \cong \hat{A}_6$, or

(b) $K^* \cong A_5$ and X^* is diagonally embedded in $K^* C_{K^* X^*}(K^*)$, with $\tilde{V}_3$ projecting nontrivially on $\tilde{U}_K$.

But now $K^* \cong A_5$ is ruled out, since in both cases (a) and (b), $\tilde{V}_3 = [\tilde{V}_3, X]$, while either X^* or its projection on K^* is a Borel subgroup of K^*, which has no such T-invariant submodule of rank 2 on the A_5-module $\tilde{U}_K$. Therefore case (a) holds and $K^* \cong L_3(2)$. Now 13.7.5.7 supplies a contradiction, completing the proof. □

LEMMA 13.7.7. $[V_H, H] \not\leq U_H$.

PROOF. If $[V_H, H] \leq U_H$, then $V_H = \langle V^H \rangle = VU_H$. Then by 13.7.3.6,

$$U_H = [V_H, Q_H] = [V, Q_H][U_H, Q_H] = V_3 V_1 = V_3,$$

contrary to 13.5.9. □

13.7.2. The elimination of the case $\langle \mathbf{V}^{\mathbf{G_1}} \rangle$ nonabelian. We come to the main result of this section, which reduces the treatment of Hypothesis 13.7.1 to the case where $\langle V^{G_1} \rangle$ is abelian. Then in the following section 13.8, that remaining case is also shown to lead to a contradiction.

THEOREM 13.7.8. *Assume Hypothesis 13.7.1. Then $\langle V^{G_1} \rangle$ is abelian.*

Until the proof of Theorem 13.7.8 is complete, assume G is a counterexample. Then the set $\mathcal{H}_1$ of those $H \in \mathcal{H}_z$ with V_H nonabelian is nonempty, since $G_1 \in \mathcal{H}_1$.

For the remainder of the section, let H denote a member of $\mathcal{H}_1$.

Then V_H is nonabelian, though U_H is abelian by 13.7.2.

LEMMA 13.7.9. *(1) $\Phi(V_H) = V_1$, $\bar{V}_H = \langle (5, 6) \rangle$, and $\bar{Q}_H = \bar{R}_1$.*
(2) $L/O_2(L) \cong A_6$ rather than $\hat{A}_6$. In particular $|L_1|_3 = 3$.
(3) H^* is faithful on $U_H/C_H(Q_H)$.

PROOF. Observe (1) follows from parts (4) and (6) of 13.7.3, and (2) follows from part (8) of 13.7.3. Finally (3) follows from (2) and 13.7.4.1. □

Pick $g \in L$ with $\bar{g}^2 = 1$ and V_1^g not orthogonal to V_1. Set $I := \langle V_H, V_H^g \rangle$ and $Z_I := V_H \cap V_H^g$. Observe Q normalizes V_H, and also V_H^g since $g \in N_G(Q)$, so Q normalizes I.

Recall that we can appeal to results in section F.9. In particular, as in F.9.6 define $D_H := U_H \cap Q_H^g$, $D_{H^g} := U_H^g \cap Q_H$, $E_H := V_H \cap Q_H^g$ and $E_{H^g} := V_H^g \cap Q_H$. Since we chose $\bar{g}^2 = 1$, F.9.6.2 says that

$$(D_H)^g = D_{H^g} \text{ and } (E_H)^g = E_{H^g}.$$

Let $A := V_H^g \cap Q$, $U_0 := C_{U_H}(Q_H)$, $U_H^+ := U_H/U_0$, and recall $H_C = C_H(U_H)$. By 13.7.9.3, H^* is faithful on U_H^+, and by 13.7.4.2, Q_H/H_C is dual to U_H^+ as an H-module.

Let $U_L := \langle U_H^L \rangle$. By 13.7.3.3, $U_L \leq Q$. In particular $U_H^g \leq Q$, so that $U_H^g \leq V_H^g \cap Q = A$.

LEMMA 13.7.10. *(1)* $V_1^g \not\leq U_H$.
(2) $O_2(I) = (V_H \cap Q)(V_H^g \cap Q)$ *and* $I/O_2(I) \cong S_3$.
(3) $O_2(I)/Z_I$ *is elementary abelian and the sum of natural modules for* $I/O_2(I)$, *and* $Z_I/V_1V_1^g$ *is centralized by* I.
(4) $\langle U_H^I \rangle Z_I = U_H U_H^g Z_I$ *and* $U_H^g Z_I = \{x \in V_H^g : [V_H, x] \leq U_H Z_I\}$.
(5) $\langle D_H^I \rangle = D_H D_{H^g} = V_1 V_1^g (D_H \cap D_{H^g}) \leq Z_I$.
(6) $[D_H, A] = 1$ *and* $[D_{H^g}, V_H] \leq V_1$.
(7) $E_H = E_{H^g} = Z_I \leq Q \cap H_C$, *so* $[E_{H^g}, V_H] \leq V_1$.
(8) L_1 *has* $m(A^*) + 2$ *noncentral 2-chief factors.*
(9) $U_H^g \cap V_3$ *is a complement to* V_1 *in* V_3, *and* $V_3 \leq Z_I$.
(10) $A \cap Q_H = E_{H^g}$.

PROOF. If $V_1^g \leq U_H$, then $V = V_3 V_1^g \leq U_H$, so $V_H \leq U_H$ is abelian, contrary to the choice of G as a counterexample to Theorem 13.7.8. Thus (1) holds.

By 13.7.9.1 and the choice of g, $\bar{I} \cong S_3$; e.g., if $\bar{g} = (4,5)$ then $\bar{I} = \langle (5,6), (4,6) \rangle$. Let $P := (V_H \cap Q)(V_H^g \cap Q)$. By 13.7.9.1, $\Phi(V_H) = V_1$, so $\Phi(V_H) \leq V_1 V_1^g \trianglelefteq I$; e.g., $V_1 V_1^g = \langle e_{5,6}, e_{4,6} \rangle$. Arguing as in G.2.3 with I, $V_1 V_1^g$ in the roles of "L, V", (2) and (3) hold. In particular $Z_I \leq O_2(I) \leq Q$, so that $Z_I \leq A$.

Let $\hat{P} := P/Z_I$. For $v \in V_H \cap Q - Z_I$, $\hat{P}_v := \langle \hat{v}^I \rangle \cong E_4$ as $\hat{P}$ is the sum of natural modules for I/P. Thus if $v \in U_H$, then $\hat{P}_v \leq \hat{U}_H \hat{U}_H^g$ and hence $\langle U_H^I \rangle Z_I = U_H U_H^g Z_I$, proving (4).

By F.9.6.3, $[D_H, U_H^g] \leq V_1^g \cap U_H = 1$ using (1). Then by symmetry, $D_{H^g} \leq H_C$, so by 13.7.3.7, $[D_{H^g}, V_H] \leq V_1$ and $[D_H, A] \leq V_1^g \cap D_H = 1$. Hence (6) is established.

By 13.7.9.1 and (6), $[I, D_H D_{H_g}] \leq V_1 V_1^g$, so

$$\langle D_H^I \rangle = D_H V_1^g = D_{H^g} V_1$$

and hence (5) holds.

By 13.7.3.6, $[E_{H^g}, V_H] \leq U_H$, so for $v \in V_H - Q$, $[E_{H^g}, v] \leq U_H$, and hence $E_{H^g} \leq U_H^g Z_I$ by (4). Thus

$$E_{H^g} = E_{H^g} \cap U_H^g Z_I = (E_{H^g} \cap U_{H^g}) Z_I = D_{H^g} Z_I$$

and $D_{H^g} \leq Z_I$ by (5), so that $E_{H^g} \leq Z_I \leq V_H$, and hence $E_{H^g} \leq Q \cap H_C$ by (2) and 13.7.3.2. But $Z_I \leq E_{H^g}$, so $E_{H^g} = Z_I$, and then by symmetry $E_{H^g} = Z_I = E_H \leq V_H$. Then $[E_{H^g}, V_H] \leq V_1$ by 13.7.3.5, completing the proof of (7).

Next there exists $l \in L$ with $\bar{L}_1^l = O^2(\bar{I})O_2(\bar{L}_1^l) \cong A_4$. We saw Q acts on I, so L_1^l has $k+1$ noncentral 2-chief factors, where k is the number of noncentral 2-chief factors of I. One of those k factors is $V_1V_1^g$, and by (2) and (3) there are $k-1 = m(O_2(I)/Z_I)/2 = m(A/Z_I)$ factors on $O_2(I)/Z_I$. Now

$$m(A/Z_I) = m(A/E_{H^g}) = m(A/(A \cap Q_H)) = m(A^*)$$

by (7), so that (8) holds. As $V_3 \cap V_3^g$ is a complement to V_1 in V_3, and $V_1 \not\leq U_H^g$ by (1), (9) holds.

Since $E_{H^g} \leq Q$ by (7), (10) is immediate from the definitions of A and E_{H^g}. □

LEMMA 13.7.11. *(1)* $D_H < U_H$.
(2) $1 \neq U_H^{g*}$ *and* $U_H^g \leq A$.

PROOF. Recall $U_H^g \leq A$, $U_{H^g} = (U_H)^g$, and $D_{H^g} = (D_H)^g$. Therefore $D_H = U_H$ iff $D_H^g = U_{H^g}$ iff $U_{H^g} \leq Q_H = C_H(\tilde{U}_H)$; and hence (1) and (2) are equivalent. Thus we may assume that $U_H = D_H$, and it remains to derive a contradiction. By 13.7.10.6, $U_H = D_H$ centralizes A, so $A \leq Q_H$. Thus by 13.7.10, $A = A \cap Q_H = E_{H^g}$, while by 13.7.10.7, $E_{H^g} = E_H \leq V_H$; so using symmetry we conclude $V_H^g \cap Q = V_H \cap Q$. Let Λ be the graph on the points of V obtained by joining non-orthogonal points. Then Λ is connected, so $V_H \cap Q = V_H^x \cap Q$ for all $x \in L$. Therefore L acts on $V_H \cap Q$. Now $\Phi(V_H) = V_1$ by 13.7.9.1, so as L does not act on V_1, $\Phi(V_H \cap Q) = 1$. Also by 13.7.9.1, $V_H = Z(V_H)V_0$ with V_0 extraspecial and $|V_H : V_H \cap Q| = 2$; so we conclude $\Phi(Z(V_H)) = 1$ and $V_0 \cong D_8$. But now, V_H has just two maximal elementary abelian subgroups, one of which is $Z(V_H)V$; so both are normal in $O^2(H)T = H$, and hence $\langle V^H \rangle = V_H = Z(V_H)V$ is abelian, contrary to our choice of G as a counterexample to Theorem 13.7.8. □

Recall that $U_0 = C_{U_H}(Q_H)$, and from 13.7.4 and 13.7.9.2 that $U_H^+ = U_H/U_0$ is H-dual to Q_H/H_C and H^* is faithful on U_H^+.

LEMMA 13.7.12. *(1)* A^* *centralizes* $(Q_H \cap Q)H_C/H_C$ *of corank* 2 *in* Q_H/H_C.
(2) $[U_H^+, A^*] \leq V_3^+$.
(3) For $F \in \{A^*, U_H^{g*}\}$, $r_{F,\tilde{U}_H} \leq 1 \geq r_{F,U_H^+}$, *so* F *contains FF***-offenders on each of these FF-modules.*

PROOF. As $g \in N_G(Q)$, Q acts on A, so by 13.7.10, $[A, Q_H \cap Q] \leq A \cap Q_H = E_{H^g}$, and $E_{H^g} \leq H_C$ by 13.7.10.7. Further by 13.7.9.1, $\bar{Q}_H = \bar{R}_1 \cong E_8$ and $\bar{V}_H$ is of order 2, so $V_H = H_C$ by parts (2) and (7) of 13.7.3. Thus $|Q_H : (Q_H \cap Q)H_C| = 4$, completing the proof of (1). Then since V_3 centralizes Q and $C_{V_3}(Q_H) = V_1$ is of index 4 in V_3, V_3^+ corresponds to $(Q_H \cap Q)H_C/H_C$ under the duality between U_H^+ and Q_H/H_C, so part (2) is the dual of (1). By 13.7.10.6, $[D_H, A] = 1$, and

$$m(U_H^{g*}) = m(U_H^g/D_{H^g}) = m(U_H^g/D_H^g) = m(U_H/D_H),$$

so $r_{F,\tilde{U}_H} \leq 1$ for $F \in \{A^*, U_H^{g*}\}$, keeping in mind that $1 \neq U_H^{g*} \leq A^*$ by 13.7.11.2. Then $r_{F,U_H^+} \leq 1$ also holds using 13.7.4.1. Thus both modules are FF-modules for H^*, and B.1.4.4 shows that F contains FF*-offenders on the modules. □

LEMMA 13.7.13. *If* $m(\tilde{U}_H) = 4$ *then* $[\tilde{U}_H, L_1] < \tilde{U}_H$.

PROOF. Assume otherwise. Observe there is $X_1 \in Syl_3(L_1)$ with $V_1V_1^g = C_V(X_1)$, and we can take $g \in N_L(X_1)$. Thus $X_1 \leq H \cap H^g$, so X_1 acts on D_H, and as $\tilde{U}_H = [\tilde{U}_H, L_1]$ is of rank 4, X_1 is irreducible on $\tilde{U}_H/\tilde{V}_3$. Thus X_1 has two nontrivial chief factors on $\tilde{U}_H = U_H^+$. By (7) and (9) of 13.7.10, $V_3 \leq Z_I = E_H$, so $V_3 \leq U_H \cap E_H = D_H$. Then as $D_H < U_H$ by 13.7.11.1, we conclude $D_H = V_3$, so that L_1 is irreducible on U_H/D_H. Then X_1 is also irreducible on $U_H^g/D_H^g \cong U_H^{g*} \leq O_2(L_1^*)$, so L_1 has a noncentral 2-chief factor not in Q_H. Also L_1 has two noncentral chief factors on each of U_H^+ and Q_H/H_C by 13.7.4.2, so L_1 has at least five noncentral 2-chief factors, with $[H_C, L_1] \leq U_H$ in case of equality. Therefore by 13.7.10.8, $m(A^*) \geq 3$, and $[H_C, L_1] \leq U_H$ in case of equality. Then as $m(U_H^+) = 4$, inspecting the subgroups H^* of $GL_4(2)$ of 2-rank at least 3 with $O_2(H^*) = 1$, we conclude $H^* \cong L_4(2)$ or S_6, with $m(A^*) = 3$ in the second case. The first case is impossible by 13.7.6.1 and 13.7.9.2. In the second, $[H_C, L_1] \leq U_H$, so $[V_H, H] \leq U_H$ in view of 13.7.3.2, contrary to 13.7.7. □

LEMMA 13.7.14. *$F(H^*)$ is centralized by each minimal FF*-offender B^* on U_H^+ contained in A^*.*

PROOF. Set $\mathcal{P} := \{C^* \in \mathcal{P}^*(H^*, U_H^+) : [F(H^*), C^*] \neq 1\}$ and suppose $B^* \in \mathcal{P}$ with $B^* \leq A^*$. Then by B.1.9, there is a normal subgroup N^* of H^* such that $N^* = H_1^* \times \cdots \times H_s^*$, $H_i^* \cong L_2(2)$, with $s = 1$ or 2 since $m_3(H) \leq 2$, $U_N^+ := [U_H^+, N^*] = U_1^+ \oplus \cdots \oplus U_s^+$ with $U_i^+ := [U_H^+, H_i^*] \cong E_4$ affording the natural module for H_i^*, and

$$\mathcal{P} = \bigcup_i Syl_2(H_i^*).$$

In particular we may take $B^* \in Syl_2(H_1^*)$. Then $[U_1^+, B^*] = [U_H^+, B^*] \leq [U_H^+, A^*] \leq V_3^+$ by 13.7.12.2. As $s \leq 2$ and $L_1 = O^2(L_1)$, L_1 acts on H_i^* for each i, and hence also on U_i^+. Then $V_3^+ = [U_1^+, B^*, L_1^*] \leq U_1^+$, so $V_3^+ = U_1^+$ as both are of rank 2. However as $A \leq Q$, $B^* \leq A^* \leq R_1^* \trianglelefteq L_1^*T^*$, so L_1^* acts on $R_1^* \cap H_1^* = B^*$ and hence also on $[U_1^+, B^*]$ of rank 1. This is impossible as L_1^* is irreducible on $V_3^+ = U_1^+$. □

Since $O_2(H^*) = 1$, by 13.7.14 some member B^* of $\mathcal{P}^*(H^*, U_H^+)$ contained in A^* acts nontrivially on $E(H^*)$. So there is $K \in \mathcal{C}(H)$ with K^* quasisimple and $[K^*, B^*] \neq 1$. Let $K_0 := \langle K^T \rangle$ and $U_K := [U_H, K]$. By B.1.5.4, B^* acts on K^*, so $K^* = [K^*, B^*]$.

LEMMA 13.7.15. *(1) $K^* \cong L_n(2)$, A_n, $SL_3(4)$, or $Sp_4(4)$.*
(2) $U_K^+ \in Irr_+(K^*, U_H^+)$.
(3) $K_0 = K$.
(4) $U_H^+ = U_K^+$.

PROOF. By B.1.5.1, $Aut_B(U_K^+)$ is an FF*-offender on U_K^+. Therefore by B.5.6 and B.5.1.1, either $U_K^+ \in Irr_+(K^*, U_H^+)$, or one of conclusions (ii)–(iv) of B.5.1.1 holds.

In the first case, (2) holds, with K^* and $\hat{U}_K := U_K^+/C_{U_K^+}(K)$ described in B.4.2. However by 13.7.12.2, $m([\hat{U}_K, B^*]) \leq 2$, so we conclude K^* is one of the groups listed in (1) in this case: Recall B.4.6.13 eliminates $K^* \cong G_2(2)'$, and K^* is not $\hat{A}_6$ since $m([\hat{U}_K, B^*]) = 4$ for the unique FF*-offender B^* in B.4.2.8.

So assume that the second case holds. As $m([U_K^+, B^*]) \leq 2$, U_K^+ has exactly two chief factors U_1 and U_2, and B^* induces a group of transvections with fixed center on U_i; but this contradicts the structure of FF*-offenders in conclusions (ii)–(iv) of B.5.1.1. This completes the proof of (1) and (2).

Suppose $K < K_0$. Then by 1.2.1.3 and (1), $K^* \cong L_3(2)$ or A_5, and by 1.2.2.1, $K_0 = O^{3'}(H)$, so $L_1 \le K_0$. Then as T acts on L_1, we conclude $K^* \cong A_5$ and L_1^* is diagonally embedded in K_0^*. Since $B^* \le R_1^*$, B^* induces inner automorphisms on K^*, so by B.4.2, $\hat{U}_K$ is the natural module for K^* rather than the A_5-module, and $V_3^+ = [U_H^+, B^*] \le U_K^+$. Now as T acts on V_3, T acts on U_K and hence also on K, contrary to assumption. This establishes (3).

Next $1 \ne [U_K^+, B^*] \le V_3^+ \cap U_K^+$ by 13.7.12.2. Then as L_1 acts on K, and acts irreducibly on V_3^+,

$$V_3^+ = [V_3 \cap U_K^+, L_1] \le U_K^+,$$

so $U_H^+ = \langle V_3^{+H} \rangle = U_K^+$ since U_K is H-invariant by (3), and hence (4) holds. □

LEMMA 13.7.16. *(1) T^* is faithful on K^*.*
(2) $[U_0, K] = 1$.
(3) $\tilde{U}_K \in Irr_+(K^*, \tilde{U}_H)$.

PROOF. By parts (2) and (4) of 13.7.15 and A.1.41, $C_{H^*}(K^*)$ is of odd order, so (1) holds. Also (3) follows from 13.7.15.2 if (2) holds, so it remains to prove (2).

Assume (2) fails. Then as $U_0 \le Z(Q_H)$, $K \in \mathcal{L}_f(G,T)$, so by 13.5.2.1, K^* is A_5, $L_3(2)$, A_6, or $\hat{A}_6$. Further K acts nontrivially on U_0 and U_H^+, so K has at least two noncentral chief factors on $\tilde{U}_H$. On the other hand by 13.7.12.3, U_H^{g*} contains an FF*-offender D^* on $\tilde{U}_H$, and by (1), D^* is faithful on K^*, so by B.1.5.1, $Aut_D(\tilde{U}_K)$ is an FF*-offender on the FF-module $\tilde{U}_K$. Then by Theorems B.5.6 and B.5.1.1, $K^* \cong L_3(2)$ and $\tilde{U}_K$ is the sum of two isomorphic natural modules for K^*. Then as $U_H^g \le Q \le O_2(L_1T)$, $L_1^*(T \cap K)^*$ is the stabilizer of a line in each irreducible and $Aut_{U_H^g}(\tilde{U}_K) = O_2(Aut_{L_1}(\tilde{U}_K))$.

By 13.5.2.3, $U_1 := [U_0, K]$ is a natural module, so U_1 is isomorphic to $\tilde{U}_1$, and so $C_{U_1}(L_1) = 1$. But $Z_1 := Z \cap U_1$ is of order 2, and as $Aut_{U_H^g}(U_1) = O_2(Aut_{L_1}(U_1))$, $Z_1 \le [U_0, U_H^g]$. Then as $U_1 \le U_H \le N_Q(U_H^g)$ by 13.7.3.3, $Z_1 \le U_H^g$, so Z_1 is centralized by $\langle V_H, V_H^g \rangle = I$ and by T. Thus $L \le \langle I, T \rangle \le C_G(Z_1)$, contrary to $C_{U_1}(L_1) = 1$. □

LEMMA 13.7.17. *(1) $Z_I \cap U_0 = V_1$.*
(2) If $U_0 \le [U_H, A]V_3$, then $U_0 = V_1$.

PROOF. First Q_H and $I = \langle V_H, V_H^g$ centralize $U_0 \cap U_H^g =: U_1$, so $L_0 := \langle I, Q_H \rangle \le C_G(U_1)$. However $\bar{Q}_H = \bar{R}_1$ by 13.7.9.1, and $\bar{L}\bar{T} \le \langle \bar{R}_1, \bar{I} \rangle$, so $LT = L_0Q$. Further Q and L_0 act on U_1, so $LT = L_0Q \le N_G(U_1)$.

If $U_1 \ne 1$ then $N_G(U_1) \le M = !\mathcal{M}(LT)$. But then by 13.7.16.2, $K \le C_G(U_1) \le M$, contrary to 13.3.9. Thus $U_1 = 1$.

Next by 13.7.10.5, $D_H = (D_H \cap D_{H^g})V_1$, and by 13.7.10.7, $E_H = Z_I$, so

$$Z_I \cap U_0 = E_H \cap U_0 = D_H \cap U_0 = (D_H \cap D_{H^g})V_1 \cap U_0$$

$$= (D_H \cap D_{H^g} \cap U_0)V_1 = (U_{H^g} \cap U_0)V_1 = U_1V_1 = V_1,$$

establishing (1). By 13.7.3.2, $U_H \le Q \cap V_H$, so $[A, U_H] \le Z_I$ by 13.7.10.3. By 13.7.10.9, $V_3 \le Z_I$. Thus if $U_0 \le [U_H, A]V_3$ then $U_0 \le Z_I$, so (1) implies (2). □

LEMMA 13.7.18. *Either*

(1) $K^* \cong L_2(4)$ *and* $\tilde{U}_K/C_{\tilde{U}_K}(K)$ *is the natural module, or*

(2) $K^* \cong A_6$ *and* $\tilde{U}_H/C_{\tilde{U}_H}(K)$ *is a natural module on which* L_1 *has two noncentral chief factors.*

PROOF. By 13.7.16.3,

$$\tilde{U}_K/C_{\tilde{U}_K}(K) \cong U_K^+/C_{U_K^+}(K)$$

is an irreducible K-module. Also $m([U_H^+, A]) \leq 2$ by 13.7.12.2, while B^* is an FF*-offender on the FF-module U_H^+. Further K^* appears in 13.7.15.1, so applying the remark before 13.7.6 to the restricted list in 13.7.15.1: either $U_K^+/C_{U_K^+}(K)$ is the natural module for $K^* \cong L_2(4)$, or K^*, $U_K^+/C_{U_K^+}(K)$ is one of the pairs considered in 13.7.6. In the former case (1) holds, so we may assume the latter. If $K^* \cong A_6$, then (2) holds by 13.7.6.3. Therefore we must eliminate the remaining cases in 13.7.15.1.

Observe that part (1) of 13.7.6 eliminates $L_3(2)$, parts (1) and (2) of that result eliminate $L_5(2)$, and $L_4(2) \cong A_8$ is eliminated by parts (1) and (3) of that result and 13.7.9.2. The natural module for A_5 is eliminated by part (3) of 13.7.6, and A_7 is eliminated by parts (3) and (4) and 13.7.9.2. Finally $SL_3(4)$ and $Sp_4(4)$ are eliminated by part (5) of 13.7.6. □

LEMMA 13.7.19. $L_1 \leq K$.

PROOF. Assume $L_1 \not\leq K$. Then case (1) of 13.7.18 holds by 13.7.18 and A.3.18. Then as $L_1 = [L_1, T]$, while $|L_1|_3 = 3$ by 13.7.9.2, we conclude $H^* \cong \Gamma L_2(4)$ and either $L_1^* = O_3(H^*)$, or L_1^* is diagonally embedded in $O_3(H^*) \times K^*$. Hence $R_1 = (T \cap K)O_2(KR_1)$, so $m_3(N_H(R_1)) > 1$. Therefore as $L_1 = O^{3'}(H \cap M)$ by 13.7.3.9, and this group has 3-rank 1 by 13.7.9.2, $N_H(R_1) \not\leq M$. Thus L is an A_6-block by 13.2.2.7. Therefore L_1 has just two noncentral 2-chief factors. But if $L_1^* = O_3(H^*)$, then L_1 has two noncentral chief factors on U_H^+, and hence also two on Q_H/H_C by the duality 13.7.4.2. Therefore L_1^* is diagonally embedded, so L_1^* has one chief factor on $O_2(L_1^*)$, plus one each on U_H^+ and Q_H/H_C, again a contradiction. □

LEMMA 13.7.20. $\tilde{U}_K = \tilde{U}_H$.

PROOF. This follows from 13.7.19 and 13.7.5.2. □

LEMMA 13.7.21. K^* *is not* $L_2(4)$.

PROOF. Assume $K^* \cong L_2(4)$. By 13.7.18.1 and 13.7.20, $\tilde{U}_H/C_{\tilde{U}_H}(K)$ is the natural module, while by 13.7.19, $L_1 \leq K$, so $\tilde{V}_3 = [\tilde{V}_3, L_1]$ is a complement to $C_{\tilde{U}_H}(K)$ in $C_{\tilde{U}_H}(T \cap K) =: \tilde{W}$. If $C_{\tilde{U}_H}(K) = 1$, then $\tilde{U}_H = [\tilde{U}_H, L_1]$ is of rank 4, contrary to 13.7.13. Thus $C_{\tilde{U}_H}(K) \neq 1$.

By B.4.2.1, $(T \cap K)^*$ is the unique FF*-offender in T^*, so $A^* = (T \cap K)^*$ by 13.7.12.3. But for each $1 \neq a^* \in A^*$, $[U_H^+, a] = V_3^+$, so $C_{U_H^+}(K) = 1$ and hence $\tilde{U}_0 = C_{\tilde{U}_H}(K)$. Thus $V_1 < U_0$ and $U_0 \leq [U_H, A]V_3$. This contradicts 13.7.17.2. □

By 13.7.18 and 13.7.21, $K^* \cong A_6$ and $\tilde{U}_H/C_{\tilde{U}_H}(K)$ is a natural module on which L_1 has two noncentral chief factors. Now L_1 has two noncentral chief factors on each of U_H and Q_H/H_C by 13.7.4.2, one on $O_2(L_1^*)$, and at least one on V_H/U_H by 13.7.7;

so L_1 has at least six noncentral 2-chief factors. Therefore $m(A^*) \geq 4$ by 13.7.10.8. On the other hand as $End_K(\tilde{U}_H/C_{\tilde{U}_H}(K)) \cong \mathbf{F}_2$, we conclude $K^* = F^*(H^*)$; so A^* acts faithfully on K^*, and hence $m(A^*) \leq m_2(Aut(K^*)) = 3$. This contradiction completes the proof of Theorem 13.7.8.

13.8. Finishing the treatment of $\mathbf{A_6}$

In this section, we complete the treatment of A_6. We prove:

THEOREM 13.8.1. *Assume Hypothesis 13.3.1 with $L/O_{2,Z}(L) \cong A_6$. Then G is isomorphic to $Sp_6(2)$ or $U_4(3)$.*

Throughout this section, we assume that G is a counterexample to Theorem 13.8.1.

Since $L/O_{2,Z}(L) \cong A_6$, we continue with the notation established in section 13.5: Namely we adopt the notational conventions of section B.3 and Notations 12.2.5 and 13.2.1.

As G is a counterexample to Theorem 13.8.1, G is not isomorphic to $U_4(3)$ or $Sp_6(2)$. Thus Hypotheses 13.5.1 and 13.7.1 hold, so we may apply results from sections 13.5 and 13.7. In particular recall from 13.5.2.3 that V is the 4-dimensional A_6-module. The main result Theorem 13.7.8 of section 13.7 has reduced us to the following situation (where $\mathcal{H}_z$ is defined below):

LEMMA 13.8.2. *$\langle V^{G_1} \rangle$ is abelian, so V_H is abelian for each $H \in \mathcal{H}_z$.*

As in the previous section, there are no quasithin examples under this restriction, so we are continuing to work toward a contradiction. Again as far as we can tell, there are not even any shadows.

LEMMA 13.8.3. *If $g \in G$ with $1 \neq V \cap V^g$, then $[V, V^g] = 1$.*

PROOF. As L is transitive on $V^{\#}$, G_1 is transitive on conjugates of V containing V_1 by A.1.7.1, so we may take $g \in G_1$. Then $\langle V, V^g \rangle \leq \langle V^{G_1} \rangle$, so the result follows from 13.8.2. □

As usual z is a generator for V_1, and as in Notation 13.5.8, $\tilde{G}_1 := G_1/V_1$. By 13.3.6, $G_1 \not\leq M$, so $\mathcal{H}_z \neq \emptyset$, where

$$\mathcal{H}_z := \{H \in \mathcal{H}(L_1T) : H \leq G_1 \text{ and } H \not\leq M\}.$$

For the remainder of the section, let H denote some member of $\mathcal{H}_z$.

By 13.5.7, Hypothesis F.9.1 is satisfied with V_3 in the role of "V_+". From Notation 13.5.8, $U_H := \langle V_3^H \rangle$, $V_H := \langle V^H \rangle$, $Q_H := O_2(H) = C_H(\tilde{U}_H)$, and $H^* := H/Q_H$ so that $O_2(H^*) = 1$. Furthermore set $H_C := C_H(U_H)$; then $H_C \leq Q_H$.

Now condition (f) of Hypothesis F.9.8 is satisfied by 13.8.3, and condition (g.i) of Hypothesis F.9.8 is satisfied since $[V, C_H(V_3)] \leq V_1$ by 13.5.4.4; indeed $C_{\bar{M}_V}(V_3) \leq \langle (5,6) \rangle$, with $(5,6)$ inducing the transvection on V with center V_1.

Thus we can appeal to the results in sections F.7 and F.9. In particular, we form the coset geometry Γ of Definition F.7.2 on the pair of subgroups LT and H, and let $b := b(\Gamma, V)$. Choose $\gamma \in \Gamma$ with $d(\gamma_0, \gamma) = b$ and $V \not\leq G_\gamma^{(1)}$. By F.9.11.1, b is odd and $b \geq 3$. Without loss γ_1 is on the geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b := \gamma$$

from γ_0 to γ.

Recall we may choose g_b with $(\gamma_0, \gamma_1)g_b = (\gamma_{b-1}, \gamma)$. Then $U_\gamma := U_H^{g_b}$, $V_\gamma := V_H^{g_b}$, $Q_\gamma := Q_H^{g_b}$, and $A_1 := V_1^{g_b}$. Further $D_H := C_{U_H}(U_\gamma/A_1)$, $E_H := C_{V_H}(U_\gamma/A_1)$, $D_\gamma := C_{U_\gamma}(\tilde{U}_H)$, and $E_\gamma := C_{V_\gamma}(\tilde{U}_H)$. We will appeal extensively to lemmas F.9.13 and F.9.16.

Set $U_L := \langle U_H^L \rangle$, and $Q := O_2(LT)$.

LEMMA 13.8.4. *(1) $b \geq 3$ is odd.*

(2) $U_L \leq Q = O_2(LT)$.

(3) If $b > 3$, then U_L is abelian.

(4) If $b = 3$, then $A_1 \leq V^h$ for some $h \in H$.

(5) $V_3 = V \cap U_H < U_H$, and V_H/U_H is a quotient of the $\mathbf{F}_2 H^*$-permutation module on $H^*/(H \cap M)^*$ with $[V_H/U_H, H] \neq 0$.

(6) V_γ^* is quadratic on V_H/U_H and $\tilde{U}_H$.

(7) $V < U_L$.

PROOF. We have already observed that (1) holds. Part (2) follows from 13.7.3.3. Parts (3) and (4) follow from parts (1) and (2) of F.9.14.

By 13.7.7, $[V_H, H] \not\leq U_H$, so $V \not\leq U_H$. Then as $V_3 \leq V \cap U_H$ with V_3 of index 2 in V, $V_3 = V \cap U_H$. By 13.7.3, $H \cap M$ acts on $VU_H/U_H \cong V/(V \cap U_H) = V/V_3 \cong \mathbf{Z}_2$, so as $V_H = \langle V^H \rangle$ and $[V_H, H] \not\leq U_H$, (5) holds.

As V_γ is abelian and V_H and V_γ normalize each other by F.9.13.2, (6) follows. As $V_3 \leq U_H$, $V = \langle V_3^L \rangle \leq U_L$, and as $V \not\leq U_H \leq U_L$, $V < U_L$. Thus (7) holds. □

LEMMA 13.8.5. *If some element of H^* induces an $\mathbf{F}_2$-transvection on $\tilde{U}_H$, then*

(1) $H = KT$ with $K \in \mathcal{C}(H)$.

(2) Either

(a) $H^* \cong S_6$, $L/O_2(L) \cong A_6$, and L_1 has two noncentral chief factors on $\tilde{U}_H$, or

(b) $H^* \cong S_7$ or $L_4(2)$, and $L/O_2(L) \cong \hat{A}_6$.

(3) $\tilde{U}_H$ is a natural module for H^* or the 5-dimensional cover of such a module for $H^* \cong S_6$.

PROOF. Let $t^* \in T^*$ induce an $\mathbf{F}_2$-transvection on $\tilde{U}_H$. If $K^* = [K^*, t^*] \neq 1$ for some $K \in \mathcal{C}(H)$, then as t^* is an $\mathbf{F}_2$-transvection, we conclude from G.6.4 that K^* is $L_n(2)$ or A_n and $\tilde{U}_K/C_{\tilde{U}_K}(K)$ is a natural module, where $U_K := [U_H, K]$. Hence the lemma follows from parts (1) and (3) of 13.7.6, using I.1.6.1 in the latter case.

So we may assume instead that $K^* := \langle t^{*H} \rangle$ is solvable, and we derive a contradiction. By B.1.8, $K^* = K_1^* \times \cdots K_s^*$, $K_i^* \cong L_2(2)$, with $s \leq 2$ since $m_3(H) \leq 2$, and $\tilde{U}_K = [\tilde{U}_H, K^*] = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_s$, where $\tilde{U}_i := [\tilde{U}_H, K_i] \cong E_4$. Then L_1 acts on each K_i. Thus if $s = 2$, then as $m_3(H) \leq 2$, $L_1^* \leq K^*$. This is impossible as T normalizes a subgroup of order 3 of L_1^*, whereas T is irreducible on $K^* = \langle t^{*H} \rangle$ by construction.

Hence $s = 1$ and $K^* = K_1^* \trianglelefteq H^*$. This time we conclude from the T-invariance of L_1 that either

(a) $L/O_2(L) \cong A_6$ so that $|L_1|_3 = 3$, and either $L_1^* = O^2(K^*)$ or $[K^*, L_1^*] = 1$, or

(b) $L/O_2(L) \cong \hat{A}_6$ so that L_1 has 3-rank 2, and hence $O^2(K) = L_0$ or $L_{1,+}$.

If $O^2(K) = L_0$, then $H \le N_G(O^2(K)) = N_G(L_0) \le M$ by 13.2.2.9, contrary to $H \not\le M$. If $O^2(K) = L_1$ or $L_{1,+}$, then $\tilde{U}_K = \tilde{V}_3$, so $H \le G_1 \cap G_3 \le M$ by 13.5.5 for the same contradiction. Thus $[K^*, L_1^*] = 1$, so

$$\tilde{V}_3 = [\tilde{V}_3, L_1] \le [\tilde{U}_H, L_1] \le C_{\tilde{U}_H}(K),$$

and then as $K^* \trianglelefteq H^*$, $\tilde{U}_H = \langle \tilde{V}_3^H \rangle \le C_{\tilde{U}_H}(K)$, contrary to $K^* \ne 1$. □

LEMMA 13.8.6. *Assume $H = KT$ with $K \in \mathcal{C}(H)$, $K^* \cong A_6$, and $\tilde{U}_H$ is a natural module for K^* or its 5-dimensional cover. Let $K_2 := O^2(C_H(V_2))$ and $U_2 := \langle V^{K_2} \rangle$. Then*

(1) $\tilde{V}_2$ is generated by a vector of weight 4 in $\tilde{U}_H$ and $K_2T/O_2(K_2T) \cong S_3$.
(2) $[K_2, L_2] \le O_2(K_2) \cap O_2(L_2)$.
(3) $U_2 = [U_2, L_2] \le U_L$.
(4) If $m(\tilde{U}_H) = 4$ and $\bar{L}\bar{T} \cong S_6$, then $m(U_2) = 6$ and U_L/V has a quotient isomorphic to the 16-dimensional Steinberg module for $\bar{L}\bar{T}$.
(5) If $m(\tilde{U}_H) = 5$ and $U_1 := C_{U_H}(K)$, then $m(U_2) = 8$, $U_0 := \langle U_1^L \rangle \le U_L$, and U_0/V is a quotient of the 15-dimensional permutation module for $\bar{L}\bar{T}$ on $\bar{L}\bar{T}/\bar{L}_1\bar{T}$.
(6) $L/O_2(L) \cong A_6$.

PROOF. Observe that (1) and (6) hold by 13.7.6.3. In particular, $K_2^*T^*$ is the parabolic of H^* stabilizing the point $\tilde{V}_2$ generated by a vector of weight 4, and $\tilde{V}_3$ is a line with all vectors of weight 4.

By (6) and parts (1) and (6) of 13.5.4, $L_2 \trianglelefteq G_2$, so $[L_2, K_2] \le C_{L_2}(V_2) = O_2(L_2)$, and hence (2) holds. Now

$$U_2 = \langle V^{K_2} \rangle = \langle V_3^{L_2K_2} \rangle = \langle V_3^{K_2L_2} \rangle \le \langle U_H^{L_2} \rangle \le U_L,$$

and as $L_2 \trianglelefteq L_2K_2$ and $V = [V, L_2]$, $U_2 = [U_2, L_2]$, so (3) holds. Set $\hat{U}_2 := U_2/V_2$; it follows that $m(\hat{U}_2) = 2m(\langle \hat{V}_3^{K_2} \rangle)$. Thus $m(U_2)$ is 6 in case (4), and 8 in case (5).

Assume the hypotheses of (4), and recall $V < U_L$ by 13.8.4.7. Let $V \le W < U_L$ with LT irreducible on U_L/W. By 13.8.5.2a, $\tilde{U}_H = [\tilde{U}_H, L_1]$, so that $U_H = [U_H, L_1]$ since $V_1 = [V_3, O_2(L_1)]$; hence $U_HV/V = [U_HV/V, L_1] \cong E_4$. As $W < U_L = \langle U_H^L \rangle$, $U_H \not\le W$, so that U_HW/W is L_1T-isomorphic to U_HV/V. Similarly by (3), $U_2 = [U_2, L_2] \le U_L$, and as we saw $m(U_2) = 6$ in this case, $U_2/V = [U_2/V, L_2] \cong E_4$, so U_2W/W is L_2T-isomorphic to U_2/V. Hence (4) holds by G.5.2.

Finally $[U_1, L_1T] \le V_1 \le V$, so (5) holds. □

LEMMA 13.8.7. *Assume $H = G_1$. Then $D_H < U_H$ iff $D_\gamma < U_\gamma$.*

PROOF. Assume the lemma fails. If $D_H = U_H$ but $D_\gamma < U_\gamma$, then $U_\gamma \not\le Q_H$, and in particular $V_\gamma \not\le Q_H$. Thus there is some $\beta \in \Gamma(\gamma)$ with $V_\beta \not\le Q_H$. By F.7.9.1, $d(\beta, \gamma_1) = b$. Thus we have symmetry (cf. the first part of Remark F.9.17) between the edges γ_0, γ_1 and β, γ, so we may assume that $D_H < U_H$ but $D_\gamma = U_\gamma$. Then case (i) of F.9.16.1 holds, so that U_H induces a nontrivial group of transvections on U_γ with center V_1. Recall there is $g \in G_0 := \langle LT, H \rangle$ with $\gamma g = \gamma_1$, and setting $\alpha := \gamma_1 g$ and $U_\alpha = U_H^g$, $U_\alpha^* \ne 1$ but $[U_H, U_\alpha] = V_1^g =: A_1$. Then U_α induces a group of transvections on $\tilde{U}_H$ with center $\tilde{A}_1$, so by 13.8.5, $H = KT$ for some $K \in \mathcal{C}(H)$, and $\tilde{U}_H$ is a natural module for $H^* \cong L_4(2)$, S_6, or S_7, or the 5-dimensional cover of a natural module for $H^* \cong S_6$.

Suppose one of the first three cases holds, namely $\tilde{U}_H$ is an irreducible module. To eliminate these cases, it will suffice to show:

$$V_1^{gh} \leq V_2 \text{ for some } h \in H. \qquad (*)$$

For if (*) holds, then $V_1^{gh} = V_1^l$ for $l \in L_2T$ with $l^2 \in H$. As $G_1 = H$, $U_H \trianglelefteq G_1$, so as $V_1^{gh} = V_1^l$, also $U_\alpha^h = U_H^{gh} = U_H^l$. Thus as $l^2 \in H$, l interchanges U_H and U_α^h, and also Q_H and Q_α^h, impossible as $U_\alpha \not\leq Q_H$ but $U_H \leq Q_\alpha$. This completes the proof of the sufficiency of (*). Now we establish (*) in each of the first three cases: If $\tilde{U}_H$ is the $L_4(2)$-module or S_6-module, then (*) holds as H is transitive on $\tilde{U}_H^\#$. If $\tilde{U}_H$ is the S_7-module, then (*) follows from 13.7.6.3b, which says $\tilde{V}_2$ is of weight 2, using the fact that the center $\tilde{V}_1^g$ of the transvection U_α^* is of weight 2.

Thus we may assume that $\tilde{U}_H$ is a 5-dimensional module for $H^* \cong S_6$. As U_α^* induces transvections on $\tilde{U}_H$ with center $\tilde{A}_1$, U_α^* has order 2, so $D_\alpha := U_\alpha \cap Q_H$ is a hyperplane of U_α; and as $D_\gamma = U_\gamma$, $[D_\alpha, U_H] = 1$ by F.9.13.7. As $U_\alpha^* \neq 1$, without loss $V_3^{g*} \neq 1$ and $[V_3^g, V_3] \neq 1$. Thus as we saw $[U_\alpha, U_H] = V_1^g$, $[V_3^g, V_3] = V_1^g \leq V_3^g$; so $V_3 \leq C_G(V_1^g) \cap N_G(V_3^g) \leq M_V^g$ by 13.5.5. Then V_3 lies in the unipotent radical of the stabilizer in M_V^g of V_1^g, and is nontrivial on the hyperplane V_3^g orthogonal to V_1^g, so $[V^g, V_3] > V_1^g$.

Define C to the preimage in U_H of $C_{\tilde{U}_H}(V_3^g)$; then $[U_\alpha, C] \leq V_1^g \cap V_1 = 1$, so $C \leq H_C^g$ and hence $[V^g, C] \leq V_1^g$ by 13.7.3.7. Thus from the action of S_6 on the core of the permutation module, $V_3^{g*} = V^{g*}$ is the group of transvections with center $\tilde{A}_1$, so $V^g = V_3^g(V^g \cap Q_H)$. Now $[V^g \cap Q_H, V_3] \leq V^g \cap V_1 = 1$ by 13.8.3. Thus $[V^g, V_3] = [V_3^g(V^g \cap Q_H), V_3] = [V_3^g, V_3] = V_1^g$, contrary to the previous paragraph. □

LEMMA 13.8.8. *Either:*

(1) $D_\gamma = U_\gamma$ or $D_H = U_H$, and U_δ or V_δ induces a nontrivial group of transvections on $\tilde{U}_H$, for $\delta := \gamma g_b^{-1}$ or γ, respectively. Hence $H = KT$ for some $K \in \mathcal{C}(H)$, and H^* and its action on $\tilde{U}_H$ are described in 13.8.5.

(2) $D_\gamma < U_\gamma$, $D_H < U_H$, and we may choose γ so that $0 < m(U_\gamma^*) \geq m(U_H/D_H)$, and $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. Further there is $h \in H$ with $\gamma_2 h = \gamma_0$, and setting $\alpha := \gamma h$, $V_\alpha = V_\gamma^h \leq O_2(L_1T) \leq R_1$.

PROOF. If $D_\gamma = U_\gamma$, then (1) holds by F.9.16.1. Similarly as in the proof of the previous lemma, (1) holds if $D_H = U_H$. Thus we may assume $D_\gamma < U_\gamma$, so by F.9.16.4, we may choose γ as in conclusion (2); then the final statement of conclusion (2) follows from parts (1) and (2) of F.9.13. □

LEMMA 13.8.9. *Assume some $F \leq U_H$ is V_γ-invariant and $G_\gamma = \langle F^{G_\gamma} \rangle G_{\gamma,\gamma_{b-1}}$. Then*

(1) $[F, V_\gamma] \not\leq U_\gamma$.

(2) If $[\tilde{F}, U_\gamma] = [\tilde{F}, V_\gamma]$, then $V_1 \not\leq U_\gamma$ and F induces a group of transvections on V_γ/U_γ with center V_1U_γ/U_γ.

PROOF. Assume $[F, V_\gamma] \leq U_\gamma$. Then F centralizes V_γ/U_γ, so $X := \langle F^{G_\gamma} \rangle$ does also. But by 13.8.4.5, $V_{\gamma_{b-1}}U_\gamma/U_\gamma$ is of order 2, so the section is centralized by $G_{\gamma,\gamma_{b-1}}$, and hence also by $G_\gamma = XG_{\gamma,\gamma_{b-1}}$. But then as $V_\gamma = \langle V_{\gamma_{b-1}}^{G_\gamma} \rangle$, G_γ centralizes V_γ/U_γ, contrary to 13.7.7. Thus (1) is established.

So assume that $[\tilde{F}, U_\gamma] = [\tilde{F}, V_\gamma]$. Then $[F, V_\gamma] \leq [F, U_\gamma]V_1 \leq U_\gamma V_1$ as U_H acts on U_γ. If $V_1 \leq U_\gamma$, then $[F, V_\gamma] \leq U_\gamma$, contrary to (1), so (2) holds. $\square$

LEMMA 13.8.10. *If $m(U_\gamma^*) = 1$ and $U_H < D_H$, then*

(1) $m(U_H/D_H) = 1$, so we have symmetry between γ_1 and γ in the sense of Remark F.9.17.

(2) Either $V_1 \leq U_\gamma$, or U_H induces transvections on U_γ with axis D_γ.

PROOF. Assume $m(U_\gamma^*) = 1$, so in particular case (2) of 13.8.8 holds. Then

$$1 = m(U_\gamma^*) = m(U_\gamma/D_\gamma) \geq m(U_H/D_H)$$

and $D_H < U_H$ by hypothesis, so we conclude that $m(U_H/D_H) = 1$, and we have symmetry between γ_1 and γ as discussed in Remark F.9.17. Now by F.9.13.6, $[D_\gamma, U_H] \leq V_1 \cap U_\gamma$, so (2) follows. $\square$

LEMMA 13.8.11. *Assume $U_\gamma^* \neq 1$ and $G_\gamma = \langle U_H^{G_\gamma} \rangle G_{\gamma,\gamma_{b-1}}$. Then*

(1) If either $V_1 \leq U_\gamma$, or no element of H induces a transvection on V_H/U_H, then $U_\gamma^* < V_\gamma^*$, so $m(V_\gamma^*) > 1$.

(2) If U_H does not induce a transvection on U_γ with axis D_γ, then $m(V_\gamma^*) > 1$.

PROOF. By hypothesis $U_\gamma^* \neq 1$ and $U_H \not\leq U_\gamma$, so that case (2) of 13.8.8 holds and $D_H \neq U_H$. If $U_\gamma^* = V_\gamma^*$, then $[\tilde{U}_H, V_\gamma] = [\tilde{U}_H, U_\gamma]$, and so 13.8.9.2 supplies a contradiction with U_H in the role of "F"; hence $U_\gamma^* < V_\gamma^*$ so that (1) holds. Assume the hypotheses of (2) but with $m(V_\gamma^*) = 1$. Then as $1 \neq U_\gamma^* \leq V_\gamma^*$, $U_\gamma^* = V_\gamma^*$ is of rank 1. Thus $V_1 \leq U_\gamma$ by 13.8.10.2, contrary to (1); hence (2) holds. $\square$

LEMMA 13.8.12. *(1) If $K \in \mathcal{C}(H)$, then $K \not\leq M$ and $\langle K, T \rangle L_1 \in \mathcal{H}_z$.*

(2) Let $X := O^2(O_{2,F}(H) \cap M)$. Then one of the following holds:

(a) $X = 1$.

(b) $L/O_2(L) \cong A_6$ and $X = L_1$.

(c) $L/O_2(L) \cong \hat{A}_6$ and $X = L_{1,+}$.

PROOF. First $K \not\leq M$ by 13.3.9 with $\langle K^T \rangle$ in the role of "Y", so (1) holds.

Now define X as in (2), and assume none of (a)–(c) holds. Then $O^2(O_{2,F}(H)) \leq M$ by 13.3.9, so $O_{2,F}(H)T \in \mathcal{H}(T, M)$. Let F denote a T-invariant subgroup of $O_{2,F}(H)$ minimal subject to $X \leq F = O^2(F)$ and $FT \in \mathcal{H}(T, M)$. Then $XO_2(F) < F$ since $F \not\leq M$, so as $F/O_2(F)$ is nilpotent, $X < O^2(N_F(XO_2(F)))$. But also $F \cap M = X$, so $O^2(N_F(XO_2(F))) = F$ by minimality of F. Then F normalizes $O^2(XO_2(F)) = X$, again contrary to 13.3.9, now with X, FT in the roles of "Y, H". $\square$

LEMMA 13.8.13. *Each solvable overgroup of L_1T in G_1 is contained in M.*

PROOF. If not, we may choose H solvable, and minimal subject to $H \in \mathcal{H}_z$. Then case (2) of 13.8.8 holds; in particular $1 \neq U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ and $1 \neq V_\alpha^* \leq R_1^*$. By 13.8.12, $O^2(O_{2,F}(H) \cap M) = 1$ or X, where $X := L_1$ if $L/O_2(L) \cong A_6$ and $X := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$.

Now as $O_2(H^*) = 1$, there exists an odd prime p with $[O_p(H^*), V_\alpha^*] \neq 1$. So by the Supercritical Subgroups Lemma A.1.21 and A.1.24, there exists a subgroup $P \cong \mathbf{Z}_p$, E_{p^2}, or p^{1+2} such that $P^* \trianglelefteq H^*$, and V_α^* is nontrivial on P^*. If $P \leq M$ then $P \leq O^2(O_{2,F}(H) \cap M) \leq X \leq L_1$, so $[V_\alpha^*, P^*] \leq O_2(L_1^*) \cap P^* = 1$, a contradiction. Thus $P \not\leq M$, so by minimality of H, $H = PL_1T$ and L_1T is irreducible on $P^*/\Phi(P^*)$. As $P \not\leq M$, $X^* \neq P^*$.

As $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, $p = 3$ or 5 by D.2.13.1.

Suppose first that $p = 3$. If P^* is of order 3, then as $m_3(H) \leq 2$ and $P \not\leq M$, $O_3(H^*) = P^* \times L_1^* \cong E_9$; hence $H^* \cong S_3 \times S_3$ as V_α^* is nontrivial on P^*. Therefore $V_\alpha^* = O_2(L_1^*T^*) \cong \mathbf{Z}_2$, so $m(V_\gamma^*) = m(U_\gamma^*) = 1$. Also $L_1 \trianglelefteq H$, so $\tilde{U}_H = [\tilde{U}_H, L_1]$, and hence $m(\tilde{U}_H) = 2m \geq 4$, where $m := m([\tilde{U}_H, U_\gamma^*])$. Now by 13.8.10.1, we have symmetry between γ and γ_1, so U_H does not induce transvections on U_γ/A_1. Hence $V_1 \leq U_\gamma$ by 13.8.10.2. Further $H = L_1T\langle U_\gamma^H\rangle$, so by symmetry, $G_\gamma = G_{\gamma,\gamma_{b-1}}\langle U_H^{G_\gamma}\rangle$, and hence $m(V_\gamma^*) > 1$ by 13.8.11.1, contradicting $|V_\gamma^*| = 2$.

Therefore $P^* \cong E_9$ or 3^{1+2}. Suppose $L_1^* \not\leq P^*$. As L_1T is irreducible on $P^*/\Phi(P^*)$, H induces $SL_2(3)$ or $GL_2(3)$ on $P^*/\Phi(P^*)$. So in particular if $P^* \cong E_9$, then $m([\tilde{U}_H, P]) \geq 8$; as $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, this contradicts D.2.17. Hence $P^* \cong 3^{1+2}$, so that $m_3(L_1P) > 2$, contradicting H an SQTK-group. Therefore $L_1^* \leq P^*$, so $L_1^* < P^*$ as $X^* \neq P^*$. Then as L_1T is irreducible on $P^*/\Phi(P^*)$, $P^* \cong 3^{1+2}$ and $L_1^* = Z(P^*)$, so that $L/O_2(L) \cong A_6$. Then $O_2(L_1^*T^*) = C_{T^*}(L_1^*)$ is of 2-rank at most 1, so $m(V_\gamma^*) = 1$. As $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, U_γ^* inverts $P^*/\Phi(P^*)$ by D.2.17.4. Now we obtain a contradiction as in the previous paragraph.

We have reduced to the case $p = 5$. As $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ and H is minimal, we conclude from D.2.17 that $P = P_1 \times \cdots \times P_s$ with $s \leq 2$, and $[P, \tilde{U}_H] = \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_s$ with $P_i^* \cong \mathbf{Z}_5$, where $\tilde{U}_i := [P_i, \tilde{U}_H]$ is of rank 4. If $s = 1$ then $U_\gamma^* \cong \mathbf{Z}_2$, while if $s = 2$, then either $U_\gamma^* \cong \mathbf{Z}_2$ with $[U_\gamma^*, P_2^*] = 1$, or $U_\gamma^* = B_1^* \times B_2^*$ with $B_i^* \cong \mathbf{Z}_2$ centralizing P_{3-i}^*. However if U_γ^* is of order 4 then $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma)) = 4 = 2m(U_\gamma^*)$, and so F.9.16.2 shows that $m(U_H/D_H) = 2$ and U_γ^* acts faithfully on $\tilde{D}_H$ as a group of transvections with center $\tilde{A}_1$. Then $m([\tilde{U}_H, U_\gamma^*]) \leq 3$, whereas this commutator space has rank 4 since $s = 2$.

Therefore $m(U_\gamma^*) = 1$, so as before we have symmetry bewtween γ_1 and γ by 13.8.10.1; and as L_1T is irreducible on P^*, $G_\gamma = G_{\gamma,\gamma_{b-1}}\langle U_H^{G_\gamma}\rangle$. As $p = 5$, no element of H^* induces a transvection on $\tilde{U}_H$ by G.6.4; hence we conclude from 13.8.11.2 that $m(V_{\gamma^*}) > 1$. In particular as V_γ^* is faithful on P^*, P^* is not cyclic, so $s = 2$ and $V_\gamma^* = U_\gamma^* \times B_2^*$ with $B_2^* \cong \mathbf{Z}_2$ centralizing P_1^*.

Let $C_H := C_{U_H}(U_\gamma)$ and $\tilde{F}_H := C_{\tilde{U}_H}(U_\gamma)$. By 13.8.10.1, $m(U_H/D_H) = 1$, and by F.9.13.6, $[U_\gamma, D_H] \leq A_1$. Thus if $F_H \not\leq D_H$, then $U_H = D_HF_H$, so $[\tilde{U}_H, U_\gamma^*] \leq \tilde{A}_1$, contrary to $m([\tilde{U}_H, U_\gamma^*]) = 2$. Hence $F_H \leq D_H$. Then by F.9.13.6,

$$[F_H, U_\gamma] \leq V_1 \cap [D_H, U_\gamma] \leq V_1 \cap A_1 = 1$$

and hence $U_2 \leq F_H = C_H$. Then $[U_2, V_\gamma] \leq [C_H, V_\gamma] \leq A_1$ by 13.7.3.7, with $\tilde{A}_1 = [\tilde{D}_H, U_\gamma] \leq \tilde{U}_1$. On the other hand, $1 \neq [\tilde{U}_2, B_2] \leq [\tilde{U}_2, V_\gamma] \cap \tilde{U}_2 \leq \tilde{A}_1 \cap \tilde{U}_2$, contrary to $\tilde{U}_1 \cap \tilde{U}_2 = 0$. □

By 13.8.13 and 13.8.12.2:

LEMMA 13.8.14. *Let $X := O^2(O_{2,F}(H))$; then one of the following holds:*

(a) $X = 1$.

(b) $L/O_2(L) \cong A_6$ and $X = L_1$.

(c) $L/O_2(L) \cong \hat{A}_6$ and $X = L_{1,+}$.

By 13.8.13, H is nonsolvable, so there exists $K \in \mathcal{C}(H)$. By 13.8.14, $F(H^*) = Z(O^2(H^*))$, so K^* is quasisimple. Then by 13.8.12.1:

LEMMA 13.8.15. *$K \not\leq M$, so $\langle K^T\rangle L_1 T \in \mathcal{H}_z$. Further K^* is quasisimple.*

LEMMA 13.8.16. *(1) $K \trianglelefteq H$, so $KL_1T \in \mathcal{H}_z$. In particular, F.9.18.4 applies.*
(2) $K/O_2(K)$ is not $Sz(2^n)$.

PROOF. Assume $K_0 = \langle K^T\rangle > K$. Then $K_0 = KK^t$ for $t \in T - N_T(K)$ by 1.2.1.3. Let $K_1 := K$ and $K_2 := K^t$. By 13.8.15, we may take $H = K_0 L_1 T$. By F.9.18.5, $K^* \cong L_2(2^n)$, $Sz(2^n)$, or $L_3(2)$. Further unless $K^* \cong Sz(2^n)$, $K_0 = O^{3'}(H)$ by 1.2.2.a so $L_1 \leq K_0$.

Suppose first that $K^* \cong L_3(2)$. Then $L_1 \leq H_1 \leq H$ where $H_1/O_2(H_1) \cong S_3$ wr $\mathbf{Z}_2$, so $L_1 = \theta(H \cap M) = O^2(H_1)$ using 13.7.3.9. As $m_3(O^2(H_1)) = 2$, $L/O_2(L) \cong \hat{A}_6$, so that $Aut_M(L_1/O_2(L_1)) \cong E_4$, whereas we have seen just above that $Aut_{H\cap M}(L_1/O_2(L_1)) \cong D_8$.

Therefore $K^* \cong L_2(2^n)$ or $Sz(2^n)$. Let B_0^* be a Borel subgroup of K_0^* containing $T_0^* := T^* \cap K_0^*$, and set $B := O^2(B_0)$. As $L_1T = TL_1$, L_1 acts on B_0. Therefore $B_0 \leq M$ by 13.8.13.

Let $\tilde{W}$ denote an H-submodule of $\tilde{U}_H$ maximal subject to $[\tilde{U}_H, K_0] \not\leq \tilde{W}$; thus $[\tilde{U}_H, K_0]\tilde{W}/\tilde{W}$ is an irreducible K_0-module. As $K_0^*T^*$ has no strong FF-modules by B.4.2, it follows from parts (5) and (6) of F.9.18 that either

(a) U_H/W and $\tilde{W}$ are FF-modules for $K_0^*T^*$, or

(b) $[\tilde{U}_H, K_0] = \tilde{I}_H = \langle \tilde{I}^H\rangle$ for some $\tilde{I} \in Irr_+(K_0, \tilde{U}_H, T)$, and $[\tilde{W}, K_0] = 0$.

Let $U := U_H/W$ or $\tilde{I}_H$ in case (a) or (b), respectively, and let V_U denote the projection of $\tilde{V}_3$ on U.

Suppose for the moment that case (a) holds. Then by Theorems B.5.1 and B.5.6, $K^* \cong L_2(2^n)$, and $U = U_1 \oplus U_2$, where U_i is the natural module or orthogonal module for K_i^*, and $[K_i, U_{3-i}] = 0$. Further as $U_H = \langle V_3^H\rangle$, $V_3 \not\leq W$, so as L_1 is irreducible on $\tilde{V}_3$, V_U is isomorphic to $\tilde{V}_3$.

Now suppose for the moment that case (b) holds. Then by F.9.18.5, either

(b1) $U = U_1 + U_2$ with $U_i := [U, K_i]$ and $U_i/C_{U_i}(K_i)$ the natural or A_5-module for K^*, or

(b2) U is the natural orthogonal module for $K_0^* \cong \Omega_4^+(2^n)$.

Here if $V_3 \leq U$, then $U = \langle V_3^H\rangle = U_H$. In particular this subcase holds when $K^* \cong L_2(2^n)$, since there we saw that $L_1 \leq K_0$, so that $V_3 \leq [U_H, L_1] \leq [U_H, K_0] = U$.

We first eliminate the case $K^* \cong L_2(2^n)$. Since $L_1 \leq K_0$, $L_1 \leq N_{K_0}(B_0) = B_0$, and hence n is even. Then $m_3(B_0) = 2$, so as $B_0 \leq M$, $L/O_2(L) \cong \hat{A}_6$ by 13.7.3.9. As $t \in T - N_T(K)$ acts on L_0 and $L_{1,+}$, these groups are diagonally embedded in K_0. Let $B := O^2(B_0)$. As $L_{1,+}/O_2(L_{1,+})$ is inverted by $s \in T \cap L$, and $[B, s] \leq L$, $[B, s]$ is a $\{2,3\}$-group. We conclude that $n = 2$ and $L_1 = B$.

Assume that case (a) or (b1) holds. Then $V_U \not\leq U_i$ as V_U is T-invariant. Thus the projections V_U^i of V_U on U_i are nontrivial. As $V_U = [V_U, L_{1,+}]$, also $V_U^i = [V_U^i, L_{1,+}]$. Similarly L_0 centralizes V_U^i. This is impossible, as $L_0K_2 = L_{1,+}K_2$ since L_0 and $L_{1,+}$ are diagonally embedded in K_0, and $[U_1, K_2] = 0$.

Therefore case (b2) holds, so $U = \tilde{U}_H$ is the orthogonal module. In particular H^* contains no $\mathbf{F}_2$-transvections, so case (2) of 13.8.8 holds. Hence the K_0-conjugate V_α^* of V_γ^* defined in that case is contained in $O_2(L_1^*)$. Further

$U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, so in particular U_α^* acts quadratically on $\tilde{U}_H$, and hence it follows from the facts that $n = 2$, U_γ^* is a 4-group, and $m(\tilde{U}_H / C_{\tilde{U}_H}(U_{\gamma^*})) = 4$. Now by F.9.16.2, $m(U_H/D_H) = 2$, which is impossible as $[\tilde{D}_H, U_\gamma^*] = \tilde{A}_1$ by F.9.13.6, whereas no 4-group in K_0^* induces a group of transvections on a subspace of codimension 2 in $\tilde{U}_H$ of dimension 8.

It remains to eliminate the case $K^* \cong Sz(2^n)$. Since $B \leq M$, $[B, L_1] \leq L_1 \cap B \leq O_2(L_1)$, so L_1^* centralizes K_0^*. However case (a) or (b1) holds, so that $U = U_1 \oplus U_2$ with U_i the natural module for K^*; then $End_{K^*}(U_i) = \mathbf{F}_{2^n}$ with n odd, and hence $[U, L_1] = 1$. This is a contradiction, since $L_1 \trianglelefteq H$ and $\tilde{V}_3 = [\tilde{V}_3, L_1]$, so $\tilde{U}_H = [\tilde{U}_H, L_1]$.

Essentially the same argument establishes (2): We conclude from parts (4) and (7) of F.9.18 that $[\tilde{U}_H, K]/C_{[\tilde{U}_H,K]}(K)$ is the natural module for $K/O_2(K) \cong Sz(2^n)$. Again L_1^* centralizes K^* and then also $[\tilde{U}_H, K]$, for the same contradiction. □

By 13.8.16 and F.9.18.4:

LEMMA 13.8.17. *$K^* \cong L_2(2^n)$, $(S)L_3(2^n)^\epsilon$, $Sp_4(2^n)'$, $G_2(2^n)'$, $L_4(2)$, $L_5(2)$, A_7, $\hat{A}_6$, M_{22}, or $\hat{M}_{22}$.*

In the remainder of the section, we successively eliminate the cases listed in 13.8.17.

Observe that the second case of 13.8.8 holds, unless K^* is one of the groups A_6, A_7, or $L_4(2)$ allowed by 13.8.5.2 in the first case.

LEMMA 13.8.18. *If $H = KL_1T$, then*

(1) $G_\gamma = \langle F^{G_\gamma} \rangle G_{\gamma,\gamma_{b-1}}$ for each $F \leq U_H$ with $F \not\leq D_H$.

(2) In case (2) of 13.8.8, the hypotheses of 13.8.11 are satisfied.

(3) If case (2) of 13.8.8 holds and U_H does not induce a transvection on U_γ, then $m(V_\gamma^*) > 1$.

(4) If no member of H^* induces a transvection on $\tilde{U}_H$, then $m(V_\gamma^*) > 1$.

PROOF. By F.9.13.2 $U_H \leq O_2(G_{\gamma,\gamma_{b-1}})$, while as $H = KL_1T$, for g_b with $(\gamma_0, \gamma_1)g_b = (\gamma_{b-1}, \gamma)$ we have $G_\gamma = K^{g_b} G_{\gamma,\gamma_{b-1}}$. Thus if $F \not\leq D_H$, then $K^{g_b} = [K^{g_b}, F]$, so (1) holds. In case (2) of 13.8.8, $D_H < U_H$, so (2) follows by an application of (1) with U_H in the role of "F". Finally 13.8.11.2 and (2) imply (3), and 13.8.8 and (3) imply (4). □

LEMMA 13.8.19. *H^* is not $L_3(2)$.*

PROOF. Assume $H^* \cong L_3(2)$. Then $L_1^* T^*$ is a maximal parabolic of H^*; let P^* be the remaining maximal parabolic of H^* containing T^*. Since $\tilde{U}_H = \langle \tilde{V}_3^H \rangle$ with L_1T inducing S_3 on $\tilde{V}_3$, H.6.5 says $\tilde{U}_H$ is one of the following: the natural module W in which P^* stabilizes a point, the core U_2 of the permutation module on H^*/P^*, the Steinberg module S, $W \oplus S$, or $U_2 \oplus S$. By 13.7.6.1, $\tilde{U}_H$ is not natural. Then since $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ by 13.8.8, it follows using B.5.1 and B.4.5 that $\tilde{U}_H = U_2$. By 13.8.8, $V_\alpha^* \leq R_1^*$, so as R_1^* is not quadratic on $\tilde{U}_H = U_2$, it follows that $m(V_\alpha^*) = 1$. This contradicts 13.8.18.4 in view of G.6.4. □

LEMMA 13.8.20. *K^* is not of Lie type over $\mathbf{F}_{2^n}$ for any $n > 1$.*

PROOF. Assume otherwise. By 13.8.16, we may take $H = KL_1T$. By 13.8.17, $K^* \cong L_2(2^n)$, $(S)L_3^\epsilon(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$. By 13.8.5, case (2) of 13.8.8 holds, so in particular $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

Let B_0^* be the Borel subgroup of K^* containing $T_0^* := T^* \cap K^*$, and let $B := O^2(B_0)$. As K is defined over $\mathbf{F}_{2^n}$ with $n > 1$, and $L_1T = TL_1$, L_1 acts on B; so by 13.8.13, $B \leq M$. Then using 13.7.3.9, $L_1 = \theta(BL_1)$, and $BL_1 = B_CL_1$, where

$$B_C := O^2(C_{BL_1}(L/O_2(L))) \leq C_M(V).$$

Let $X := L_1$ if $L/O_2(L) \cong A_6$, and $X := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$. If $L/O_2(L) \cong A_6$ then $B_C = O^3(B)$, while if $L/O_2(L) \cong \hat{A}_6$, then $B_C = O^3(B)L_0$. In either case, $|BX : B_CO_2(B)| = 3$.

Next $X/O_2(X)$ is inverted by some $t \in T \cap L$, and $[B_C, t] \leq O_2(B_C)$. Now from the structure of $Aut(K^*)$, one of the following holds:

(i) $C_{T^*}(O^3(B^*)O_2(B^*)/O_2(B^*)) = O_2(B_0^*)$.

(ii) $n = 2$ or 6, and K^* is not $U_3(2^n)$.

(iii) $K^* \cong (S)U_3(8)$.

In case (i) as $[t^*, O^3(B^*)] \leq O_2(B^*)$, $t^* \in O_2(B^*)$, a contradiction as t^* inverts $X^*/O_2(X^*)$. In case (ii) if t^* induces an outer automorphism on K^*, then $|[B^*, t^*]/O_2([B^*, t^*])| > 3$ unless K^* is $(S)L_3(4)$ or $L_2(4)$. Therefore we conclude that either:

(a) $X \not\leq K$, $X^* \leq C_{H^*}(K^*)$ so that $X \trianglelefteq KL_1T = H$, and X^* is inverted in $C_{H^*}(K^*)$, or

(b) $K^* \cong L_2(4)$, $(S)U_3(8)$, or $(S)L_3(4)$, and t^* induces an outer automorphism on K^*.

Assume first that (a) holds. Then as H is an SQTK-group, $m_3(K) = 1$, so that $K^* \cong L_2(2^n)$, $L_3(2^n)$ for $n > 1$ odd, or $U_3(2^n)$ for n even. Further $X \trianglelefteq H$ and $\tilde{V}_3 = [\tilde{V}_3, X]$, so $\tilde{U}_H = [\tilde{U}_H, X]$. Hence as X^* is inverted in $C_{H^*}(K^*)$, each noncentral chief factor for H on $\tilde{U}_H$ is the sum of a pair of isomorphic K^*-modules. Then case (ii) of F.9.18.4 holds, so that each $\tilde{I} \in Irr_+(K, \tilde{U}_H, T)$ is a T-invariant FF-module for KT. Therefore $\tilde{I}_H := \langle \tilde{I}^H \rangle$ is the sum of two X-conjugates of $\tilde{I}$, and K^* is not $(S)U_3(2^n)$.

Suppose K^* is $L_2(2^n)$. Observe that if n is even, then $m_3(XB) > 1$, so we conclude from 13.7.3.9 that $L/O_2(L) \cong \hat{A}_6$. We saw earlier that $B_C = O^3(B)L_0$, with $|BX : O_2(B)B_C| = 3$; then since $X \not\leq K$ as case (a) holds, we conclude that $B = B_C$ centralizes V. On the other hand if n is odd, then B is a $3'$-group, so again $B = B_C$ centralizes V.

Next as K^*T^* has no strong FF-modules by B.4.2, applying F.9.18.6 to $\tilde{I}_H$ in the role of "$\tilde{W}$", we conclude $[\tilde{U}_H, K] = \tilde{I}_H$. As $\tilde{I}/C_{\tilde{I}}(K)$ is an FF-module, by B.4.2 it is either the natural $L_2(2^n)$-module or the A_5-module. In the first case as B centralizes V, $\tilde{V}_3 \leq C_{\tilde{U}_H}(BT_0^*) = C_{\tilde{U}_H}(K)$, a contradiction since $U_H = \langle V_3^H \rangle$ and $K^* \neq 1$. Thus $\tilde{I}$ is the A_5-module, so that $J(H^*) \cong S_5$ by B.4.2.5; hence $H^* \cong S_5 \times S_3$ and $\tilde{U}_H$ is the tensor product of the S_5-module and S_3-module. Since case (2) of 13.8.8 holds, there is an H-conjugate α of γ such that $V_\alpha^* \leq O_2(L_1^*T^*) = T_0^* \leq K^*$. Then as V_γ^* is quadratic on $\tilde{U}_H$, $|\tilde{V}_\gamma^*| = 2$, contrary to 13.8.18.4.

This leaves the case $K^* \cong L_3(2^n)$, $n > 1$ odd. This time the FF-module $\tilde{I}$ is natural by B.4.2, so $\tilde{I}_H$ is the tensor product of natural modules for K^* and S_3.

As n is odd, $B = B_C$, so B centralizes $\tilde{V}_3$. Therefore as $C_{\tilde{I}_H}(B) = 0$, we conclude $V_3 \not\leq I_H$. If $I_H = [U_H, K]$, then $V_3 I_H$ is invariant under $KL_1T = H$, so $U_H = V_3 I_H$. Then as T_0 centralizes $\tilde{V}_3$, $\tilde{U}_H = \tilde{I}_H \oplus C_{\tilde{U}_H}(K)$ and $C_{\tilde{U}_H}(K) = C_{\tilde{U}_H}(B) = \tilde{V}_3$. But now $U_H = \langle V_3^H \rangle = V_3$, contrary to 13.5.9. Hence K^* is faithful on U_H/I_H, so case (b) or (c) of F.9.18.6 holds with $\tilde{I}_H$ in the role of "$\tilde{W}$". Therefore $[U_H, K]/I_H$ is an FF-module for K^*T^*, and hence this quotient is also the tensor product of natural modules for K^* and S_3. Then again B centralizes $\tilde{V}_3$, but is fixed-point-free on $[\tilde{U}_H, K]$, so that $V_3 \not\leq [U_H, K]$. Now we obtain a contradiction as in the earlier case, arguing on $[U_H, K]$ in place of I_H.

Therefore (b) holds. As $q(H^*, \tilde{U}_H) \leq 2$, K^* is not $L_3(4)$ by B.4.5. Thus $K^* \cong L_2(4)$, $SL_3(4)$, or $(S)U_3(8)$. We claim $L_1 \leq K$; so assume otherwise. As $1 \neq O^{3'}(B) \leq L_1$ but $L_1 \not\leq K$, it follows that $L/O_2(L) \cong \hat{A}_6$ and $|B|_3 = 3$. Hence $K^* \cong U_3(8)$ or $L_2(4)$. In the first case, A.3.18 supplies a contradiction as $L_1/O_2(L_1) \cong E_9$ and T acts on L_1 but does not permute with the subgroup generated by the element x^* in A.3.18.b. Thus $K^* \cong L_2(4)$, and as L_0 and $L_{1,+}$ are the only proper T-invariant subgroups of L_1 which are not 2-groups, $K^*L_1^* = K^* \times L_C^*$, where $L_C = L_0$ or $L_{1,+}$. In the former case, $K \leq N_G(L_0) = M$ by 13.2.2.9, a contradiction. In the latter case, as $[L_0, t] \leq O_2(L_0)$ and case (a) fails, we have a contradiction. Thus the claim is established.

By the claim and 13.7.5.2, $U_H = [U_H, K]$. We next observe that K^* is not $(S)U_3(8)$: For otherwise we may apply F.9.18.7 and B.4.5 to conclude that $\tilde{U}_H$ is the natural module for $K^* \cong SU_3(8)$, defined over $\mathbf{F}_8$. But then there is no B-invariant subspace $\tilde{V}_3 = [\tilde{V}_3, L_1]$ of 2-rank 2.

Suppose K^* is $SL_3(4)$. By B.4.5, any $I \in Irr_+(K, \tilde{U}_H, T)$ is the natural module. Further $B = \theta(B) \leq L_1$ by 13.7.3.9, and B is of 3-rank 2, so $L/O_2(L) \cong \hat{A}_6$. Then as $X = L_{1,+}$ is inverted by $t \in C_T(L_0/O_2(L_0))$, we conclude that either t induces a graph-field automorphism on K^* with $L_0^* = C_{L_1^*}(t^*) = Z(K^*)$, or t induces a graph automorphism on K^* and $X^* = [L_1^*, t^*] = Z(K^*)$. In the first case, $H \leq N_G(L_0) \leq M$ by 13.2.2.9, contrary to $H \not\leq M$; so the second case holds. Now case (iii) of F.9.18.4 holds, with $\tilde{I}_H = \tilde{I} \oplus \tilde{I}^t$, where $\tilde{I} \in Irr_+(\tilde{U}_H, K, T)$ is a natural module for K^* and $\tilde{I}^t$ is its dual. By F.9.18.7, $\tilde{I}_H = [\tilde{U}_H, K]$, so $I_H = U_H$ by the previous paragraph. Further $U_{\gamma^*} \in \mathcal{Q}(H^*, \tilde{U}_H)$ is either a root group of K^* of rank 2 with $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma)) = 4$, or $m(U_\gamma^*) \geq 3$ with $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma)) = 6$. In the first case by F.9.16.2, U_γ^* is faithful on $\tilde{D}_H$ of corank 2 in $\tilde{U}_H$; and in the second, at least $m(U_H/D_H) \leq m(U_\gamma^*) \leq m_2(H^*) = 4$. In either case, no subspace $\tilde{D}_H$ of this corank in $\tilde{U}_H$ satisfies the requirement $[U_\gamma^*, \tilde{D}_H] = \tilde{A}_1$ of F.9.13.6.

We are left with the case $L_1 \leq K^* \cong L_2(4)$. Thus $L/O_2(L) \cong A_6$ by 13.7.5.5. As $L_1 = [L_1, T]$ and $H = KL_1T = KT$, $H^* \cong S_5$. Then as case (2) of 13.8.8 holds, $V_\alpha^* \leq R_1^* \in Syl_2(K^*)$, and $m_2(R_1^*) = 2$, so $V_\alpha^* = R_1^*$ by 13.8.18.4. Now by 13.8.4.5, V_H/U_H is a nontrivial quotient of the 5-dimensional permutation module for $H^* \cong S_5$. Then as $V_\alpha^* = R_1^*$, V_γ^* is not quadratic on V_H/U_H, contrary to 13.8.4.6. □

LEMMA 13.8.21. *(1)* $L_1 \leq K$.
(2) $K/O_2(K)$ *is* $L_n(2)$, $3 \leq n \leq 5$, A_6, A_7, *or* $G_2(2)'$.
(3) $H = KT$. *In particular if* $K \cong L_3(2)$, *then* $H^* \cong Aut(L_3(2))$.
(4) $U_H = [U_H, K]$.

PROOF. We begin with the proof of (2); as usual, we may take $H = KL_1T$. By 13.8.17 and 13.8.20, $K^* \cong L_4(2)$, $L_5(2)$, A_6, A_7, $G_2(2)'$, $\hat{A}_6$, M_{22}, or $\hat{M}_{22}$. Thus to establish (2) we may assume $K/O_2(K) \cong \hat{A}_6$, M_{22}, or $\hat{M}_{22}$, and it remains to derive a contradiction.

By A.3.18, $L_1 \leq \theta(H) = K$. Then L_1 is solvable and normal in $J := K \cap M$. It follows when $K/O_2(K) \cong \hat{A}_6$ that $J/O_{2,Z}(K)$ is a maximal parabolic subgroup of $K/O_{2,Z}(K)$, and when $K/O_{2,Z}(K) \cong M_{22}$ that $J/O_{2,Z}(K)$ is a maximal parabolic of the subgroup $K_1/O_{2,Z}(K) \cong A_6/E_{2^4}$ of $K/O_{2,Z}(K)$.

Assume $K/O_2(K) \cong M_{22}$. By the previous paragraph, $|L_1|_3 = 3$, so $L/O_2(L) \cong A_6$ rather than $\hat{A}_6$. Further case (i) of F.9.18.4 holds with $\tilde{I} \in Irr(K, \tilde{U}_H)$, and $\tilde{I}$ is the code module in view of F.9.18.2 and B.4.5. As M_{22} has no FF-modules by B.4.2, $\tilde{I} = [\tilde{U}_H, K]$ by F.9.18.7, so that $\tilde{V}_3 = [\tilde{V}_3, L_1] \leq \tilde{I}$. By the previous paragraph, $C_{\tilde{I}}(O_2(L_1T)) \leq C_{\tilde{I}}(O_2(K_1T))$, while $m(C_{\tilde{I}}(O_2(K_1T))) = 1$ by H.16.2.1. This is a contradiction, since L_1T induces $GL(\tilde{V}_3)$ on $\tilde{V}_3$ of rank 2 in $\tilde{I}$, so that $O_2(L_1T)$ centralizes $\tilde{V}_3$.

Thus we may assume that $K/O_2(K) \cong \hat{A}_6$ or $\hat{M}_{22}$. Then $Y := O^2(O_{2,Z}(K)) \neq 1$; by 13.8.13, $Y \leq M$, so $Y \leq \theta(H \cap M) = L_1$ by 13.7.3.9. Then if $Y = L_1$, each solvable overgroup of YT in H is contained in M by 13.8.13. However there is $K_1 \in \mathcal{L}(KT, T)$ with $K_1/O_{2,Z}(K_1) \cong A_6$, so either $K_1 \in \mathcal{C}(H \cap M)$ or $K = K_1$ and T is nontrivial on the Dynkin diagram of K^*. In the former case $K_1 = L$, contradicting $M = !\mathcal{M}(LT)$. As $q(H^*, \tilde{U}_H) \leq 2$, the latter is impossible by B.4.5. Thus $Y < L_1$, so $L/O_2(L) \cong \hat{A}_6$. Then as $N_G(L_0) = M$ using 13.2.2.9, $Y \neq L_0$, so $Y = L_{1,+}$. Further if $K/O_2(K) \cong \hat{M}_{22}$, replacing K by K_1, we reduce to the case $K/O_2(K) \cong \hat{A}_6$.

Now $L_1 = \theta(H \cap M)$ by 13.7.3.9, and $H \cap M$ is a maximal parabolic of H. As $\tilde{V}_3 = [\tilde{V}, L_{1,+}]$ and $Y = L_{1,+} \trianglelefteq H$, $\tilde{U}_H = [\tilde{U}_H, Y]$.

Since K^* is $\hat{A}_6$, case (2) of 13.8.8 holds, so that $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. Let $\tilde{I} \in Irr_+(K, \tilde{U}_H, T)$; by B.4.5, $\tilde{I}$ is a 6-dimensional module for H^*. Further as H^* has no faithful strong FF-modules by B.4.2.8, F.9.18.6 says that either $\tilde{I} = \tilde{U}_H$ or $\tilde{U}_H/\tilde{I}$ is 6-dimensional. Set $W := \tilde{U}_H$ or $\tilde{U}_H/\tilde{I}$ in the respective cases. Now L_1T acts on $\tilde{V}_3$ and hence also on its image in W, so $L_1^*T^*$ is the stabilizer of an $\mathbf{F}_4$-point in W. Choose α as in case (2) of 13.8.8; then $V_\alpha^* \leq O_2(L_1^*T^*) \cong E_4$, so by 13.8.18.4, $V_\alpha^* = O_2(L_1^*T^*)$. This is a contradiction as V_α^* is quadratic on U_H by 13.8.4.6. Thus (2) is established.

We next prove (1); we may continue to assume $H = KL_1T$ but $L_1 \not\leq K$. Therefore $m_3(K) = 1$ by (2) and A.3.18, so $K^* \cong L_3(2)$. Let $X := L_{1,+}$ if $L/O_2(L) \cong \hat{A}_6$, and $X := L_1$ if $L/O_2(L) \cong A_6$. As $X = [X, T]$, either $X \leq K$ or $[X, K] \leq O_2(K)$.

Assume first that $L/O_2(L) \cong A_6$. Then $L_1^* = X^*$ centralizes K^* by the previous paragraph, so that $F^*(H^*) = K^* \times X^*$. As $\tilde{V}_3 = [\tilde{V}_3, X^*]$ and $X^* \trianglelefteq H^*$, $\tilde{U}_H = [\tilde{U}_H, X]$. If $H^*/C_{H^*}(K^*)$ is not $Aut(L_3(2))$, then KX is generated by a pair of solvable overgroups of X, so that $KX \leq M$ by 13.8.13, contrary to 13.8.12.1. On the other hand if $H^*/C_{H^*}(K^*) \cong Aut(L_3(2))$, then since $\tilde{U}_H = [\tilde{U}_H, X]$, for each chief factor W for H^* on $\tilde{U}_H$ with $[K^*, W] \neq 1$, W consists of either a pair of Steinberg modules, or a pair of natural modules and a pair of duals for K^*, contradicting $q(H^*, \tilde{U}_H) \leq 2$ by B.4.5.

Thus $L/O_2(L) \cong \hat{A}_6$, so that L_0 and $L_{1,+} = X$ are the two T-invariant subgroups of 3-rank 1 in L_1. As usual $K \not\leq M$ by 13.8.12.1, so that K does not act on L_0 in view of 13.2.2.9. Then $C_{L_1}(K^*) = X$ rather than L_0, so that $L_0 \leq K$. Now $X = [X, t]$ for $t \in T \cap L \leq C_T(L_0/O_2(L_0))$, so $[K^*, t^*] = 1$. Also T acts on L_0, and hence is trivial on the Dynkin diagram of K^*, so $H^* \cong L_3(2) \times S_3$. As earlier, $\tilde{U}_H = [\tilde{U}_H, X]$, so an H-chief factor W in $\tilde{U}_H$ is the tensor product of natural modules for the factors, as usual using B.4.5 and the fact that $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. As L_0 centralizes $\tilde{V}_3$, $L_0^*T^*$ is the stabilizer of a point in these natural modules. Then as $V_\alpha^* \leq O_2(L_1^*T^*)$ and V_α^* is quadratic on U_H, V_α^* is of order 2, contrary to 13.8.18.4. So (1) is established.

By (1), L_1 is contained in each $K \in \mathcal{C}(H)$, so there is a unique $K \in \mathcal{C}(H)$. Then by 13.8.14, $K^* = F^*(H^*)$, so (3) holds as $Out(K^*)$ is a 2-group for each of the groups listed in (2); if $K^* \cong L_3(2)$ that $H^* \cong Aut(L_3(2))$ by 13.8.19. Part (4) follows from (1) and 13.7.5.2. □

Let $\tilde{W}$ be a proper H-submodule of $\tilde{U}_H$ and set $\hat{U}_H := \tilde{U}_H/\tilde{W}$. As $\tilde{U}_H = \langle V_3^H \rangle$ and L_1 is irreducible on $\tilde{V}_3 \cong E_4$, it follows that $\hat{V}_3 \cong E_4$ is L_1T-isomorphic to $\tilde{V}_3$. By 13.8.21, $\hat{U}_H = [\hat{U}_H, K]$ and $K^* = F^*(H^*)$ is simple, so that H^* is faithful on $\hat{U}_H$.

LEMMA 13.8.22. *Assume K is nontrivial on $\tilde{W}$. Then H^* is faithful on $\tilde{W}$, case (2) of 13.8.8 holds, and either*

(1) $A_1 \leq W$, U_γ^* contains an FF-offender on the FF-module $\hat{U}_H$, and either U_γ^* contains a strong FF*-offender on $\hat{U}_H$, or $W \leq D_H$ and $[\tilde{W}, U_\gamma^*] = \tilde{A}_1$.*

(2) $A_1 \not\leq W$, U_γ^* contains an FF-offender on the FF-module $\tilde{W}$, and either U_γ^* contains a strong FF*-offender on $\tilde{W}$, or $U_H = WD_H$ and $[\hat{U}_H, U_\gamma] = \hat{A}_1$, so that $A_1 \leq U_H$.*

PROOF. As K^* is nontrivial on $\tilde{W}$ and $K^* = F^*(H^*)$ is simple, H^* is faithful on $\tilde{W}$. As H^* is also faithful on $\hat{U}_H$, no member of H^* induces a transvection on $\tilde{U}_H$, so case (2) of 13.8.8 holds.

Suppose $A_1 \leq W$. Then using F.9.13.6, $[\hat{D}_H, U_\gamma] \leq \hat{A}_1 = 1$, so $\hat{D}_H < \hat{U}_H$ and

$$m(\hat{U}_H/C_{\hat{U}_H}(U_\gamma)) \leq m(\hat{U}_H/\hat{D}_H) \leq m(U_H/D_H) \leq m(U_\gamma^*),$$

so by B.1.4.4, U_γ^* contains an FF*-offender on the FF-module $\hat{U}_H$. Indeed either U_γ^* contains a strong FF*-offender, or all inequalities are equalities, so that $m(\hat{U}_H/\hat{D}_H) = m(U_H/D_H)$, and hence $W \leq D_H$. In the latter case, $[\tilde{W}, U_\gamma] \leq [\tilde{D}_H, U_\gamma] = \tilde{A}_1$, so that (1) holds.

So assume $A_1 \not\leq W$. Then $[D_H \cap W, U_\gamma] \leq W \cap A_1 = 1$, so

$$m(W/C_W(U_\gamma)) \leq m(W/(D_H \cap W)) \leq m(U_H/D_H) \leq m(U_\gamma^*)$$

since case (2) of 13.8.8 holds. So by B.1.4.4, U_γ^* contains an FF*-offender on the FF-module $\tilde{W}$. Further if U_γ^* does not contain a strong FF*-offender, then all inequalities are equalities, so that $m(W/(D_H \cap W)) = m(U_H/D_H)$ and hence $U_H = WD_H$. But then

$$[\hat{U}_H, U_\gamma] \leq [\hat{D}_H, U_\gamma] \leq \hat{A}_1,$$

so that (2) holds. □

LEMMA 13.8.23. *Assume $m(U_\gamma^*) = 1$, and K is nontrivial on $\tilde{W}$. Then*

(1) U_γ induces transvections on $\tilde{W}$ and $\hat{U}_H$, D_H is a hyperplane of U_H, $C_H := C_{U_H}(U_\gamma)$ is a hyperplane of D_H, and $\hat{C}_H = C_{\hat{U}_H}(U_\gamma^*)$.

(2) $U_\gamma^* < V_\gamma^*$.

(3) Either $A_1 \not\leq W$ and $[C_{\tilde{W}}(U_\gamma^*), V_\gamma^*] = 1$, or $A_1 \leq W$ and $[C_{\hat{U}_H}(U_\gamma^*), V_\gamma^*] = 1$.

(4) $U_\gamma^* < C_{H^*}(C_E(U_\gamma^*))$ for at least one of $E := \tilde{W}$ or $\hat{U}_H$.

PROOF. By 13.8.22, H^* is faithful on $\tilde{W}$ and case (2) of 13.8.8 holds, so $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. Therefore as $m(U_\gamma^*) = 1$ it follows that $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma)) \leq 2$; then since K is nontrivial on $\tilde{W}$, equality holds and U_γ^* induces transvections on both $\tilde{W}$ and $\hat{U}_H$. Therefore by 13.8.10, D_H is a hyperplane of U_H.

Let $C_H := C_{U_H}(U_\gamma)$. By F.9.13.7, $[D_\gamma, D_H] = 1$, so as $m(U_\gamma^*) = 1$ and $[D_H, U_\gamma] \leq A_1$ by F.9.13.6, C_H is a hyperplane of D_H. Therefore $\tilde{C}_H = C_{\tilde{U}_H}(U_\gamma)$ as both subgroups are of codimension 2 in $\tilde{U}_H$. Hence (1) holds.

Part (2) follows from 13.8.18.4. Next $[\tilde{C}_H, V_\gamma] \leq \tilde{A}_1$ by 13.7.3.7. Further by (1), U_γ^* is not a strong FF*-offender on $\hat{U}_H$ or $\tilde{W}$. Assume $A_1 \leq W$. Then $W \leq D_H$ by 13.8.22.1, so $\hat{D}_H = C_{\hat{U}_H}(U_\gamma^*) = \hat{C}_H$ by (1). Thus if $[C_{\hat{U}_H}(U_\gamma^*), V_\gamma^*] \neq 1$, $D_H > WC_H$, and hence $W \leq C_H$ as $|D_H : C_H| = 2$ by (1). However this contradicts $[\tilde{W}, U_\gamma] \neq 1$. So suppose instead $A_1 \not\leq W$. Then by F.9.13.6, $[D_H \cap W, U_\gamma] \leq A_1 \cap W = 1$, so $D_H \cap W \leq C_H$, and hence

$$[\widetilde{D_H \cap W}, V_\gamma] \leq [\tilde{C}_H, V_\gamma] \cap \tilde{W} \leq \tilde{A}_1 \cap \tilde{W} = 1.$$

Since $C_{\tilde{W}}(U_\gamma^*) \leq \widetilde{D_H \cap W}$, this establishes (3).

Finally (2) and (3) imply (4). □

LEMMA 13.8.24. *K^* is not isomorphic to A_7.*

PROOF. Assume $K^* \cong A_7$. We adopt the notational conventions of section B.3, and represent H^* on $\Omega := \{1, \ldots, 7\}$, so that T^* has orbits $\{1, 2, 3, 4\}$, $\{5, 6\}$, and $\{7\}$. Let $\beta := \gamma g_b^{-1}$ for g_b as defined earlier, and let $\delta \in \{\beta, \gamma\}$. By (1) and (2) of F.9.13, $V_\delta^{*y} \leq O_2(L_1^* T^*)$ for some $y \in H$.

Suppose first that case (1) of 13.8.8 holds, and pick δ as in that case. Then V_δ or U_δ induces a nontrivial group of transvections on $\tilde{U}_H$, so in particular $K^*T^* \cong S_7$. But as case (2b) of 13.8.5 holds, $L/O_2(L) \cong \hat{A}_6$ so $|L_1|_3 = 3^2$, and hence $L_1^*T^* \cong S_4 \times S_3$ is the stabilizer of the partition $\{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$ of Ω. Thus $O_2(L_1^*T^*)$ contains no transvections, whereas we showed $V_\delta^{*y} \leq O_2(L_1^*T^*)$ and $U_\delta \leq V_\delta$.

Therefore case (2) of 13.8.8 holds. Define α as in that case; thus $V_\alpha^* \leq O_2(L_1^*T^*)$ and $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

Pick $\tilde{W}$ maximal in $\tilde{U}_H$, so that $\hat{U}_H$ is an irreducible H^*-module. It will suffice to show $m(U_\gamma^*) = 1$ and $[\tilde{W}, K] \neq 1$: for then 13.8.23.4 supplies a contradiction, since for each faithful $\mathbf{F}_2H^*$-module E on which some $h^* \in H^*$ induces a transvection (that is, with $[E, H]$ the A_7-module), $\langle h^* \rangle = C_{H^*}(C_E(h^*))$.

As $L_1T = TL_1$, L_1T stabilizes either $\{1, 2, 3, 4\}$ or a partition of type $2^3, 1$. Assume the first case holds. Then the stabilizer S in H of $\{1, 2, 3, 4\}$ is solvable, so $S \leq M$ by 13.8.13. Thus $S = H \cap M$; hence by 13.7.3.9, $L_1 = \theta(S)$ is of 3-rank 2, so that $L/O_2(L) \cong \hat{A}_6$. Next either $L_0^* = \langle (5, 6, 7) \rangle$ and $L_{1,+}^* = O^2(K_{5,6,7}^*)$, or vice versa. As $L_{1,+}^*$ is inverted in $T \cap L \leq C_T(L_0^*)$, $H^* \cong S_7$. As $q(H^*, \hat{U}_H) \leq 2$

and $H^* \cong S_7$, B.4.2 and B.4.5 say that $\hat{U}_H$ is either a natural module or the sum of a 4-dimensional module and its dual. As $\hat{V}_3$ is of rank 2 and T-invariant, with $\hat{V}_3 = [\hat{V}_3, L_{1,+}] \leq C_{\hat{U}_H}(L_0)$, we conclude that $\hat{U}_H$ is natural, and $L^*_{1,+} = \langle(5,6,7)\rangle$. Recall $V^*_\alpha \leq O_2(L^*_1T^*) = O_2(L^*_{1,+})$. As V^*_α is quadratic on $\hat{U}_H$ by 13.8.4.6, it follows that $m(V^*_\alpha) = 1$, so U_H induces transvections on U_γ by 13.8.18.3. But then V^{*y}_β induces transvections on $\tilde{U}_H$, whereas $V^{*y}_\beta \leq O_2(L^*_1T^*)$, which contains no transvections.

Thus L_1T is the stabilizer of a partition of type $2^3, 1$. In particular $m_3(L_1) = 1$, so $L/O_2(L) \cong A_6$ as $L_1 = \theta(H \cap M)$ by 13.7.3.9. As $U^*_\gamma \in \mathcal{Q}(H^*, \hat{U}_H)$, B.4.2 and B.4.5 say that $\hat{U}_H$ is either of dimension 4 or 6, or else the sum $4 + 4'$ of 4 and its dual $4'$. But L^*_1 stabilizes the T^*-invariant line $\hat{V}_3 \leq \hat{U}_H$, so as L_1T is the stabilizer of a partition of type $2^3, 1$, $\dim(\hat{U}_H) \neq 4$ or 8, and hence $\dim(\hat{U}_H) = 6$. If $[\tilde{W}, K] = 1$, then $[\tilde{U}_H, K] \cong \hat{U}_H$ is the natural module for K^*, so as $L/O_2(L) \cong A_6$, 13.7.6.3 supplies a contradiction. Thus $[\tilde{W}, K] \neq 1$, and so we may apply 13.8.22. As H^* has no strong FF-modules, we conclude from 13.8.22 that U^*_γ induces a group of transvections on $\hat{U}_H$ or $\tilde{W}$. Therefore $m(U^*_\gamma) = 1$, and we saw this suffices to complete the proof. □

Lemma 13.8.25. *If $K/O_2(K) \cong L_n(2)$ for $3 \leq n \leq 5$, then $n = 4$ and*

(1) $L/O_2(L) \cong \hat{A}_6$, and $\tilde{U}_H$ is a 4-dimensional natural module for $H^* \cong L_4(2)$.
(2) $H = G_1$.

Proof. Assume otherwise. If case (1) of 13.8.8 holds, then conclusion (1) holds by 13.8.5. In particular $n = 4$, and we will see below that this implies conclusion (2); so we may assume that case (2) of 13.8.8 holds.

Then $U^*_\gamma \in \mathcal{Q}(H^*, \tilde{U}_H)$. Let $T^*_K := T^* \cap K^*$. As $L_1 \leq K$ by 13.8.21.1, $L^*_1T^*_K$ is a T^*-invariant parabolic of K^*. Indeed $L^*_1T^*_K$ is a minimal parabolic when $L/O_2(L) \cong A_6$, since $|L_1|_3 = 3$ in that case, whereas $L^*_1T^*/O_2(L^*_1T^*) \cong S_3 \times S_3$ when $L/O_2(L) \cong \hat{A}_6$.

If $L^*_1T^*_K$ is a minimal parabolic, then as $L^*_1T^*_K$ is T^*-invariant, either $T^* = T^*_K$, or $n = 4$ and $L^*_1T^*_K$ is the middle-node parabolic of K^*. This allows us to eliminate the case $n = 3$: For if $n = 3$, then $m_3(K) = 1$, so $L^*_1T^*_K$ is a minimal parabolic and hence $T^* = T^*_K$. contrary to 13.8.21.3.

Further if $n = 5$, then $L/O_2(L) \cong \hat{A}_6$: For otherwise we have seen that $L^*_1T^*_K$ is a minimal parabolic and $T^* = T^*_K$. Therefore $L_1T \leq H_1 \leq H$ with $H_1/O_2(H_1) \cong S_3 \times S_3$. But now $H_1 \leq M$ by 13.8.13, so $L_1 = \theta(H \cap M)$ is of 3-rank 2 by 13.7.3.9, contradicting $L/O_2(L) \cong A_6$.

In the next few paragraphs, we assume $L/O_2(L) \cong A_6$ and derive a contradiction. Here the arguments above have reduced us to the case $n = 4$.

Suppose $T_K = T$. Then $L_1 \leq K_1 \in \mathcal{L}(K, T)$ with $K_1/O_2(K_1) \cong L_3(2)$. But now $K_1T \in \mathcal{H}_z$, a case already eliminated. Thus $T_K < T$, so we have seen that $L^*_1T^*_K$ is the middle-node parabolic.

Let $\tilde{W}$ be a maximal H^*-submodule of $\tilde{U}_H$, so that $\hat{U}_H$ is irreducible. As $U^*_\gamma \in \mathcal{Q}(H^*, \hat{U}_H)$ and $T^*_K < T^*$, B.4.2 and B.4.5 say that either $m(\hat{U}_H) = 6$, or $\hat{U}$ is the sum of a natural K^*-module and its dual. The latter is impossible, as $L^*_1T^*_K$ is a middle-node minimal parabolic and $\hat{V}_3 = [\hat{V}_3, L_1]$ is an L_1T-invariant line in $\hat{U}_H$.

Thus $m(\hat{U}_H) = 6$. Since the case with a single nontrivial 2-chief factor which is an A_8-module is excluded by 13.7.6.3, $[\tilde{W}, K] \neq 1$, so we can appeal to 13.8.22.

If U_γ^* contains a strong FF*-offender on $\hat{U}_H$, then B.3.2.6 says that $U_\gamma^* \cong E_{16}$ is generated by the transpositions in T^* and $m(\hat{U}_H/C_{\hat{U}_H}(U_\gamma)) = 3$. Thus as $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$,

$$m(\tilde{W}/C_{\tilde{W}}(U_\gamma)) \leq 2m(U_\gamma^*) - 3 = 5;$$

so as $\tilde{W}$ is a faithful module for $H^* \cong S_8$, we conclude $[\tilde{W}, H]$ is the 6-dimensional module or its 7-dimensional cover, and $m(\tilde{W}/C_{\tilde{W}}(U_\gamma)) = 3$. Thus

$$m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*) \geq m(\hat{U}_H/C_{\hat{U}_H}(U_\gamma^*)) + m(\tilde{W}/C_{\tilde{W}}(U_\gamma^*)) = 6.$$

Now by 13.8.8, $m(U_H/D_H) \leq m(U_\gamma^*) = 4$, so U_γ^* does not centralize D_H. Then by F.9.13.6, $A_1 = [D_H, U_\gamma] \leq U_H$. So as the transpositions in U_γ^* induce transvections on $\hat{U}_H$ with distinct centers, we conclude $C_{U_\gamma^*}(D_H)$ is a hyperplane of U_γ^*, so $m(U_H/C_{U_H}(U_\gamma)) \leq 5$, contrary to our previous calculation.

Therefore U_γ^* contains no strong FF*-offender on $\hat{U}_H$, so by 13.8.22 either

(i) $U_\gamma^* \cong \mathbf{Z}_2$ induces a transvection with center $\tilde{A}_1$ on $\tilde{W}$, or a transvection with center $\hat{A}_1$ on $\hat{U}_H$, or

(ii) $A_1 \not\leq W$, and U_γ^* contains a strong FF*-offender on $\tilde{W}$.

In case (ii), as in the previous paragraph, we conclude $U_\gamma^* \cong E_{16}$ and $\tilde{W}$ is the orthogonal module, leading to the same contradiction.

So case (i) holds. Then $m(U_\gamma^*) = 1$ and $[\tilde{W}, K] \neq 1$, so 13.8.23.4 supplies a contradiction, since $C_{H^*}(C_E(h^*)) = \langle h^* \rangle$ for each faithful $\mathbf{F}_2 H^*$-module E (namely with $[E, K]$ of dimension 6 or 7) on which some $h^* \in H^*$ induces a transvection. This contradiction completes the elimination of the case $L/O_2(L) \cong A_6$.

Therefore $L/O_2(L) \cong \hat{A}_6$. We eliminated $n = 3$ earlier, so $n = 4$ or 5. As T acts on the two minmal parabolics determined by L_0 and $L_{1,+}$, $T_K^* = T^*$. Further as $L_1 \trianglelefteq H \cap M$, $L_1 T = H \cap M$. Observe that $L_1^* T_K^*$ is a parabolic of rank 2 determined by two nodes not adjacent in the Dynkin diagram.

Suppose $n = 5$. By 13.2.2.9, $N_K(L_0) \leq K \cap M \leq N_K(L_{1,+})$, the node β determined by L_0 is an interior node, and the node δ determined by $L_{1,+}$ is the unique node not adjacent to β. Thus we may take δ and β to be the first and third nodes of the diagram for H^*. Then $L_1 T \leq H_2 \leq H$ with $H_2/O_2(H_2) \cong S_3 \times L_3(2)$. As $L_1 T = H \cap M$, $H_2 \not\leq M$, so $H_2 \in \mathcal{H}_z$, contrary to 13.8.21.1.

Therefore we have established that $n = 4$ in each case of 13.8.8. As mentioned earlier, we can now show that (2) holds: For $H \leq G_1$, so $K \leq K_1 \in \mathcal{C}(G_1)$ by 1.2.4. Now $K_1 T \in \mathcal{H}_z$ and we have shown $K_1/O_2(K_1)$ is not $L_5(2)$. Hence $K_1 = K \in \mathcal{C}(G_1)$ by 13.8.21.2 and A.3.12. As $G_1 \in \mathcal{H}_z$, we conclude from 13.8.21.3 that $H = KT = K_1 T = G_1$, so that (2) holds.

Thus (2) is established, and we've shown that $L/O_2(L) \cong \hat{A}_6$ and $H^* \cong L_4(2)$. We may assume (1) fails, so that $\tilde{U}_H$ is not the natural module for H^*. Now $U_\gamma^* \in \mathcal{Q}(H^*, \hat{U}_H)$, so $\hat{U}_H$ is of dimension 4 or 6 by B.4.2 and B.4.5. Then as the maximal parabolic $L_1^* T^*$ determined by the end nodes stabilizes the line $\hat{V}_3$, $\dim(\hat{U}_H) = 4$. As (1) fails, $[\tilde{W}, K] \neq 1$; hence we can appeal to 13.8.22 and 13.8.23. Moreover $m(V_\gamma^*) > 1$ by 13.8.18.4.

We claim $\tilde{U}_H$ has a unique maximal submodule $\tilde{W}$. Assume not; then (writing $J(\tilde{U}_H)$ for the Jacobson radical of $\tilde{U}_H$)

$$\dot{U}_H := \tilde{U}_H/J(\tilde{U}_H) = \dot{U}_1 \oplus \cdots \oplus \dot{U}_s$$

is the sum of $s > 1$ four-dimensional irreducibles. Further the projection $\dot{V}_3^i$ of $\tilde{V}_3$ on $\dot{U}_i$ is faithful for each i and centralized by L_0, so the $\dot{U}_i$ are isomorphic natural modules. As $C_{\dot{U}_i}(T^*)$ is a point, each L_1T-invariant line is contained in a member of $Irr_+(H, \dot{U}_H)$, so $\dot{V}_3 \leq \dot{U}_0$ for some irreducible H-submodle $\dot{U}_0$. But then $\dot{U}_H = \langle \dot{V}_3^H \rangle = \dot{U}_0$, contrary to $s > 1$. Thus the claim is established.

Define α as in case (2) of 13.8.8; thus $V_\alpha^* \leq O_2(L_1^*T^*) = O_2(L_1^*)$ and $m(V_\alpha^*) = m(V_\gamma^*) > 1$.

Let B be a noncentral chief factor for H on $\tilde{W}$. We claim $m(B) = 4$. For otherwise, as $q(H^*, \tilde{U}_H) \leq 2$, B is of rank 6 by B.4.2 and B.4.5. Thus as V_α^* is a noncyclic subgroup of the unipotent radical $O_2(L_1^*)$ of the parabolic $L_1^*T^*$ stabilizing a point of B and acting quadratically on B, it follows that $m(V_\alpha^*) = 2$ and V_α^* contains no FF*-offender on B by B.3.2.6. Therefore case (1) of 13.8.22 holds, so $A_1 \leq W$. As $T^* = T_K^*$, no member of H^* induces a transvection on B, so $[\tilde{W}, U_\gamma^*] > \tilde{A}_1$ and hence $W \not\leq D_H$ by F.9.13.6. Thus we conclude from 13.8.22 that U_γ^* contains a strong FF*-offender on $\hat{U}_H$. As $m(V_\gamma^*) = 2$ and U_γ^* contains a strong FF*-offender, we conclude $V_\gamma^* = U_\gamma^* \cong E_4$. Then $m(U_H/D_H) \leq m(U_\gamma^*) = 2$, so as $W \not\leq D_H$, $m(\hat{U}_H/\hat{D}_H) \leq 1$, with $m(U_H/D_H) = 2$ in case of equality. However U_γ^* centralizes $\hat{D}_H$ by F.9.13.6 as $A_1 \leq W$. Therefore $m(\hat{U}_H/\hat{D}_H) = 1$ and $m(U_H/D_H) = 2 = m(U_\gamma^*)$. Thus we have symmetry between γ_1 and γ. In particular as $A_1 \leq W \leq U_H$, $V_1 \leq U_\gamma$; further in view of 13.8.18.2, we may apply 13.8.11.1 to conclude $U_\gamma^* < V_\gamma^*$, contrary to an earlier remark. This establishes the latest claim that $m(B) = 4$.

Thus we have shown that all noncentral chief factors of $\tilde{U}_H$ are 4-dimensional. Then as the 1-cohomology of 4-dimensional modules is trivial by I.1.6.6, and $\tilde{W}$ is the unique maximal submodule of $\tilde{U}_H$, all chief factors are 4-dimensional.

Observe next that no noncyclic subgroup of V_γ^* centralizes a hyperplane of $\hat{U}_H$: For otherwise as V_γ^* is quadratic on $\tilde{U}_H$ by 13.8.4.6, the quotient module $\tilde{U}_H$ splits over the submodule $\tilde{W}$ by B.4.9.1, contradicting $\tilde{W}$ the unique maximal submodule of $\tilde{U}_H$. So as V_α^* lies in the unipotent radical $O_2(L_1^*)$ of the stabilizer of a line in the natural module for $L_4(2)$, it follows that $m(U_\gamma^*) \leq m(V_\gamma^*) \leq 3$.

Now let $\tilde{I}$ denote any member of $Irr_+(H^*, \tilde{W})$, so that in particular $\tilde{I}$ is 4-dimensional. Applying the dual of B.4.9.1, we conclude similarly that no noncyclic subgroup of V_γ^* acts as a group of transvections with a fixed center on $\tilde{I}$.

Suppose next that $A_1 \leq I$. Then $C_H(A_1)^*$ is the maximal parabolic fixing $\tilde{A}_1$. Then as $H = G_1$, $V_\gamma \trianglelefteq N_G(A_1)^*$, so $V_\gamma^* = O_2(C_H(A_1))^*$ as $N_G(A_1)^*$ is irreducible on $O_2(C_H(A_1))^*$. This is impossible as $V_\gamma^* \leq O_2(L_1)^*$ where $L_1^*T^*$ stabilizes a line of $\tilde{I}$.

Therefore $A_1 \not\leq I$, so $[I \cap D_H, U_\gamma] \leq I \cap A_1 = 1$; then by 13.7.3.7, $[I \cap D_H, V_\gamma] \leq A_1 \cap I = 1$. In particular $I \not\leq D_H$.

Suppose next that $m(U_\gamma^*) = 1$. We saw V_γ centralizes $D_H \cap I$, which is a hyperplane of I by 13.8.23.1. Then as $V_\alpha^* \leq O_2(L_1^*)$ and $L_1^*T^*$ is the parabolic stabilizing a line in I, we conclude $m(V_\alpha^*) \leq 2$, and hence $m(V_\alpha^*) = 2$ as $m(V_\gamma^*) > 1$

by 13.8.18.4. Also by 13.8.23.1, U_γ^* induces transvections on $\tilde{W}$ and $\hat{U}_H$, so H^* has a unique noncentral chief factor on W, and hence $W = I$. Again by 13.8.23.1, D_H is a hyperplane of U_H, so as $W = I \not\leq D_H$, $\hat{U}_H = \hat{D}_H$ and hence $\hat{A}_1 = [\hat{U}_H, U_\gamma]$ and $\tilde{A}_1[\tilde{W}, U_\gamma^*] = [\tilde{U}_H, U_\gamma^*]$ is of rank 2. Now $U_\alpha^* = Z(T^*)$, so T^* acts on $[\tilde{U}_H, U_\alpha^*]$ and centralizes $\tilde{W}_1\tilde{V}_2$ where $\tilde{W}_1 := [\tilde{W}, U_\alpha^*]$. Thus

$$\tilde{W}_1\tilde{A}_1^h = [\tilde{U}_H, U_\alpha^*] = \tilde{W}_1\tilde{V}_2,$$

where $h \in H$ with $\gamma h = \alpha$. Thus the middle-node minimal parabolic H_0^* of H^* containing T^* centralizes $[\tilde{U}_H, U_\alpha^*]$, and in particular $\tilde{A}_1^h$, so H_0^* acts on V_α^* since $G_1 = H$ by (2). This is impossible as $H_0^* \cong S_3/D_8^2$ has no normal E_4-subgroup.

So $m(U_\gamma^*) > 1$. Now $V_\alpha^* \leq O_2(L_1^*)$, and we've seen that U_γ^* is noncyclic and U_α^* does not induce a group of transvections with fixed center on $\tilde{I}$; thus $[\tilde{I}, U_\alpha^*]$ is the line in $\tilde{I}$ fixed by L_1^*, and hence $[\tilde{I}, U_\alpha^*] = [\tilde{I}, V_\alpha^*]$. Therefore by 13.8.9.2 applied to I in the role of "F", $V_1 \not\leq U_\gamma$. Also we saw V_γ centralizes $D_H \cap I$, so $m(I/D_H \cap I) \geq 2$.

Suppose $m(U_\gamma^*) = m(U_H/D_H)$. Then we have symmetry between γ_1 and γ, so $A_1 \not\leq U_H$, and hence $[D_H, U_\gamma] \leq A_1 \cap U_H = 1$. Further as U_γ^* is noncyclic and we saw earlier that U_γ^* does not centralize any hyperplane of $\hat{U}_H$, $m(\hat{U}_H/\hat{D}_H) \geq 2$. Hence as $m(I/I \cap D_H) \geq 2$, $m(U_\gamma^*) = m(U_H/D_H) \geq 4$, contrary to our earlier observation that $m(U_\gamma^*) \leq 3$.

Therefore $m(U_\gamma^*) > m(U_H/D_H)$. So as $m(U_\gamma^*) \leq 3$, we conclude

$$3 \geq m(U_\gamma^*) > m(U_H/D_H) \geq 2,$$

where the final inequality holds since we saw $m(I/D_H \cap I) \geq 2$. Thus $m(U_\gamma^*) = 3$ and $m(U_H/D_H) = 2$. Hence $\hat{U}_H = \hat{D}_H$ since $m(I/D_H \cap I) \geq 2$, so $[\hat{U}_H, U_\gamma] = [\hat{D}_H, U_\gamma] = \hat{A}_1$ by F.9.13.6. This is impossible as $U_\alpha^* \leq O_2(L_1^*)$ with $m(U_\alpha) = 3$. Thus the proof of 13.8.25 is at last complete. □

LEMMA 13.8.26. *If $K^* \cong A_6$, then $\tilde{U}_H$ is the natural module for K^* on which L_1 has two noncentral chief factors or its 5-dimensional cover.*

PROOF. In case (1) of 13.8.8, this holds by 13.8.5, so we may assume case (2) of 13.8.8 holds. Then $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, so each noncentral chief factor for K^* on $\tilde{U}_H$ is of rank 4 by B.4.2 and B.4.5. Suppose K has more than one such factor, and pick $\tilde{W}$ as in 13.8.22.

First assume U_γ^* contains a strong FF*-offender on $N := \hat{U}_H$ or $\tilde{W}$. Then by B.3.4.2i, $U_\alpha^* = R_1^* \cong E_8$ is generated by the transvections on N in T^*. But by 13.8.4.5, V_H/U_H has a quotient B which is the 4-dimensional H^*-module on which L_1T fixes a point. Then as $U_\alpha^* = R_1^*$, U_α^* is not quadratic on B, contrary to 13.8.4.6.

Thus U_γ^* contains no strong FF*-offender on either $\hat{U}_H$ or $\tilde{W}$, so by 13.8.22, U_γ^* induces transvections on $E =: \hat{U}_H$ or $\tilde{W}$, and hence $m(U_\gamma^*) = 1$. This is a contradiction to 13.8.23.4, as $U_\gamma^* = C_{H^*}(C_E(U_\gamma^*))$ for any transvection.

Thus $\tilde{U}_H$ has a unique noncentral chief factor. Since $U_H = \langle V_3^H \rangle$ with $\tilde{V}_3 = [\tilde{V}_3, L_1]$ a nontrivial irreducible for L_1, $\tilde{U}_H/C_{\tilde{U}_H}(K)$ is the 4-dimensional natural module on which L_1 has two noncentral chief factors. Then by I.1.6.1, $\tilde{U}_H$ is either natural or a 5-dimensional cover, completing the proof. □

LEMMA 13.8.27. *(1) Either*

(a) $L/O_2(L) \cong A_6$, $H^* \cong A_6$ or S_6, and $\tilde{U}_H$ is the natural module for K^* on which L_1 has two noncentral chief factors or its 5-dimensional cover, or

(b) $L/O_2(L) \cong \hat{A}_6$, $H^* \cong L_4(2)$, and $\tilde{U}_H$ is a 4-dimensional natural module for H^*.

(2) $G_1 = H = KT$.

(3) If case (1) of 13.8.8 holds then $D_H = U_H$, $D_\gamma = U_\gamma$, V induces a group of transvections on U_γ with center V_1, and $V_1 \leq U_\gamma$. Further $V_\gamma \not\leq Q_H$, so we have symmetry between γ and γ_1.

PROOF. By 13.8.24 and 13.8.25, the list of 13.8.21.2 has been reduced to $K^* \cong A_6$, $L_4(2)$, or $G_2(2)'$. Further $K \leq K_1 \in \mathcal{C}(G_1)$ by 1.2.4, and as $G_1 \in \mathcal{H}_z$, $K_1/O_2(K_1) \cong A_6$, $L_4(2)$, or $G_2(2)'$. So as A.3.12 contains no inclusions between any pair on this list, we conclude that $K = K_1$. Thus $G_1 = K_1T = KT = H$ by 13.8.21.3, so (2) holds.

By (2) and 13.8.7, $D_H = U_H$ and $D_\gamma = U_\gamma$. Thus V induces a group of transvections on U_γ with center V_1 by F.9.16.1, so $V_1 \leq U_\gamma$. Thus to complete the proof of (3), we assume $V_\gamma \leq Q_H$ and derive a contradiction. Then $[U_H, V_\gamma] \leq V_1 \cap A_1 = 1$. Thus $V^g \leq C_G(V_3) \leq M_V$, so that $[V, V_3^g] = V_1 = [V, V^g]$. Then $C_{V^g}(V)$ is a hyperplane of V^g and hence conjugate to V_3^g, so $V \leq C_G(C_{V^g}(V)) \leq M_V^g$ by 13.5.4.4. Then $V_1 = [V, V^g] \leq V \cap V^g$, contrary to 13.8.3.

It remains to prove (1). However if $K^* \cong L_4(2)$ or A_6, then (1) holds by 13.8.25 or 13.8.26, so we may assume that $K^* \cong G_2(2)'$ and derive a contradiction. Thus case (2) of 13.8.8 holds as $G_2(2)'$ does not appear in 13.8.5. As H^* has no strong FF-modules and no transvection modules by B.4.2, 13.8.22 and 13.8.21.4 say $\tilde{U}_H \in Irr_+(K, \tilde{U}_H)$. So as $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ by 13.8.8, B.4.2, B.4.5, and I.1.6.5 say that $\tilde{U}_H$ is the 7-dimensional Weyl module or its 6-dimensional quotient module. Thus $m(U_H) \leq 8$.

By 13.8.18.2 and 13.8.11.1, $U_\gamma^* < V_\gamma^*$. Thus $m(U_\gamma^*) < m_2(H^*) = 3$, so by B.4.6.13, $r_{U_\gamma^*, \tilde{U}_H} > 1$. But by the choice of γ in case (2) of 13.8.8, $m(U_\gamma^*) \geq m(U_H/D_H)$, and $[U_\gamma, D_H] \leq A_1$ by F.9.13.6, so we conclude $A_1 \leq U_H$. Thus $H_1^* := C_{H^*}(\tilde{A}_1) = C_H(A_1)^*$ is a maximal parabolic of H^*, and U_γ^* is elementary abelian and normal in $C_H(A_1)^*$. Therefore as $m(U_\gamma^*) < 3$, $U_\gamma^* \cong E_4$ (cf. B.4.6.3). Thus $m(D_\gamma \cap U_H) \geq m([U_\gamma, U_H]) \geq 3$. Next by 13.7.4.2, Q_H/H_C is H^*-isomorphic to $U_H/C_{U_H}(Q_H)$, so $1 \neq [C_{Q_H}(A_1)/H_C, U_\gamma] \leq D_\gamma H_C/H_C$, so $D_\gamma \not\leq H_C$. Finally as V_H is abelian, $V_H \leq H_C$, and by (2) and 13.8.4.5, $H^* \cong G_2(2)'$ or $G_2(2)$ is faithful on V_H/U_H; so as $U_\gamma^* \cong E_4$, $m((D_\gamma \cap V_H)U_H/U_H) \geq m([V_H/U_H, U_\gamma]) \geq 3$. Thus as $r_{U_\gamma^*, \tilde{U}_H} > 1$,

$$m(U_\gamma) > m(U_\gamma^*) + m((D_\gamma \cap V_H)U_H/U_H) + m(D_\gamma \cap U_H) \geq 2 + 3 + 3 = 8,$$

contrary to the previous paragraph. □

THEOREM 13.8.28. *$K^* \cong A_6$.*

Until the proof of Theorem 13.8.28 is complete, assume G is a counterexample. Then $H^* \cong L_4(2)$, $L/O_2(L) \cong \hat{A}_6$, and $m(U_H) = 5$ by 13.8.27.1. Recall $G_2 = N_G(V_2)$, and set $K_2 := O^2(N_H(V_2))$, $Q_2 := O_2(G_2)$, and $\dot{G}_2 := G_2/Q_2$. Set $U_0 := \langle U_H^{G_2} \rangle$ and $V_0 := \langle V_H^{G_2} \rangle$.

Since $L/O_2(L) \cong \hat{A}_6$, by 13.5.4, $I_2 = O_2(G_1)L_{2,+} \trianglelefteq G_2$ with $O_2(I_2) = C_{I_2}(V_2) = I_2 \cap Q_2$ and $\dot{I}_2 \cong S_3$. Let $g \in L_{2,+} - H$, so that $\widetilde{V_1^g} = \tilde{V}_2$.

LEMMA 13.8.29. *(1)* $K_2 \in \mathcal{C}(G_2)$ *with* $K_2/O_2(K_2) \cong L_3(2)$.

(2) $G_2 = K_2L_{2,+}T$ *and* $\dot{G}_2 = \dot{K}_2 \times \dot{I}_2 \cong L_3(2) \times S_3$.

(3) $U_0 = V_H \cap V_H^g = U_H U_H^g$ *and* U_0/V_2 *is the tensor product of the natural modules* V/V_2 *and* U_H/V_2 *for* $\dot{I}_2$ *and* $\dot{K}_2$.

(4) $V_0 = V_H V_H^g V_H^{g^2}$, *and* V_0/U_0 *is the tensor product of* V/V_2 *or* $V/V_2 \oplus \mathbf{F}_2$ *with the dual of* U_H/V_2.

(5) V_H/U_H *is the* 6*-dimensional orthogonal module for* $H^* \cong L_4(2)$.

PROOF. As $\tilde{U}_H$ is the natural module for $H^* \cong L_4(2)$, $N_H(V_2)^* = C_{H^*}(\tilde{V}_2)$ is the parabolic subgroup $L_3(2)/E_8$ of H^* stabilizing the point $\tilde{V}_2$, so $K_2 \in \mathcal{C}(H \cap G_2)$ with $K_2/O_2(K_2) \cong L_3(2)$. As $I_2 \trianglelefteq G_2$, and I_2 acts transitively on $V_2^{\#}$, $G_2 = I_2(H \cap G_2)$ with $H \cap G_2 = K_2T$ and $[\dot{K}_2, \dot{I}_2] = 1$. Thus (1) and (2) hold.

Next U_H/V_2 is the natural module for $\dot{K}_2 \cong L_3(2)$. Thus as $\dot{K}_2 \trianglelefteq \dot{G}_2$, U_0/V_2 is the direct sum of I_2-conjugates of U_H/V_2. Further $U_H = \langle V_3^{K_2} \rangle$ with V/V_2 the natural module for $\dot{I}_2$, so as $I_2 \trianglelefteq G_2$, U_0/V_2 is the direct sum of conjugates of V/V_2. Thus $U_0/V_2 = U_H U_H^g/V_2$ is the tensor product of V/V_2 and U_H/V_2. Further $U_H^g = \langle V_3^{gK_2} \rangle \le V_H$, so $U_0 = U_H U_H^g \le V_H$ and so $U_0 = U_0^g \le V_H \cap V_H^g$.

Let $\hat{V}_H := V_H/U_H$. Then $\hat{V}_H = \langle \hat{V}^H \rangle$ with the maximal parabolic $L_1^*T^*$ of H^* centralizing the point $\hat{V}$, and $\langle \hat{V}^{K_2} \rangle \cong U_H^g/V_2 \cong U_H/V_2$ as a K_2-module, so we conclude from B.4.13 that (5) holds. In particular K_2 is irreducible on V_H/U_0, so either $U_0 = V_H \cap V_H^g$ or $V_H = V_H^g$. In the latter case, both $LT = \langle L_{2,+}, L_1T \rangle$ and H act on V_H, contrary to $H \not\le M = !\mathcal{M}(LT)$. This completes the proof of (3).

By (5), V_H/U_0 is isomorphic to the dual of U_0/U_H as a K_2-module, and by (3), $V_H < V_0$. Thus (4) holds. □

LEMMA 13.8.30. L_0 *has at least* 9 *noncentral* 2*-chief factors.*

PROOF. Recall $V < U_L = \langle U_H^L \rangle \le O_2(LT) = Q$ by (7) and (2) of 13.8.4. Let W be a normal subgroup of L maximal subject to being proper in U_L, and set $\hat{U}_L := U_L/W$. As U_H/V_3 is a 2-dimensional irreducible for $L_0 \trianglelefteq L$, and $U_L = \langle U_H^L \rangle$, $\hat{U}_L = \langle \hat{U}_H^L \rangle = [\hat{U}_L, L_0]$ is a faithful irreducible for $L^+ := L/O_2(L) \cong \hat{A}_6$, and so may be regarded as an $\mathbf{F}_4$-module on which $L_0^+ \cong \mathbf{Z}_3$ acts by scalar multiplication. In particular from the 2-modular character table for $\hat{A}_6$, $\dim_{\mathbf{F}_4}(\hat{U}_L) = 3$ or 9, so to complete the proof, it suffices to show $\dim_{\mathbf{F}_4}(\hat{U}_L) > 3$.

From 13.8.29.3, $\hat{S}_2 := \langle \hat{U}_H^{L_2} \rangle \cong \langle U_H^{L_2} \rangle/V$ is of $\mathbf{F}_4$-dimension 2. Let $\hat{S}_3 := \langle \hat{S}_2^{L_1} \rangle$; from 13.8.29.5, $\hat{S}_3/\hat{U}_L \cong \langle U_0^{L_1} \rangle/U_H V$ is of $\mathbf{F}_4$-dimension 2, so $\dim_{\mathbf{F}_4}(\hat{S}_3) = 3$. Finally by 13.8.29.4, $L_{2,+}$ does not act on $\langle U_0^{L_1} \rangle/U_0$, so $\hat{U}_L > \hat{S}_3$, completing the proof. □

LEMMA 13.8.31. *(1)* $A_1 \not\le U_H$.

(2) Case (2) of 13.8.8 holds.

(3) $[H_C, K] \not\le V_H$; *and if* K *has a unique noncentral chief factor on* H_C/V_H, *it is not a* 4*-dimensional module for* $H^* \cong L_4(2)$.

PROOF. In case (1) of 13.8.8, $A_1 \le U_H$ by 13.8.27.3, so to prove (2), it will suffice to establish (1).

Assume (1) fails, so that $A_1 \le U_H$. Then as H is transitive on $\tilde{U}_H^{\#}$, there is $k \in H$ with $\tilde{A}_1^k = \tilde{V}_2 = V_1^g$; and since $[V_3, Q_H] = V_1$ by 13.7.3.6, we may assume $A_1^k = V_1^g$. Then as $G_1 = H$ by 13.8.27.2, $\gamma k = \gamma_1 g$, so that by 13.8.29.3,

$U_\gamma^k = U_H^g \leq V_H$. Thus as V_H is abelian, V_H centralizes U_γ^k, and hence also U_γ. Therefore $[U_H, U_\gamma] = 1$, and hence case (1) of 13.8.8 holds.

Next $V_\gamma^* \neq 1$, so $1 \neq V_\gamma^{*k} = V_H^{g*}$. However as $\widetilde{V_1^g} = \tilde{V}_2$, $V_H^{g*} \trianglelefteq C_{H^*}(\tilde{V}_2) = K_2^*$, so $V_H^g/E \cong V_H^{g*} = O_2(K_2^*) \cong E_8$, where $E := Q_H \cap V_H^g$. But $U_0 = V_H \cap V_H^g \leq E$, so $E = U_0$ as $m(V_H^g/E) = 3 = m(V_H^g/U_0)$ by parts (3) and (5) of 13.8.29. Also $H_C \leq C_G(A_1^k) \leq N_G(V_H^g)$, so $[H_C, V_H^g] \leq H_C \cap V_H^g \leq E = U_0 \leq V_H$, and hence $K = [K, V_H^g]$ centralizes H_C/V_H. Thus to complete the proof of (1) and hence of (2), it will suffice to establish (3).

Appealing to 13.8.29.5 and the duality in 13.7.4.2, K has the following noncentral 2-chief factors on Q_H/H_C and V_H: The natural module $\tilde{U}_H$, its dual Q_H/H_C, and the orthogonal module V_H/U_H. Therefore L_0 has six noncentral 2-chief factors not in H_C/V_H: two on $O_2(L_0^*)$, one each on Q_H/H_C and $\tilde{U}_H$, and two on V_H/U_H. Therefore by 13.8.30, L_0 has at least three noncentral chief factors on H_C/V_H, so (3) holds and the proof of the lemma is complete. □

LEMMA 13.8.32. *(1)* $m(U_\gamma^*) = 1$.
(2) $m(U_\gamma \cap V_H) \geq 3$.
(3) $A_1 \leq V_H$.

PROOF. By 13.8.31.2, $U_\gamma^* \neq 1$; thus $m(U_H \cap U_\gamma) \geq m([U_H, U_\gamma]) > 0$. Further by 13.8.29.5, no member of H^* induces a transvection on V_H/U_H, so

$$m((U_\gamma \cap V_H)/(U_\gamma \cap U_H)) = m((U_\gamma \cap V_H)U_H/U_H) \geq m([V_H/U_H, U_\gamma] \geq 2, \quad (*)$$

with equality only if (1) holds. In particular this establishes (2), and moving on to the proof of (1), we may assume that $m(U_\gamma \cap V_H) \geq 4$. But then as $m(U_\gamma^*) = m(U_\gamma/(U_\gamma \cap Q_H)) \leq m(U_\gamma/(U_\gamma \cap V_H))$ and $m(U_\gamma) = 5$, it follows again that $m(U_\gamma^*) = 1$, completing the proof of (1). Thus (1) and (2) are established.

By (1) and 13.8.10, $m(U_H/D_H) = 1$, and we have symmetry between γ_1 and γ in the sense of Remark F.9.17. By 13.8.31.1, $A_1 \not\leq U_H$, so by symmetry and 13.8.10.2, $V_1 \not\leq U_\gamma$, and U_γ induces transvections on U_H with axis D_H.

Let $\beta \in \Gamma(\gamma)$; by F.7.3.2 there is $y \in G$ with $\gamma_1 y = \gamma$ and $V^y = V_\beta$. By 13.5.4.4, $[C_G(V_3^y), V^y] \leq A_1$, so as $A_1 \not\leq U_H$,

$$[C_{U_H}(V_3^y), V^y] \leq U_H \cap A_1 = 1. \quad (**)$$

But $V_3^y \leq U_\gamma \leq C_H(D_H)$, so $V_\beta = V^y$ centralizes D_H by (**). As this holds for each $\beta \in \Gamma(\gamma)$, V_γ centralizes D_H. Therefore V_γ^* induces a group of transvections on $\tilde{U}_H$ with axis $\tilde{D}_H$. We saw that no member of H^* induces a transvection on V_H/U_H, so we conclude from 13.8.18.2 and 13.8.11.1 that $U_\gamma^* < V_\gamma^*$. By parts (1) and (2) of F.9.13, $V_\gamma^* \leq O_2(L_1^*T^*)^x$ for some $x \in H$. So as $L_1^*T^*$ is the parabolic subgroup of H^* stabilizing the 2-subspace $\tilde{V}_3$ of the 4-dimensional module $\tilde{U}_H$, while V_γ^* centralizes the hyperplane $\tilde{D}_H$ of $\tilde{U}_H$, we conclude that $m(V_\gamma^*) = 2$. By symmetry, $E_H = V_H \cap Q_H^y$ is of corank 2 in V_H, so as $|U_H : D_H| = 2$, $E_H U_H/U_H$ is a hyperplane of V_H/U_H. Thus $1 \neq [E_H U_H/U_H, U_\gamma] \leq A_1 U_H/U_H$ by F.9.13.6, establishing (3). □

We are now in a position to obtain a contradiction, and hence establish Theorem 13.8.28. Recall $H^* \cong L_4(2)$, $m(U_H) = 5$, and $L/O_2(L) \cong \hat{A}_6$. Now $|Q_H : (Q \cap Q_H)| = |Q_H Q : Q| \leq |O_2(L_1 T) : Q|$, and as $L/O_2(L) \cong \hat{A}_6$, $|\overline{O_2(L_1 T)}| = 4$. Next by 13.7.4.2, $|Q_H/H_C : C_{Q_H}(V_3)/H_C| = |V_3/V_1| = 4$. So as $Q \cap Q_H \leq$

$C_{Q_H}(V_3)$, we conclude that $H_C \leq C_{Q_H}(V_3) = Q \cap Q_H \leq Q$. Thus H_C centralizes V, and hence H_C also centralizes $\langle V^H \rangle = V_H$. Therefore as $A_1 \leq V_H$ by 13.8.32.3, $H_C \leq C_G(A_1) = G_\gamma$, since $H = G_1$ by 13.8.27.2; thus $[H_C, U_\gamma] \leq H_C \cap U_\gamma$. But $m(U_\gamma \cap Q_H) = 4$ by 13.8.32.1, and by 13.8.32.2, $m(U_\gamma \cap V_H) \geq 3$. So $m((U_\gamma \cap H_C)V_H/V_H) \leq 1$. Thus as $[H_C, U_\gamma] \leq H_C \cap U_\gamma$, K has at most one noncentral chief factor on H_C/V_H, and by G.6.4, that factor is 4-dimensional if it exists. But this contradicts 13.8.31.3. This completes the proof of Theorem 13.8.28.

By Theorem 13.8.28, case (a) of 13.8.27.1 holds: Namely $L/O_2(L) \cong A_6$, $H^* \cong A_6$ or S_6, and $\tilde{U}_H$ is the natural module for K^* on which L_1 has two noncentral chief factors, or its 5-dimensional cover.

LEMMA 13.8.33. *Case (2) of 13.8.8 holds; that is, $D_\gamma < U_\gamma$.*

PROOF. Assume instead that case (1) of 13.8.8 holds. By 13.8.27.3, $D_H = U_H$, $D_\gamma = U_\gamma$, V induces a group of transvections with center V_1 on U_γ, $V_\gamma \not\leq Q_H$, and we have symmetry between γ_1 and γ, (cf. the first part of Remark F.9.17), so V_γ^* induces a transvection on $\tilde{U}_H$ with center $\tilde{A}_1$, and $A_1 \leq U_H$. As usual choose $g := g_b \in \langle LT, H \rangle$ with $\gamma_1 g = \gamma$. By F.9.13.7, $[U_H, U_\gamma] = 1$. Therefore $U_\gamma \leq C_G(V_3) \leq M_V$ by 13.5.4.4. By 13.8.5, $H = KT$ and $H^* \cong S_6$. In particular, we can appeal to 13.8.6 and adopt the notation of that lemma. As $[V, U_\gamma] \neq 1$, we may pick g so that $[V_3^g, V] \neq 1$. Thus as $[V_3, V_3^g] \leq [U_H, U_\gamma] = 1$, 13.5.4.4 says $V_1 = [V, V_3^g]$ and $\bar{V}_3^g = \langle (5,6) \rangle$, so that $\bar{L}\bar{T} \cong S_6$.

Notice that if $m(\tilde{U}_H) = 4$ then $\tilde{A}_1 \leq \tilde{V}_3^h$ for some $h \in H$. Assume instead for the moment that $m(\tilde{U}_H) = 5$. Then $\tilde{A}_1$ is of weight 2, while by 13.8.6.1, $\tilde{V}_3$ consists of vectors of weight 4, so A_1 is not contained in an H-conjugate of V_3. Thus as $V_3 = V \cap U_H$ by 13.8.4.5, we conclude from 13.8.4.4 that $b > 3$ when $m(\tilde{U}_H) = 5$.

We claim that U_L is abelian; the proof will require several paragraphs. Assume U_L is nonabelian. Then $b = 3$ by (1) and (3) of 13.8.4, so by the previous paragraph, $m(\tilde{U}_H) = 4$ and $A_1 \leq V_3^h$ for some $h \in H$. Thus $V_1 = V_1^h$ is orthogonal to A_1 in V^h, so V_1 is orthogonal to $A_1^{h^{-1}}$ in V, and $1 = [U_H, U_\gamma] = [U_H, U_H^g] = [U_H, U_H^{gh^{-1}}]$. Now $V_1^{gh^{-1}} = A_1^{h^{-1}} = V_1^y$ for some $y \in L$, so as $H = G_1$ by 13.8.27.2, we conclude that $U_H^{gh^{-1}} = U_H^y$. Finally L_1T is transitive on the points of V distinct from V_1 and orthogonal to V_1, and T is transitive on the points of V not orthogonal to V_1; so since we are assuming U_L is nonabelian, $[U_H, U_H^l] \neq 1$ for some $l \in L$ with V_1^l not orthogonal to V_1, and hence for all such V_1^l by transitivity of T on this set. Therefore $U_H^{l*} \neq 1$: for otherwise $U_L \leq Q_H$, and hence $U_L \leq Q_H^l$, so that by 13.7.3, $[U_H, U_H^l] \leq V_1 \cap V_1^l = 1$, contrary to our choice of l.

Choose l with $l^2 \in Q$. Then as $V_3 = V \cap U_H$, $W_2 := V \cap U_H \cap U_H^l$ is a complement to V_1 in V_3, and to $V_1 V_1^l$ in V. Further $X_1 := O^2(C_{L_1}(V_1 V_1^l))$ acts on U_H and U_H^l, with $W_2 = [W_2, X_1]$. By 13.8.27, L_1 has two nontrivial chief factors on U_H, so $[U_H V/V, X_1] = U_H V/V \cong E_4$, and hence $[U_H^l V/V, X_1] = U_H^l V/V \cong E_4$. Then X_1 is irreducible on $U_H^l V/V$, so as $U_H^{l*} \neq 1$, $U_H^{l*} = O_2(L_1^*) \cong E_4$ and $U_H^l \cap Q_H = V_3^l = U_H^l \cap V$.

Next by 13.7.3.7, $|H_C : (H_C \cap H^l)| \leq 2$, and as $U_L \leq Q \leq N_G(H_C)$ by 13.7.3,

$$[U_H^l, H_C \cap H^l] \leq U_H^l \cap H_C \leq U_H^l \cap Q_H = U_H^l \cap V \leq V_H.$$

Since $U_H^{l*} = O_2(L_1^*) \cong E_4$ does not centralize a hyperplane in any nontrivial irreducible for K^*, we conclude that $[H_C, K] \leq V_H$. Further $V_H \leq H_C$ by 13.7.3.2, so that $|V_H : V_H \cap H^l| \leq 2$, and $[U_H^l, V_H \cap H^l] \leq U_H^l \cap Q_H = U_H^l \cap V$ with VU_H/U_H of rank 1 by 13.8.4.5. Thus $O_2(L_1^*)$ induces transvections with a common center on $(V_H \cap H^l)U_H/U_H$ of index at most 2 in V_H/U_H. So we conclude that K has at most one nontrivial chief factor on V_H/U_H, and such a factor must be the natural module on which L_1 has one noncentral chief factor. So since L_1 has two noncentral chief factors on U_H, and Q_H/H_C is H-isomorphic to $\tilde{U}_H$ of rank 4 by 13.7.4.2, we conclude that L_1 has at most six noncentral 2-chief factors. However by 13.8.6.4, L_1 has at least six noncentral chief factors on U_L/V, and hence at least eight noncentral 2-chief factors including those on $O_2(\bar{L}_1)$ and V. This contradiction establishes the claim that U_L is abelian.

Since U_L is abelian, $U_L \leq H_C$. Also we saw $A_1 \leq U_H$, so $U_L \leq C_G(A_1) = H^g \leq N_G(U_\gamma)$ as $H = G_1$. But also $U_\gamma \leq M$, so U_L and U_γ act quadratically on each other. In particular, $U_L^{g^{-1}*} \leq Q_H^{g^{-1}*} \leq O_2(C_{H^*}(\tilde{V}_1^{g^{-1}}))$, so $m(U_L^{g^{-1}*}) \leq 2$, as $O_2(C_{H^*}(\tilde{V}_1^{g^{-1}})) \cong E_8$ is not quadratic on $\tilde{U}_H$. Indeed as $V \not\leq Q_H^g$, $1 \neq V^{g^{-1}*} \leq U_L^{g^{-1}*}$, so $|U_L : V(U_L \cap Q_H^g)| \leq 2$. Thus as $[U_L \cap Q_H^g, V_3^g] \leq A_1$, there is a subgroup B/V of index at most 2 in U_L/V such that $[V_3^g, B/V] \leq A_1V/V$. In particular, $C_{U_L/V}(V_3^g)$ is of codimension at most 2 in U_L/V, so as $m(H^*, S) = 8$ for the Steinberg module S for H^*, we conclude from 13.8.6.4 that $m(\tilde{U}_H) = 5$.

Define U_1 and U_0 as in 13.8.6.5, and recall that $V \leq U_0$. Assume first that $U_0 < U_L$. Then as L_1 is irreducible on U_H/V_3U_1, $V_3U_1 = U_H \cap U_0$, so the image of U_H in U_L/U_0 is a T-invariant 4-group. Similarly define U_2 and K_2 as in 13.8.6.5, and set $U_{2,1} := \langle V_3^{K_2} \rangle$. Then $\tilde{U}_{2,1} = \tilde{V}_2^{\perp}$ in the 5-dimensional orthogonal space $\tilde{U}_H$, so $U_{2,1}/V_2 \cong E_8$ with $|U_{2,1} : U_1V_3| = 2$ and $U_H = \langle U_{2,1}^{L_1} \rangle$. By 13.8.6.3, $U_2 = [U_2, L_2]$, and by 13.8.6.5, $m(U_2) = 8$, so $U_2/V_2 = U_{2,1}/V_2 \oplus U_{2,1}^l/V_2$ for $l \in L_2 - H$ and $U_1U_1^lV \leq U_0 \cap U_2$ with L_2 irreducible on $U_2/U_1U_1^lV \cong E_4$. As $U_H = \langle U_{2,1}^{L_1} \rangle$ and $U_L > U_0$, $U_2 \not\leq U_0$; so $U_0 \cap U_2 = U_1U_1^lV$, and hence $U_2/(U_2 \cap U_0)$ is also a T-invariant 4-group. We conclude just as in 13.8.6.4 that U_L/U_0 has a Steinberg module as a quotient, and then obtain a contradiction as in the previous paragraph.

Therefore $U_0 = U_L$. As $U_H = [U_H, L_1]$, $U_L = [U_L, L]$. By 13.8.6.5, $U_L/V = U_0/V$ is a quotient of the 15-dimensional permutation module on L/L_1T; so as $U_L/V = [U_L/V, L]$, G.5.3.3 says that either U_L/V is L-isomorphic to V, or U_L/V has a quotient U_L/E isomorphic to the 5-dimensional cover of V. Indeed as V_3^g centralizes a subspace of U_L/V of codimension at most 2, in the latter case G.5.3 implies that $V = E$.

So $m(U_L/V) = 4$ or 5, and hence $m(U_L) = 8$ or 9. If $m(U_L) = 8$, then $U_L = U_2$ by 13.8.6.5, so $K_2 \leq N_G(U_L) \leq M = !\mathcal{M}(LT)$, and then $H = \langle K_2, L_1T \rangle \leq M$, contrary to $H \in \mathcal{H}_z$. Therefore $m(U_L/V) = 5$.

Let $u_1 \in U_1 - V$. Suppose $U_1 \leq Z(Q)$. Then $U_L = U_0 \leq Z(Q)$. Also $W_1 := \langle u_1^L \rangle$ is a quotient of the 15-dimensional permutation module on $L/L_1(T \cap L)$ with $W_1/(W_1 \cap V) \cong U_L/V$ of rank 5, so we conclude from G.5.3 that W_1 is the 5-dimensional cover of a copy of V. This is contrary to Theorem 13.4.1 and our choice of G as a counterexample to Theorem 13.8.1.

Thus $U_1 \not\leq Z(Q)$, so that $|Q : C_Q(u_1)| = 2$. Now U_L is generated by V and a set I of 5 conjugates of u_1, so

$$C_Q(U_L) = \bigcap_{i \in I} C_Q(i).$$

Therefore as $|Q : C_Q(u_1)| = 2$, $m(Q/C_Q(U_L)) \leq 5$. Also $C_L(u_1V/V) = L_1T$, so $C_{LT}(u_1)$ is of index 2 in L_1T. We conclude from G.5.3.3 that $Q/C_Q(U_L)$ is a copy of V as a $\bar{L}$-module, or its 5-dimensional cover. Therefore L_1 has one noncentral 2-chief factor on each of $O_2(\bar{L}_1)$, $Q/C_Q(U_L)$, U_L/V, and V. Let k and j be the number of noncentral chief factors of L_1 on $C_Q(U_L)/U_L$ and H_C/U_H, respectively; thus L_1 has $n := 4 + k$ noncentral 2-chief factors. Next $C_Q(U_L) \leq H_C$, while the two noncentral chief factors for L_1 on U_L are the two contained in U_H, so $k \leq j$. On the other hand, L_1 has two noncentral 2-chief factors on U_H, and hence also two on Q_H/H_C by 13.7.4.2, and one on $O_2(L_1^*)$, so that $5 + j = n = 4 + k$. But now $j + 1 = k \leq j$, a contradiction. This contradiction completes the proof of 13.8.33. $\square$

LEMMA 13.8.34. *(1)* $A_1 \leq U_H$ *and* $V_1 \leq U_\gamma$.
(2) $m(U_\gamma^*) = 1$.
(3) $O_2(L_1^*) \not\leq V_\alpha^*$, *so* $m(V_\gamma^*) \leq 2$.

PROOF. By 13.8.33, case (2) of 13.8.8 holds. Then $1 \neq V_\alpha^* \leq R_1^* \cong E_4$ or E_8. By 13.8.4.5, V_H/U_H is a nontrivial quotient of the 15-dimensional $\mathbf{F}_2H^*$-permutation module on $H^*/L_1^*T^*$, and by 13.8.4.6, V_α^* is quadratic on V_H/U_H. So $O_2(L_1^*) \not\leq V_\alpha^*$, and hence (3) holds. Further $R_1^* = C_{H^*}(\tilde{U}_1\tilde{V}_3)$, where $\tilde{U}_1 := C_{\tilde{U}_H}(H)$ and $m(U_H/U_1V_3) = 2$.

Suppose that $m(U_\gamma^*) > 1$. Then as $O_2(L_1^*) \not\leq V_\alpha^*$, $m(U_\alpha^*) = 2$ and $[\tilde{U}_H, U_\alpha] = \tilde{U}_1\tilde{V}_3$, so as $[U_H, U_\alpha] \leq U_\alpha$, $U_1V_3 \leq U_\alpha V_1$. Thus $U_H \cap U_\gamma$ is of codimension at most 3 in U_H, so

$$m(D_\gamma U_H/U_H) = m(D_\gamma/(D_\gamma \cap U_H)) = m(D_\gamma) - m(U_\gamma \cap U_H)$$

$$= m(U_H) - m(U_\gamma^*) - m(U_\gamma \cap U_H) \leq 1.$$

But $[V_H, U_\gamma] \leq V_H \cap U_\gamma \leq D_\gamma$, and hence $m([V_H/U_H, U_\gamma]) \leq 1$. However by 13.7.7, H^* is faithful on V_H/U_H, whereas by G.6.4, U_γ^* does not induce transvections with a common center on any faithful $\mathbf{F}_2H^*$-module. This contradiction shows that $m(U_\gamma^*) = 1$, so (2) holds.

Assume that (1) fails. By 13.8.10.1, $m(U_H/D_H) = 1$, and we have symmetry between γ_1 and γ in the sense of Remark F.9.17. Therefore interchanging γ and γ_1 if necessary, we may assume that $A_1 \not\leq U_H$, and hence by 13.8.10.2 that U_γ induces transvections on U_H with axis D_H. Then as $A_1 \not\leq U_H$, 13.7.3.7 says $[V_\gamma, D_H] \leq A_1 \cap U_H = 1$. Thus V_γ^* induces transvections on $\tilde{U}_H$ with axis $\tilde{D}_H$, so V_γ^* is of rank 1, and hence $V_\gamma^* = U_\gamma^*$. Then by 13.8.9.2, $V_1 \not\leq U_\gamma$ and U_H induces transvections on V_γ/U_γ with center V_1U_γ/U_γ. Since $V_1 \not\leq U_\gamma$, U_H induces transvections on U_γ/A_1 by 13.8.10.2. However by 13.8.4.5, V_γ/U_γ is a nontrivial quotient of the 15-dimensional $\mathbf{F}_2H^g$-permutation module for $G_\gamma/Q_\gamma \cong A_6$ or S_6 on $H^g/L_1^gT^g$. Thus as $L_1^gT^g$ is not the stabilizer of a point in U_γ/A_1, V_γ/U_γ has a quotient which is the conjugate of U_γ/A_1 by an outer automorphism of G_γ/Q_γ. Therefore as U_H induces transvections on U_γ/A_1, it does not induce a transvection on V_γ/U_γ, a contradiction. This completes the proof of (1) and of the lemma. $\square$

We are now ready to complete the proof of Theorem 13.8.1. As $A_1 \leq U_H$ by 13.8.34.1, $\tilde{A}_1 \leq Z(\tilde{T}^h)$ for some $h \in H$, and $C_{H^*}(\tilde{A}_1)$ is a maximal parabolic of H^* stabilizing a point of $\tilde{U}_H$. Next by 13.8.34.1 we may apply 13.8.11.1 to conclude that $U_\gamma^* < V_\gamma^*$. Thus as V_γ^* and U_γ^* are normal in $C_H(A_1)^* = C_{H^*}(\tilde{A}_1)$, it follows that $O_2(L_1^{h*}) \leq V_\gamma^*$. But this contradicts 13.8.34.3.

This final contradiction establishes Theorem 13.8.1.

Observe in fact that Theorems 13.3.16, 13.6.1, and 13.8.1 complete the treatment of Hypothesis 13.3.1 for all possibilities for $L/O_2(L)$ (cf. 13.3.2.1) except $L_3(2)$.

13.9. Chapter appendix: Eliminating the $\mathbf{A_{10}}$-configuration

This section eliminates the shadow of the group A_{10}, by ruling out the existence of $M \in \mathcal{M}$ with $M \cong S_4$ wr $\mathbf{Z}_2$. We prove:

Theorem 13.9.1. *There is no simple QTKE-group G such that there exists $T \in Syl(G)$ and $M \in \mathcal{M}(T)$ satisfying $M \cong S_4$ wr $\mathbf{Z}_2$.*

Throughout the section, we assume G, T, M is a counterexample to Theorem 13.9.1. As usual we will begin with a number of preliminary lemmas describing the structure of M.

Observe that $J(M) = M_1 \times M_2$ with $M_i \cong S_4$, and $M_1^s = M_2$ for s an involution in $M - J(T)$. Let $T_i := T \cap M_i$ and $\langle t \rangle := Z(T_1)$. Notice $Z(T) = \langle z \rangle$ is of order 2 where $z := tt^s$. Let $A := O_2(M)$, so that $A \cong E_{16}$. For $X \subseteq G$, let $G_X := C_G(X)$, and set $\tilde{G}_z := G_z/\langle z \rangle$.

Let $\hat{G} := A_{10}$ be the alternating group on $\Omega := \{1, \ldots, 10\}$, and $\hat{M}$ the subgroup of $\hat{G}$ permuting

$$\{\{1,2,3,4\},\{5,6,7,8\},\{9,10\}\}.$$

There is an isomorphism $\alpha : M \to \hat{M}$ such that $\hat{T} := \alpha(T) \in Syl_2(\hat{G})$ and $\hat{M} \in \mathcal{M}(\hat{T})$. Let $\hat{M}_i := \alpha(M_i)$, $\hat{z} := \alpha(z)$, etc. We may choose our isomorphism α so that $\hat{M}_1 = \hat{G}_{5,\ldots,10}$ and $\hat{z} = (1,2)(3,4)(5,6)(7,8)$.

We will show that the 2-local subgroups and 2-fusion in G are the same as that of $\hat{G}$; this is a contradiction since G is quasithin while $\hat{G}$ is not. From time to time, we use the identification α of M with $\hat{M}$ to compute facts about M and its subgroup T.

Lemma 13.9.2. *(1) $\mathcal{A}(T) = \{A_i : 1 \leq i \leq 4\}$, with $A_1 := A$, and $B := A_2$ both normal in T, while $A_3^s = A_4$. Further $J(T) = T_1 \times T_2 = AB$ and $A \cap B = Z(J(T))$.*

(2) $N_G(J(T)) = T$.

(3) A and B are weakly closed in T with respect to G. Hence fusion in A is controlled by $M = N_G(A)$, and in B by $N_G(B)$.

(4) $M = N_G(A)$, $a^G \cap A = a^M$ for each $a \in A$, $|z^M| = 9$, and $|t^M| = 6$.

(5) $t \notin z^G$.

(6) $J(T) \in Syl_2(G_t)$.

Proof. Part (1) is an easy calculation. As $M \in \mathcal{M}$, $M = N_G(A)$. Let $X := N_G(J(T))$ and $X^* := X/J(T)$. Then X acts on $\mathcal{A}(T)$, and as $M = N_G(A)$, $T = N_M(J(T)) = N_X(A)$, so $J(T)$ is the kernel of the action of X on $\mathcal{A}(T)$. Thus $X^* \leq Sym(\mathcal{A}(T)) \cong S_4$ with $\mathbf{Z}_2 \cong T^* \in Syl_2(X^*)$, so either $X = T$, or $X^* \cong S_3$. The latter is impossible, as $Aut(J(T))$ is a 2-group. Thus (2) holds.

As $J(T)$ is weakly closed in T, and each member of $\mathcal{A}(T)$ is normal in $J(T)$, we may apply the Burnside Fusion Lemma A.1.35 to these normal subsets to conclude for each $D \in \mathcal{A}(T)$ that $D^G \cap J(T) = D^{N_G(J(T))}$, and hence $D^G \cap J(T) = D^T$ by (2). In particular as A and B are normal in T, they are weakly closed in T.

Hence (3) holds by application of the Burnside Fusion Lemma to the elements of A and B. Next as $M = N_G(A)$, (4) follows from (3) and the identification α given after the statement of Theorem 13.9.1, which says A is the orthogonal module for $M/A \cong O_4^+(2)$ with z^M the singular points and t^M the nonsingular points. Now (4) implies (5), and then as $Z(T) = \langle z \rangle$ has order 2, t is not 2-central by (5), so (6) holds. □

From now on let B be the group defined in 13.9.2.1, and set $K := N_G(B)$. As $B \trianglelefteq T$ by that result, $K \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ by 1.1.4.6.

In the following lemma, "diagonal involutions" in $J(M)$ are those projecting nontrivially on both factors of the decomposition $J(M) = M_1 \times M_2$. The next two lemmas follow from straightforward calculations.

LEMMA 13.9.3. *M has 6 classes of involutions Δ_i, $1 \leq i \leq 6$, where*

(1) $\Delta_1 := z^M$ consists of the diagonal involutions in A.

(2) $\Delta_2 := t^M = (A \cap T_1)^\# \ \cup \ (A \cap T_2)^\#$.

(3) Δ_3 consists of the involutions in $M_1 - A$ and $M_2 - A$.

(4) Δ_4 consists of the diagonal involutions $i_1 i_2$ with $i_k \in M_k \cap \Delta_3$, $k = 1, 2$.

(5) Δ_5 consists of the diagonal involutions ij with $i \in M_k \cap \Delta_3$ and $j \in M_{3-k} \cap \Delta_2$, $k = 1, 2$.

(6) $\Delta_6 := s^M$ consists of the involutions in $M - J(M)$.

LEMMA 13.9.4. *$B \cap \Delta_1 = \{z\}$, $|B \cap \Delta_2| = 2$, and $|B \cap \Delta_i| = 4$ for $i = 3, 4, 5$. Further each set is an orbit under T.*

LEMMA 13.9.5. *$G_z > T$, so $G_z \not\leq M$.*

PROOF. Assume that $G_z = T$. We will obtain a contradiction using Thompson Transfer A.1.36 on s, based on an analysis of fusion which will eventually include the explicit identification of $O^2(G_t)$.

First $C_M(s) \cong \mathbf{Z}_2 \times S_4$ is not a 2-group, so $s \notin z^G$. Suppose that z is weakly closed in B with respect to G. Then $z^G \cap M = \Delta_1 = z^M$ by 13.9.4 and 13.9.3, and $C_G(z) = T \leq M$ by hypothesis. Therefore by 7.3.1 in [**Asc94**], M is the unique fixed point of z on G/M. Hence by 7.4.2 in [**Asc94**], $s^G \cap M = s^M$. Therefore as $s^M \subseteq M - J(M)$, $s \notin O^2(G)$ by Thompson Transfer, contradicting G simple.

Therefore z is not weakly closed in B with respect to G. On the other hand, $C_M(i)$ is not a 2-group for $i \in \Delta_2 \cup \Delta_3$, so

$$z^G \cap M \subseteq \Delta_1 \cup \Delta_4 \cup \Delta_5 \qquad (*)$$

by 13.9.3. Also by 13.9.2.3, $z^G \cap B = z^K$. Thus by (*), and since the sets in 13.9.4 are T-orbits, z^K is of order 5 or 9, so in particular, $T < K$. Set $V := \langle z^K \rangle$ and $K^* := K/C_K(V)$. Then $O_2(K^*) = 1$ by B.2.14, so that $O_2(K) \leq C_K(V)$. Also $C_K(V)$ is a 2-group as $T = G_z$, so $C_K(V) = O_2(K)$. On the other hand if $A \leq O_2(K)$, then $J(T) = AB \leq O_2(K)$, so $K \leq N_G(J(T)) = T$ by 13.9.2.2, contrary to $T < K$. Thus $A \not\leq C_K(V)$, and hence by B.2.5, V is an FF-module for K^*. Then $3 \in \pi(K^*)$ by Theorem B.5.6, while $C_K(z) = T$ is a 2-group, so

$$|K : T| = |z^K| = 9$$

rather than 5. Hence the inclusion in (*) is an equality, and as $B = \langle \Delta_4 \cap B, \Delta_5 \cap B \rangle$, $V = B \cong E_{16}$. Then as $B = C_T(B)$, we conclude that $O_2(K) = B$. Inspecting the subgroups of $GL_4(2)$ with Sylow group D_8 and of order 72, we conclude B is the orthogonal module for $K/B \cong O_4^+(2)$ and $|t^K| = 6$. Therefore $t^G \cap J(M) = \Delta_2 \cup \Delta_3$ in view of 13.9.4. Thus all involutions in $J(T)$ are fused to t or z. So writing $I(S)$ for the set of involutions in a subgroup S of G:

(!) $t^G \cap J(T) = I(T_1) \cup I(T_2)$. So T_1 and T_2 are the subgroups S of $J(T)$ maximal subject to the property that $I(S) \subseteq t^G$. In particular each 4-subgroup of $J(T)$ consisting of members of t^G contains either t or t^s, and lies in either T_1 or T_2.

Set $X_1 := O^2(C_M(t)) = O^2(M_2)$, $X_2 := O^2(C_K(t))$, $X := O^2(G_t)$, and $\bar{G}_t := G_t/\langle t\rangle$. Thus $X_i \leq X$. We begin the explicit determination of X mentioned earlier. Recall $J(T) \in Syl_2(G_t)$ by 13.9.2.6.

For $i = 1, 2$, A_i is the orthogonal module for $N_G(A_i)/A_i$, with $t^{N_G(A_i)}$ the set of nonsingular vectors in A_i, so $X_i \cong A_4$ with $O_2(X_i) = A_i \cap X_i$ and $t \notin O_2(X_i)^\# \subseteq t^G$. Thus we conclude from (!) that $O_2(X_i) \leq T_2$, so that $O_2(X_i) = A_i \cap T_2 = X_i \cap T_2$, and hence $T_2 = O_2(X_1)O_2(X_2) \leq X$.

If U is a 4-subgroup of T_1 and $g \in G_t$ with $U^g \leq J(T)$, then $t \in U^g$, so $U^g \leq T_1$ by (!). Hence $I(T_1)$ is strongly closed in the Sylow group $J(T)$ with respect to G_t, so as $\bar{T}_1 \leq Z(\overline{J(T)})$, $N_{\bar{G}_t}(\bar{T}_1)$ controls fusion in $\overline{J(T)}$ by the Burnside Fusion Lemma A.1.35. Then as $Aut(T_1)$ is a 2-group and $\bar{T}_1$ is central in the Sylow group $\overline{J(T)}$, each element of $\bar{T}_1$ is strongly closed in $\overline{J(T)}$ with respect to $\bar{G}_t$; so by Thompson Transfer, $\bar{T}_1 \cap \bar{X} = 1$ as $X = O^2(X)$. Then $\langle t\rangle T_2 \in Syl_2(\langle t\rangle X)$, so by Thompson Transfer, $t \notin X$, and we conclude $T_1 \cap X = 1$. Hence $T_2 \in Syl_2(X)$ as $T_2 \leq X$ and $J(T) = T_1T_2$ is Sylow in G_t. Further $C_X(tz) = C_X(z) = T \cap X = T_2 \cong D_8$, and from the action of the X_i on the $O_2(X_i)$, X has one class of involutions represented by tz. Thus by I.4.1, $X \cong L_3(2)$ or A_6. In particular the involutions in $X = O^2(G_t)$ are in t^G.

As G is simple, by Thompson Transfer, $s^G \cap J(T) \neq \emptyset$. We showed that z, t are representatives for the G-classes of involutions in $J(T)$, and that $s \notin z^G$. Thus $s \in t^G$. We also saw that $O^2(C_M(s)) \cong A_4$, with $I(O^2(C_M(s))) \subseteq z^G$. This is impossible, as we saw $I(O^2(G_t)) \subseteq t^G$. This contradiction completes the proof of 13.9.5. □

Recall $\hat{G} := A_{10}$ and set $\hat{Q} := O_2(O^2(\hat{G}_{\hat{z}}))$. We may check directly from the structure of A_{10} that $\hat{Q} \cong Q_8^2$ and $J(\hat{Q}/\langle\hat{z}\rangle) = \hat{Q}/\langle\hat{z}\rangle \cong E_{16}$. Let $Q := \alpha^{-1}(\hat{Q})$. Since $\alpha : M \to \hat{M}$ is an isomorphism:

LEMMA 13.9.6. *$Q \cong Q_8^2$ and $\tilde{Q} = J(\tilde{T}) \cong E_{16}$.*

Furthermore from the structure of A_{10}, $\hat{G}_{\hat{z}}/\hat{Q} \cong S_3 \times \mathbf{Z}_2$. We wish to establish analogous statements in G, starting with:

LEMMA 13.9.7. *$G_{z,t} = J(T)$.*

PROOF. First by 13.9.2.6, $J(T) \in Syl_2(G_t)$. As $z \in Z(T)$, $F^*(G_z) = O_2(G_z)$ by 1.1.4.6, so $F^*(G_{t,z}) = O_2(G_{t,z})$ by 1.1.3.2; then setting $G^*_{t,z} := G_{t,z}/\langle t, z\rangle$, we obtain $F^*(G^*_{t,z}) = O_2(G^*_{t,z})$ from A.1.8. As the Sylow group $J(T)^*$ of G^*_t is abelian,

$$J(T)^* \leq C_{G^*_{t,z}}(O_2(G^*_{t,z})) \leq O_2(G^*_{t,z}),$$

so $J(T) = O_2(G_{t,z})$. Then the lemma follows from 13.9.2.2. □

LEMMA 13.9.8. *(1) $Q \trianglelefteq G_z$ and $G_z/Q \cong S_3 \times \mathbf{Z}_2$.*
(2) $B \trianglelefteq G_z$, and hence $G_z \leq K$.

PROOF. We claim first that $Q \trianglelefteq G_z$. Let $Q_z := O_2(G_z)$. By G.2.2 with $\langle z\rangle$, $\langle t, z\rangle$, 1 in the roles of "V_1, V, L", $\tilde{U} := \langle \tilde{t}^{G_z}\rangle \leq \Omega_1(Z(\tilde{Q}_z))$ and $\tilde{U} \in \mathcal{R}_2(\tilde{G}_z)$. Now $C_{G_z}(\tilde{U})/C_{G_{z,t}}(\tilde{U})$ is of order at most 2, so by 13.9.7, $C_{\tilde{G}_z}(\tilde{U})$ is a 2-group, and hence $\tilde{Q}_z = C_{\tilde{G}_z}(\tilde{U})$. Let $G_z^* := G_z/Q_z$, so that $O_2(G_z^*) = 1$ and $G_z^* \leq GL(\tilde{U})$.

If $Q \leq Q_z$, then $\tilde{Q} = J(\tilde{Q}_z)$ by 13.9.6, so that $Q \trianglelefteq G_z$, as claimed. Thus we may assume $Q \not\leq Q_z$, so in particular $m(\tilde{U}) < m(J(\tilde{T})) = 4$ by 13.9.6. Further using the identification α, no element of $\tilde{T}$ induces a transvection on $\tilde{Q}$; so if $|Q : Q \cap Q_z| = 2$, then $\tilde{U} \leq C_{\tilde{T}}(\tilde{Q} \cap \tilde{Q}_z) \leq C_{\tilde{T}}(\tilde{Q})$, and then $\tilde{Q} \leq C_{\tilde{T}}(\tilde{U}) = \tilde{Q}_z$ by the first paragraph, contrary to assumption. Thus $|Q^*| > 2$, so $m(\tilde{U}) > 2$ and hence $m(\tilde{U}) = 3$. Then $G_z^* \leq GL(\tilde{U}) = L_3(2)$, with Sylow group T^* of order at least 4 and $O_2(G_z^*) = 1$, so we conclude $G_z^* = GL(\tilde{U})$. Then $G_{t,z}$ has order divisible by 3, contrary to 13.9.7, completing the proof of the claim.

By the claim, $Q \trianglelefteq G_z$. In particular as $t \in Q$, we have $U \leq Q \leq Q_z$. Now $C_{\tilde{G}_z}(\tilde{Q}) \leq C_{\tilde{G}_z}(\tilde{U}) = \tilde{Q}_z$, so as $\tilde{Q}$ is self-centralizing in $\tilde{T}$, we conclude $C_{\tilde{G}_z}(\tilde{Q}) = \tilde{Q}$. Hence $G_z' := G_z/Q$ lies in the orthogonal group $O(\tilde{Q}) \cong O_4^+(2)$ with Sylow group $T' \cong E_4$. As $T < G_z$ by 13.9.5, we conclude that either (1) holds, or $G_z^* \cong S_3 \times S_3$. In the latter case G_z is transitive on the involutions in $Q - \langle z\rangle$. This is impossible as $A \cap Q \cong E_8$ contains an element of Δ_1, and hence t is fused into z^G in G_z, contrary to 13.9.2.5. This completes the proof of (1).

Let $b \in B - Q$. As $B \trianglelefteq T$ and $m([\tilde{Q}, b]) = 2$, $[b, Q] = B \cap Q \cong E_8$. Similarly for $a \in A - Q$, $[Q, a] = A \cap Q \cong E_8$. Thus a and b interchange the two Q_8-subgroups Q_1 and Q_2 of Q. Now $T/Q \cong E_4$ has three subgroups E_i/Q, $1 \leq i \leq 3$, of order 2, with $E_1 := N_T(Q_1)$. Then $Q_z \neq E_1$, since (1) shows that Q_z/Q centralizes an a-invariant subgroup of G_z/Q of order 3, whereas E_1/Q does not. Furthermore E_1 is not AQ or BQ as we saw these subgroups interchange Q_1 and Q_2. Let $AQ =: E_2$; then $C_{AQ}([Q, a]) \cong E_{16}$ and $[Q, a] = [Q, i]$ for each $i \in AQ - Q$. Therefore $A = C_{AQ}([Q, a])$ and $BQ \neq AQ$. Thus $BQ = E_3$. As $M = N_G(A)$ and $T = C_M(z) < G_z$ by 13.9.5, A is not normal in G_z; therefore $A \not\leq Q_z$ as A is weakly closed in T by 13.9.2.3. Thus $BQ = E_3 = Q_z$, and then $B \trianglelefteq G_z$, as B is weakly closed in T by 13.9.2.3. This completes the proof of 13.9.8. □

LEMMA 13.9.9. *(1) B is the natural module for $K/B \cong O_4^-(2)$.*

(2) z^K and t^K are of order 5 and 10, respectively, and afford the set of singular and nonsingular points in the orthogonal space B.

(3) $z^G \cap M = \Delta_1 \cup \Delta_3 \cup \Delta_6$.

(4) $t^G \cap M = \Delta_2 \cup \Delta_4 \cup \Delta_5$.

(5) G has two classes of involutions with representatives z and t.

PROOF. First $Q = \langle s\rangle[J(T), s]$ with $[J(T), s] = C_{J(T)}(s)\langle t\rangle$. This allows us to calculate that T has four orbits Γ_i, $1 \leq i \leq 4$, on the set Γ of 18 involutions in $Q - \langle z\rangle$: $\Gamma_1 := \{t, tz\} \subseteq \Delta_2$, $\Gamma_2 \subseteq \Delta_1$ of order 4, $\Gamma_3 := s^T$ of order 8 containing $s \in \Delta_6$, and $\Gamma_4 := B \cap \Delta_4$ of order 4. On the other hand, from 13.9.8.1, G_z has two orbits on Γ: Γ^1 of length 6, and Γ^2 of length 12, with $t \in \Gamma^1$ as $\langle \tilde{t}\rangle = Z(\tilde{T})$. As $t \notin z^G \supseteq \Delta_1 \supseteq \Gamma_2$, we conclude

$$\Gamma^1 = \Gamma_1 \cup \Gamma_4 = t^G \cap Q \text{ and } \Gamma^2 = \Gamma_2 \cup \Gamma_3 = z^G \cap Q - \{z\}. \qquad (*)$$

In particular $\Delta_4 \subseteq t^G$. Next $M/O^2(M) \cong D_8$; let M_0 be the subgroup of M of index 2 with $M_0/O^2(M) \cong \mathbf{Z}_4$. Then $\Delta_1 \cup \Delta_2 \cup \Delta_4$ is the set of involutions in M_0, so each is in $z^G \cup t^G$. Hence (5) follows from Thompson Transfer.

Next by 13.9.8 and (*), $G_z \le K$ and $B \cap Q = \langle z, t^{G_z} \rangle$. We conclude using 13.9.8.1 that the orbits of G_z on $B^\#$ are $\Sigma_0 := \{z\}$, $\Sigma_1 := \Gamma^1$ of length 6, and two orbits Σ_i, $i = 2, 3$, on $B - Q$ of length 4. Then since $\Gamma_4 = B \cap \Delta_4 \subseteq Q$, appealing to 13.9.4, we may choose notation so that $\Sigma_2 = B \cap \Delta_3$ and $\Sigma_3 = B \cap \Delta_5$.

By 13.9.2.3, K controls fusion in B, so it follows from (5) that the G_z-orbit Σ_3 is fused to z or t under K. In particular, $G_z < K$. Thus there are three possibilities for z^K: $\{z\} \cup \Sigma_3$, $\{z\} \cup \Sigma_2$, or $\{z\} \cup \Sigma_2 \cup \Sigma_3$—of order 5, 5, or 9, respectively. Now by 13.9.7, $C_G(B) = C_{J(T)}(B) = B$, so $K/B \le GL(B)$. As $|GL_4(2)|$ is not divisible by 27, while 3 divides the order of $C_K(z)$ by 13.9.8, we conclude $|z^K| = 5$ rather than 9. Set $K^* := K/B$; then $C_K(z)^* = G_z^* \cong S_4$ by 13.9.8. Further $B = \langle \Sigma_i \rangle$ for $i = 2, 3$, so B is the kernel of the action of $C_K(z)$ on Σ_2 and Σ_3. We conclude K^* acts faithfully as S_5 on z^K. As K has orbits of length 5 and 10 on $B^\#$, it follows that B is the natural module for $K^* \cong O_4^-(2)$, with z^K the singular points of the orthogonal space B and t^K the nonsingular points. This establishes (1) and (2). Also if $k \in K - G_z$ then $zz^k \in t^K$, while if $z^k \in \Delta_5$, then $zz^k \in \Delta_5$. Thus $\Sigma_3 = B \cap \Delta_5 \not\subseteq z^K$, so $z^K = \{z\} \cup \Sigma_2 = \{z\} \cup (B \cap \Delta_3)$, and $\Delta_5 \cap B = \Sigma_3 \subseteq t^K$. Now it follows using (*) that (3) and (4) hold. □

LEMMA 13.9.10. *Let $E := T_1 \cap A = O_2(M_1)$. Then E centralizes $O^2(C_K(t)) \cong A_4$.*

PROOF. From the structure of K described in 13.9.9, and as $J(T) \in Syl_2(G_t)$ by 13.9.2.6, $C_K(t) = R_1 \times X$, where $t \in R_1 \cong D_8$, and $X \cong S_4$ with $O_2(X)^\# \subseteq t^K$. Let $R_2 := T \cap X$. Then $T_1 \times T_2 = J(T) = R_1 \times R_2$ with $\langle t \rangle = [R_1, R_1] = [T_1, T_1]$, $\langle t^s \rangle = [R_2, R_2] = [T_2, T_2]$, and $z = tt^s$. Now by the Krull-Schmidt Theorem A.1.15, $T_i\langle z \rangle = R_i\langle z \rangle$ for each $i = 1, 2$. Therefore $E \le T_1 \le R_1\langle z \rangle$.

Suppose $E \not\le R_1$. Then there is $e \in E - \langle t \rangle$ with $ez \in R_1$. As $E \cap B = \langle t \rangle$, $e \notin B$ and hence $ez \notin B$. Further $ez \in z^M$ by 13.9.3.1, so $ez = z^g \in (R_1 \cap z^G) - B$ for some $g \in G$. Then as X centralizes z^g, from the description of G_z in 13.9.8, $O_2(X) = [O_2(X), O^2(X)] \le Q^g$. So as $O_2(X)^\# \subseteq t^K$, it follows from (*) in the proof of 13.9.9.1 that

$$U := \langle t^G \cap Q^g \rangle = \langle z^g \rangle \times O_2(X).$$

By 13.9.8, $\langle t^{G_z} \cap Q \rangle = B \cap Q$, while $C_G(B \cap Q) \le G_{z,t} = J(T)$ by 13.9.7, so we conclude $U = B^g \cap Q^g$ and $C_G(U) \le C_{J(T)^g}(U) = B^g$. Now $C_{R_1 O_2(X)}(U) = C_{R_1 O_2(X)}(z^g) \cong E_{16} \cong C_G(U)$, so $B^g = C_{R_1 O_2(X)}(U) \le K = N_G(B)$. Hence $B^g = B$ as B is weakly closed in T by 13.9.2.3, contradicting $z^g \notin B$.

This contradiction shows that $E \le R_1$. Hence E centralizes $O^2(X) = O^2(C_K(t)) \cong A_4$. □

We are now in a position to prove Theorem 13.9.1. The argument will be much like that in the proof of 13.9.5.

Let E be as in 13.9.10, and recall $G_E = C_G(E)$. As $J(T) \in Syl_2(G_t)$ by 13.9.2.6 and $t \in E \trianglelefteq J(T)$, $E \times T_2 = C_{J(T)}(E) \in Syl_2(G_E)$. Let $H := O^2(G_E)$, $H_1 := O^2(M_2)$, and $H_2 := O^2(C_K(t))$. Thus $H_i \cong A_4$ centralizes E, using 13.9.10 in the case of H_2. Therefore $H_i \le H$. Furthermore $O_2(H_1) = T_2 \cap A$, while $O_2(H_2) = H_2 \cap B$, with $O_2(H_1) \cap O_2(H_2) = \langle tz \rangle$. Therefore as $tz \in O_2(H_2)^\# \subseteq t^G$, $O_2(H_2) \le T_2$, so as $H_2 \cap B \not\le A$, $O_2(H_2)$ normalizes but does not centralize, $O_2(H_1)$, and then $T_H := O_2(H_1)O_2(H_2) \cong D_8$. As $H_i \le H$, $T_H \le H$. As $E \le Z(H)$, by Thompson Transfer, $E \cap H = 1$, so that $T_H \in Syl_2(H)$. As all involutions in T_H are fused under H_1 and H_2, H has one class of involutions. Further $C_{G_t}(z) = J(T)$ by 13.9.7, so $C_H(tz) = C_H(z) = J(T) \cap H = T_H \cong D_8$. Therefore by I.4.1, $H \cong L_3(2)$ or A_6.

Now $M_1 \leq N_G(E) \leq N_G(H)$, and M_1 centralizes $O^2(M_2) = H_1 \cong A_4$, so from the structure of $Aut(H)$, $O^2(M_1) \leq C_G(H)$, and indeed M_1 centralizes H if $H \cong L_3(2)$. Therefore as $m_3(M_1H) \leq 2$ since $N_G(H)$ is an SQTK-group, $H \cong L_3(2)$ and $M_1H = M_1 \times H$. But by 13.9.9.3, there is $z^g \in M_1 - O^2(M_1)$, so $L_3(2) \cong H \leq G_z^g$, contrary to 13.9.8.1. This contradiction completes the proof of Theorem 13.9.1.

CHAPTER 14

$\mathbf{L_3(2)}$ in the FSU, and $\mathbf{L_2(2)}$ when $\mathcal{L}_\mathbf{f}(\mathbf{G},\mathbf{T})$ is empty

The previous chapter reduced the treatment of the Fundamental Setup (3.2.1) to the case $\bar{L} \cong L_3(2)$—which we handle in this chapter. This in turn reduces the proof of the Main Theorem to the case $\mathcal{L}_f(G,T) = \emptyset$.

Recall that the case in the FSU where $\bar{L} \cong A_5$ is actually treated last in the natural logical order, but because of similarities with the case $\bar{L} \cong A_6$, those cases were treated together in the previous chapter; this was accomplished by introducing assumption (4) in Hypothesis 13.3.1.

In this chapter it will again be convenient to take advantage of some similarities in the treatment of two small linear groups: namely between the case $\bar{L} \cong L_3(2)$ for $L \in \mathcal{L}_f(G,T)$, and suitable $L \in \mathcal{M}(T)$ such that $LT/O_2(LT) \cong L_2(2)$ acts naturally on some 2-dimensional member of $\mathcal{R}_2(LT)$. The latter situation is the most difficult subcase of the case $\mathcal{L}_f(G,T) = \emptyset$, which of course remains after the Fundamental Setup is treated. As a result, we begin this chapter with several sections providing preliminary results on the case $\mathcal{L}_f(G,T) = \emptyset$, and in particular on the subcase with $L/C_L(V) \cong L_2(2)$.

14.1. Preliminary results for the case $\mathcal{L}_\mathbf{f}(\mathbf{G},\mathbf{T})$ empty

As usual, $T \in Syl_2(G)$ and $Z := \Omega_1(Z(T))$.

This chapter includes the beginning of the treatment of the case $\mathcal{L}_f(G,T) = \emptyset$. The first few results below are based only on that assumption, but afterwards we will assume the stronger Hypothesis 14.1.5.

We use the following notation through the section:

NOTATION 14.1.1. Let $E := \Omega_1(Z(J(T)))$, $M \in \mathcal{M}(T)$, $V \in \mathcal{R}_2(M)$, and $\bar{M} := M/C_M(V)$.

Recall from section A.5 of Volume I that for $H \in \mathcal{H}(T)$, in this section we deviate from our usual meaning of $V(H)$ in definition A.4.7, instead using the meaning in notation A.5.1, namely

$$V(H) := \langle Z^H \rangle.$$

Recall the partial ordering on $\mathcal{M}(T)$ given by $M_1 \lesssim M_2$ whenever

$$M_1 = C_{M_1}(V(M_1))(M_1 \cap M_2).$$

Recall $V(H) \in \mathcal{R}_2(H)$ by B.2.14.

The first result below does not even require the hypothesis $\mathcal{L}_f(G,T) = \emptyset$:

LEMMA 14.1.2. *Assume $J(T) \leq C_M(V)$ and set $S := Baum(T)$. Then*
(1) $V \leq E$ and $S = Baum(C_T(V))$.
(2) Assume either

(a) M is maximal in $\mathcal{M}(T)$ under $\lesssim$ and $V = V(M)$, or

(b) $V = \langle (V \cap Z)^M \rangle$, and M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.

Then $M = !\mathcal{M}(N_M(S))$ and $C(G,S) \leq M$.

PROOF. Part (1) follows from B.2.3.5. Assume one of the hypotheses of (2). Then $M = !\mathcal{M}(N_M(C_T(V)) = !\mathcal{M}(N_M(S))$ by A.5.7.2, so that $C(G,S) \leq M$. □

The next two preliminary results do assume $\mathcal{L}_f(G,T) = \emptyset$:

LEMMA 14.1.3. *Assume $\mathcal{L}_f(G,T) = \emptyset$. Then $H^\infty \leq C_H(U)$ for each $H \in \mathcal{H}(T)$ and $U \in \mathcal{R}_2(H)$.*

PROOF. By 1.2.1.1, H^∞ is the product of groups $L \in \mathcal{C}(H)$. Then $L \in \mathcal{L}(G,T)$. By hypothesis, $L \notin \mathcal{L}_f(G,T)$, so by 1.2.10, $[U,L] = 1$. □

LEMMA 14.1.4. *Assume $\mathcal{L}_f(G,T) = \emptyset$, M is maximal in $\mathcal{M}(T)$ under $\lesssim$, and $J(T) \leq C_M(V(M))$. Then M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.*

PROOF. Let $S := \mathrm{Baum}(T)$. By 14.1.2.2, $C(G,S) \leq M$ and $M = !\mathcal{M}(N_M(S))$. In particular $N_G(J(T)) \leq M$.

Let $M_1 \in \mathcal{M}(T) - \{M\}$, and $V := V(M_1)$. If $[V, J(T)] = 1$, then by a Frattini Argument, $M_1 = C_{M_1}(V)N_{M_1}(J(T))$, so we conclude $M_1 \lesssim M$ as $N_G(J(T)) \leq M$.

Hence we may assume $[V, J(T)] \neq 1$. Set $M_1^* := M_1/C_{M_1}(V)$ and $I := J(M_1)$. By a Frattini Argument, $M_1 = IN_{M_1}(J(T)) = I(M_1 \cap M)$, so it will suffice to show that $I = C_{M_1}(V)(I \cap M)$. By 14.1.3, M_1^* is solvable, so by Solvable Thompson Factorization B.2.16, $I^* = I_1^* \times \cdots \times I_s^*$ with $I_i^* \cong S_3$, and $s \leq 2$ by A.1.31.1. Now $I^* \leq O^{2'}(I^*T^*)$, and by B.6.5, $O^{2'}(IT)$ is generated by minimal parabolics H above T, so it will suffice to show that $H \leq M$ for those H with $H^* \neq 1$. We apply Baumann's Lemma B.6.10 to H to conclude $S \in Syl_2(O^2(H)S)$. Then we apply Theorem 3.1.1 with $N_M(S)$, S in the roles of "M_0, R", and as $M = !\mathcal{M}(N_M(S))$, we conclude that $H \leq M$ as required. □

We next discuss the basic hypothesis which we will use during the bulk of our treatment of the case $\mathcal{L}_f(G,T)$ empty:

The final result in this chapter, Theorem D (14.8.2), determines the QTKE-groups G in which $\mathcal{L}_f(G,T) \neq \emptyset$. Then in the following chapter we determine those QTKE-groups G such that $\mathcal{L}_f(G,T) = \emptyset$. As in the previous chapters on the Fundamental Setup (3.2.1), we may also assume that $|\mathcal{M}(T)| > 1$, since Theorem 2.1.1 determined the groups for which that condition fails. Finally we divide our analysis of groups G with $\mathcal{L}_f(G,T) = \emptyset$ into two subcases: the subcase where $|\mathcal{M}(C_G(Z))| = 1$, and the subcase where $|\mathcal{M}(C_G(Z))| > 1$. The second subcase is comparatively easy to handle, perhaps because all the examples other than $L_3(2)$ and A_6 occur in the first subcase.

Thus in this section, and indeed in most of those sections in this and the following chapter which are devoted to the case $\mathcal{L}_f(G,T)$ empty, we assume the following hypothesis:

HYPOTHESIS 14.1.5. *G is a simple QTKE-group, $T \in Syl_2(G)$, and*

(1) $\mathcal{L}_f(G,T) = \emptyset$.

(2) There is $M_c \in \mathcal{M}(T)$ satisfying $M_c = !\mathcal{M}(C_G(Z))$.

(3) $|\mathcal{M}(T)| > 1$.

LEMMA 14.1.6. *(1)* $M^\infty \leq C_M(V)$.
(2) $\mathcal{L}^*(G,T) = \mathcal{C}(M_c)$*, so that* $M_c = !\mathcal{M}(\langle K,T\rangle)$ *for each* $K \in \mathcal{C}(M_c)$.
(3) For each $H \in \mathcal{H}(T)$, $H^\infty \leq C_G(Z) \leq M_c$.

PROOF. Part (1) follows from 14.1.3, and (3) follows from 14.1.3 applied to $V(H)$, using Hypothesis 14.1.5.2.

Let $L \in \mathcal{L}(G,T)$. Then $\langle L,T\rangle \in \mathcal{H}(T)$, so $L \leq M_c$ by (3). Therefore if $L \in \mathcal{L}^*(G,T)$, then by 1.2.7.3, $N_G(\langle L^T\rangle) = !\mathcal{M}(\langle L,T\rangle) = M_c$, and hence $L \in \mathcal{C}(M_c)$. Conversely let $L \in \mathcal{C}(M_c)$ and embed $L \leq K \in \mathcal{L}^*(G,T)$. We just showed $K \in \mathcal{C}(M_c)$, so $L = K$ and hence $L \in \mathcal{L}^*(G,T)$. Thus (2) is established. □

LEMMA 14.1.7. *Assume* $J(T) \not\leq C_M(V)$*, and either* $M \neq M_c$ *or* $|Z| = 2$*. Then either*

(1) $m(V) = 2$, $\bar{M} = GL(V) \cong L_2(2)$*, and* $E \cap V = Z$ *is of order* 2*, or*
(2) $m(V) = 4$ *and* $\bar{M} \cong O_4^+(V)$*. Thus* $\bar{M} = (\bar{Y}_1 \times \bar{Y}_2)\langle\bar{t}\rangle$*, where* $\bar{Y}_i \cong L_2(2)$*,* $\bar{t}$ *is an involution interchanging* $\bar{Y}_1$ *and* $\bar{Y}_2$*, and* $V = V_1 \times V_2$*, where* $V_i := [V,Y_i] \cong E_4$*, and* $E \cap V$ *of order* 4 *contains* Z *of order* 2.

PROOF. Set $Y := J(M)$. By 14.1.6.1, $\bar{M}$ is solvable; so by Solvable Thompson Factorization B.2.16 $\bar{Y} = \bar{Y}_1 \times \cdots \times \bar{Y}_r$ with $\bar{Y}_i \cong L_2(2)$ and $V = V_1 \times \cdots \times V_r \times C_V(Y)$, where $V_i := [V,Y_i] \cong E_4$ for the preimage Y_i of $\bar{Y}_i$. As M is an SQTK-group, $r \leq 2$ by A.1.31.1. Thus either $\bar{M} = \bar{Y} \times C_{\bar{M}}(\bar{Y})$, or $r = 2$ and $\bar{M} = (\bar{Y} \times C_{\bar{M}}(\bar{Y}))\langle\bar{t}\rangle$, where t interchanges $\bar{Y}_1$ and $\bar{Y}_2$. Then as $\mathrm{End}_{\bar{Y}}(V_i) = \mathbf{F}_2$, $C_M(\bar{Y})$ centralizes $[V,Y]$.

Next $Z \cap [V,Y] \neq 1$ and $[V,Y]$ is T-invariant, so by 14.1.5.2,

$$C_M(\bar{Y}) \leq C_M([V,Y]) \leq C_M(Z \cap [V,Y]) \leq M_c.$$

Suppose that $C_V(Y) \neq 1$. Then $C_Z(Y) \neq 1$, so $|Z| > 2$, and $Y \leq C_G(C_Z(Y)) \leq M_c$. Therefore $M = C_M(\bar{Y})YT \leq M_c$, and hence $M = M_c$, contrary to our hypothesis that $M \neq M_c$ when $|Z| > 2$.

Therefore $C_V(Y) = 1$ so that $[V,Y] = V$. Then $C_{\bar{M}}(\bar{Y})$ centralizes V, so that $C_{\bar{M}}(\bar{Y}) = 1$. Hence if $r = 1$, then (1) holds, so we may assume $r = 2$. If $\bar{Y} < \bar{M}$, then (2) holds, so we may assume $\bar{M} = \bar{Y} = \bar{Y}_1 \times \bar{Y}_2$. But then $Z_i := Z \cap V_i \neq 1$, so $|Z| = 4$ and $Y_{3-i} \leq C_G(Z_i) \leq M_c$, so that $M = C_M(V)Y_1Y_2 \leq M_c$, and thus $M = M_c$, again contrary to our choice of M when $|Z| > 2$. □

LEMMA 14.1.8. *Assume* $M \neq M_c$*,* $\bar{X} \leq \bar{M}$ *is* T*-invariant of odd order, and* $X \not\leq M_c$*. Then* $V = [V,X]$.

PROOF. As $\bar{X}$ is of odd order, $V = [V,X] \times C_V(X)$. Suppose $C_V(X) \neq 1$. As $\bar{X}$ is T-invariant, $C_Z(X) \neq 1$. But then by 14.1.5.2, $X \leq C_G(C_Z(X)) \leq M_c$, contrary to hypothesis. □

LEMMA 14.1.9. *If* M *is the unique maximal member of* $\mathcal{M}(T)$ *under* $\lesssim$*, then* $M \neq M_c$.

PROOF. Assume $M = M_c$. By uniqueness of M and the definition of $\lesssim$, for each $M_1 \in \mathcal{M}(T)$ we have

$$M_1 = C_{M_1}(V(M_1))(M \cap M_1) \leq M$$

since $C_G(V(M_1)) \leq C_G(Z) \leq M_c = M$. This is impossible as $|\mathcal{M}(T)| > 1$ by 14.1.5.3. □

LEMMA 14.1.10. *Assume M has a subnormal A_3-block X, and $O_2(M) \leq R \leq T$ such that $X = [X, J(R)]$. Then $M = M_c$ and $|Z| > 2$.*

PROOF. Let $X_0 := \langle X^M \rangle$. Thus as $m_3(M) \leq 2$, either $X_0 = X$, or $X_0 = X_1 \times X_2$ with $X = X_1$ while $X_2 = X^t$ for $t \in T - N_M(X)$. Set $K := C_M(X_0)$. As $X = [X, J(R)]$ and $O_2(M) \leq R$, $Aut_{X_0T}(X_0) = Aut(X_0)$, so as $Z(X_0) = 1$ we conclude

$$M = (K \times X_0)T. \qquad (*)$$

Since $Z_0 := Z \cap [O_2(X_0), X_0] \neq 1$, $K \leq C_G(Z_0) \leq M_c = !\mathcal{M}(C_G(Z))$ by 14.1.5.2. Now if $C_T(X_0) \neq 1$, then $C_Z(X_0) \neq 1$, so $|Z| > 2$ and $X_0 \leq C_G(C_Z(X_0)) \leq M_c = !\mathcal{M}(C_G(Z))$. Then $M = M_c$ by (*), so the lemma holds.

Therefore we may assume that $C_T(X_0) = 1$. Then as $F^*(M) = O_2(M)$, we conclude from (*) that $K = 1$. Thus as $Aut_{X_0T}(X_0) = Aut(X_0)$, $M = X_0T \cong S_4$ or S_4 wr $\mathbf{Z}_2$. In the first case, $T \cong D_8$, so $G \cong L_3(2)$ or A_6 by I.4.3. But then $T = C_G(Z)$, contrary to Hypothesis 14.1.5. In the second case, Theorem 13.9.1 supplies a contradiction. □

LEMMA 14.1.11. *There exists a nontrivial characteristic subgroup $C_2 := C_2(T)$ of $Baum(T)$, such that for each $M \in \mathcal{M}(T)$, either*

(1) $M = C_M(V(M))N_M(C_2)$, or

(2) $M = M_c$ and $|Z| > 2$.

PROOF. Let $V := V(M)$. By 14.1.5.2, $M_c = !\mathcal{M}(C_G(Z))$, so $C_M(V) \leq C_M(Z) \leq M_c$. Let $S := \mathrm{Baum}(T)$ and choose $C_i := C_i(T)$ for $i = 1, 2$ as in the Glauberman-Niles/Campbell Theorem C.1.18. Thus $1 \neq C_2$ char S and $1 \neq C_1 \leq Z$. In particular $C_G(C_1) \leq M_c = !\mathcal{M}(C_G(Z))$.

Suppose first that $[V, J(T)] = 1$. Then $S = \mathrm{Baum}(C_T(V))$ by 14.1.2.1, so (1) holds by a Frattini Argument since C_2 is characteristic in S.

Thus we may assume that $[V, J(T)] \neq 1$, and that (2) fails, so that one of the conclusions of 14.1.7 holds. In either case $|Z| = 2$, so as $1 \neq C_1 \leq Z$ we conclude $C_1 = Z$. Further from the structure of $\bar{M}$, $C_M(C_1) = C_M(Z) = C_M(V)T \leq C_M(V)N_M(C_2)$. Therefore as we also may assume that conclusion (1) fails, $\langle C_M(C_1), N_M(C_2)\rangle < M$. Thus conclusion (2) of C.1.28 holds; in particular, there is a χ-block X of M with $X = [X, J(T)]$ such that X does not centralize V. Therefore as $\bar{M}$ is solvable by 14.1.6.1, we conclude that each such X is an A_3-block of M, and then 14.1.10 contradicts our assumption that (2) fails. □

In the remainder of the section let $C_2 := C_2(T)$ be the subgroup defined as in 14.1.11 and its proof.

LEMMA 14.1.12. *Let $M_f \in \mathcal{M}(N_G(C_2))$ and $V(M_f) \leq V_f \in \mathcal{R}_2(M_f)$. Then*

(1) M_f is maximal in $\mathcal{M}(T)$ under $\lesssim$, M_f is the unique maximal member of $\mathcal{M}(T) - \{M_c\}$ under $\lesssim$, and if $|Z| = 2$ then M_f is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.

(2) $M_f = !\mathcal{M}(N_{M_f}(C_T(V_f)))$.

(3) $C_{M_f}(V_f) \leq M$ for each $M \in \mathcal{M}(T)$.

(4) $M_f \neq M_c$.

PROOF. If $M \in \mathcal{M}(T)$ and either $M \neq M_c$ or $|Z| = 2$, then by 14.1.11, $M = C_M(V(M))N_M(C_2)$; so as $N_M(C_2) \leq M \cap M_f$, $M \lesssim M_f$. In particular if

$M_c = M_f$, then M_c is the unique maximal member of $\mathcal{M}(T)$ under $\stackrel{<}{\sim}$, contrary to 14.1.9. Thus $M_c \neq M_f$, proving (4). So as $C_{M_f}(V(M_f)) \leq M_c = !\mathcal{M}(C_G(Z))$, $M_f \not\stackrel{<}{\sim} M_c$, and then (1) follows from the first sentence of the proof.

As $V(M_f) \leq V_f$, $C_{M_f}(V_f) \leq C_{M_f}(V(M_f))$. By a Frattini Argument,

$$M_f = C_{M_f}(V_f)N_{M_f}(C_T(V_f)) = C_{M_f}(V(M_f))N_{M_f}(C_T(V_f)),$$

so (2) follows from A.5.7.1 and (1).

Next $C_{M_f}(V_f) \leq C_G(Z) \leq M_c$, while by (1) we may apply A.5.3.3 to each $M \in \mathcal{M}(T) - \{M_c\}$ to conclude that

$$C_{M_f}(V_f) \leq C_{M_f}(V(M_f)) \leq C_M(V(M)),$$

so (3) holds. □

LEMMA 14.1.13. *Assume $T \leq H \leq M$ with $R := O_2(H) \neq 1$ and $C(M,R) \leq H$. Then either*

(1) $O_{2,F^*}(M) \leq H$ and $O_2(H) = O_2(M)$, or

(2) $|Z| > 2$, and $M = M_c = !\mathcal{M}(H)$.

PROOF. Observe that the triple R, H, M satisfies Hypothesis C.2.3 in the roles of "R, M_H, H". Thus we can appeal to results in section C.2, and in particular we conclude from C.2.1.2 that $O_2(M) \leq R$.

Suppose $L \in \mathcal{C}(M)$ with $L/O_2(L)$ quasisimple and $L \not\leq H$. By 14.1.5.1, $L \notin \mathcal{L}_f(G,T)$, so L is not a block. Thus $R \cap L \notin Syl_2(L)$ by C.2.4.1, so by C.2.4.2, $R \leq N_M(L)$. Then by C.2.2.3, $R \in \mathcal{B}_2(LR)$, so that $O_2(LR) \leq R$ by C.2.1.2. Further $Z(R) \leq O_2(LR)$ as $F^*(LR) = O_2(LR)$. Then as $L \notin \mathcal{L}_f(G,T)$, L centralizes $\Omega_1(Z(O_2(LR))) \geq \Omega_1(Z(R)) =: Z_R$, so

$$L \leq C_M(Z_R) \leq C(M,R) \leq H.$$

Thus we conclude $O_{2,E}(M) \leq H$.

Next set $F := O_{2,F}(M)$. By C.2.6, $R \in Syl_2(FR)$ and either

(i) $FR \leq H$, and hence also $O_{2,F^*}(M) \leq H$, or

(ii) $FR = (FR \cap H)X_0$, where X_0 is the product of A_3-blocks X subnormal in M with $X = [X, J(R)]$.

If (i) holds, then $O_2(M) = O_2(M \cap H) = O_2(H)$ by A.4.4.1, so that conclusion (1) holds. Thus we may assume (ii) holds. Therefore $M = M_c$ and $|Z| > 2$ by 14.1.10, so it remains to show that $M_c = !\mathcal{M}(H)$. Let $K := C_M(X_0)$. Then from (ii) and (*) in the proof of 14.1.10, we see that $KT = C_G(Z)$ and $O_{2,F^*}(K) = O_{2,F^*}(M) \cap H \leq H$, so that $O_{2,F^*}(KH) \leq H$.

We conclude from A.4.4.1 that $O_2(KH) = O_2(H) = R$. Thus $C_G(Z) = KT \leq C(M,R) \leq H$, so that $M_c = !\mathcal{M}(H)$ by 14.1.5.2. This completes the proof that (2) holds. □

LEMMA 14.1.14. *If $M_1 \in \mathcal{M}(T) - \{M\}$, then $O_2(M) < O_2(M_1 \cap M) > O_2(M_1)$.*

PROOF. Let $H := M \cap M_1$, $R := O_2(H)$, $\{M_2, M_3\} = \{M, M_1\}$, and assume that $R = O_2(M_2)$. Then $C(G,R) \leq M_2$, so that $C(M_3,R) \leq H$. Thus by 14.1.13, either $R = O_2(M_3)$, or $M_3 = M_c = !\mathcal{M}(H)$. In the first case, $O_2(M_2) = R = O_2(M_3)$, so $M_1 = M$, contrary to the choice of M_1. In the second case as $H \leq M_2$, $M = M_1$ for the same contradiction. □

LEMMA 14.1.15. $M = !\mathcal{M}(O_{2,F^*}(M)T)$.

PROOF. Suppose $M_1 \in \mathcal{M}(O_{2,F^*}(M)T)$ and let $H := M \cap M_1$. As $O_{2,F^*}(M) \leq H$, $O_2(M) = O_2(H)$ by A.4.4.1. Thus $M = M_1$ by 14.1.14. □

LEMMA 14.1.16. *If $T \leq H \leq M$ with $1 \neq O_2(H)$ and $C(M, O_2(H)) = H$, then $M = !\mathcal{M}(H)$.*

PROOF. Suppose $M_1 \in \mathcal{M}(H) - \{M\}$. Then $|\mathcal{M}(H)| > 1$, so $O_{2,F^*}(M) \leq H$ by 14.1.13, contrary to 14.1.15. □

LEMMA 14.1.17. *Let $M_1 \in \mathcal{M}(T) - \{M\}$ and assume either*

(a) $M_1 \lesssim M$ and $V = V(M)$, or

(b) $M_1 = M_c$.

Let $R := O_2(M_1 \cap M)$, assume there is T-invariant subgroup Y_0 of M with $\bar{Y}_0$ of odd order, and set $Y := O^2(\langle R^{Y_0T}\rangle)$ and $M^* := M/O_2(M)$. Then

(1) $\bar{R} \neq 1$.

(2) $\bar{Y} = [\bar{Y}_0, \bar{R}]$.

(3) $[C_M(V)^*, Y^*R^*] = 1$ and $C_Y(V)^* \leq O(Z(C_M(V)^*R^*))$.

(4) If $1 \neq r^* \in R^*$ is faithful on $O_p(M^*)$ for some odd prime p, then $C_{M^*}(r^*)$ has cyclic Sylow p-groups, so $m_p(C_M(V)) \leq 1$.

(5) $R = O_2(C_M(V)R)$, so $N_{\bar{M}}(\bar{R}) = \overline{N_M(R)}$.

PROOF. In case (a), $M_1 \lesssim M$ and $V = V(M)$, so $C_M(V) \leq M_1$ by A.5.3.3. In case (b), $M_1 = M_c$ and $C_M(V) \leq C_M(Z \cap V) \leq M_c = !\mathcal{M}(C_G(Z))$. So in either case, $C_M(V) \leq M \cap M_1 \leq N_M(R)$. Since $V \in \mathcal{R}_2(M)$ by 14.1.1, it follows that $C_R(V) = O_2(C_M(V)) = O_2(M)$. Then $R = O_2(C_M(V)R)$, and hence (5) holds. Further if $\bar{R} = 1$, then $R = C_R(V) = O_2(M)$, contrary to 14.1.14. Hence (1) is established.

Next by Coprime Action, $\bar{Y}_0 = \bar{Y}_+\bar{Y}_-$, where $\bar{Y}_+ := C_{\bar{Y}_0}(\bar{R})$ and $\bar{Y}_- := [\bar{Y}_0, \bar{R}]$ are T-invariant since Y_0 and R are T-invariant. By (5), $\bar{Y}_+ \leq \overline{N_M(R)}$, so $\bar{Y}_R := \langle \bar{R}^{Y_0T}\rangle = \langle \bar{R}^{\bar{Y}_-}\rangle$ and $\bar{Y}_R = \bar{R}\bar{Y}_-$ with $\bar{Y}_- = \bar{Y}$. In particular (2) holds.

Also $[C_M(V), R] \leq C_R(V) = O_2(M)$, so $[C_M(V)^*, R^*] = 1$, and hence $Y^* = [Y^*, R^*]$ centralizes $C_M(V)^*$, so that (3) holds. Part (4) follows from A.1.31.1 applied to the product of a Sylow p-subgroup of $O_p(M^*)C_{M^*}(r^*)$ with $\langle r^*\rangle$. □

LEMMA 14.1.18. *Let $M := M_f$ as in 14.1.12, and assume $V := V(M)$ is of rank 2 with $\bar{M} \cong S_3$. Let $R_c := O_2(M \cap M_c)$, $Y := O^2(\langle R_c^M\rangle)$, $R := C_T(V)$, and $M^* := M/O_2(M)$. Then*

(1) M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.

(2) $R_cR = T$ and $M \cap M_c = C_M(V)R_c$.

(3) $\bar{Y} = O^2(\bar{M}) \cong \mathbf{Z}_3$ and $O_2(Y) = C_Y(V)$.

(4) $M = !\mathcal{M}(YT)$.

(5) $M^* = Y^*R_c^* \times C_M(V)^*$ with $Y^*R_c^* \cong S_3$ and $m_3(C_M(V)) \leq 1$.

(6) Z is of order 2 and $M_c = C_G(Z)$.

(7) $\mathcal{M}(T) = \{M, M_c\}$.

PROOF. As $V = V(M)$ is of rank 2 and $\bar{M} \cong S_3$, Z is of order 2, so (1) follows from part (1) of 14.1.12. Further $M_c \neq M$ by part (4) of that result, so case (b) of the hypothesis of 14.1.17 holds. By 14.1.17.1, $\bar{R}_c \neq 1$, so as $\bar{T}$ is of order 2, $\bar{R}_c = \bar{T}$ and hence $T = RR_c$. As $C_M(V) \leq C_G(Z) \leq M_c = !\mathcal{M}(C_G(Z))$, but $M \not\leq M_c$, it follows that $M \cap M_c = C_M(V)R_c$, so that (2) holds. Further applying 14.1.17 to the preimage Y_0 in M of $O(\bar{M})$, we conclude $\bar{Y} = \bar{Y}_0$ and $C_Y(V)^* \leq O(Z(C_M(V)^* R_c^*))$. By the first remark, $\bar{M} = \bar{Y}\bar{T}$, so (4) holds by A.5.7.1. By (2) there is $r \in R_c$ inverting $\bar{Y}$, so as $[C_Y(V)^*, R_c^*] = 1$, r inverts y of order 3 in $Y - C_Y(V)$, and $Y^* = [Y^*, R_c^*] = \langle y^* \rangle$. Therefore $Y^* \cong \mathbf{Z}_3$, completing the proof of (3). As $[C_M(V), YR_c] \leq O_2(M)$, the first two statements in (5) hold, while the third follows from 14.1.17.4.

Let $K \in \mathcal{M}(T)$. By (1) and A.5.3.1, $V(K) \leq V$; so as $|V| = 4$, it follows that $V(K) = Z$ or V. In the latter case $K = M$ by A.5.4; in the former, $K \leq C_G(Z) \leq M_c$ so that $K = M_c$, completing the proof of (6) and (7). □

14.2. Starting the $\mathbf{L_2(2)}$ case of $\mathcal{L}_\mathbf{f}$ empty

We now state Hypothesis 14.2.1, which in effect is the special case of Hypothesis 14.1.5 where $V(M_f)$ is of of rank 2, for M_f the member of $\mathcal{M}(T)$ defined in 14.1.12. Namely Hypothesis 14.2.1 implies Hypothesis 14.1.5, and conversely when Hypothesis 14.1.5 holds and $V(M_f)$ is of rank 2, then Hypothesis 14.2.1 is satisfied with M_f in the role of "M" by 14.1.18. Indeed 14.1.18 supplies a normal subgroup Y of M with $YT/O_2(YT) \cong L_2(2)$ and $M =!\mathcal{M}(YT)$. Thus we view Y as a solvable analogue of $L \in \mathcal{L}_f^*(L,T)$, and then Hypothesis 14.2.1 allows us to treat the case $LT/O_2(LT) \cong L_2(2)$ in parallel with the final case in the Fundamental Setup where $L/O_2(L) \cong L_3(2)$.

Thus in this section, and as appropriate in the later sections of this chapter, we assume:

HYPOTHESIS 14.2.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, $Z := \Omega_1(Z(T))$, and*

(1) $\mathcal{L}_f(G,T) = \emptyset$.
(2) $M_c := C_G(Z) \in \mathcal{M}(T)$.
(3) There exists a unique maximal member M of $\mathcal{M}(T)$ under $\lesssim$.
(4) $V := V(M) = \langle Z^M \rangle$ is of rank 2, and $\bar{M} := M/C_M(V) \cong Aut(V) \cong L_2(2)$.
(5) $|\mathcal{M}(T)| > 1$.

We observe that by parts (1), (2), and (5) of Hypothesis 14.2.1, Hypothesis 14.1.5 is satisfied. Indeed by 14.2.1.3 and 14.1.12.1, M is the maximal 2-local M_f containing $N_G(C_2(T))$ appearing in 14.1.12. Then by 14.2.1.4, the hypotheses of 14.1.18 are satisfied. As in 14.1.18, we set

$$R_c := O_2(M \cap M_c) \text{ and } Y := O^2(\langle R_c^M \rangle).$$

Then applying 14.1.18 we conclude:

LEMMA 14.2.2. *(1) $T = C_T(V)R_c$, and $M \cap M_c = C_M(V)R_c$ so that $O^2(M \cap M_c) \leq C_M(V)$.*
(2) $\bar{Y} = O^2(\bar{M}) \cong \mathbf{Z}_3$ and $O_2(Y) = C_Y(V)$.
(3) $M = !\mathcal{M}(YT)$.
(4) $M/O_2(M) = YR_c/O_2(M) \times C_M(V)/O_2(M)$ with $YR_c/O_2(M) \cong L_2(2)$ and $m_3(C_M(V)) \leq 1$.
(5) $\mathcal{M}(T) = \{M, M_c\}$.
(6) $|Z| = 2$, and hence $C_T(Y) = 1$.
(7) $N_G(T) \leq M \cap M_c$.

(8) For each $H \in \mathcal{H}_*(T,M)$, $H \cap M$ is the unique maximal subgroup containing T, and H is a minimal parabolic described in B.6.8, and in E.2.2 when H is nonsolvable.

PROOF. Parts (1)–(6) follow from 14.1.18, so it remains to prove (7) and (8). As $Z = \Omega_1(Z(T))$ is of order 2, $N_G(T) \leq C_G(Z) = M_c$. As $N_G(T)$ preserves $\lesssim$, $N_G(T) \leq M$ by 14.2.1.3, completing the proof of (7). Then (8) follows from (7) just as in the proof of 3.3.2.4. □

For the remainder of the section, H will denote a member of $\mathcal{H}(T,M)$.

Set $M_H := M \cap H$, $U_H := \langle V^H \rangle$, $Q_H := O_2(H)$, and $H^* := H/Q_H$. Let $\tilde{M}_c := M_c/Z$. Since $T \leq H \not\leq M$, we conclude from 14.2.2.5 that:

LEMMA 14.2.3. $C_G(Z) = M_c = !\mathcal{M}(H)$.

In particular $\tilde{H} := H/Z$ makes sense. Next observe using 14.2.2 that:

LEMMA 14.2.4. *Case (2) of Hypothesis 12.8.1 is satisfied with Y in the role of "L".*

In Notation 12.8.2, we have $V_2 = V$, $L_2 = Y$, $V_1 = Z$, and $L_1 = 1$. Defining $\mathcal{H}_z$ as in Notation 12.8.2, 14.2.3 says:

LEMMA 14.2.5. $\mathcal{H}_z = \mathcal{H}(T,M)$.

By 14.2.5, results from section 12.8 apply to H. In particular recall from 12.8.4 that:

LEMMA 14.2.6. *(1) Hypothesis G.2.1 is satisfied.*
(2) $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$ and $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$.
(3) $\Phi(U_H) \leq V_1$.
(4) $Q_H = C_H(\tilde{U}_H)$.

Part (2) of Hypothesis 14.2.1 excludes the quasithin examples $L_3(2)$ and A_6, which will be treated in the final section of the next chapter. In the remainder of this section, we will identify the other quasithin examples corresponding to $\bar{L} \cong L_2(2)$, which do satisfy Hypothesis 14.2.1. These examples arise in the cases where some $H \in \mathcal{H}_*(T,M)$ has one of three possible structures: $n(H) > 1$; $H/O_2(H) \cong D_{10}$ or $Sz(2) \cong F_{20}$; or $H/O_2(H) \cong L_2(2)$. In each case we will show that G possesses a weak BN-pair of rank 2, as discussed in section F.1; then we appeal to section F.1 and the subsequent sections in chapter F of Volume I, to identify G. Then in later sections we show that no further quasithin groups arise under Hypothesis 14.2.1, although certain shadows are eliminated in those sections.

14.2.1. The treatment of $\mathbf{n(H) > 1}$. The first major result of this section is:

THEOREM 14.2.7. *Either*
(1) $n(H) = 1$ for each $H \in \mathcal{H}_*(T,M)$, or
(2) G is ${}^3D_4(2)$, J_2, or J_3.

Until the proof of Theorem 14.2.7 is complete, we assume $H \in \mathcal{H}_*(T,M)$ with $n(H) > 1$. By 14.2.2.8 and E.1.13, the structure of H is described in E.2.2. As $n(H) > 1$, only cases (1a), (2a), or (2b) of E.2.2 can hold. Set $K := O^2(H)$. In each case, we next define a Bender subgroup K_1 of K which, together with Y, will be used to construct our weak BN-pair:

NOTATION 14.2.8. One of the following holds:

(1) $K/O_2(K)$ is $L_2(2^n)$ or $Sz(2^n)$, and we set $K_1 := K$.

(2) $K/O_2(K)$ is the product of two commuting Bender groups interchanged by T, and we choose $K_1 \in \mathcal{C}(H)$.

(3) $K/O_2(K)$ is $(S)L_3(2^n)$ or $Sp_4(2^n)$ for $n \geq 2$, with T inducing an automorphism nontrivial on the Dynkin diagram of $K/O_2(K)$, and we set $K_1 := P_1^\infty$, where $P_i/O_2(K)$, $i = 1, 2$, are the maximal parabolics of $K/O_2(K)$ with $T \cap K \leq P_i$.

Let $S := N_T(K_1)$. In each case in 14.2.8, $K_1/O_2(K_1)$ is a Bender group with $K_1 \in \mathcal{C}(K_1 S)$ and $K_1 \not\leq M$. In case (1), $K = K_1$ and $S = T$, while in cases (2) and (3), $K_1 < K$ and $|T : S| = 2$.

By 14.2.3, $H \leq M_c = C_G(Z)$.

LEMMA 14.2.9. *Assume $S < T$. Then K_1 is contained in some $K_c \in \mathcal{C}(M_c)$, and one of the following holds:*

(1) Case (3) of 14.2.8 holds, and $K = K_c$ is of 3-rank 2, with $K = \langle K_1^R \rangle$ for each $R \in Syl_2(M_c)$ with $S \leq R$.

(2) Case (2) of 14.2.8 holds, $K_1 = K_c$, $K = \langle K_1^T \rangle$, and either K has 3-rank 2, or $K_1/O_2(K_1) \cong Sz(2^n)$.

(3) Case (2) of 14.2.8 holds, $K_1/O_2(K_1) \cong L_2(4)$, $K_c/O_2(K_c) \cong J_1$ or $L_2(p)$ for p an odd prime with $p^2 \equiv 1 \mod 5$, $S = N_T(K_c)$, and $\langle K_1^R \rangle$ is of 3-rank 2 for each $R \in Syl_2(M_c)$ with $S \leq R$.

PROOF. The existence of K_c follows from 1.2.4. In case (3) of 14.2.8, $K/O_2(K) \cong (S)L_3(2^n)$ or $Sp_4(2^n)$, so that $K \in \mathcal{L}^*(G,T)$ by 1.2.8.4—except when $K/O_2(K) \cong L_3(4)$, where $K \in \mathcal{L}^*(G,T)$ by 1.2.8.3, since T is nontrivial on the Dynkin diagram of $K/O_2(K)$. Thus $K \in \mathcal{C}(M_c)$ by 14.1.6.2, so that $K = K_c$, and conclusion (1) holds in this case. In case (2) of 14.2.8, $K_1 \in \mathcal{L}(G,T)$, so by 1.2.8.2, either $K_1 \in \mathcal{L}^*(G,T)$ so that $K_1 = K_c$ and (2) holds; or else (3) holds. □

LEMMA 14.2.10. *If $S < T$, then $M_c = !\mathcal{M}(\langle K_1, T_1 \rangle)$ for each $T_1 \in Syl_2(M_c)$ containing S.*

PROOF. By Sylow's Theorem, $T_1 = T^g$ for some $g \in M_c$. If $K_1 = K_c$, the result follows from 14.1.6.2 applied to T_1 in the role of "T". Thus we may assume $K_1 < K_c$, so that conclusion (1) or (3) of 14.2.9 holds.

Let $H_1 := \langle K_1, T_1 \rangle$ and $M_1 \in \mathcal{M}(H_1)$. By 14.2.2.5, $M_1 = M_c$ or M^g, and we may assume the latter. As case (1) or (3) of 14.2.9 holds, $\langle K_1^R \rangle$ is of 3-rank 2 for each $R \in Syl_2(M_c)$ containing S, so in particular $H_1 = \langle K_1, T_1 \rangle$ is of 3-rank 2. Then $O^2(H_1) \leq O^2(M_c \cap M^g) \leq C_{M^g}(V^g)$ by 14.2.2.1, contrary to 14.2.2.4. □

Let B be a Hall $2'$-subgroup of $K \cap M$, and set $B_1 := B \cap K_1$.

LEMMA 14.2.11. *B acts on K_1, $BT = TB$, $BS = SB$, and $B \leq C_M(V)$.*

PROOF. As $M_H = BT$, $BT = TB$. Then as B acts on K_1, $BS = SB$. As $H \leq M_c$, $B \leq O^2(M \cap M_c) \leq C_G(V)$ by 14.2.2.1. □

LEMMA 14.2.12. *Either $O_2(M) \leq S$, so that $S \in Syl_2(YS)$, or the following hold:*

(1) $K/O_2(K) \cong L_3(4)$, *and some element of* T *induces a graph automorphism on* $K/O_2(K)$.

(2) $B = B_1$ *is of order* 3 *and* $B \leq C_M(V)$.

(3) $K = O^{3'}(M_c^\infty)$.

PROOF. Assume $Q_M := O_2(M) \not\leq S$; in particular, $S < T$, so one of the cases of 14.2.9 holds. Now $Q_M = [Q_M, B]C_{Q_M}(B)$ by Coprime Action, and using A.1.6, $[Q_M, B] \leq [O_2(BT), B] \leq S$. Thus if $C_T(B) \leq S$, then $Q_M \leq S$, contrary to assumption; so $C_T(B) \not\leq S$, and then of the cases in 14.2.9, only conclusion (1) of the present result can hold.

As $K/O_2(K) \cong L_3(4)$ by (1), $B = B_1$ is of order 3. By 14.2.11, $B \leq C_M(V)$, so (2) holds. By 14.2.9, $K = K_c \in \mathcal{C}(M_c)$. By A.3.18, $C_{M_c}(K/O_2(K))$ is a $3'$-group, so (3) holds. □

LEMMA 14.2.13. $O_2(Y) \leq S$.

PROOF. Assume not. Then as $O_2(Y) \leq O_2(M)$, $O_2(M) \not\leq S$, so conclusions (1)–(3) of 14.2.12 are satisfied. In particular $K \in \mathcal{C}(M_c)$ and $K/O_2(K) \cong L_3(4)$. By 14.2.5, $M_c \in \mathcal{H}_z$. Let $U := \langle V^{M_c} \rangle$.

We first show that U is abelian. Suppose not. Let $y \in Y$ be of order 3 and set $I := \langle U^Y \rangle$. We appeal to 12.8.9; recall $V_2 = V$, and Y, I play the roles of "$O^2(P)$, I_2". Thus by 12.8.9.2, $O_2(I) = U_I U_I^y$, where $U_I := U \cap O_2(I)$. By 12.8.9.1, $Y = O^2(I)$ and T acts on I. Thus T acts on $O_2(I) = U_I U_I^y$, so that as $U \leq Q_H$ by 14.2.6.2, U_I^{y*} is a normal elementary abelian subgroup of T^*. Thus as $K^*T^* \leq Aut(L_3(4))$, we conclude $U_I^{y*} \leq K^*$. But then $O_2(Y) \leq O_2(I) = U_I U_I^y \leq S$, contrary to our hypothesis.

Therefore U is abelian. So by 12.8.6.5, Hypothesis F.9.8 is satisfied, for each $H \in \mathcal{H}_z$, with Z, V in the roles of "V_1, V_+". As $K^* \cong L_3(4)$ and T^* is nontrivial on the Dynkin diagram of K^*, H^* has no FF-modules by Theorem B.4.2, so we conclude from (3) and (4) of F.9.18 that there is $\tilde{I} \in Irr_+(K, \tilde{U}_H)$ with $I \trianglelefteq H$ and $q(Aut_H(\tilde{I}), \tilde{I}) \leq 2$. This contradicts B.4.2 and B.4.5. □

Set $S_2 := O_2(Y)(T \cap K)$. We begin to verify the hypotheses of F.1.1 with K_1, YS_2, S in the roles of "L_1, L_2, S": By 14.2.13, $O_2(Y) \leq S$, while $T \cap K \leq S$ by definition, so that $S_2 \leq S$ and $S_1 := S \cap K_1 \in Syl_2(K_1)$. By construction $O_2(Y) \leq S_2$, so that $S \cap YS_2 = S_2 \in Syl_2(YS_2)$. Thus hypothesis (b) of F.1.1 holds. By definition, S acts on K_1. As S acts on K and Y, S acts on YS_2. Thus hypothesis (a) of F.1.1 holds. Next $K_1/O_2(K_1)$ is a Bender group by construction, and so satisfies (c) of F.1.1. Since $Y/O_2(Y) \cong \mathbf{Z}_3 \cong L_2(2)'$, to verify (c) for YS_2 we must show:

LEMMA 14.2.14. $Y = [Y, S_2]$.

PROOF. If not, then $S_2 \trianglelefteq YT$, so Theorem 3.1.1 applied to S_2, YT in the roles of "R, M_0" says $O_2(\langle YT, H \rangle) \neq 1$, contrary to 14.2.2.3. □

Next $N_{K_1}(S_1) = S_1 B_1 =: C_1$ lies in M and so normalizes Y, and hence normalizes YS_2 by construction, and $C_2 := N_{YS_2}(S_2) = S_2$ normalizes K_1. Thus (d) of F.1.1 holds with C_1, C_2 in the roles of "B_1, B_2" ; and (f) of F.1.1 also follows by construction. Therefore it remains to establish hypothesis (e) of F.1.1.

Let $G_1 := K_1S$, $G_2 := B_1YS$, and $G_{1,2} := G_1 \cap G_2 = SB_1$. Consider the amalgam $\alpha := (G_1, G_{1,2}, G_2)$, and let $G_0 := \langle G_1, G_2\rangle$. To establish hypothesis (e) of F.1.1, we need to show:

LEMMA 14.2.15. $O_2(G_0) = 1$.

PROOF. Assume $O_2(G_0) \neq 1$, and let $M_1 \in \mathcal{M}(G_0)$. Then $T \not\leq M_1$, since otherwise by 14.2.2.3, $M = !\mathcal{M}(YT) = M_1$, contrary to $\langle K_1, T\rangle = H \not\leq M$. Thus $S < T$, and hence one of the cases of 14.2.9 holds.

Let $Z_S := \Omega_1(Z(S))$. As T normalizes S, $Z \leq Z_S$. By 14.2.9, $K_1 \leq K_c \in \mathcal{C}(M_c)$. As $O_2(\langle K_c, T\rangle) \leq N_T(K_1) = S$, $Z_S \leq \Omega_1(Z(O_2(\langle K_c, T\rangle)))$. Thus as $\mathcal{L}_f(G,T) = \emptyset$ by 14.2.1.1,

$$K \leq \langle K_c^T\rangle \leq C_G(Z_S), \tag{!}$$

so that $Z_S \trianglelefteq \langle K_c, T\rangle$. Hence $N_G(Z_S) \leq M_c = !\mathcal{M}(\langle K_1, T\rangle)$ by 14.2.10. As $|T : S| = 2$, S is normal in a Sylow 2-subgroup T_1 of M_1, and hence $T_1 \leq N_{M_1}(Z_S) \leq M_c$. If $S < T_1$, then $T_1 \in Syl_2(M_c)$, so $M_c = !\mathcal{M}(\langle K_1, T_1\rangle) = M_1$ by 14.2.10, a contradiction as $M_c \neq M = !\mathcal{M}(YT)$.

So $S \in Syl_2(M_1)$. Therefore we can embed K_1 in some $L \in \mathcal{C}(G_0)$ by 1.2.4. Now $Y = O^2(Y)$ normalizes L by 1.2.1.3, and $S \leq N_G(K_1) \leq N_G(L)$, so $G_0 = \langle K_1S, Y\rangle = LYS$.

Suppose that $L \leq C_G(Z)$. Then L centralizes $V = \langle Z^Y\rangle$, so $\langle L, T\rangle \leq N_G(V) = M$, a contradiction as $M \neq M_c = !\mathcal{M}(\langle K_1, T\rangle)$.

Therefore $[L, Z] \neq 1$. In particular, $K_1 < L$, so as $G_0 = LYS$, $L = [L, Y]$. Let $R := O_2(YS)$. Then $R \trianglelefteq YT$, so $C(G, R) \leq M = !\mathcal{M}(YT)$. Moreover if $Y \not\leq L$ then $YS \cap L = S \cap L$ is Y-invariant, so $S \cap L \leq R$ and hence $R \in Syl_2(LR)$.

Next $C_T(O_2(M_1)) \leq M_1$ as $M_1 \in \mathcal{M}$, and as $S \in Syl_2(M_1)$ and $S \leq M_c$, $O_2(M_1) \leq O_2(G_0) \leq O_2(M_c \cap G_0) \leq S$ by A.1.6, so that

$$C_{O_2(M_c)}(O_2(M_c \cap G_0)) \leq C_T(O_2(G_0)) \leq C_T(O_2(M_1)) \leq S \leq G_0, \tag{$*$}$$

and hence G_0, M_c, S satisfy the hypotheses of 1.1.5 in the roles of "H, M, T_H". By (*), hypothesis (b) of 1.2.11 is satisfied, and since a generator z for Z is in $V = [V, Y]$, hypothesis (a) of 1.2.11 is also satisfied. Thus by 1.2.11, either $G_0 \in \mathcal{H}^e$, or L is quasisimple.

Assume first that L is quasisimple. Then L is described in 1.1.5.3, and $L = [L, z]$. As $K_1/O_2(K_1)$ is a Bender group over $\mathbf{F}_{2^n}$ with $n > 1$, and $K_1 \in \mathcal{L}(C_L(z), S)$, comparing the list of 1.1.5.3 with that of A.3.12, we conclude that either $L/Z(L)$ is $Sp_4(2^n)$, $G_2(2^n)$, ${}^2F_4(2^n)$, or ${}^3D_4(2^{n/3})$, or else $K_1/O_2(K_1) \cong L_2(4)$ and $L \cong J_2$, J_4, HS, or Ru. Then by A.3.18, $O^{3'}(G_0) = L$, so that $G_0 = LYS = LS$.

Suppose first that L is of Lie type. As $YS = SY$, either $L \cong {}^3D_4(2)$, or n is even and Y is contained in the Borel subgroup of L over S. But in the latter case, Y lies in the parabolic P of L with $K_1 = P^\infty$, so $G_0 = \langle K_1S, Y\rangle \leq P < L$, contrary to $G_0 = LS$.

Therefore $L \cong {}^3D_4(2)$ with $K_1/O_2(K_1) \cong L_2(8)$, or J_2, J_4, HS, or Ru with $K_1/O_2(K_1) \cong L_2(4)$; note that case (2) of 14.2.8 holds. Now $1 \neq O_2(G_0) \leq C_S(L)$, so $1 \neq C_{Z_S}(L) =: Z_L$ is in the center of $G_0 = LS$. Thus we may assume $M_1 \in \mathcal{M}(C_G(Z_L))$. Then by (!), $K \leq C_G(Z_L) \leq M_1$. Further $K = O^{3'}(K)$ since $K_1/O_2(K_1) \cong L_2(2^m)$ for some m. By 1.2.4 and 1.2.8.4, $L \in \mathcal{C}(M_1)$, and by A.3.18, $L = O^{3'}(M_1)$. Thus $K \leq L$. However, when L is ${}^3D_4(2)$, J_2, J_4, HS, or Ru, $C_L(z)$ has no subgroup isomorphic to K satisfying 14.2.9—namely containing the product of two conjugates of K_1, since case (2) of 14.2.8 holds—see e.g. 16.1.4 and 16.1.5. This contradiction completes the treatment of the case that L is quasisimple.

Therefore $G_0 \in \mathcal{H}^e$, so $V_0 := \langle Z^{G_0}\rangle \in \mathcal{R}_2(G_0)$ by B.2.14. Then $[V_0, L] \neq 1$ since we saw $[L, Z] \neq 1$. If C is a nontrivial characteristic subgroup of S with $L \leq N_G(C)$, then $H = KT \leq \langle L, T\rangle \leq N_G(C)$, so $L \leq N_G(C) \leq M_c = !\mathcal{M}(H)$ by 14.2.3, contradicting $[L, Z] \neq 1$. Hence no such C exists, so as $L/O_{2,F}(L)$ is quasisimple by 1.2.1.4, $L = [L, J(S)]$. Then appealing to Thompson Factorization B.2.15, V_0 is an FF-module for $LS/C_{LS}(V_0)$, so by Theorems B.5.1 and B.5.6, $L/C_L(V_0) \cong L_2(2^n)$, $SL_3(2^n)$, $Sp_4(2^n)$, $G_2(2^n)$, $L_n(2)$, $\hat{A}_6$, or A_7. As $K_1 < L$ and S acts on K_1 with $n(K_1) > 1$, $L/C_L(V_0)$ is not $L_n(2)$ or a group over $\mathbf{F}_2$ or $\hat{A}_6$, and also L is not a χ_0-block. Further $L/O_2(L)$ is not A_7, since the FF-modules in Theorem B.5.1 do not satisfy the condition $[K_1, Z_S] = 1$ in (!). Therefore $L/C_L(V_0)$ is $SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$, and $K_1/O_2(K_1) \cong L_2(2^n)$ for $n > 1$. Recall $R = O_2(YS)$. If $Y \not\leq L$, then as we observed earlier, $R \in Syl_2(LR)$; while if $Y \leq L$ then Y is contained in a Borel subgroup of L, and then once again, R is Sylow in LR. We also saw $C(G, R) \leq M$, while $L \not\leq M$ as $K_1 \not\leq M$; thus L is a χ_0-block by C.1.29, contrary to an earlier observation. This contradiction completes the proof of 14.2.15. □

LEMMA 14.2.16. *α is a weak BN-pair of rank 2, $K = K_1$, $T = S$, $Q := O_2(K) = O_2(M_c)$ is extraspecial, and either*

(1) α is isomorphic to the ${}^3D_4(2)$-amalgam, $|Q| = 2^{1+8}$, and $K/Q \cong L_2(8)$, or

(2) α is parabolic isomomorphic to the J_2-amalgam or $Aut(J_2)$-amalgam, $|Q| = 2^{1+4}$, and $K/Q \cong L_2(4)$.

PROOF. Recall 14.2.15 completed the verification of Hypothesis F.1.1 with K_1, YS_2, S in the roles of "L_1, L_2, S". Then by F.1.9, α is a weak BN-pair of rank 2. Furthermore we saw $B_2 = S_2$, so α appears in the list of F.1.12. Since $G_2/C_{G_2}(V) \cong S_3$, while K_1 is nonsolvable and centralizes Z, we conclude that α is either isomorphic to the ${}^3D_4(2)$-amalgam, or is parabolic-isomorphic to the J_2-amalgam or the $Aut(J_2)$-amalgam. In each case $Z_S \cong \mathbf{Z}_2$, $\langle Z_S^Y\rangle \cong E_4$, and $Q = O_2(K_1) = O_2(K_1S)$ is extraspecial of order 2^{1+8} or 2^{1+4}, while $K_1/Q \cong L_2(8)$ or $L_2(4)$.

As Z_S is of order 2, $Z = Z_S$. Also K_1 is irreducible on Q/Z, so $Q = O_2(M_c)$ using A.1.6. Further the action of K_1 on Q/Z does not extend to $(S)L_3(2^n)$, $Sp_4(2^n)$, or $L_2(2^n) \times L_2(2^n)$, so as $K = \langle K_1^T\rangle$, case (1) of 14.2.8 holds, so $K = K_1$ and $T = S$. □

We say G is of *type* J_3 or J_2 if α is parabolic isomorphic to the J_2-amalgam, and G has 1 or 2 classes of involutions, respectively.

LEMMA 14.2.17. *Assume α is parabolic isomorphic to the J_2-amalgam or the $Aut(J_2)$-amalgam. Then*

(1) α is parabolic-isomorphic to the J_2-amalgam, and G is of type J_2 or J_3.

(2) If G is of type J_2, then $G \cong J_2$.

(3) If G is of type J_3, then $G \cong J_3$.

PROOF. By 14.2.16, $Q = O_2(M_c)$, so as $Out(Q) \cong S_5$, $KT = M_c = C_G(Z)$.

Assume first that α is parabolic isomorphic to the $Aut(J_2)$-amalgam. Then by 46.1 and 46.11 in [**Asc94**], K has three orbits on involutions in K, with representatives $z \in Z$, $s \in V - Z$, and $t \in K - Q$ with $C_T(t) \in Syl_2(C_{KT}(t))$. Then $s \in z^Y$, so as J_2 has two classes of involutions, $C_T(t)$ is isomorphic to a Sylow 2-subgroup of the centralizer in $Aut(J_2)$ of a non-2-central involution of J_2. Hence $C_T(t) = A\langle k\rangle$, where $A := C_{T\cap K}(t) \cong E_{16}$ and k is an involution acting freely on A. Next as α is parabolic isomorphic to the amalgam of $Aut(J_2)$, there is $j \in T - K$ with $\mathbf{Z}_2 \times D_{16} \cong C_T(j) \in Syl_2(C_{KT}(j))$. Now $N_G(C_T(j))$ normalizes $\Omega_1(\Phi(C_T(j))) = Z$ and hence lies in $C_G(z) = KT$; it follows that $C_T(j)$ is Sylow in $C_G(j)$—for otherwise $C_T(j) < X \in Syl_2(C_G(j))$ so that $C_T(j) < N_X(C_T(j)) \leq C_{KT}(j)$, contrary to $C_T(j) \in Syl_2(C_{KT}(j))$. But $C_T(j)$ does not contain a copy of $C_T(t)$ or T, so $j^G \cap K = \emptyset$. Therefore by Thompson Transfer, $j \notin O^2(G)$, contrary to the simplicity of G.

Therefore α is not parabolic isomorphic to the $Aut(J_2)$-amalgam, so α is parabolic isomorphic to the J_2-amalgam, and $K = C_G(z)$. G is of type J_2 or J_3, completing the proof of (1). Then (2) and (3) follow from existing classification theorems which we have stated in Volume I as I.4.7. □

In view of 14.2.17, to complete the proof of Theorem 14.2.7, it remains to treat the ${}^3D_4(2)$-case. So assume α is the ${}^3D_4(2)$ amalgam. Let $Z = \langle z\rangle$, $\hat{G} := {}^3D_4(2)$, and $\dot{G} := Aut(\hat{G})$.

LEMMA 14.2.18. *Assume α is the ${}^3D_4(2)$-amalgam. Then $M_c = C_G(z)$ and either*

(1) $M_c = K$, or

(2) $M_c = KA$, where $A \leq M_c \cap M$ is of order 3 and induces field automorphisms on K/Q. Moreover $\dot{\alpha} := (M_c, M_c \cap M, M)$ is the $\dot{G}$-extension of α, in the sense of Definition F.4.3.

PROOF. By 14.2.1.2, $M_c = C_G(z)$. By 14.2.16, $Q = O_2(K) = O_2(M_c)$, so M_c/Q is faithful on $\tilde{Q}$ by A.1.8. Now $K \in \mathcal{L}(M_c, T)$ with $K/O_2(K) \cong L_2(8)$, and $T/Q \cong E_8$ is Sylow in M_c/Q, so we conclude from 1.2.4 and A.3.12 that $K \in \mathcal{C}(M_c)$. Then $K \trianglelefteq M_c$ by 1.2.1.3 since $T \leq K$. As the normalizer in $GL(\tilde{Q})$ of K/Q is isomorphic to $Aut(K/Q)$, either (1) holds or $M_c/Q \cong Aut(L_2(8))$, and we may assume the latter. Thus $M_c = KA$ where $A \leq N_G(T)$ is of order 3 and induces field automorphisms on K/Q. Then A acts on $C_{\tilde{Q}}(T) = \tilde{V}$, so $A \leq N_G(V) = M$. As $M_c = KA$, $M \cap M_c = B_1A$, so $M = Y(M \cap M_c) = YTA$. Then $\dot{\alpha} := (M_c, M \cap M_c, M)$ satisfies Hypothesis F.1.1 just as α did, and hence by F.1.9, $\dot{\alpha}$ is a weak BN-pair of rank 2. Then $\dot{\alpha}$ is an extension of its sub-amalgam α, which we have already identified; so $\dot{\alpha}$ is the $\dot{G}$-extension of α. □

LEMMA 14.2.19. *If α is the ${}^3D_4(2)$ amalgam, then $G \cong {}^3D_4(2)$.*

PROOF. Let $\gamma := \alpha$ in case (1) of 14.2.18, and $\gamma := \dot{\alpha}$ in case (2) of 14.2.18. In either case, by 14.2.18, γ is an extension of the ${}^3D_4(2)$-amalgam, with the role of "G_1" played by $M_c = C_G(z)$. Thus the hypotheses of Theorem F.4.31 are satisfied since $G = O^2(G)$, so by that Theorem, G is an extension of ${}^3D_4(2)$ of odd degree, and hence isomorphic to ${}^3D_4(2)$ since G is simple. □

Observe that 14.2.17 and 14.2.19 establish Theorem 14.2.7.

14.2.2. The treatment of certain cases where H is solvable. We next analyze the case where for some $H \in \mathcal{H}_*(T, M)$, $H/O_2(H)$ is either a group of Lie rank 1 over $\mathbf{F}_2$ isomorphic to $L_2(2)$ or $Sz(2) \cong F_{20}$, or $H/O_2(H) \cong D_{10}$. We do not treat the case where $H/O_2(H)$ is $U_3(2)$.

We prove:

THEOREM 14.2.20. *Let* $H \in \mathcal{H}_*(T, M)$. *Then*

(1) If $H/O_2(H) \cong D_{10}$ *or* $Sz(2)$*, then* $H/O_2(H) \cong Sz(2)$ *and* $G \cong {}^2F_4(2)'$.

(2) If $H/O_2(H) \cong L_2(2)$ *then* $G \cong M_{12}$ *or* $G_2(2)'$.

Again we assume that H satisfies one of the hypotheses of Theorem 14.2.20, and we begin a series of reductions.

Let $G_1 := H$, $G_2 := YT$, and $G_0 := \langle G_1, G_2 \rangle$. Then $G_1 \cap G_2 = T$. We check easily that G_1, G_2, T satisfy Hypothesis F.1.1 in the roles of "L_1, L_2, S": For example since $H \not\leq M$, $O_2(G_0) = 1$ by 14.2.2.3. By F.1.9, $\alpha := (G_1, T, G_2)$ is a weak BN-pair of rank 2. Set $K := O^2(H)$.

LEMMA 14.2.21. $H = M_c$, $M = YT$, *and one of the following holds:*

(1) α *is the amalgam of* ${}^2F_4(2)$ *or of the Tits group* ${}^2F_4(2)'$.

(2) α *is the amalgam of* M_{12} *or of* $Aut(M_{12})$.

(3) α *is the amalgam of* $G_2(2)'$ *or of* $G_2(2)$.

PROOF. Since $T = N_{G_i}(T)$, the hypothesis of F.1.12 holds. Since $G_2/C_{G_2}(V) \cong L_2(2)$, while $G_1/O_2(G_1)$ is D_{10}, $Sz(2)$, or $L_2(2)$ with G_1 centralizing Z, we conclude from the list of F.1.12 that either α appears in conclusions (1)–(3) of 14.2.21, or α is the amalgam of $Sp_4(2)$. However in the latter case, $|Z| = 4$, contrary to 14.2.2.6.

Thus it remains to show $M_c = H$ and $M = YT$. If $M_c = H$, then $M \cap M_c = T$, so $C_M(V) = C_T(V)$, and then $M = YT$ by 14.2.2.1. So it suffices to show $M_c = H$.

Let $K_c := O^2(M_c)$. If $K = K_c$, then $H = KT = K_cT = M_c$, so we may assume $K < K_c$, and it remains to derive a contradiction.

Let $Q := O_2(M_c)$. Then $Q \leq Q_H$ by A.1.6, and $F^*(\tilde{M}_c) = \tilde{Q}$ by A.1.8, so

$$Z(\tilde{Q}_H) \leq C_{\tilde{M}_c}(\tilde{Q}) \leq \tilde{Q} \leq \tilde{Q}_H. \qquad (*)$$

Suppose first that α is the amalgam of $G_2(2)'$, $G_2(2)$, or M_{12}. Then $\tilde{Q}_H$ is abelian, so $Q_H = Q$ by (*). Hence if α is the $G_2(2)'$-amalgam, then $Q \cong Q_8 * \mathbf{Z}_4$ is the central product of Q_8 and $\mathbf{Z}_4$. Therefore $O^2(Aut(Q)) \cong A_4 \cong Aut_K(Q)$, so $K = K_c$, contrary to our assumption. Hence α is the amalgam of $G_2(2)$ or M_{12}, so $Q = Q_H \cong Q_8^2$, and hence $Out(Q) \cong O_4^+(2)$. Then as $K < K_c$, $K_c \cong SL_2(3) * SL_2(3)$. Next $YT \cong D_{12}/\mathbf{Z}_4^2$, so $V \leq E \leq Q$, where $E_8 \cong E \trianglelefteq YT$. Hence $N_G(E) \leq M$ by 14.2.2.3. But $\tilde{E}$ is a maximal totally singular subspace of $\tilde{Q}$, so from the structure of K_c, $S_4 \cong Aut_{M_c}(E)$ is the stabilizer in $GL(E)$ of z. Then since Y does not centralize Z, $Aut_G(E) \cong L_3(2)$, contradicting $N_G(E) \leq M = N_G(Y)$.

Assume next that α is the $Aut(M_{12})$-amalgam. Then $Q_K := [Q_H, K] \cong Q_8^2$ and $\tilde{Q}_H \cong E_4$ wr $\mathbf{Z}_2$. Therefore we conclude from (*) that either Q is Q_H or Q_K, or else $\tilde{Q} \cong E_8$ is the maximal abelian subgroup of $\tilde{Q}_H$ distinct from $\tilde{Q}_K$.

Assume this last case holds. Then $Q \cong E_{16}$ and $S_4 \cong H/Q \leq M_c/Q$, with M_c/Q contained in the stabilizer $L_3(2)/E_8$ in $GL(Q)$ of the point Z of Q. Further $T/Q \cong D_8$ is Sylow in M_c/Q, so as $K < K_c$, we conclude $M_c/Q \cong L_3(2)$ acts indecomposably on Q. But then $M_c \in \mathcal{L}_f(G, T)$, contrary to 14.2.1.1.

So $Q = Q_H$ or Q_K, and therefore $\tilde{Q}_K = J(\tilde{Q}) \trianglelefteq \tilde{M}_c$. In either case, $Q_8^2 \cong Q_K \trianglelefteq M_c$. Then as above, $K < K_c$ implies $K_c \cong SL_2(3) * SL_2(3)$. Now from the structure of $Aut(M_{12})$, $J(T) \cong E_{16}$ is normal in YT, so $N_G(J(T)) \leq M = !\mathcal{M}(YT)$. But $N_{K_c}(J(T))$ does not act on V, a contradiction.

Thus it remains to deal with the case where α is the ${}^2F_4(2)$-amalgam or the Tits amalgam. The subgroups G_1 and G_2 are described in section 3 of [**Asc82b**]. In particular $E := [Q_H, Q_H] \cong E_{32}$, and $Z(\tilde{Q}_H) = \tilde{F}$, where $F := C_H(E)$. Further $F = E$ if α the Tits amalgam, while if α is the ${}^2F_4(2)$ amalgam, then $F = \langle v_5 \rangle E$ with $\langle v_5 \rangle := C_{Q_H}(K) \cong \mathbf{Z}_4$. In particular $F \leq Q$ by (*). Next H is irreducible on Q_H/F of rank 4, so $Q = F$ or Q_H. In the former case, F and $E = \Omega_1(F)$ are normal in M; in the latter, $E = [Q, Q]$ and $F = C_Q(E)$ are normal in M.

Now $H/F \leq M_c/F$, with M_c/F contained in the stabilizer $\Lambda \cong L_4(2)/E_{16}$ of Z in $GL(E)$, and $H/F \cong Sz(2)/E_{16}$ or D_{10}/E_{16} contains a Sylow 2-group T/F of M_c/F, with $Q_H/F = O_2(\Lambda)$. Thus $Q_H \trianglelefteq M_c$, so $Q = Q_H$ using (*). Further the Sylow 2-group T/Q of M_c/Q is cyclic, so by Cyclic Sylow-2 Subgroups A.1.38, M_c/Q is 2-nilpotent. Therefore $K_c/Q = O(M_c/Q)$ is of odd order and contains $K/Q \cong \mathbf{Z}_5$; then as $K < K_c$, $K_c/Q \cong \mathbf{Z}_{15}$ from the structure of $L_4(2)$. But by 3.2.11 in [**Asc82b**], H is transitive on the involutions in $Q - F$, so if j is such an involution, then $M_c = HC_{M_c}(j)$ by a Frattini Argument. In particular, j centralizes an element of order 3 in M_c, impossible as K_c/Q of order 15 is regular on $(Q/F)^\#$. This completes the proof of 14.2.21. □

By 14.2.21, α is isomorphic to the amalgam of $\hat{G}$, where $\hat{G}$ is ${}^2F_4(2)$, the Tits group ${}^2F_4(2)'$, $G_2(2)$, $G_2(2)' \cong U_3(3)$, M_{12}, or $Aut(M_{12})$. As G and $\hat{G}$ are both faithful completions of the amalgam α, there exist injections $\beta_J : \hat{G}_J \to G_J$ of the parabolics $\hat{G}_J, G_J$ for each $\emptyset \neq J \subseteq \{1, 2\}$, such that $\beta_{1,2}$ is the restriction of β_i to $\hat{G}_{1,2}$ and $\beta_i(\hat{G}_i) = G_i$ for $i = 1, 2$. We abuse notation and write β for each of the maps β_J. Let $\hat{T} := \beta^{-1}(T)$.

LEMMA 14.2.22. *(1) α is not the amalgam of ${}^2F_4(2)$, $G_2(2)$, or $Aut(M_{12})$.*
(2) If α is the amalgam of $G_2(2)'$, M_{12}, or ${}^2F_4(2)'$, then $G \cong \hat{G}$.

PROOF. First if α is of type ${}^2F_4(2)'$, ${}^2F_4(2)$, or $G_2(2)$, then $G_1 = H = M_c = C_G(Z)$ by 14.2.21, so that the hypotheses of Theorem F.4.31 are satisfied. Then $G \cong \hat{G}$ by F.4.31, and hence as G is simple, α is the amalgam of ${}^2F_4(2)'$ and $G \cong {}^2F_4(2)'$, so that (2) holds.

Thus we may assume that α is of type $G_2(2)'$, M_{12}, or $Aut(M_{12})$.

Suppose first that α is of type $Aut(M_{12})$. Let $R := \beta(\hat{T} \cap O^2(\hat{G}))$. Then $J(T) \cong E_{16}$ is normal in YT and $M = YT$ by 14.2.21, so M controls fusion in $J(T)$ by Burnside's Fusion Lemma A.1.35. Thus for $j \in J(T) - R$, $j^G \cap J(T) \cap R = \emptyset$. But each involution in R is fused into $J(T) \cap R$ under $G_1 \cup G_2$, so $j^G \cap R = \emptyset$, and hence $j \notin O^2(G)$ by Thompson Transfer, contrary to the simplicity of G.

In the remaining cases we appeal to existing classification theorems stated in Volume I: If α is of type M_{12}, then $G \cong M_{12}$ by I.4.6, and if α is of type $G_2(2)'$, then $G \cong G_2(2)'$ by I.4.4. □

Notice 14.2.21 and 14.2.22 establish Theorem 14.2.20.

14.3. First steps; reducing $\langle \mathbf{V^{G_1}} \rangle$ nonabelian to extraspecial

As mentioned at the beginning of the chapter, the work of the previous two sections allows us to treat the most important subcase of the case $\mathcal{L}_f(G, T) = \emptyset$ where $M_f/C_{M_f}(V(M_f)) \cong L_2(2)$ in parallel with the final case $L/O_2(L) \cong L_3(2)$ in the Fundamental Setup (3.2.1). As usual we define an appropriate hypothesis, which excludes the quasithin examples characterized in earlier sections.

Thus in this section, and indeed for the remainder of the chapter, we assume:

HYPOTHESIS 14.3.1. *Either*

(1) Hypothesis 13.3.1 holds with $L/O_2(L) \cong L_3(2)$, *and* G *is not* $Sp_6(2)$ *or* $U_4(3)$*; or*

(2) Hypothesis 14.2.1 holds, and G *is not* J_2, J_3, ${}^3D_4(2)$, *the Tits group* ${}^2F_4(2)'$, $G_2(2)' \cong U_3(3)$, *or* M_{12}.

Observe that in case (1) of Hypothesis 14.3.1, parts (4) and (5) of 13.3.2 say that Hypotheses 13.1.1, 12.2.1, and 12.2.3 are satisfied, and 13.3.1 is satisfied for any $K \in \mathcal{L}_f(G,T)$ with $K/O_2(K) \cong L_3(2)$. Thus we may make use of appropriate results from the previous chapters 12 and 13, including (in view of the exclusions in 14.3.1.1) results depending on Hypotheses 13.5.1 and 13.7.1. Similarly the exclusions in case (2) allow us to make use of results from the previous section 14.2.

As usual, we let $Z := \Omega_1(Z(T))$, $M_V := N_M(V)$, and $\bar{M}_V := M_V/C_M(V)$.

NOTATION 14.3.2. In case (1) of 14.3.1, L is the member of $\mathcal{L}_f^*(G,T)$ appearing in Hypothesis 13.3.1, while in case (2), take $L := O^2(\langle O_2(M \cap M_c)^M \rangle)$. (Thus L plays the role of the group "Y" in section 14.2.)

Observe:

LEMMA 14.3.3. *(1)* $L \trianglelefteq M$.

(2) $M = !\mathcal{M}(LT)$.

(3) $N_G(T) \leq M$, *and each* $H \in \mathcal{H}_*(T,M)$ *is a minimal parabolic described in B.6.8, and in E.2.2 when* H *is nonsolvable.*

(4) V *is a TI-set in* M.

(5) $N_G(V) = M_V$.

(6) If $H \leq N_G(U)$ *for some* $1 \neq U \leq V$, *then* $H \cap M = N_H(V)$.

PROOF. Part (1) follows from 13.3.2.2 in case (1) of 14.3.1, and by construction in case (2). Part (2) follows from 1.2.7.3 or 14.2.2.3, and (3) follows either from Theorem 3.3.1 together with 3.3.2.4, or from parts (7) and (8) of 14.2.2. Further (5) follows from (2); and (4) follows by construction of $M = N_G(V)$ in case (2) of Hypothesis 14.3.1, and from 12.2.2.3 in case (1). Finally as in the proof of 12.2.6, (6) follows from (4) using 3.1.4.1. □

We typically distinguish the two cases of Hypothesis 14.3.1 by writing $L/O_2(L) \cong L_3(2)$ or $L_2(2)'$.

14.3.1. Preliminary results under Hypothesis 14.3.1.

LEMMA 14.3.4. *If there exists* $K \in \mathcal{L}_f(G,T)$, *then*

(1) $K/O_2(K) \cong A_5$ *or* $L_3(2)$.

(2) $K \trianglelefteq KT$ *and* $K \in \mathcal{L}^*(G,T)$.

(3) Each $V_K \in Irr_+(K, R_2(KT), T)$ *is* T*-invariant,* K, V_K *satisfies the FSU, and* V_K *is the natural module for* $K/O_2(K) \cong A_5$ *or* $L_3(2)$.

(4) Case (1) of 14.3.1 holds, so that $L/O_2(L) \cong L_3(2)$.

PROOF. First case (1) of 14.3.1 must hold, since in case (2), $\mathcal{L}_f(G,T) = \emptyset$ by Hypothesis 14.2.1.1. In particular, (4) holds. Further 14.3.1.1 excludes $G \cong Sp_6(2)$ or $U_4(3)$, so $K/O_{2,Z}(K)$ is not A_6 by Theorem 13.8.1. Also we saw Hypothesis 13.5.1 holds, so (1)–(3) follow from 13.5.2. □

LEMMA 14.3.5. *Assume $L/O_2(L) \cong L_2(2)'$ and $H \in \mathcal{H}(T)$ with $|H : T| = 3$ or 5. Then $H \leq M$.*

PROOF. Assume $H \not\leq M$. By 14.3.3.3, $H \not\leq N_G(T)$ so that $H/O_2(H) \cong S_3$, D_{10}, or $Sz(2)$. But the groups G appearing as conclusions in Theorem 14.2.20 are excluded by Hypothesis 14.3.1.2, so we conclude that the lemma holds. □

LEMMA 14.3.6. *Assume $L/O_2(L) \cong L_2(2)'$ and $H \in \mathcal{H}(T, M)$ such that $K := O^2(H) = \langle K_1^T \rangle$ for some $K_1 \in \mathcal{L}(G, T)$. Then*

(1) If $K/O_2(K)$ is of Lie type over $\mathbf{F}_{2^n}$ of Lie rank 1 or 2, then either

(i) $n = 1$, $K/O_2(K) \cong L_3(2)$ or A_6, and T is nontrivial on the Dynkin diagram of $K/O_2(K)$, or

(ii) M does not contain the Borel subgroup of K over $T \cap K$.

(2) If $K/O_2(K)$ is of Lie type over $\mathbf{F}_2$ of Lie rank 2, then $K/O_2(K) \cong L_3(2)$ or A_6, and T is nontrivial on the Dynkin diagram of $K/O_2(K)$.

(3) If $K/O_2(K)$ is of Lie type over $\mathbf{F}_4$, then $KT/O_2(KT) \cong Aut(Sp_4(4))$ or S_5 wr $\mathbf{Z}_2$.

(4) If $K/O_2(K) \cong L_4(2)$ or $L_5(2)$, then T is nontrivial on the Dynkin diagram of $K/O_2(K)$.

(5) $K/O_2(K)$ is not A_7.

(6) $K/O_2(K)$ is not M_{12}, M_{22}, or $\hat{M}_{22}$.

PROOF. Assume that K either satisfies the hypotheses of one of (1)–(4) or is a counterexample to (5) or (6). Then $K/O_2(K)$ is either quasisimple, or else semisimple of Lie type in characteristic 2, and of Lie rank 1 or 2 using Theorem C (A.2.3). Thus as $K = \langle K_1^T \rangle$ with $K \in \mathcal{L}(G, T)$, using 1.2.1.3 we conclude that either $K/O_2(K)$ is quasisimple, or K is the product of two T-conjugates of $K_1 < K$ with $K_1/O_2(K_1) \cong L_2(2^n)$ or $Sz(2^n)$ and $n > 1$.

Assume the hypotheses of (1). We may assume that (ii) fails, so that $M \cap K$ contains the Borel subgroup B of K over $T \cap K$. Let $\mathcal{H}_0$ be the set of subgroups $\langle P, T \rangle$, such that P is a rank one parabolic of K over B. Then $H = \langle \mathcal{H}_0 \rangle$. So as $H \not\leq M$, there exists $H_0 \in \mathcal{H}_0$ with $H_0 \not\leq M$. Then $H_0 = H_2 B$ where $H_2 \in \mathcal{H}_*(T, M)$. Since Hypothesis 14.3.1 excludes the groups in Theorem 14.2.7, we conclude that $n(H_2) = 1$. Hence $K/O_2(K)$ is defined over $\mathbf{F}_2$. Then from the first paragraph, $K/O_2(K)$ is quasisimple. If T is trivial on the Dynkin diagram of K, then H_0 is a rank one parabolic, so as $K/O_2(K)$ is quasisimple and defined over $\mathbf{F}_2$, $|H_2 : T| = 3$ or 5 from the list of such groups $K/O_2(K)$ in Theorem C, contrary to 14.3.5. Thus T is nontrivial on the diagram, so again from that list, conclusion (i) of (1) holds. This completes the proof of (1).

If (2) fails, then conclusion (ii) of (1) must hold, so $B \not\leq M$. In particular, a Cartan subgroup of B is nontrivial, so as $K/O_2(K)$ is defined over $\mathbf{F}_2$, we conclude from the list of Theorem C that $K/O_2(K) \cong {}^3D_4(2)$ and $|B : T \cap K| = 7$. Now $B \leq N_G(T)$ since $Out(K/O_2(K))$ is of odd order, so $B \leq M$ by 14.3.3.3, contrary to the first sentence of this paragraph. Thus (2) is established.

Assume the hypotheses of (3); then by the first paragraph, $K/O_2(K)$ is either quasisimple of Lie rank at most 2, or $L_2(4) \times L_2(4)$. Let B be the T-invariant Borel subgroup of K. By (1), $B \not\leq M$, so there exists $H_2 \in \mathcal{H}_*(T, M)$ with $H_2 \leq BT$. Inspecting the groups in Theorem C defined over $\mathbf{F}_4$, either $B/O_2(B) \cong \mathbf{Z}_3$ or E_9; or $K/O_2(K) \cong U_3(4)$ with $B/O_2(B) \cong \mathbf{Z}_{15}$; or $K/O_2(K) \cong {}^3D_4(4)$ with $B/O_2(B) \cong$

$\mathbf{Z}_3 \times \mathbf{Z}_{63}$. By 14.3.5, any subgroup of order 3 or 5 permuting with T is contained in M, so as $H_2 \leq BT$ but $B \not\leq M$, we conclude that either $K/O_2(K) \cong {}^3D_4(4)$ with $(B \cap M)/O_2(B \cap M) \cong E_9$, or $B/O_2(B) \cong E_9$ and T is irreducible on $B/O_2(B)$. In the latter case, the irreducible action of T implies that (3) holds. In the former, $m_3(K \cap M) = 2$. However by 14.2.2.5, $K \leq M_c$, so $O^2(K \cap M) \leq C_M(V)$ by Coprime Action, whereas $m_3(C_M(V)) \leq 1$ by 14.2.2.4. This completes the proof of (3).

Finally suppose $K/O_2(K)$ is one of the groups in (4)–(6), and T is trivial on the Dynkin diagram of $K/O_2(K)$ in (4). Then in each case H is generated by the set $\mathcal{H}_1$ of T-invariant subgroups H_2 with $H_2/O_2(H_2) \cong L_2(2)$. Thus $H \leq M$ by 14.3.5, completing the proof of 14.3.6. □

Next recall from our discussion at the beginning of the section that in case (1) of Hypothesis 14.3.1, Hypotheses 12.2.3 and 13.3.1 hold, so case (1) of Hypothesis 12.8.1 holds. Further by 14.2.4, case (2) of Hypothesis 12.8.1 holds in case (2) of Hypothesis 14.3.1. Thus we can appeal to the results in section 12.8, and we adopt Notation 12.8.2 from that section. In particular V_i is the T-invariant subspace of V of dimension i for $i \leq \dim(V)$, $G_i := N_G(V_i)$, $L_i := O^2(N_L(V_i))$, $R_i := O_2(L_iT)$, etc.

Notice $V_1 = Z \cap V$, and indeed in case (2) of 14.3.1, $V_1 = Z$ by 14.2.1.4, and so $G_1 = M_c$ by 14.2.1.2. Recall $\tilde{G}_1 := G_1/V_1$, and by 12.8.3.4,

$$G_1 \not\leq M, \text{ so } G_1 \in \mathcal{H}(T, M).$$

Observe since LT induces $GL(V)$ on V that:

LEMMA 14.3.7. *$M_V = LC_M(V) = L(M \cap G_1)$. In particular if $M \cap G_1 = L_1T$ and $V \trianglelefteq M$, then $M = LT$.*

LEMMA 14.3.8. *Assume $L/O_2(L) \cong L_3(2)$. If $H \leq G_1$ with $HL_i = L_iH$ for $i = 1, 2$, then $H \leq M$.*

PROOF. First $V_1^{L_2H} = V_1^{HL_2} = V_1^{L_2}$, so H acts on $\langle V_1^{L_2} \rangle = V_2$. Similarly $V_2^{L_1H} = V_2^{HL_1} = V_2^{L_1}$, so H acts on $\langle V_2^{L_1} \rangle = V$, so $H \leq N_G(V) \leq M$ by 14.3.3.5. □

LEMMA 14.3.9. *Assume $L/O_2(L) \cong L_3(2)$. Then*

(1) If $J(R_1) \not\leq O_2(LT)$ then there exists $A \in \mathcal{A}(R_1)$ and $g_i \in L$ with $A^{g_i} \leq T$ but $A^{g_i} \not\leq R_i$ for $i = 1, 2$.

(2) If $J(T) \leq R_1$ then $J(T) \trianglelefteq LT$.

(3) If $J(T) \not\leq O_2(LT)$ then $J_1(T) \not\leq R_i$ for $i = 1, 2$.

PROOF. Notice (1) implies (2): For if $J(T) \leq R_1$, then $J(T) = J(R_1)$ by B.2.3.3, so $J(R_1) \leq O_2(LT)$ assuming (1), and hence $J(T) = J(R_1) = J(O_2(LT)) \trianglelefteq LT$.

Assume $J(R_1) \not\leq O_2(LT)$. Then there is $A \in \mathcal{A}(R_1)$ with $\bar{A} \neq 1$, and either $\bar{A}$ has rank 1, or $\bar{A} = \bar{R}_1$ has rank 2. Since $\bar{R}_1$ is not a strong FF*-offender on V, in the latter case B.2.9.2 says we may make a new choice of A so that $\bar{A}$ has rank 1. Then there exists g_i as claimed. Thus (1) and hence (2) are established, so it remains to prove (3).

Assume the hypothesis of (3), so there is $D \in \mathcal{A}(T)$ with $\bar{D} \neq 1$. Now as $m(\bar{D}) \leq 2$, we may choose B of index at most 2 in D, with $C_D(V) \leq B$ and $\bar{B}$ of rank 1. Thus for either choice of $i = 1, 2$, there exists $g_i \in L$ with $B^{g_i} \leq T$ but $B^{g_i} \not\leq R_i$. Hence (3) holds. □

14.3.2. Preliminary results for the case $\langle \mathbf{V^{G_1}} \rangle$ is nonabelian. When $\langle V^{G_1} \rangle$ is nonabelian, we will concentrate on G_1, as opposed to an arbitrary member of $\mathcal{H}_z$; recall the latter set was defined in Notation 12.8.2.3. Thus in the remainder of this section, and indeed in the subsequent section 14.4, we assume:

HYPOTHESIS 14.3.10. *Assume Hypothesis 14.3.1 with $U := \langle V^{G_1} \rangle$ nonabelian. Take $H := G_1$.*

Observe that U plays the role of "U_H" in Notation 12.8.2; in particular by 12.8.4.2, $\tilde{U}$ is elementary abelian.

Since U is nonabelian, we also adopt the notation of the second subsection of section 12.8. Since $H \not\leq M$, $V < U$. Write $Q := O_2(H)$, rather than Q_H as in section 12.8, set $H^* := H/Q$, $Z_U := Z(U)$, $\hat{H} := H/Z_U$, $\dot{H} := H/C_H(\hat{U})$, pick $g \in N_L(V_2) - H$, let $I_2 := \langle U^{L_2} \rangle$, $W := C_U(V_2)$, and $E := W \cap W^g$. Let $d := m(\hat{U})$.

By 12.8.8.1, $U = U_0 Z_U$ with U_0 extraspecial and $\Phi(U_0) = V_1$, and $\dot{H}$ preserves a symplectic form on $\hat{U}$ of dimension d. By 12.8.12, this action satisfies Hypothesis G.10.1, with $\dot{H}$, $\hat{U}$, $\hat{V}_2$, $\hat{E}$, $\dot{W}^g$, $\dot{Z}_U^g$ in the roles of "G, V, V_1, W, X, X_0", and Hypothesis G.11.1 is also satisfied. Thus we may make use of results from sections G.10 and G.11. Recall also from G.10.2 that the bound (*) of sections G.7 and G.9 holds, so that we may apply the results of section G.9.

By 12.8.8.3:

LEMMA 14.3.11. *$m(\hat{V}) = m(\tilde{V})$.*

LEMMA 14.3.12. *Assume $m(\dot{W}^g) \leq d/2 - 1$. Then*

(1) $m(\dot{W}^g) = d/2 - 1$.

(2) $m(\hat{E}) = d/2$.

(3) $Z_U = V_1$, so U is extraspecial, $\hat{U} = \tilde{U}$, and $\dot{H} = H^*$.

(4) H preserves a quadratic form on $\tilde{U}$ of maximal Witt index in which $\hat{E}$ is totally singular.

PROOF. As $m(\dot{W}^g) \leq d/2 - 1$, the first inequality in G.10.2 is an equality with $\dot{Z}_U^g = 1$. Thus (1) holds. Then (1) and 12.8.11.5 imply (2). As $\dot{Z}_U^g = 1$, $Z_U = V_1$ by 12.8.13.4. Thus U is extraspecial by 12.8.8.1, so $\hat{U} = \tilde{U}$. By 12.8.4.4, $Q = C_H(\tilde{U})$, so $H^* = \dot{H}$. Thus (3) holds. Also $\Phi(Z_U) = \Phi(V_1) = 1$, so by 12.8.8.2, H preserves a quadratic form $q(\tilde{u}) := u^2$ on $\tilde{U}$. By 12.8.11.2, $\Phi(E) = 1$, so $\tilde{E}$ is a totally singular subspace of the orthogonal space $\tilde{U}$, of rank $d/2$ by (2). Thus $\tilde{U}$ is of maximal Witt index, completing the proof of (4). □

LEMMA 14.3.13. *$\dot{H}$ and its action on $\hat{U}$ satisfy one of the conclusions of Theorem G.11.2, but not conclusion (1), (4), (5), or (12).*

PROOF. By 12.8.12.4, $\dot{H}$ and its action on $\hat{U}$ satisfy one of the conclusions of G.11.2. By (6) and (7) of 12.8.13, conclusions (4) and (12) are not satisfied. If conclusion (5) is satisfied, then by 12.8.13.5, there is $K \in \mathcal{L}_f(G, T)$ with $K/O_2(K) \cong A_8$, contrary to 14.3.4.1.

Assume conclusion (1) is satisfied. Then $d = 2$ and $\dot{H} \cong S_3$. By 14.3.11, $m(\hat{V}) = m(\tilde{V})$, so if $L/O_2(L) \cong L_3(2)$, then $m(\hat{U}) = 2 = m(\hat{V})$, and hence

$U = VZ_U$, contradicting U nonabelian. Thus $L/O_2(L) \cong L_2(2)'$. Here by 12.8.13.3, $m(\tilde{Z}_U) = m(\dot{Z}_U^g) \leq m_2(\dot{H}) = 1$, so by Coprime Action, $O^2(C_H(\hat{U}))$ centralizes $\tilde{U}$, and then by (2) and (4) of 12.8.4, $C_H(\hat{U}) = C_H(\tilde{U}) = Q$. Then $H^* \cong \dot{H} \cong S_3$, so that $|H : T| = 3$, and then 14.3.5 contradicts $H \not\leq M$. □

LEMMA 14.3.14. *One of the following holds:*
(1) $m(\dot{W}^g) = d/2 - 1$, so that the conclusions of 14.3.12 hold.
(2) $d = 4$ and $m(\dot{W}^g) \geq 2$. Further $\dot{H}$ contains A_5 or $S_3 \times S_3$.
(3) $d = 6$, $\dot{H} \cong G_2(2)$, and $m(\dot{W}^g) = 3$.

PROOF. If $m(\dot{W}^g) \leq d/2 - 1$, then (1) holds by 14.3.12. Thus we may assume $m(\dot{W}^g) \geq d/2$. But by 14.3.13, $\dot{H}$ and $\hat{U}$ appear in one of the cases of G.11.2 other than (1), (4), (5), and (12). Thus as $m_2(\dot{H}) \geq d/2$, case (2), (6), or (7) of G.11.2 holds. Case (6) of G.11.2 gives conclusion (3), and case (2) gives conclusion (2) as $m_2(\dot{H}) \geq 2$ and $\dot{H}$ is a subgroup of $Sp_4(2)$ whose order is divisible by 10 or 18. Finally in case (7) of G.11.2, $\dot{W}^g \not\leq E(\dot{H})$, so $m(\dot{W}^g) \leq 3 < d/2$, contrary to assumption. □

LEMMA 14.3.15. *Assume $L/O_2(L) \cong L_2(2)'$. Then $U \cong Q_8^2$ and $H^* \cong O_4^+(2)$ with $\tilde{E}$ totally singular.*

PROOF. Suppose first that $\dot{H}$ is not solvable. Then appealing to 14.3.13, and inspecting the list of G.11.2, there exists a component $\dot{K}_1$ of $\dot{H}$ isomorphic to $L_2(4)$, A_6, $G_2(2)'$, A_7, $L_2(8)$, or $\hat{M}_{22}$. By 1.2.1.4 we may choose $K \in \mathcal{L}(G,T)$ with $K/O_2(K)$ quasisimple and $\dot{K} = \dot{K}_1$, although K may not be in $\mathcal{C}(H)$; set $K_0 := \langle K^T \rangle$. From G.11.2, either $K = K_0$, or conclusion (7) of G.11.2 holds and $K_0/O_2(K_0) \cong \Omega_4^+(4)$. Further if $\dot{K} \cong A_6$, then from G.11.2, T is trivial on the Dynkin diagram of $K/O_2(K)$. Finally if $\dot{K}_0 \cong \Omega_4^+(4)$, then $K_0T/O_2(K_0T)$ is not S_5 wr $\mathbf{Z}_2$ since $N_{Sp(\hat{U})}(\dot{K}_0)$ is a proper subgroup of index 2 in S_5 wr $\mathbf{Z}_2$. We conclude using 14.3.6 that $K = K_0 \cong L_2(8)$, and $K \cap M = T$. However $Out(L_2(8))$ is of odd order, so $N_K(T)$ is a Borel subgroup of K. Then as $N_G(T) \leq M$ by 14.3.3.3, $K \cap M > T$, contrary to the previous remark.

This contradiction shows that $\dot{H}$ is solvable. Thus in view of 14.3.13, $\dot{H}$ and its action on $\hat{U}$ are described in conclusion (2) or (3) of G.11.2. Indeed $\dot{H}$ and $\hat{U}$ are described in Theorem G.9.4 if H is irreducible on $\hat{U}$, and in G.10.5.2 if H is not irreducible on $\hat{U}$.

Suppose first that $V_1 = Z_U$. Then arguing as in the proof of (3) and (4) of 14.3.12, U is extraspecial with $\hat{U} = \tilde{U}$, and $\dot{H} = H^*$ preserves the quadratic form on $\tilde{U}$ in which $\tilde{E}$ is totally singular. In particular if $d = 4$ and $\dot{H}$ has order divisible by 9, then as 9 does not divide $|O_4^-(2)|$, $U \cong Q_8^2$ and so $\dot{H} = H^*$ lies in $O_4^+(2)$; further by 12.8.9.2, W^g/E is a natural $L_2(2)$-module, so that $\dot{W}^g = W^{*g}$ has rank 2. So since $H \not\leq M$, 14.3.5 reduces cases (1)–(4) of G.9.4 and G.10.5.2 to $H^* \cong O_4^+(2)$, so that the lemma holds. Otherwise we have case (5) of G.9.4, with H^* a subgroup of $SD_{16}/3^{1+2}$ acting irreducibly on $O_3(H^*)/Z(O_3(H^*))$. Let X be the preimage in H of $Z(O_3(H^*))$; again $X \leq M$ by 14.3.5. This is impossible, since $\tilde{U} = [\tilde{U}, X]$ in G.9.4.5, so that X does not act on the subspace $\tilde{V}$ of rank 1.

Thus $V_1 < Z_U$. Hence by 14.3.12.3, $m(\dot{W}^g) \geq d/2$, so case (2) of 14.3.14 holds as $\dot{H}$ is solvable; that is, $d = 4$, $m(\dot{W}^g) \geq 2$, and $\dot{H}$ contains $S_3 \times S_3$. It follows that $\dot{H} \cong S_3 \times S_3$ or $O_4^+(2)$, and $m(\dot{W}^g) = 2 = m_2(O_4^+(2))$.

Next by 12.8.13.3, $\dot{Z}_U^g \cong \tilde{Z}_U$, so $\dot{Z}_U^g \neq 1$. Let $K := O^2(\langle Z_U^{gH} \rangle)$. By 12.8.13.3, Z_U^g centralizes Z_U, so K centralizes Z_U. Thus using Coprime Action, $O^2(C_K(\hat{U}))$ centralizes $\tilde{U}$, and hence $1 \neq \dot{K} \cong K^*$ by parts (2) and (4) of 12.8.4. So if $\dot{H} \cong S_3 \times S_3$ then $K \leq M_V$ by 14.3.5 and 14.3.3.6, so K centralizes $\hat{V}$ of order 2. But then as $K \trianglelefteq H$, K centralizes $\hat{U} = \langle \hat{V}^H \rangle$, contrary to $\dot{K} \neq 1$.

Therefore $\dot{H} \cong O_4^+(2)$. By 12.8.12.2, $\dot{Z}_U^g \trianglelefteq \dot{T}$, so $O_3(\dot{H}) = [O_3(\dot{H}), \dot{Z}_U^g]$, and hence $O_3(\dot{H}) \leq \dot{K}$. Thus $K^* \cong E_9$, so $\hat{U} = [\hat{U}, K]$. Then as K centralizes Z_U, $\tilde{U} = [\tilde{U}, K] \oplus \tilde{Z}_U$ with $\tilde{Z}_U \neq 0$. Further as $\dot{Z}_U^g \trianglelefteq \dot{T}$ and $W \leq C_G(V) \leq C_G(Z_1^g)$, $[Z_U^g, W] \leq Z_U^g \cap W$. As $L/O_2(L) \cong L_2(2)'$, $Z(I_2) = 1$ by 12.8.13.3. Thus $Z_U^g \cap W = V_1^g$ by 12.8.10.3, so as $\dot{Z}_U^g$ acts nontrivially on the hyperplane $\hat{W}$ of $\hat{U}$ and centralizes $\tilde{Z}_U$,

$$\tilde{V} = \tilde{V}_1^g \leq [\tilde{W}, Z_U^g] \leq [\tilde{U}, K],$$

so $\tilde{U} = \langle \tilde{V}^H \rangle \leq [\tilde{U}, K]$, contradicting $0 \neq \tilde{Z}_U \not\leq [\tilde{U}, K]$. Thus the proof of 14.3.15 is complete. □

14.3.3. Eliminating $\mathbf{L_2(2)}$ when $\langle \mathbf{V}^{\mathbf{G_1}} \rangle$ is nonabelian. Recall that $\langle V^{G_1} \rangle$ is nonabelian in the quasithin examples for $L/O_2(L) \cong L_2(2)'$ characterized in section 14.2; but of course those groups are now excluded in Hypothesis 14.3.1.

Thus in this subsection we prove:

THEOREM 14.3.16. *Assume Hypothesis 14.3.10. Then case (1) of Hypothesis 14.3.1 holds, namely $L/O_2(L) \cong L_3(2)$.*

REMARK 14.3.17. In proving Theorem 14.3.16, we will be dealing in effect only with the shadows of extensions of $U_4(3)$ which interchange the two classes of 2-locals isomorphic to A_6/E_{16}. These extensions satisfy our hypotheses except they are not simple, and sometimes not quasithin. Thus we construct 2-local subgroups which appear in those shadows, and eventually achieve a contradiction by showing $O^2(G) < G$ using transfer.

Until the proof of Theorem 14.3.16 is complete, assume G is a counterexample. Thus case (2) of 14.3.1 holds, so $V_1 = Z = \langle z \rangle$, $V = V_2$ and $G_2 = N_G(V) = M$. Recall $Q = O_2(H)$. By 14.3.15, $U \cong Q_8^2$, and $\tilde{U}$ has an orthogonal structure over $\mathbf{F}_2$ preserved by $H = G_1$, with $H^* = H/Q = O(\tilde{U}) \cong O_4^+(2)$ and $\tilde{E}$ totally singular. Thus H is a $\{2,3\}$-group, so in particular, H is solvable.

Recall $I_2 = \langle U^{L_2} \rangle = \langle U^L \rangle$, and by 12.8.9.1, $I_2 \trianglelefteq G_2 = M$ and $L = O^2(I_2)$.

LEMMA 14.3.18. *(1) $V = Q \cap U^g$.*

(2) $I_2 \trianglelefteq M$, $R := O_2(I_2) = O_2(L)$.

(3) R^* is the 4-subgroup of T^* containing no transvections, and hence lying in $\Omega_4^+(\tilde{U})$; so $|T : RQ| = 2$.

(4) $R = AA^t$, where $A \cong E_{16}$ and A^t are the maximal elementary abelian subgroups of R, $|R| = 2^6$, $t \in T - RQ$, $V = A \cap A^t$, and $A \trianglelefteq I_2Q$.

(5) $U = O_2(O^2(H))$.

(6) $N_H(A)^* = C_{H^*}(A^*) \cong \mathbf{Z}_2 \times S_3$.

PROOF. First I_2 plays the role of "I" in 12.8.8; then by 12.8.8.4 we may apply G.2.3.4 to conclude that $E = W \cap W^g$ is T-invariant. But we saw $\tilde{E}$ is totally singular and $H^* = O(\tilde{U}) \cong O_4^+(2)$, so T acts on no totally singular 2-subspace of

$\tilde{U}$; hence $\tilde{E}$ has rank 1, and so $V = E$. On the other hand, $U^g \cap Q \leq U^g \cap G_1 = W^g$, and $Q \cap W^g = E$ by 12.8.9.5. Thus (1) holds.

Recall $I_2 \trianglelefteq G_2 = M$ and $L = O^2(I_2)$. As $V = E$ by (1) and $|Q| = 2^5$, 12.8.9.2 says $R/V = W/V \oplus W^g/V$ is the sum of $m(W/V) = 2$ natural modules for $I_2/R \cong L_2(2)$. Therefore $R^* = W^{g*} \cong E_4$ and $R = [R, L] \leq L$, so that $R = O_2(L)$ as $L \trianglelefteq I_2$. Thus (2) holds. Recall Hypothesis G.10.1 is satisfied; then (3) follows from part (d) of G.10.1 and the fact that transvections in $O(\tilde{U})$ have nonsingular centers.

As R/V is the sum of two natural modules for I_2/R, R has order 2^6, and I_2 has three irreducibles $R(i)/V$, $1 \leq i \leq 3$, on R/V. As $W/V = C_{R/V}(U)$, each $R(i)$ contains some $r_i \in W - V$. Since $U \cong Q_8^2$, $W \cong \mathbf{Z}_2 \times D_8$. Thus we can choose notation so that $\langle r_i \rangle V \cong E_8$ for $i = 1$ and 2, and $\mathbf{Z}_4 \times \mathbf{Z}_2$ for $i = 3$. Then as I_2 is transitive on $(R(i)/V)^\#$, $R(1) \cong R(2) \cong E_{16}$ and $V = \Omega_1(R(3))$. It follows that $A := R(1) \cong E_{16}$ and $A' := R(2)$ are the maximal elementary abelian subgroups of R, $AA' = R$, and A^* is of order 2 in T^*, with $C_{\tilde{U}}(A^*) = \widetilde{A \cap U}$ a totally singular line. Thus $A^* \neq Z(T^*)$, so $A^t = A'$ for $t \in T - RQ$. Therefore (4) holds as A is I_2-invariant by construction, and $C_{H^*}(A^*) \cong \mathbf{Z}_2 \times S_3$.

Next $[W^g, Q] \leq W^g \cap Q = V$ using (1), so $O^2(H) = [O^2(H), W^g]$ centralizes Q/U, and hence (5) holds. Thus if H_A is the preimage of $C_{H^*}(A^*)$, $O^2(H_A)$ acts on AU and hence on $A = J(AU)$, completing the proof of (6). □

From now on, let A be defined as in 14.3.18.4. We will show next that $A_6/E_{16} \leq N_G(A) \leq S_6/E_{32}$. Set $D := C_Q(U)$.

LEMMA 14.3.19. *Let* $K := \langle O^2(N_H(A)), L\rangle$. *Then*

(1) $Q = UD$.

(2) Either

(i) $[A, D] = 1$ *with* $Aut_T(A) \cong D_8$, *or*

(ii) D *induces the transvection on* A *with axis* $A \cap U$ *and center* V_1.

(3) $Aut_{RQ}(A) \in Syl_2(Aut_G(A))$, *and* $Aut_{RQ}(A) \cong D_8$ *or* $\mathbf{Z}_2 \times D_8$.

(4) K *is an* A_6*-block and* $A = O_2(K)$.

(5) $C_G(K) = 1$.

(6) $N_G(K) = KD$, *and* D *is a subgroup of* D_8, *with* $D \cong D_8$ *iff* $|N_G(K) : K| = 4$ *and* $A < C_G(A)$.

(7) $N_G(A) = N_G(K)$.

(8) $RU \in Syl_2(K)$.

(9) K *splits over* A.

(10) $Aut(K) = K\langle \alpha, \beta\rangle$, *with* $A\langle\alpha\rangle = C_{Aut(K)}(A) \cong E_{32}$ *the quotient of the permutation module for* K/A *modulo the fixed space of* K/A, β *is an involution inducing a transposition on a complement to* A *in* K, *and* $D_8 \cong \langle\alpha, \beta\rangle = C_{Aut(K)}(U)$.

PROOF. By 12.8.4.2, Q centralizes $\tilde{U}$, while as $U \cong Q_8^2$, $Inn(U) = C_{Aut(U)}(\tilde{U})$ by A.1.23, so (1) holds. Next D centralizes the hyperplane $A \cap U$ of A, and $[A, D] \leq C_A(U) = V_1$, so (2) holds.

As $A \cong E_{16}$, $Aut_K(A) \leq Aut_G(A) \leq GL(A) \cong L_4(2)$. As $Z = V_1 = C_A(U)$ and $RQ \in Syl_2(N_H(A))$, $Z = C_A(RQ)$, and hence $N_{N_G(A)}(RQ) \leq H$, so that $RQ = N_T(A) \in Syl_2(N_G(A))$. From 14.3.18.6, $Aut_{RU}(A) \cong D_8$ and $C_H(A)$ is a 2-group, so (2) implies (3), and as $Z \leq A$, $C_G(A) = C_H(A)$ is a 2-group.

By 14.3.18.4, $A \trianglelefteq I_2Q$, so that as $L = O^2(I_2)$, $K \leq N_G(A)$. Indeed from 14.3.18, $Aut_{I_2}(A) \cong S_4$, $A = [A, L]$, and setting $Y_A := O^2(N_H(A))$, $Aut_{RY_A}(A) \cong S_4$ is of index at most 2 in the stabilizer in $Aut_G(A)$ of V_1. We conclude from the structure of $L_4(2)$ that $Aut_K(A)$ is A_6. Hence as $C_G(A)$ is a 2-group, and as $K = O^2(K)$ and K is $C_G(A)$-invariant by definition, it follows that $K = O^2(KC_G(A))$ and $K/O_2(K) \cong A_6$. Then as $[R, C_G(A)] \leq C_R(A) = A$, $K = [K, R]$ centralizes $C_G(A)/A$, so K is an A_6-block. Next $R = [R, L]$ and $U = [U, Y_A]$, so $RU \leq K$. Then as $R/A \cong E_4$ is elementary abelian, K/A does not involve the double cover of A_6, so $A = O_2(K)$, and $N_G(K) \leq N_G(A)$, completing the proof of (4). As $RU \leq K$ and $|RU| = 2^7 = |K|_2$, (8) is established. For $u \in U - R$ an involution, u acts on a complement B to V in A^t, so $B\langle u \rangle$ is a complement to A in RU, and hence (9) holds using Gaschütz's Theorem A.1.39. Let K_0 be a complement to A in K.

Let $J := Aut(K)$ and $A_0 := C_J(A)$. By (9), with 17.2 and 17.6 in [**Asc86a**], A_0 is elementary abelian with $A_0/A \cong H^1(K_0, A)$. Hence $A_0 \cong E_{32}$ by I.1.6.1. Further by 17.7 in [**Asc86a**] and a Frattini Argument, $J = A_0J_0$, where $J_0 := N_J(K_0)$, and of course J_0 is the subgroup of $Aut(K_0)$ stabilizing the representation of K_0 on A, so $J_0 \cong S_6$. Thus (10) holds.

Recall $N_T(A) \in Syl_2(N_G(A))$, so $N_T(A) \in Syl_2(N_G(K))$. As $L = O^2(P)$ where P is the minimal parabolic of K with $A = [A, P]$, $C_T(K) = C_T(L)$ from the structure of $Aut(K)$ described in (10). Further $C_T(L) = 1$, since $Z \cong \mathbf{Z}_2$ by 14.2.2.6, while Z is not centralized by L. Thus $C_T(K) = 1$, so (5) holds since $C_G(K) \leq C_H(A)$ and we saw $C_H(A)$ is a 2-group.

As $N_G(K) \leq N_G(A)$ with K transitive on $A^\#$, by a Frattini Argument, $N_G(K) = KN_H(A) = KQR$. Thus $N_G(K) = KD$ by (1) and (8). Now (6) follows from (10).

As $N_T(A)$ acts on K and is Sylow in $N_G(A)$, $K \leq K_1 \in \mathcal{C}(N_G(A))$. Then we conclude from the structure of $GL_4(2)$ and A.3.12 that either $K = K_1$ or $Aut_{K_1}(A) \cong A_7$, and the latter case is impossible as we saw $Aut_H(A)$ is solvable. Thus $K \trianglelefteq N_G(A)$ by 1.2.1.3, so (7) holds as we saw $N_G(K) \leq N_G(A)$. □

LEMMA 14.3.20. $A = C_G(A)$.

PROOF. Let $A_1 := C_G(A)$ and suppose $A < A_1$. Let $G_A := N_G(A)$. Then $G_A \leq Aut(K)$ by (5) and (7) of 14.3.19, so we conclude from the structure of $Aut(K)$ described in (10) of 14.3.19 that $E_{32} \cong A_1 = O_2(G_A)$. As the element t defined in 14.3.18.4 acts on $N_T(A)$, $A_1^t \leq G_A$. As $[A, A^t] \neq 1$, $[A, A_1^t] \neq 1$. By B.3.2.4, $K/O_2(K) \cong A_6$ contains no FF*-offenders on A_1, so $A_1 = J(KA_1)$. Thus $A_1^t \not\leq KA_1$, so $|G_A : K| = 4$, and hence $G_A = KA_1A_1^t \cong Aut(K)$ and $D = C_Q(U) \cong D_8$ using 14.3.19.6.

Next by 14.3.19.10, the action of G_A/A_1 on A_1 is described in section B.3. In the notation of that section, $z = e_{5,6}$, so as $UA/A = O_2(C_K(z)/A)$, $D \cap A_1 = C_{A_1}(U) = \langle e_5, e_6 \rangle$. In particular $d := e_6 \in A_1 \cap D$ with $K_d := C_K(d)$ an A_5-block, and $C_{G_A}(d) \cong S_5/E_{32}$. Further $O^2(H)$ centralizes D as $D \cong D_8$. As $[d, RQ] = V_1$, $RQ = DC_{RQ}(d)$, so $C_{RQO^2(H)}(d) = C_{RQ}(d)O^2(H) \cong (S_3 \times S_3)/(Q_8^2 \times \mathbf{Z}_2)$. Let $T_d := C_T(d)$ and $S_d := RQ \cap T_d$. As $O^2(H)$ centralizes d, $T_d \in Syl_2(C_H(d))$. Further $Z(T_d) = Z(S_d) = \langle z, d \rangle$, and $|T_d : S_d| \leq |T : RQ| = 2$. As H is a $5'$-group by 14.3.15, $d \notin z^G$. Thus as $dz \in d^K$, z is weakly closed in $Z(T_d)$, so that $N_G(T_d) \leq G_1 = H$, and hence T_d is Sylow in $G_d := C_G(d)$.

Now $z \notin O_2(G_d)$: for otherwise $A = \langle z^{K_d} \rangle \leq O_2(G_d)$, impossible as $A \not\leq O_2(C_H(d))$. Thus as K_d is irreducible on A, $A_1 \cap O_2(G_d) = \langle d \rangle$. Now $T_d \in Syl_2(G_d)$,

$O_2(G_d) \leq S_d$, and $A_1 = O_2(K_dS_d)$, so we conclude that $\langle d\rangle = O_2(G_d)$. Let $\check{G}_d := G_d/\langle d\rangle$. Next $K_d \in \mathcal{L}(G_d, S_d)$ and $|T_d : S_d| \leq 2$, so $K_d \leq L_d \in \mathcal{C}(G_d)$ by 1.2.5. As $O_2(G_d) = \langle d\rangle$, $G_d \notin \mathcal{H}^e$, so L_d is quasisimple by 1.2.11 applied with V, G_d in the roles of "U, H". As the hypotheses of 1.1.6 are satisfied with G_d in the role of "H", L_d is described in 1.1.5.3. As $C_{\check{G}_d}(\check{z})$ has a subgroup of index at most 2 isomorphic to $(S_3 \times S_3)/Q_8^2$, we have a contradiction to the 2-local structure of the groups on that list. □

LEMMA 14.3.21. *(1) If $|T : RU| = 2$, then there exist involutions in $T - RU$.*
(2) No involution in $T - RQ$ is in z^G.
(3) All involutions in RU are in z^G.

PROOF. First $RU \in Syl_2(K)$ by 14.3.19.8, and K is transitive on $A^\#$, while all involutions in $K - A$ are fused into A^t, so (3) holds.

Assume $|T : RU| = 2$. As $I_2 = LU$ by G.2.3.2 and $I_2/R \cong S_3$, $LT/R \cong S_3 \times \mathbf{Z}_2$. Further for X of order 3 in I_2, $C_R(X) = 1$. Thus $C_{O_2(LT)}(X) = \langle t_X\rangle$ with t_X an involution in $T - RU$, proving (1).

It remains to prove (2). So suppose some $t \in T - RQ$ is of the form $t = z^y$ for some $y \in G$. Let $I_t := C_{I_2}(t)$, $R_t := R\langle t\rangle$, and $R_t^+ := R_t/V$. By 14.3.18, $A \cap A^t = V$, and A, A^t are the maximal elementary abelian subgroups of R, so that $V = \Omega_1([R,t]) \geq \Omega_1(C_R(t))$ and R is transitive on $[A^+, t^+]t^+$; hence

(*) Each coset of V in $[R,t]\langle t\rangle$ not contained in $[R,t]$ contains a conjugate of t.

We claim that $z \in Q^y$. First consider the case where $[V,t] = 1$. Here $R_t = C_{I_2R_t}(V) \trianglelefteq I_2R_t$. Further by (*), each element of $[R,t]t$ is an involution, so that t inverts $[R,t]$; hence $C_R(t) = V$ and R is transitive on $[R,t]t$. Thus R is transitive on the involutions in Rt, so that $I_t/C_R(t) \cong S_3$. As $C_R(t) = V$, we conclude $I_t \cong S_4$. Therefore $V = [V, O^2(I_t)] \leq U^y$. In particular, $z \in Q^y$, as claimed. Now consider the case where $[V,t] \neq 1$. Then by Exercise 2.8 in [**Asc94**], R is transitive on involutions in Rt and $|C_R(t)| = 8$, so since $\Omega_1(C_R(t)) \leq V$, we conclude $C_R(t) \cong Q_8$. Then as H/Q has no Q_8-subgroup, $z \in Q^y$, completing the proof of the claim.

By the claim, $z \in Q^y$. Thus $t \in \Phi(C_{U^y}(z))$. This is a contradiction as $t \notin RQO^2(H)$ which is of index 2 in H. □

LEMMA 14.3.22. *$D = Z$, so $U = Q$.*

PROOF. Notice by 14.3.19.1 that $U = Q$ if $D = Z$. So we assume $Z < D$, and will derive a contradiction.

As $A = C_G(A)$ by 14.3.20, $|D| \leq 4$ by 14.3.19.6. So $|D| = 4$, and we take $d \in D - Z$. By 14.3.19.10, $C_{Aut(K)}(U) \leq \langle \alpha, \beta\rangle$ where $\langle\alpha\rangle A = C_{Aut(K)}(A)$ and $\langle\beta\rangle A^t = C_{Aut(K)}(A^t)$. Thus $d \neq \alpha$ or β as A is self-centralizing in G, so d induces $\alpha\beta$ on K, and hence $D = \langle d\rangle \cong \mathbf{Z}_4$.

As $D \trianglelefteq H = C_G(d^2)$, D is a TI-set in G. Then the standard result I.7.5 from the theory of TI-sets says that $X := \langle D^G \cap T\rangle$ is abelian. Now L is transitive on $V^\#$ and $D \leq C_T(V) = O_2(LT)$, so $V \leq \Omega_1(\langle D^L \cap T\rangle) \leq X$. Then $X \leq C_T(V)$ so X is weakly closed in $O_2(LT)$, and hence $X \trianglelefteq LT$. Then as $M = !\mathcal{M}(LT)$, $N_G(X) = N_M(X) = LN_{H\cap M}(X)$ using 14.3.7. As X is abelian and weakly closed, we may apply Burnside's Fusion Lemma A.1.35 to conclude $D^G \cap T = D^{N_G(X)} = D^L$ is of order 3.

Let $G_A^+ := G_A/A$. From the structure of $Aut(K)$ in 14.3.19.10, since $A = C_G(A)$, $G_A^+ \cong S_6$ with $d^+ = (5,6)$. Recall $g \in N_L(V_2) - H$, so that $w := dd^g d^{g^2}$ is an involution with $w^+ = (1,2)(3,4)(5,6)$, and hence $X \cong \mathbf{Z}_4^2 \times \mathbf{Z}_2$, with $\Omega_1(X \cap U) = V$. Now I_2 acts on $\Omega_1(X) = V \times \langle w \rangle$, so as $[A, w] = V$ and $A \leq R$, $[R, w] = V$. Therefore R is transitive on Vw, so by a Frattini Argument, $I_2 = RC_{I_2}(w)$, and hence $C_{I_2}(w)/C_R(w) \cong S_3$. Further $|C_R(w)| = |R|/4 = |X \cap R|$, so $C_R(w) = X \cap R \cong \mathbf{Z}_4^2$. Also for $u \in U - R$, $d^{+g} d^{+g^2} = [d^{+g}, u]$, so $d^g d^{g^2} \equiv [d^g, u] \mod V$ since $V = X \cap A$. Then $[d^g, u] \in (X \cap U) - V$, so $[d^g, u]$ is of order 4 as $\Omega_1(X \cap U) = V$. Thus $d^g d^{g^2} \in U$ has order 4 and hence as $O^2(H)$ centralizes d,

$$C_{O^2(H)}(w) = C_{O^2(H)}(d^g d^{g^2}) \cong \mathbf{Z}_4 * SL_2(3).$$

Further choosing T so that $T_w := C_T(w) \in Syl_2(C_H(w))$, $\Omega_1(Z(T_w)) = \langle w, z \rangle$ and $wz \in w^U$.

Set $G_w := C_G(w)$. As $C_R(w) \cong \mathbf{Z}_4^2$, $O^2(C_{I_2}(w)) \cong \mathbf{Z}_3/\mathbf{Z}_4^2$, while by (5) of 14.3.18, $O_2(O^2(H)) = U \cong Q_8^2$ has no $\mathbf{Z}_4^2$-subgroup, so we conclude $w \notin z^G$. Thus as $\Omega_1(Z(T_w)) = \langle w, z \rangle$ and $wz \in w^G$, z is weakly closed in $Z(T_w)$, so that $N_G(T_w) \leq H$ and hence $T_w \in Syl_2(G_w)$.

If $z \in O_2(G_w)$, then $V = \langle z^{C_{I_2}(w)} \rangle \leq Z(O_2(G_w))$, impossible since $V \not\leq Z(\langle V^{C_H(w)} \rangle)$. Thus $z \notin O_2(G_w)$; now $T_w \in Syl_2(G_w)$, $O_2(C_H(w)) \leq G_A$, and z is contained in each nontrivial normal subgroup of $G_w \cap G_A$ other than $\langle w \rangle$, so we conclude that $O_2(G_2) = \langle w \rangle$. As in the the proof of 14.3.20, we appeal to 1.2.11, 1.1.6, and 1.1.5.3; this time from the structure of $C_H(w) = C_{G_w}(z)$ and $C_{I_2}(w)$, we conclude $G_w/\langle w \rangle \cong G_2(2)$, so $G_w \cong \mathbf{Z}_2 \times G_2(2)$ since $G_2(2)$ has trivial Schur multiplier by I.1.3. Set $L_w := G_w^\infty$, and observe that L_w has one class z^{L_w} of involutions, and so the set $\{w\} \cup (zw)^{L_w}$ of involutions in wL_w is contained in w^G since we saw w is conjugate to zw. Also $T_w \cap \langle w \rangle L_w = XC_U(t) \leq RQ$, so $T_w \cap \langle w \rangle L_w = T_w \cap RQ$. By 14.3.21.2, no involution in $T - RQ$ is in z^G, so $z^G \cap G_w = z^{G_w}$, and hence $w^G \cap H = w^H$ since G is transitive on commuting pairs from $z^G \times w^G$. But then as $H/O^2(H)R$ is of order 4 and $w \notin RU$, it follows that $w \notin O^2(G)$ from Generalized Thompson Transfer A.1.37.2, contrary to the simplicity of G. □

We are now in a position to derive a contradiction, and hence establish Theorem 14.3.16. By 14.3.22, $Q = U$, so $|T : RU| = 2$. Thus by 14.3.21.1, there is an involution $t \in T - RU$. By 14.3.21.2, $t \notin z^G$, while by 14.3.21.3, all involutions in RU are in z^G. Thus $t^G \cap RU = \emptyset$, so $t \notin O^2(G)$ by Thompson Transfer, contrary to the simplicity of G.

14.3.4. Characterizing HS by $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ nonabelian but not extraspecial. In this subsection we continue to assume Hypothesis 14.3.10. By Theorem 14.3.16, case (1) of Hypothesis 14.3.1 holds. Thus in the remainder of our treatment of the case U nonabelian in this section and the next, we have $L/O_2(L) \cong L_3(2)$.

In this final subsection, we first prove several more preliminary results, and then reduce to the case where U is extraspecial, by showing HS is the only quasithin example with $V_1 < Z_U$. The treatment of the extraspecial case occupies the following section 14.4.

LEMMA 14.3.23. *$d \geq 4$. If $d = 4$, then*

(1) $\hat{V} = \hat{E} \cong E_4$.

(2) One of the following holds:

(i) $\dot{H} \cong S_3 \times S_3$, *with* $\dot{L}_1 \trianglelefteq \dot{H}$. *Further if* $\dot{Z}_U^g \neq 1$ *then* $\dot{Z}_U^g = C_{\dot{H}}(\hat{V})$ *is of order* 2, *and setting* $K := \langle Z_U^{gH} \rangle$, $\dot{K} \cong S_3$, $\dot{H} = \dot{K}\dot{L}_1\dot{T}$, *and* $K \not\leq M$.

(ii) $\dot{H} \cong S_5$, $\hat{U}$ *is the* $L_2(4)$*-module, and* $\mathbf{Z}_2 \cong \dot{Z}_U^g \leq E(\dot{H})$.

(iii) $\dot{H} \cong A_6$ *or* S_6, *and* $m(\dot{Z}_U^g) = 1$ *or* 2.

(iv) $\dot{H}$ *is* E_9 *extended by* $\mathbf{Z}_2$, $\dot{L}_1 \trianglelefteq \dot{H}$, *and* $U \cong Q_8^2$.

(3) $m(\dot{W}^g/\dot{Z}_U^g) = 1$ *and* Z_U^g *centralizes* $\hat{V}$.

(4) $H > (H \cap M)C_H(\hat{U})$.

PROOF. By 12.8.13.1, $V \leq E$. By 14.3.11, $m(\hat{V}) = m(\tilde{V}) = 2$. But by 12.8.11.2, $\hat{E}$ is totally isotropic in the symplectic space $\hat{V}$, so $2 = m(\hat{V}) \leq m(\hat{E}) \leq d/2$, and hence $d \geq 4$. Further if $d = 4$, these inequalities are equalities, so (1) holds.

Assume $d = 4$. By 12.8.11.5 and (1), $m(\dot{W}^g/\dot{Z}_U^g) = 1$, while by 12.8.13.2, Z_U^g centralizes V, and then by 12.8.11.3, Z_U^g is the kernel of the action of W^g on $\hat{V}$. Thus (3) is established. By 14.3.3.6, $H \cap M$ acts on V; so if (4) fails, then $\dot{H}$ acts on $\hat{V}$, contrary to $\hat{U} = \langle \hat{V}^H \rangle$ and $d = 4$. Thus (4) holds.

Observe that if $\dot{H} \leq O_4^+(2)$, then $O^2(\dot{H})$ is abelian, so $\dot{L}_1 \trianglelefteq O^2(\dot{H})$. Thus $\dot{L}_1 \trianglelefteq O^2(\dot{H})\dot{T} = \dot{H}$. If $O^2(\dot{H})$ is of order 3, then $\dot{H} = \dot{L}_1\dot{T}$, contrary to (4). Thus $O^2(\dot{H}) \cong E_9$, so as $\dot{L}_1 \trianglelefteq \dot{H}$, we conclude $\dot{H} < O_4^+(2)$ in this case.

Suppose first that $m_2(\dot{H}) = 1$. Then by (3) and (4) of 14.3.12, $U \cong Q_8^2$, so $\dot{H} \leq O_4^+(2)$. Then by the previous paragraph, $O^2(\dot{H}) \cong E_9$, so as $m_2(\dot{H}) = 1$, (2iv) holds.

Thus we may assume $m_2(\dot{H}) \geq 2$. Suppose first that $\dot{H} \leq O_4^+(2)$. Then $\dot{H} \cong S_3 \times S_3$ by remarks in paragraph three. Assume that $\dot{Z}_U^g \neq 1$. Then as Z_U^g centralizes $\hat{V}$ by (3), and as $\hat{V} = [\hat{V}, \dot{L}_1]$, $\dot{Z}_U^g$ is of order 2, $\dot{L}_1 = C_{O^2(\dot{H})}(\dot{Z}_U^g)$, $\dot{K} := \langle \dot{Z}_U^{gH} \rangle \cong S_3$, and $\dot{H} = \dot{L}_1\dot{K}\dot{T}$, and so $K \not\leq M$ by (4). This completes the proof that (2i) holds

Thus we may assume that $\dot{H}$ is not contained in $O_4^+(2)$. But by 14.3.14.2, $\dot{H}$ is a subgroup of $Sp_4(2)$ containing $S_3 \times S_3$ or A_5, so we conclude $F^*(\dot{H}) \cong L_2(4)$ or A_6.

Suppose $Z_U = V_1$. Then U is extraspecial, so $\dot{H} \leq O_4^\epsilon(2)$, and by the assumption in previous paragraph, $\epsilon = -1$. This is impossible, as $\tilde{U}$ contains the totally singular line $\tilde{V}$. We conclude $Z_U > V_1$, so $\dot{Z}_U^g \neq 1$ by 12.8.13.4.

Suppose $F^*(\dot{H}) \cong L_2(4)$. As $\dot{L}_1 \trianglelefteq \dot{L}_1\dot{T}$ and $\hat{V} = [\hat{V}, \dot{L}_1] \cong E_4$, it follows that $\hat{U}$ is the $L_2(4)$-module, and $\hat{V}$ is the $\mathbf{F}_4$-line invariant under T. As $\dot{W}^g$ is nontrivial on $\hat{V}$ by 12.8.11.3, $\dot{H} \cong S_5$. Then as $\dot{W}^g$ is elementary abelian and we saw that $1 \neq \dot{Z}_U^g < \dot{W}^g$ and $\dot{Z}_U^g$ centralizes $\hat{V}$, (2ii) holds. A similar argument shows (2iii) holds if $F^*(\dot{H}) \cong A_6$. □

LEMMA 14.3.24. *Assume* $V_1 < Z_U$, *so that* U *is not extraspecial. Then either:*

(1) $d = 6$ *and* $\hat{U}$ *is the natural module for* $\dot{H} \cong G_2(2)$, *or*

(2) $d = 4$ *and one of conclusions (i)—(iii) of 14.3.23.2 holds.*

PROOF. By 14.3.12.3, $m(\dot{W}^g) \geq d/2$, so case (1) of 14.3.14 does not hold. Case (3) of 14.3.14 is conclusion (1), and in case (2) of 14.3.14, $d = 4$ so one of the conclusions of 14.3.23.2 holds, with conclusion (iv) ruled out as there U is extraspecial. □

LEMMA 14.3.25. $Z(LT) \cap U = 1$.

PROOF. Assume $Z_L := Z(LT) \cap U \neq 1$. Set $V_H := \langle Z_L^H \rangle$; then $V_H \leq Z_U$, and as usual $V_H \in \mathcal{R}_2(H)$ by B.2.14. As $Z_L \trianglelefteq LT$ and $M = !\mathcal{M}(LT)$, $C_G(V_H) \leq C_G(Z_L) \leq M$. As $Z_L \neq 1$, $Z_U > V_1$, so by 14.3.24, either $d = 4$ and one of conclusions (i)–(iii) of 14.3.23.2 holds, or $d = 6$ and $\hat{U}$ is the natural module for $\dot{H} \cong G_2(2)$. In any case, $\dot{Z}_U^g \neq 1$ by 12.8.13.4.

Assume first that $\dot{H}$ is not solvable. Then from the previous paragraph, $F^*(\dot{H})$ is quasisimple, so there is $K \in \mathcal{C}(H)$ with $\dot{K} = F^*(\dot{H})$. As K is irreducible on $\hat{U}$ and $\hat{U} > \hat{V}$ in each case, K does not act on $\hat{V}$. Then as $K \cap M \leq M_V$ by 14.3.3.6, $K \not\leq M$. Thus as $C_G(V_H) \leq M$, $[V_H, K] \neq 1$. Therefore $K \in \mathcal{L}_f(G,T)$ by 1.2.10, and then as $\dot{K}$ is not $L_3(2)$ from 14.3.24, $K/O_2(K) \cong L_2(4)$ by 14.3.4.1. Thus case (ii) of 14.3.23.2 holds, so that $\mathbf{Z}_2 \cong \dot{Z}_U^g \leq \dot{K}$; in particular $K = [K, Z_U^g]$. As $m(\dot{Z}_U^g) = 1$, $C_{Z_U^g}(\hat{U})$ is a hyperplane of Z_U^g, so $Z_0 := (Z_U^g \cap Z_U)V_1$ is a hyperplane of Z_U by 12.8.10.6. Thus Z_U^g induces transvections on V_H with axis $Z_0 \cap V_H$. This is impossible, as Z_U^g induces inner automorphisms on $\dot{K}$ and we saw $K = [K, Z_U^g]$.

Therefore $\dot{H}$ is solvable. Hence by the first paragraph, case (i) of 14.3.23.2 holds, so $d = 4$, $\dot{H} \cong S_3 \times S_3$, $\dot{L}_1 \trianglelefteq \dot{H}$, and setting $K := \langle Z_U^{gH} \rangle$, $\dot{K} \cong S_3$, $\dot{H} = \dot{K}\dot{L}_1\dot{T}$, and $K \not\leq M$. Then $K \cap M \leq (K \cap T)C_K(\hat{U})$, since the latter group is maximal in $KC_H(\tilde{U})$. Set $H^+ := H/C_H(V_H)$. As in the previous paragraph, Z_U^g induces transvections on V_H with axis $Z_0 \cap V_H$. By the first paragraph of the proof, $C_K(V_H) \leq M$, so that $C_K(V_H) \leq (K \cap T)C_K(\hat{U})$. Therefore K^+ has the quotient group $\dot{K} \cong S_3$ and $C_K(V_H) \leq C_K(\hat{U})$. Thus we conclude from the structure of SQTK-groups generated by transvections (e.g., G.6.4) that $K^+ \cong S_3$, and hence $C_K(V_H) = C_K(\hat{U})$ and $[V_H, K]$ is of rank 2. Indeed as Z_0 is a hyperplane of Z_U centralized by Z_U^g, $Z_U = [V_H, K] \times C_{Z_U}(K)$ and $C_{Z_U}(K) \trianglelefteq H$. Set $\check{H} := H/C_{Z_U}(K)$ and $H^! := H/C_H(\tilde{U})$; observe that $C_H([V_H, K]) \leq C_H(\check{Z}_U)$, and $\check{U}$ is a quotient of $\tilde{U}$ and so elementary abelian. As $\check{U} = \langle \check{V}_2^H \rangle$ and $\check{V}_2 \leq \Omega_1(Z(\check{T}))$, $O_2(H^!) = 1$ by B.2.13. As $C_K(\hat{U}) = C_K(V_H) \leq C_K([V_H, K]) \leq C_K(\check{Z}_U)$, $C_K(\hat{U})^! \leq O_2(K^!) = 1$. Therefore $C_K(\check{U}) = C_K(\hat{U})$, so $K^! \cong \dot{K} \cong S_3$. Next $[C_H(\hat{U}), K] \leq C_K(\hat{U}) = C_K(V_H)$, so that $[C_H(\hat{U})^+, K^+] = 1$. Then as $End_{K^+}([V_H, K]) \cong \mathbf{F}_2$, $C_H(\hat{U}) \leq C_H([V_H, K]) \leq C_H(\check{Z}_U)$, so $C_H(\hat{U})^! \leq O_2(H^!) = 1$. Therefore $C_H(\hat{U}) = C_H(\check{U})$, and hence $\dot{H} \cong H^!$. Next $\check{U} = \langle \check{V}^H \rangle$, while $\check{V} = [\check{V}, L_1]$ and $L_1^! \trianglelefteq H^!$ as $H^! \cong \dot{H}$, so we conclude $\check{U} = [\check{U}, L_1]$, contrary to $1 \neq [\check{V}_H, K] \leq C_{\check{U}}(L_1)$ since $\check{U}$ is elementary abelian. This contradiction completes the proof of 14.3.25. □

THEOREM 14.3.26. *Assume Hypothesis 14.3.10. Then either $Z_U = V_1$ so that U is extraspecial, or $G \cong HS$.*

REMARK 14.3.27. If Hypothesis 14.3.1 did not exclude the possibility that $K/O_2(K) \cong A_6$ for some $K \in \mathcal{L}_f(G,T)$, then $Sp_6(2)$ would also appear as a conclusion in Theorem 14.3.26. Its shadow will be eliminated during the proof of lemma 14.3.31. Recall that the case leading to $Sp_6(2)$ was treated in Theorem 13.4.1.

Until the proof of Theorem 14.3.26 is complete, assume G is a counterexample. Thus $V_1 < Z_U$. Then by 12.8.13.4, $\dot{Z}_U^g \neq 1$.

Recall $V_2 = V_1V_1^g$. As $L/O_2(L) \cong L_3(2)$ by Theorem 14.3.16, we may choose $l \in C_L(V_1^g)$ with $V = V_2V_1^l$. In particular $V_2^l = V_1^gV_1^l$, so we may apply results from section 12.8 with V_2^l in the role of "V_2". Similarly $V_1V_1^l$ can play the role of "V_2".

LEMMA 14.3.28. *(1)* $Z(I_2^l) = Z_U^g \cap Z_U^l$.
(2) $Z_U \cap Z(I_2^l) = 1$.

PROOF. As $(U, U^g)^l = (U^l, U^g)$, part (1) follows from 12.8.10.2. Then by (1) and 12.8.10.2,

$$Z_U \cap Z(I_2^l) = Z_U \cap Z_U^g \cap Z_U^l = Z(I_2) \cap Z(I_2^l) \le C_U(L) = 1,$$

since $L = \langle L_2, L_2^l\rangle$, and $C_U(L) = 1$ by 14.3.25. □

LEMMA 14.3.29. *Assume there exists* $1 \ne e \in Z(I_2) \cap Z$*, and let* $V_e := \langle e^L\rangle$*. Then*

(1) V_e *is of dimension* 3, 4, 6, *or* 7*, and* V_e *has an quotient* L*-module isomorphic to the dual of* V*.*
(2) $J(T) \trianglelefteq LT$*.*

PROOF. By 12.8.10.2, $Z(I_2) \le Z_U$, so by choice of e and 14.3.25, $[L, e] \ne 1$. Thus $I_2T = C_{LT}(e)$, so $|e^{LT}| = 7$. Thus (1) follows from H.5.3. As usual $VV_e \in \mathcal{R}_2(LT)$ by B.2.14, so as there is a quotient of V_e isomorphic to the dual of V as an LT-module, (2) follows from Theorem B.5.6. □

LEMMA 14.3.30. *(1)* $|Z(I_2)| \le 2$.
(2) If $Z(I_2) \ne 1$*, then the image of* $Z(I_2^l)$ *in* $\dot H$ *is the subgroup of order* 2 *generated by an involution of type* a_2 *in* $Sp(\hat U)$ *with* $[\hat U, Z(I_2^l)] = \hat V$*.*

PROOF. We may assume $Z(I_2) \ne 1$. By 14.3.28.1, $Z(I_2^l) = Z_U^g \cap Z_U^l$, so by 12.8.10.4,

$$[Z(I_2^l), W] \le [Z_U^g, W] \le Z_UV_2 = Z_UV_1^g \quad\text{and}\quad [Z(I_2^l), U \cap H^l] \le Z_UV_1^l.$$

By 12.8.4.1 and G.2.5.1, $\bar U = O_2(\bar L_1)$, so from the action of $\bar L$ on V, $U = C_U(V_1^g)C_U(V_1^l) = W(U \cap H^l)$, and hence $[U, Z(I_2^l)] \le Z_UV$, with $\hat V \cong VZ_U/Z_U$ of rank 2. Thus the image of $Z(I_2^l)$ in $\dot H$ is either trivial, or is $\langle \dot a\rangle$ of order 2, where $\dot a$ is the element of $Sp(\hat U)$ of type a_2 with $[\hat U, \dot a] = \hat V$, and in the latter case (2) holds. We will show that $Z(I_2^l)$ is faithful on $\hat U$. This will prove (1), and complete the proof of (2).

So let $A := C_{Z(I_2^l)}(\hat U)$; we must show $A = 1$. Applying 12.8.10.6 with $V_2 = V_1V_1^g$ and $V_1V_1^l$ in the role of "V_2", $A \le V_1^gZ_U \cap V_1^lZ_U = Z_U$, so $A \le Z_U \cap Z(I_2^l) = 1$ by 14.3.28.2, completing the proof. □

LEMMA 14.3.31. $Z(I_2) = 1$.

PROOF. Assume $Z(I_2) \ne 1$. Then by 14.3.30.1, $Z(I_2) = \langle e\rangle$ is of order 2, and $e \in Z_U$ by 12.8.10.2. Further as T normalizes I_2, $e \in Z$. Let $a := e^l$. By 14.3.30.2, $\dot a$ is the involution in $Sp(\hat U)$ of type a_2 with $[\hat U, \dot a] = \hat V$.

Let $K := \langle a^H\rangle$. Then $[a, Z_U] \le Z_U \cap Z(I_2^l) = 1$ by 14.3.28.2. Thus a centralizes Z_U, so K does too. In particular $\langle K, I_2T\rangle \le C_G(e) =: G_e$. Also then $C_K(\hat U) = C_K(\tilde U) = O_2(K)$ using 12.8.4.4, so $K/O_2(K) \cong \dot K \cong K^*$.

By 14.3.24, either $\hat U$ is the natural module for $\dot H \cong G_2(2)$, or $d = 4$ and one of conclusions (i)–(iii) of 14.3.23.2 holds.

Assume first that one of the cases other than case (i) of 14.3.23.2 holds. Then $F^*(\dot H)$ is simple, so $F^*(\dot H) \le \dot K$ and $K_1 := K^\infty \in \mathcal{C}(H)$ with $\dot K_1 = \dot F^*(\dot H)$. If

$\dot{K}_1$ is $G_2(2)'$ or A_6, then K_1 contains all elements of order 3 in H by A.3.18, so $L = \langle L_1, L_2 \rangle \leq \langle K_1, I_2 \rangle \leq G_e$, contrary to 14.3.25. On the other hand if $\dot{H} \cong S_5$, then $\dot{H}$ contains no involution of type a_2, contrary to 14.3.30.2.

Therefore case (i) of 14.3.23.2 holds. so $\dot{H} \cong S_3 \times S_3$. Since $\dot{a}$ has type a_2, $\hat{U} = [\hat{U}, K]$, and since $[\hat{U}, \dot{a}] = \hat{V}$, $\dot{a}$ centralizes $\hat{V}$, so $\langle \dot{a} \rangle = \dot{Z}_U^g$, $\dot{K} \cong S_3$, $\dot{H} = \dot{K}\dot{L}_1\dot{T}$, and $K \not\leq M$ by 14.3.23.2.

We saw earlier that $K_e := \langle KT, I_2T \rangle \leq G_e$; set $U_e := \langle V_1^{K_e} \rangle$, $K_e^+ := K_e/C_{K_e}(U_e)$, and $\check{K}_e := K_e^+/O_{3'}(K_e^+)$. Then $O_2(K_e^+) = 1$ by B.2.14, so $\alpha := (I_2^+T^+, T^+, K^+T^+)$ is a Goldschmidt amalgam in the sense of Definition F.6.1. Observe that $V_2 = \langle V_1^{I_2} \rangle \leq U_e$, so $U_1 := \langle V_2^K \rangle \leq U_e$. Now $K/O_2(K) \cong S_3$, $\hat{U} = [\hat{U}, K]$, and $[V_2, U] = V_1$; so $F^*(K/C_K(U_1)) = O_2(K/C_K(U_1))$ and hence $F^*(K^+) = O_2(K^+)$.

By 14.3.29.2, $J(T) \trianglelefteq LT$. Hence $J(T) \leq O_2(I_2T)$, and as $K \not\leq M = !\mathcal{M}(LT)$, $J(T) \not\leq O_2(KT)$ in view of B.2.3.3, so $O^2(K) = [O^2(K), J(T)]$. Thus $O_2(K^+T^+) \neq O_2(I_2^+T^+)$, and U_e is an FF-module for K_e^+. By F.6.11.1, $O_{3'}(K_e^+)$ is of odd order, so $K^+T^+ \cong \check{K}\check{T}$ and $I_2^+T^+ \cong \check{I}_2\check{T}$, and hence $F^*(\check{K}) = O_2(\check{K})$. Then as $O_2(K^+T^+) \neq O_2(I_2^+T^+)$, F.6.11.2 says $K_e^+ \cong \check{K}_e$ is described in Theorem F.6.18. As $F^*(\check{K}) = O_2(\check{K})$, cases (1) and (2) of F.6.18 are ruled out. In the remaining cases, $K_e^+ \cong \check{K}_e$ is not solvable, so $K_0 := K_e^\infty \in \mathcal{L}_f(G, T)$ by 1.2.10. Then by 14.3.4.1, $K_0/O_2(K_0) \cong L_3(2)$ since A_5 is not a composition factor of any group in F.6.18. Then $\check{K}_e$ appears in case (6) of F.6.18, so $K_e = K_0Y$ with Y the preimage in K_e of $O_{3'}(K_e^+)$. As $O^2(\check{K}) = [O^2(\check{K}), T \cap K_0]$ and $O^2(K)$ is T-invariant, $O^2(K) \leq K_0$. Similarly $O^2(I_2) \leq K_0$, so $K_0 = O^2(K_e)$ using F.6.6, and hence $K_e = K_0T$. Also $V_2 = \langle V_1^{I_2} \rangle$ and KT centralizes V_1, so by H.5.5, $U_e = \langle V_1^{K_e} \rangle$ is a 3-dimensional natural module for $K_e^+ \cong L_3(2)$. Thus $U_e = \langle V_2^K \rangle$. We saw earlier that $\hat{U} = [\hat{U}, K]$, K centralizes Z_U, and $C_K(\hat{U}) = O_2(K)$. Therefore $\tilde{U} = [\tilde{U}, K] \oplus \tilde{Z}_U$. Now as $\dot{H} = \dot{K}\dot{L}_1\dot{T}$, $U = \langle V_2^{L_1K} \rangle$, so $V_2 \not\leq [K, U]$ and $U_e = \langle V_2^K \rangle$ has rank greater than 3, contradicting $m(U_e) = 3$. $\square$

LEMMA 14.3.32. *(1) $\hat{U}$ is the $L_2(4)$-module for $\dot{H} \cong S_5$.*

(2) $U \cong Q_8^2 * \mathbf{Z}_4$.

(3) $Q = C_H(\hat{U})$, so that $\dot{H} \cong H^*$.

(4) $H = KT$ with $K \in \mathcal{C}(H)$, $U = [O_2(K), K]$, and K acts indecomposably on $\tilde{U}$.

PROOF. By 14.3.31, $Z(I_2) = 1$, so that by 12.8.10.6,

$$C_{Z_U^g}(\hat{U}) = V_1^g \text{ , so } \dot{Z}_U^g \cong \tilde{Z}_U \neq 1. \qquad (*)$$

By 14.3.24, either case 14.3.14.3 holds with $\dot{H} \cong G_2(2)$, or $\dot{H}$ is described in one of cases (i)–(iii) of 14.3.23.2. Then $d = 6$ or 4, respectively. By 12.8.11.2, $m(\hat{E}) \leq d/2$. Then we can use 12.8.11.5 to show $m(\hat{E}) = d/2$ and $m(\dot{W}^g/\dot{Z}_U^g) = d/2 - 1$: For when $d = 4$, $\hat{E} = \hat{V} \cong E_4$ by 14.3.23.1, and when $d = 6$, $m(\dot{W}^g) = 3$ by 14.3.14.3 and $\dot{Z}_U^g \neq 1$ by (*). This also shows $m(\dot{Z}_U^g) = 1$ when $\dot{H} \cong G_2(2)$. When $d = 4$, $m(\dot{W}^g/\dot{Z}_U^g) = 1$ by 14.3.23.3, so as $\dot{Z}_U^g \neq 1$ by (*), either $m_2(\dot{H}) = 2$, $m(\dot{W}^g) = 2$, and $m(\dot{Z}_U^g) = 1$, or case (iii) of 14.3.23.2 holds with $\dot{H} \cong S_6$, $m(\dot{W}^g) = 3$, and $m(\dot{Z}_U^g) = 2$.

Thus in view of (*), we have shown that either $|\tilde{Z}_U| = 2$, or $\dot{H} \cong S_6$ and $|\tilde{Z}_U| = 4$. In either case, H^∞ centralizes Z_U by Coprime Action, and in the former H centralizes $\tilde{Z}_U$. Thus as $H = H^\infty T$ in the latter case, (3) holds by 12.8.4.4.

Suppose next that case (i) of 14.3.23.2 does not hold; we will eliminate that case at the end of the proof. Then there is $K \in \mathcal{C}(H)$ with $K^* = F^*(H^*)$. As K centralizes Z_U and T acts on V_2 with $[V_2, Q] = V_1$, $C_K(\hat{V}_2)^* = C_K(V_2)^*$ by Coprime Action. Then as $H^* \cong \dot{H}$ by (3), $C_K(\hat{V}_2)^*$ acts on Z_U^{g*} and W^{g*} by 12.8.12.2. But when $H^* \cong G_2(2)$, we saw $\dot{Z}_U^g$ has order 2, whereas $C_K(\hat{V}_2)^*$ is the stabilizer of a 4-subgroup of T^*, and in particular does not normalize $Z(T^*)$ of order 2.

Thus $d = 4$, so $\hat{V} = \hat{E}$ by 14.3.23.1. Further since $I_2 \trianglelefteq G_2$ by 12.8.9.1, $C_K(\hat{V}_2)^* = C_K(V_2)^*$ normalizes $E = U \cap U^g$. But in case (iii) of 14.3.23.2, the maximal parabolic $C_K(\hat{V}_2)^*$ does not normalize $\hat{V}$.

Thus we have reduced to case (ii) of 14.3.23.2, so that (1) holds, and also $|Z_U| = 4$. If $Z_U \cong E_4$ then $\dot{H}$ preserves a quadratic form on $\hat{U}$ by 12.8.8.2, which is not the case as here $\hat{U}$ is a natural $L_2(4)$-module. Thus (2) holds.

Next as Q normalizes V_2 with $[V_2, U] = V_1$, $Q = UC_Q(V_1^g)$. By 12.8.9.5, $W^g \cap Q = E$. Thus

$$[Q, W^g] = [U, W^g][C_Q(V_1^g), W^g] \leq U(W^g \cap Q) = U.$$

Then as $K = [K, W^g]$, $[Q, K] \leq U$. If $[U, K] < U$, then $[U, K]$ is extraspecial by (2), impossible as $\hat{U}$ is the $L_2(4)$-module for $\dot{K}$. Thus $U = [U, K] = [O_2(K), K]$, so K is indecomposable on $\tilde{U}$. By (1) and (3), $H = KT$, completing the proof of (4).

Finally we must eliminate case (i) of 14.3.23.2. Here $\dot{L}_1 \trianglelefteq \dot{H}$, so as $\dot{H} \cong H^*$ by (3), $L_1 \trianglelefteq H$, and hence $\tilde{U} = [\tilde{U}, L_1]$ by 12.8.5.1. This is a contradiction as we saw H centralizes $\tilde{Z}_U$. □

LEMMA 14.3.33. *(1)* $P := O_2(L) = \langle Z_U^L \rangle \cong \mathbf{Z}_4^3$, *with* P/V *isomorphic to* V *as an* L*-module.*

(2) $U = O_2(K)$ *and* $PU \in Syl_2(K)$.

(3) $M = L$ *and* $H = KT$ *with* $U = O_2(H)$.

PROOF. By 14.3.32.2, $C_U(V) = VZ_U$, and $Z_U \cong \mathbf{Z}_4$ is centralized by L_1. By 14.3.23.1, $\hat{E} = \hat{V}$, so $V \leq U \cap U^g = E \leq VZ_U$ and hence $E = V(Z_U \cap U^g)$. By (*) in the proof of 14.3.32 and symmetry, $Z_U \cap U^g = V_1$, so $E = V$. By 12.8.8.4, $O_2(LU)/V$ is described in G.2.5; thus as $E = V$ and $m(W/V) = 1$, we conclude that $O_2(LU)/V$ is isomorphic to V as an L-module, and hence $O_2(LU) = \langle Z_U^L \rangle$ and $O_2(LU) = [O_2(LU), L] = O_2(L) = P$. As Z_U is a cyclic normal subgroup of $H = C_G(\Omega_1(Z_U))$, Z_U is a TI-set in G. Further $Z_U \leq C_T(V)$, so $[Z_U, Z_U^y] = 1$ for $y \in L$ by I.7.5, and hence (1) holds.

By (1), $V = \Omega_1(O_2(L)) \trianglelefteq M$. By 14.3.32, $H = KT$, with $KQ/Q \cong A_5$, so $H \cap M = L_1T$, and hence $M = LT$ by 14.3.7.

From the structure of L, $PU = O_2(L_1)$; so as $L_1 \leq O^2(H) = K$, $PU \leq K$. By 14.3.32.4, $U = [O_2(K), K]$, so if $U < O_2(K)$, then $K/U \cong SL_2(5)$; but this is impossible, as the central 2-chief factors of L_1 are in Z_U by (1). Thus $U = O_2(K)$. Then $|PU| = |K|_2$, so (2) holds.

Now $[K, C_T(U)] \leq C_K(U) = Z_U$ with Z_U centralized by K, so $K = O^2(K)$ centralizes $C_T(U)$ by Coprime Action. In particular $C_T(U) = C_T(K)$ since $U \leq K$. Then by (2), $C_T(L) \leq C_T(PU) \leq C_T(U) = C_T(K)$. But $K \not\leq LT = M$, while if $C_T(L) \neq 1$, then $N_G(C_T(L)) \leq M = !\mathcal{M}(LT)$; so we conclude $C_T(L) = 1$. By (2), $C_T(K)$ centralizes PU; so as $PU = O_2(L_1)$, from the structure of $Aut(L)$, $C_T(K) \leq C_T(PU) \leq C_T(L)Z_U = Z_U$. Thus $C_T(U) = C_T(K) = Z_U$.

Let X_1 be of order 3 in L_1. Then $Q = [Q, X_1]C_Q(X_1)$ with $[Q, X_1] = \widetilde{[U, X_1]} \cong Q_8^2$ by 14.3.32. Now if Q_1 is the preimage of an irreducible X_1-submodule of $\widetilde{[Q, X_1]}$, then by 12.8.4.2, $C_Q(X_1)$ normalizes Q_1; further $C_{Q_1}(C_Q(X_1)) > V_1 = C_{Q_1}(X_1)$ by the Thompson $A \times B$-Lemma, so $C_Q(X_1)$ centralizes Q_1 as X_1 is irreducible on $\tilde{Q}_1$. Thus $C_Q(X_1)$ centralizes $[Q, X_1] = [U, X_1]$, so $Z_U = C_T(U) = C_Q(X_1) \cap C_Q(Z_U)$ is of index at most 2 in $C_Q(X_1)$ as $Z_U \cong \mathbf{Z}_4$. Thus either $C_Q(X_1) = Z_U$, or $C_Q(X_1)$ is dihedral or quaternion of order 8.

Suppose first that $C_Q(X_1) = Z_U$. Then $Q = U$, so as $H = KT$, $|H|_2 = 2^9$ by 14.3.32. Hence as we saw $M = LT$, $M = L$ using (1), so (3) holds.

So we assume $C_Q(X_1)$ is of order 8, and it remains to derive a contradiction. [1] Now $C_Q(X_1) \leq O_2(LT)$, so $O_2(LT) = PC_Q(X_1)$. Then as $M = LT$, $M = LC_Q(X_1)$.

For $r \in C_Q(X_1) - U$, r centralizes the supplement $[U, X_1]$ to P in $O_2(L_1)$, so from the structure of $Aut(L_3(2))$, r centralizes L/P. Then by Gaschütz's Theorem A.1.39, we may choose r so that $[r, L] \leq V$. Now as L is irreducible on V, r is an involution, and as $C_T(L) = 1$, P induces the full group of transvections on $V\langle r \rangle$ with axis V. So $L = PC_L(r)$ by a Frattini Argument, and r inverts P.

Let $T_L := T \cap L$, so that T_L is of index 2 in T. As G is simple, Thompson Transfer says there is $g \in G$ with $r^g \in T_L$. We show that any such r^g is not extremal in M; then the standard transfer result Exercise 13.1 in [**Asc86a**] contradicts $r \in O^2(G)$.

As H contains no $L_3(2)$-section, $r^G \cap V = \emptyset$. Thus $r^g \in T_L - P$ as $V = \Omega_1(P)$, and conjugating in L, we may take $r^g \in O_2(L_1) = PU$. By 14.3.32.2, each nontrivial coset of Z_U in U contains exactly two involutions fused under U, and by 14.3.32.1, K is transitive on $\hat{U}^\#$, so K is transitive on involutions in $U - V_1$. Thus as $r^G \cap V = \emptyset$, $r^g \notin U$. Then as $P^* \in Syl_2(K^*)$, $\hat{V} = C_{\hat{U}}(r^g)$; so as PU centralizes Z_U, $C_U(r^g) = Z_U C_V(r^g) = Z_U V_2$. Thus $|U : C_U(r^g)| = 2^3$, so $|C_T(r^g)| \leq 2^7$ as $|T| = 2^{10}$. Therefore if r^g is extremal in M, then $C_T(r^g) = C_T(r)^g$. As V is the natural module for $C_L(r)/V \cong L_3(2)$, $V_1 = Z(C_T(r)) \cap \Phi(C_T(r))$. Then as $V_1 \leq Z(C_T(r^g)) \cap \Phi(C_T(r^g))$, we conclude $g \in H$. This is impossible as $r \in Q = O_2(H)$, while $r^g \in PU$ but $r^g \notin U = Q \cap PU$. This contradiction completes the proof of (3), and hence of 14.3.33. □

At this point, we can complete the identification of G as HS, and hence establish Theorem 14.3.26. Namely by 14.3.33 and 14.3.32, $U = Q = O_2(H) \cong \mathbf{Z}_4 * Q_8^2$ with $H/U \cong S_5$. By 14.3.32.4, $\tilde{U}$ is an indecomposable module under the action of H. Further by 14.3.33, $F^*(M) = P \cong \mathbf{Z}_4^3$, and $M/P \cong L_3(2)$. Thus G is of type HS in the sense of section I.4 of Volume I, so we quote the classification theorem stated there as I.4.8 to conclude that $G \cong HS$.

14.4. Finishing the treatment of $\langle \mathbf{V^{G_1}} \rangle$ nonabelian

In this section, we assume Hypothesis 14.3.1 holds, and continue the notation of section 14.3. In addition, we assume $U := \langle V^{G_1} \rangle$ is extraspecial. In particular, Hypothesis 14.3.10 holds, and we can appeal to results in the later subsections of section 14.3.

[1] Notice we are here eliminating the shadow of $Aut(HS)$.

Theorem 14.3.26 handled the case where U is nonabelian but not extraspecial, so this section will complete the treatment of the case U nonabelian. Recall also by Theorem 14.3.16 that $L/O_2(L) \cong L_3(2)$.

Recall that in Hypothesis 14.3.10, $H := G_1$, $U = \langle V^{G_1}\rangle$, and we can appeal to results in both subsections of section 12.8. Also $g \in N_L(V_2) - H$ and $W := C_U(V_2)$. Let s be the generator of V_1^g. As U is extraspecial, $Z_U = V_1$, so that $\hat{U} = \tilde{U}$, $\dot{Z}_U^g = 1$, $\dot{H} = H^*$, and $d := m(\hat{U}) = m(\tilde{U})$. By 12.8.4.4, $Q := O_2(H) = C_H(\tilde{U})$. Let $K := O^2(H)$. By 12.8.8.2, H^* preserves a quadratic form on $\tilde{U}$, so $H^* \leq O(\tilde{U}) \cong O_d^\epsilon(2)$ for $\epsilon := \pm 1$. Notice $C_H(\tilde{V}_2) = N_H(V_2) \leq G_2$, so since $I_2 \trianglelefteq G_2$ by 12.8.9.1, $C_{H^*}(\tilde{V}_2)$ acts on W^{g*} by 12.8.12.2, and on $\tilde{E}$ since $E = W \cap W^g = W \cap W^l$, where V_1^l is the point of V_2 distinct from V_1 and V_1^g.

As $Z_U^{g*} = 1$, 12.8.11.5 becomes:

LEMMA 14.4.1. $m(\tilde{E}) + m(W^{g*}) = m(\tilde{U}) - 1 = d - 1$.

We next obtain a list of possiblities for H^* and U from G.11.2; all but the second case will eventually be eliminated, although several correspond to shadows which are not quasithin.

LEMMA 14.4.2. *$m(\tilde{E}) = d/2$, so $m(W^{g*}) = d/2 - 1$, $U \cong Q_8^{d/2}$, and one of the following holds:*

(1) $d = 4$ and $H^* \cong S_3 \times S_3$.

(2) $d = 4$ and H^* is E_9 extended by $\mathbf{Z}_2$.

(3) $d = 8$, $\tilde{U}$ is the natural module for $K^* \cong \Omega_4^+(4)$, and $W^{g*} = C_{T^* \cap K^*}(x^*)\langle x^*\rangle$, where $x^* \in W^{g*} - K^*$ interchanges the two components of K^*, and $m([\tilde{U}, x^*]) = 4$.

(4) $d = 8$, $H^* \cong S_7$, $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$, where $\tilde{U}_i$ is a totally singular K-module of rank 4, and $U_1^x = U_2$ for $x \in W^g - N_H(U_1)$.

(5) $d = 8$, $H^* \cong S_3 \times S_5$ or $S_3 \times A_5$, and $\tilde{U}$ is the tensor product of the natural module for S_3 and the natural or A_5-module for $L_2(4)$.

(6) $d = 12$ and $H^* \cong \mathbf{Z}_2/\hat{M}_{22}$.

PROOF. Notice the assertion that $m(W^{g*}) = d/2 - 1$ will follow from 14.4.1 once we show $m(\tilde{E}) = d/2$, as will the assertion that $U \cong Q_8^{d/2}$.

By 14.3.13, H^* and its action on $\tilde{U}$ satisfy one of the conclusions of G.11.2, but not conclusion (1), (4), (5), or (12). Further by 14.3.23: $d \geq 4$, and if $d = 4$ then $\tilde{E} = \tilde{V}$ is of rank $2 = d/2$, so that either (1) or (2) of 14.4.2 holds, since in conclusions (ii) and (iii) of 14.3.23.2, $1 \neq Z_U^{g*}$, contrary to an earlier remark.

Suppose $d = 6$. Then conclusion (3) or (6) of G.11.2 holds. In either case, 27 divides the order of H^*, so $\epsilon = -1$ as 27 does not divide the order of $O_6^+(2)$. Therefore $m(E) \leq m_2(U) = 3$, so $\tilde{E} = \tilde{V}$ is of rank 2 as $V \leq E$ by 12.8.13.1, and hence $m(W^{g*}) = 3$ by 14.4.1. Thus conclusion (3) of G.11.2 does not hold, as there $m_2(H^*) = 2$. In conclusion (6), $C_{H^*}(\tilde{V}_2)$ acts on $\tilde{E}$ of rank 2, impossible as $C_{H^*}(\tilde{V}_2)$ is the stabilizer in H^* of a point of $\tilde{U}$, and so acts on no line of $\tilde{U}$.

In the remaining cases of G.11.2, we have $d = 8$ or 12. So $m(W^{g*}) = d/2 - 1$ by 14.3.14, and thus $m(\tilde{E}) = d/2$ by 14.4.1, completing the proof of the initial conclusions of the lemma as mentioned earlier.

If $d = 12$, then conclusion (13) of G.11.2 holds, and hence conclusion (6) of 14.4.2 holds. Thus we may assume one of conclusions (7)–(11) of G.11.2 holds, where $d = 8$.

Conclusion (10) of G.11.2 is impossible, as $m_3(H^*) = m_3(H) \le 2$ since H is an SQTK-group. As $L_1^*T^* \le H^*$ with $L_1^*T^*/O_2(L_1^*T^*) \cong S_3$, conclusion (11) of G.11.2 does not hold. Conclusions (8) and (9) of G.11.2 appear as conclusion (4) and (5) of 14.4.2. So it remains to show that conclusion (7) of G.11.2 leads to conclusion (3) of 14.4.2. In case (7) of G.11.2, $W^{g^*} \not\le K^*$. Then as we saw $W^{g*} \trianglelefteq C_{H^*}(\tilde{V}_2)$, it follows that $W^{g*} = \langle x^*\rangle Y^*$, where x^* is an involution interchanging the two components of H^*, $m([\tilde{U}, x^*]) = 4$, and $Y^* = C_{T^*\cap K^*}(x^*)$, as desired. □

14.4.1. Characterizing $\mathbf{G_2(3)}$ when $d = 4$. The only quasithin example satisfying Hypotheses 14.3.1 with U extraspecial is $G_2(3)$, occurring when $d = 4$, so our first main result treats this case:

THEOREM 14.4.3. *Assume Hypothesis 14.3.10 with U extraspecial. If $d = 4$, then $G \cong G_2(3)$.*

Until the proof of Theorem 14.4.3 is complete, assume G is a counterexample.

LEMMA 14.4.4. *(1) $K^* \cong E_9$.*

(2) $V = E$ and W^{g*} is of order 2, inverts K^*, and is generated by an involution of type c_2 on $\tilde{U}$.

(3) Either $H^* = K^*W^{g*}$, or $H^* \cong S_3 \times S_3$.

(4) L is an $L_3(2)$-block with $V = O_2(L)$.

(5) $UW^g \in Syl_2(L)$ and $U = O_2(L_1)$.

(6) L does not split over V, and $m_2(UW^g) = 3$.

(7) $H = KT$ and $M = LT$.

PROOF. As $d = 4$, case (1) or (2) of 14.4.2 holds, establishing (1) and (3) since $m(W^{g*}) = 1$ by 14.4.2. By 14.3.23.1, $V = E$. Thus the first two statements in (2) are established. By (1), H is a $\{2,3\}$-group. As $L_1^* \trianglelefteq H^*$ by (1), $\tilde{U} = [\tilde{U}, L_1]$ by 12.8.5.1, so that $U = [U, L_1] \le L$. By 12.8.8.4, $O_2(LU) = O_2(L)$ is described in G.2.5. Therefore since $E = V$ and $m(U/V) = 2 = m_2(O_2(\bar{L}_1))$, $V = O_2(L)$, giving (4); and $\bar{U} = O_2(\bar{L}_1)$ so $U = O_2(L_1)$, and hence $T_L := T \cap L = UW^g$, giving (5).

Let $a \in W^g - U$. Then a inverts L_1^* with $\tilde{U} = [\tilde{U}, L_1]$, so using the structure of $O_4^+(2)$, either the remaining two statements of (2) hold, or a^* is of type a_2, $A := \langle a, [U, a]\rangle \cong E_{16}$, and $H^* = N_H(A)^*L_1^*$. In the latter case, a acts on a complement to V in U, so that UW^g splits over V; then by Gaschütz's Theorem A.1.39, L splits over V. Conversely if L splits over V, then from the structure of the split extension of E_8 by $L_3(2)$, $J(T_L) \cong E_{16}$, so a^* is of type a_2 and $A = J(T_L)$. Thus to complete the proof of (2) and (6), it remains to assume L splits over V, and obtain a contradiction. Set $N^+ := N_G(A)/C_G(A)$; then $[O_2(L_2), L_2] = A$, so that $L_2^+T_L^+ \cong \mathbf{Z}_2 \times S_3$ while $N_K(A)^+ \cong A_4$. Then from the structure of $Aut(A) \cong GL_4(2)$, the subgroup of N^+ generated by L_2T and $N_K(A)$ is isomorphic to A_7. But then the stabilizer of z in this subgroup is $L_3(2)$, contradicting H a $\{2,3\}$-group.

For (7), observe $V = O_2(L) \trianglelefteq M$ by (4). Then as $H = KT$ and $KQ/Q \cong E_9$ by (1), $H \cap M = L_1T$, so that $M = LT$ by 14.3.7. □

LEMMA 14.4.5. *(1) $L = M$.*

(2) $U = O_2(H)$ and $H^* \cong \mathbf{Z}_2/E_9$.

(3) $T = UW^g$.

PROOF. Let $K_1 := \langle W^{gH}\rangle$. By (1) and (2) of 14.4.4, K_1^* is $K^* \cong E_9$ extended by $W^{g*} \cong \mathbf{Z}_2$; so as $V \le W^g$, $U = \langle V^H\rangle \le K_1$. Hence using 14.4.4.5, UW^g of order

2^6 is Sylow in both L and K_1. Then $[K_1, C_T(L)] \leq [K_1, C_T(U)] \leq C_{K_1}(U) = V_1$, so $K \leq C_G(C_T(L))$ by Coprime Action; therefore $C_T(L) = 1$ as $K \not\leq M = !\mathcal{M}(LT)$. Let $A := O_2(M)$. As $M = LT$ and L is an $L_3(2)$-block with $V = O_2(L)$ by parts (4) and (7) of 14.4.4, A is elementary abelian by C.1.13.1, while $m(A/V) \leq \dim H^1(L/V, V) = 1$ by C.1.13.b and I.1.6.4. Thus either (1) holds, or $A \cong E_{16}$ and T/A is regular on $A - V$ from the structure in B.4.8.3 of the unique indecomposable A with $[A, L] = V$. But in the latter case, $A = J(T)$ using 14.4.4.6, and all involutions in $T - L$ are in A. However as $J(T) = A$, $N_G(A) = M$ controls fusion in A by Burnside's Fusion Lemma A.1.35, so $a^G \cap L = \emptyset$ for $a \in A - L$, and then Thompson Transfer contradicts the simplicity of G. [2]

Therefore (1) is established. Now (3) follows from (1) and 14.4.4.5. Then $H = KT = K_1$, and (2) holds. □

We are now in a position to complete the proof of Theorem 14.4.3. We will show G is of $G_2(3)$-type in the sense of section I.4, and then conclude $G \cong G_2(3)$ by the classification theorem stated in Volume I as I.4.5.

First by 14.4.4.4 and 14.4.5.1, $F^*(M) = V \cong E_8$ and $M/V \cong L_3(2)$. Second $U = O_2(H)$ by 14.4.5.2, and as $d = 4$, $U \cong Q_8^2$ by 14.4.2. By 14.4.4.1, $K^* \cong E_9$, so $K = K_1K_2$, where $K_i \cong SL_2(3)$, $[K_1, K_2] = 1$, and $K_1 \cap K_2 = V_1$. By 14.4.5.2, $|H : K| = 2$. Further by 14.4.4.2, W^{g*} inverts K^*; so W^g, and hence also H, acts on K_i. Thus G is of $G_2(3)$-type, completing the proof of Theorem 14.4.3.

14.4.2. Eliminating the case $\mathbf{d > 4}$. Having established Theorem 14.4.3, we may assume for the remainder of this section that $d > 4$; as no quasithin examples arise, we are working toward a contradiction. In fact $d = 8$ or 12 since one of cases (3)–(6) of 14.4.2 holds.

LEMMA 14.4.6. *If a^* is an involution in H^* then either*

(1) $m([\tilde{U}, a^*]) > 2$, or

(2) $H^* \cong S_3 \times S_5$ or $F^*(H) \cong \Omega_4^+(4)$, and in either case $\tilde{V}_2 \not\leq [\tilde{U}, a^*]$ and $m([\tilde{U}, a^*]) = 2$.

PROOF. Assume (1) fails. Then by inspection of cases (3)–(6) in 14.4.2, either:

(a) conclusion (5) of 14.4.2 holds, with $H^* = H_1^* \times H_2^*$ where $H_1^* \cong S_3$, $H_2^* \cong S_5$, $\tilde{U}$ is the tensor product of the natural module for H_1^* and the A_5-module for H_2^*, and a^* is a transposition in H_2^*, or

(b) conclusion (3) of 14.4.2 holds, with a^* inducing an $\mathbf{F}_4$-transvection on $\tilde{U}$.

In case (a), $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$ is the sum of two irreducible H_2^*-modules with $C_{\tilde{U}_i}(T^* \cap H_2^*) = \langle \tilde{u}_i \rangle$ and $\tilde{u}_i$ singular in the orthogonal space $\tilde{U}_i$. Therefore as the generator $\tilde{s}$ of $\tilde{V}_2$ centralizes T^*, $\tilde{s} = \tilde{u}_1 + \tilde{u}_2$. However $[\tilde{U}_i, a^*] = \langle \tilde{v}_i \rangle$ with $\tilde{v}_i$ nonsingular, so $\tilde{s} \notin [\tilde{U}, a^*]$, and hence (2) holds.

Similarly in case (b), $\tilde{V}_2$ is contained in a singular $\mathbf{F}_4$-point of $\tilde{U}$, while $[\tilde{U}, a^*]$ is a nonsingular $\mathbf{F}_4$-point, so again (2) holds. □

LEMMA 14.4.7. *$U = O_2(H) = Q$.*

PROOF. As U is extraspecial, $O_2(H^g) = U^gD$, where $D := C_{H^g}(U^g)$. Now as $g \in N_L(V_2)$, $V_2 \leq U^g$, so $[D, W] \leq C_W(U^g)$. But $C_W(U^g) \leq U^g$ by 12.8.9.5, so that $C_W(U^g) = V_1^g$. Therefore if $D^* \neq 1$, either D induces transvections on $\tilde{U}$ with

[2]Notice here we are eliminating the shadow of $Aut(G_2(3))$.

axis $\tilde{W}$, or $[\tilde{W}, D] = \tilde{V}_1^g = \tilde{V}_2$ with $m([\tilde{U}, d]) \le 2$ for each $d \in D$. This contradicts 14.4.6, so D centralizes $\tilde{U}$, and hence $[D, W] \le V_1 \cap V_1^g = 1$. Recall that W^{g*} is elementary abelian of rank $d/2 - 1 > 1$ by 14.4.2, and this forces $K^* = [K^*, W^{g*}]$ in each of cases (3)–(6) of 14.4.2. Thus by symmetry $K^g = [K^g, W] \le C_G(D)$. But $K^g \not\le M$ and $M = !\mathcal{M}(LT^g)$, so as T^g acts on $C_D(L)$, it follows that $C_D(L) = 1$.

Next we saw D centralizes $\tilde{U}$ so that $[D, U] \le V_1 \le V$, and hence by symmetry, $[D, U^x] \le V_1^x \le V$ for each $x \in L$. Thus $L \le \langle U^x : x \in L \rangle =: I$ centralizes DV/V. Further by 12.8.8.4, I is described by G.2.5, so $S := U^g W C_{U^l}(V)$ is Sylow in LS, for $l \in L - L_2T$. As W centralizes D, so does $C_{U^l}(V)$ by symmetry, so that S centralizes D; then we conclude from Gaschütz's Theorem A.1.39 that $DV = V \times C_D(L)$ with $C_D(L)$ a complement to V_1^g in D. Then as $C_D(L) = 1$, $D = V_1^g$. Then $Q^g = U^g$, so $Q = U$, completing the proof of the lemma. □

We now define certain $\{2,3\}$-subgroups X of H, which are analogous to L_1: for example, 14.4.8 will show that $\langle X, L_2 \rangle =: L_X$ satisfies the hypotheses of L. Then 14.4.13 will show that $\langle LT, L_X \rangle \cong L_4(2)$, leading to our final contradiction.

So let $\mathcal{X}$ consist of the set of T-invariant subgroups $X = O^2(X)$ of H such that $|X : O_2(X)| = 3$. Let $\mathcal{Y}$ consist of those $X \in \mathcal{X}$ such that $V_X := [V_2, X]$ is of rank 3 and contained in E, and set $L_X := \langle L_2, X \rangle$.

LEMMA 14.4.8. *(1) $L_1 \in \mathcal{Y}$, with $V_{L_1} = [V_2, L_1] = V$ and $L_{L_1} = L$.*

(2) If $X \in \mathcal{Y}$ then $L_X \in \mathcal{L}_f^*(G, T)$, $L_X/O_2(L_X) \cong L_3(2)$, L_XT induces $GL(V_X)$ on V_X with kernel $O_2(L_XT)$, and I_2 and XT are the maximal parabolics of L_XT over T.

PROOF. By construction, $L_1 \in \mathcal{X}$ with $V = [V_2, L_1]$, and $V \le E$ by 12.8.13.1. Thus (1) holds.

Assume $X \in \mathcal{Y}$. Then $V_2 \le V_X \le E \le U \cap U^g$, so U and U^g act on V_X, and hence also $I_2 = \langle U, U^g \rangle$ acts on V_X. Then $Aut_{I_2}(V_X)$ is the maximal subgroup of $GL(V_X)$ stabilizing the hyperplane V_2 of V_X, and X does not act on that hyperplane as $V_X = [V_X, X]$, so $L_X/C_{L_X}(V_X) = GL(V_X)$. Thus there is $L_+ \in \mathcal{C}(L_X)$ with $L_+C_{L_X}(V_X) = L_X$, so $L_+ \in \mathcal{L}_f(G, T)$. Then by 14.3.4.1, $L_+ \in \mathcal{L}_f^*(G, T)$ and $L_+/O_2(L_+) \cong L_3(2)$. The projection P of L_2 on L_+ satisfies $P = [P, T \cap L_+]$, so as T acts on L_2, $L_2 = [L_2, T \cap L_+] \le L_+$. Similarly $X \le L_+$, so $L_X = L_+$, and (2) holds. □

The shadow of the Harada-Norton group F_5 is eliminated in the proof of the next lemma. We obtain a contradiction in the 2-local which would correspond to the local subgroup $\Omega_6^-(2)/E_{2^6}$ in F_5.

LEMMA 14.4.9. *Case (3) of 14.4.2 does not hold.*

PROOF. Assume case (3) of 14.4.2 holds. Then we can view $\tilde{U}$ as a 4-dimensional orthogonal space over $\mathbf{F}_4$ preserved by K^*. In particular $\tilde{V}_2 = C_{\tilde{U}}(W^{g*})$ lies in some totally singular $\mathbf{F}_4$-point $\tilde{U}_2$ of $\tilde{U}$. Further $\mathcal{X} = \{X_1, X_2\}$, where a subgroup of order 3 in each X_i^* is diagonally embedded in K^*, and we may choose notation so that $[X_2, \tilde{U}_2] = 1$ and $\tilde{U}_2 = [X_1, \tilde{U}_2]$. Thus $L_1 = X_1$ and $V = U_2$ by 14.4.8.1.

Therefore the subspace $\tilde{V}^{\perp_2}$ orthogonal to $\tilde{V}$ in the $\mathbf{F}_2$-orthogonal space $\tilde{U}$ is the same as the subspace $V^{\perp_4} =: \tilde{W}_1$ orthogonal to $\tilde{V}$ in $\mathbf{F}_4$-orthogonal space $\tilde{U}$. Choose $k \in K$ so that $\tilde{V}^k \not\le \tilde{W}_1$. As $\tilde{W}_1$ is an $\mathbf{F}_2$-hyperplane of $\tilde{W}$ and L_1 is transitive on $\tilde{V}^\#$, we can choose k so that $s^k \in W$ (recall s is the generator of V_1^g).

As the preimage W_1 of $\tilde{W}_1$ satisfies $W_1 = C_U(V)$, $W_1^{g*} = C_{W^{g*}}(V) = W^{g*} \cap K^*$ since $C_{H^*}(V) \leq K^* \cong L_2(4) \times L_2(4)$ as case (3) of 14.4.2 holds. Therefore as $s^k \in W - W_1$, for some $x \in G$ there is $i := z^x \in W^g$ with $i^* \notin K^*$.

Then $K = K_1K_1^i$, where $K_1 \in \mathcal{C}(H)$ and $K_1^* \cong L_2(4)$. As case (3) of 14.4.2 holds, $m([\tilde{U}, i^*]) = 4$. Thus by Exercise 2.8 in [**Asc94**], $C_{H^*}(\tilde{i}) = C_{H^*}(i^*)$. Let $K_0 := O^2(C_H(\tilde{i}))$. Then $K_0^* \cong L_2(4)$ is diagonally embedded in K^*, and K_0 centralizes $\langle i, z\rangle$, so $K_0 = O^2(C_H(i))$. Of course K_0 acts on $[\tilde{U}, i]$, and since diagonal subgroups of K^* of order 3 centralize a subspace of $\tilde{U}$ of rank exactly 4, it follows that $[\tilde{U}, i]$ is the A_5-module for K_0^*.

Let $D := [U, i]\langle i, z\rangle$. Then $D \cong E_{64}$ since K_0 is irreducible on $[\tilde{U}, i]$ of rank 4. Further as U is extraspecial, K_0U acts on D with $C_D(u) \leq [U, i]V_1$ for each $u \in U - [U, i]V_1$, and $U/[U, i]V_1$ induces the full group of transvections on $[U, i]V_1$ with center V_1. In particular $C_D(U) = \langle z\rangle$, $D = C_{UD}(D)$, and $U/[U, i]V_1$ is also the A_5-module for K_0U/U. Further as $Q = U$ by 14.4.7, $D = O_2(C_H(i)) = O_2(C_G(\langle z, i\rangle))$ and $UD = O_2(K_0UD)$.

Next $K_0 = O^2(C_{H^x}(z))$, so we conclude that z interchanges the two members of $\mathcal{C}(H^x)$. Thus we have symmetry between i and z, and so U^x acts on D with $C_D(U^x) = \langle i\rangle$. Therefore as D is an indecomposable K_0U-module with chief series $1 < V_1 < [U, i]V_1 < D$, it follows that $Y := \langle K_0U, U^x\rangle$ is irreducible on D.

Now let $T_D := N_T(D) \in Syl_2(N_H(D))$, and $G_D := N_G(D)$. As Y is irreducible on D, $D \leq Z(O_2(G_D))$, so as $D = C_{UD}(D)$, $D = UD \cap O_2(G_D)$. As $C_D(T_D) \leq C_D(U) = \langle z\rangle$, $N_G(T_D) \leq H$ so that $T_D \in Syl_2(G_D)$.

Next $K_0 \in \mathcal{L}(G_D, T_D)$, so $K_0 \leq K_+ \in \mathcal{C}(G_D)$ by 1.2.4, and as $D = DU \cap O_2(G_D)$, $K < K_+$. However A.3.14 contains no "B" with $O_2(B)$ the A_5-module UD/D for K_0D/UD. This contradiction completes the proof of 14.4.9. □

The elimination of the A_5-module in part (3) of the next lemma 14.4.10 rules out the shadow of the non-quasithin group $\Omega_8^-(2)$. Again we obtain a contradiction working in the 2-local corresponding to the local $\Omega_6^-(2)/E_{2^6}$ in the shadow.

LEMMA 14.4.10. *Assume case (5) of 14.4.2 holds. Then*

(1) $\mathcal{X} = \{X_1, X_2\}$ *where* $X_1 := O^2(O_{2,3}(H))$ *and* $X_2 := O^2(B)$ *for* B *a* T-*invariant Borel subgroup of* $K_0 := H^\infty$.

(2) There is a unique T-*invariant chief factor* $\tilde{U}_1$ *for* K_0, *and* $\tilde{V}_2 \leq C_{\tilde{U}}(T) \leq \tilde{U}_1$.

(3) $\tilde{U}_1$ *is the* $L_2(4)$-*module for* K_0^*.

(4) $\mathcal{X} = \mathcal{Y}$.

(5) $H^* \cong S_3 \times S_5$ *and* $X_1X_2T/O_2(X_1X_2T) \cong S_3 \times S_3$.

PROOF. Assume conclusion (5) of 14.4.2 holds. Let $K_0 := H^\infty$. It is easy to check that (1) and (2) hold, with $\tilde{U}_1 := [\tilde{U}, x]$ for $x^* \in C_{W^{g*}}(K_0^*)$; such an x^* exists since W^{g*} is of rank 3 by 14.4.2. Also $[\tilde{U}, X_1] = \tilde{U}$, so V_{X_1} is of rank 3, and there is a K_0-complement $\tilde{U}_2$ to $\tilde{U}_1$. Recall that $[W^g, W] \leq E$ by 12.8.11.1, and that $\tilde{W} = \tilde{V}_2^\perp$. Then computing in either module for A_5 in case (5) of 14.4.2, we obtain

$$\tilde{V}_{X_1} \cap \tilde{U}_2 \leq [\tilde{V}_2^\perp \cap \tilde{U}_2, W^{g*} \cap K_0^*] \leq \tilde{E}.$$

So as $V_{X_1} = \langle V_2, V_{X_1} \cap U_2\rangle$, $X_1 \in \mathcal{Y}$.

Assume first that $\tilde{U}_1$ is the $L_2(4)$-module for K_0^*. Then (3) holds, and $\tilde{V}_{X_2} = [\tilde{V}_2, X_2]$ is the $\mathbf{F}_4$-point in $\tilde{U}_1$ containing $\tilde{V}_2$. Now $[\tilde{U}, x] \leq \tilde{W}$ as x acts on the hyperplane $\tilde{W}$ of $\tilde{U}$, so $\tilde{V}_{X_2} \leq [\tilde{U}_1, W^g] \leq \tilde{E}$. Thus $X_2 \in \mathcal{Y}$, and hence (4) holds. As $X_2 \in \mathcal{Y}$, $X_2T/O_2(X_2T) \cong S_3$ by 14.4.8.2, so (5) holds, completing the proof of the lemma for the $L_2(4)$-module.

Thus as conclusion (5) of 14.4.2 holds, we may assume instead that $\tilde{U}_1$ is the A_5-module, and it remains to derive a contradiction.

As $\tilde{U}_1$ is the A_5-module, X_2 centralizes $\tilde{V}_2$, so that $X_2 \notin \mathcal{Y}$. Hence we conclude from (1) and 14.4.8.1 that $L_1 = X_1$ and $V = V_{X_1}$. Then X_2 centralizes $\langle \tilde{V}_2^{X_1} \rangle = \tilde{V}$. Thus $X_2 \leq C_G(V) \leq M$ using Coprime Action, and then $[L, X_2] \leq C_L(V) = O_2(L)$, so that X_2 acts on L_2 and hence on $\langle U^{L_2} \rangle = I_2$. Let $G_0 := \langle I_2, K_0 \rangle$, $V_0 := \langle z^{G_0} \rangle$, and $G_0^+ := G_0/C_{G_0}(V_0)$.

Suppose $O_2(G_0) = 1$. Then Hypothesis F.1.1 is satisfied with K_0, I_2, T in the roles of "L_1, L_2, S"; for example we just saw that $B_1 := N_{K_0}(T \cap K_0) = X_2(T \cap K_0)$ normalizes I_2. Thus $\alpha := (K_0T, TX_2, I_2X_2)$ is a weak BN-pair by F.1.9. Further $B_2 := N_{I_2}(K_0) = T \cap I_2$, so $T \trianglelefteq TB_2$, and hence the hypotheses of F.1.12 are satisfied. Therefore α is described in F.1.12. This is a contradiction as $U = O_2(K_0) \cong Q_8^4$ and $K_0/U \cong L_2(4)$, a configuration not appearing in F.1.12.

Thus $O_2(G_0) \neq 1$, so $G_0 \in \mathcal{H}(T)$, and $V_0 \in \mathcal{R}_2(G_0)$ by B.2.14. By 1.2.4, $K_0 \leq J \in \mathcal{C}(G_0)$. Then $1 \neq [V_2, K_0] \leq [V_0, J]$, so that $J \in \mathcal{L}_f(G,T)$ by 1.2.10. Then $J \in \mathcal{L}_f^*(G,T)$ by 14.3.4, so that $K_0 = J$ by 13.1.2.5. Now $I_2 = O^2(I_2)$ normalizes K_0 by 1.2.1.3, and hence acts on $Z(O_2(K_0)) = V_1$, contradicting $I_2 \not\leq G_1$. □

LEMMA 14.4.11. *Assume case (4) of 14.4.2 holds. Then*

(1) We can represent $H^* \cong S_7$ on $\Omega := \{1, \dots, 7\}$ so that T preserves the partition $\{\{1,2,3,4\},\{5,6\},\{7\}\}$ of Ω.

(2) $\mathcal{Y} = \{X_1, X_2\}$, where $X_1 := O^2(H_{1,2,3,4})$ and $X_2 = O^2(H_{5,6,7})$. In particular, $X_1X_2T/O_2(X_1X_2T) \cong S_3 \times S_3$.

PROOF. Part (1) is trivial; cf. the convention in section B.3. Further $\mathcal{X} = \{X_1, X_2, X_3\}$, where X_1 and X_2 are defined in (2), $X_3 := O^2(P)$ for P the stabilizer of the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}$, and $X_1X_2T/O_2(X_1X_2T) \cong S_3 \times S_3$.

Next $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$, where $\tilde{U}_1$ is a 4-dimensional irreducible for K, and $\tilde{U}_2 = \tilde{U}_1^x$ for $x^* \in W^{g*} - K^*$ is dual to $\tilde{U}_1$. Now $C_{\tilde{U}_i}(N_T(U_1)) = \langle \tilde{u}_i \rangle$ for suitable $\tilde{u}_i$, so $\tilde{s} = \tilde{u}_1\tilde{u}_2$, with $C_{H^*}(\tilde{s}) = P^* = X_3^*T^*$ from the structure of the sum of $\tilde{U}_1$ and its dual. In particular, $X_3 \notin \mathcal{Y}$. Recall $W^{g*} \trianglelefteq C_{H^*}(\tilde{V}_2) = P^*$ and $m(W^{g*}) = 3$ by 14.4.2, so $W^{g*} = O_2(P^*) = \langle x_i^* : 1 \leq i \leq 3 \rangle$, where $x_i^* := (2i-1, 2i)$ on Ω. Let $[U, x_i] =: D_i$. Then $D_i \leq W$ and $\tilde{D}_i$ is of rank 4, so $\tilde{D}_i$ is the $L_2(4)$-module for $Y_i^* := C_{K^*}(x_i^*) \cong S_5$ since elements of order 3 in Y_i^* are fixed-point-free on $\tilde{U}$. As such elements lie in X_i for $i = 1, 2$, V_{X_i} is of rank 3. Further $X_2^* \leq Y_3^*$ with $V_{X_2} \leq [D_3, x_1^*x_2^*] \leq [D_3, W^g] \leq E$ by 12.8.11.1, so $X_2 \in \mathcal{Y}$. Similarly a Sylow 3-group B^* of X_1^* is contained in Y_1^* with $\tilde{V}_{X_1} = [\tilde{V}_2, B] = [\tilde{V}_2, x_2^*x_3^*] \leq [\tilde{D}_1, W^g] \leq \tilde{E}$, so $X_1 \in \mathcal{Y}$, completing the proof of (2). □

LEMMA 14.4.12. *Assume case (6) of 14.4.2 holds. Then $\mathcal{Y} = \{X_1, X_2\}$ where $X_1 := O^2(O_{2,3}(H))$ and $X_1X_2T/O_2(X_1X_2T) \cong S_3 \times S_3$.*

PROOF. First (cf. H.12.1.5) $C_{\tilde{U}}(T) = \tilde{V}_2$ and $C_{H^*}(\tilde{V}_2) \cong S_5/E_{32}$. We have seen that $W^{g*} \trianglelefteq C_{H^*}(\tilde{V}_2)$, and by 12.8.1, $m(W^{g*}) = 5$, so $W^{g*} = O_2(C_{H^*}(\tilde{V}_2))$.

Next we calculate that $\mathcal{X} = \{X_1, X_2, X_3\}$, where $X_1 := O^2(O_{2,3}(H))$, $X_3 := O^2(B)$ where B^* is a Borel subgroup of $C_{H^*}(\tilde{V}_2)$, and X_2T is a minimal parabolic in the remaining maximal 2-local $D^* := \langle TX_2, X_3\rangle^* \cong S_6/E_{16}/\mathbf{Z}_3$ of H^* over T^*, which does not centralize $\tilde{V}_2$. As X_3 centralizes $\tilde{V}_2$, $X_3 \notin \mathcal{Y}$.

Let $\tilde{U}_D := \langle \tilde{V}_2^D\rangle$; then $O_2(D^*)$ centralizes $\tilde{U}_D$, and $\tilde{U}_D$ is the 6-dimensional irreducible for $D^+ := D^*/O_2(D^*) \cong \hat{S}_6$. Now $\tilde{U}$ has the structure of a 6-dimensional $\mathbf{F}_4$-space preserved by K^*, with the $\mathbf{F}_4$-points the irreducibles for X_1^*. This $\mathbf{F}_4$-space structure restricts to $\tilde{U}_D$ of $\mathbf{F}_4$-dimension 3, and $\tilde{V}_{X_1}$ is the $\mathbf{F}_4$-point containing $\tilde{V}_2$, so that $V_{X_1} \cong E_8$. Further $T^*X_1^*X_2^*$ is the stabilizer of an $\mathbf{F}_4$-line $\tilde{U}_0$ of $\tilde{U}_D$ containing $\tilde{V}_{X_1}$, with $X_1X_2T/O_2(X_1X_2T) \cong S_3 \times S_3$. In particular X_2 is fixed-point-free on $\tilde{U}_0$, so $V_{X_2} \cong E_8$. Thus to complete the proof, it remains to show that $V_{X_i} \leq E$ for $i = 1, 2$. Now $\tilde{U}_D$ is totally singular, since $\tilde{U}_D$ is not self-dual as a D-module. Thus $\tilde{U}_D \leq \tilde{V}_2^\perp = \tilde{W}$, so by 12.8.11.1 it suffices to show $\tilde{V}_{X_i} \leq [\tilde{U}_D, W^g]$. But there is $x \in W^g$ inverting X_1^+ with x^+ centralizing X_2^+. As x^+ inverts X_1^+, $C_{\tilde{U}_D}(x) = [\tilde{U}_D, x]$ is of rank 3, and $\tilde{V}_2 \leq C_{\tilde{U}_D}(x)$. Then $\tilde{V}_{X_2} = [\tilde{V}_2, X_2] \leq [\tilde{U}_D, x] \leq [\tilde{U}_D, W^g]$, as required. As $W^{g*} \cap K^*$ induces a group of $\mathbf{F}_4$-transvections on $\tilde{U}_D$ with center $\tilde{V}_{X_1}$, $\tilde{V}_{X_1} \leq [\tilde{U}_D, W^{g*} \cap K^*] \leq \tilde{E}$. This completes the proof. □

By 14.4.9–14.4.12, we have reduced to the situation where one of cases (4)–(6) of of 14.4.2 holds, and in case (5) the chief factors for H^∞ on $\tilde{U}$ are $L_2(4)$-modules. By 14.4.8.1, $L_1 \in \mathcal{Y}$; hence 14.4.10–14.4.12 show that in each case $\mathcal{Y} = \{L_1, X\}$ is of order 2, with $XL_1T/O_2(XL_1T) \cong S_3 \times S_3$.

Let $H_1 := LT$, $H_2 := L_1XT$, and $H_3 := L_XT$. Set $\mathcal{F} := \{H_1, H_2, H_3\}$ and $G_0 := \langle \mathcal{F}\rangle$.

Lemma 14.4.13. $G_0 \cong L_4(2)$.

Proof. We show that $(G_0, \mathcal{F})$ is an A_3-system as defined in section I.5. Then the lemma follows from Theorem I.5.1. We just observed that $H_2/O_2(H_2) \cong S_3 \times S_3$ and $H_i/O_2(H_i) \cong L_3(2)$ for $i = 1, 3$ by 14.4.8.2, so (D1) and (D2) hold. As L_2T is maximal in H_1 and H_3 but $X \neq L_1$, $L_2T = H_1 \cap H_3$, so $L_2 = L \cap H_3 \trianglelefteq M \cap H_3$ and hence $L_2T = M \cap H_3$. Thus as $M = !\mathcal{M}(LT)$, $O_2(G_0) = 1$, [3] so hypothesis (D4) holds. Similarly $L_1T = H_1 \cap H_2$ and $XT = H_2 \cap H_3$, so (D3) holds. Finally (D5) is vacuous for a system of type A_3. □

We are now in a position to obtain a contradiction to our assumption that $d > 4$. Namely as $|T| \geq |U| > 2^9$, G_0 is not $L_4(2)$, contrary to 14.4.13. This contradiction shows:

Theorem 14.4.14. *Assume Hypothesis 14.3.1 holds with $\langle V^{G_1}\rangle$ nonabelian. Then $L/O_2(L) \cong L_3(2)$ and G is isomorphic to HS or $G_2(3)$.*

Proof. By assumption, Hypothesis 14.3.10 holds. Thus $L/O_2(L) \cong L_3(2)$ by Theorem 14.3.16. Then by Theorem 14.3.26, either $U = \langle V_1^{G_1}\rangle$ is extraspecial or $G \cong HS$, and we may assume the former. Hence if $d = m(\tilde{U}) = 4$, then $G \cong G_2(3)$ by Theorem 14.4.3. Finally we just obtained a contradiction under the assumption that U is extraspecial and $d > 4$, so the proof of Theorem 14.4.14 is complete. □

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[3] The group J_4 has the involution centralizer appearing in case (6) of 14.4.2, and there is $L \in \mathcal{L}_f(G,T)$ with $L/O_2(L) \cong L_3(2)$, but the condition $O_2(G_0) = 1$ fails as $L \notin \mathcal{L}_f^*(G,T)$.

14.5. Starting the case $\langle \mathbf{V}^{\mathbf{G_1}} \rangle$ abelian for $\mathbf{L_3(2)}$ and $\mathbf{L_2(2)}$

In this section, and indeed in the remainder of the chapter, we assume:

HYPOTHESIS 14.5.1. *Hypothesis 14.3.1 holds and* $U := \langle V^{G_1} \rangle$ *is abelian.*

As U is abelian and $\Phi(V) = 1$, U is elementary abelian. Recall from the discussion after 14.3.6 that Hypothesis 12.8.1 holds. In particular $G_1 \not\leq M$ by 12.8.3.4, so that $V < U$. Recall also the definitions of G_i, L_i, and V_i, for $i \leq \dim(V)$, from Notation 12.8.2.

LEMMA 14.5.2. *If* $g \in G$ *with* $1 \neq V \cap V^g$, *then* $[V, V^g] = 1$.

PROOF. As $\langle V^{G_1} \rangle$ is abelian by Hypothesis 14.5.1, the results follows from the equivalence of (2) and (3) in 12.8.6. □

14.5.1. A result on $\mathbf{X} \in \mathcal{H}(\mathbf{T})$ with $\mathbf{X}/\mathbf{O_2}(\mathbf{X}) = \mathbf{L_2}(\mathbf{2})$.

Recall that under case (2) of Hypothesis 14.3.1 where $L/O_2(L) \cong L_2(2)'$, 14.3.5 says there exists no $X \in \mathcal{H}(T, M)$ such that $X/O_2(X) \cong L_2(2)$. In this subsection, we establish a result providing some restrictions on such subgroups in case (1) of Hypothesis 14.3.1, where $L/O_2(L) \cong L_3(2)$. Namely we prove:

THEOREM 14.5.3. *Suppose* $Y = O^2(Y) \leq G_1$ *is* T*-invariant with* $YT/O_2(YT) \cong L_2(2)$. *Then*

(1) Either $Y \leq M$, *or case (1) of Hypothesis 14.3.1 holds and* $[V_2, Y] = 1$.
(2) If $YL_1 = L_1Y$, *then* $Y \leq M$.
(3) $\langle \tilde{V}_2^Y \rangle$ *is not isomorphic to* E_8.

Until the proof of Theorem 14.5.3 is complete, assume Y is a counterexample.

LEMMA 14.5.4. *(1)* $Y \not\leq M$.
(2) Case (1) of Hypothesis 14.3.1 holds, namely $L/O_2(L) \cong L_3(2)$.

PROOF. Assume (1) fails, so that $Y \leq M$. Then conclusions (1) and (2) of Theorem 14.5.3 are satisfied. Further Y acts on V by 14.3.3.6. Thus as $V_2 \leq V$, $m(\langle \tilde{V}_2^Y \rangle) \leq m(\tilde{V}) \leq 2$, so that conclusion (3) of 14.5.3 holds. This contradicts our assumption that we are working in a counterexample.

Thus (1) is established. Then (1) and 14.3.5 imply (2). □

Set $X := L_2$, and $H := \langle X, Y, T \rangle$. Notice that $H \not\leq G_1$ since $X \not\leq G_1$. Set $V_H := \langle V_1^H \rangle$, $Q_H := O_2(H)$, $\dot{H} := H/Q_H$, and $H^* := H/C_H(V_H)$. Observe that (H, XT, YT) is a Goldschmidt triple (in the language of Definition F.6.1), so by F.6.5.1, $\alpha := (\dot{X}\dot{T}, \dot{T}, \dot{Y}\dot{T})$ is a Goldschmidt amalgam, and so is described in F.6.5.2.

LEMMA 14.5.5. $Q_H \neq 1$.

PROOF. Assume $Q_H = 1$. By 1.1.4.6, XT and YT are in $\mathcal{H}^e$, and so satisfy Hypothesis F.1.1 in the roles of "L_1, L_2", with T in the role of "S". Then α is a weak BN-pair of rank 2 by F.1.9, and the hypothesis of F.1.12 is satisfied, so that α is described in case (vi) of F.6.5.2. Then as X has at least two noncentral 2-chief factors (from V and the image of $O_2(L_2)$ in $L/O_2(L) \cong L_3(2)$), by inspection of that list, α is isomorphic to the amalgam of $G_2(2)'$, $G_2(2)$, M_{12}, or $Aut(M_{12})$, and X has exactly two such factors. In each case, $Z = \Omega_1(Z(T))$ is of order 2, so $V_1 = Z$.

Next we saw $V < \langle V^{G_1}\rangle = U \trianglelefteq YT$, so $m_2(U) \geq 4$ since $m(V) = 3$ by 14.5.4.2. As the 2-rank of $G_2(2)'$, $G_2(2)$, and M_{12} is at most 3, it follows that α is the $Aut(M_{12})$-amalgam and $m(U) = 4$. Thus $A := Aut(M_{12})$ is a faithful completion of α, so identifying YT with its image under this completion, we may assume $YT \leq A$. As $m_2(M_{12}) = 3$, there is an involution $u \in U - M_{12}$. Thus $C_A(u) \cong \mathbf{Z}_2/(E_4 \times J)$ where $J := C_G(u)^\infty \cong A_5$. Therefore $U = J(C_T(u)) = O_2(C_A(u)) \times (U \cap J)$. Then from the structure of A, $N_A(U) = T(N_A(U) \cap C_A(u))$, and $V_1 = Z = C_U(T) \leq J$. Thus $|YT| = 2^7{\cdot}3 = |N_A(U)|$, so $N_A(U) = YT$ as $U \trianglelefteq YT$. This is a contradiction as YT centralizes V_1 but $Z(N_J(U)) = 1$. $\square$

By 14.5.5 and 1.1.4.6, $H \in \mathcal{H}(T) \subseteq \mathcal{H}^e$.

LEMMA 14.5.6. *Y^* does not act on X^*.*

PROOF. Assume otherwise. Then $V_1^{X^*Y^*} = V_1^{Y^*X^*} = V_1^{X^*}$ as $Y \leq G_1$. Therefore Y acts on $\langle V_1^X\rangle = V_2$, and hence $[Y, V_2] = 1$ by Coprime Action. Thus Y is not a counterexample to conclusion (1) or (3) of 14.5.3, so Y must be a counterexample to conclusion (2). Therefore $YL_1 = L_1Y$, and hence Y acts on $\langle V_2^{L_1}\rangle = V$, contradicting 14.5.4.1. $\square$

Set $H^+ := H/O_{3'}(H)$.

LEMMA 14.5.7. *(1) $C_X(V_H) \leq O_2(X)$ and $C_Y(V_H) \leq O_2(Y)$.*
(2) $Q_H = C_T(V_H)$.
(3) $V_H \leq Z(Q_H)$ and $O_2(H^*) = 1$.
(4) $Q_H \in Syl_2(O_{3'}(H))$, so $O_{3'}(H)$ is 2-closed and in particular solvable.
(5) Either
(i) H^+ is described in Theorem F.6.18, or
(ii) $O_2(XT) = O_2(YT) = Q_H$, and $H^+ \cong S_3$.

PROOF. We saw $H \in \mathcal{H}^e$, so as $V_1 \leq Z$, part (3) follows from B.2.14. Next $C_X(V_H) \leq O_2(X)$ as $X \not\leq G_1$. Thus if $Y \leq C_H(V_H)$, then $Y^* = 1$, so Y^* acts on X^*, contrary to 14.5.6. Hence (1) holds. By (3), $Q_H \leq C_T(V_H)$, while by (1), we may apply F.6.8 to $C_H(V_H)$ in the role of "X" to conclude that $C_T(V_H) \leq Q_H$, so (2) holds. Similarly F.6.11.1 implies (4), and F.6.11.2 implies (5) as H is an SQTK-group. $\square$

LEMMA 14.5.8. *H is solvable.*

PROOF. Assume H is nonsolvable. Then by 1.2.1.1 there is $K \in \mathcal{C}(H)$, and by 14.5.7.2, $C_H(V_H)$ is 2-closed and hence solvable, so $K^* \neq 1$. Then $K \in \mathcal{L}_f(G,T)$ by 1.2.10, so by 14.3.4.1, $K/O_2(K) \cong A_5$ or $L_3(2)$. Now $O_2(K) = O_{3'}(K) = C_K(V_H)$, so $K^+ \cong K/O_2(K) \cong K^*$. By 14.5.7.5, H^+ is described in F.6.18, so we conclude that case (6) of F.6.18 holds, with $H^+ = K^+ \cong L_3(2)$. Hence $K = O^{3'}(H) = \langle X, Y\rangle$. Then $K = O^2(H)$ by F.6.6.3, so that $H = KT$. Now as $[V_1, Y] = 1$ and $\langle V_1^X\rangle = V_2 \cong E_4$, V_H is the natural module for K^* by H.5.5. In particular $V_2 = \langle V_1^X\rangle \leq V_H$ and $V_H = \langle V_2^Y\rangle$.

By 14.3.4.2, $K \in \mathcal{L}_f^*(G,T)$, so by our discussion after Hypothesis 14.3.1, part (1) of that Hypothesis holds with K in the role of "L". Thus by Theorem 14.4.14, either $\langle V_H^{G_1}\rangle$ is abelian, or $G \cong G_2(3)$ or HS. However in the latter two cases, L is the unique member of $\mathcal{L}_f^*(G,T)$, so $K = L \leq M$, contrary to 14.5.4.1. Therefore

$\langle V_H^{G_1} \rangle$ is abelian, so we have symmetry between LT, V and $H = KT$, V_H; that is, Hypothesis 14.5.1 holds with H, V_H in the roles of "LT, V".

Now $Y \not\leq M = !\mathcal{M}(LT)$, so that $O_2(\langle LT, H \rangle) = 1$. Hence Hypotheses F.7.1 and F.7.6 are satisfied with LT and H in the roles of "G_1" and "G_2", so we can form the coset geometry Γ of Definition F.7.2 with respect to this pair. Similarly we can form the dual geometry Γ' where the roles of LT and H are reversed. Let $\gamma_0 := LT$, $\gamma_1 := H$, and for $g, h \in \langle LT, H \rangle$ let $V_{\gamma_0 g} := V^g$ and $V_{\gamma_1 h} := V_H^h$. Also for $\sigma \in \Gamma$ let $Q_\sigma := O_2(G_\sigma)$. Observe $G_{\gamma_0,\gamma_1} = LT \cap H = XT$ and $\ker_{XT}(G_i) = O_2(G_i)$ for $i = 1, 2$, is the centralizer in G_i of V or V_H, respectively, so

$$Q_\sigma = G_\sigma^{(1)} = C_{G_\sigma}(V_\sigma).$$

Next as usual choose a geodesic

$$\alpha := \alpha_0, \ldots, \alpha_b =: \beta$$

in Γ of minimal length b, subject to $V_\alpha \not\leq Q_\beta$. Then $b = \min\{b(\Gamma, V), b(\Gamma', V_H)\}$, so by F.7.9.1, $V_\alpha \leq G_\beta$ and $V_\beta \leq G_\alpha$, and hence

$$1 \neq [V_\alpha, V_\beta] \leq V_\alpha \cap V_\beta. \qquad (*)$$

Thus by 14.5.2 and the corresponding result for V_H, β is not conjugate to α, so b is odd. Replacing Γ by Γ' if necessary, we may assume $V_\alpha = V$, and we may assume $z \in V \cap V_\beta$ by transitivity of L on $V^\#$. As H is also transitive on $V_H^\#$, $V_\beta = V_H^g$ for some $g \in G_1$ by A.1.7.1, so

$$\langle V_\beta^{G_1} \rangle = \langle V_H^{G_1} \rangle = \langle \langle V_2^Y \rangle^{G_1} \rangle = \langle V_2^{G_1} \rangle = \langle V^{G_1} \rangle$$

since $V = \langle V_2^{L_1} \rangle$. Then as $\langle V^{G_1} \rangle$ is abelian, V_β centralizes $V = V_\alpha$, contrary to (*). □

Lemma 14.5.9. *$[V_H, J(T)] = 1$ and $J(T) \trianglelefteq H$.*

Proof. If $J(T)$ centralizes V_H, then $J(T) = J(Q_H)$ by 14.5.7.2 and B.2.3.5, so the lemma holds. Thus we assume $[V_H, J(T)] \neq 1$, and derive a contradiction. By 14.5.8, we may apply Solvable Thompson Factorization B.2.16 to conclude that $J(H)^* = K_1^* \times \cdots \times K_s^*$, with $K_i^* \cong S_3$ and $V_i := [V_H, K_i] \cong E_4$. Notice $s \leq 2$ by A.1.31.1. As $X = [X, T]$ either $X^* = O^2(K_i^*)$ for some i, or $[X^*, J(H)^*] = 1$. The same holds for Y as $Y = [Y, T]$. Thus if $X^* = O^2(K_i^*)$, then Y^* normalizes X^*, contrary to 14.5.6. Therefore X^* centralizes $J(H)^*$, so that $J(H) \cap X \leq O_2(X)$. Similarly $J(H) \cap Y \leq O_2(Y)$. Then we may apply F.6.8 to $J(H)$, to conclude that $J(T) \leq T \cap J(H) \leq Q_H \leq C_H(V_H)$, contrary to our assumption. □

Lemma 14.5.10. *$J(T) = J(O_2(XT)) \not\leq O_2(LT)$ and $X = [X, J_1(T)]$.*

Proof. By 14.5.9 and 14.5.7.2, $J(T) \leq C_T(V_H) = Q_H \leq O_2(XT)$, so $J(T) = J(O_2(XT))$ by B.2.3.3. If $J(T) \not\leq O_2(LT)$, then $J_1(T) \not\leq R_2$ by 14.3.9.3, and hence the lemma holds. On the other hand if $J(T) = J(O_2(LT))$ then by 14.5.9, $H \leq N_G(J(T)) \leq M = !\mathcal{M}(LT)$, contradicting 14.5.4.1. □

Lemma 14.5.11. *(1) H^* is a $\{2, 3\}$-group.*
(2) $O_{3'}(H)) \leq C_H(V_H)$, so H^* is a quotient of H^+.

Proof. Assume $[O_{3'}(H^*), X^*] \neq 1$. Then as $O_{3'}(H^*)$ is solvable of odd order by (2) and (4) of 14.5.7, $[R^*, X^*] \neq 1$ for some prime $p > 3$ and some supercritical subgroup R^* of $O_p(H^*)$ by A.1.21. As $X^* = [X^*, T^*]$, R^* is not cyclic, so $R^* \cong E_{p^2}$

or p^{1+2} by A.1.25. As $m_2(Aut(R^*)) \leq 2$ and $X^* = [X^*, J_1(T)]$ by 14.5.10, the hypothesis of D.2.17 is satisfied for each indecomposable pair in a decomposition of $(R^*X^*J_1(T)^*, V_H)$. So as $p > 3$ and R^* is not cyclic, we conclude from D.2.17 that $p = 5$, and that there are two indecomposable components: that is, $R^* = R_1^* \times R_2^*$ with $R_i^* \cong \mathbf{Z}_5$, $[V_H, R] = V_{H,1} \oplus V_{H,2}$, and $V_i := [V_H, R_i]$ is of rank 4. But by definition of the decomposition, X^* acts on each component, contradicting $[R^*, X^*] \neq 1$.

Therefore $[O_{3'}(H^*), X^*] = 1$, so (1) follows from F.6.9. Of course (1) implies (2). □

LEMMA 14.5.12. *(1) H^+ is described in Theorem F.6.18.*
(2) $O_2(XT) \neq O_2(YT)$; in particular, case (2) of F.6.18 holds.

PROOF. By 14.5.11, H^* is a quotient of H^+, and by 14.5.7.1, $X^* \neq 1 \neq Y^*$. Thus if $H^+ \cong S_3$, then $H^* \cong S_3$, so that $Y^* = X^*$, contrary to 14.5.6. Thus (1) follows from 14.5.7.5. As H^+ is solvable by 14.5.8, case (1) or (2) of F.6.18 holds. As $O_{3'}(H^+) = 1$ by definition, H^+ is a $\{2,3\}$-group by F.6.9.

Assume (2) fails; then $O_2(XT) = O_2(YT) = Q_H$, so $X^+T^+ \cong Y^+T^+ \cong S_3$. As T^+ is of order 2, case (1) of F.6.18 holds, and we may apply Cyclic Sylow 2-Subgroups A.1.38 and F.6.6 to conclude that

$$\langle X^+, Y^+\rangle = O^2(H^+) = O(H^+).$$

Then as H^+ is a $\{2,3\}$-group, $O^2(H^+) =: P^+$ is a 3-group. Furthermore P^+ is noncyclic in case (1) of F.6.18, so that $m_3(P^+) = 2$ as H is an SQTK-group.

We claim $P^+ \cong 3^{1+2}$; the proof will require several paragraphs. By 14.5.6, P^+ is nonabelian with X^+ and Y^+ of order 3, so $\Omega_1(P^+)$ is nonabelian. Thus as we saw $m_3(P^+) = 2$, if P^+ is of symplectic type (cf. p. 109 in [**Asc86a**]), then $\Omega_1(P^+) \cong 3^{1+2}$ and the claim holds.

So assume P^+ is not of symplectic type. Then P^+ has a characteristic subgroup $E^+ \cong E_9$. If X^+ or Y^+ is contained in E^+, say X^+, then $P^+ = \langle X^+, Y^+\rangle = E^+Y^+ \cong 3^{1+2}$, and again the claim holds, so we may assume neither X^+ nor Y^+ is contained in E^+. Now $F^+ := C_{P^+}(E^+)$ is of index 3 in P^+, and $E^+ = \Omega_1(F^+)$.

Let $T^+ = \langle t^+\rangle$. Then t^+ inverts X^+, so as $X^+E^+ \cong 3^{1+2}$, $B^+ := C_{E^+}(t^+) \cong \mathbf{Z}_3$, and hence $N_E(T)^+ \neq 1$. But $N_G(T) \leq M$ by 14.3.3.3, so either $E = \langle N_E(T)^X\rangle \leq M$, or $B^+ = \Omega_1(Z(P^+))$. The former case is impossible, as $X \trianglelefteq H \cap M$, whereas E^+ does not normalize X^+. Thus the latter case holds, and we let $B_0 \in Syl_3(B \cap M)$, where B is the preimage of B^+, and set $B_M := O^2(B_0Q_H)$. Observe that $O_2(B_M) \neq 1$ since $H \in \mathcal{H}^e$. By a Frattini Argument, $H = O_{3'}(H)N_H(B_0)$, so $H^+ = N_H(B_M)^+$. As $X \not\leq B_M$, with $X/O_2(X)$ inverted in $T \cap L$ and $TB_M = B_MT$, we conclude $B_M \leq C_M(L/O_2(L))$, so L normalizes $O^2(B_MO_2(L)) = B_M$. Hence $N_G(B_M) \leq M = !\mathcal{M}(LT)$. As $H^+ = N_H(B_M)^+$ and $X \trianglelefteq H \cap M$ but $E^+ \not\leq N_{H^+}(X^+)$, this is a contradiction.

This establishes the claim that $P^+ \cong 3^{1+2}$. Thus t^+ inverts $P^+/Z(P^+)$ as t^+ inverts X^+ and Y^+. Hence t^+ centralizes $Z(P^+)$. Then we obtain a contradiction as in the previous paragraph. □

LEMMA 14.5.13. *(1) $\langle X^*, Y^*\rangle = P^* = O_3(H^*) \cong 3^{1+2}$ and $H = PT$, where $P \in Syl_3(H)$.*
(2) $T^* \cong E_4$.
(3) $C_H(V_H) = O_{3'}(H)$.

PROOF. By 14.5.11.2, H^* is a quotient of H^+, while by 14.5.12.2, H^+ is described in case (2) of Theorem F.6.18. Thus $\langle X^+, Y^+ \rangle = O_3(H^+) \cong 3^{1+2}$ or E_9, and $T^+ \cong E_4$. Then 14.5.6 completes the proof. □

We are now in a position to obtain a contradiction, and hence establish Theorem 14.5.3. Let $B^* := Z(P^*)$. By 14.5.10, $J_1(H)^* \neq 1$, so as $T^* \cong E_4$, the hypothesis of D.2.17 holds. Thus in view of 14.5.13, case (4) of D.2.17 holds, with $[V_H, P^*] = [V_H, B^*]$ of rank 6. Then $V_2 = [V_2, X] \leq [V_H, P^*]$, so $V_H = \langle V_1^H \rangle \leq [V_H, P^*]$ and hence $V_H = [V_H, P^*]$.

In particular, $V_H = V_X \oplus V_X^y \oplus V_X^{y^2}$, where $\langle y^* \rangle = Y^*$ and $V_X := C_{V_H}(X)$ is of rank 2. Further $C_{V_H}(T) = \langle w, z \rangle$ where $\langle w \rangle = C_{V_X}(T)$ and $z := ww^y w^{y^2}$. Thus $\langle z \rangle = C_{V_H}(YT)$, so $V_1 = \langle z \rangle$. On the other hand,

$$z \in V_2 = [V_2, X] \leq [V_H, X],$$

and X acts on $V_X^{y^i}$, since $V_X^{y^i} = C_{V_H}(X^{*y^i})$ and X^{*y^i} is contained in the abelian group X^*B^*. Therefore $[V_H, X] = V_X^y \oplus V_X^{y^2}$. This is a contradiction as $z \notin V_X^y \oplus V_X^{y^2}$ but we saw $z \in [V_H, X]$.

This contradiction completes the proof of Theorem 14.5.3.

14.5.2. Further preliminaries for the case U abelian. Recall we have adopted Notation 12.8.2, including: $V_1 = \langle z \rangle$, and

$$\mathcal{H}_z := \{H \leq G_1 : L_1 T \leq H \text{ and } H \not\leq M\}.$$

In the remainder of this section, H denotes a member of $\mathcal{H}_z$.

In contrast to the case where $\langle V^{G_1} \rangle$ was non-abelian, when $\langle V^{G_1} \rangle$ is abelian we work with members H of $\mathcal{H}_z$ possibly smaller than G_1.

Recall $U_H = \langle V^H \rangle$, $Q_H = O_2(H)$, and $\tilde{G}_1 = G_1/V_1$.

LEMMA 14.5.14. *(1) Hypothesis F.8.1 is satisfied in H.*

(2) Hypothesis F.9.8 is satisfied in H, with V in the role of "V_+".

PROOF. In view of Hypothesis 14.5.1, this follows from the list of equivalences in 12.8.6. □

By 14.5.14, we may appeal to the results of sections F.8 and F.9.

LEMMA 14.5.15. *(1) $\tilde{U}_H \leq Z(\tilde{Q}_H))$, and $\tilde{U}_H \in \mathcal{R}_2(\tilde{H})$.*

(2) U_H is elementary abelian.

(3) Assume $L/O_2(L) \cong L_3(2)$, and $L_1 \trianglelefteq H$. Then $\tilde{U}_H$ is the direct sum of isomorphic natural modules for $L_1/O_2(L_1) = L_1/C_{L_1}(U_H) \cong \mathbf{Z}_3$.

(4) $Q_H = C_H(\tilde{U}_H)$.

PROOF. Parts (1) and (4) follow from 12.8.4, (2) follows from Hypothesis 14.5.1 and 12.8.6, and (3) follows from 12.8.5.1. □

NOTATION 14.5.16. By 14.5.14, Hypotheses F.8.1 and F.9.8 are satisfied in H, so we can form the coset geometry Γ with respect to LT and H. Let $b := b(\Gamma, V)$, and choose a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b =: \gamma$$

as in section F.9. Define U_H, U_γ, D_H, D_γ, etc., as in section F.9; in particular set $A_1 := V_1^{g_b}$, recalling b is odd by F.9.11.1.

Since V plays the role of "V_+" in 14.5.14.2 in the notation of section F.9, $U_H = \langle V^H \rangle =: V_H$, and hence $D_H = E_H$. These identifications simplify the statements of various results in section F.9. In particular:

LEMMA 14.5.17. *$D_H < U_H$.*

PROOF. By F.9.13.5, $V \not\leq D_H$, so the remark follows as $V \leq V_H = U_H$. □

LEMMA 14.5.18. *(1) If $U_\gamma = D_\gamma$, then U_H induces a nontrivial group of transvections with center V_1 on U_γ.*

(2) If $m(U_\gamma^*) \geq m(U_H/D_H)$, then $U_\gamma^* \neq 1$ and $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$. In case

$$2m(U_\gamma^*) = m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*)),$$

then also $m(U_\gamma^*) = m(U_H/D_H)$, and U_γ^* acts faithfully on $\tilde{D}_H$ as a group of transvections with center $\tilde{A}_1$.

(3) $q(H^*, \tilde{U}_H) \leq 2$.

(4) If we can choose γ with $D_\gamma < U_\gamma$, then we can choose γ with

$$0 < m(U_\gamma^*) \geq m(U_H/D_H),$$

in which case $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

(5) Let $h \in H$ with $\gamma_0 = \gamma_2 h$ and set $\alpha := \gamma h$. Then $U_\alpha \leq R_1$ and if $D_\gamma < U_\gamma$ then $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

PROOF. Part (3) holds by F.9.16.3, while (1), (2), and (4) follow from 14.5.17 and the corresponding parts of F.9.16. Assume the hypotheses of (5). By parts (1) and (2) of F.9.13, $U_\alpha \leq R_1$, and if $U_\gamma^* \neq 1$, then since we can choose γ so that $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ in (4), also $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, completing the proof of (5). □

LEMMA 14.5.19. *If $K \in \mathcal{C}(H)$ then $K \not\leq M$, so $K_0 L_1 T \in \mathcal{H}_z$, where $K_0 := \langle K^T \rangle$.*

PROOF. This follows from 13.3.8.2 applied to L, K_0 in the roles of "K, Y". □

LEMMA 14.5.20. *Assume $Y \trianglelefteq H$ with $Y/O_2(Y)$ a p-group of exponent p. Then either*

(1) $Y \cap M = O_2(Y)$, or

(2) $p = 3$, $L/O_2(L) \cong L_3(2)$, $L_1 \leq Y$, and one of the following holds:

(i) $L_1 = Y \trianglelefteq H$.

(ii) $Y/O_2(Y) \cong 3^{1+2}$, $L_1 = O^2(O_{2,Z}(Y)) = O^2(Y \cap M)$, and T is irreducible on $Y/L_1O_2(Y)$.

(iii) $Y/O_2(Y) \cong E_9$ and there exists $Y_0 \leq H$ such that $L_1 \leq Y_0 \trianglelefteq Y_0T$ with $Y_0/O_2(Y_0) \cong \mathbf{Z}_9$ and $Y_0 \not\leq M$.

PROOF. We may assume that (1) fails, so that $Y_M := O^2(Y \cap M) \neq 1$.

Let $\mathcal{X}$ be the set of T-invariant subgroups X of H such that $1 \neq X = O^2(X) \leq C_M(L/O_2(L))$. Then using the T-invariance of X, L normalizes $O^2(XO_2(L)) = X$, so $N_G(X) \leq M = !\mathcal{M}(LT)$. In particular as $H \not\leq M$:

$$\text{For each } \ X \in \mathcal{X}, \ N_G(X) \leq M, \text{ so } X \text{ is not normal in } \ H. \qquad (!)$$

Set $Y_Z := O^2(O_{2,Z}(Y))$, and $Y_C := O^2(C_{Y_M}(L/O_2(L))$. By (!), $Y_Z \notin \mathcal{X}$, so $Y_Z \not\leq Y_C$. On the other hand if $Y_C = Y_M$ then $Y_M \in \mathcal{X}$, so $Y_Z \leq N_G(Y_M) \leq M$ by (!); then $Y_Z \leq Y_M = Y_C$, contrary to the previous remark, so:

$$Y_C < Y_M. \tag{$*$}$$

It follows that $p = 3$: For if $p > 3$ then T permutes with no p-subgroup of $L/O_2(L)$, so that $Y_M \leq Y_C$, contrary to (*). If $L/O_2(L) \cong L_2(2)'$, then Y_M centralizes V/V_1 and V_1 of order 2, and hence centralizes V by Coprime Action, so again Y_M centralizes $L/O_2(L)$, contrary to (*). Therefore $L/O_2(L) \cong L_3(2)$. Next we claim:

$$L_1 \leq Y. \tag{!!}$$

For if $L_1 \not\leq Y$, then

$$[Y_M, T \cap L] \leq C_L(V_1) \cap Y_M = L_1 O_2(L) \cap Y_M \leq O_2(Y_M),$$

so Y_M centralizes $(T \cap L)/O_2(L)$ and hence also $L/O_2(L)$ by the structure of $Aut(L_3(2))$, again contrary to (*). We have established the first three statements in (2), so it remains to show that one of cases (i)–(iii) holds.

If $Y/O_2(Y)$ is cyclic then $Y = L_1$ by (!!) since $Y/O_2(Y)$ is of exponent 3, so conclusion (i) of (2) holds. Therefore by A.1.25.1, we may assume $Y/O_2(Y) \cong E_9$ or 3^{1+2}. In the latter case, Y_Z satisfies the hypotheses of "Y", so we conclude $L_1 = Y_Z$ from (!!). Thus in either case, L_1 is normal in Y.

Let $H^* := H/Q_H$. As $M = LC_M(L/O_2(L))$ and $L_1 \not\leq Y_C$:

$$Y_M^* = L_1^* \times Y_C^*. \tag{$**$}$$

In particular if $Y^* \cong 3^{1+2}$ then $Y \not\leq M$ by (**).

Next we claim that if $Y_1 = O^2(Y_1) \leq Y$ is T-invariant with $Y_1/O_2(Y_1)$ of order 3, then $Y_1 \leq M$: For if $Y_1 \not\leq M$, then as $N_G(T) \leq M$ by 14.3.3.3, $Y_1T/O_2(Y_1T) \cong S_3$. Then as L_1 is normal in Y, the claim follows from 14.5.3.2. It then follows from the claim that if T acts reducibly on $Y/O_{2,\Phi}(Y)$, then $Y \leq M$. Now if $Y^* \cong 3^{1+2}$ we saw $Y \not\leq M$ and $L_1 = Y_Z$, so T acts irreducibly on Y^*/L_1^* and $L_1 = Y_M$, so that conclusion (ii) of (2) holds.

Thus we may assume that $Y^* \cong E_9$. Then $L_1 < Y$ so that T acts reducibly on Y^*, and hence $Y \leq M$ by an earlier remark. Then $Y^* = L_1^* \times Y_C^*$ by (**), with Y_C^* of order 3. Then $Y_C \in \mathcal{X}$, so Y_C is not normal in H by (!). Therefore as $Aut(Y^*) \cong GL_2(3)$ with $Aut_T(Y^*)$ normalizing Y_C, there is some 3-element $y \in H - Y$ inducing an automorphism of order 3 on Y^* centralizing L_1^*, with T acting on $Y_+ := Y\langle y \rangle$. As $M = LC_M(L/O_2(L))$, $Y_+ \not\leq M$, so $Y_+T \in \mathcal{H}_z$, and then we may assume $H = Y_+T$. If y^* has order 3, then $Y_+^* \cong 3^{1+2}$. As T is not irreducible on Y_+^*/L_1^*, this is contrary to an earlier reduction. Hence y has order 9, and we may choose y so that $Y_0 := \langle y, L_1 \rangle \trianglelefteq H$ with $Y_0/O_2(Y_0) \cong \mathbf{Z}_9$, and thus conclusion (iii) of (2) holds. □

LEMMA 14.5.21. *(1) The map φ defined from $Q_H/C_{Q_H}(U_H)$ to the dual space of $U_H/C_{U_H}(Q_H)$ by $\varphi : xC_{Q_H}(U_H) \mapsto C_{U_H}(x)/C_{U_H}(Q_H)$ is an H-isomorphism.*

(2) $[U_H, Q_H] = V_1$.

(3) $C_H(V_2)$ acts on L_2, and $m_3(C_H(V_2)) \leq 1$.

PROOF. Part (1) is F.9.7.

Assume case (1) of Hypothesis 14.3.1 holds. Then (2) follows from 13.3.14 and 14.5.15.1, while (3) follows from parts (1), (2), and (5) of 13.3.15.

Assume case (2) of Hypothesis 14.3.1 holds. Then (3) follows from 14.2.2.4. Assume $[U_H, Q_H] = 1$. Then by 14.5.15.4, $Q_H = C_H(U_H)$. By 14.5.15.1, $O_2(H/Q_H) = 1$, so that $U_H \in \mathcal{R}_2(H)$. Suppose there exists $K \in \mathcal{C}(H)$. As $Q_H = C_H(U_H)$, $K \in \mathcal{L}_f(G,T)$, contradicting 14.3.4.4. So H is solvable by 1.2.1.1, and hence $O(H^*) \neq 1$. Then $U_H = [U_H, O(H^*)] \oplus C_{U_H}(O(H^*))$ by Coprime Action. As $H > Q_H = C_H(U_H)$, $[U_H, O(H^*)] \neq 0$. Then $Z \cap [U_H, O(H^*)] \neq 0$, contradicting $H \leq C_G(V_1)$ since $Z = V_1$ when $L/O_2(L) \cong L_2(2)'$. □

14.6. Eliminating $\mathbf{L_2(2)}$ when $\langle \mathbf{V^{G_1}} \rangle$ is abelian

In this section we assume Hypothesis 14.5.1 holds with $L/O_2(L) \cong L_2(2)'$; in particular, $U := \langle V^{G_1} \rangle$ is abelian. Also Hypotheses 14.3.1.2 and 14.2.1 are satisfied, so we can appeal to results in sections 14.2 (with L in the role of "Y"), 14.3, and 14.5.

We will see in Theorem 14.6.25 that no further quasithin examples arise beyond those which we characterized earlier in Theorems 14.2.7 and 14.2.20, where U was nonabelian. Thus in this section we will be working toward a contradiction. Indeed as far as we can tell, there are no shadows.

As usual $Z := \Omega_1(Z(T))$ for $T \in Syl_2(G)$. Recall that by Hypothesis 14.2.1.4, V is of rank 2 with $V \trianglelefteq M$. Recall also that $C_T(L) = 1$ by 14.2.2.2.

We also adopt Notation 12.8.2: Thus $V_1 := Z \cap V = Z$ since Z is of order 2 by 14.2.2.6, and $G_1 = N_G(V_1) = C_G(Z) = M_c \in \mathcal{M}(T)$. Recall also that $L_1 := O^2(C_L(V_1)) = 1$; this simplifies the application of results from sections 14.3 and 14.5 involving L_1. For example as $L_1 = 1$, 14.2.5 says that:

$$\mathcal{H}(T,M) = \mathcal{H}_z.$$

For the remainder of this section, H denotes a member of $\mathcal{H}(T,M)$.

Recall $\tilde{G}_1 := G_1/V_1$ and notice $\tilde{H}$ makes sense as $H \leq G_1$ by definition of $\mathcal{H}_z$. As U is elementary abelian and $H \leq G_1$, $U_H := \langle V^H \rangle \leq U$ is also elementary abelian (cf. 14.5.15.2).

LEMMA 14.6.1. *(1) $G_1 = !\mathcal{M}(H)$.*
(2) $O_{2,p}(H) \cap M = O_2(H)$ for each odd prime p.
(3) If $K \in \mathcal{C}(H)$, then $K \not\leq M$.
(4) If $1 \neq X = O^2(X) \trianglelefteq H$, then $XT \in \mathcal{H}(T,M)$.
(5) $O_{2,F^*}(H)$ centralizes $\Omega_1(Z(O_2(H)))$.
(6) If $O_2(H) \leq T_1 \trianglelefteq T$, then $N_G(T_1) \leq N_G(\Omega_1(Z(T_1))) \leq G_1$.

PROOF. Part (1) is 14.2.3, part (3) is 14.5.19, and part (2) follows as case (1) of 14.5.20 holds because $L/O_2(L) \not\cong L_3(2)$. Under the hypotheses of (4), $O_2(X) < O_{2,F^*}(X)$, and $O_{2,F^*}(X) \not\leq M$ by (2) and (3), so (4) holds.

Let $R := O_2(H)$, $W := \Omega_1(Z(R))$, and $\hat{H} := H/C_H(W)$. Suppose there is $K \in \mathcal{C}(H)$ with $[W,K] \neq 1$. Then as R centralizes W, $K \in \mathcal{L}_f(G,T)$ by A.4.9, contrary to 14.3.4.4. This contradiction shows that $O_{2,E}(H)$ centralizes W.

Suppose (5) fails. Then by the previous remark, for some odd prime p, $X := O^2(O_{2,p}(H))$ is nontrivial on W. As $O_2(X) \leq R \leq C_H(W)$, $\hat{X}$ is of odd order, so $W = [W,X] \oplus C_W(X)$ by Coprime Action. Then as $[W,X] \neq 0$, $Z \leq [W,X]$ since Z has order 2. However $X \leq H \leq G_1$ by (1), so also $Z \leq C_W(X)$. This contradiction establishes (5).

Assume the hypotheses of (6), and let $Z_1 := \Omega_1(Z(T_1))$. As $R \leq T_1$ by hypothesis, and $H \in \mathcal{H}^e$, $Z_1 \leq W$, so that $[O_{2,F^*}(H), Z_1] = 1$ by (5). As $T_1 \trianglelefteq T$, T acts on Z_1, so $H_1 := O_{2,F^*}(H)T \leq N_G(Z_1)$. Now $H_1 \in \mathcal{H}(T, M)$ by (4), so $G_1 = !\mathcal{M}(H_1)$ by (1), and then (6) follows. $\square$

LEMMA 14.6.2. *If $1 \neq X = O^2(X) \leq O_{2,F^*}(H)$ with $Q_H \leq N_H(X)$, then $Z \leq [U_H, X]$.*

PROOF. As $C_H(\tilde{U}_H) = Q_H$ while $X = O^2(X) \neq 1$, $U_X := [U_H, X] \neq 1$. As Q_H acts on X, U_X is normal in Q_H, but U_X is not central in Q_H by 14.6.1.5. Then $1 \neq [U_X, Q_H] \leq U_X \cap Z$ using 14.5.15.1, so as $|Z| = 2$ we conclude that $Z \leq U_X$. $\square$

14.6.1. Preliminary results on suitable involutions in $\mathbf{U_H}$. In the proof of Theorem 14.6.18 and also at the end of the section, we will need to control the centralizers of involutions in U_H which satisfy certain special conditions (cf. 14.6.17.3 and 14.6.24.1). Thus we are led to define $\mathcal{U}(H)$ to consist of those u satisfying

(U0) $u \in U_H$,
(U1) $T_u := C_T(u) \in Syl_2(C_H(u))$, and $T_0 := C_T(\tilde{u})$ is of index 2 in T,
(U2) $[O_2(G_1), u] \neq 1 \neq [O_2(G_1), uu^t]$ for $t \in T - T_0$, and
(U3) $T = N_{G_1}(T_0)$.

LEMMA 14.6.3. *Assume $u \in \mathcal{U}(H)$. Then*

(1) $|T : T_u| = 4$, $|T_0 : T_u| = 2$, $T_0 = N_T(T_u)$, and $T_0 = O_2(G_1)T_u = Q_H T_u$.
(2) $N_G(T_0) = T$.
(3) $C_{Q_H}(u) \not\leq C_{Q_H}(V)$ and $L = [L, C_{O_2(G_1)}(u)] = [L, C_{Q_H}(u)]$.
(4) $N_G(T_u) = T_0$, $T_u \in Syl_2(C_G(u))$, and $u \notin z^G$.

PROOF. Set $Q_1 := O_2(G_1)$. First $[Q_1, u] \neq 1$ by (U2) and $u \in U_H \leq U$ by (U0), so $[Q_H, u] = [Q_1, u] = V_1$ is of order 2 by 14.5.15.1. Hence $C_{Q_1}(u) = Q_1 \cap T_u$ is of index 2 in Q_1, T_u is of index 2 in T_0, and $T_0 = Q_1 T_u = Q_H T_u$ as $C_T(\tilde{u}) = T_0$ and $C_T(u) = T_u$ by (U1).

Pick $t \in T - T_0$. If t normalizes $C_{Q_1}(u)$, then

$$C_{Q_1}(u) = C_{Q_1}(u)^t = C_{Q_1}(u^t).$$

Therefore for $x \in Q_1 - C_{Q_1}(u)$, $z = [x, u] = [x, u^t]$, and hence $Q_1 = \langle x, C_{Q_1}(u) \rangle$ centralizes uu^t, contrary to (U2). Thus t does not normalize $C_{Q_1}(u)$, so as $N_T(T_u)$ normalizes $T_u \cap Q_1 = C_{Q_1}(u)$, $t \notin N_T(T_u)$. As $|T_0 : T_u| = 2$ we conclude that $T_0 = N_T(T_u)$, and as $|T : T_0| = 2$ by (U1), $|T : T_u| = 4$, completing the proof of (1).

As $Q_H \leq T_0$ by (1), and $T_0 \trianglelefteq T$ by (U1), we may apply 14.6.1.6 to conclude that $N_G(T_0) \leq G_1$. Then as $N_{G_1}(T_0) = T$ by (U3), (2) holds. By (1), $T_0 = N_T(T_u)$, so $T_0 \in Syl_2(N_G(T_u))$ by (2).

As $[U, Q_1] = V_1$ by 14.5.21.2, and $U = \langle V^{G_1} \rangle$, also $[V, Q_1] = V_1$, so that $C_{Q_1}(V)$ is of index 2 in Q_1 since $m(V) = 2$. Suppose that $C_{Q_1}(u) \leq C_{Q_1}(V)$. Then $C_{Q_1}(u) = C_{Q_1}(V)$, as both are of index 2 in Q_1, so $\langle u \rangle C_U(Q_1) = V C_U(Q_1)$ by the duality in 14.5.21.1. Thus for $t \in T - T_0$, $\langle u^t \rangle C_U(Q_1) = V C_U(Q_1)$, so that $uu^t \in C_U(Q_1)$. This is impossible since Q_1 does not centralize uu^t by (U2), so

$C_{Q_1}(u) \not\leq C_{Q_1}(V)$. Since $Q_1 \leq Q_H$, $C_{Q_H}(u) \not\leq C_{Q_H}(V)$, and since $L = O^2(L)$ induces $\mathbf{Z}_3$ on V, also

$$L = [L, C_{Q_1}(u)] = [L, C_{Q_H}(u)],$$

completing the proof of (3).

Next $N_{G_1}(T_u)$ normalizes $T_uQ_1 = T_0$ using (1), so $N_{G_1}(T_u) \leq T$ by (2); hence again using (1), $N_{G_1}(T_u) = N_T(T_u) = T_0$.

We now show that to prove (4) it will suffice to establish that $I := N_G(T_u) \leq G_1$: For in that case $I = T_0$ by the previous paragraph, establishing the first assertion of (4). Next let $T_u \leq S \in Syl_2(C_G(u))$. Then $N_S(T_u) \leq I = T_0 \leq H$, so as $T_u \in Syl_2(C_H(u))$ by (U1), $S = T_u$. In particular $u \notin z^G$ as $|T_u| < |T|$. This completes the proof that (4) holds if $I \leq G_1$, so we may assume that $I \not\leq G_1$, and it remains to establish a contradiction. We saw earlier that $T_0 \in Syl_2(I)$, so in particular $T_0 < I$ as $T_0 \leq G_1$.

We claim that $N_I(C) = T_0$ for each $1 \neq C \leq T_0$ with $C \trianglelefteq T$, so we assume that $T_0 < N_I(C)$ and derive a contradiction. We saw that $T_0 = N_{G_1}(T_u)$, so $N_I(C) \not\leq G_1$. Hence as $\mathcal{M}(T) = \{M, G_1\}$ by 14.2.2.5, we must have $N_G(C) \leq M$. Therefore as $|M : M \cap G_1| = 3$ by 14.2.2.1, and $N_I(C) \not\leq G_1$, $M = (M \cap G_1)N_I(C)$; hence as I normalizes $Z(T_u)$,

$$V = \langle Z^M \rangle = \langle Z^{N_I(C)} \rangle \leq Z(T_u).$$

But then $C_{Q_H}(u) = T_u \cap Q_H \leq C_{Q_H}(V)$, contrary to (3), so the claim is established. In particular $C(I, T_0) = T_0$ as $T_0 \trianglelefteq T$.

We have seen that $T_0 \in Syl_2(I)$, with $|T_0 : T_u| = 2$, so that I/T_u and hence also I is solvable by Cyclic Sylow 2-Subgroups A.1.38. Also $F^*(I) = O_2(I)$ by 1.1.4.3 as $Z \leq T_u$. So since $C(I, T_0) = T_0 < I$, we may apply the Local $C(G,T)$-Theorem C.1.29 to conclude that $I = T_0B$, where B is the product of $s := 1$ or 2 blocks of type A_3 which are not contained in G_1. Further $N_I(J(T_0)) = T_0$ as $C(I, T_0) = T_0$, so Solvable Thompson Factorization B.2.16 says that $I/O_2(I)$ contains the direct product of s copies of S_3. Therefore if $s = 2$, then $I/O_2(I)$ contains $S_3 \times S_3$, contradicting $T_u \trianglelefteq I$ and $|T_0 : T_u| = 2$. Thus $s = 1$, so $B \cong A_4$ by C.1.13.c.

Now the hypotheses of Theorem C.6.1 are satisfied with I, T, T_0 in the roles of "H, Λ, T_H"; for example, part (iv) of that hypothesis follows from the claim and the facts that $T_0 < I$ and $|T : T_0| = 2$. Therefore case (a) or (b) of Theorem C.6.1.6 holds since $s = 1$; thus $I \cong S_4$ or $\mathbf{Z}_2 \times S_4$, and in particular $T_u = O_2(I)$. By C.6.1.1, $T_0 = J(T_0) = O_2(I)O_2(I)^x$ for each $x \in T - T_0$, and hence $T_0 = T_uT_u^x$. However by (3), T_u is nontrivial on V, so that $T = T_0C_T(V)$ since $|T : C_T(V)| = 2$; thus we may take $x \in C_T(V)$. Next as $T_0 = O_2(I)O_2(I)^x$, $C_{\tilde{T}_0}(x) = \tilde{Z}_0\langle \tilde{b}\tilde{b}^x \rangle$, where $Z_0 := Z(T_0)$ and $O_2(O^2(I)) =: \langle b, z \rangle$. Further if $I \cong \mathbf{Z}_2 \times S_4$, then $Z_0 \cong E_4$, and hence $[Z_0, x] = Z$ as Z has order 2. However in either case, bb^x is of order 4, so that $\Omega_1(C_{T_0}(x)) = Z$; this is a contradiction, as $x \in C_T(V)$ and $V \leq T_u \leq T_0$. This contradiction completes the proof of (4), and hence of 14.6.3. □

For the remainder of this subsection, u denotes a member of $\mathcal{U}(H)$.

Define $\mathcal{I} := \mathcal{I}(T, u)$ to be the set of $I \in \mathcal{H}(T_u)$ such that I is contained in neither G_1 nor M. We will see later (cf. 14.6.17.5 and 14.6.24.4) that for suitable $u \in \mathcal{U}(H)$, $C_G(u) \in \mathcal{I}$, so that $\mathcal{I}$ is nonempty.

Let $\mathcal{I}^*$ consist of those $I \in \mathcal{I}$ such that $T \cap I$ is not properly contained in $T \cap J$ for any $J \in \mathcal{I}$. Finally let $\mathcal{I}_*$ be the minimal members of $\mathcal{I}^*$ under inclusion.

For $I \in \mathcal{I}$, set $T_I := T \cap I$ and $I_z := I \cap G_1$.

The next two observations are straightforward from the definitions:

LEMMA 14.6.4. *If $I \in \mathcal{I}^*$ and $T_I \leq J \in \mathcal{I}$, then $J \in \mathcal{I}^*$ and $T_J = T_I$.*

LEMMA 14.6.5. *If $I \in \mathcal{I}$ and $I \leq J \in \mathcal{H}$, then $J \in \mathcal{I}$. If further $I \in \mathcal{I}^*$, then $J \in \mathcal{I}^*$ and $T_J = T_I$.*

Recall from Definition F.6.1 the discussion of Goldschmidt triples.

LEMMA 14.6.6. *Assume $I \in \mathcal{I}^*$, and let $L_I := O^2(L \cap I)$. Then*

(1) T_I is either T_u or T_0.

(2) $T_I \in Syl_2(I)$.

(3) If $I \cap M \not\leq G_1$ then $L = L_I O_2(L)$ and $LT = L_I T_I O_2(LT)$.

(4) Either $C(I, T_I) \leq I_z$, or $L = L_I O_2(L)$ and $LT = L_I T_I O_2(LT)$.

(5) If $I \in \mathcal{I}_*$ then either I_z is the unique maximal subgroup of I containing T_I, or $L = L_I O_2(L)$ and $LT = L_I T_I O_2(LT)$.

(6) Assume $|T| > 2^9$ and $L = L_I O_2(L)$. Assume further that there exists H_2 with $T_0 \leq H_2 \leq C_H(\tilde{u})$, $H_2/O_2(H_2) \cong S_3$, $H_2 \not\leq M$, and H_2 has at least two noncentral 2-chief factors. Then setting $I_2 := O^2(H_2)T_I$, $I_1 := L_I T_I$, and $I_0 := \langle I_1, I_2 \rangle$, we have $I_0 \in \mathcal{I}^*$ and (I_0, I_1, I_2) is a Goldschmidt triple.

(7) T_I is not normal in I.

PROOF. We first establish (1) and (2). Let $T_I \leq S \in Syl_2(I)$ and set $Q_1 := O_2(G_1)$. Now $N_G(T_u) = T_0$ by 14.6.3.4, so as T_u is of index 2 in T_0 by 14.6.3.1, $N_S(T_u) = T_u$ or T_0. In the first case, $S = T_u = T_I$, so that (1) and (2) hold. In the second case I is not contained in M or G_1, so that $T_I < T$ by 14.2.2.5, and hence $T_I = T_0$ since $|T : T_0| = 2$ by (U1). Then as $N_G(T_0) = T$ by 14.6.3.2, $N_S(T_I) \leq N_{T \cap I}(T_0) = T_I$, so that $S = T_I = T_0$, and so (1) and (2) hold in this case also.

Next we prove (3), so assume $X := I \cap M \not\leq G_1$. As L is transitive on $V^\#$, $M = L(M \cap G_1)$ and $|M : M \cap G_1| = 3$ is prime, so $M = X(M \cap G_1)$. Next $T_u \leq T_I$, so by 14.6.3.3, $L = [L, a]$ for some $a \in Q_1 \cap T_I \leq X$. As $LQ_1 \trianglelefteq L(M \cap G_1) = M$, $\langle a^X \rangle \leq LQ_1 \cap X$. If $a^X \subseteq Q_1$, then as $M = X(G_1 \cap M)$ and $Q_1 \trianglelefteq G_1$, $a^M \subseteq Q_1$ so that $\langle a^M \rangle$ is a 2-group and hence $a \in O_2(M)$, contradicting $L = [L, a]$. Thus $a^X \not\subseteq Q_1$, so as Q_1 is of index 3 in LQ_1 and $L = O^2(LQ_1)$, $L \leq \langle a^X \rangle O_2(LQ_1)$. Then as $L = O^2(LQ_1)$, $O^2(\langle a^X \rangle) \leq L \cap X$, so that $L = (L \cap X)O_2(L) = L_I O_2(L)$, and as $L = [L, a]$, $LT = L_I T_I O_2(LT)$. Hence (3) holds.

Next suppose there is $1 \neq C$ char T_I with $N_I(C) \not\leq I_z$. As $T_I < T$ by (1), T_I is proper in $N_T(T_I) \leq N_G(C)$. Then as $I \in \mathcal{I}^*$, $N_G(C) \notin \mathcal{I}$ by 14.6.4, and hence $N_G(C) \leq M$ since $N_I(C) \not\leq I_z$. Therefore $I \cap M \not\leq G_1$, so (4) follows from (3).

Next assume $I \in \mathcal{I}_*$ and let Y be a maximal subgroup of I containing T_I. Then by minimality of I, Y is contained in G_1 or M, so that Y is I_z or $I \cap M$ by maximality of Y. Thus (5) also follows from (3).

Assume the hypotheses of (6), and set $I_1 := L_I T_I$. By (2), $T_I \in Syl_2(I)$, so that $T_I \in Syl_2(I_1)$. As $L = L_I O_2(L)$, we conclude from 14.6.3.3 that $I_1/O_2(I_1) \cong S_3$.

Next since $T_I \leq T_0 \leq H_2$ using (1) and the hypothesis for (6), $I_2 := O^2(H_2)T_I$ is a subgroup of H_2 with $O^2(I_2) = O^2(H_2)$. Also $O^2(H_2)$ centralizes $\tilde{u}$ and hence also u, so as $T_u \in Syl_2(C_H(u))$ by 14.6.3.4, $T_u \in Syl_2(O^2(H_2)T_u)$. Thus as $T_u \leq T_I$, $T_I \in Syl_2(I_2)$. By (U1), $T_0 \in Syl_2(C_H(\tilde{u}))$ so that $H_2 = O^2(H_2)T_0$, while

$H_2/O_2(H_2) \cong S_3$ by the hypothesis of (6). Then since $T_0 = Q_1T_u$ by 14.6.3.1, and Q_1 is normal in H, we conclude $I_2/O_2(I_2) \cong S_3$.

Suppose first that $O_2(I_0) \neq 1$. Since $L = L_IO_2(L)$, we have $I_1 \not\leq G_1$, while $H_2 \not\leq M$ by hypothesis, so $I_2 \not\leq M$ since we saw $H_2 = O^2(H_2)T_0$. Thus $I_0 \in \mathcal{I}$, and indeed as $T_I \leq I_0$, $I_0 \in \mathcal{I}^*$ and $T_{I_0} = T_I$ by 14.6.4, so that $T_I \in Syl_2(I_0)$ by (2). We conclude that (I_0, I_1, I_2) is a Goldschmidt triple in the sense of Definition F.6.1, so that (6) holds in this case.

So we suppose instead that $O_2(I_0) = 1$, and it remains to derive a contradiction. By construction, Hypothesis F.1.1 is satisfied with I_1, I_2, T_I in the roles of "L_1, L_2, S". So by F.1.9, $\alpha := (I_1, T_I, I_2)$ is a weak BN-pair of rank 2, and as T_I plays the role of "B_j" for $j = 1, 2$, α appears on the list of F.1.12. Since $I_i/O_2(I_i) \cong S_3$, and $I_2/O_2(I_2)$ has at least two noncentral chief factors by hypothesis, it follows that α is of type $G_2(2)'$, $G_2(2)$, M_{12} or $Aut(M_{12})$. But then $|T_I| \leq 2^7$, so as $|T : T_I| \leq 4$ by (1) and 14.6.3.1, $|T| \leq 2^9$, contrary to the hypothesis for (6). This contradiction completes the proof of (6).

Finally observe that as $T_I = T_u$ or T_0 by (1), $N_G(T_I) \leq T$ by (2) or (4) of 14.6.3. Thus (7) holds since $I \not\leq M$. This completes the proof of (7), and hence of 14.6.6. □

LEMMA 14.6.7. *Assume $I \in \mathcal{I}^*$. Then*

(1) The hypotheses of 1.1.5 are satisfied with I, G_1 in the roles of "H, M".
(2) $F^*(I_z) = O_2(I_z)$.
(3) $O(I) = 1$.
(4) If K is a component of I, then $K \not\leq I_z$ and $\langle K, T_I\rangle \in \mathcal{I}^*$.

PROOF. As $u \in U_H \leq U \leq O_2(G_1)$, $u \in O_2(I \cap G_1)$. Therefore

$$C_{O_2(G_1)}(O_2(I \cap G_1)) \leq C_{O_2(G_1)}(u) \leq T_u \leq I,$$

so (1) holds; hence we may apply 1.1.5. Then 1.1.5.1 implies (2). In view of (2), to prove (3) it suffices to show that $O(I) \leq G_1$. But as L is transitive on $V^\#$, $V \leq O_2(C_G(v))$ for each $v \in V^\#$ since $V \leq O_2(G_1)$ by 14.5.15.1. Therefore $[V, C_{O(I)}(v)] \leq O(I) \cap O_2(C_G(v)) = 1$. Then using Generation by Centralizers of Hyperplanes A.1.17, $O(I) \leq C_I(V) \leq G_1$, establishing (3).

Suppose K is a component of I. By 1.1.5.3, $K \not\leq I_z$. Further if $K \leq M$, then as $m(V) = 2$ by 14.2.1.4, $K \leq C_I(V) \leq I_z$, contrary to the previous remark; so also $K \not\leq M$. Thus $\langle K, T_I\rangle \in \mathcal{I}$, so that $\langle K, T_I\rangle \in \mathcal{I}^*$ by 14.6.4, completing the proof of (4). □

LEMMA 14.6.8. *Assume $I \in \mathcal{I}^*$ and $F^*(I) \neq O_2(I)$. Then $m_2(I/O_2(I)) \geq m(U_HO_2(I)/O_2(I)) \geq m(U_H/C_{U_H}(Q_H))$.*

PROOF. By 14.6.7.3, $O(I) = 1$, so as $F^*(I) \neq O_2(I)$ by hypothesis, we conclude there is a component K of I. By 14.6.7.4, z does not centralize K, so that $Z \cap O_2(I) = 1$ as Z has order 2. Set $P := C_{Q_H}(u)$. By 14.5.15.1, $[U_H \cap O_2(I), P] \leq Z \cap O_2(I) = 1$. So since $Z \not\leq O_2(I)$, using the duality in 14.5.21.1 we obtain

$$U_H \cap O_2(I) < C_{U_H}(P) = \langle u\rangle C_{U_H}(Q_H).$$

Therefore

$$m_2(I/O_2(I)) \geq m(U_HO_2(I)/O_2(I)) = m(U_H/(U_H \cap O_2(I)))$$
$$> m(U_H/C_{U_H}(P)) = m(U_H/C_{U_H}(Q_H)) - 1,$$

so the lemma is established. □

LEMMA 14.6.9. *Assume* $I \in \mathcal{I}^*$, $|T : Q_H| > 4$, *and* $m(U_H/C_{U_H}(Q_H)) \geq 4$. *Then* $|T| > 2^{11}$, *and*

$$LT = O^2(L \cap I)T_I O_2(LT).$$

PROOF. Observe first that by the duality in 14.5.21.1,

$$m(Q_H/C_{Q_H}(U_H)) = m(U_H/C_{U_H}(Q_H)) =: m,$$

with $Z \leq C_{U_H}(Q_H)$, so that $|Q_H| \geq 2^{2m+1} \geq 2^9$ since $m \geq 4$ by hypothesis. As we also assume $|T : Q_H| > 4$, $|T| > 2^{11}$, establishing the first conclusion of 14.6.9. By 14.6.6.1, $T_I = T_u$ or T_0, so $|T : T_I| \leq 4$ by 14.6.3.1, and hence $|T_I| > 2^9$.

Thus we may assume that $LT > O^2(L \cap I)T_I O_2(LT)$, and it remains to derive a contradiction. Then by 14.6.6.4, $C(I, T_I) \leq I_z$. As we are working toward a contradiction, we may also assume that I is minimal under inclusion; that is, $I \in \mathcal{I}_*$. Then by 14.6.6.5, I_z is the unique maximal subgroup of I containing T_I. Since T_I is not normal in I by 14.6.6.7, I is a minimal parabolic in the sense of Definition B.6.1.

We first treat the lengthier case where $F^*(I) = O_2(I)$. Here since $T_I \in Syl_2(I)$ by 14.6.6.2, and I is a minimal parabolic, we may apply C.1.26: Since $C(I, T_I) \leq I_z < I$, we conclude that $I = T_I K_1 \cdots K_s$, where K_i is a χ_0-block of I not contained in I_z, and T_I is transitive on the K_i. Further $s = 1$ or 2 as I is an SQTK-group, and the action of $J(T_I)$ on $O_2(K)$ is described in E.2.3. Also K_1 is not an $L_2(2^n)$-block for $n > 1$, as $I_z = C_I(z)$ is the unique maximal overgroup of T_I in I, whereas when K_1 is an $L_2(2^n)$-block, the center of that overgroup is $Z(I)$. Thus K_1 is a block of type A_3 or A_5.

Observe using 14.6.5 and 14.6.6.2 that:

(a) If $1 \neq S \leq T_I$ with $S \trianglelefteq I$, then $N_G(S) \in \mathcal{I}^*$ and $N_T(S) = T_I \in Syl_2(N_G(S))$.

Since $T_I < T$ by 14.6.6.1, we may choose $r \in N_T(T_I) - T_I$ with $r^2 \in T_I$. Then by (a),

(b) r acts on no nontrivial subgroup S of T_I normal in I.

Set $K := K_1 \cdots K_s$, so that $I = KT_I$. Assume first that K is not the product of two A_5-blocks. As $F^*(I) = O_2(I)$, this assumption establishes part (i) of the hypothesis of Theorem C.6.1, with I, $T_I\langle r \rangle$, T_I in the roles of "H, Λ, T_H", while (a) gives part (iv) of that hypothesis, and (ii) and (iii) are immediate. If K is an A_3-block then $|T_I| \leq 16$ since case (a) or (b) of C.6.1.6 must hold, contrary to $|T_I| > 2^9$ in the first paragraph of the proof. Therefore K is an A_5-block or a product of two A_3-blocks. In either case by C.1.13.c, $O_2(I) = D \times O_2(K)$, where $D := C_{T_I}(K)$, and by C.6.1.4, D is elementary abelian, so that $D \leq D_I := \Omega_1(Z(J(T_I))$. Then we conclude from the action of $J(T_I)$ on $O_2(K)$ described in E.2.3, that $|D_I : D| = 4$. As $D \cap D^r$ is normalized by $KT_I = I$ and r, $D \cap D^r = 1$ by (b), so that $|D| \leq 4$. But now $|T_I| \leq 4|Aut(K)|_2 \leq 2^9$, again contrary to the first paragraph.

Therefore $K = K_1 \times K_2$ is the product of two A_5-blocks. Set $K_z := O^2(I_z)$ and $R_z := O_2(I_z)$. Then $K_z T_I / R_z \cong S_3$ wr $\mathbf{Z}_2$, and $J(R_z) = J(O_2(I))$ using E.2.3.3 and B.2.3.3. So applying (a) to $J(R_z)$ in the role of "S", we obtain $T_I \in Syl_2(N_G(J(R_z)))$; hence $T_I \in Syl_2(N_{G_1}(R_z))$. Thus as $R_z = O_2(I_z)$,

$R_z = O_2(N_{G_1}(R_z))$ by A.1.6—that is $R_z \in \mathcal{B}_2(G_1)$, so setting $Q_1 := O_2(G_1)$, we conclude from C.2.1.2 that

(c) $Q_1 \leq R_z$.

But $T_0 = T_u Q_1$ by 14.6.3.1, so as $R_z \leq T_I$, we conclude from (c) and 14.6.6.1 that

(d) $T_I = T_0$.

By (U1), $|T : T_0| = 2$, so $T = T_I\langle r\rangle$ by (d). Further as $G_1 \in \mathcal{H}^e$, (c) says

(e) $Z_z := \Omega_1(Z(R_z)) \leq \Omega_1(Z(Q_1)) =: Z_1$.

By 14.6.1.5 and (e):

(f) $Y := O^2(O_{2,F^*}(G_1))$ centralizes Z_1 and Z_z.

Next $K_z = X_1 \times X_2$ where $X_i := K_z \cap K_i$, $R_i := O_2(X_i) \cong Q_8^2$, and $|X_i : R_i| = 3$. Further as T_I is of index 2 in T, Hypothesis C.5.1 is satisfied with I, T_I, T_I, T in the roles of "H, T_H, R, M_0". Similarly Hypothesis C.5.2 is satisfied using (b), as is the hypothesis $|T : T_I| = 2$ in C.5.6.7. So by C.5.6.7, $D := C_{T_I}(K) \leq Z(\mathrm{Baum}(T_I))$ is elementary abelian, and $O_2(I) = DO_2(K)$. Hence setting $Z_0 := Z(R_1R_2)$, we have

(g) $O_2(I) = DO_2(K)$ and $Z_z = DZ_0$.

Observe since $|T : T_0| = 2 = |Z|$ that z is diagonally embedded in $Z(R_1) \times Z(R_2) = Z_0$.

We claim that $D = 1$. Suppose instead that $D \neq 1$. Then as D is normal in $KT_I = I$, (a) and (d) say that $I_D := N_G(D) \in \mathcal{I}^*$, and $T_I = T_0 \in Syl_2(I_D)$.

Assume first that $LT = L_D T_I O_2(LT)$, where $L_D := O^2(L \cap I_D)$. As $T_I = T_0$ and

$$|O_2(L) : O_2(L) \cap T_0| \leq |T : T_0| = 2,$$

L_D centralizes $O_2(L)/(O_2(L) \cap T_0)$, and hence $L = O^2(L) = L_D$. Next $K \leq C_G(D) \leq I_D$, and indeed $K_1 \in \mathcal{L}(I_D, T_0)$, so that $K_1 \leq K_1^+ \in \mathcal{C}(I_D)$ with K_1^+ described in 1.2.8.2. Then using 1.2.2.a, $L \leq O^{3'}(I_D) = \langle K_1^{+T_0}\rangle \leq C_G(D)$. Therefore $C_T(L) \neq 1$, contrary to 14.2.2.6 as we mentioned at the start of the section.

This contradiction shows that $LT > L_D T_I O_2(LT)$. Next assume $F^*(I_D) \neq O_2(I_D)$. Since $O(I_D) = 1$ by 14.6.7.3, I_D has a component K_D. By 14.6.7.1, K_D appears in the list of 1.1.5.3. As that list does not contain the possible proper overgroups of KT_I in 1.2.8.2, we conclude K centralizes K_D. But each component in that list has order divisible by 3 or 5, so $m_p(KK_D) > 2$ for $p = 3$ or 5, contrary to I_D an SQTK-group. Thus $F^*(I_D) = O_2(I_D)$.

Since $L_D T_I O_2(LT) < LT$, 14.6.6.4 says that $C(I_D, T_I) \leq I_{D,z} := I_D \cap G_1$. Thus as $F^*(I_D) = O_2(I_D)$, we may apply the local $C(G,T)$-Theorem C.1.29 to conclude that I_D is the product of $I_{D,z}$ with one or two χ_0-blocks. Since I_D contains $I = KT_I$, where K is the product of two A_5-blocks not in $I_{D,z}$, and no A_5-block is contained in a larger χ_0-block, we conclude that the blocks in K are the blocks in I_D, and $K \trianglelefteq I_D = KI_{D,z}$. By (e) and (f), YQ_1 centralizes Z_z, so $YQ_1 \leq I_{D,z}$ by (g). Then by A.4.4.1 applied with G_1, I_D, $I_{D,z}$, Q_1Y in the roles of "H, K, $H \cap K$, X", we conclude that $Q_1 = O_2(I_{D,z})$. Using A.1.6, $O_2(I_D) \leq O_2(I_{D,z}) = Q_1$ and $O_2(I_D) \leq O_2(I)$. Further $O_2(I) = O_2(K)D$ by (g), and $O_2(K) \leq O_2(I_D)$ as $K \trianglelefteq I_D$, so we conclude that $O_2(I_D) = O_2(I)$. Therefore $O_2(I) = O_2(I_D) \leq Q_1 \leq R_z$ by (c). As K_i is an A_5-block, $J(R_z) = J(O_2(I))$, so $J(O_2(I)) = J(Q_1)$ by B.2.3.3. Therefore $I \leq N_G(J(Q_1)) = G_1$ as $G_1 \in \mathcal{M}$ by 14.6.1.1, contrary to $I \in \mathcal{I}$.

This contradiction establishes the claim that $D = 1$. Now by (e) and (g):

(h) $O_2(I) = O_2(K)$ and $Z_0 = Z_z \leq Z_1$.

Thus $I \cong (S_5/E_{16})$ wr $\mathbf{Z}_2$. It follows also that $I_z = C_I(z) \cong (S_4/E_{16})$ wr $\mathbf{Z}_2$, and:

(i) $K_z = O^2(I_z) \cong \mathbf{Z}_3/Q_8^2 \times \mathbf{Z}_3/Q_8^2$, $C_{R_z}(O_2(K_z)) = Z_0$, and $C_{R_z}(O_2(K_z)/Z_0) = O_2(K_z)$.

Set $G_1^+ := G_1/Z_1$ and $C_1 := C_G(Z_1)$. As $C_1 \trianglelefteq G_1 \in \mathcal{H}^e$, $C_1 \in \mathcal{H}^e$ by 1.1.3.1, so that $Q_1 = O_2(C_1) = F^*(C_1)$. Then $Q_1^+ = F^*(C_1^+)$ by A.1.8. Let X be the preimage in G_1 of $F^*(G_1^+)$; as Y centralizes Z_1 by (f), $O^2(X) \leq Y \leq C_1$; so as $Q_1^+ = F^*(C_1^+)$, $O^2(X) = 1$ and hence $F^*(G_1^+) = Q_1^+$. Thus using B.2.14:

(j) $E^+ := \Omega_1(Z(T^+)) \cap R_1^+ R_2^+ \leq \Omega_1(Z(Q_1^+)) =: F^+$.

Next $O_2(K) = U_1 \times U_2$, where $U_i := O_2(K_i)$, and $E_i := U_i \cap R_i$ is a hyperplane of U_i. Let $E_0 := E_1 E_2$. Then as $I \cong S_5/E_{16}$ wr $\mathbf{Z}_2$, $Z(T_0/Z_0) \leq E_0/Z_0$ and I_z is irreducible on E_0/Z_0; so as $Z_0 \leq Z_1$ by (h), we conclude from (j) that

(k) $E_0 \leq E \leq F \leq Q_1 \leq O_2(K_z T_I)$,

where E and F are the preimages of E^+ and F^+ in G_1.

Recall $r \in T - T_I$, $T_I = T_0$, and case (iii) of C.5.6.7 holds. Hence by C.5.6.7, $A := O_2(K)$ and A^r are the two T_0-invariant members of $\mathcal{A}(T_0)$, and $A \cap A^r = [A, A^r]$ is of rank 4. Thus as E_0 is of rank 6, $E_0 \not\leq A \cap A^r$, so as $E_0^r \leq A^r$, $E_0^r \not\leq A$. Now F is normal in G_1, so $E_0^r \leq F \leq O_2(K_z T_I)$ by (k). Then as $E_0^r \not\leq A$ and $K_z T$ is irreducible on $O_2(K_z T_I)/A$:

(l) $O_2(K_z) = E_0[E_0^r, K_z] \leq F \leq Q_1 \leq R_z$.

It follows from (l) that $Z_1 = \Omega_1(Z(Q_1)) \leq C_{R_z}(O_2(K_z))$, so we conclude from (h) and (i) that:

(m) $Z_1 = Z_0$.

Then as $F^+ = \Omega_1(Z(Q_1^+))$, (l) says $Q_1 \leq C_{R_z}(O_2(K_z)^+)$, while as $Z_1 = Z_0$ by (m), $C_{R_z}(O_2(K_z)^+) = O_2(K_z)$ by (i). So we conclude from (l) that:

(n) $Q_1 = O_2(K_z)$.

From (i), $\widetilde{O_2(K_z)} \cong Q_8^4$, so by 14.5.15.1 and (n), $\tilde{V} \leq Z(\tilde{Q}_1) = Z(\widetilde{O_2(K_z)}) = \tilde{Z}_0$. Thus $V = Z_0 = Z_1 \trianglelefteq G_1$, contrary to $G_1 \not\leq M = N_G(V)$. This contradiction finally completes the treatment of the case $F^*(I) = O_2(I)$.

Thus it remains to treat the case $F^*(I) \neq O_2(I)$. As $O(I) = 1$ by 14.6.7.3, there is a component K of I. As I_z is the unique maximal overgroup of T_I in the minimal parabolic I, I and I_z are described in E.2.2, and in particular $I = K_0 T_I$, where $K_0 := \langle K^{T_I} \rangle$. On the other hand by 14.6.7.1, K is described in 1.1.5.3; in particular $K = [K, z]$ with z 2-central in I.

We consider the possibilities from the intersection of the lists of E.2.2 and 1.1.5.3: First suppose $K/O_2(K)$ is a Bender group. Then by E.2.2, I_z is the normalizer of a Borel subgroup B of K_0, and centralizes no element of $(O_2(B)/O_2(K_0))^\#$, whereas I_z centralizes the projection of z on $O_2(B)/O_2(K_0)$. Similarly if $K/O_2(K) \cong Sp_4(2^n)'$ or $L_3(2^n)$, then $N_{T_I}(K)$ is nontrivial on the Dynkin diagram of $K/O_2(K)$ by E.2.2, so again I_z is the normalizer of a Borel subgroup B of K_0, and hence $n = 1$ since I_z centralizes the projection of z on $O_2(B)/O_2(K_0)$. Thus $K/O_2(K)$ is $L_2(p)$ with $p > 7$ a Fermat or Mersenne prime, or $K/O_2(K)$ is $L_3(2)$ or A_6 with $N_{T_I}(K)$ nontrivial on the Dynkin diagram of $K/O_2(K)$.

Set $I^* := I/O_2(I)$. Then $U_H^* \trianglelefteq T_I^*$, while $m(U_H/C_{U_H}(Q_H)) \geq 4$ by hypothesis. Therefore by 14.6.8, $m_2(I^*) \geq m(U_H^*) \geq 4$, so from the previous paragraph, $K_0 > K$ and $K/O_2(K) \cong L_2(p)$ for $p \geq 7$ a Fermat or Mersenne prime, with $Aut_I(K^*) \cong PGL_2(7)$ if $p = 7$. But in these groups T_I^* has no normal elementary abelian subgroup of rank at least 4. This contradiction completes the proof of 14.6.9. □

This subsection culminates in the technical lemma 14.6.10. In each of the subsequent two subsections, the final contradiction will be to part (5) of 14.6.10.

LEMMA 14.6.10. *Assume the hypotheses of 14.6.9 and let $L_I := O^2(L \cap I)$. Assume that $I = \langle I_1, I_2 \rangle$, where $I_1 := L_I T_I$ and $T_I \leq I_2 \leq H$ with $I_2/O_2(I_2) \cong S_3$. Set $R_i := O_2(I_i)$. Then*

(1) $C(G, R_1) \leq M$.

(2) $R_1 \neq R_2$.

(3) If P is an I_1-invariant subgroup of I, then either $L_I \leq P$ or $P \leq C_M(V)$.

(4) $F^*(I) = O_2(I)$.

(5) If $T_I = T_u$, assume further that $I \leq C_G(u)$. Then $m(\langle V^{I_2} \rangle) = 3$.

PROOF. As the hypotheses of 14.6.9 hold, by that result $LT = L_I T_I O_2(LT) = I_1 O_2(LT)$. In particular $L_I \not\leq G_1$, $I_1/O_2(I_1) \cong S_3$, $L_I/O_2(L_I) \cong \mathbf{Z}_3$, and $R_1 = O_2(LT) \cap T_I$. By 14.6.6.1, $T_I < T$, so $T_I < N_T(T_I) \leq N_T(R_1)$ since $R_1 = O_2(LT) \cap T_I$. Then as $I \in \mathcal{I}^*$ and $N_{LT}(R_1)$ contains $I_1 \not\leq G_1$, we conclude from 14.6.4 that $M = !\mathcal{M}(N_{LT}(R_1))$, so (1) holds. Since $I \not\leq M$ but $I_1 \leq M$, $I_2 \not\leq M$, so (1) implies (2).

Assume P is a counterexample to (3). If $P \leq G_1$, then as P is I_1-invariant, P centralizes $\langle Z^{I_1} \rangle = V$, so that $P \leq C_G(V) = C_M(V)$, contrary to the choice of P as a counterexample; thus $P \not\leq G_1$. Set $M^+ := M/O_2(M)$. By 14.2.2.4, $M^+ = L^+ R_c^+ \times C_M(V)^+$, where $R_c := O_2(M \cap G_1)$. As $L_I \neq 1$ while $L = [L, C_{O_2(G_1)}(u)]$ by 14.6.3.3, $L^+ R_c^+ = I_c^+$, where $I_c := I_1 \cap L R_c$. As we are assuming that P is I_1-invariant with $L_I \not\leq P$, $P \cap L_I \leq O_2(L_I)$, so as $O^2(L \cap P) \leq O^2(L \cap I) = L_I$, $O^2(L \cap P) = 1$. If $P \leq M$ then $[P, I_c] \leq P \cap L R_c \leq O_2(L \cap P) R_c = R_c$, so $P^+ \leq C_{M^+}(O^2(I_c^+)) \leq C_M(V)^+$, again contrary to the choice of P since $O_2(M) \leq C_M(V)$. Therefore P is contained in neither M nor G_1, and as $PT_I \leq I$, $PT_I \in \mathcal{H}(T_u)$. Hence $PT_I \in \mathcal{I}^*$ by 14.6.4. Thus we may apply 14.6.9 to PT_I in the role of "I", to conclude that $O^2(L \cap P) \neq 1$, contrary to an earlier observation. So (3) is established.

Set $I^* := I/O_{3'}(I)$. Observe as $T_I \in Syl_2(I)$ by 14.6.6.2, that (I, I_1, I_2) is a Goldschmidt triple in the sense of Definition F.6.1. In view of (2), case (i) of F.6.11.2 holds, so I^* is a Goldschmidt amalgam, and hence as I is an SQTK-group, I^* is described in Theorem F.6.18.

To prove (4), we assume $F^*(I) \neq O_2(I)$, and derive a contradiction. By hypothesis $m(U_H/C_{U_H}(Q_H)) \geq 4$, so since $O_2(I) \in Syl_2(O_{3'}(I))$ by F.6.11.1, $m(U_H^*) \geq 4$ by 14.6.8. Now the only case of Theorem F.6.18 in which $m_2(I^*) \geq 4$ is case (13), where $I^* \cong Aut(M_{12})$. Thus $|I_z^* : T_I^*| = 3 = |I_2^* : T_I^*|$, so $I_2^* = I_z^*$. Thus as $I_2 \leq H$, $U_H^* \trianglelefteq I_z^*$ with $m(U_H^*) \geq 4$, whereas in $Aut(M_{12})$ (as we saw during the proof of 14.5.5), I_z^* has no such normal subgroup. This contradiction establishes (4).

Assume the hypotheses of (5). By (2), conclusion (1) of Theorem F.6.18 does not hold. If either case of conclusion (2) of F.6.18 holds, then there is a normal subgroup P of I with $I = PI_1$ and $P \cap L = O_2(L)$. But then by (3), $P \leq C_M(V)$, so $I = I_1 P \leq M$, contrary to $I \in \mathcal{I}$.

In the remaining conclusions of F.6.18, there is $K \in \mathcal{C}(I)$ with $K \trianglelefteq I$, and either $I = KT_I$, or case (3) of F.6.18 holds with KT_I of index 3 in I. Since $O_{3'}(I)$ is 2-closed by F.6.11.1, $K/O_2(K)$ is quasisimple by 1.2.1.4. Next by (1), $C(I, R_1) \leq M_I := I \cap M$. Further $L_I \trianglelefteq M_I$ and $R_1 = T_I \cap O_2(L_I T_I) \in Syl_2(C_{M_I}(L_I/O_2(L_I)))$, so $R_1 \in \mathcal{B}_2(M_I)$ by C.1.2.4; then as $N_I(R_1) \leq M_I$, $R_1 \in \mathcal{B}_2(I)$. Now Hypothesis C.2.3 is satisfied with I, R_1, M_I in the roles of "H, R, M_H". As K appears in F.6.18, $K/O_2(K)$ is not $L_2(2^n)$, so that K is not a χ_0-block. Now as K is T_I-invariant and $K/O_2(K)$ is quasisimple, we may apply C.2.7 to conclude that K is described in C.2.7.3. Comparing the lists of C.2.7.3 and F.6.18, we conclude that $O_2(I) = O_{3'}(I)$, $I^* = I/O_2(I) \cong L_3(2)$, $\hat{A}_6$, A_7, S_6, S_7, or $G_2(2)$, and except possibly in the first case, K is a block. In particular case (3) of F.6.18 is now ruled out, so $I = KT_I$. Then again using F.6.6, $K = O^2(I) = \langle K_1, K_2 \rangle$, where $K_i := O^2(I_i)$. Thus $L_I = K_1 \leq K$, so

$$V = [Z, L_I] \leq [\Omega_1(Z(O_2(K))), K] =: W.$$

To prove (5), we must show that $V_0 := \langle V^{I_2} \rangle$ is of rank 3, so we assume $m(V_0) \neq 3$, and it remains to derive a contradiction.

Suppose first that $K^* \cong L_3(2)$. Then case (g) of C.2.7.3 occurs, so we may apply C.1.34 to conclude that W is either a natural module, the sum of two isomorphic natural modules, or a 4-dimensional indecomposable module with a 1-dimensional submodule. As $V = [V, L_I]$ is a T_I-invariant projective line in W, it follows that $m(W) \neq 4$, and that $\langle V^K \rangle$ is an irreducible K-submodule of W of rank 3, so $V_0 = \langle V^{I_2} \rangle = \langle V^K \rangle$ is of rank 3, contrary to assumption. Therefore K is a block.

Suppose first that K is an $\hat{A}_6$-block. Then since $K = \langle K_1, K_2 \rangle$, $K_1 = L_I \not\leq X := O^2(O_{2,Z}(K))$, and of course X is normalized by $K_1 = I_1$. Thus $X \leq C_M(V)$ by (3), impossible as $C_W(X) = 1$ in an $\hat{A}_6$-block.

Next $V = [V, L_I]$ is a T_I-invariant line and I_2 stabilizes the point Z on that line. In particular if K is a $G_2(2)$-block then V is a doubly singular line in the language of [**Asc87**], and so V_0 is of rank 3, contrary to assumption. Similarly when $m(W) = 4$ and $K^* \cong A_6$ or A_7, we compute that Z and V_0 have ranks 1 and 3, respectively—again contrary to assumption.

Thus K is an A_n-block for $n := 6$ or 7, $I^* \cong A_n$ or S_n, with $m(W) = 5$ when $n = 6$, and we can represent I on $\Omega := \{1, \ldots, n\}$ as in section B.3, so that W is the core of the permutation module on Ω. Further M_I is the stabilizer in I of the T_I-invariant line V. So when $n = 6$, $M_I^* = I_1^*$ is the stabilizer of the partition $\Lambda := \{\{1,2\},\{3,4\},\{5,6\}\}$, $V = \langle e_{1,2,3,4}, e_{1,2,5,6} \rangle$, $z = e_{1,2,3,4}$, and $V_0 = \{e_J : |J \cap \{1,2,3,4\}| \equiv 0 \mod 2\}$, while $I_2^* = I_z^*$ is the stabilizer of the partition $\{\{1,2,3,4\},\{5,6\}\}$. Next assume for the moment that $n = 7$. Then I_1^* and I_2^* are (in some order) the stabilizers of the partitions $\Lambda' := \Lambda \cup \{7\}$ and $\theta := \{\{1,2\},\{3,4\},\{5,6,7\}\}$. However if I_1^* is the stabilizer of θ then $V = \langle e_{5,6}, e_{5,7} \rangle$ and $z = e_{5,6}$, impossible as I_2 centralizes z but the stabilizer of Λ' does not. Thus $M_I^* = I_1^*$ is the stabilizer of Λ', I_2^* is the stabilizer of θ, and as before $V = \langle e_{1,2,3,4}, e_{1,2,5,6} \rangle$ and $z = e_{1,2,3,4}$, while now $V_0 = \langle V, e_{5,6}, e_{5,7} \rangle$. Observe in this case that I_2^* is a proper subgroup of the stabilizer I_z^* of the partition $\{\{1,2,3,4\},\{5,6,7\}\}$.

Suppose first that $T_I = T_0$. We saw earlier that $|LT : L_I T_I| = |T : T_I|$, so as $|T : T_0| = 2$, $L_I = O^2(L_I T_I) = O^2(LT) = L$. Further $C_T(L) = 1$ by 14.2.2.6. However by the previous paragraph, $L = L_I$ centralizes $e_{1,2,3,4,5,6}$, contrary to $C_T(L) = 1$.

Thus $T_I = T_u$ by 14.6.6.1. Therefore by the hypothesis of part (5), $I \leq C_G(u)$. Further as $I_2 \leq H$, $V_+ := V_0\langle u\rangle \leq U_H$, and then from the discussion above, $V_- := C_{V_+}(T_u) = \langle z, e_{5,6}, u\rangle$.

Suppose that $u \notin W$. Then $W \cap V_- = \langle z, e_{5,6}\rangle$, so that $m(V_-) = 3$. Therefore as $[U_H, Q_H] \leq Z$ by 14.5.15.1, while $T_0 = Q_H T_u > T_u$ by 14.6.3.1, we conclude $V_\# := C_{V_-}(Q_H) = C_{V_-}(T_0)$ is a hyperplane of V_- with $u \notin V_\#$, so that $V_- = V_\#\langle u\rangle$. Let v_- be the projection on $V_\#$ of $e_{1,2,3,4,5,6}$, and set $J := C_K(v_-)$; then $v_- \neq 1$ as $u \notin W$. Now $J^* \cong A_6$, so J^* is contained in neither M_I^* nor I_z^* which are solvable from the discussion above, and hence J is contained in neither M nor G_1. But then $\langle T_0, J\rangle \leq C_G(v_-) \in \mathcal{I}$, contrary to 14.6.4 since $T_0 > T_u = T_I$.

Therefore $u \in W$. Since $I \leq C_G(u)$, $C_W(K) \neq 1$, and hence $n = 6$ and $\langle u\rangle = C_W(K)$, so that $u = e_{1,2,3,4,5,6}$. Let $Q_I := O_2(I)$. Since T_u is nontrivial on V by 14.6.3.3, and $|T : C_T(V)| = 2$, $T_0 = T_u C_{T_0}(V)$, so we may choose $t \in C_{T_0}(V) - T_u$. Since t normalizes T_u and $W \trianglelefteq T_u$, both $B := W^t$ and $WW^t = WB$ are normal in $T_u = T_I$. If $B \leq Q_I$, then as $[Q_I, K] = W$ since K is a block, $J := \langle K, T_I, t\rangle$ acts on WB, and J contains $KT_I = I$ and $T_0 \geq T_J > T_I$, contradicting 14.6.5. Thus $B \not\leq Q_I$, so that $B^* \neq 1$. By (U1), T_0 acts on $\langle z, u\rangle$, so as $V \trianglelefteq T$, T_0 acts on $V_u := V\langle u\rangle$. Therefore as $V \leq W$ and $\langle u\rangle = C_W(K)$, $V_u \leq W \cap B$. Similarly $u^t \in V_u \cap Z(T_u)$, and this latter group is generated by $z = e_{1,2,3,4}$ and $u = e_{1,2,3,4,5,6}$. Therefore as $u^t \notin z^K$ by 14.6.3.4, we conclude that $u^t = e_{5,6}$.

Notice for $v \in V^\#$ that $W_v := \langle V^{C_K(v)}\rangle$ is a hyperplane of W, and if $V = \langle v, w\rangle$, then $W = W_v W_w$. For example $W_z = V_0$. Thus $B^* = W^{t*} = W_v^{t*} W_w^{t*}$. Now $\langle V^{C_G(v)}\rangle$ is abelian by Hypothesis 14.5.1 and the transitivity of L on $V^\#$, and we chose t to centralize V, so $W_v^t \leq \langle V^{C_G(v)}\rangle \leq C_G(W_v)$. Therefore from the action of S_6 on its permutation module, $W_v^{t*} = \langle (i,j)\rangle$, where $v := e_{\Omega - \{i,j\}}$. Then as $W_v^{t*} W_w^{t*} = B^* \trianglelefteq T_I^*$, and the only normal subgroup of T_I^* containing $W_z^{t*} = \langle (5,6)\rangle$ generated by at most two transpositions is $\langle (5,6)\rangle$, we conclude that $B^* = W_z^{t*} = \langle (5,6)\rangle$. Thus $[W, W_z^t] = \langle e_{5,6}\rangle = \langle u^t\rangle$. This is impossible, as $C_I(W/\langle u\rangle) = C_I(W)$, so that $C_{I^t}(W_z^t/\langle u^t\rangle) = C_{I^t}(W_z^t)$.

This contradiction completes the proof of (5), and hence of 14.6.10. □

14.6.2. Showing $\mathbf{O(H/O_2(H)) = 1}$. Recall that H denotes a member of $\mathcal{H}(T, M) = \mathcal{H}_z$, and we have adopted Notation 12.8.2. In the remaining two subsections we adopt Notation 14.5.16 and use notation and results from section F.9. For example Γ is the coset geometry determined by LT and H as in section F.7, with the parameter b, the geodesic $\gamma_1, \ldots, \gamma = \gamma_b$, the element g_b taking γ_1 to γ, and the subgroups U_H, U_γ, D_H, D_γ etc., as well as $Z_\gamma := Z^{g_b}$ defined in section F.9—where Z_γ was often denoted by A_1.

This second subsection is devoted to the proof of a key intermediate result:

THEOREM 14.6.11. *$O(H^*) = 1$ for each $H \in \mathcal{H}(T, M)$.*

Until Theorem 14.6.11 is established, assume H is a counterexample. Thus H is a member of $\mathcal{H}(T, M)$ with $O(H^*) \neq 1$, and we must derive a contradiction from the existence of such an H.

Let P_0^* be a minimal normal subgroup of H^* contained in $O(H^*)$; then P_0^* is an elementary abelian p-group and $P_0^* = P^*$ for $P \in Syl_p(P_0)$. Indeed $PT \in \mathcal{H}(T, M)$ by 14.6.1.4; so replacing H by PT, we may assume $H = PT$ with P^* a minimal normal subgroup of H^*. Thus T is maximal in $PT = H$, and $P \cong \mathbf{Z}_p$ or E_{p^2}, since H is an SQTK-group.

Set $K := O^2(H)$, so that $K^* = P^*$.

LEMMA 14.6.12. *(1) $p = 3$ or 5.*

(2) There is a subgroup H_0 of index 2 in H such that $H_0^* = H_1^* \times H_2^*$, $H_i^* \cong D_{2p}$, $H_2 = H_1^t$ for $t \in T - N_T(H_1)$ and H_i the preimage of H_i^* in H, and $[\tilde{U}_H, H] = \tilde{U}_{H,1} \oplus \tilde{U}_{H,2}$, where $\tilde{U}_{H,i} := [\tilde{U}_H, H_i]$ is of rank 4 when $p = 5$, and of rank 2 or 4 when $p = 3$.

(3) $Z \leq [U_H, O^2(H_i)]$ for each i.

PROOF. By 14.5.18.3, $q(H^*, \tilde{U}_H) \leq 2$. Let $H_0^* := \langle \mathcal{Q}_*(H^*, \tilde{U}_H) \rangle$; as T is maximal in H, $H = H_0T$. By D.2.17, $H_0^* = H_1^* \times \cdots \times H_s^*$ and $[\tilde{U}_H, H_0] = \tilde{U}_{H,1} \oplus \cdots \oplus \tilde{U}_{H,s}$, where $(H_i^*, \tilde{U}_{H,i})$ are indecomposables in the sense of D.2.17. In particular $p = 3$ or 5 by D.2.17, so that (1) holds. Further $O_p(H_0)^*$ is not of order p by 14.3.5. Hence $P^* \cong E_{p^2}$, and as T is irreducible on P^*, our indecomposables appear only in conclusions (1) or (2) of D.2.17, so that (2) holds. Finally (3) follows from 14.6.2. □

During the remainder of the proof of Theorem 14.6.11, we adopt the notation of 14.6.12.2, with $U_{H,i}$ the preimage in U_H of $\tilde{U}_{H,i}$. Also set $U_K := [U_H, H]$.

LEMMA 14.6.13. *Either*

(1) $p = 3$, $\tilde{U}_K$ is a 4-dimensional orthogonal space over $\mathbf{F}_2$ for

$$H^* = O(\tilde{U}_K) \cong O_4^+(2),$$

and $[\tilde{U}_{H,1}, U_H^g] \neq 1$ for some $g \in G - G_1$ such that $U_H^g \leq N_H(U_{H,1})$ and $U_H \leq H^g$, or

(2) $p = 3$ or 5, $m(\tilde{U}_K) = 8$, $D_\gamma < U_\gamma$, and we may choose γ so that $U_\gamma^* \leq H_1^*$, $Z_\gamma \leq U_{H,1}$, and $Z \leq U_{H,1}^g$, for $g \in G - G_1$ with $\gamma_1 g = \gamma$.

PROOF. Suppose first that $D_\gamma = U_\gamma$. Then by 14.5.18.1, U_H induces a nontrivial group of transvections on U_γ with center Z, so by 14.6.12, $p = 3$, and H^* acts as $O_4^+(2)$ on $\tilde{U}_K$ of rank 4. Since $b \geq 3$ is odd by F.9.11.1, in this case there is $g \in \langle LT, H \rangle$ with $\gamma_1 = \gamma g$. Then U_H^g induces a nontrivial group of transvections on U_H with center Z^g, so $U_H^g \leq N_H(U_{H,1})$, and we may choose notation so that $[\tilde{U}_{H,1}, U_H^g] \neq 1$. By F.9.13.2, $U_\gamma \leq H$, so $U_H = U_\gamma^g \leq H^g$. Thus (1) holds when $D_\gamma = U_\gamma$.

Hence we may suppose instead that $D_\gamma < U_\gamma$. So by 14.5.18.4, we may choose γ with $m(U_\gamma^*) \geq m(U_H/D_H) > 0$ and $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$; in particular U_γ is quadratic on U_H, and hence either U_γ acts on $U_{H,1}$, or else the quadratic action forces $U_\gamma^* = \langle x^* \rangle$ to be of order 2 with $U_{H,1}^x = U_{H,2}$. Let $g \in \langle LT, H \rangle$ with $\gamma_1 g = \gamma$.

Suppose first that $U_\gamma^* = \langle x^* \rangle$ is of order 2 with $U_{H,1}^x = U_{H,2}$. As $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$,

$$m(\tilde{U}_H / C_{\tilde{U}_H}(U_\gamma)) \leq 2m(U_\gamma^*) = 2,$$

while $C_{\tilde{U}_H}(U_\gamma) = [\tilde{U}_H, U_\gamma]$ since x^* is an involution with $U_{H,1}^x = U_{H,2}$. Therefore $m(\tilde{U}_H) = 4$, and the inequality is an equality. Again by 14.6.12, $p = 3$, $\tilde{U}_K$ is a 4-dimensional orthogonal space over $\mathbf{F}_2$, and $H^* = O(\tilde{U}_K)$. Further $Z_\gamma = [U_\gamma, D_H]$ by F.9.13.6, so $\tilde{z}^g$ is a singular vector in $\tilde{U}_K$ since $\tilde{U}_{H,1}^\# \cup \tilde{U}_{H,2}^\#$ is the set of nonsingular vectors of $\tilde{U}_K$. Then $S^* := C_S(z^g)^*$ for some Sylow 2-subgroup S of H containing U_γ. Now $H \leq G_1$ with $U_H = \langle V^H \rangle \leq \langle V^{G_1} \rangle = U$; thus $U_\gamma = U_H^g \leq U^g = \langle V^{gG_1^g} \rangle$, so since U is abelian by Hypothesis 14.5.1, $[U^g, C_S(z^g)] \leq U^g \leq C_G(U_\gamma)$. Now as $U_\gamma^* \leq S^* \cong D_8$ with $U_{H,1}^x = U_{H,2}$, $[U_\gamma, C_S(z^g)]$ contains s with $s^* = Z(S^*)$. Then $s \in [U^g, C_S(z^g)] \leq C_G(U_\gamma)$, whereas s^* does not centralize $[U_K, x] \leq U_\gamma$.

Therefore U_γ acts on $U_{H,1}$. Suppose first that $m(\tilde{U}_K) = 4$. Then by 14.6.12.2, $p = 3$ and $\tilde{U}_K$ is a 4-dimensional orthogonal space for H^*. This time choose g so that $U_\gamma = U_H^g$, and choose notation so that $[\tilde{U}_{H,1}, U_H^g] \neq 1$; now $U_H \leq H^g$ by F.9.13.2, completing the verification that (1) holds.

Thus it remains to treat the case in 14.6.12 with $m(\tilde{U}_K) = 8$. By the choice of γ:

$$0 < m(U_H/D_H) \leq m(U_\gamma^*) \leq m_2(H/C_H(\tilde{U}_K)) = 2. \qquad (*)$$

As $m(\tilde{U}_K) = 8$, if $U_\gamma^* \not\leq H_i^*$ for $i = 1$ or 2, then $m(\tilde{U}_K/C_{\tilde{U}_K}(u_\gamma^*)) = 4$ for suitable $1 \neq u_\gamma^* \in U_\gamma^*$; this is a contradiction as $[D_H, U_\gamma] \leq Z_\gamma$ by F.9.13.2, while $m(Z_\gamma) = 1$ and $m(U_H/D_H) \leq 2$ by (*). Therefore we may assume $U_\gamma \leq H_1$, so that $m(U_\gamma^*) = 1$ from the structure of H^* in 14.6.12, and hence $m(U_H/D_H) = 1$ by (*). Then since $[\tilde{U}_{H,1}, U_\gamma]$ has rank 2, $1 \neq [D_H \cap U_{H,1}, U_\gamma]$, so that $Z_\gamma \leq U_{H,1}$ by F.9.13.6. Also $m(U_\gamma/D_\gamma) = 1 = m(U_H/D_D)$, so our hypotheses are symmetric in γ and γ_1, as discussed in Remark F.9.17. Hence we may choose notation so that $Z \leq U_{H,1}^g$, so that (2) holds, completing the proof of 14.6.13. □

Recall that G_1 is a member of $\mathcal{H}(T, M)$, so that the notational conventions of section 14.5 apply also to G_1 in the role of "H". Our convention in this subsection is to define $U := U_{G_1} = \langle V^{G_1} \rangle$, and set $Q_1 := O_2(G_1)$. Set $\hat{G}_1 := G_1/Q_1$ and $K_z := \langle K^{G_1} \rangle$.

Now we further specify our choice of $H \in \mathcal{H}(T, M)$, so that the odd prime $p \in \pi(H)$ is maximal over odd primes such that $O_p(H_0^*) \neq 1$ for some $H_0 \in \mathcal{H}(T, M)$; that is, in view of 14.6.12.1, we choose H with $p := 5$ if $O_5(H_0^*) \neq 1$ for some $H_0 \in \mathcal{H}(T, M)$, and otherwise $p := 3$.

LEMMA 14.6.14. *One of the following holds:*

(1) $K_z = K$, and if $p = 3$ then G_1 is a $\{2,3\}$-group.

(2) $p = 3$, $K_z \in \mathcal{C}(G_1)$, and $\hat{G}_1 \cong Aut(L_n(2))$ for $n := 4$ or 5.

(3) $p = 3$, $K_z = K_1 K_1^s$ for $s \in T - N_T(K_1)$ with $K_1 \in \mathcal{C}(G_1)$, and $\hat{G}_1 \cong S_5$ wr $\mathbf{Z}_2$ or $L_3(2)$ wr $\mathbf{Z}_2$.

(4) $p = 5$, $K_z = K_1 K_1^s$ for $s \in T - N_T(K_1)$ with $K_1 \in \mathcal{C}(G_1)$, and $\hat{G}_1 \cong Aut(L_2(16))$ wr $\mathbf{Z}_2$.

PROOF. First suppose H_+ is a solvable overgroup of H in G_1. If $X \trianglelefteq H_+$ with $X/O_2(X)$ a q-group for some odd prime q, then $XT \in \mathcal{H}(T, M)$ by 14.6.1.4, and so $q \leq p \leq 5$ by 14.6.12 and our maximal choice of p. Thus setting $\dot{H}_+ := H_+/O_2(H_+)$,

$$F^*(\dot{H}_+) = \prod_{q \leq p} O_q(\dot{H}_+),$$

with $m_q(O_q(\dot{H}_+)) \leq 2$ since H_+ is an SQTK-group. Therefore using A.1.25 and inspecting the order of $GL_2(q)$, we conclude H_+ is a $\{2,3\}$-group if $p = 3$, and a $\{2,3,5\}$-group if $p = 5$.

We claim next that for $J \in \mathcal{C}(G_1)$, $\hat{J}$ is not a Suzuki group: For if $\hat{J} \cong Sz(2^m)$ for some odd $m > 1$, then the T-invariant Borel subgroup B of $J_0 := \langle \hat{J}^T \rangle$ has order divisible by each prime dividing $2^m - 1$, and one of these primes is larger than 5. On the other hand $H \cap J_0$ is a solvable overgroup of $T \cap J_0$ in J_0, and hence is 2-closed, so H acts on $N_{J_0}(T \cap J_0) = B$, contrary to the previous paragraph applied to HB in the role of "H_+".

Now $K \in \Xi(G,T)$ by 14.6.12, so by 1.3.4, either $K = K_z$, or $K_z = \langle K_1^T \rangle$ for some $K_1 \in \mathcal{C}(G_1)$ with $K_1/O_2(K_1)$ quasisimple, and K_z is described in 1.3.4.

Supose $K = K_z$. If $p = 5$ then (1) holds, so we may assume that $p = 3$. Now $\hat{K}$ contains all elements of order 3 in $C_{\hat{G}_1}(\hat{K})$ since $m_3(\hat{K}) = 2$ and G_1 is an SQTK-group. Thus if $J \in \mathcal{C}(G_1)$ then J is a $3'$-group, which is impossible by the claim, so we conclude from 1.2.1.1 that G_1 is solvable. Then G_1 is a $\{2,3\}$-group by the first paragraph applied to G_1 in the role of "H". Therefore (1) holds when $K = K_z$.

Thus we may assume that $K < K_z = \langle K_1^T \rangle$ with $K_1 \in \mathcal{C}(G_1)$. As $K_1/O_2(K_1)$ is quasisimple, K_z is described in part (4) or (5) of F.9.18. Comparing the lists of 1.3.4 and F.9.18, we conclude that either:

(i) $K_z = K_1K_2$ with $K_2 := K_1^s$ for $s \in T - N_T(K_1)$, and either $K_1^* \cong L_2(2^m)$ with $2^m \equiv 1 \mod p$, or $p = 3$ and $K_1^* \cong L_3(2)$.

(ii) $p = 3$ and $K_1T/O_2(K_1T) \cong Aut(L_n(2))$, $n = 4$ or 5.

Notice that the $Sp_4(2^n)$-case in 1.3.4.3 is excluded, as here $Aut_T(P)$ is noncyclic by 14.6.12.

Observe that $K_z = O^{p'}(G_1)$: in case (i), this follows from 1.2.2, and in case (ii) from A.3.18. Furthermore when $p = 5$ we have case (i) with $\hat{K}_1 \cong L_2(2^m)$ for m divisible by 4, so that $K_z = O^{3'}(G_1)$ by 1.2.2. Thus $C_{\hat{G}_1}(\hat{K}_z)$ is a $3'$-group, and if $p = 5$, then $C_{\hat{G}_1}(\hat{K}_z)$ is a $\{3,5\}'$-group. Therefore applying the first paragraph to $HO_{2,F}(G_1)$ in the role of "H_+", we conclude $F(\hat{G}_1) = 1$, and by the second paragraph, $\hat{G}_1$ has no Suzuki components. Therefore as $C_{\hat{G}_1}(\hat{K}_z)$ is a $3'$-group, $\hat{K}_z = F^*(\hat{G}_1)$.

As $F^*(\hat{G}_1) = \hat{K}_z$, conclusion (2) of the lemma holds in case (ii), so we may assume case (i) holds. Similarly conclusion (3) holds if $\hat{K}_1 \cong L_3(2)$, since $N_T(K_1)$ is trivial on the Dynkin diagram of $\hat{K}_1$ because T acts on K. Thus we may assume that $\hat{K}_1 \cong L_2(2^m)$. Applying the first paragraph to BT in the role of "H_+", where B is a Borel subgroup of K_z over $T \cap K_z$, we conclude that $m = 2$ if $p = 3$, and that $m = 4$ if $p = 5$. If $p = 3$, then as T^* induces D_8 on K^*, $\hat{G}_1 \cong S_5$ wr $\mathbf{Z}_2$, so conclusion (3) holds. Finally if $p = 5$ we showed $BT \in \mathcal{H}(T,M)$, so for $B_3 \in Syl_3(B)$, $B_3T \in \mathcal{H}(T,M)$ by 14.6.1.4. Applying 14.6.12 to B_3T in the role of "H", we conclude $B_3T/O_2(B_3T) \cong O_4^+(2)$. Therefore $\hat{G}_1 \cong Aut(L_2(16))$ wr $\mathbf{Z}_2$, so that conclusion (4) holds. This completes the proof of 14.6.14. □

LEMMA 14.6.15. *$p = 3$.*

PROOF. Assume otherwise. Then by 14.6.12.1 we may assume $p = 5$, and it remains to derive a contradiction. As $p = 5$, conclusion (2) of 14.6.13 holds, so we may choose γ as in 14.6.13.2; in particular $Z_\gamma \leq U_{H,1}$. Also since $p = 5$, case (1) or (4) of 14.6.14 holds. Let $U_z := [U, K_z]$. As $U_H \leq U$ and $K \leq K_z$, $U_K = [U_H, K] \leq U_z$.

In the next several paragraphs we assume $K < K_z$ and establish some preliminary results in that case. First case (4) of 14.6.14 holds, so $G_1 = K_zT$ and $K_z = K_1K_1^s$ for $K_1 \in \mathcal{C}(G_1)$ with $K_1/O_2(K_1) \cong L_2(16)$ and $s \in T - N_T(K_1)$. We now apply F.9.18 to K_1, G_1 in the roles of "K,H": As the $O_4^+(16)$-module in case (i) of F.9.18.5 does not extend to $\hat{G}_1 \cong Aut(L_2(16))$ wr $\mathbf{Z}_2$, case (iii) of F.9.18.5 holds. Indeed as $\hat{K}_1 \cong L_2(16)$, subcase (a) of case (iii) holds, so for $\tilde{I} \in Irr_+(K_z, \tilde{U}, T)$, $I_0 := \langle I^T \rangle = II^s$, and we may choose notation so that $\tilde{I}/C_{\tilde{I}}(K_1)$ is the natural or orthogonal module for $\hat{K}_1$ and $[\tilde{I}, K_1^s] = 1$.

We claim that $U_z = I_0$. For if not, case (a) of F.9.18.6 does not hold and G_1^* has no strong FF-modules, so that case (c) of F.9.18.6 does not hold. Thus case (b) of F.9.18.6 holds, so that $W_z := U_z/I_0$ and $\tilde{I}_0/C_{\tilde{I}_0}(K_z)$ are nontrivial FF-module for G_1, and hence $W_z/C_{W_z}(K_z)$ and $\tilde{I}_0/C_{\tilde{I}_0}(K_z)$ are both natural modules for $L_2(16)$ by Theorem B.4.2. Indeed since $D_\gamma < U_\gamma$ in case (2) of 14.6.13, we may choose α as in 14.5.18.5; then $\hat{U}_\alpha \in \mathcal{Q}(\hat{G}_1, \tilde{U})$, so since $\hat{G}_1$ has no strong FF-modules by Theorem B.4.2, $\hat{U}_\alpha^*$ is an FF*-offender on both $\tilde{I}_0$ and W_z. Therefore either $\hat{U}_\alpha$ is Sylow in $\hat{K}_z$, or interchanging $\hat{K}_1$ and $\hat{K}_1^s$ if necessary, we may assume that $\hat{U}_\alpha$ is Sylow in $\hat{K}_1$. In either case, $m(\tilde{U}/C_{\tilde{U}}(\hat{U}_\alpha)) = 2\ m(\hat{U}_\alpha)$, so we conclude from 14.5.18.2 that $m(U/D) = m(\hat{U}_\alpha)$ where $D := D_{G_1}$, and that $\hat{U}_\alpha$ acts faithfully on $\tilde{D}$ as a group of $\mathbf{F}_2$-transvections with center $\tilde{Z}_\alpha$. As $\hat{U}_\alpha$ is Sylow in $\hat{K}_1$ or $\hat{K}_z$, and $\tilde{I}/C_{\tilde{I}}(\hat{K}_1)$ is the natural $\hat{K}_1$-module, $\hat{U}_\alpha$ does not induce a nontrivial group of $\mathbf{F}_2$-transvections on any subspace of $\tilde{I}_0$, so $\tilde{D} \cap \tilde{I}_0 = C_{\tilde{I}_0}(\hat{U}_\alpha)$ is of codimension $m(\hat{U}_\alpha)$ in $\tilde{I}_0$, and hence $U = I_0 D$. But this is impossible as $\hat{U}_\alpha$ does not induce a nontrivial group of $\mathbf{F}_2$-transvections on W_z. Thus the claim is established.

Set $K_2 := K_1^s$. Since case (iii.a) of F.9.18.5 holds, with $I_0 = U_z$ by the claim, $\tilde{U}_z = \tilde{U}_1 + \tilde{U}_2$ with $U_i := [U, K_i]$, and $\tilde{U}_i/C_{\tilde{U}_i}(K_i)$ the 2-dimensional natural or 4-dimensional orthogonal module for $K_i/O_2(K_i)$. Then as $U_H \leq U_z$, we can choose notation so that $O^2(H_i) \leq K_i$, and hence $U_{H,i} \leq U_i$.

This completes our preliminary treatment of the case $K < K_z$. In the case where $K = K_z$ we establish a similar setup: Namely in this case we set $K_i := O^2(H_i)$ and $U_i := U_{H,i}$.

Thus in any case $Z_\gamma \leq U_{H,1} \leq U_1$, so that Z_γ centralizes K_2. Choose g as in case (2) of 14.6.13, and for $X \leq G$, let $\theta(X)$ be the subgroup generated by the elements of order 5 in X. Then $K_2 \leq \theta(C_G(Z_\gamma)) = K_z^g$, and by 14.6.13.2, $Z \leq U_{H,1}^g \leq U_1^g$, so $K_2 \leq \theta(C_{K_z^g}(Z)) = K_2^g$. Therefore $K_2 = K_2^g$, so $g \in N_G(K_2)$.

Set $G_2 := N_G(K_2)$; since $g \in G - G_1$ in 14.6.13.2, $G_2 \not\leq G_1$. Set $T_2 := N_T(K_2)$ and $G_{1,2} := G_1 \cap G_2$, so that $|G_1 : G_{1,2}| = |T : T_2| = 2$, and in particular $G_{1,2} \trianglelefteq G_1$. As $Q_1 = O_2(K_z T_2)$, and $K_z T_2 \leq G_{1,2}$, we conclude $Q_1 = O_2(G_{1,2})$. Then as $G_1 \in \mathcal{M}$ by 14.6.1.1, $C(G_2, Q_1) \leq G_{1,2} = N_{G_2}(Q_1)$, so $Q_1 \in \mathcal{B}_2(G_2)$. Thus Hypothesis C.2.3 is satisfied with G_2, Q_1, $G_{1,2}$ in the roles of "H, R, M_H". As $Z \leq [U, K_2] \leq O_2(K_2)$ using 14.6.12.3, $F^*(G_2) = O_2(G_2)$ by 1.1.4.3.

Suppose $O_{2,F^*}(G_2) \leq G_{1,2}$. Then $O_2(G_2) = O_2(G_{1,2})$ by A.4.4.1, and we saw $G_{1,2} \trianglelefteq G_1$, so $G_2 \leq N_G(O_2(G_{1,2})) = G_1$ as $G_1 \in \mathcal{M}$, contrary to an earlier remark. Thus $O_{2,F^*}(G_2) \not\leq G_{1,2}$.

Next $G_1 = N_G(K_z)$ as $G_1 \in \mathcal{M}$. If X is an A_3-block of G_2, then as G_2 is an SQTK-group, $|X^{G_2}| \leq 2$; hence $K_z = O^{5'}(K_z)$ normalizes X, and then centralizes X as $Aut(X) \cong S_4$. Thus $X \leq C_{G_2}(K_z) \leq G_{1,2}$. Therefore $O_{2,F}(G_2) \leq G_{1,2}$ by C.2.6, so there is $J \in \mathcal{C}(G_2)$ with $J/O_2(J)$ quasisimple and $J \not\leq G_{1,2}$. If K_z centralizes $J/O_2(J)$, then J normalizes $O^2(K_z O_2(J)) = K_z$, contrary to $J \not\leq G_1 = N_G(K_z)$, so we conclude $J = [J, K_z]$. Furthermore $[J, K_2] \leq O_2(J)$ by 1.2.1.2, so as

$K_z = K_1K_2$, we obtain $J = [J, K_1]$. As $m_5(G_2) \leq 2$ and $m_5(K_2) = 1$, $m_5(J) \leq 1$, with $J = O^{5'}(C_{G_2}(K_2)) \trianglelefteq G_2$ in case of equality.

Now either Q_1 does not normalize J, so that C.2.4.1 holds, or Q_1 normalizes J, so that C.2.7.3 holds. Set $J_+ := \langle J^{Q_1} \rangle$, and observe that $B := J_+ \cap G_{1,2}$ normalizes K_2 and hence also K_1.

Suppose first that $K_1 \not\leq J$. Then an element k of K_1 of order 5 induces an outer automorphism of order 5 on J, since we saw that $J = O^{5'}(C_{G_2}(K_2))$ when $5 \in \pi(J)$. Inspecting C.2.4.1 and C.2.7.3 for cases where $J/O_2(J)$ admits an outer automorphism of order 5, we conclude $J/O_2(J)$ is $L_2(2^n)$ or $SL_3(2^n)$ with 5 dividing n, k induces a field automorphism on $J/O_2(J)$, and B is a Borel subgroup or a minimal parabolic of J_+, respectively. But then B does not normalize K_1, contrary to the previous paragraph.

Thus $K_1 \leq J$, so that $m_5(J) = 1$ and $J \trianglelefteq G_2$. Now we examine the list of C.2.7.3 for those $\hat{J}$ of 5-rank 1 with a subgroup $\hat{K}_1$ normalized by $\hat{B}$, such that $\hat{K}_1/O_2(\hat{K}_1) \cong \mathbf{Z}_5$ or $L_2(16)$. We conclude that $K_1/O_2(K_1) \cong \mathbf{Z}_5$, J is an $L_2(2^m)$-block, and B is a Borel subgroup of J. But then as $Z \leq B \leq G_{1,2} \leq C_G(Z)$, $Z \leq Z(B) = Z(J)$ using the structure of an $L_2(2^m)$-block; so $J \leq C_G(Z) = G_1$, contrary to $J \not\leq G_{1,2}$. This contradiction completes the proof of 14.6.15. □

We will see shortly in 14.6.17 that the group T_0 in the following result can play the role of "T_0" in (U1) in the first subsection.

LEMMA 14.6.16. *Let $T_0 := N_T(H_1)$. Then $|T : T_0| = 2$ and $N_{G_1}(T_0) = T$.*

PROOF. From 14.6.12, $|T : T_0| = 2$ and $T = N_H(T_0)$. Further $p = 3$ by 14.6.15, and in particular case (4) of 14.6.14 does not hold.

Suppose case (2) or (3) of 14.6.14 holds. Then $G_1 = K_zT$ and $\hat{B} := N_{\hat{K}_z}(\hat{Q}_H)$ is a parabolic subgroup of $\hat{K}_z$ with unipotent radical $\hat{Q}_H$ and $\hat{H} = \hat{B}\hat{T} = N_{\hat{G}_1}(\hat{Q}_H)$. Thus Q_H is weakly closed in T with respect to G_1 by I.2.5, so $N_{G_1}(T_0) \leq N_{G_1}(Q_H) = H$, and hence $N_{G_1}(T_0) = N_H(T_0) = T$.

Finally assume case (1) of 14.6.14 holds. Then $\hat{K} = \hat{K}_1 \times \hat{K}_2$ where $K_i := O^2(H_i)$ and $\hat{K}_1$ and $\hat{K}_2$ are the two T_0-invariant subgroups of $\hat{K}$ of order 3. Thus $X := O^2(N_{G_1}(T_0))$ acts on $\hat{K}_i$ and hence X centralizes $\hat{K}$. Then as $C_{\hat{K}}(T_0) = 1$ and $m_3(\hat{G}_1) = 2$, X is a $3'$-group. However as case (1) of 14.6.14 holds, G_1 is a $\{2, 3\}$-group, so again we conclude that $N_{G_1}(T_0) = T$. □

We can now determine H^*, and show that the set $\mathcal{U}(H)$ of involutions discussed in the first subsection is nonempty.

LEMMA 14.6.17. *(1) $\tilde{U}_K$ is a 4-dimensional orthogonal space over $\mathbf{F}_2$ for $H^* = O(\tilde{U}_K) \cong O_4^+(2)$.*

(2) $Z_\gamma \not\leq U_{H,1}$.

(3) Let $u \in U_K$ with $\tilde{u}$ nonsingular in the orthogonal space $\tilde{U}_K$ and centralized by $N_T(H_1)$. Then $u \in \mathcal{U}(H)$.

(4) $C_G(u) \in \mathcal{I}$, so $\mathcal{I}^*$ is nonempty.

(5) $m(\langle V^{O^2(H_2)} \rangle) = 4$.

PROOF. Set $T_0 := N_T(H_1)$ and let $u_1 \in U_{H,1} - Z$ with $\tilde{u}_1 \in Z(\tilde{T}_0)$. We first show that $u_1 \in \mathcal{U}(H)$ in the sense of Subsection 1. By choice of u_1, (U0) and (U1) are satisfied, and (U3) holds by 14.6.16. Next for $t \in T - T_0$, $u_1^t \in U_{H,2}$, so

$[K,u_1] \neq 1 \neq [K,u_1u_1^t]$. Further in all cases of 14.6.14, $K \leq O^2(O_{2,F^*}(G_1)) =: X$, so $[X,u_1] \neq 1 \neq [X,u_1u_1^t]$. Now by 14.6.1.5 applied to G_1 in the role of "H", X centralizes $\Omega_1(Z(Q_1))$, so as $F^*(G_1) = Q_1$, Q_1 does not centralize u_1 or $u_1u_1^t$. Thus (U2) holds, completing the proof that $u_1 \in \mathcal{U}(H)$.

Assume for the moment that $Z_\gamma \leq U_{H,1}$, and if case (2) of 14.6.13 holds, assume further that $U_\gamma^* \leq H_1^*$; we will show that these assumptions lead to a contradiction. Let $Z_\gamma = \langle u_\gamma \rangle$. We claim first that $u_\gamma \in \mathcal{U}(H)$. Suppose that case (2) of 14.6.13 holds, so that $U_\gamma^* \leq H_1^*$ by assumption. By 14.6.15, $p = 3$, so $U_\gamma^*(T^* \cap H_2^*)$ and T_0^* are Sylow in H_0^*, and therefore conjugating in H_1, we may take $T_0^* = U_\gamma^*(T^* \cap H_2^*)$. Then $\tilde{u}_\gamma$ is centralized by $\tilde{T}_0$, so by the previous paragraph, $u_\gamma \in \mathcal{U}(H)$, establishing the claim in this case. Suppose instead that case (1) of 14.6.13 holds. Then each member of $U_{H,1} - Z$ is conjugate to an element in $Z(\tilde{T}_0)$, so as before $u_\gamma \in \mathcal{U}(H)$, completing the proof of the claim. But then by by the claim, we may apply 14.6.3.4 to conclude that $u_\gamma \notin z^G$, contrary to $\langle u_\gamma \rangle = Z_\gamma = Z^{g_b}$. Thus the hypotheses of the first sentence of this paragraph lead to a contradiction.

If case (2) of 14.6.13 holds, that result shows we may choose γ so that $U_\gamma^* \leq H_1^*$ and $Z_\gamma \leq U_{H,1}$, contrary to the previous paragraph. Thus case (1) of 14.6.13 holds, establishing (1). Then (2) follows from the previous paragraph.

Next by (1), H has two orbits on $\tilde{U}_K$: the singular and nonsingular vectors, with $\tilde{U}_{H,1}^{\#} \cup \tilde{U}_{H,2}^{\#}$ the set of nonsingular vectors. Thus (3) follows from the first paragraph.

Choose u as in (3). By 14.6.3.4, $T_u \in Syl_2(C_G(u))$, and by 14.6.3.1, $|T : T_u| = 4$. But if $w \in U_K$ with $\tilde{w}$ singular, then $|C_H(w)| = |T|/2 > |T_u|$, so that $w \notin u^G$. Therefore $u^G \cap U_K = u^H$, so using A.1.7.1:

$$C_G(u) \text{ is transitive on the } G\text{-conjugates of } U_K \text{ containing } u. \qquad (*)$$

As case (1) of 14.6.13 holds, $[\tilde{U}_{H,1}, U_H^g] \neq 1$ for some $g \in G$ with $U_H^g \leq N_H(U_{H,1})$ and $U_H \leq H^g$. In particular by (1) we may take $u \in [U_{H,1}, U_H^g] \leq U_K^g$. By (*), $U_K^g = U_K^h$ for some $h \in C_G(u)$. Therefore as $[U_{H,1}, U_K^h] \neq 1$, while $U_K \leq \langle V^{G_1} \rangle$ and $\langle V^{G_1} \rangle$ is abelian, $h \notin G_1$. Thus $C_G(u) \not\leq G_1$. Finally as $u \in U_{H,1}$, u is centralized by K_2, so $C_H(u) \not\leq M$. Thus $C_G(u)$ is in the set $\mathcal{I} = \mathcal{I}(T,u)$ defined in the first subsection, so (4) holds.

As $\tilde{V} =: \langle \tilde{v} \rangle \leq Z(\tilde{T})$, $\tilde{v} = \tilde{u}_1\tilde{u}_2\tilde{c}$, where $\langle \tilde{u}_i \rangle = C_{\tilde{U}_{H,i}}(T_0)$ and $\tilde{c} \in C_{\tilde{U}_H}(H)$. Therefore $\langle V^{O^2(H_2)} \rangle = \langle u_1c, U_{H,2} \rangle$ is of rank 4 since $Z \leq U_{H,2}$ by 14.6.12.3, so (5) holds, completing the proof of 14.6.17. □

We are now in a position to derive a contradiction, and hence establish Theorem 14.6.11.

Let T_0 and u be defined as in 14.6.17. By 14.6.17.4, $C_G(u) \in \mathcal{I}$, so $\mathcal{I}^*$ is nonempty, and if $T_u = T_I := T \cap I$ for some $I \in \mathcal{I}^*$, then also $C_G(u) \in \mathcal{I}^*$ by 14.6.4. By 14.6.17.1, $|T : Q_H| > 4$ and $m(U_H/C_{U_H}(Q_H)) = 4$. Thus the hypotheses of 14.6.9 are satisfied for any $I \in \mathcal{I}^*$, so by that result $|T| > 2^{11}$, and for any such I, setting $L_I := O^2(L \cap I)$ we have $LT = L_IT_IO_2(LT)$. Pick $I \in \mathcal{I}^*$, choosing $I := C_G(u)$ if $T_I = T_u$ for some $I \in \mathcal{I}^*$. Set $I_2 := O^2(H_2)T_I$, $I_1 := L_IT_I$, and $I_0 := \langle I_1, I_2 \rangle$. Observe H_2 has a noncentral 2-chief factor on U_H and on $Q_H/C_{Q_H}(U_H)$ by the duality in 14.5.21.1. Therefore $I_0 \in \mathcal{I}^*$ by 14.6.6.6. Further $O^2(H)$ centralizes u by Coprime Action; so if $T_I = T_u$, then $O^2(H_2)T_u = I_2$ centralizes u, while $I_1 \leq C_G(u)$ by our choice of I, so that $I_0 \leq C_G(u)$. Thus I_0

satisfies the hypotheses of 14.6.10.5, so $m(\langle V^{I_2}\rangle) = 3$ by that lemma, contrary to 14.6.17.5.

This contradiction completes the proof of Theorem 14.6.11.

As a corollaries to Theorem 14.6.11 we have:

THEOREM 14.6.18. *Each solvable member of $\mathcal{H}(T)$ is contained in M.*

LEMMA 14.6.19. *Let $H \in \mathcal{H}(T, M)$. Then*
(1) $O_{2,2'}(H) = O_2(H)$.
(2) If $K \in \mathcal{C}(H)$, then $K/O_2(K)$ is simple, and hence is described in F.9.18.

PROOF. Part (1) follows from 14.6.1.4 in view of 14.6.18. Then (1) implies (2). □

14.6.3. The final elimination of U abelian when $\mathbf{L/O_2(L)}$ is $\mathbf{L_2(2)}$.

LEMMA 14.6.20. *If $H \in \mathcal{H}(T, M)$ and $K \in \mathcal{C}(H)$, then $K/O_2(K) \cong L_3(2)$ or A_6, and $N_T(K)$ is nontrivial on the Dynkin diagram of $K/O_2(K)$.*

PROOF. Let $K_0 := \langle K^T\rangle$. As $L_1 = 1$, $K_0T \in \mathcal{H}(T, M)$ by 14.5.19, so without loss $H = K_0T$. By 14.6.19.2, $K/O_2(K)$ is simple, and is described in (4) or (5) of F.9.18, so $K/O_2(K)$ is a group of Lie type and characteristic 2, A_7, or M_{22}. If $K/O_2(K) \cong A_7$, then KT is generated by solvable overgroups of T, which lie in M by 14.6.18, contrary to $H \not\leq M$. If $K/O_2(K) \cong M_{22}$, solvable overgroups of T generate a subgroup J of KT with $O^2(J)/O_2(K) \cong A_6/E_{2^4}$, so that $J \leq K \cap M$; then $J \leq C_M(V)$, impossible as $m_3(C_M(V)) \leq 1$ by 14.2.2.4. Thus $K/O_2(K)$ is of Lie type and characteristic 2. Set $B := N_K(T \cap K)$; then B is a Borel subgroup of K, so BT is solvable, and hence $BT \leq M$ by Theorem 14.6.18.

Suppose first that $K = K_0$. If $K/O_2(K)$ is of Lie rank at most 2, then as $B \leq M$ by the previous paragraph, the lemma follows from 14.3.6.1. Thus we may assume $K/O_2(K)$ is of higher Lie rank, and hence $K/O_2(K)$ is $L_4(2)$ or $L_5(2)$ by F.9.18. Let P be the product of the end-node minimal parabolics of K. Then $PT \leq H \cap M$ by Theorem 14.6.18, so $P \leq C_M(V)$ by 14.2.2.1, contrary to 14.2.2.4.

Therefore we may assume $K < K_0$. By F.9.18.5, $K/O_2(K)$ is either a Bender group or $L_3(2)$. In the former case, since $B \leq M$, we contradict 14.3.6.1.ii; so $K/O_2(K) \cong L_3(2)$. Further by Theorem 14.6.18, K_0 is not generated by T-invariant solvable parabolics, so $N_T(K)$ is nontrivial on the Dynkin diagram of $K/O_2(K)$. This completes the proof of the lemma. □

In the remainder of the section, we fix G_1 as our choice for $H \in \mathcal{H}(T, M)$, and use the symbol H to denote this group. As in the previous subsection, we adopt the setup of Notation 14.5.16, including the notation reviewed in that subsection involving the coset geometry Γ determined by LT and H, the vertex γ at distance b from γ_0, and the subgroups $U = U_H$, $D := D_H$, U_γ, etc. By Theorem 14.6.18, G_1 is not solvable, so there is $K \in \mathcal{C}(H)$. Then K is described in 14.6.20. Set $U_K := [U, K]$. Recall $H^* = H/Q_H$; as $H = G_1$ in this subsection, we do not require the convention $\hat{G}_1 = G_1/O_2(G_1)$ of the previous subsection.

LEMMA 14.6.21. *One of the following holds:*
(1) $H^\infty = K$, with $K/O_2(K) \cong L_3(2)$ or A_6.
(2) $H^\infty = KK^t$ for some $t \in T - N_G(K)$, with $K/O_2(K) \cong L_3(2)$.

(3) $H^\infty = KK_+$ *with* K, K_+ *normal* $\mathcal{C}$*-components of* H*, and* $K/O_2(K) \cong K_+/O_2(K_+) \cong L_3(2)$.

PROOF. If $H^\infty = K$ then (1) holds by 14.6.20, so assume $H^\infty > K$. Then by 1.2.1.1, there is $K_+ \in \mathcal{C}(H) - \{K\}$, and K_+ is also described in 14.6.20. As $m_3(H) \leq 2$, we conclude from 14.6.20 that $K/O_2(K) \cong K_+/O_2(K_+) \cong L_3(2)$ and $H^\infty = KK_+$. Then by 1.2.1.3, either (2) or (3) holds. □

LEMMA 14.6.22. *(1)* $\tilde{U}_K = \tilde{U}_{K,1} + \tilde{U}_{K,2}$*, where* $\tilde{U}_{K,1}$ *is a natural module for* $K/O_2(K)$ *or the* 5*-dimensional cover of a natural module for* $K/O_2(K) \cong A_6$*, and* $U_{K,2} = U_{K,1}^s$ *for* $s \in N_T(K)$ *acting nontrivially on the Dynkin diagram of* $K/O_2(K)$.

(2) If there exists $K_+ \in \mathcal{C}(H) - \{K\}$*, then* $[U_K, K_+] = 1$.

(3) $Z \leq U_{K,i}$ *for* $i = 1, 2$.

PROOF. Let $K_0 := \langle K^T \rangle$ and $\tilde{I} \in Irr_+(K_0, \tilde{U})$. As $T_K := N_T(K)$ is nontrivial on the Dynkin diagram of $K/O_2(K)$ by 14.6.20, $KT_K/O_2(KT_K)$ has no FF-modules by Theorem B.5.1. By 14.6.19.2, we may apply F.9.18, so $[\tilde{U}, K_0] = \langle \tilde{I}^H \rangle$ by part (7) of that result. Next as T is nontrivial on the Dynkin diagram of $K/O_2(K)$, F.9.18 says $[\tilde{U}, K_0]$ is described in case (iii) of part (4) of F.9.18 if $K = K_0$, and in case (iii.b) of part (5) if $K < K_0$. Next if $C_{\tilde{I}}(K) \neq 1$, then $\tilde{I}$ is described in I.1.6.1; in particular $\tilde{I}$ is 5-dimensional when $K^* \cong A_6$. On the other hand if $K^* \cong L_3(2)$, then $\tilde{I}$ is the extension in B.4.8.2, and that result says $q(H^*, \tilde{U}_H) > 2$, contrary to part (2) of F.9.18. This completes the proof of (1). Also (2) follows, since $\tilde{U}_{K,1}$ is not K-isomorphic to $\tilde{U}_{K,2}$ and $End_K(U_{K,i}/C_{U_{K,i}}(K))$ is a field by A.1.41. Finally (3) follows from 14.6.2. □

LEMMA 14.6.23. *(1)* $H^\infty = O^2(H)$*, so* $H = H^\infty T$.

(2) $M = LT$ *and* $T = M \cap H$.

(3) We have

$$\tilde{U} = \Big(\bigoplus_{K \in \mathcal{C}(H)} \tilde{U}_K \Big) + C_{\tilde{U}}(H).$$

PROOF. In view of 14.6.19.1, we obtain $F^*(H^*) = H^{\infty *}$ from 1.2.1.1. By 14.6.21, $Out(H^{\infty *})$ is a 2-group, so (1) holds.

Let $\tilde{U}_0 := [\tilde{U}, H^\infty]$. By 14.6.21 and 14.6.22, $\tilde{U}_0 = \bigoplus_{K \in \mathcal{C}(H)} \tilde{U}_K$. Now T centralizes $\tilde{V}$ of order 2, and $\tilde{U} = \langle \tilde{V}^H \rangle$, while $H = H^\infty T$ by (1), so (3) follows using Gaschütz's Theorem A.1.39. Further the projection $\tilde{V}_K$ of $\tilde{V}$ on $\tilde{U}_K$ is of order 2 and centralized by $N_T(K)$ for each $K \in \mathcal{C}(H)$. By 14.6.22.1, $C_{K^*}(\tilde{V}_K) = T^* \cap K^*$, so $T = C_H(\tilde{V})$ by (1). Therefore $T = M \cap H$, so $M = LT$ by 14.3.7. Thus (2) holds. □

We next choose an element $u \in U$, which we will show lies in the set $\mathcal{U}(G_1)$ of the first subsection. In cases (1) and (3) of 14.6.21, pick $u \in U_{K,1}$ such that $[\tilde{u}, K] \neq 1$ and $N_T(U_{K,1})$ centralizes $\tilde{u}$. (This choice is possible when $U_{K,1}$ is the 5-dimensional cover of a natural module for $K/O_2(K) \cong A_6$ by I.2.3.1ia). In case (2) of 14.6.21, pick $u \in U_K - Z$ such that $N_T(K)$ centralizes $\tilde{u}$.

LEMMA 14.6.24. *(1)* $u \in \mathcal{U}(H)$.

(2) $K/O_2(K) \cong L_3(2)$.

(3) $K = H^\infty$.

(4) $C_G(u) \in \mathcal{I}$, so that $\mathcal{I}^*$ is nonempty.

PROOF. Set $T_0 := C_T(\tilde{u})$. To prove (1) we must verify (U1), (U2), and (U3). By construction T_0 is $N_T(U_{K,1})$ or $N_T(K)$, and so is of index 2 in T. Then as $T_0 \in Syl_2(C_H(\tilde{u}))$, (U1) holds. By 14.6.21, $N_{H^\infty T_0}(T_0) = T_0$, so as $G_1 = H = H^\infty T$ by 14.6.23.1, $N_{G_1}(T_0) = T$, establishing (U3). As $u \in U_K - Z$, $[K, u] \neq 1$, and for $t \in T - T_0$, u^t lies in either $U_{K,2}$ or U_{K^t}, so $1 \neq [K, uu^t]$. By 14.6.1.5, K centralizes $\Omega_1(Z(O_2(G_1)))$, so as $F^*(G_1) = Q_H$, neither u nor uu^t centralizes Q_H, establishing (U2). This completes the proof of (1).

In view of 14.6.22.1 and 14.6.23, we conclude from Theorem B.5.6 that for any $K \in \mathcal{C}(H)$,

$$\tilde{U} \text{ is not an FF-module for } H^*, \text{ and } \tilde{U}_K \text{ is not an FF-module for } Aut_H(\tilde{U}_K). \tag{a}$$

In particular, no member of H^* induces a transvection on $\tilde{U}$, so by 14.5.18.1, $D_\gamma < U_\gamma$. Therefore by 14.5.18.4, we can choose γ as in 14.5.18.4; in particular $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U})$, and from that choice and (a):

$$0 < m(U/D) \leq m(U_\gamma^*) < m(\tilde{U}/C_{\tilde{U}}(U_\gamma^*)). \tag{b}$$

In view of (b), $[\tilde{D}, U_\gamma^*] \neq 1$ by (a), so $\tilde{Z}_\gamma = [\tilde{D}, U_\gamma^*]$ by F.9.13.6. Then $\tilde{Z}_\gamma \leq [\tilde{U}, H^\infty]$ by 14.6.23.3. Set $g := g_b$, so that $\gamma_1 g = \gamma$ and $Z_\gamma := Z^g$ plays the role of "A_1" of section F.9.

Now we begin the proof of (3), which will be lengthier. Thus we assume that $K < H^\infty$ and derive a contradiction. Observe that case (2) or (3) of 14.6.21 holds, so that $\mathcal{C}(H) = \{K, K_+\}$ with $K/O_2(K) \cong K_+/O_2(K_+) \cong L_3(2)$. By 14.6.22.1 and 14.6.23.3, $\tilde{U} = \tilde{U}_K \oplus \tilde{U}_+ \oplus C_{\tilde{U}}(H)$, where $U_+ := [K_+, U]$. By 14.6.22.1, $\tilde{U}_K$ and $\tilde{U}_+$ have rank 6.

Assume that some $a^* \in U_\gamma^*$ does not normalize K^*. Then $C_{H^*}(a^*) \cong \mathbf{Z}_2 \times L_3(2)$, and

$$m(\tilde{U}/C_{\tilde{U}}(U_\gamma)) \geq m(\tilde{U}/C_{\tilde{U}}(a)) = m(\tilde{U}_K) = 6 = 2m_2(C_{H^*}(a^*)),$$

with $\langle a^* \rangle$ the kernel of the action of $C_{H^*}(a^*)$ on $C_{\tilde{U}}(a)$ of corank 6 in $\tilde{U}$. Thus $m(U_\gamma^*) \leq m_2(C_{H^*}(a^*)) = 3$, while $\tilde{U}/C_{\tilde{U}}(U_\gamma)$ is of rank 6 if $U_\gamma^* = \langle a^* \rangle$ is of rank 1, and rank greater than 6 if $m(U_\gamma) = 2$ or 3, contrary to $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U})$.

Thus U_γ normalizes K and K_+, and hence also U_K and U_+. So as U_γ^* is faithful on $F^*(H^*) = K^* K_+^*$ we may choose notation so that $K^* = [K^*, U_\gamma^*]$.

We claim that $\tilde{Z}_\gamma \leq \tilde{U}_K$ or $\tilde{U}_+$. Suppose otherwise. Then as $[D, U] \leq Z_\gamma$ by F.9.13.6, U_γ centralizes $\tilde{D} \cap \tilde{U}_K$ and $\tilde{D} \cap \tilde{U}_+$. Then by (a),

$$m(\tilde{U}_K/(\tilde{U}_K \cap \tilde{D})) \geq m(\tilde{U}_K/C_{\tilde{U}_K}(U_\gamma)) \geq m_2(Aut_{U_\gamma}(K^*)) + 1. \tag{c}$$

Set $U^+ := \tilde{U}/(\tilde{U}_K + C_{\tilde{U}}(H))$. As U_γ^* is faithful on $K^* K_+^*$ and normalizes both factors,

$$m(U_\gamma^*) \leq m(Aut_{U_\gamma}(K^*)) + m(Aut_{U_\gamma}(K_+^*)), \tag{d}$$

so using (b)–(d):

$$m := m(Aut_{U_\gamma}(K_+^*)) \geq m(U_\gamma^*) - m(Aut_{U_\gamma}(K^*)) \geq m(U/D) - (\, m(\tilde{U}_K/(\tilde{U}_K \cap \tilde{D})) - 1\,)$$

$$\geq m(U^+/D^+) + 1 \geq m(U^+/C_{U^+}(U_\gamma)) - m + 1, \tag{e}$$

where the last inequality follows from the fact that U_γ induces a group of transvections on D^+ with center Z_γ^+.

By (e),

$$2m \geq m(U^+/C_{U^+}(U_\gamma)) + 1. \qquad (f)$$

In particular $m > 0$, so that U_γ^* is nontrivial on $\tilde{U}_+$, and hence also on K_+^*. Therefore $m(U^+/C_{U^+}(U_\gamma)) > 1$ by (a), so that (f) now gives $m > 1$. Hence $m = 2$ as $m_2(Aut_H(K_+^*)) = 2$. As $m = 2$, we conclude from (e) and the structure of $\tilde{U}_+$ that U_γ^* induces inner automorphisms on K_+^*, and

$$m(U^+/C_{U^+}(U_\gamma)) = m(\tilde{U}_+/C_{\tilde{U}_+}(U_\gamma)) = 3. \qquad (g)$$

Therefore (f) is an equality, and hence all inequalities in (c)–(f) are equalities. Then (d) becomes:

$$m(U_\gamma^*) = m(Aut_{U_\gamma}(K^*)) + m(Aut_{U_\gamma}(K_+^*)). \qquad (h)$$

As the inequalities in (c) are equalities,

$$\tilde{D} \cap \tilde{U}_K = C_{\tilde{U}_K}(U_\gamma) \text{ is of codimension } m_2(Aut_{U_\gamma}(K^*)) + 1 \text{ in } \tilde{U}_K, \qquad (i)$$

and since $m = 2$ and the inequalities in (e) are equalities, $m(U^+/D^+) = m(U^+/C_{U^+}(U_\gamma)) - 2$. Thus by (g):

$$D^+ \text{ is a hyperplane of } U^+. \qquad (j)$$

We had chosen notation so that $K^* = [K^*, U_\gamma^*]$, but we also saw after (f) that $K_+^* = [K_+^*, U_\gamma^*]$. Thus we have symmetry between K and K_+, so we conclude $m(Aut_{U_\gamma}(K^*)) = 2$ and U_γ^* induces inner automorphisms on K^*. Then by (h), $U_\gamma^* = A^* \times A_+^*$, where A^* and A_+^* are 4-subgroups of K^* and K_+^*, respectively. Since U_γ induces a group of transvections on D^+ with center Z_γ^+, and we are assuming that $\tilde{Z}_\gamma$ is not contained in $\tilde{U}_K$ or $\tilde{U}_+$, it follows from (j) that Z_γ is generated by $z^g = z_1 z_2$, where $1 \neq \tilde{z}_1 \in \tilde{U}_{K,1}$ and $1 \neq \tilde{z}_2 \in \tilde{U}_{K_+,1}$. Therefore from the structure of U_K in 14.6.22, $C_H(z^g) = C_G(ZZ_\gamma)$ has a Sylow 3-subgroup P isomorphic to E_9. However by 14.5.21.2 and 14.5.15.1, Q_H and Q_H^g induce transvections on ZZ_γ with centers Z and Z_γ, respectively, so that $m_3(C_G(V)) \leq 1$ by A.1.14.4. This contradiction finally completes the proof of the claim.

By the claim we may choose notation so that $\tilde{Z}_\gamma \leq \tilde{U}_K$, and hence $Z_\gamma = Z^g$ centralizes K_+. Set $\hat{H}^g := H^g/Q_H^g$; then $K_+ \leq C_G(Z^g) = H^g = N_G(U_\gamma)$ since $H = G_1 \in \mathcal{M}$ by 14.6.1.1. Therefore U_γ^* centralizes K_+^*, and hence $m(U_\gamma^*) \leq m_2(C_{H^*}(K_+^*)) = 2$. Also $H^g = K^g K_+^g T^g$ by 14.6.23.1, so either $\hat{K}_+$ is $\hat{K}^g$ or $\hat{K}_+^g$, or else $\hat{K}_+$ is a full diagonal subgroup of $\hat{K}^g \times \hat{K}_+^g$. Suppose this last case holds. Then as $\hat{K}_+$ also acts on $\hat{U}$, $\hat{U} = \langle \hat{w} \rangle$ for $\hat{w}$ an involution interchanging K^g and K_+^g, and hence $U_K^{gw} = U_+^g$. Then as $[D_\gamma, U] \leq Z$ by F.9.13.6 and $C_{U_\gamma}(w)$ is of codimension 6 in U_γ, $m(U_\gamma^*) = m(U_\gamma/D_\gamma) \geq m(\tilde{U}_K) - 1 = 5 > 2$, contradicting $m(U_\gamma^*) \leq 2$. Thus $\hat{K}_+ = \hat{J}$ where $J := K^g$ or K_+^g. Hence $K_+ \leq JQ_H^g$, so that $K_+ = K_+^\infty \leq (JQ_H^g)^\infty = J$.

Suppose $K_+ = J$. Then by 14.6.22.3, $z^g \in [U^g, J] \leq O_2(J)$. Then $U_K = \langle z^{gN_G(K)} \rangle \leq O_2(J) \leq Q_H^g$, so by 14.5.15.1, $[U_K, U_\gamma] \leq \langle z^g \rangle$, contrary to (a). Hence $K_+ < J$, so in particular $|K_+| < |J|$, and hence $J = K^g$. Further K and K_+ have different orders and so are normal in H, so that case (3) of 14.6.21 holds. As $K_+ < J = K^g$ with $J/O_2(J) \cong L_3(2) \cong K/O_2(K)$ by 14.6.21.3, K_+ has a noncentral chief factor on $O_2(J)$ not in $O_2(K_+)$, and this factor has dimension at least 3. However $K_+ = C_G(\langle z, z^g \rangle)^\infty$ is invariant under $S \in Syl_2(C_G(\langle z, z^g \rangle))$, so $S \cap J \notin Syl_2(J)$ and hence z does not centralize J, and $|Q_H^g S : S| \geq 8$, so that $|C_H(z^g)|_2 = |S| \leq |T|/8$.

Suppose first that $m(U/D) = m(U_\gamma^*)$. Then, as in Remark F.9.17, we have symmetry of hypotheses between γ_1 and γ, so there exists a unique $J_1 \in \mathcal{C}(H^g)$ such that $J_1 = C_{J_1}(z)O_2(J_1)$. Thus as z centralizes $K_+ \leq J$, $J = J_1$, whereas we saw $[J, z] \neq 1$. Therefore $m(U/D) < m(U_\gamma^*)$, and we saw earlier that $m(U_\gamma^*) \leq 2$, so we conclude from (b) that $m(U_\gamma^*) = 2$ and $m(U/D) = 1$. Thus $m(U_{K,i}/(D \cap U_{K,i})) \leq 1$, and as U_γ^* is of rank 2, U_γ^* does not centralize a hyperplane of both $U_{K,1}$ and its dual $U_{K,2}$. Therefore as $[D, U_\gamma] \leq Z_\gamma$ by F.9.13.6, we may take $z^g \in U_{K,1}$. But then as $K \trianglelefteq H$, $|C_H(z^g)|_2 = |T|/4$, contrary to the previous paragraph. This contradiction finally completes the proof of (3).

Suppose next that (2) fails. Then $K/O_2(K) \cong A_6$ by 14.6.21.1, and $m(U_{K,i}) = 4$ or 5 by 14.6.22.1. Then for i^* an involution in H^*, $m([\tilde{U}_{K,1}, i^*]) \geq 1$, and in case of equality, i^* induces a transposition on K^*. But also by 14.6.22.1, $\tilde{U}_{K,2} = U_{K,1}^s$ for $s \in N_T(K)$ nontrivial on the Dynkin diagram of K^*, so if i^* acts as a transvection on $\tilde{U}_{K,1}$, then $m([\tilde{U}_{K,2}, i^*]) = 2$. We conclude $m(U_\gamma^*) > 1$, since $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U})$. Next as U_γ^* is quadratic on $\tilde{U}_{K,1}$, from the action of $N_H(U_{K,1})$ on $U_{K,1}$, U_γ^* is contained in a 2-subgroup of H^* generated by transpositions. Then again as $\tilde{U}_{K,2} = U_{K,1}^s$ and U_γ^* is quadratic on $\tilde{U}_{K,2}$, U_γ^* is a 4-group F^* generated by a transposition and the product of three commuting transpositions. Then $m(\tilde{U}/C_{\tilde{U}}(U_\gamma^*)) = 4 = 2m(U_\gamma^*)$. This contradicts 14.5.18.2, as F^* does not induce a group of transvections on any subspace of $\tilde{U}_K$ of codimension 2. This establishes (2).

By (2) and (3), $H^\infty = K$ with $H^* \cong Aut(L_3(2))$, so by 14.6.22.1, $\tilde{U}_K = \tilde{U}_{K,1} \oplus \tilde{U}_{K,2}$ with $\tilde{U}_{K,1}$ natural and $\tilde{U}_{K,2}$ its dual. Thus H has three orbits on $U_K - Z$: $U_{K,1}^\# \cup U_{K,2}^\# = u^H$ plus two diagonal classes, one of which is 2-central in $\tilde{H}$. Denote this latter 2-central class by $\mathcal{C}$. Recall that $u \in \mathcal{U}(H)$ by (1), so that $u \notin z^G$ by 14.6.3.4. As $C_K(u)$ is not a 2-group, $C_K(u) \not\leq M$ since $H \cap M = T$ by 14.6.23.2. Thus to prove (4), we must also show that $C_G(u) \not\leq G_1 = H$.

Now U_γ^* is of rank 1 or 2 as $m_2(H^*) = 2$. Suppose first that $m(U_\gamma^*) = 1$. Then $[\tilde{U}_K, U_\gamma] = \langle \tilde{u}_1, \tilde{u}_2 \rangle$ with $u_i \in U_{K,i}$ and $u_1u_2 \in \mathcal{C}$. Conjugating in H, we may take $u = u_1$. Recall $Z_\gamma \leq [U, U_\gamma] \leq \langle u_1, u_2, z \rangle$, with $u_i \notin z^G$ as $u \notin z^G$, so that Z_γ is generated by u_1u_2 or u_1u_2z, and hence $\mathcal{C} \subseteq z^G$. Since $m(U_\gamma^*) = 1$, $m(U/D) = 1$ by (b), and so our hypotheses are symmetric between γ and γ_1. Thus $[U, U_\gamma] = \langle u_1', u_2', z^g \rangle$, with $u_i' \in U_{K,i}^g$ and $u_1'u_2' \in \mathcal{C}^g$. As $\mathcal{C} \subseteq z^G$, $\mathcal{C}^g \subseteq z^G$, so $u \notin \mathcal{C}^g$, and hence $u \in U_{K,1}^g$ or $U_{K,2}^g$. So since H is transitive on $U_{K,1}^\# \cup U_{K,2}^\#$, we may take $g \in C_G(u)$. Thus as $Z \neq Z^g$, $C_G(u) \not\leq H$. Hence $C_G(u) \in \mathcal{I}$, and so (4) holds.

So suppose instead that $m(U_\gamma^*) = 2$. Then $[\tilde{U}_K, U_\gamma] = \langle \tilde{U}_1, \tilde{u}_2 \rangle$ where $\tilde{U}_1$ is a hyperplane of $\tilde{U}_{K,1}$ and $u_2 \in U_{K,2}$, with $U_1^\# u_2 \subseteq \mathcal{C}$. If $m(U/D) = 2$, we again have symmetry between γ and γ_1, so the argument of the previous paragraph establishes (4) in this case also. Thus by (b) we may take $m(U/D) = 1$. As U_γ^* is of rank 2, U_γ^* does not centralize a hyperplane of both $U_{K,1}$ and its dual $U_{K,2}$, so $Z_\gamma = [U_\gamma, D \cap U_{K,i}] \leq U_{K,i}$ for $i = 1$ or 2, contrary to $u \notin z^G$. This contradiction completes the proof of (4), and of 14.6.24. $\square$

We now derive a contradiction, hence showing that no examples satisfy the hypotheses of this section.

By 14.6.24.4, $C_G(u) \in \mathcal{I}$, so that $\mathcal{I}^* \neq \emptyset$, and if $T_u = T_I := T \cap I$ for some $I \in \mathcal{I}^*$, then also $C_G(u) \in \mathcal{I}^*$ by 14.6.4. By 14.6.20, $|T : O_2(H)| > 4$, and by 14.6.22.1, $m(U/C_U(Q_H)) \geq 4$. Thus the hypotheses of 14.6.9 are satisfied for any $I \in \mathcal{I}^*$, and that result shows that $|T| > 2^{11}$ and $LT = L_I T_I O_2(LT)$, where $L_I := O^2(L \cap I)$. Set $H_2 := C_H(\tilde{u})$. By 14.6.24.2 and 14.6.22.1, $H_2/O_2(H_2) \cong S_3$, with $H_2 \not\leq M$ as $H \cap M = T$ by 14.6.23.2. By construction, $T_0 := C_T(\tilde{u}) = N_T(U_{K,1}) \in Syl_2(H_2)$, and H_2 has nontrivial chief factors on each $\tilde{U}_{K,i}$. Pick $I \in \mathcal{I}^*$, choosing $I := C_G(u)$ if $T_I = T_u$ for some $I \in \mathcal{I}^*$, and let $I_2 := O^2(H_2)T_I$, $I_1 := L_I T_I$, and $I_0 := \langle I_1, I_2 \rangle$. Then $I_0 \in \mathcal{I}^*$ by 14.6.6.6. Further $O^2(H_2)$ centralizes u by Coprime Action, so if $T_I = T_u$, then $I_2 = O^2(H_2)T_u$ centralizes u, while $I_1 \leq C_G(u)$ by our choice of I, so that $I_0 \leq C_G(u)$. Thus I_0 satisfies the hypotheses of 14.6.10.5, and hence $m(\langle V^{I_2} \rangle) = 3$ by that result. However as $\tilde{V} = \langle \tilde{v} \rangle \leq Z(\tilde{T})$, from the module structure in 14.6.22.1, $\tilde{v} = \tilde{u}\tilde{u}_2\tilde{c}$, where $\tilde{u}_2$ generates $C_{\tilde{U}_{K,2}}(T)$ and $\tilde{c} \in C_{\tilde{U}}(H)$. Therefore $\langle V^{I_2} \rangle = \langle uc \rangle [U_{K,2}, I_2]$ is of rank 4, since $Z \leq [U_{K,2}, O^2(H_2)]$ by 14.6.2.

This contradiction completes our analysis of the $L_2(2)$-case under Hypothesis 14.2.1; namely we have now proved:

THEOREM 14.6.25. *Assume Hypothesis 14.2.1. Then G is isomorphic to J_2, J_3, ${}^3D_4(2)$, the Tits group ${}^2F_4(2)'$, $G_2(2)' \cong U_3(3)$, or M_{12}.*

PROOF. We may assume that the Theorem fails, so that case (2) of Hypothesis 14.3.1.2 is satisfied. By Theorem 14.3.16, $U = \langle V^{G_1} \rangle$ is abelian, so that the hypotheses of this section are satisfied. Finally, as we just saw, those hypotheses lead to a contradiction, so the Theorem is established. □

14.7. Finishing $\mathbf{L_3(2)}$ with $\langle \mathbf{V^{G_1}} \rangle$ abelian

In this section we continue to assume Hypothesis 14.5.1, but now assume that $L/O_2(L) \cong L_3(2)$; that is, we treat case (1) of Hypothesis 14.3.1, so in particular Hypothesis 13.3.1 holds, with $G \not\cong Sp_6(2)$ or $U_4(3)$, and $U := \langle V^{G_1} \rangle$ is abelian. Further by 13.3.2.4, Hypothesis 12.2.3 holds, and hence so does case (1) of Hypothesis 12.8.1. Thus we can appeal to results in sections 12.8, 13.3, 14.3, and 14.5.

We will see in Theorem 14.7.75 that the Rudvalis group Ru is the only quasithin example which arises under the hypothesis of this section; as far as we can tell, there are no shadows.

We adopt Notation 12.8.2, including the T-invariant subspaces V_i of V for $i = 1, 2$, and the subgroups $G_i := N_G(V_i)$, $M_i := N_M(V_i)$, $L_i := O^2(N_L(V_i))$, and $R_i := O_2(L_i T)$. In particular $V_1 = V \cap Z$ where as usual $Z := \Omega_1(Z(T))$, z is the generator for V_1, $\tilde{G}_1 := G_1/V_1$, and $\mathcal{H}_z$ consists of the members of $\mathcal{H}(L_1T, M)$ which lie in G_1.

In this section, H denotes a member of $\mathcal{H}_z$.

By 14.5.14 we may adopt Notation 14.5.16; in particular, form the coset geometry Γ of Hypothesis F.9.1 with respect to LT and H, set $b := b(\Gamma, V)$, choose a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b =: \gamma,$$

define U_H, U_γ, D_H, D_γ, etc. as in section F.9, and set $A_1 := V_1^{g_b}$ where $\gamma_1 g_b = \gamma$, using the fact from F.9.11 that b is odd.

Often we can show that $D_\gamma < U_\gamma$, and in those situations we also adopt:

NOTATION 14.7.1. If $D_\gamma < U_\gamma$, choose γ as in 14.5.18.4, so that

$$m(U_\gamma^*) \geq m(U_H/D_H) > 0,$$

and $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, and (as in 14.5.18.5) choose $h \in H$ with $\gamma_0 = \gamma_2 h$, and set $\alpha := \gamma h$ and $Q_\alpha := O_2(G_\alpha)$; then $U_\alpha \leq R_1$ and $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$.

Set $Q := O_2(LT) = C_T(V)$ and

$$S := \langle U_H^L \rangle.$$

Since $Out(L_3(2))$ is a 2-group and T induces inner automorphisms on $L/O_2(L)$ (because T acts on V):

$$M = LC_M(L/O_2(L)).$$

14.7.1. Preliminary reductions.

LEMMA 14.7.2. *Let $\tilde{I}$ be a proper H-submodule of $\tilde{U}_H$, and assume that $Y = O^2(Y) \leq H$ with $YT/O_2(YT) \cong S_3$. Set $\hat{U}_H := U_H/I$. Then*

(1) $\hat{V}$ is isomorphic to $\tilde{V}$ as an L_1T-module.

(2) $\langle \hat{V}_2^Y \rangle$ is of rank 1 or 2.

(3) If $[\hat{V}_2, Y] = 1$, then $[V_2, Y] = 1$.

PROOF. Observe as $I < U_H$ that $V \not\leq I$ since $U_H = \langle V^H \rangle$. Then as L_1 is irreducible on $\tilde{V}$, $V \cap I = V_1$, so part (1) follows. Next as $\tilde{V}_2$ is centralized by T of index 3 in YT, $\tilde{E} := \langle \tilde{V}_2^Y \rangle$ is of rank $\tilde{e} = 1$, 2, or 3, with $\hat{E} := \langle \hat{V}_2^Y \rangle$ of rank $\hat{e} \leq \tilde{e}$. By Theorem 14.5.3.3, $\tilde{e} < 3$, so that (2) holds. If $[\hat{V}_2, Y] = 1$, then $\hat{e} = 1$ so $\tilde{E}$ has the 1-dimensional quotient $\hat{E}$, and therefore $\tilde{e} = 1$ or 3. But we just saw $\tilde{e} < 3$, so $\tilde{e} = 1$, and hence (3) holds. □

LEMMA 14.7.3. *(1) $b \geq 3$ is odd.*

(2) $S \leq Q$.

(3) S is abelian iff $b > 3$.

(4) If $H = G_1$ and $A_1^h \leq V$ for some $h \in H$, then $b = 3$ and $U_{\gamma h} \in U_H^L$.

PROOF. Part (1) is F.9.11.1. As U_H is abelian, $U_H \leq C_{LT}(V) = Q$, so (2) holds. Part (3) is F.9.14.1, and part (4) follows from F.9.14.3 as L is transitive on $V^\#$ since $\bar{L} = GL(V)$. □

LEMMA 14.7.4. *(1) $[V_2, O_2(G_1)] = V_1$.*

(2) $I_2 := \langle O_2(G_1)^{G_2} \rangle \trianglelefteq G_2$, $I_2 = L_2 O_2(G_1)$, $I_2/O_2(I_2) \cong S_3$, and $L_2 = O^2(I_2)$.

(3) $m_3(C_G(V_2)) \leq 1$.

(4) $QQ_H = R_1$, so $R_1^* = Q^*$.

PROOF. Part (1) follows from 14.5.21.2, and 13.3.15 implies (2) and (3). By (1), $1 \neq \bar{Q}_H \leq \bar{R}_1$, so as L_1 is irreducible on $\bar{R}_1$, (4) holds. □

LEMMA 14.7.5. *Assume $L_1^* \trianglelefteq H^*$. Then*

(1) $D_\gamma < U_\gamma$, so we may adopt Notation 14.7.1.

(2) $QQ_H = R_1$.

(3) $L_1^* \cong \mathbf{Z}_3$, and $R_1^* = Q^* = C_{T^*}(L_1^*)$ is of index 2 in T^*.

(4) $[U_\gamma^*, L_1^*] = 1$.

(5) $\tilde{U}_H = [\tilde{U}_H, L_1]$.

PROOF. As $L_1^* \trianglelefteq H^*$, H normalizes $O^2(L_1Q_H) = L_1$. Then 14.5.15.3 says that $L_1^* \cong \mathbf{Z}_3$ and (5) holds. As $L_1T/R_1 \cong S_3$, it follows that $|T^* : C_{T^*}(L_1^*)| = 2$. Further $L_1 \trianglelefteq G_{\gamma_0,\gamma_1}$, so as γ_2 is conjugate to γ_0 in $H \leq G_1$, $L_1 \trianglelefteq G_{\gamma_1,\gamma_2}$. Then as $L_1^* \cong \mathbf{Z}_3$, L_1^* centralizes $O_2(G^*_{\gamma_1,\gamma_2})$. Thus (4) follows as $U_\gamma \leq O_2(G_{\gamma_1,\gamma_2})$ by F.9.13.2, and similarly $[U_H, L_\gamma] \leq O_2(G_\gamma)$, where $L_\gamma := L_1^{g_b}$. Therefore as $U_\gamma/A_1 = [U_\gamma/A_1, L_\gamma]$ by (5) where L_γ has action of order 3 commuting with that of U_H, $m([U_\gamma/A_1, u])$ is even for each $u \in U_H$, so u does not induce a transvection on U_γ/A_1. Thus $D_\gamma < U_\gamma$ by 14.5.18.1, establishing (1).

Part (2) is contained in 14.7.4.4. Then by (2), $Q^* = C_{T^*}(L_1^*)$, completing the proof of (3). □

LEMMA 14.7.6. *$F(H^*)$ is a 3-group.*

PROOF. Suppose H is a minimal counterexample, let $p > 3$ be prime with $H_1^* := \Omega_1(Z(O_p(H^*)) \neq 1$, and pick $P \in Syl_p(H_1)$ where H_1 is the preimage of H_1^*. Since $p > 3$, $H_1 \cap M = Q_H$ by 14.5.20, so $H = PL_1T$ by minimality of H. Similarly H^* is irreducible on P^*. By 14.5.18.3, $q(H^*, \tilde{U}_H) \leq 2$, so by D.2.17, $p = 5$ and $P \leq K \trianglelefteq H$ with $K^* = K_1^* \times \cdots \times K_s^*$, $K_i^* \cong D_{10}$, and $\tilde{U}_i := [\tilde{U}_H, K_i]$ of rank 4. As usual $s = m_5(H^*) \leq 2$ as H is an SQTK-group, so $L_1^* = O^2(L_1^*)$ normalizes K_i^*. Then $[K_i^*, L_1^*] = 1$, so that $L_1 \trianglelefteq KL_1T = H$. Hence 14.7.5 says we may adopt Notation 14.7.1, $Q^* = C_{T^*}(L_1^*)$, U_γ^* centralizes L_1^* of order 3, and $\tilde{U}_H = [\tilde{U}_H, L_1]$. In particular, U_γ^* is faithful on P^* since $F^*(H^*) = L_1^*P^*$. Then since $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{Q}_H)$, either $\mathbf{Z}_2 \cong U_\gamma^* \leq K_i^*$ for some i, or $s = 2$ and $E_4 \cong U_\gamma^* \leq K_1^*K_2^*$. In either case $2m(U_\gamma^*) = m(\tilde{U}_H/C_{\tilde{U}_H}(U_\gamma^*))$, so by 14.5.18.2, U_γ^* induces a faithful group of transvections with center $\tilde{A}_1$ on a subspace $\tilde{D}_H$ of $\tilde{U}_H$ of codimension $m(U_\gamma^*)$. But if U_γ^* is of rank 2, this is not the case, so we may choose notation so that $\mathbf{Z}_2 \cong U_\gamma^* \leq K_1^*$. Therefore $A := [U_H, U_\gamma] \leq U_1$. Since L_1^* centralizes U_γ^*, L_1^* normalizes $\tilde{A}$.

Now by the choice of γ in Notation 14.7.1, $1 = m(U_\gamma^*) \geq m(U_H/D_H) \geq 1$, so $m(U_\gamma^*) = 1 = m(U_H/D_H)$, and hence as discussed in Remark F.9.17, our hypotheses are symmetric between γ_1 and γ. As U_γ centralizes no hyperplane of $\tilde{U}_H$, $A_1 = [D_H, U_\gamma] \leq A$ by F.9.13.6. Thus by the symmetry, $V_1 \leq A$, so that $m(A) = 3$ as $m(\tilde{A}) = 2$, and L_1 acts on A as L_1^* acts on $\tilde{A}$.

Assume first that $s = 2$, and let $T_1 := N_T(K_1)$. Then T_1 is of index 2 in a Sylow 2-subgroup of G, $T_1 \in Syl_2(N_H(A))$, and by 14.5.21.1, L_1T_1 induces A_4 or S_4 on A and centralizes V_1. Again by the symmetry between γ and γ_1, $N_{G_\gamma}(A)$ induces A_4 or S_4 on A and centralizes $A_1 \neq V_1$, so we conclude that $N_G(A)$ induces $GL(A) \cong L_3(2)$ on A. Therefore by 1.2.1.1, $N_G(A) = L_AC_G(A)$ for some $L_A \in \mathcal{C}(N_G(A))$ with $L_A/C_{L_A}(A) \cong L_3(2)$. By 1.2.1.4, $L_A/O_2(L_A) \cong L_3(2)$ or $SL_2(7)/E_{49}$, and in either case $Aut(L_A/O_2(L_A))$ is a $5'$-group. Thus as $K_0 := O^{5'}(K_2)$ acts on A, $[L_A, K_0] \leq O_2(L_A)$. Also L_1 is nontrivial on A, so either $L_1 \leq L_A$ or L_1 is diagonally embedded in $L_AC_G(A)$. As $[L_1^*, K_2^*] = 1$, L_1 acts on K_0.

Next $[K_0, T_1 \cap L_A] \leq K_0 \cap O_2(L_A) \leq O_2(K_0) \leq Q_H$, so $(T_1 \cap L_A)^*$ centralizes K_0^*. Therefore as $C_{GL(\tilde{U}_2)}(K_2^*) \cong \mathbf{Z}_{15}$, $T_1 \cap L_A$ centralizes $\tilde{U}_2$, and hence also centralizes L_1^* and $L_1/O_2(L_1)$.

Let L_0 be the preimage in L_A of $Aut_{L_1}(A)$. As $Aut_{L_1}(A) \cong A_4$, $N_G(L_0) \cap N_G(A)$ contains a Sylow 2-group of $N_G(A)$. Thus $T_1 \leq T_A \in Syl_2(N_G(A))$ with T_A acting on L_0. Further each $t \in T_A - O_2(L_0T_A)$ inverts $L_0/O_2(L_0)$, so $t \notin T_1$ by the previous paragraph. Therefore as $|T_A : T_1| \leq |T : T_1| = 2 = |T_A : O_2(L_0T_A)|$, we conclude $T_A \in Syl_2(G)$. As $A \cap Z(T_A) \neq 1$, $L_A \in \mathcal{L}_f(G, T_A)$, so $L_A \in \mathcal{L}_f^*(G, T_A)$ by 14.3.4.2 and $T_1 \cap L_A = O_2(L_0T_A) \cap L_A$. In particular $O_2(L_A) \leq T_1 \leq N_G(K_0)$, so as $[L_A, K_0] \leq O_2(L_A)$, L_A normalizes $O^2(K_0O_2(L_A)) = K_0$, as does $T_1(T_A \cap L_A) = T_A$. Then as $N_G(L_A) = !\mathcal{M}(N_G(L_AT_A))$ by 1.2.7.3, $K_1 \leq N_G(K_0) \leq N_G(L_A)$. Therefore as $[V_1, K_1] = 1$, K_1 acts on $A = \langle V_1^{L_A} \rangle$, so as $m(A) = 3$, we conclude $K_1 = O^{5'}(K_1)$ centralizes A, whereas $K_1/O_2(K_1)$ is fixed-point-free on $\tilde{U}_1 \geq \tilde{A}$.

This contradiction shows that $s = 1$. Thus $H = K_1TL_1$, so $K_1 \not\leq M$. As $L_1/O_2(L_1)$ is inverted in T, and involutions in $GL(\tilde{U}_1)$ normalizing K_1^* centralize L_1^*, we conclude that $T^* \cong \mathbf{Z}_4$. Thus $\Omega_1(T) \leq Q_HQ$ using parts (2) and (3) of 14.7.5. Then as all involutions in LT/Q are fused to involutions in T/Q inverting $L_1/O_2(L_1)$, $\Omega_1(T) \leq Q$. Therefore $J(T) \leq J_1(T) \leq Q$, so using B.2.3.3 we conclude that $N_G(J(T))$ and $N_G(Z(J_1(T)))$ lie in $M = !\mathcal{M}(LT)$. Therefore as $K_1 \not\leq M$, $K_1 = [K_1, J(T)]$. Then as $p > 3$, a standard result of Thompson (see 26.18.a in [**GLS96**]) shows that $K_1 \leq N_G(J(T))N_G(Z(J_1(T))) \leq M$, a contradiction. □

LEMMA 14.7.7. *If* $L_1^* \trianglelefteq H^*$, *then* $O_{3'}(H^*) = 1$.

PROOF. Suppose H is a counterexample. Then $O_{3'}(E(H^*)) \neq 1$ by 14.7.6, so there is $K \in \mathcal{C}(H)$ with $K^* \cong Sz(2^n)$ for some odd $n \geq 3$. Let $K_1 := \langle K^T \rangle$; by 14.5.19, $K_1L_1T \in \mathcal{H}_z$, so without loss $H = K_1L_1T$.

As $L_1^* \trianglelefteq H^*$, 14.7.5 says that $L_1^* \cong \mathbf{Z}_3$ and $\tilde{U}_H = [\tilde{U}_H, L_1]$. As $Sz(2^n)$ has no FF-module by Theorem B.4.2, examining parts (4)–(6) of F.9.18 we conclude that $W := [\tilde{U}_H, K]/C_{[U_H,K]}(K)$ is the natural module for K^*. This is impossible as $[\tilde{U}_H, L_1] = \tilde{U}_H$ and $[L_1^*, K^*] = 1$, whereas $End_{\mathbf{F}_2K^*}(W) \cong \mathbf{F}_{2^n}$ has multiplicative group of order coprime to 3 since n is odd. □

LEMMA 14.7.8. *There is no* $H \in \mathcal{H}_z$ *with* $O^2(H^*)$ *a cyclic* 3*-group.*

PROOF. Assume $O^2(H^*)$ is a cyclic 3-group. Then as $L_1 \leq H$, $H = PT$ with $P \cong \mathbf{Z}_{3^n}$, and $L_1 = O^2(\Omega_1(P)O_2(H)) \trianglelefteq H$. Furthermore $n > 1$ since $H \not\leq M$. But then $Q_H = O_2(L_1T) = R_1$, so $U_\gamma \leq Q_H$ by 14.7.5, whereas $U_\gamma^* \neq 1$ by 14.7.5.1. □

Observe that 14.7.8 eliminates case (2.iii) of 14.5.20, so we may strengthen 14.5.20 to read:

LEMMA 14.7.9. *Assume* $Y = O^2(Y) \trianglelefteq H$ *with* Y^* *a* p*-group of exponent* p*, and* $O_2(Y) < Y \cap M$*. Then* $p = 3$*, and either*

(1) $Y = L_1$*, or*

(2) $Y^* \cong 3^{1+2}$, $L_1^* = Z(Y^*)$, T *is irreducible on* Y^*/L_1^**, and* $L_1 = O^2(Y \cap M)$.

LEMMA 14.7.10. *Either*

(1) L_1 *has at most three noncentral* 2*-chief factors, or*

(2) $N_G(Baum(R_1)) \leq M$.

PROOF. Let $S_1 := \mathrm{Baum}(R_1)$. We apply the Baumann Argument C.1.37 to the action of LT on V. If (1) fails, then by C.1.37 there is a nontrivial characteristic subgroup C of S_1 normal in LT. Thus as $M = !\mathcal{M}(LT)$, $N_G(S_1) \leq N_G(C) \leq M$, so (2) holds. □

LEMMA 14.7.11. H^* *is not* $L_3(2)$, A_6, *or* S_6.

PROOF. Assume otherwise and let $H_1 := L_1T$ and H_2 the minimal parabolic of H over T distinct from H_1. Set $Y := O^2(H_2)$. Then $H = \langle L_1T, Y\rangle \not\leq M$, so $Y \not\leq M$. Thus $[V_2, Y] = 1$ by 14.5.3.2, so $YL_2/O_2(YL_2) \cong E_9$ by 14.7.4.2.

Let $D_0 := H$, $D_1 := H_2L_2$, $D_2 := LT$, $\mathcal{F} := (D_0, D_1, D_2)$, and $D := \langle\mathcal{F}\rangle$. We will show $(D,\mathcal{F})$ is an A_3-system or C_3-system in the sense of section I.5.

Set $P_i := O_2(D_i)$ and $\dot{D}_1 := D_1/P_1$. We saw $O^2(\dot{D}_1) = \dot{L}_2 \times \dot{Y} \cong E_9$. Further $O_2(G_1) \leq P_0$ by A.1.6, so $\dot{L}_2 = [\dot{L}_2, \dot{P}_0]$ by 14.7.4.2. On the other hand, $Y \leq H \leq N_G(P_0)$, so $\dot{P}_0$ centralizes $\dot{Y}$, and hence $\dot{P}_0$ is of order 2. Next from the structure of H^* under our hypothesis, $Y^* = [Y^*, T^*]$, so $\dot{Y} = [\dot{Y}, \dot{T}]$, and hence $\dot{D}_1 \cong L_2(2) \times L_2(2)$. Of course $D_2/Q_2 = LT/O_2(LT) \cong L_3(2)$, so hypothesis (D2) of section I.5 holds. By the hypotheses of this lemma, hypothesis (D1) holds, and by construction hypothesis (D3) holds. By definition, $D = \langle\mathcal{F}\rangle$. As $H \not\leq M = !\mathcal{M}(LT)$, $\ker_T(D) = 1$, so hypothesis (D4) is satisfied. As $V_1 \leq Z(H)$, hypothesis (D5) holds. This completes the verification that $(D,\mathcal{F})$ is an A_3-system or C_3-system.

As $(D,\mathcal{F})$ is an A_3-system or C_3-system, $D \cong L_4(2)$ or $Sp_6(2)$ by Theorem I.5.1. But then $O_2(H)$ is abelian, contrary to 14.7.4.1 as $O_2(G_1) \leq Q_H$. □

Recall that when $D_\gamma < U_\gamma$, we adopt Notation 14.7.1, and in particular we obtain α with $U_\alpha \leq R_1$.

LEMMA 14.7.12. *(1) L acts 2-transitively on the subgroups U_H^L generating S.*
(2) Assume $D_\gamma < U_\gamma$ and $b = 3$. Then $U_H^L = \{U_H\} \cup U_\alpha^{L_1T}$.

PROOF. As $N_L(U_H) = H \cap L$ is a maximal parabolic of L and $L/O_2(L) \cong L_3(2)$, L is 2-transitive on $L/N_L(U_H)$, so that (1) holds.

Assume the hypotheses of (2). As $b = 3$, $\gamma \in \Gamma(\gamma_2)$, so $\alpha = \gamma h \in \Gamma(\gamma_2 h) = \Gamma(\gamma_0)$, and hence $U_\alpha \in U_H^L$. Therefore (2) follows from (1). □

LEMMA 14.7.13. *(1) Set $E := [U_H, Q]$ and $R := \langle E^L\rangle$. Then $[S,Q] = R$.*
(2) Assume $[\tilde{E}, Q] = \tilde{V}$. Then $[R,Q] = V$.

Assume further that $D_\gamma < U_\gamma$ and $b = 3$.
(3) If $[E, U_\alpha] = 1$ then $R \leq Z(S)$.
(4) Set $A := [U_H, U_\alpha]$ and $B := \langle A^L\rangle$. Then $\Phi(S) = [S,S] = B$.

PROOF. Observe that (1) and (2) follow directly from the definitions of $S = \langle U_H^L\rangle$ and $R = \langle E^L\rangle$. Now assume that $D_\gamma < U_\gamma$ and $b = 3$, so that in particular Notation 14.7.1 holds. Suppose E commutes with U_α. As E also commutes with U_H and $E \trianglelefteq L_1T$, $E \leq Z(S)$ by 14.7.12.2. Thus (3) holds. Similarly 14.7.12.1 implies (4). □

We close Subsection 1 with a brief overview of an argument used to analyze the most difficult configurations in Subsections 2 and 5:

(a) Begin with a particular structure for H^*, and possibly $\tilde{U}_H$.

(b) Determine the structure of Q_H, and hence of H—cf. 14.7.20 and 14.7.71.1.

(c) Determine the structure of S, and hence of LT—cf. 14.7.24, 14.7.25, and 14.7.71–14.7.72.

In Subsection 2 we will obtain a contradiction from this analysis, while in Subsection 5 we will determine G_1 and M, and this information is sufficient to identify G as *Ru*.

14.7.2. Eliminating solvable members of $\mathcal{H}_\mathbf{z}$. As was the case in Theorem 14.6.18 where $LT/O_2(LT) \cong L_2(2)$, in Theorem 14.7.29 of this subsection we will be able to show that no member of $\mathcal{H}_z$ is solvable. The most complicated configuration we must treat is that of case (2) of 14.7.9. We eliminate that case in the following result:

THEOREM 14.7.14. *There exists no $H \in \mathcal{H}_z$ such that $O^2(H^*) \cong 3^{1+2}$.*

Until the proof of Theorem 14.7.14 is complete, assume H is a counterexample. Let $K := O^2(H)$ and $P := O_2(K)$. By hypothesis, $K^* \cong 3^{1+2}$.

LEMMA 14.7.15. *(1) $L_1 \trianglelefteq H$ with $L_1^* = Z(K^*)$.*
(2) $R_1^* = Q^* = C_{T^*}(L_1^*) \cong \mathbf{Z}_4$ or Q_8.
(3) $H = G_1 = KT$ is the unique member of $\mathcal{H}_z$.
(4) $N_G(K) = !\mathcal{M}(KT)$.
(5) $D_\gamma < U_\gamma$, so that $U_\gamma^* \neq 1$.

PROOF. Let K_Z be the preimage of $Z(K^*)$ in K, and $K_0 := O^2(K_Z)$. Then L_1 acts on K_0, so if $K_0 = [K_0, T]$ then $K_0 \leq M$ by 14.5.3.2. If $K_0 > [K_0, T]$, then $K_0 \leq N_G(T) \leq M$ by Theorem 3.3.1.

Thus in any case, $K_0 \leq M$, so we may apply 14.7.9 to K in the role of "Y" to conclude that $L_1 = K_0$, T is irreducible on K^*/L_1^*, and $L_1 = O^2(K \cap M)$. Thus (1) holds, and by (1) we can apply 14.7.5. By 14.7.5.1, (5) holds. By 14.7.5.3, $R_1^* = Q^* = C_{T^*}(L_1^*)$ is of index 2 in T^*, while as T is irreducible on K^*/L_1^*, the remainder of (2) follows from the structure of $Out(K^*) \cong GL_2(3)$; and we also conclude that $K \in \Xi(G,T)$ in the sense of chapter 1. Then by 1.3.6, $K \in \Xi^*(G,T)$, so (4) follows from 1.3.7. In particular $K \trianglelefteq G_1$, so also $L_1 \trianglelefteq G_1$.

Let $\dot{G}_1 := G_1/O_2(G_1)$, $C_1 := C_{G_1}(\dot{K})$, and $Y_1 \in Syl_3(C_1)$. As $m_3(G_1) \leq 2$, $\dot{Y}_1$ is cyclic. Thus $\Omega_1(\dot{Y}_1) = \dot{L}_1 \leq Z(\dot{C}_1)$, and hence $\dot{Y}_1 \leq Z(N_{\dot{C}_1}(\dot{Y}_1))$, so $\dot{C}_1$ is 3-nilpotent by Burnside's Normal p-complement Theorem 39.1 in [**Asc86a**]. As $L_1 \trianglelefteq G_1$, we may apply 14.7.7 with G_1 in the role of "H" to conclude that $O_{3'}(\dot{G}_1) = 1$, so that $\dot{C}_1 = \dot{Y}_1$ is a cyclic 3-group. Thus $Y_1 \leq M$ by 14.7.8. Then as $M = LC_M(L/O_2(L))$, $Y_1 = (Y_1 \cap L_1) \times C_{Y_1}(L/O_2(L))$, so we conclude $|Y_1| = 3$ as Y_1 is cyclic. Then as $\dot{Y}_1 = \dot{C}_1$, $\dot{K} = F^*(\dot{G}_1)$. Therefore as $O^2(\dot{G}_1) \leq GL_3(4)$, either $G_1 = KT$, or $O^2(\dot{G}_1)$ is the split extension of 3^{1+2} by $SL_2(3)$ in view of (2). In the latter case, $m_3(G_1) = 3$, contradicting G_1 an SQTK-group, so the former case holds with $H = KT = G_1$, completing the proof of (3). □

By 14.7.15.3, $G_1 = H$ is the unique member of $\mathcal{H}_z$, so $U_H = \langle V^{G_1} \rangle = U$. Similarly set $D := D_H$. Also in view of 14.7.15.5 and 14.7.5.1:

During the remainder of the proof of Theorem 14.7.14, we adopt Notation 14.7.1.

LEMMA 14.7.16. *(1) $\tilde{U} = [\tilde{U}, L_1]$ is a 6-dimensional faithful irreducible module for K^*.*
(2) $U_\alpha^* = Z(T^*)$ is of order 2.
(3) $[U, U_\alpha] = V$ and $\tilde{V} = C_{\tilde{U}}(Q^*)$.
(4) $U^L = \{U\} \cup U_\alpha^{L_1 T}$.
(5) $b = 3$.
(6) $m(U/D) = 1 = m(U_\alpha^*)$.
(7) $N_G(Baum(R_1)) \leq M$.

REMARK 14.7.17. Notice (6) shows that our hypotheses are symmetric between γ_1 and γ, in the sense discussed in Remark F.9.17; therefore if a result $S(\gamma_1, \gamma)$ (proved under the choice $m(U_\gamma^*) \geq m(U_H/D_H) > 0$ made in Notation 14.7.1) holds, then $S(\gamma, \gamma_1)$ also holds. Similarly as α is an H-translate of γ, $S(\gamma_1, \alpha)$ and $S(\alpha, \gamma_1)$ hold too.

PROOF. (of 14.7.16) By 14.7.15.1, we may apply 14.7.5 to H, so 14.7.5.5 says that $\tilde{U} = [\tilde{U}, L_1]$.

From Notation 14.7.1, $U_\alpha \leq R_1$, so as U is elementary abelian, (2) follows from 14.7.15.2. Then from our choice of γ,

$$1 = m(U_\gamma^*) \geq m(U/D) > 0,$$

and hence $m(U/D) = 1$. Thus we have established (6), and hence also the symmetry between γ_1 and γ discussed in Remark 14.7.17. As $\mathbf{Z}_2 \cong U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U})$, $m(\tilde{U}/C_{\tilde{U}}(U_\alpha)) \leq 2$. Then as $\tilde{U} = [\tilde{U}, L]$, (1) holds by D.2.17.

From 14.7.15.2 and the action of $Aut(K^*)$ on the module $\tilde{U}$ for K^* in (1), $[\tilde{U}, U_\alpha] = C_{\tilde{U}}(R_1)$ is of rank 2; then as $\tilde{V}$ is of rank 2 and centralizes $R_1^* = Q^*$, we conclude that $[\tilde{U}, U_\alpha] = \tilde{V} = C_{\tilde{U}}(Q^*)$. Therefore U_γ^* does not induce transvections on $\tilde{U}$, so U_γ^* does not centralize D, and hence $A_1 \leq [U, U_\gamma]$ by F.9.13.6. Thus by symmetry between γ_1 and α, $V_1 \leq [U, U_\alpha]$, so that $[U, U_\alpha] = V$, completing the proof of (3). In particular $A_1^h \leq V$, so as $H = G_1$, (5) follows from 14.7.3.4, and (4) follows from (5) and 14.7.12.2.

By (1) and 14.5.21.1, L_1 has at least six noncentral 2-chief factors, so (7) follows from 14.7.10. □

Let $E := [U, Q]$ and $R := \langle E^L \rangle$. By 14.7.16.1, $\tilde{U}$ has the structure of a 3-dimensional $\mathbf{F}_4$-module preserved by K^*Q^*, with the 1-dimensional $\mathbf{F}_4$-subspaces the L_1-irreducibles since $L_1^* = Z(K^*)$. Thus $\tilde{V}$ is a 1-dimensional $\mathbf{F}_4$-subspace, and from the action of Q^* on $\tilde{U}$, $\tilde{E} = C_{\tilde{U}}(U_\alpha^*)$ is a 2-dimensional $\mathbf{F}_4$-subspace. Then as $V_1 \leq [U, U_\alpha]$ by 14.7.16.3, $m(E) = 5$. Set $E_H := E^{h^{-1}}$, so that $\tilde{E}_H = C_{\tilde{U}}(U_\gamma^*)$. Define E_γ by $E_\gamma/A_1 = C_{U_\gamma/A_1}(U)$, and set $D_\alpha := D_\gamma^h$ and $E_\alpha := E_\gamma^h$. Observe that these definitions of "E_H, E_γ" differ from those in section F.9, but the latter notation is unnecessary here, since U, D play the role of the groups "V_H, E_H" of section F.9.

LEMMA 14.7.18. *$E_H = C_U(U_\gamma)$ is of index 2 in D, $E_\gamma = C_{U_\gamma}(U)$, $E = C_U(U_\alpha)$, and $E_\alpha = C_{U_\alpha}(U)$ is of rank 5.*

PROOF. By F.9.13.7, $[D, D_\gamma] = 1$, while $m(U/D) = 1 = m(U_\gamma/D_\gamma)$ by 14.7.16.6. Also for $x \in U_\gamma - D_\gamma$, $[x, D] \leq A_1$ by F.9.13.6, so $m(U/C_U(U_\gamma)) \leq m(D/C_D(x)) + 1 \leq 2$. Thus as $C_U(U_\gamma) \leq E_H$ and $m(U/E_H) = m(U/E) = 2$, these inequalities are equalities, and so the first statement of the lemma follows. Then the second statement follows from the symmetry between γ_1 and γ in Remark 14.7.17, and then the third and fourth statements follow from the first and second via conjugation by h. □

LEMMA 14.7.19. *(1) $[S, Q] = R$.*
(2) $[R, Q] = V$.
(3) $R \leq Z(S)$; in particular, R is abelian and $R \leq C_H(U)$.
(4) $\Phi(S) = [S, S] = V$.

PROOF. Recall $E = [U, Q]$ by definition, while $[\tilde{E}, Q] = \tilde{V}$ from the action of Q^* on the module $\tilde{U}$. Then (1) and (2) follow from the corresponding parts of 14.7.13. Furthermore $[E, U_\alpha] = 1$ by 14.7.18, and $[U, U_\alpha] = V$ by 14.7.16.3; then in view of 14.7.15.5 and 14.7.16.5, (3) and (4) follow from the corresponding parts of 14.7.13. □

Recall that $K = O^2(H)$, $P = O_2(K)$, and $C_H(U) = C_{Q_H}(U)$ as $Q_H = C_H(\tilde{U})$.

LEMMA 14.7.20. *(1) $Q_H/C_H(U)$ is isomorphic to $P/C_P(U)$ and to the dual of $\tilde{U}$ as an H-module.*

(2) Either

(i) $[C_H(U), K] \leq U$, or

(ii) H has a unique noncentral chief factor W on $C_H(U)/U$, W is of rank 6, and H^* is faithful on W.

(3) $O_2(L_1) = O_2(K) = P$.

(4) $|P : P \cap Q| = 4$ and $(P \cap Q)/C_P(U) = [P/C_P(U), Q]$.

PROOF. By 14.7.4.1, $[U, Q_H] \neq 1$, so as H is irreducible on $\tilde{U}$ by 14.7.16.1, $C_U(Q_H) = V_1$. Next $Q_H/C_H(U)$ is dual to $\tilde{U}$ as an H-module by 14.5.21.1, so as $\tilde{U} = [\tilde{U}, K]$, also $Q_H/C_H(U) = [Q_H/C_H(U), K]$. Thus $Q_H = PC_H(U)$, so that (1) holds. As $m(\tilde{V}) = 2$, the duality shows that $C_P(V) = P \cap Q$ is of corank 2 in P, and also that (4) holds, since $\tilde{V} = C_{\tilde{U}}(Q)$ by 14.7.16.3.

By 14.7.18, $E_\alpha = C_{U_\alpha}(U)$ is of rank 5, and $V \leq U_\alpha \cap U \leq E_\alpha$ using 14.7.16.3, so $m(E_\alpha U/U) \leq 2$ as $m(V) = 3$. By 14.7.16.4, $U_\alpha = U^y$ for some $y \in L$; thus $V_1^y \leq V \leq U$. Then $C_H(U) \leq C_H(V_1^y) \leq N_H(U_\alpha)$ since $H = C_G(V_1)$, so $[C_H(U), U_\alpha] \leq C_{U_\alpha}(U) = E_\alpha$. Hence if $W := W_1/W_2$ with $U \leq W_1 \leq W_2 \leq C_H(U)$ is a noncentral chief factor for H on $C_H(U)/U$, then $m([W, U_\alpha]) \leq 2$ as we saw $m(E_\alpha U/U) \leq 2$. Therefore as U_α^* has rank 1 by 14.7.16.2, $\hat{\mathcal{Q}}(H^*, W)$ is nonempty. Then by D.2.17, W is a 6-dimensional faithful module, and $[C_H(U), U_\alpha] \leq W_2$, so W is the unique noncentral chief factor for K^* on $C_H(U)/U$. Therefore conclusion (ii) of (2) holds in this case, while conclusion (i) holds if no such chief factor exists; hence (2) is established.

By (1) and (2), all noncentral chief factors X for K on P satisfy $X = [X, L_1]$, so $O_2(L_1) = O_2(K) = P$, establishing (3). □

LEMMA 14.7.21. *(1) $H = C_G(z) \in \mathcal{M}$.*

(2) $Z(P) \leq Z(K)$.

PROOF. By 14.7.15.4, $H_K := N_G(K) = !\mathcal{M}(H)$. By 14.7.15.1, $L_1 \trianglelefteq H_K$. Set $C_K := C_{H_K}(K/O_2(K))$, and $Y_K := C_{H_K}(L_1/O_2(L_1))$, so that Y_K is of index 2 in $H_K = Y_K T$. Then as $R_1 = O_2(L_1 T)$, R_1 is Sylow in Y_K, and hence in $C_K R_1$. As

$$C_{Aut(K^*))}(L_1^*))/Aut_K(K^*) \cong SL_2(3) \text{ is 2-closed,}$$

$C_K R_1 \trianglelefteq H_K$. Let $B \in Syl_3(H_K)$, $B_K := B \cap K$, and $B_C := B \cap C_K$. As $m_3(H_K) \leq 2 = m_3(K)$, B_C is cyclic with $B_1 := \Omega_1(B_C) = B_K \cap B_C$ Sylow in L_1. Then as $R_1 = O_2(L_1 T)$, $[B_1, R_1] \leq O_2(L_1) \cap C_K \leq O_2(C_K)$; so as R_1 is Sylow in $C_K R_1$, B_C is not inverted in its normalizer in $C_K R_1$. Therefore by Burnside's Normal p-complement Theorem 39.1 in [**Asc86a**], $C_K R_1$ has a normal 3-complement. Then by a Frattini Argument, we may take $B = B_K B_M$, where $B_C \leq B_M := N_B(R_1)$, and $B_M \leq M$ by 14.7.16.7. Therefore as $M = LC_M(L/O_2(L))$, $B_M = B_1 \times B_0$

where $B_0 := C_{B_M}(L/O_2(L))$. As B_1 is Sylow in L_1 and $B_1 \leq B_C \leq B_M$ with B_C cyclic, $B_C = B_1$. Further if $B_0 \neq 1$, then B_0 centralizes some a of order 3 in B_K which is inverted by $r \in R_1$ inverting B_K/B_1. But then $m_3(B_0B_1\langle a\rangle) = 3$, contradicting $m_3(H_K) = 2$.

Thus $B_0 = 1$, so $B = B_K \cong K^* \cong 3^{1+2}$ and $H_K = HO_{3'}(H_K)$ with $C_K = L_1O_{3'}(H_K)$. Let $W := \langle z^{C_K}\rangle$, so that $W \in \mathcal{R}_2(C_KR_1)$ by B.2.14. As L_1 centralizes z and $L_1 \trianglelefteq H_K$, L_1 centralizes W, so that $C_K/C_{C_K}(W)$ is a $3'$-group. Since a $3'$-group has no FF-modules by Theorem B.4.2, and R_1 is Sylow in C_KR_1, $J(R_1)$ centralizes W by Thompson Factorization B.2.15. Also as $H = G_1 = C_G(z)$ by 14.7.15.3, $C_{C_K}(W) \leq C_K \cap H \leq L_1O_2(H)$, so $C_{C_K}(W) = L_1O_2(C_K)$. Then $\mathrm{Baum}(R_1) = \mathrm{Baum}(C_{R_1}(W)) = \mathrm{Baum}(O_2(C_KR_1))$ by B.2.3.5. Therefore $C_K \leq N_G(\mathrm{Baum}R_1) \leq M$ by 14.7.16.7. Then by 13.3.8 with L in the role of "K", $O^2(C_K)$ is a $\{2,3\}$-group; so as B_1 is Sylow in C_K, $O^2(C_K) = L_1$. Thus $H = HC_K = H_K \in \mathcal{M}$, and (1) is established.

Let $Z_0 := \Omega_1(Z(P))$ and assume (2) fails, so that $Z_P := [Z_0, K] \neq 1$. Now $Z_0 \in \mathcal{R}_2(K)$, so $Z_0 = Z_P \times C_{Z_0}(K)$ by Coprime Action. But $U \not\leq Z_0$ by 14.7.20.1, so Z_P is a faithful irreducible of rank 6 for K^* by 14.7.20.2. Hence $Z_P \in \mathcal{R}_2(KR_1)$ is not an FF-module for $K^*R_1^*$ by Theorem B.5.6; so as $R_1 \in Syl_2(KR_1)$, $\mathrm{Baum}(R_1) \trianglelefteq KR_1$ by Solvable Thompson Factorization B.2.16 and B.2.3.5. Thus $\mathrm{Baum}(R_1) \trianglelefteq KT = H$, contradicting 14.7.16.7. $\square$

LEMMA 14.7.22. *(1) R/V is isomorphic as an LT/Q-module to one of: the dual of V; the 6-dimensional core of the permutation module on $L/N_L(V_2)$, which we will denote by Core; the direct sum of the 8-dimensional Steinberg module with either Core or the dual of V; or the Steinberg module.*

(2) L_1 has three noncentral chief factors on the Steinberg module, two on Core, and one on the dual of V.

PROOF. As E/V is the natural module for $L_1T/O_2(L_1T)$ and $R = \langle E^L\rangle$, (1) follows from H.6.5. Part (2) follows from H.6.3.3 and H.5.2. $\square$

LEMMA 14.7.23. *$E = U \cap R \leq P$, and either*

(1) Case (i) of 14.7.20.2 holds, R/V is isomorphic to the dual of V as an L-module, and E/V is the unique noncentral chief factor for L_1 on R/V; or

(2) Case (ii) of 14.7.20.2 holds, and $R/V \cong$ Core.

PROOF. By (3) and (4) of 14.7.19, S is nonabelian while $R \leq Z(S)$; so $U \not\leq R$ as $S = \langle U^L\rangle$. Then as L_1 is irreducible on U/E and $R = \langle E^L\rangle$, $E = U \cap R$. Further $E = [E, L_1]$ in view of 14.7.16.1, so $E \leq P$. Thus the noncentral L_1-chief factors of R contained in U are the two in E, so E/V is the unique noncentral chief factor on R/V contained in U/V. Therefore if case (i) of 14.7.20.2 holds, then as $R \leq Z(S) \leq C_H(U)$, E/V is the unique noncentral L_1-chief factor on R/V, and hence R/V is dual to V by 14.7.22, so that (1) holds.

Thus we may assume instead that case (ii) of 14.7.20.2 holds. Then H has a unique noncentral chief factor W on $C_H(U)/U$, and H^* is faithful and irreducible on W of rank 6. Now $[R, Q] = V \leq U$ by 14.7.19.2, so that $[R \cap W, Q] = 1$, and hence $m(R \cap W) \leq 2$ from the action of Q^* on the 6-dimensional faithful irreducible W for K^*. As $U_\alpha \leq S$ by 14.7.16.4, $[Q, U_\alpha] \leq R$ by 14.7.19.1. Therefore as $C_H(U) \leq C_T(V) = Q$, $[W, U_\alpha] \leq R \cap W$. Thus as L_1 acts nontrivially on $[W, U_\alpha]$ in the 6-dimensional module W, we conclude that $R \cap W$ has rank 2, and is the unique noncentral L_1-chief factor on W contained in R/V. So as $R \le C_H(U)$ by 14.7.19.3, and the first paragraph showed that E/V is the unique noncentral L_1-chief factor on R/V contained in U/V, we conclude there are exactly two noncentral L_1-chief factors on R/V. Then it follows from 14.7.22 that (2) holds. □

Let $P_C := C_P(U)$ and $\hat{H} := H/U$.

LEMMA 14.7.24. *Assume case (1) of 14.7.23 holds. Then*
(1) $S/R \cong Core$.
(2) $V_1 < Z(K)$.

PROOF. As we are case in (1) of 14.7.23, case (i) of 14.7.20.2 holds, so that K centralizes $\hat{P}_C$. By 14.7.20.1, $P^+ := P/P_C$ is a 6-dimensional irreducible for H^*. Thus L_1 has exactly six nontrivial 2-chief factors, three each from $\tilde{U}$ and P^+. We next locate these factors relative to the series $Q > S > R > V$. By 14.7.20.4, one of the factors is $P^+/(P \cap Q)^+$, leaving five in $P \cap Q$. By 14.7.23, $E = U \cap R \le P$, and the two factors in E are the factors appearing in V and R/V since case (1) of 14.7.23 holds; this leaves three factors to be located in Q/R. Further $UR/R \cong U/E$ is the natural module for $L_1T/O_2(L_1T)$. Therefore applying H.6.5 as in the proof of 14.7.22, S/R has one of the structures listed in 14.7.22.1. We will show that L_1 has exactly two noncentral chief factors in S/R, so that (1) will hold by applying 14.7.22.2 to the possibilities in 14.7.22.1.

First U/E is the only factor in S/R contained in UR/R. This leaves just the two factors from $(P \cap Q)^+$ to be located in Q/R. Now using (1) and (3) of 14.7.19, $[Q, S \cap P] \le R \cap P \le C_P(U) = P_C$, so that $(S \cap P)^+ \le C_{P^+}(Q^*) =: A_0^+$. Observe $m(A_0^+) = 2$ by applying the duality in 14.7.19.1 to $C_{\tilde{U}}(Q^*) = \tilde{V}$ in view of 14.7.16.3. By 14.7.16.4, $U_\alpha \le S$, and by 14.7.16.2, $U_\alpha^* = Z(T^*)$, so again applying 14.7.19.1, $[P^+, U_\alpha] = A_0^+ \le (S \cap P)^+$. Hence $A_0^+ = (S \cap P)^+$ is of rank 2, so that L_1 has exactly two noncentral chief factors on S/R, given by A_0^+ and UR/R. As indicated earlier, this completes the proof of (1).

Define S_1 as the preimage in S of $Soc(S/R)$. Then $S_1/R \cong V$ as $S/R \cong Core$ by (1), so that $S_1/R = [S_1/R, L_1]$. Observe $U \not\le S_1$ since S is generated by the L-conjugates of U, so we conclude from the proof of (1) that the noncentral L_1-chief factor in S_1/R comes from A_0^+ rather than from UR/R. Thus $S_1 = P_1R$, where $P_1 := P \cap S_1$ and $P_1^+ = A_0^+$ is of rank 2. So setting $C_1 := P_C \cap S_1$, $C_1R/R = C_{S_1/R}(L_1)$ has rank 1. But $C_{\tilde{U}}(L_1) = 1$, so $C_1 \not\le U$, and hence $\hat{C}_1 \ne 1$.

Next as we are in case (i) of 14.7.20.2, $P \le K \le C_H(\hat{P}_C) \le C_H(\hat{C}_1)$, so that $[C_1, P] \le U$. Then as $C_1R/R = C_{S_1/R}(L_1)$ and $P \le L_1$ by 14.7.20.3, $[C_1, P] \le U \cap R = E$, so $[C_1U, P] \le E$. Since K centralizes $\hat{C}_1$, K normalizes C_1U and hence also $[C_1U, P]$, so as K is irreducible on $\tilde{U}$, we conclude $[C_1, P] \le V_1$—that is, P centralizes $\tilde{C}_1$. Let D_1 be the preimage of $C_{\tilde{C}_1\tilde{U}}(L_1)$. As P centralizes $\tilde{C}_1\tilde{U}$, by Coprime Action we have an L_1-module decomposition $\tilde{C}_1\tilde{U} = \tilde{D}_1 \times \tilde{U}$, and then $L_1 = O^2(L_1)$ centralizes D_1. In particular $D_1 \le Z(P)$, and hence $D_1 \le Z(K)$ by 14.7.21.2. As $\hat{C}_1 \ne 1$, $V_1 < D_1$, so (2) is established. □

LEMMA 14.7.25. $V_1 < Z(K)$.

PROOF. In case (1) of 14.7.23 we obtained this result in 14.7.24, so we may assume we are in case (2) of 14.7.23. The proof proceeds much as did the proof of 14.7.24.2, except we analyze P_-/C_- rather than P/P_C, where $P_- := [P_C, L_1]$, and

C_- is the preimage in P_- of $C_{\hat{P}_-}(L_1)$. As case (ii) of 14.7.20.2 holds, P_-/C_- is a 6-dimensional faithful irreducible module for H^*. This time we work modulo V rather than modulo R, so we let R_0 denote the preimage in R of $Soc(R/V)$. Since we are in case (2) of 14.7.23, R_0/V is isomorphic to V as an L-module, so that $R_0/V = [R_0/V, L_1]$. Since R is generated by the L-conjugates of E, $E \not\leq R_0$, so from the analysis of case (2) in the proof of 14.7.23, the noncentral L_1-chief factor in R_0/V is $(R_0 \cap P_-)/(R_0 \cap C_-)$. Thus $R_0 = P_0V$, where $P_0 := [P \cap R_0, L_1] \leq P_-$. This time we set $C_0 := P_0 \cap C_-$, so that $C_0V/V = C_{R_0/V}(L_1)$ is of rank 1. Now $V \leq C_0$, but as L_1 is fixed-point-free on $\tilde{U}$, $C_0 \not\leq U$ and hence $\hat{C}_0 \neq 1$. As K is trivial on $\hat{C}_-$, K acts on C_0U, and hence again K acts on $[C_0U, P] = [C_0, P]$. Now $[C_0, P] \leq V$ as C_0V/V is T-invariant of rank 1, so as K is irreducible on $\tilde{U}$, we conclude P centralizes $\tilde{C}_0$. Let D_0 denote the preimage in C_0U of $C_{\tilde{C}_0\tilde{V}}(L_1)$; just as at the end of the proof of 14.7.24.2, $D_0 \leq Z(K)$, so as $\hat{C}_0 \neq 1$, $V_1 < D_0$, completing the proof. □

As $H = KT$, by 14.7.25 there is a subgroup D of order 4 in $Z(K)$ containing V_1 and normal in H.

LEMMA 14.7.26. *D is a TI-subgroup of G.*

PROOF. If D is cyclic, then $V_1 = \Omega_1(D)$, so as $D \trianglelefteq H = G_1$, the lemma holds. Thus we may assume $D \cong E_4$. If $D \leq Z(T)$ then $D \leq Z(H)$, so as $H \in \mathcal{M}$ by 14.7.21.1, the lemma follows from I.6.1.2. Therefore we may assume that $[D, T] = V_1$.

Let $d \in D - V_1$, and set $G_d := C_G(d)$, and $H_d := H \cap G_d$. Then $T_d := T \cap G_d$ is Sylow in H_d and of index 2 in T, so that $H_d := KT_d$ is of index 2 in $KT = H$; hence $H_d \trianglelefteq H$, and so $H_d \in \mathcal{H}^e$ by 1.1.3.1. Then $Z(T_d) \leq Z(O_2(H_d)) =: Z_d$. As $P = O_2(K)$ and $K \trianglelefteq H_d$, $P \leq O_2(H_d)$, so $[Z_d, K] \leq Z_d \cap P \leq Z(P) \leq Z(K)$ by 14.7.21.2. Therefore K centralizes Z_d by Coprime Action, and so $Z(T_d) \trianglelefteq KT = H$. Thus $N_G(T_d) \leq N_G(Z(T_d)) = H$ as $H \in \mathcal{M}$, so that $T_d \in Syl_2(G_d)$. In particular $d \notin z^G$, so that H controls fusion in D. So appealing to I.6.1.1, it suffices to show that $G_d \leq H$. Thus we assume $G_d \not\leq H$, and it remains to derive a contradiction. As $G_d \not\leq H$,

$$\mathcal{G}_0 := \{G_0 \leq G_d : H_d < G_0\}$$

is nonempty. The bulk of the proof consists of an analysis of $\mathcal{G}_0$.

Let $G_0 \in \mathcal{G}_0$; then $G_0 \in \mathcal{H}(T_d)$ as $d \in O_2(G_0)$. As $L_1 \trianglelefteq H \in \mathcal{M}$, $H = N_G(L_1)$, so $H_d = N_{G_d}(L_1)$ and in particular L_1 is not normal in G_0.

Suppose first that T_d is irreducible on K/L_1. Then $H_d \in \Xi(G_0, T_d)$, so the conclusions of 1.3.2 hold with T_d in the role of "T", and we may apply the proof of 1.3.4 to G_0 in the role of "H" (as that argument uses only 1.3.2 and the fact that G_0 is an SQTK-group, and does not actually require T to be Sylow in G_0) to conclude since $K/O_2(K)$ is not elementary abelian that $K \trianglelefteq G_0$, and hence $L_1 = O^2(O_{2,Z}(K)) \trianglelefteq G_0$, contrary to the previous paragraph.

Thus T_d is reducible on K/L_1, so as R_1 is irreducible on K/L_1 by 14.7.15.2, $R_1^* \not\leq T_d^*$ and hence $R_1 \not\leq T_dQ_H$. So $T_dQ_H < T$, and then as $|T : T_d| = 2$, $Q_H \leq T_d$. Therefore $Q_H = O_2(H_d)$. Further $H = N_G(Q_H)$ as $H \in \mathcal{M}$, so that $H_d = N_{G_0}(Q_H)$, and hence $C(G_0, Q_H) = H_d$. Therefore Hypothesis C.2.3 is satisfied with G_0, H_d, Q_H in the roles of "H, M_H, R".

Let $Y \in Syl_3(K)$, set $X := Y \cap L_1$, and let $\mathcal{I}$ consist of the Y-invariant subgroups I of G_0 with $3 \in \pi(I)$. Then for $I \in \mathcal{I}$, there is a Y-invariant Sylow 3-subgroup Y_I of I, and $X_I := \Omega_1(Z(YY_I) \cap I) = X$ since $m_3(Y) = 2$ and $m_3(G_0) \leq 2$. Thus $X \leq Z(Y_I)$ for each $I \in \mathcal{I}$.

Suppose next that $O_2(G_0) < O_{2,3}(G_0)$. Then $O_{2,3}(G_0) \in \mathcal{I}$, so X is in the center of a Sylow 3-subgroup of $O_{2,3}(G_0)$ by the previous paragraph. Then as $L_1 = X[O_2(G_0), X]$, $O_{2,F^*}(G_0) \leq N_{G_0}(L_1) = H_d$ using an earlier observation. Hence as H is a $\{2,3\}$-group by 14.7.15.3, $O_{2,F^*}(G_0)$ is a $\{2,3\}$-group. Then using A.1.25.3, G_0 is a $\{2,3\}$-group, so $G_0 \in \mathcal{I}$. Therefore X is in the center of a Sylow 3-group Y_I of G_0 containing Y, so that Y_I acts on $X[X, O_2(G_0)] = L_1$. Then $G_0 = Y_I T_d \leq N_{G_0}(L_1) = H_d$, contrary to $G_0 \in \mathcal{G}_0$. This contradiction shows that $O_{2,3}(G_0) = O_2(G_0)$, so that $O_3(G_0/O_2(G_0)) = 1$.

Now suppose J is a subnormal subgroup of G_0 contained in H_d. As H_d is a $\{2,3\}$-group, so is J, so as $O_{2,3}(J) \leq O_{2,3}(G_0)$, J is a 2-group by the previous paragraph. Hence $O_2(G_0)$ is the largest subnormal subgroup of G_0 contained in H_d.

Suppose that $L_0 \in \mathcal{C}(G_0)$ with $3 \in \pi(L_0)$. Then $Y = O^2(Y)$ acts on L_0 by 1.2.1.3, so $L_0 \in \mathcal{I}$, and hence $L_1 = X[X, O_2(G_0)] \leq L_0$. Therefore L_0 is the unique member of $\mathcal{C}(G_0)$ with $3 \in \pi(L_0)$.

Suppose next that $F^*(G_0) = O_2(G_0)$. Set $J := O_{2,3'}(G_0)$. Then $T_d \cap J \leq O_{2,3'}(H_d) = Q_H$, so Q_H is Sylow in JQ_H. Therefore as Hypothesis C.2.3 holds in G_0, we conclude from C.2.5 that $J \leq H_d$, so as J is normal in G_0, $J = O_2(G_0)$ by an earlier reduction. Thus $O_{3'}(G_0/O_2(G_0)) = 1 = O_3(G_0/O_2(G_0))$, so $O_{2,F^*}(G_0)$ is a product of $O_2(G_0)$ with members of $\mathcal{C}(G_0)$ whose order is divisible by 3. Then we conclude from the previous paragraph that $O^2(O_{2,F^*}(G_0)) =: L_0$ is the unique member of $\mathcal{C}(G_0)$ and $L_1 \leq L_0$. In particular $L_0 \trianglelefteq G_0$, so that L_0 is described in C.2.7.3. As $L_1 \leq L_0$ and $O_3(G_0/O_2(G_0)) = 1$, Y acts faithfully on $L_0/O_2(L_0)$. However no group K listed in C.2.7.3 has a group of automorphisms A containing $Inn(K)$ and a subgroup H_A of odd index in A with $O^2(H_A/O_2(H_A)) \cong 3^{1+2}$. Therefore $O_2(G_0) < F^*(G_0)$, so

$$H_d \text{ is maximal in } \{G_+ \leq G_d : F^*(G_+) = O_2(G_+)\}. \qquad (*)$$

Observe that by 1.1.6, Hypothesis 1.1.5 is satisfied with G_d, T_d, H in the roles of "H, S, M". However $U = [U, L_1]$ by 14.7.16.1, so U centralizes $O(G_d)$ by A.1.26. Then as $z \in U$, $O(G_d) = 1$ by 1.1.5.2. Thus there is a component L_d of G_d, and $L_d \not\leq H$ by 1.1.5.3.

Suppose first that L_d is a Suzuki group and set $L_0 := \langle L_d^{H_d} \rangle$. As H_d is a $\{2,3\}$-group, $H_d \cap L_0 = T_d \cap L_0$, so H_d acts on the Borel subgroup $B := N_{L_0}(T_d \cap L_0)$ of L_0. Therefore as $F^*(BH_d) = O_2(BH_d)$, $B \leq H_d$ by (*), impossible as B is not a $\{2,3\}$-group.

Thus $3 \in \pi(L_d)$, so by an earlier reduction, L_d is the unique component of G_d and $L_1 \leq L_d$. Similarly $O_3(G_d/O_2(G_d)) = 1 = O_{3'}(G_d/O_2(G_d))$, so as before Y acts faithfully on L_d. This time L_d is described in 1.1.5.3, and again no subgroup A satisfying $Inn(L_d) \leq A \leq Aut(L_d)$ contains a subgroup H_A of odd index in A with $O^2(H_A/O_2(H_A)) \cong 3^{1+2}$. This contradiction finally completes the proof of 14.7.26. □

We are now in a position to obtain a contradiction, and thus establish Theorem 14.7.14. To obtain our contradiction, we will show that the weak closure $X :=$

$W_0(R_1, D)$ of D in R_1 is normal in both LT and H, so that $H \leq N_G(X) \leq M = !\mathcal{M}(LT)$, contrary to $H \not\leq M$.

It suffices to show that X centralizes U: For then as $V \leq U$, $X \leq C_T(V) = Q$ and $X \leq C_T(U) \leq Q_H$, so $X = W_0(Q, D) = W_0(Q_H, D)$ is normal in LT and H using E.3.15. Thus we may assume there is $g \in G$ such that $A := D^g \leq R_1$, but A does not centralize U. By 14.7.26, A is a TI-subgroup of G, so:

(!) $[C_U(a), A] \leq A \cap U$ for each $a \in A^\#$.

Suppose first that $A \cap U = 1$. Then by (!),

(*) $C_U(a) = C_U(A)$ for each $a \in A^\#$.

In particular if $1 \neq a \in C_A(U)$, then A centralizes U, contrary to our assumption, so A is faithful on U. Thus A is not cyclic of order 4 by (*), so $A \cong E_4$. Now as $m_2(R_1^*) = 1$ by 14.7.15.2, $A \cap Q_H \neq 1$. Then as A is faithful on U, for each $b \in A \cap Q_H^\#$, $C_U(b)$ is a hyperplane of U in view of 14.7.4.1. However no element of $H - Q_H$ centralizes a hyperplane of $\tilde{U}$, and elements of $Q_H - bC_{Q_H}(U)$ centralize hyperplanes of U distinct from $C_U(b)$ by the duality in 14.5.21.1, so again using (*), we conclude $A^\# \subseteq bC_{Q_H}(U)$, a contradiction as A is faithful on U.

Therefore $A \cap U \neq 1$. Then as $|A| = 4$, $|A \cap U| = 2$, and hence A induces a group of transvections on U with center $A \cap U$ by (!). As no element of $H - Q_H$ centralizes a hyperplane of $\tilde{U}$, $A \leq Q_H$; hence $[A, U] = V_1$ by 14.7.4.1, so $A \cap U = V_1$. Therefore as D is a TI-subgroup of G by 14.7.26, $A = D \leq Z(K) \leq C_G(U)$ since $U = [U, K] \leq K$, contrary to our assumption that A does not centralize U.

Thus the proof of Theorem 14.7.14 is complete.

In the remainder of the subsection, H again denotes an arbitrary member of $\mathcal{H}_z$. We deduce various consequences of Theorem 14.7.14 for members of $\mathcal{H}_z$.

LEMMA 14.7.27. *For each $H \in \mathcal{H}_z$, either $O_3(H^*) = 1$ or $O_3(H^*) = L_1^*$.*

PROOF. Suppose H is a minimal counterexample, and let $P^* := O_3(H^*)$ with P a Sylow 3-group of the preimage of P^*. Let P_0 be a supercritical subgroup of P, so that $P_0 \cong \mathbf{Z}_3$, E_9, or 3^{1+2} by A.1.25.1. Further by definition, P_0 contains each subgroup of order 3 in $C_P(P_0)$, so if $|P_0| = 3$, then P is cyclic.

Suppose first that $P_0 \leq M$. Applying 14.7.9 with $O^2(P_0Q_H)$ in the role of "Y" we conclude that $L_1^* = P_0^*$ is of order 3, so P is cyclic. But then $P \leq M$ by 14.7.8, so as $M = LC_M(L/O_2(L))$, $P = C_P(L/O_2(L)) \times (P \cap L_1)$; then as P is cyclic, $P^* = L_1^*$, contrary to the choice of H as a counterexample.

Thus $P_0 \not\leq M$, so by minimality of H, $H = P_0L_1T$. Let B be of order 3 in L_1; we may assume B acts on P.

Assume first that $B \not\leq P$. Then $L_1^* \not\leq O_3(H^*)$, so since L_1 is T-invariant in $H = P_0L_1T$, we conclude that $1 \neq O_2(L_1^*)$. Then by A.1.21.3, L_1^* is faithful on $P_0^*/\Phi(P_0^*)$, so H^* is the split extension of P_0^*, isomorphic to E_9 or 3^{1+2}, by $L_1^*T^* \cong GL_2(3)$. However if P_0^* is 3^{1+2}, then this split extension is of 3-rank 3, contradicting G quasithin. Therefore $P_0^* \cong E_9$. Now $q(H^*, \tilde{U}_H) \leq 2$ by 14.5.18.3, and the normal subgroup $J^* := \langle \mathcal{Q}(H^*, \tilde{U}_H) \rangle$ is either $H^* \cong GL_2(3)$ or $O_{3,Z}(H^*)$. But the first does not appear in D.2.17, and the second does not satisfy conclusion (3) of D.2.17, since irreducibles for H^* faithful on P_0^* have dimension 8 rather than 4.

Therefore $B \leq P$. If $P_0 \cap M \neq 1$, we may apply 14.7.9 to $O^2(P_0Q_H)$ in the role of "Y" to conclude that $P_0^* \cong 3^{1+2}$. But now Theorem 14.7.14 supplies a contradiction. Hence $P_0 \cap M = 1$, so in particular $B \not\leq P_0$. Then as P_0 contains each subgroup of order 3 in $C_P(P_0)$, $P_1 := C_{P_0}(B) < P_0$. Now TL_1 acts on P_1^*, so as $P_1 < P_0$, $P_1 \leq M$ by minimality of H, contradicting $P_0 \cap M = 1$. This completes the proof of 14.7.27. □

LEMMA 14.7.28. *For each $H \in \mathcal{H}_z$, either $O(H^*) = 1$ or $O(H^*) = L_1^*$.*

PROOF. Suppose H is a counterexample. Then by 14.7.27, $O_p(H^*) \neq 1$ for some prime $p > 3$. But this contradicts 14.7.6. □

As a corollary to 14.7.28 we have

THEOREM 14.7.29. *Each solvable subgroup of G_1 containing L_1T is contained in M.*

PROOF. Assume $L_1T \leq H \not\leq M$ is solvable. Then $1 \neq O(H^*) = F^*(H^*) = L_1^* \cong \mathbf{Z}_3$ by 14.7.28, and hence $|H^* : C_{H^*}(F^*(H^*))| \leq 2$. Then $H = Q_H L_1 T \leq M$, contrary to assumption. □

14.7.3. Reducing to $\mathbf{O^2(H^*)}$ isomorphic to $\mathbf{G_2(2)'}$ or $\mathbf{A_5}$. Let $H \in \mathcal{H}_z$. By Theorem 14.7.29 and 1.2.1.1, H contains $\mathcal{C}$-components. In this subsection, we establish restrictions on the $\mathcal{C}$-components of H: For example, 14.7.48 will show that H contains a unique $\mathcal{C}$-component K, and that $H = KT$. Then Theorem 14.7.52 will reduce our analysis to the cases where $K/O_2(K) \cong A_5$ or $G_2(2)'$.

Let $K \in \mathcal{C}(H)$. By 14.7.28, $|O(K^*)| \leq 3$, so $K/O_2(K)$ is quasisimple by 1.2.1.4. Also $K \not\leq M$ and $\langle K^T \rangle L_1 T \in \mathcal{H}_z$ by 14.5.19, and hence:

LEMMA 14.7.30. *For each $K \in \mathcal{C}(H)$, $K/O_2(K)$ is quasisimple, $K \not\leq M$, $\langle K^T \rangle L_1 T \in \mathcal{H}_z$, and $K/O_2(K)$ is described in F.9.18.*

LEMMA 14.7.31. *Suppose $C_G(V_2) \leq M$. Then*

(1) $W_0(R_1, V) \trianglelefteq LT$, so $N_G(W_0(R_1, V)) \leq M$.

(2) Let $U := \langle V^{G_1} \rangle$ and assume there is $Y \in \mathcal{H}^e$, $T_Y \in Syl_2(Y)$, and $V_Y \in \mathcal{R}_2(Y)$ with $Y/O_2(Y) \cong S_3$, $O_2(Y) = C_Y(V_Y)$, and $U^g \leq C_Y(V_Y)$ for each $V_1^g \leq V_Y$. Then $W_0(T_Y, V) \trianglelefteq Y$.

PROOF. Observe first that as $C_G(V_2) \leq M$ by hypothesis, $C_G(V_2) \leq M_V$ by 14.3.3. Thus as L is transitive on hyperplanes of V:

(*) $C_G(A) \leq N_G(V^g)$ for each $g \in G$ and each hyperplane A of V^g.

Suppose $V^g \leq R_1$ with $\bar{V}^g \neq 1$. Then

$$V = \langle C_V(A) : m(V^g/A) = 1 \rangle,$$

while for each hyperplane A of V^g, $[C_V(A), V^g] \leq V \cap V^g = 1$ by (*) and 14.5.2. Thus $[V, V^g] = 1$, contrary to assumption. We conclude $W_0(R_1, V) \leq C_T(V) = O_2(LT)$, so by E.3.15 and E.3.16, $W_0(R_1, V) = W_0(O_2(LT), V) \trianglelefteq LT$ and also $N_G(W_0(R_1, V)) \leq M = !\mathcal{M}(LT)$. Thus (1) holds.

Assume the hypotheses of (2), and suppose $V^g \leq T_Y$ with $[V_Y, V^g] \neq 1$. Then $A := V^g \cap O_2(Y)$ is a hyperplane of V^g, so by (*), $[V_Y, V^g] \leq V_Y \cap V^g$, and hence by transitivity of L on $V^\#$, we may take $V_1^g \leq V_Y$. Then $V^g \leq U^g \leq C_Y(V_Y)$ by hypothesis, contrary to assumption. Thus $W_0(T_Y, V) = W_0(O_2(Y), V) \trianglelefteq Y$ using E.3.15 just as in the proof of (1). □

LEMMA 14.7.32. *T normalizes each $K \in \mathcal{C}(H)$, so $KL_1T \in \mathcal{H}_z$.*

PROOF. Let $K_0 := \langle K^T \rangle$. By 14.7.30, $K_0L_1T \in \mathcal{H}_z$, so without loss $H = K_0L_1T$. We assume T does not act on K and derive a contradiction. By 1.2.1.3, $K_0 = KK^t$ for $t \in T - N_T(K)$. Then by 14.7.30 we may apply F.9.18.5 to conclude that $K/O_2(K)$ is $L_2(2^n)$, $Sz(2^n)$, or $L_3(2)$.

Suppose first that $K^* \cong L_3(2)$. Then by 1.2.2, $K_0 = O^{3'}(H)$, and so $L_1 \leq K_0$. Therefore there is an overgroup H_1 of L_1T in H with $H_1/O_2(H_1) \cong S_3$ wr $\mathbf{Z}_2$, and hence by Theorem 14.7.29, $H_1 \leq M$. But then $O^2(H_1) = [L_1, H_1] \leq L$, so that $m_3(H_1 \cap L) = 2$, contrary to $m_3(L) = 1$.

So K^* is $L_2(2^n)$ or $Sz(2^n)$. Let B_0 be the preimage of the Borel subgroup of K_0^* containing $T_0^* := T^* \cap K_0^*$, and $B := O^2(B_0)$. Then B_0 is the unique maximal overgroup of $L_1T \cap K_0$ in K_0, so L_1T normalizes B. Hence as B is solvable, $B \leq M$ by Theorem 14.7.29, so B acts on L_1. However if K^* is $Sz(2^n)$, then B^* acts on no subgroup L_1^* of $Aut(K^*)$ with $|L_1^* : O_2(L_1^*)| = 3$, so that $[K_0^*, L_1^*] = 1$. Hence $L_1^* \trianglelefteq H^*$, contrary to 14.7.7.

We now interrupt the proof of 14.7.32 briefly, to observe that we can use the previous argument to establish three further results:

LEMMA 14.7.33. *If $H \in \mathcal{H}_z$ and $K \in \mathcal{C}(H)$, then $K/O_2(K)$ is not $Sz(2^n)$.*

PROOF. By the reduction above, we may assume that T normalizes K, and take $H = KL_1T$ using 14.7.30; then we repeat the argument for that reduction essentially verbatim. □

Then using 14.7.33 and 1.2.1.4:

LEMMA 14.7.34. *If $H \in \mathcal{H}_z$ and $K \in \mathcal{C}(H)$, then $m_3(K) = 1$ or 2.*

By 14.7.28 and 14.7.34:

LEMMA 14.7.35. *For each $H \in \mathcal{H}_z$, $O_{3'}(H) = Q_H$.*

Now we return to the proof of 14.7.32. Recall we had reduced to the case where $K/O_2(K) \cong L_2(2^n)$ and $B_0 = K_0 \cap M$. Then 3 divides the order of K^*, so $L_1 \leq O^{3'}(H) = K_0$ by 1.2.2, and hence $L_1 \leq O^2(M \cap K_0) = B$, so n is even. As L_1 is T-invariant, L_1 is diagonally embedded in KK^t. Also $L_1/O_2(L_1)$ is inverted by a suitable $t_L \in T \cap L$, so either t_L induces a field automorphism on both K^* and K^{*t}, or t_L interchanges K^* and K^{*t}.

By 1.2.4, $K \leq K_1 \in \mathcal{C}(G_1)$; then as $K < K_0$, 1.2.8.2 says that K_1 is not T-invariant and either $K = K_1$, or $n = 2$ and $K_1/O_2(K_1) \cong J_1$ or $L_2(p)$ for $p^2 \equiv 1 \mod 5$. In the latter cases we replace H by $H_1 := \langle K_1, L_1T \rangle$ and obtain a contradiction from the reductions above. Therefore $K \in \mathcal{C}(G_1)$ and $K^* \cong L_2(2^n)$ with n even. Again by 1.2.2, $K_0 = O^{3'}(G_1)$, so $C := C_{G_1}(K_0/O_2(K_0)) = O_2(G_1)$ by 14.7.35.

Next as $M = LC_M(L/O_2(L))$, $B = L_1B_C$, where $B_C := O^2(C_B(L/O_2(L)))$ is of index 3 in B. Further $[B_C, t_L] \leq O_2(B_C)$, so that $n = 2$ and t_L does not induce a field automorphism on both K^* and K^{*t}; hence t_L interchanges K^* and K^{*t}.

As $n = 2$, $Out(K_0^*)$ is a 2-group, so as $C = O_2(G_1)$ we conclude

$$G_1 = K_0T = H, \qquad (*)$$

and hence $U_H = \langle V^{G_1} \rangle = U$. As before, our convention will be to also abbreviate D_H by D, but we continue to write H for G_1.

As L_1 is T-invariant and diagonally embedded in K_0, no involution in H^* induces an outer automorphism on K^* centralizing K^{t*}. Thus H^* is A_5 wr $\mathbf{Z}_2$ or

A_5 wr $\mathbf{Z}_2$ extended by an involution inducing a field automorphism on both K^* and K^{*t}. In either case no element of H^* induces a transvection on $\tilde{U}$. Therefore $D_\gamma < U_\gamma$ by 14.5.18.1, so we may adopt Notation 14.7.1.

Let $\tilde{I}$ be a maximal H-submodule, and set $W := \tilde{U}/\tilde{I}$, so that W is H-irreducible. Let V_W denote the image of V in W. By 14.7.2.1 applied to L_1 in the role of "Y", V_W is isomorphic to $\tilde{V}$ as an L_1-module. Next as H is irreducible on W, either W is the tensor product $W_1 \otimes W_2$ of irreducibles W_i for $K_1 := K$ and $K_2 := K^t$, or $W = W_1 \oplus W_2$ with $W_i := [W, K_i]$ a K_i-irreducible. But in the latter case there is no BT-invariant line V_W of W with $V_W = [V_W, L_1]$. Thus $W = W_1 \otimes W_2$, and a similar argument shows that each W_i is the $L_2(4)$-module, so that W is the orthogonal module for $K_0^* \cong \Omega_4^+(4)$, and V_W is the T-invariant singular $\mathbf{F}_4$-point. Let $T_0 := T \cap K_0$. By (*), $H^* = K_0^* T^*$, so as K_0^* is faithful on W, so is H^*; then as $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U})$, $U_\gamma^* \in \mathcal{Q}(H^*, W)$. If a^* is an involution in H^* then either $C_{\tilde{U}}(a^*) = [\tilde{U}, a^*]$ or a^* induces an $\mathbf{F}_4$-transvection. Thus as U_γ^* is quadratic on $\tilde{U}$, $C_{\tilde{U}}(U_\gamma^*) = C_{\tilde{U}}(a^*)$ for each $a^* \in U_\gamma^{*\#}$ which is not an $\mathbf{F}_4$-transvection, and in particular for each $a^* \in K_0^*$. Then calculating in the orthogonal module, we conclude that one of the following holds:

(i) $U_\gamma^* = \langle t^* \rangle$, t^* an $\mathbf{F}_4$-transvection, and $[W, t]$ is a nonsingular $\mathbf{F}_4$-point of W.

(ii) U_γ^* is a 4-group with $[W, U_\gamma] = C_W(U_\gamma)$ of rank 4.

(iii) $U_\gamma^* = \langle t^* \rangle F^*$, where $F^* := C_{T_0^*}(t^*) \cong E_4$, and $[W, U_\gamma] = C_W(U_\gamma)$ is of rank 4.

Suppose case (iii) holds. Then $3 = m(U_\gamma^*) \geq m(U/D)$ by choice of γ in Notation 14.7.1, while by F.9.13.6, $[\tilde{D}, U_\gamma] \leq \tilde{A}_1$ with $m(\tilde{A}_1) = 1$. But then the image D_W of D has corank at most 3 in W, so D_W is not $C_W(U_\gamma)$, and we compute in the orthogonal module W that $[D_W, U_\gamma]$ has rank at least 2. This contradiction eliminates case (iii).

Thus case (i) or (ii) holds. Then as $U_\gamma^* \in \mathcal{Q}(H^*, \tilde{U})$ with $m(W/C_W(U_\gamma)) = 2m(U_\gamma^*)$, we conclude that W is the unique noncentral H-chief factor on $\tilde{U}$, and $W = [\tilde{U}, K_0]$. Further as $L_1 \leq K_0$ with $\tilde{V} = [\tilde{V}, L_1]$, $\tilde{U} = \langle \tilde{V}^H \rangle = W$. By 14.5.18.2, $m(U_\gamma^*) = m(U/D)$, so we have symmetry between γ_1 and γ (cf. Remark 14.7.17), and U_γ^* acts faithfully as a group of $\mathbf{F}_2$-transvections on $\tilde{D}$ with center $\tilde{A}_1$. This eliminates case (ii), for there D has corank 2 in $\tilde{U} = W$, while as U_γ^* contains a free involution, U_γ does not induce a 4-group of $\mathbf{F}_2$-transvections with fixed center on any subspace of corank 2. It also shows $A_1 \leq U$, and hence by symmetry, $V_1 \leq U_\gamma$.

Thus case (i) holds. Recall that under Notation 14.7.1, we choose α and h so that $U_\alpha \leq R_1$, and as in Remark 14.7.17, we also have symmetry between γ_1 and α. Then $U_\alpha^* = \langle t^* \rangle$, where $t^* \in T^*$ induces an $\mathbf{F}_4$-transvection on $\tilde{U} = W$, and $[\tilde{U}, t]$ is a nonsingular $\mathbf{F}_4$-point. We also saw that $m(U/D) = 1$ and that t^* induces an $\mathbf{F}_2$-transvection on the $\mathbf{F}_2$-hyperplane $\tilde{D}$ of $\tilde{U}$ with $[\tilde{D}, t] = \tilde{A}_1^h$.

To complete the proof, we will define subgroups Y, V_Y to which we apply 14.7.31.2, to construct a 2-local I, which we then use to derive a contradiction. We saw that $\tilde{V}$ is the T-invariant singular $\mathbf{F}_4$-point in $\tilde{U}$ containing $\tilde{V}_2$, and $H = G_1$, so $C_G(V_2) = C_H(V_2) \leq N_H(V) \leq M$.

Set $V_Y := V_1 A_1^h \cong E_4$. As H is irreducible on $\tilde{U}$, $[A_1^h, Q_H] = V_1$ by 14.5.21.1, and then by symmetry between γ_1 and α, also $[V_1, Q_\alpha] = A_1^h$. Thus Q_H and $O_2(G_\alpha)$ induce groups of transvections on V_Y with centers V_1 and A_1^h, so by A.1.14,

$Y_0 := \langle Q_H, O_2(G_\alpha)\rangle$ induces $GL(V_Y)$ with kernel $O_2(Y_0) = C_{Q_H}(V_Y)C_{O_2(G_\alpha)}(V_Y)$, and

$$N_G(V_Y) \leq N_G(Y_0). \qquad (**)$$

Set $T_Y := U_\alpha Q_H C_{T\cap K_0 Q_H}(U_\alpha^*)$ and $Y := \langle T_Y, O_2(G_\alpha)\rangle$. Then T_Y centralizes U_α^* and preserves the $\mathbf{F}_4$-structure on $\tilde{U}$, so T_Y centralizes the $\mathbf{F_4}$-point $[\tilde{U}, U_\alpha^*]$ containing $\tilde{A}_1^h$, and hence acts on V_Y. Then by (**), $Y_0 \trianglelefteq Y = Y_0T_Y$, and Y acts on V_Y.

As $U_\alpha \leq R_1$, from the structure of H^*:

$$\langle t^{*L_1^*T^*}\rangle = \langle t^{*T^*}\rangle = \langle t^*\rangle C_{O_2(L_1^*)}(t^*) = T_Y^*;$$

that is, $T_Y \trianglelefteq L_1T$.

Recall that $O_2(Y_0) = C_{Y_0}(V_Y)$, so $O_2(Y_0) \leq O_2(C_H(V_Y))$ by (**), while $O_2(C_{H^*}(\tilde{V}_Y)) = U_\alpha^*$ from the action of H^* on the orthogonal module $\tilde{U}$, so $O_2(Y_0) \leq Q_HU_\alpha \leq T_Y$. Thus $T_Y \in Syl_2(Y)$, and as Y_0 induces $GL(V_Y)$ on V_Y, $C_Y(V_Y) = O_2(Y)$. Further $V_Y \leq U$, so $V_Y \leq U^y$ for each $y \in Y$. This completes the verification of the hypotheses for part (2) of 14.7.31, so we conclude from 14.7.31.2 that $W_0(T_Y, V) \trianglelefteq Y$.

Set $I := \langle L_1T, Y\rangle$. We saw earlier that L_1T acts on T_Y, so I acts on $W_0(T_Y, V)$. Set $V_I := \langle V_1^I\rangle$ and $I^+ := I/C_I(V_I)$; as usual $V_I \in \mathcal{R}_2(I)$ by B.2.14. Also $V_Y \leq V_I$ as $Y \leq I$. We claim that L_1^+ is not subnormal in I^+: For otherwise $O_2(L_1)^+ = 1$, so that $O_2(L_1)$ centralizes V_Y. This is impossible, as $O_2(L_1)$ does not act on V_Y since $O_2(L_1^*) \in Syl_2(K_0^*)$ and $\tilde{A}_1^h$ is nonsingular. This completes the proof of the claim. By the claim, $L_1^+ \neq 1$ and also $Y_0 \not\leq N_G(L_1)$.

Now $L_0 := C_{L_1}(V_Y) \leq N_G(Y_0)$ by (**), so $[Y_0, L_0] \leq C_{Y_0}(V_Y) = O_2(Y_0) \leq T_Y \leq N_G(L_0)$. Thus Y_0 acts on $O^2(L_0) =: L_Y$. Also $L_1^* = L_0^*O_2(L_1^*)$ from the action of H on $\tilde{U}$, so $L_1 = L_YO_2(L_1)$.

Suppose next that $Y_0 \leq M$. Then as Y_0 normalizes L_Y, we conclude from the structure of $Aut(L_3(2))$ that $O^2(Y_0)$, and hence also $O^2(Y_0)T_Y = Y$, acts on L_1, whereas we saw that $Y_0 \not\leq N_G(L_1)$.

Therefore $Y_0 \not\leq M$. We claim next that $[V_I, J(T)] \neq 1$. For otherwise $J(T) \leq C_T(V_I) \leq C_T(V_Y) \leq R_1$ from the action of H^* on $\tilde{U}$. Then $J(T) = J(O_2(Y))$ by B.2.3.3, so that $Y_0 \leq N_G(J(T)) \leq M = !\mathcal{M}(LT)$ using 14.3.9.2, a contradiction establishing the claim.

By the claim, $J(I)^+ \neq 1$. If $J(I)^+$ is solvable, then by Solvable Thompson Factorization B.2.16, $J(I)^+$ has a direct factor $K_I^+ \cong S_3$, and there are at most two such factors by Theorem B.5.6, so that K_I^+ is normalized by $O^2(I^+)$ and L_1^+. If $J(I)^+$ is nonsolvable, then there is $K_I \in \mathcal{C}(J(I))$ with $K_I^+ \neq 1$, so $K_I \in \mathcal{L}_f(G,T)$ by 1.2.10—and then by parts (1) and (2) of 14.3.4, K_I^+ is A_5 or $L_3(2)$, and $K_I \trianglelefteq I$.

We saw that $L_Y = O^2(L_0)$ contains a Sylow 3-subgroup P_L of L_1, and that L_0 acts on Y_0. Since $V_Y = [V_Y, Y_0]$, $P_L \leq P \in Syl_3(Y_0L_Y)$ with $P \cong E_9$. As $L_1^+ \neq 1$, $P_L^+ \neq 1$, and then as $P_L^+ = C_{P^+}(V_Y)$, $P^+ \cong E_9$. From the previous paragraph, P^+ normalizes K_I^+ and $Out(K_I^+)$ is a 2-group, so $P = P_K \times P_C$, where $P_K := P \cap K_I$ and $P_C := C_P(K_I^+)$. Now P_K has order at most 3 by the structure of K_I^+, and P_C has order at most 3 by A.1.31.1, so we conclude both P_K and P_C are of order 3. As $Y = (P \cap Y_0)T_Y$, $I = \langle L_1T, Y\rangle = \langle L_1T, P\rangle$, so as L_1^+ is not normal in I^+, P^+ does not act on L_1^+. Finally one of the following holds:

(a) $L_1^+ \leq K_I^+$.

(b) L_1^+ centralizes K_I^+.

(c) P_L^+ projects faithfully on both P_K^+ and P_C^+.

In case (a), $P_K^+ = P_L^+ \leq L_1^+ \leq K_I^+$, and hence P_C^+ centralizes L_1^+, so that $P^+ \leq L_1^+ P_C^+ \leq N_{I^+}(L_1^+)$, contrary to an earlier observation. In case (b), $P_C^+ = P_L^+ \leq L_1^+$, and P_K^+ centralizes L_1^+, so $P^+ \leq P_K^+ L_1^+ \leq N_{I^+}(L_1^+)$, for the same contradiction. Therefore case (c) holds. We saw that either T normalizes K_I^+ or $\langle K_I^{+T} \rangle \cong S_3 \times S_3$. However the latter case is impossible, as then by A.1.31.1, $P^+ \leq O^{3'}(I^+) = O(\langle K_I^{+T} \rangle)$, contradicting L_1^+ not subnormal in I^+. Thus T acts on K_I, so as T acts on L_1, it acts on the projections L_K^+ and L_C^+ of L_1^+ on K_I^+ and $C_{I^+}(K_I^+)$, respectively. Then as T acts on $L_1 = P_L O_2(L_1)$, and $P_L^+ \leq L_K^+ L_C^+ = O_2(L_K^+) O_2(L_C^+) P^+$, P^+ normalizes $O_2(L_K^+) O_2(L_C^+) P_L^+ = L_1^+$, for the same contradiction yet again. This finally completes the proof of 14.7.32. □

LEMMA 14.7.36. *If $K \in \mathcal{C}(H)$, then $K/O_{2,Z}(K)$ is not sporadic.*

PROOF. Assume $K/O_{2,Z}(K)$ is sporadic. By 14.7.32, $KTL_1 \in \mathcal{H}_z$, so without loss $H = KTL_1$. We conclude from 14.7.30 and F.9.18.4 that $K^* \cong M_{22}$ or $\hat{M}_{22}$. As M_{22} and $\hat{M}_{22}$ have no FF-modules by B.4.2, $\tilde{I} := [\tilde{U}_H, K]$ is irreducible under K using F.9.18.7. As $q(H^*, \tilde{U}_H) \leq 2$ by 14.5.18.3, B.4.2 and B.4.5 say that $\tilde{I}$ is either the code module for M_{22} or the 12-dimensional irreducible for $\hat{M}_{22}$. In either case $\tilde{V}$ of rank 2 lies in $\tilde{I}$.

We first eliminate the case $K^* \cong M_{22}$, as in the proof of 13.8.21: First $L_1 \leq O^{3'}(H) = K$ by A.3.18. Since L_1 is solvable and normal in $J := K \cap M$, $J/O_2(K)$ is a maximal parabolic of $N/O_2(K) \cong A_6/E_{2^4}$. Then $C_V(O_2(L_1(T \cap K))) \leq C_V(O_2(NT))$, with $m(C_V(O_2(NT))) = 1$ by H.16.2.1. This is a contradiction, since $L_1 T$ induces $GL(\tilde{V})$ on $\tilde{V}$ of rank 2 in $\tilde{I}$, so that $O_2(L_1 T)$ centralizes $\tilde{V}$.

Thus we may assume that $K^* \cong \hat{M}_{22}$. By 14.7.28, $L_1^* = Z(K^*) \trianglelefteq H^*$, so $H = KT$, and $\tilde{I} = \tilde{U}_H = [\tilde{U}_H, L_1] = [\tilde{U}_H, K]$ by 14.7.5.5. As L_1^* is inverted in T^*, $H^* = K^* T^* \cong Aut(\hat{M}_{22})$. By 14.7.5.3, $Q^* \in Syl_2(K^*)$. By H.12.1.9, $m(C_{\tilde{U}_H}(T^*)) = 1$, so $\tilde{V}_2 = C_{\tilde{U}_H}(T^*)$, and then $\tilde{V} = [\tilde{V}_2, L_1]$. Now $H \cap M = N_H(V)$ by 14.3.3.6, so using H.12.1.7,

$$(H \cap M)^* = N_{H^*}(\tilde{V}) \cong S_5/E_{32}/\mathbf{Z}_3.$$

However from the structure of $Aut(\hat{M}_{22})$, there is an overgroup H_1 of $L_1 T$ in H (arising from the maximal parabolic of $A_6/E_{16}/\mathbf{Z}_3$ which is not contained in $S_5/E_{32}/\mathbf{Z}_3$) with $H_1/O_2(H_1) \cong S_3 \times S_3$ and $H_1^* \not\leq N_{H^*}(\tilde{V}) = (H \cap M)^*$, contrary to Theorem 14.7.29. □

LEMMA 14.7.37. *(1) $\tilde{U}_H > \langle \tilde{V}^{C_{H^*}(\tilde{V}_2)} \rangle$.*

(2) $\tilde{U}_H$ is not the natural module for $O^2(H^*) \cong L_n(2)$, with $3 \leq n \leq 5$.

(3) $\tilde{U}_H$ is not the natural module for $H^* \cong S_7$.

PROOF. Set $H_0 := O^2(C_H(V_2))$; by Coprime Action, $H_0^* = O^2(C_{H^*}(\tilde{V}_2))$. Assume that (1) fails; then $U_H = \langle V^{H_0} \rangle$. By 14.7.4.2, H_0 acts on L_2, so $[L_2, H_0] \leq C_{L_2}(V_2) = O_2(L_2)$, and then L_2 acts on $O^2(H_0 O_2(L_2))) = H_0$. So as L_2 also acts on V, it acts on on $\langle V^{H_0} \rangle = U_H$. But then $LT = \langle L_1 T, L_2 \rangle$ acts on U_H, so as $M = !\mathcal{M}(LT)$, $H \leq N_G(U_H)) \leq M$, contrary to $H \in \mathcal{H}_z$. This contradiction establishes (1).

If (2) fails, then $C_{H^*}(\tilde{V}_2)$ is irreducible on U_H/V_2, contrary to (1); so (2) holds.

Assume (3) fails, and adopt the notation of section B.3 to describe $\tilde{U}_H$. Now L_1T induces $L_2(2)$ on $\tilde{V} \cong E_4$, so as we saw in the proof of 14.6.10, either

(i) $L_1^*T^*$ is the stabilizer in H^* of the partition $\Lambda := \{\{1,2\},\{3,4\},\{5,6\},\{7\}\}$, $\tilde{V}_2 = \langle e_{1,2,3,4}\rangle$, and $\tilde{V} = \langle e_{1,2,3,4}, e_{1,2,5,6}\rangle$, or

(ii) $L_1^*T^*$ is the stabilizer of the partition $\theta := \{\{1,2\},\{3,4\},\{5,6,7\}\}$, $\tilde{V}_2 = \langle e_{5,6}\rangle$, and $\tilde{V} = \langle e_{5,6}, e_{5,7}\rangle$.

However in case (i), $m_3(C_H(V_2)) = 2$, contrary to 14.7.4.3, so case (ii) must hold. Here $H_0^* \cong A_5$ stabilizes $\{5,6\}$, and $\tilde{U}_H = \langle \tilde{V}^{H_0}\rangle$, contrary to (1). □

LEMMA 14.7.38. $U_\gamma > D_\gamma$.

PROOF. Assume $U_\gamma = D_\gamma$. By 14.5.18.1, U_H induces a nontrivial group of transvections on U_γ with center V_1. Recall that b is odd by 14.7.3.1, so by edge-transitivity in F.7.3.2, we may pick $g = g_b \in \langle LT, H\rangle$ such that $g : (\gamma_{b-1},\gamma) \mapsto (\gamma_0,\gamma_1)$. Let $\beta := \gamma_1 g$, so that U_β induces a group of transvections with center $B_1 := V_1^g$ on U_H. By (1) and (2) of F.9.13, $U_\beta \leq O_2(G_{\gamma_0,\gamma_1}) = R_1$. Set $H_1 := \langle U_\beta^H\rangle$.

If H_1^* is solvable then by G.6.4, H_1^* is a product of copies of S_3, so by 14.7.28, $L_1^* = O^2(H_1^*)$ and hence $H_1^* = L_1^*U_\beta^* \cong S_3$, contradicting $U_\beta \leq R_1$. Therefore H_1^* is not solvable. Thus by 1.2.1.1, $K^* = [K^*, U_\beta^*]$ for some $K \in \mathcal{C}(H)$. Let $U_K := [U_H, K]$. As U_β^* induces transvections on $\tilde{U}_H$, G.6.4 says $\tilde{U}_K/C_{\tilde{U}_K}(K)$ is a natural module for $K^*U_\beta^*/C_{K^*U_\beta^*}(\tilde{U}_K) \cong S_n$ or $L_n(2)$.

Suppose first that $K^* \cong A_5$ or $L_3(2)$, and let L_K^* be the projection of L_1^* in K^* with respect to the decomposition $K^* \times C_{H^*}(K^*)$. As L_1 is T-invariant, L_K^* is T^*-invariant; so either $L_K^* \cong A_4$, or $L_K^* = 1$ so that $[L_1^*, K^*] = 1$. In case $K^* \cong A_5$, as U_β^* induces a transposition on K^* and $U_\beta \leq R_1$, $L_K^* = 1$, so $[L_1^*, K^*] = 1$. In case $K^* \cong L_3(2)$, as L_1 is T-invariant and $L_K^* = [L_K^*, T^* \cap K^*]$, either $L_1^* = L_K^* \leq K^*$ or $[L_1^*, K^*] = 1$. However if $[L_1^*, K^*] = 1$, then $[\tilde{U}_K, L_1] = 1$ since $End_{K^*}(\tilde{U}_K) \cong \mathbf{F}_2$, so $\tilde{V} = [\tilde{V}, L_1] \leq C_{\tilde{U}_H}(K)$, and then $\tilde{U}_H = \langle \tilde{V}^H\rangle \leq C_{\tilde{U}_H}(K)$, contradicting $K^* \neq 1$.

Therefore $L_1^* \leq K^* \cong L_3(2)$. Further $\tilde{V} = [\tilde{V}, L_1] \leq \tilde{U}_K$, so that $\tilde{U}_K = \tilde{U}_H$. Then as $End_{K^*}(\tilde{U}_H) \cong \mathbf{F}_2$, $C_{H^*}(K^*) = 1$ as H^* is faithful on $\tilde{U}_H$, so that $H^* = K^*T^*$. Then as the natural module $\tilde{U}_H$ is T-invariant, we conclude that $H^* \cong L_3(2)$, contrary to 14.7.11.

Therefore $K^*U_\beta^* \cong S_6$, S_7, S_8, $L_4(2)$, or $L_5(2)$. In particular by A.3.18, $K = O^{3'}(H)$, so $L_1 \leq K$, and then as above, $U_K = U_H$ and $H^* = K^*U_{\beta^*}$. By 14.7.11, H^* is not S_6, and by 14.7.37, H^* is not $L_n(2)$ or S_7.

Thus it remains to eliminate the case $H^* \cong S_8$. Here $\tilde{V}$ projects on a singular line in the orthogonal space $\tilde{U}_H/C_{\tilde{U}_H}(H)$, so $\tilde{V}_2$ projects on a singular point; hence

$$C_{H^*}(\tilde{V}_2)/O_2(C_{H^*}(\tilde{V}_2)) \cong S_3 \ \text{ wr } \ \mathbf{Z}_2,$$

contrary to 14.7.4.3. □

In view of 14.7.38, we establish the following convention:

In the remainder of the section, we adopt Notation 14.7.1.

REMARK 14.7.39. Whenever we can show that $m(U_\gamma^*) = m(U_H/D_H)$, our hypotheses are symmetric in γ and γ_1; see Remarks 14.7.17 and F.9.17 for a more extended discussion of this point.

THEOREM 14.7.40. *Assume $H \in \mathcal{H}_z$ such that $H = KL_1T$ for some $K \in \mathcal{C}(H)$ with $K/O_{2,Z}(K)$ of Lie type over $\mathbf{F}_{2^n}$ for some $n > 1$. Then $H^* \cong S_5$ and $\tilde{U}_H/C_{\tilde{U}_H}(K)$ is the $L_2(4)$-module.*

Until the proof of Theorem 14.7.40 is complete, assume the hypotheses of the Theorem. By 14.7.30, we may apply F.9.18.4 to conclude that

(*) K^* is a Bender group, $(S)L_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$.

Let B_0^* be the Borel subgroup of K^* containing $T_0^* := T^* \cap K^*$ and let $B := O^2(B_0)$. As K is defined over $\mathbf{F}_{2^n}$ with $n > 1$, and $L_1T = TL_1$, L_1 acts on B, so by Theorem 14.7.29, $B \leq H \cap M \leq N_H(L_1)$. Then as $M = LC_M(L/O_2(L))$, $BL_1 = B_CL_1$, where $B_C := O^2(C_{BL_1}(L/O_2(L)))$. Also $L_1/O_2(L_1)$ is inverted by some $t \in T \cap L$, and $[t, B_C] \leq O_2(L) \cap B_C \leq O_2(B_C)$, so $B_CO_2(L_1B)/O_2(L_1B)$ is the unique t-invariant complement to $L_1O_2(L_1B)/O_2(L_1B)$ in $L_1B/O_2(L_1B)$. Choose $X_1 \in Syl_3(L_1)$ with X_1 inverted by t.

LEMMA 14.7.41. *Either*

(1) $L_1 \not\leq K$, $m_3(K) = 1$, $B_C = B$, $L_1 \trianglelefteq H$, and L_1^* is inverted in $C_{H^*}(K^*)$, or

(2) $L_1^* \leq K^* \cong L_2(4)$, $U_3(8)$, or $(S)L_3(4)$.

PROOF. Suppose first that $L_1 \not\leq K$. Then $B^*/O_2(B^*)$ is a t-invariant complement to X_1^* in $X_1^*B^*/O_2(B^*)$, so as $B_C^*O_2(B^*)/O_2(B^*)$ is the unique such complement, $B_C = B$. Thus $X_1\langle t \rangle$ centralizes $B^*/O_2(B^*)$, so from the structure of $Aut(K^*)$ for K^* on the list in (*), either X_1 induces inner automorphisms on K^*, or $K^*X_1^* \cong PGL_3(4)$. As $L_1 \not\leq K$, K^* is not $GL_3(4)$ by 14.7.28. As $q(H^*, \tilde{U}_H) \leq 2$ by Notation 14.7.1, Theorems B.4.2 and B.4.5 eliminate the case $K^*X_1^* \cong PGL_3(4)$. Thus $L_1^* \leq K^*C_{H^*}(K^*/O_2(K^*)) =: Y^*$, and as $L_1 \not\leq K$, $\theta(Y) \not\leq K$, where Y is the preimage of Y^* in H. Therefore $m_3(K) < 2$ by A.3.18, so that $m_3(K) = 1$ by 14.7.34. Then as t centralizes $B^*/O_2(B^*)$, t also induces an inner automorphism on K^*, from the structure of $Aut(K^*)$ for K^* in (*) of 3-rank 1. Indeed the projection of t on K^* then lies in $O_2(B^*) \leq R_1^*$, so we conclude $L_1 = [L_1, t_C]$ for some $t_C \in C_T(K^*)$, and hence L_1 centralizes K^*. Therefore $L_1^* \trianglelefteq H^*$ as $H = KL_1T$, so H normalizes $O^2(L_1Q_H) = L_1$, and hence (1) holds.

So assume instead that $L_1 \leq K$. As T acts on L_1, $L_1 \leq B$ and $B/O_2(B) = L_1O_2(B)/O_2(B) \times B_CO_2(B)/O_2(B)$. Then as t inverts $L_1/O_2(L_1)$ but $[t, B_C] \leq O_2(B_C)$ with B_C of index 3 in B, we conclude (2) holds from the structure of $Aut(K^*)$ for K^* on the list of (*). □

LEMMA 14.7.42. *If $L_1 \not\leq K$ then $H^* \cong S_5 \times S_3$, $\tilde{U}_H = [\tilde{U}_H, K] \oplus C_{\tilde{U}_H}(K)$, and $[\tilde{U}_H, K]$ is the tensor product of the S_3-module and the S_5-module.*

PROOF. Assume $L_1 \not\leq K$. Then by 14.7.41, $L_1 \trianglelefteq H$, L_1^* is inverted in $C_{H^*}(K^*)$, $m_3(K) = 1$, and $B = B_C$. Also $B \leq H \cap M = N_H(V)$ by 14.3.3.6, so B centralizes V as $End_{L/O_2(L)}(V) \cong \mathbf{F}_2$.

As L_1^* is inverted in $C_{H^*}(K^*)$, each H-chief factor W on $\tilde{U}_H$ is the sum $W = W_1 \oplus W_2$ of a pair of isomorphic K^*-modules W_i. Indeed since $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ by Notation 14.7.1, arguing as in the proof of F.9.18.6, U_α^* is an FF*-offender on W_1 and W_2, so that K^* is $L_2(2^n)$, $SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ by Theorem B.4.2. As $m_3(K) = 1$, the last two cases are eliminated, and n is odd if $K^* \cong SL_3(2^n)$.

Pick $\tilde{I}$ to be an H-submodule of $\tilde{U}_H$ maximal subject to $[\tilde{U}_H, K] \not\leq \tilde{I}$, and let $\hat{U}_H := U_H/I$; then we may take $W = [\hat{U}_H, K]$. As $L_1 \trianglelefteq H$, Q^* is Sylow in $C_{H^*}(L_1^*)$ by 14.7.5.3, so as Q centralizes V, and $\hat{U}_H = \langle \hat{V}^{C_{H^*}(L_1^*)} \rangle$, $\hat{U}_H = [\hat{U}_H, K] = W$ by Gaschütz's Theorem A.1.39. Now by B.4.2, W is either the sum of two natural modules for K^*, or the sum of two A_5-modules for $K^* \cong L_2(4)$. In the first case, as B centralizes V, $\hat{V} \leq C_W(B^*) = 1$, contradicting $W = \langle \hat{V}^H \rangle$.

Thus the second case holds. As K^* has no strong FF-modules by B.4.2, $\tilde{I} = C_{\tilde{U}_H}(K)$ by 14.7.30 and F.9.18.6. Then $\tilde{U}_H = [\tilde{U}_H, K] \oplus C_{\tilde{U}_H}(K)$ as the A_5-module is K-projective, so the lemma holds. $\square$

LEMMA 14.7.43. *K^* is not $U_3(8)$.*

PROOF. Assume otherwise. By 14.7.42, $L_1 \leq K$. By Theorems B.5.1 and B.4.2, H^* has no FF-modules, so by 14.7.30 we may apply parts (7) and (4) of F.9.18 to conclude that $\tilde{U}_H \in Irr_+(K, \tilde{U}_H)$. As $q(H^*, \tilde{U}_H) \leq 2$ by Notation 14.7.1, we conclude from B.4.2 and B.4.5 that $\tilde{U}_H$ is the natural module for H^*. But then there is no B-invariant 2-subspace over $\mathbf{F}_2$ satisfying $\tilde{V} = [\tilde{V}, L_1]$. $\square$

LEMMA 14.7.44. *K^* is not $(S)L_3(4)$.*

PROOF. Assume otherwise. Again $L_1 \leq K$ by 14.7.42.

Suppose first that $K^* \cong SL_3(4)$. By 14.7.28, $L_1^* = Z(K^*)$. Recall L_1^* is inverted in $C_{T\cap L}(B_C^*/O_2(B_C^*))$; thus from the structure of $Aut(SL_3(4))$, there is $t^* \in T^*$ inducing a graph automorphism on K^*. Choose I and I_H as in F.9.18.4; because t induces a graph automorphism, H^* has no FF-modules by Theorem B.5.1, so $U_H = I_H$ by F.9.18.7, and case (iii) of F.9.18.4 holds. Then as the 1-cohomology of the natural module is zero by I.1.6.4, $\tilde{U}_H = \tilde{I} \oplus \tilde{I}^t$, where $\tilde{I}$ is a natural module for K^* and $\tilde{I}^t$ is its dual. Further as $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, either U_α^* is a root group of K^* of rank 2 with $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\alpha)) = 4$, or $m(U_\alpha^*) \geq 3$ and $m(\tilde{U}_H/C_{\tilde{U}_H}(U_\alpha)) = 6$. If $m(U_\alpha^*) = 2$ or 3, we get a contradiction from 14.5.18.2, since U_α^* does not induce $\mathbf{F}_2$-transvections on a subspace of $\tilde{U}_H$ of codimension $m(U_\alpha^*)$. If $m(U_\alpha^*) = 4$ at least $m(U_H/D_H) \leq 4$ by Notation 14.7.1, whereas no subspace of $\tilde{U}_H$ of corank at most 4 satisfies the requirement $[U_\alpha^*, \tilde{D}_H] = \tilde{A}_1^h$ of F.9.13.6.

Thus $K^* \cong L_3(4)$, and hence H^* has no module $\tilde{U}_H$ with $q(H^*, \tilde{U}_H) \leq 2$ by Theorems B.4.2 and B.4.5. This contradiction completes the proof. $\square$

LEMMA 14.7.45. *(1) $K^* \cong A_5$.*

(2) Either

(a) $K \in \mathcal{C}(G_1)$, or

(b) $L_1 \leq K$ and $K \leq K_1 \in \mathcal{C}(G_1)$ with $K_1/O_2(K_1) \cong A_7$.

PROOF. Conclusion (1) holds if $L_1 \not\leq K$ by 14.7.42. If $L_1 \leq K$, it holds since 14.7.43 and 14.7.44 eliminate the other possibilities in 14.7.41.2. Thus (1) is established.

Next as $K \in \mathcal{L}(G_1, T)$, $K \leq K_1 \in \mathcal{C}(G_1)$ by 1.2.4, so $H_1 := K_1 L_1 T \in \mathcal{H}_z$ by 14.7.32. By 14.7.30, $K_1/O_2(K_1)$ is quasisimple. Applying 14.7.36 to G_1 in the role of "H", $K_1/O_{2,Z}(K_1)$ is not sporadic. Applying F.9.18.4 to H_1, either $K_1/O_{2,Z}(K_1)$ is of Lie type in characteristic 2 or $K_1/O_2(K_1) \cong A_7$. If $K = K_1$ then (2a) holds, so we may assume $K < K_1$. Then from the list of possible proper overgroups of A_5 in A.3.14 with $K_1/O_2(K_1)$ quasisimple, either $K_1/O_{2,Z}(K_1)$ is of Lie type over $\mathbf{F}_4$ of Lie rank 2, or $K_1/O_2(K_1) \cong A_7$. In the first case since $K_1/O_2(K_1)$ is defined over $\mathbf{F}_4$, we may apply (1) to H_1 to obtain a contradiction. In the second case $K_1 = O^{3'}(G_1)$ by A.3.18, so $L_1 \leq K_1$. Then as $K = O^2(N_{K_1}(K))$, $L_1 \leq K$, and (2b) holds. □

LEMMA 14.7.46. *$L_1 \leq K$.*

PROOF. Assume $L_1 \not\leq K$; then H and its action on U_H are described in 14.7.42, and $K \in \mathcal{C}(G_1)$ by 14.7.45.2. Let B be the Borel subgroup of K containing $T \cap K$. Then $BT = C_K(\tilde{V}_2)$ from the module structure in 14.7.42, so B normalizes I_2 by 14.7.4.2. Further $L_1 \trianglelefteq H$ since case (1) of 14.7.41 holds, so B also centralizes $\tilde{V} = \langle \tilde{V}_2^{L_1} \rangle$. By 14.7.4.2, $I_2/O_2(I_2) \cong S_3$. Set $G_0 := \langle I_2, K, T \rangle$.

Suppose first that $O_2(G_0) = 1$. Then Hypothesis F.1.1 is satisfied with K, I_2, T in the roles of "L_1, L_2, S", so $\beta := (KT, BT, I_2BT)$ is a weak BN-pair of rank 2 by F.1.9. Further $T \trianglelefteq TN_{I_2}(T \cap I_2)$, so β is described in F.1.12. Then as KT centralizes V_1 with $KT/O_2(KT) \cong S_5$, and $I_2T/O_2(I_2T) \cong S_3$, it follows that β is parabolic isomorphic to the $Aut(J_2)$-amalgam. This is impossible, since in that amalgam, $O_2(KT) \cong Q_8D_8$ while $U_H \leq O_2(KT)$ is of 2-rank 9 by 14.7.42.

Thus $G_0 \in \mathcal{H}(T)$, so $K \leq K_0 \in \mathcal{C}(G_0)$ by 1.2.4. If $K = K_0$, then $L_2 = O^2(I_2)$ acts on K by 1.2.1.3, so $LT = \langle L_1T, L_2 \rangle$ acts on K; then as $M = !\mathcal{M}(LT)$, $K \leq N_G(K) \leq M$, contrary to 14.7.30. Thus $K < K_0$, so since $L_1 \not\leq K$, $K_0 \not\leq G_1$ by 14.7.45.2. Then $K_0 \in \mathcal{L}_f(G,T)$, so that $K_0/O_2(K_0) \cong A_5$ or $L_3(2)$ by 14.3.4.1, contrary to A.3.14. □

We are now in a position to complete the proof of Theorem 14.7.40.

By 14.7.46, $L_1 \leq K$, so $H = KL_1T = KT$. Further $L_1T/O_2(L_1T) \cong S_3$. Therefore $H^* \cong S_5$ by 14.7.45.1.

As $L_1 \leq K$, $V = [V, L_1] \leq [U_H, K]$, so $U_H = [U_H, K]$. Suppose $\tilde{U}_H \in Irr_+(K, \tilde{U}_H)$. As $\tilde{V}$ is an L_1T-invariant line in $\tilde{U}_H$, $\tilde{U}_H$ is not the A_5-module. Then $\tilde{U}_H/C_{\tilde{U}_H}(K)$ is the $L_2(4)$-module, and hence Theorem 14.7.40 holds in this case.

Thus we may assume $\tilde{U}_H \notin Irr_+(K, \tilde{U}_H)$, and it remains to derive a contradiction. By Notation 14.7.1, $U_\alpha^* \leq R_1^*$ with R_1^* Sylow in K^*. Further $m(U_\alpha^*) =: k = 1$ or 2, $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, and $k \geq m(U_H/D_H)$ by choice of γ in 14.7.1. As $U_\alpha^* \leq R_1^* \leq K^*$, $m(W/C_W(U_\alpha^*)) \geq 2$ for each noncentral chief factor W for K on $\tilde{U}_H$, and as $\tilde{U}_H \notin Irr_+(K, \tilde{U}_H)$, there are at least two such chief factors. On the other hand, as $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$, $2k \geq m(\tilde{U}_H/C_{\tilde{U}_H}(U_\alpha))$, so we conclude $k = 2$, and there are exactly two noncentral chief factors, both $L_2(4)$-modules. Further $2m(U_\gamma^*) = m(\tilde{U}/C_{\tilde{U}_H}(U_\alpha))$ so by 14.5.18.2, $m(U_H/D_H) = 2$, and U_γ^* acts as a group of transvections on $\tilde{D}_H$ with center $\tilde{A}_1$. This is impossible as $\tilde{U}_H$ has two $L_2(4)$-chief factors.

Thus Theorem 14.7.40 is at last established.

LEMMA 14.7.47. *Let $K \in \mathcal{C}(H)$. Then*

(1) $L_1 \leq K$, and

(2) $K/O_2(K) \cong L_n(2)$ or A_n for suitable n, or $G_2(2)'$.

PROOF. As $KTL_1 \in \mathcal{H}_z$ by 14.7.32, we may take $H = KTL_1$. By 14.7.36, $K/O_{2,Z}(K)$ is not sporadic, so by 14.7.30 we may apply F.9.18.4 to conclude that either $K/O_{2,Z}(K)$ is of Lie type in characteristic 2, or $K/O_2(K) \cong A_7$. Assume the first case holds. If $K/O_{2,Z}(K)$ is not defined over $\mathbf{F}_2$, then $H^* \cong S_5$ by Theorem

14.7.40, so the lemma holds. On the other hand, if $K/O_{2,Z}(K)$ is defined over $\mathbf{F}_2$, then from F.9.18.4, either (2) holds, or $K/O_2(K) \cong \hat{A}_6$, and T is trivial on the Dynkin diagram of $K/O_{2,Z}(K)$ from the possible modules listed in that result. However in the latter case, $L_1^* = Z(K^*)$ by 14.7.28, so as T is trivial on the Dynkin diagram of $K/O_{2,Z}(K)$, KT is generated by solvable overgroups of L_1T, which lie in M by Theorem 14.7.29, contrary to $H \not\leq M$. Thus (2) is established, and it remains to establish (1) when $K/O_2(K)$ is not A_5.

If $m_3(K) > 1$, then $K = O^{3'}(H)$ by A.3.18, so (1) holds. Thus we may assume $m_3(K) = 1$, so as K^* is not A_5, $K^* \cong L_3(2)$ by (2). Assume $L_1 \not\leq K$. Then L_1^* centralizes K^* as $Out(L_3(2))$ is of order 2 and $L_1 = [L_1, T]$. Now if $H^*/C_{H^*}(K^*) \not\cong Aut(L_3(2))$, then H is generated by a pair of solvable subgroups containing L_1T which lie in M by Theorem 14.7.29, contrary to $H \not\leq M$. Therefore $H^*/C_{H^*}(K^*) \cong Aut(L_3(2))$, so K^*T^* has no FF-modules by Theorem B.4.2. Therefore by parts (7) and (4) of F.9.18, either $\tilde{U}_H \in Irr_+(K, \tilde{U}_H)$ or $\tilde{U}_H = \tilde{I} + \tilde{I}^t$ with $\tilde{I}$ a natural K^*-module and t inducing an outer automorphism of K^*. In either case, $C_{GL(\tilde{U}_H)}(K^*) = 1$, impossible as L_1^* centralizes K^*. □

LEMMA 14.7.48. *(1) There is a unique $K \in \mathcal{C}(H)$, and $H = KT$.*
(2) $U_H = [U_H, K]$.

PROOF. By Theorem 14.7.29, H is not solvable, so there exists $K \in \mathcal{C}(H)$. By 14.7.47.1, L_1 is contained in each $K \in \mathcal{C}(H)$, so K is unique. Then $C_{H^*}(K^*)$ is solvable by 1.2.1.1, and hence $C_{H^*}(K^*) = 1$ by 14.7.28, since $L_1 \leq K$ but $L_1^* \not\leq Z(K^*)$ by 14.7.47.2. So (1) holds as $Out(K^*)$ is a 2-group in each case listed in 14.7.47.2.

As $L_1 \leq K$, $V = [V, L_1] \leq [U_H, K]$, so $U_H = \langle V^H \rangle = [U_H, K]$, and (2) holds. □

LEMMA 14.7.49. *K^* is not $L_3(2)$ or A_6.*

PROOF. Assume otherwise. First $H = KT$ by 14.7.48.1. By 14.7.11, H^* is not $L_3(2)$, A_6, or S_6. Thus T is nontrivial on the Dynkin diagram of K^*, a contradiction as $H = KT$ and T acts on L_1. □

LEMMA 14.7.50. *K^* is not A_7.*

PROOF. Let $\tilde{I}$ be a maximal submodule of $\tilde{U}_H$, and $\hat{U}_H := \tilde{U}_H/\tilde{I}$. As $U_H = [U_H, K]$ by 14.7.48.2, $\hat{U}_H$ is a nontrivial irreducible for K. As $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U}_H)$ by Notation 14.7.1, $\hat{U}_H$ is of rank 4 or 6 by Theorems B.4.2 and B.4.5.

We first eliminate the case $\dim(\hat{U}_H) = 4$. Notice $H^* \cong A_7$ since $\hat{U}_H$ is not invariant under S_7. By 14.7.2.1, $\hat{V}$ is isomorphic to $\tilde{V}$ as an L_1T-module, so from the action of H^* on $\hat{U}_H$, $N_{H^*}(\hat{V})$ is the stabilizer $H_{4,3}^*$ in $H^* \cong A_7$ of a partition of type $4, 3$. Set $H_M := H \cap M$; by 14.3.3.6, $H_M = N_H(V)$. As $H_{4,3}^*$ is solvable and maximal in H^*, we conclude from Theorem 14.7.29 that $H_M^* = H_{4,3}^*$. Since $M = LC_M(L/O_2(L))$, an element $t \in T \cap L$ inverts $L_1/O_2(L_1)$ and centralizes $O^2(C_{H_M}(L/O_2(L)))$ modulo $O_2(M)$. This is a contradiction as $H^* \cong A_7$ rather than S_7, so elements of $H_{4,3}^* - O^2(H_{4,3}^*)$ invert $O^2(H_{4,3}^*)/O^2(O_2(H_{4,3}^*))$.

Thus $\dim(\hat{U}_H) = 6$, and if $H_M^* = H_{4,3}^*$ then $H^* \cong S_7$. As usual, we use the notation of section B.3 for the module $\hat{U}_H$. Since H_M normalizes L_1, H_M^* is a solvable overgroup of $L_1^*T^*$ in S_7, rather than one of the overgroups of T^*

containing a subgroup isomorphic to A_6 or $L_3(2)$. Thus either $H^*_M = H^*_{4,3}$, or H^*_M is the stabilizer $H^*_{2^3,1}$ of a partition of type $2^3, 1$.

Assume first that $H^*_M = H^*_{4,3}$, so that $H^* \cong S_7$. As $\hat{V} = [\hat{V}, L_1]$ is a T-invariant line, $L_1^* \cong \mathbf{Z}_3$ fixes 4 points, and $\hat{V}_2 = \langle e_{5,6} \rangle$. Set $Y := O^2(H_{2^3,1})$; then $\langle \hat{V}_2^Y \rangle$ is of rank 3, contrary to 14.7.2.2.

Finally assume that $H^*_M = H^*_{2^3,1}$. This time as $\hat{V}$ is a line, $\hat{V}_2 = \langle e_{1,2,3,4} \rangle$, so that $[\hat{V}_2, O^2(H_{4,3})] = 1$, and then $[V_2, O^2(H_{4,3})] = 1$ by 14.7.2.3. But then $m_3(C_G(V_2)) > 1$, contrary to 14.7.4.3. □

Lemma 14.7.51. *K^* is not $L_n(2)$.*

Proof. Assume otherwise. In view of 14.7.49 and Theorem C (A.2.3), $n = 4$ or 5. Observe $P^* := L_1^*(T^* \cap K^*)$ is a T-invariant minimal parabolic of K^*.

Assume first that T^* is nontrivial on the Dynkin diagram of K^*. Then $n = 4$, P^* is the middle-node parabolic, and $H^* \cong S_8$. Define $\tilde{I}$ and $\hat{U}_H$ as in the proof of 14.7.50. Again using Theorems B.4.2 and B.4.5, we conclude that $m(U_H) = 4$ or 6, and since P^* acts on the T-invariant line $\hat{V}$, that $\hat{U}_H$ is the 6-dimensional orthogonal module for H^*, and $\hat{V}$ is a totally singular line. Thus $m_3(C_{H^*}(\hat{V}_2)) = 2$, so $m_3(C_H(V_2)) = 2$ by 14.7.2.3, again contrary to 14.7.4.3.

Thus T is trivial on the Dynkin diagram of K^*, so $K^* = H^*$. Thus H^* is generated by rank-2 parabolics H_1^* containing P^* which satisfy $H_1/O_2(H_1) \cong L_3(2)$ or $S_3 \times S_3$. Therefore $H_1 \leq M$ by 14.7.49 or Theorem 14.7.29, contrary to $H \not\leq M$. □

Theorem 14.7.52. *(1) $H = KT = G_1$ is the unique member of $\mathcal{H}_z$, and $U_H = U$.*

(2) $K^* \cong A_5$ or $G_2(2)'$.

Proof. Part (2) follows since 14.7.49–14.7.51 eliminate all other possibilities from 14.7.47.2. As $K \in \mathcal{L}(G_1, T)$, $K \leq K_1 \in \mathcal{C}(G_1, T)$ by 1.2.4. But $G_1 \in \mathcal{H}_z$, so since $K_1/O_2(K_1)$ is quasisimple by 14.7.30, (2) shows there is no proper containment $K < K_1$ in A.3.12, and hence $K = K_1 \in \mathcal{C}(G_1)$. Then by 14.7.48.1 applied to both H and G_1, $H = KT = G_1$, so (1) holds. □

14.7.4. Eliminating the case $\mathbf{O^2(H^*)}$ isomorphic to $\mathbf{G_2(2)'}$. In the remainder of this section, set $M_1 := H \cap M$. Thus $M_1 = C_M(z)$ as $H = G_1$ by Theorem 14.7.52. Further $M_1 = N_H(V)$ by 14.3.3.6. Abbreviate U_H by U. Since in this subsection we use α in preference to γ, we will reserve the abbreviation D not for $D_H = U \cap Q_\gamma$ but instead for $U \cap Q_\alpha$.

In this subsection we show $K^* \cong A_5$ by proving:

Theorem 14.7.53. *$K/O_2(K)$ is not $G_2(2)'$.*

Until the proof of Theorem 14.7.53 is complete, assume H is a counterexample. Recall we are operating under Notation 14.7.1, so we choose γ as in 14.5.18.4 and α as in 14.5.18.5, and in particular $U_\alpha^* \in \mathcal{Q}(H^*, \tilde{U})$.

Lemma 14.7.54. *$\tilde{U}$ is either the 7-dimensional indecomposable Weyl module for $K^* \cong G_2(2)'$, or its 6-dimensional irreducible quotient.*

Proof. By 14.7.48.2, $U = [U, K]$. By Theorems B.4.2 and B.4.5, the 6-dimensional module for K^* is the unique irreducible $\mathbf{F}_2H^*$-module W satisfying

$q(H^*, W) \leq 2$, and that module is not a strong FF-module. By B.4.6.1, the Weyl module is the unique indecomposable extension of that irreducible by a module centralized by K^*. By 14.7.30, we may apply parts (6) and (4) of F.9.18, so if the lemma does not hold there are exactly two noncentral chief factors W_1 and W_2 for H^* on U, and each is of dimension 6. Indeed as in the proof of F.9.18.6, $m(\tilde{U}/C_{\tilde{U}}(U_\gamma) = 2m(U_\gamma^*) = 6$ and U_γ^* is an FF^*-offender on both W_1 and W_2. Then 14.5.18.2 supplies a contradiction, as U_γ^* does not act as a group of transvections on any subspace of corank 3 in $\tilde{U}$. □

In view of 14.7.54, we now appeal to B.4.6 and [**Asc87**] for the structure of $\tilde{U}$, and we use the terminology in [**Asc87**], such as "doubly singular line". As $\tilde{V} = [\tilde{V}, L_1]$ is T-invariant, we have:

LEMMA 14.7.55. *(1) $\tilde{V}$ is a doubly singular line of $\tilde{U}$.*

(2) $\tilde{V}_2$ is a singular point of $\tilde{U}$.

(3) The set $\mathcal{V}(V_1, V_2)$ of doubly singular lines in $\tilde{U}$ through $\tilde{V}_2$ is of order 3, and generates a subspace $\tilde{U}(V_1, V_2)$ of rank 3.

(4) $C_{K^*}(\tilde{U}(V_1, V_2)) =: B^* = B^*(V_1, V_2) \cong E_4$, $[\tilde{U}, b] \in \mathcal{V}(V_1, V_2)$ for each $1 \neq b^* \in B^*$, and

$$\tilde{W} := \tilde{W}(V_1, V_2) := \langle C_{\tilde{U}}(b^*) : 1 \neq b^* \in B^* \rangle = \tilde{V}_2^{\perp}$$

is a hyperplane of $\tilde{U}$. If $H^* \cong G_2(2)$, then $C_{H^*}(\tilde{U}(V_1, V_2)) =: A^* = A^*(V_1, V_2) \cong E_8$, and $\tilde{U}(V_1, V_2)C_{\tilde{U}}(H) = C_{\tilde{U}}(a^*) = C_{\tilde{U}}(B^*) = [\tilde{U}, A^*]$ for each $a^* \in A^* - B^*$.

(5) If $\tilde{U}$ is an FF-module for H^* then $H^* \cong G_2(2)$ and $A^*(V_1, V_2)^H$ is the set of FF-offenders in H^*.*

(6) Let $Y := O^2(C_H(V_2))$. Then $YT/O_2(YT) \cong S_3$, $\tilde{U}(V_1, V_2) = [\tilde{U}(V_1, V_2), Y]$, and Y is transitive on $\mathcal{V}(V_1, V_2)$.

(7) The geometry $\mathcal{G}(\tilde{U})$ of singular points and doubly singular lines in $\tilde{U}$ is the generalized hexagon for $G_2(2)$. In particular, there is no cycle of length 4 in the collinearity graph of $\mathcal{G}(\tilde{U})$.

(8) $\{[\tilde{U}, b^*] : b^* \in B^*\} = \{[\tilde{w}, A^*] : \tilde{w} \in \tilde{W}\}$.

In the remainder of this subsection, we adopt the notation in 14.7.55.

LEMMA 14.7.56. *Let $\mathcal{V}(V_2)$ be the set of preimages in U of members of $\mathcal{V}(V_1, V_2)$, and $U(V_2)$ the preimage of $\tilde{U}(V_1, V_2)$. Then*

(1) $\mathcal{V}(V_2) = V^Y$ is the set of G-conjugates of V containing V_2.

(2) Y centralizes $L_2/O_2(L_2)$, and $G_2 = L_2YT$ acts on $U(V_2)$, with L_2 fixing $\mathcal{V}(V_2)$ pointwise and $G_2/C_G(U(V_2))$ the stabilizer in $GL(U(V_2))$ of V_2.

PROOF. By parts (1) and (6) of 14.7.55, $V^Y = \mathcal{V}(V_2)$. Then $U(V_2) = \langle V^Y \rangle$, so as $[V, L_2] = V_2$, while $[L_2, Y] \leq C_{L_2}(V_2) = O_2(L_2)$ by 14.7.4.2, we have $[U(V_2), L_2] = V_2$, and hence L_2 fixes V^Y pointwise. Further as $H = G_1$, $C_G(V_2) = C_H(V_2) = YC_T(V_2)$, so since L_2T induces $GL(V_2)$ on V_2, $G_2 = L_2YT$. Then $V^{G_2} = V^Y$, so as L is transitive on the hyperplanes of V, (1) follows from A.1.7.1. Finally $P_0 := Aut_{G_2}(U(V_2)) \leq N_{GL(U(V_2))}(V_2) =: P$ with $P = P_0O_2(P)$ and $1 \neq O_2(Aut_Y(U(V_2))) \leq O_2(P_0)$, so $P = P_0$ as P is irreducible on $O_2(P)$. This completes the proof of (2). □

LEMMA 14.7.57. *(1) $M_1 = N_H(V) = L_1T$ and $L_1^*T^*$ is the minimal parabolic of H^* over T^* other than Y^*T^*.*

(2) $QQ_H = R_1$. *Thus* $Q^* = R_1^* = O_2(M_1^*)$ *is the unipotent radical of* M_1^*.

(3) If $H^* \cong G_2(2)$ *then* M_1 *is transitive on the three conjugates in* R_1^* *of* $A^* := A^*(V_1, V_2)$ *in 14.7.55.4.*

PROOF. Recall $M_1 = N_H(V)$, so M_1^* is the minimal parabolic $N_{H^*}(\tilde{V}) = L_1^* T^*$ of H^*, and hence (1) holds. Next $R_1 = QQ_H$ by 14.7.4, so (2) follows from (1). Finally (3) follows from (1) and B.4.6.13. □

Recall from Notation 14.7.1 that $h \in H$ with $\gamma_0 = \gamma_2 h$, $\alpha := \gamma h$, and $U_\alpha \leq R_1$. Set $Z_\alpha := A_1^h$, and let U_0 denote the preimage in U of $C_{\tilde{U}}(H)$. Let $D := U \cap Q_\alpha$.

LEMMA 14.7.58. *Assume* $Z_\alpha \leq V$. *Then*

(1) There exists $g \in G$ *interchanging* γ_1 *and* α.

(2) $V_1 \leq U_\alpha$ *and* $m(U_\alpha^*) = m(U/D)$.

PROOF. Part (1) follows as L is 2-transitive on $V^\#$. Then (1) implies (2). □

Set $H_\alpha := C_H(Z_\alpha)$ and $U_- := U(V_1, V_2)U_0$. As $H = G_1$ by Theorem 14.7.52, H_α acts on U_α and hence on U_α^*, so that:

LEMMA 14.7.59. $O_2(H_\alpha^*) \neq 1$.

LEMMA 14.7.60. *Assume* $Z_\alpha \leq U$. *Then*

(1) Replacing α *by a suitable conjugate under* M_1, *we may assume* $Z_\alpha U_0 = V_2 U_0$.

(2) $H_\alpha^* = C_{H^*}(\tilde{V}_2)$.

(3) Either

(a) $U_\alpha^* = A^* := A^*(V_1, V_2)$, $D \leq U_-$, *and* $H^* \cong G_2(2)$*), or*

(b) $U_\alpha^* = B^* := B^*(V_1, V_2)$, *and either* $D \leq U_-$ *or* $\tilde{D}\tilde{U}_- = \tilde{V}_2^\perp$.

PROOF. As $Z_\alpha \leq U$, H_α is a subgroup of index at most 2 in $C_H(\tilde{Z}_\alpha)$, so that $O_2(C_{H^*}(\tilde{Z}_\alpha)) \neq 1$ by 14.7.59. Therefore as $O_2(H^*) = 1$, $Z_\alpha \not\leq U_0$. It follows that there is $a \in H$ with $Z_\alpha U_0 = V_2^a U_0$. Indeed by 14.5.21.2, $[Q_H, Z_\alpha] = V_1$, so $H_\alpha^* = C_{H^*}(\tilde{Z}_\alpha)$ is the parabolic subgroup of H^* centralizing $\tilde{Z}_\alpha$. Thus $U_\alpha^* \trianglelefteq H_\alpha^*$ with $\Phi(U_\alpha^*) = 1$, so it follows from the structure of the parabolic H_α^* that U_α^* is one of the two subgroups $B^*(V_1, V_2^a)$ or $A^*(V_1, V_2^a)$ described in 14.7.55. In either case, 14.7.55.4 says that $C_{\tilde{U}}(U_\alpha^*) = \tilde{U}(V_1, V_2^a)\tilde{U}_0 =: \tilde{U}_-^a$. But $U_\alpha^* \leq R_1^* = C_{H^*}(\tilde{V})$ using 14.7.57, so the doubly singular line $\tilde{V}$ is contained in $\tilde{U}_-^a$. Therefore $V_2^a \leq V$ by 14.7.56.1 and the fact that the generalized hexagon $\mathcal{G}(\tilde{U})$ contains no cycle of length 3. Then as $M_1 = N_H(V)$ is transitive on $\tilde{V}^\#$, conjugating in M_1, we may take $V_2^a = V_2$, and maintain the constraint $U_\alpha \leq R_1$. Hence (1) holds. We saw $[Q_H, Z_\alpha] = V_1$, so (2) holds. Further H_α acts on $U \cap Q_\alpha = D$, so from the action of H_α^* on $\tilde{U}$, $\tilde{D}\tilde{U}_-$ is U_-, $\tilde{V}_2^\perp$, or $\tilde{U}$. As $[\tilde{D}, U_\alpha] \leq \tilde{Z}_\alpha$ by F.9.13.6, the third case is impossible as H^* induces no transvections on the module $\tilde{U}_H$ in 14.7.54. In the second $U_\alpha^* = B^*(V_1, V_2)$ by 14.7.55.4. Thus (3) is established. □

LEMMA 14.7.61. *(1)* $H^* \cong G_2(2)$, *and replacing* α *by a suitable* M_1*-conjugate, we may assume* $U_\alpha^* = A^* := A^*(V_1, V_2)$.

(2) $[\tilde{U}, U_\alpha] = \widetilde{U \cap U_\alpha} = C_{\tilde{U}}(A^*) = \tilde{U}_-$.

(3) $D = U_-$.

(4) $m(U_\alpha^*) = 3 = m(U/D)$.

(5) We have symmetry between γ_1 *and* α, *as discussed in Remark 14.7.39.*

PROOF. Suppose first that $\tilde{U}_\alpha^*$ is not an FF*-offender on $\tilde{U}_H$. As $m(U_\alpha^*) \geq m(U/D)$, U_α does not centralize D, so that $Z_\alpha = [D, U_\alpha] \leq U$ using F.9.13.6. Therefore by 14.7.60.1, we may take $Z_\alpha U_0 = V_2 U_0$, and by 14.7.60.3, U_α^* is $B^* := B^*(V_1, V_2)$ or $A^* := A^*(V_1, V_2)$. As U_α^* is not an FF*-offender on $\tilde{U}$, $U_\alpha^* = B^*$, so $[U_\alpha, \tilde{U}] = \tilde{U}(V_1, V_2)$ by 14.7.55.4. Also by 14.7.60.3, either $\tilde{D} \leq \tilde{U}_-$ or $\tilde{D}\tilde{U}_- = \tilde{V}_2^\perp$. The first case is impossible, as $m(U/D) \leq m(U_\alpha^*) = 2$, whereas $m(\tilde{U}/\tilde{U}_-) = 3$. Thus $\tilde{D}\tilde{U}_- = \tilde{V}_2^\perp$, and hence $\tilde{Z}_\alpha = [\tilde{D}, U_\alpha] = [\tilde{V}_2^\perp, B^*] = \tilde{V}_2$, so that $Z_\alpha \leq V_2 \leq V$. Therefore $V_1 \leq U_\alpha$ and $m(U_\alpha^*) = 2 = m(U/D)$ by 14.7.58.2. Then as $\tilde{D}$ lies in the hyperplane $\tilde{D}\tilde{U}_- = \tilde{V}_2^\perp$ of $\tilde{U}$, while $m(\tilde{U}/\tilde{U}_-) = 3$, we obtain $m(\tilde{D}\tilde{U}_-/\tilde{D}) = 1$, and in particular $\tilde{U}_- \not\leq \tilde{D}$. Since $U_- = [U, U_\alpha]U_0$ and $[U, U_\alpha] \leq U \cap Q_\alpha = D$, we conclude there is $u_0 \in U_0 - D$. But this is impossible, as then $[U_\alpha, u_0] \leq V_1$, whereas no nontrivial element of G_α/Q_α induces a transvection on U_α/Z_α.

Thus U_α^* is an FF*-offender on $\tilde{U}$, so $H^* \cong G_2(2)$ by 14.7.55.5. As $U_\alpha^* \leq R_1^*$ by Notation 14.7.1, (1) follows from 14.7.57.3. Then (2) follows from 14.7.55.4.

Suppose $D \not\leq U_-$. Then as $\widetilde{C_U(U_\alpha)} = C_{\tilde{U}}(A^*) = \tilde{U}_-$, $1 \neq [D, U_\alpha]$, so as in the previous paragraph, $Z_\alpha \leq U$ by F.9.13.6. However this contradicts 14.7.60.3a since $U_\alpha^* = A^*$. Therefore $D \leq U_-$, so as $3 = m(U_\alpha^*) \geq m(U/D) \geq m(U/U_-) = 3$, we conclude (3) and (4) hold. Then (4) implies (5), completing the proof. □

Set $H_\alpha^+ := H_\alpha/Q_\alpha$ and let W denote the preimage of $\tilde{W}(V_1, V_2) = \tilde{V}_2^\perp$ in U.

LEMMA 14.7.62. *(1) $V_1 \not\leq U_\alpha$ and $Z_\alpha \not\leq U$.*
(2) $U^+ = A^*(Z_\alpha, V_{2,\alpha})$ for a suitable conjugate $V_{2,\alpha}$ of V_2 in U_α containing Z_α.
(3) $W^+ = B^*(Z_\alpha, V_{2,\alpha})$.

PROOF. Recall that $U \leq G_\gamma \leq C_G(A_1)$, so that $A_1 \leq C_{U_\gamma}(U) \leq Q_H$ and hence also $Z_\alpha \leq Q_H$.

Suppose first that $V_1 \leq U_\alpha$. Then by 14.7.61.2, $U \cap U_\alpha = U_-$ is of codimension 3 in U_α, so as $m(U_\alpha^*) = 3$, $Q_H \cap U_\alpha = U \cap U_\alpha$. Thus $Z_\alpha \leq Q_H \cap U_\alpha \leq U$. Then $C_{Q_H}(Z_\alpha)$ is of index 2 in Q_H by 14.5.21.1, with $[C_{Q_H}(Z_\alpha), U_\alpha] \leq Q_H \cap U_\alpha \leq U$, so U_α^* centralizes a hyperplane of $Q_H/C_H(U)$. But this is impossible since by 14.5.21.1, $Q_H/C_H(U)$ is H^*-dual to $\tilde{U}$, and no member of H^* acts as a transvection on $\tilde{U}$.

Therefore $V_1 \not\leq U_\alpha$. Then by the symmetry in 14.7.61.5, $Z_\alpha \not\leq U$, so (1) holds.

By 14.7.61.1, U_α^* is an FF*-offender on $\tilde{U}$, so by symmetry U/D is also an FF*-offender on U_α/Z_α. In particular (2) holds.

As $V_1 \not\leq U_\alpha$, $C_{\tilde{U}}(a) = \widetilde{C_U(a)}$ for each $a \in U_\alpha$, so as each $w \in W$ is centralized by some $1 \neq b^* \in B^*$ by 14.7.55.4, $m(U_\alpha/C_{U_\alpha}(w)) \leq 2$. Thus W^+ is a hyperplane of U^+ such that $m(U_\alpha/C_{U_\alpha}(w^+)) \leq 2$ for each $w \in W$, so (3) follows from 14.7.55.4. □

We now enter the last stages of our proof of Theorem 14.7.53.

From 14.7.62, in the symmetry between γ_1 and α appearing in 14.7.61.5, the tuple H, U, V_1, V_2, W, U_α^*, B^*, γ_1 corresponds to the tuple G_α, U_α, Z_α, $V_{2,\alpha}$, B, U^+, W^+, α, where B is the preimage in U_α of B^*.

Now since $V_1 \not\leq U_\alpha$, using 14.7.55.4 we see that

$$\mathcal{F} := \{[U, b] : 1 \neq b^* \in B^*\}$$

consists of three 4-subgroups, with

(a) $\mathcal{V}(V_2) = \{FV_1 : F \in \mathcal{F}\}$.

Pick $F_0 \in \mathcal{F}$. As V_2 and F_0 are distinct hyperplanes of $F_0V_1 \cong E_8$, $V_2 \cap F_0 = V_1^l$ for a suitable $l \in L_2T$ interchanging V_1 and V_1^l. Set $\beta := \gamma_0 l$. Then $Z_\beta \leq F_0 \leq [U, U_\alpha] \leq U_\alpha$, and as $V_1 \not\leq U_\alpha$:

(b) For each $F \in \mathcal{F}$, $Z_\beta = V_2 \cap U_\alpha = V_2 \cap F$ is a complement to V_1 in V_2.

Set $\hat{U}_\beta := U_\beta/Z_\beta$ and consider the generalized hexagon $\mathcal{G}(\hat{U}_\beta)$. Since $\tilde{V}_2 = \tilde{Z}_\beta$ is a singular point of $\tilde{U}$, conjugating by l it follows that $\hat{V}_1$ is a singular point of $\hat{U}_\beta$.

Next $\tilde{F}$ is the set of lines in $\mathcal{G}(\tilde{U})$ through $\tilde{V}_2 = \tilde{Z}_\beta$, while by 14.7.56.2, L_2 fixes $\mathcal{V}(V_2) = \{FV_1 : F \in \mathcal{F}\}$ pointwise. Therefore conjugating by l, we conclude:

(c) $\{\hat{F}\hat{V}_1 : F \in \mathcal{F}\}$ is the set of lines through $\hat{V}_1$ in $\mathcal{G}(\hat{U}_\beta)$.

Now by 14.7.55.8,

$$\tilde{\mathcal{F}} = \{[\tilde{w}, U_\alpha] : 1 \neq \tilde{w} \in \tilde{W}\}.$$

Therefore as $[U_\alpha, w] \leq U_\alpha$ and $F = U_\alpha \cap FV_1$ for each $F \in \mathcal{F}$,

(d) $\{[U_\alpha, w] : w \in W\} = \mathcal{F} = \{[U, b] : b \in B\}$.

Applying symmetry to (a), and using (d) to conclude that $\mathcal{F}$ is invariant when interchanging γ_1 and α, it follows that

(a$'$) $\mathcal{V}(Z_\alpha, V_{2,\alpha}) = \{FZ_\alpha : F \in \mathcal{F}\}$,

and then from (a') and (c) that:

(c$'$) $\{\hat{F}\hat{Z}_\alpha : F \in \mathcal{F}\}$ is the set of lines through $\hat{Z}_\alpha$ in $\mathcal{G}(\hat{U}_\beta)$.

But now choosing F_1 and F_2 to be distinct members of $\mathcal{F}$, it follows from (c) and (c$'$) that $\hat{Z}_\alpha$, $\hat{F}_1$, $\hat{V}_1$, $\hat{F}_2, \hat{Z}_\alpha$ is a 4-cycle in the collinearity graph of $\mathcal{G}(\hat{U}_\beta)$, contrary to 14.7.55.7.

This contradiction completes the proof of Theorem 14.7.53.

14.7.5. Identifying Ru when $\mathbf{O^2(H^*) = A_5}$. We summarize the major reductions achieved so far in this section:

THEOREM 14.7.63. *$H = C_G(z)$ is the unique member of $\mathcal{H}_z$, $H = KT$ where $K := O^2(H) \in \mathcal{C}(H)$, $H/O_2(H) \cong S_5$, $\tilde{U}$ is an indecomposable K-module, and $\tilde{U}/C_{\tilde{U}}(K)$ is the $L_2(4)$-module for $K/O_2(K)$.*

PROOF. By Theorem 14.7.52.1, $C_G(z) = H$ is the unique member H of $\mathcal{H}_z$. By 14.7.48.1, $H = KT$ for some $K \in \mathcal{C}(H)$; thus $K = O^2(H)$. By Theorem 14.7.52.2 and Theorem 14.7.53, $K/O_2(K)$ is A_5. Then Theorem 14.7.40 says $H^* \cong S_5$ and U/U_0 is the $L_2(4)$-module. Thus $\tilde{U}$ is indecomposable as $U_H = [U_H, K]$ by 14.7.48.2. □

REMARK 14.7.64. We will be working with the following special case of I.1.6.1: Let $\check{U}$ be the largest $\mathbf{F}_2H^*$-module such that $\check{U} = [\check{U}, H^*]$ and $\check{U}/C_{\check{U}}(H^*) \cong N := \check{U}/C_{\check{U}}(K^*)$. (cf. 17.12 in [**Asc86a**]) As N is the natural module for $K^* \cong L_2(4)$, $\check{U}$ has the structure of an $\mathbf{F}_4K^*$-module, and as $\dim_{\mathbf{F_4}}(H^1(K^*, N)) = 1$, $\dim_{\mathbf{F_4}}(\check{U}) = 3$. Set $\check{U}_0 := C_{\check{U}}(K^*)$. There exists a 4-dimensional orthogonal space $\check{U}_1$ over $\mathbf{F}_4$ with $H^* \leq \Gamma O(\check{U}_1)$ such that $\check{U}_0$ is a nonsingular point of $\check{U}_1$ and $\check{U} = \check{U}_0^\perp$. This facilitates later calculations in the image $\tilde{U}$ of $\check{U}$.

Observe that by Theorem 14.7.63 and 14.3.3.6, $M_1 = H \cap M = N_H(V) = L_1T$, and $R_1^* = O_2(L_1^*) \in Syl_2(K^*)$. Let U_0 be the preimage in U of $C_{\tilde{U}}(K)$. As $\tilde{V} = [\tilde{V}, L_1] \cong E_4$ and $\tilde{U}$ is a quotient of the module $\check{U}$ in Remark 14.7.64:

LEMMA 14.7.65. *$\tilde{V}\tilde{U}_0 = C_{\tilde{U}}(R_1)$ and $\tilde{V} = [C_{\tilde{U}}(R_1), L_1]$.*

In Notation 14.7.1 we chose $h \in H$ with $\gamma_0 = \gamma_2 h$, $\alpha := \gamma h$, and $U_\alpha \leq R_1$. Let $Z_\alpha := A_1^h$.

LEMMA 14.7.66. *$U_\alpha^* \leq Q^* \in Syl_2(K^*)$.*

PROOF. By 14.7.4.4, $Q^* = R_1^*$, so $Q^* \in Syl_2(K^*)$, and the lemma follows as $U_\alpha \leq R_1$. □

LEMMA 14.7.67. *(1) $G_2 \leq M$.*

(2) If F is a hyperplane of V, then V is the unique member of V^G containing F.

(3) $K \in \mathcal{L}^*(G,T)$.

(4) $N_G(K) = H \in \mathcal{M}$.

(5) $LT = N_G(V)$.

PROOF. First as $H = C_G(z)$, $C_G(V_2) = C_H(V_2) \leq T$ from the action of H on U, so (1) holds since L_2T induces $GL(V_2)$ on V_2. Then as L is transitive on hyperplanes of V, (1) and A.1.7.1 imply (2). Similarly $Aut_{LT}(V) = GL(V)$, so $N_G(V) = LTC_G(V)$ with $C_G(V) = C_H(V) \leq C_H(V_2) \leq T$, so (5) holds.

Suppose $K < I \in \mathcal{L}(G,T)$. As $K = O^2(C_G(z))$, $[z, I] \neq 1$, so $I \in \mathcal{L}_f(G,T)$, and hence $I/O_2(I)$ is A_5 or $L_3(2)$ by 14.3.4.1. But then A.3.14 supplies a contradiction, establishing (3).

Let $M_K := N_G(K)$; by (3) and 1.2.7.3, $M_K = !\mathcal{M}(H)$. It remains only to prove (4), so we may assume $H < M_K$, and we must derive a contradiction. Let $D := C_{M_K}(K/O_2(K))$; then $M_K = KDT$ so $O^2(D) \neq 1$.

Set $D_1 := O^2(D \cap M)$. Then KT normalizes $O^2(D_1O_2(K)) = D_1$, and D_1 normalizes $O^2(L_1O_2(K)) = L_1$. Thus D_1 centralizes $L_1/O_2(L_1)$, and $D \cap L_1 \leq O_2(L_1)$ as $L_1 \leq K$, so as D_1 is T-invariant and $L_1 = [L_1, T \cap L]$, we conclude that D_1 centralizes $L/O_2(L)$. Thus LT normalizes $O^2(D_1O_2(L)) = D_1$, so if $D_1 \neq 1$ then $K \leq N_G(D_1) \leq M = !\mathcal{M}(LT)$, a contradiction.

Therefore $D_1 = 1$, so that $D \cap M \leq T$. Also $D \cap H \leq C_H(K/O_2(K)) \leq T$ as $H = KT$. As $[D, L_1] \leq O_2(L_1)$, $D \cap T \leq R_1$, and hence $R_1 \in Syl_2(DR_1)$. Let $S_1 := \mathrm{Baum}(R_1)$. Now L_1 has two noncentral chief factors on $\tilde{U}$, and hence also two on $Q_H/C_H(U)$ by the duality in 14.5.21.1. Thus L_1 has at least four noncentral 2-chief factors, so $N_G(S_1) \leq M$ by 14.7.10.

Let $E := \langle V_1^D \rangle$; then $E \in \mathcal{R}_2(DR_1)$ by B.2.14, since $D \in \mathcal{H}^e$ by 1.1.3.1. Further $C_D(E) \leq D \cap H \leq T$, so $C_{DR_1}(E) = O_2(DR_1) = C_{R_1}(E)$. Thus if $J(R_1)$ centralizes E, then $S_1 = \mathrm{Baum}(O_2(DR_1))$ by B.2.3.5, and then $1 \neq O^2(D) \leq N_G(S_1) \leq M$, contrary to $D \cap M \leq T$. Therefore $J(R_1)$ does not centralize E, so by Thompson Factorization B.2.15, E is an FF-module for $(DR_1)^+ := DR_1/O_2(DR_1)$.

Suppose there exists $K_D \in \mathcal{C}(J(DR_1))$. Then as $[E, K_D] \neq 1$, $K_D \in \mathcal{L}_f(G,T)$ by 1.2.10, so we conclude from 14.3.4.1 that $K_D/O_2(K_D) \cong L_3(2)$ or A_5, $K_D \trianglelefteq M_K$, and for each $V_K \in Irr_+(K_D, E, T)$, V_K is the $L_3(2)$-module or A_5-module and is T-invariant. As $K_D = [K_D, J(R_1)]$, we conclude using Theorem B.5.1 and B.2.14 that $E = [E, K_D] \oplus C_E(K_DR_1)$, and $[E, K_D]$ is the A_5-module or the sum of at most two isomorphic $L_3(2)$-modules. Thus $O^2(C_{K_D}(V_1)) \neq 1$, impossible as $O^2(C_{K_D}(V_1)) \leq D \cap H \leq T$.

Thus $J := J(DR_1)$ is solvable by 1.2.1.1. As D centralizes $K/O_2(K)$ and $m_3(M_K) \leq 2$, $m_3(J) = 1$ and hence $J/O_2(J) \cong S_3$ by Solvable Thompson Factorization B.2.16. Let $W_0 := W_0(R_1, V)$. By (1) and 14.7.31.1, $N_G(W_0) \leq M$.

Next suppose $g \in G$ with $V_1^g \leq E$. As K centralizes V_1 and D normalizes K, K centralizes $\langle V_1^D \rangle = E$, so $K \leq O^2(C_G(V_1^g)) = K^g$, and hence $g \in N_G(K) = M_K$. Thus as $U \leq O_2(K)$, $U^g \leq O_2(K) \leq O_2(JR_1)$. Also using an earlier remark, $C_{JR_1}(E) = JR_1 \cap C_{DR_1}(E) = C_{R_1}(E) = O_2(JR_1)$. Therefore we may apply 14.7.31.2 with JR_1, E in the roles of "Y, V_Y", to conclude that $W_0 \trianglelefteq JR_1$. But then $O^2(J) \leq N_D(W_0) \leq D \cap M \leq T$, a contradiction which completes the proof of (4), and hence of 14.7.67. □

LEMMA 14.7.68. *(1)* $z^G \cap U_0 = \{z\}$.

(2) If $u \in U_0^\#$ *with* $[\tilde{u}, T] = 1$, *then* $C_G(u) \leq H$, *and* U_0 *is the unique member of* U_0^G *containing* u.

PROOF. Assume u satisfies the hypotheses of (2) and set $G_u := C_G(u)$. Notice $T_u := C_T(u)$ is of index at most 2 in T and $K \leq G_u$ by Coprime Action.

Suppose first that $G_u \leq H$ holds; we will show that (1) and the remaining statement in (2) follow. Assume that u lies in some conjugate U_0^g. Then $K^g \leq O^2(G_u) \leq O^2(H) = K$, so that $K^g = K$. Thus $g \in N_G(K) = H$ by 14.7.67.4, so in particular g normalizes U_0, completing the proof of (2) in this case. Further as z satisfies these hypotheses in the role of "u", z is in a unique G-conjugate of U_0, so $z^G \cap U_0 = z^{N_G(U_0)}$ by A.1.7.1. But then as $H \in \mathcal{M}$ by 14.7.67.4, $N_G(U_0) = H = C_G(z)$ so that (1) also holds.

So to complete the proof of the lemma, we assume $G_u \not\leq H$, and it remains to derive a contradiction. As K has more than one noncentral 2-chief factor by 14.5.21.1, KT_u is not a block, so by C.1.26 there is $1 \neq C$ char T_u with $C \trianglelefteq KT_u$. But then as T_u is of index at most 2 in T, $H = KT \leq N_G(C)$ so that $N_G(C) = H$ since $H \in \mathcal{M}$. Thus if $T_u \leq T_0 \in Syl_2(G_u)$, then $N_{T_0}(T_u) \leq N_G(C) = H$, so that $T_u = N_{T_0}(T_u)$ and hence $T_u = T_0$. Therefore $K \leq L_u \in \mathcal{C}(G_u)$ by 1.2.4, and $L_u \trianglelefteq G_u$ by 1.2.1.3 since T normalizes K. Thus $K < L_u$ as $G_u \not\leq H = N_G(K)$. In particular $L_u \not\leq H$ since $K \trianglelefteq H$, so as $H = C_G(z)$, $[z, L_u] \neq 1$. Observe further as U_α is elementary abelian and contained in R_1 with $R_1^* = Q^* \in Syl_2(K^*)$ that $L_u/O_2(L_u)$ does not involve $SL_2(5)$ on a group of odd order, and so is quasisimple by 1.2.1.4.

Observe that the hypotheses of 1.1.6 are satisfied with G_u, H in the roles of "H, M", so that we may apply 1.1.5. Suppose first that L_u is quasisimple, and hence a component of G_u. As $u \in [U, K] \leq L_u \leq G_u$, $Z(L_u)$ is of even order. On the other hand z is in the center of the Sylow 2-subgroup T_u of G_u, and $KT_u = C_{G_u}(z)$. Inspecting the list of possiblities for L_u in 1.1.5.3, we conclude from this structure of KT_u (in particular from the two noncentral 2-chief factors) that L_u is the covering group of Ru. Next V is the unique L_1-invariant complement to $\langle u \rangle$ in $\langle u \rangle V$, so as $L_u/\langle u \rangle \cong Ru$, $N_{L_u}(V) =: L_0$ satisfies $L_0/O_2(L_0) \cong L_3(2)$. Thus $L_0 \leq O^2(N_G(V)) = L$ by 14.7.67.5, so as $|T : T_u| = 2$ and $L = O^2(L)$, we conclude $L = L_0$. Then as $Z(L_0)$ is of order 2 by I.1.3, $\langle u \rangle = Z(L) \cap T_u$, so $\langle u \rangle$ is T-invariant, contrary to $T_u \in Syl_2(G_u)$.

Therefore L_u is not quasisimple, so $F^*(L_u) = O_2(L_u)$ by 1.2.11. Let $R_u := O_2(KT_u)$. As $KT_u \trianglelefteq H$, $R_u = O_2(H) \cap KT_u \trianglelefteq H$, so since $H \in \mathcal{M}$, $C(G_u, R_u) \leq H_u := H \cap G_u$ and $R_u = O_2(H_u)$. Thus Hypothesis C.2.3 is satisfied with G_u, R_u, H_u in the roles of "H, R, M_H". Then as $L_u \trianglelefteq G_u$, while $L_u \not\leq H$ and $L_u/O_2(L_u)$ is quasisimple, L_u is described in C.2.7.3; and comparing the list in C.2.7.3 to the embeddings in A.3.14, we conclude that either L_u is a block with $L_u/O_2(L_u) \cong A_7$

or $Sp_4(4)$, or else $L_u/O_2(L_u) \cong SL_3(4)$. The first case is impossible as K has two noncentral 2-chief factors. In the remaining two cases, there is Y of order 3 in $C_{L_u}(K/O_2(K))$, so $Y \leq N_G(K) = H$, a contradiction as $C_H(K/O_2(K)) = Q_H$ by Theorem 14.7.63. □

LEMMA 14.7.69. *U_α^* is of order 2.*

PROOF. Assume otherwise. Then as $U_\alpha^* \leq Q^* \cong E_4$ by 14.7.66, $U_\alpha^* = Q^*$. Therefore using Remark 14.7.64,

$$C_{\tilde{U}}(U_\alpha) = [\tilde{U}, U_\alpha] = \tilde{U}_0\tilde{V}, \tag{a}$$

so as

$$[U, U_\alpha] \leq U \cap U_\alpha =: F \leq C_U(U_\alpha), \tag{b}$$

we conclude

$$[U, U_\alpha]V_1 = (U \cap U_\alpha)V_1 = FV_1 = C_U(U_\alpha) = U_0V. \tag{c}$$

From the action of H^* on $\tilde{U}$, for $u \in U - U_0V$ we have $m([u, U_\alpha]) \geq 2$, so $[u, U_\alpha] \not\leq Z_\alpha$. Thus we conclude from 14.7.4.1 and (c) that

$$U \cap Q_\alpha = U_0V = C_U(U_\alpha). \tag{d}$$

Then $m(U_\alpha^*) = 2 = m(U/U_0V) = m(U/U \cap Q_\alpha)$, so that we have symmetry between γ_1 and α as discussed in Remark 14.7.39. As $U \leq G_\alpha = C_G(Z_\alpha)$ and $C_G(U) \leq Q_H$:

$$Z_\alpha \leq Q_H \cap U_\alpha. \tag{e}$$

Suppose first that $V_1 \leq U_\alpha$. Then $V_1 \leq U \cap U_\alpha = F$, so $F = U_0V$ by (c). Hence $m(U_\alpha^*) = 2 = m(U/F) = m(U_\alpha/F)$, so

$$Q_H \cap U_\alpha = F \leq U.$$

Then using (e), $Z_\alpha \leq U$. It now follows from 14.7.4.1 that $m(Q_H/C_{Q_H}(Z_\alpha)) \leq 1$. But $C_{Q_H}(Z_\alpha) \leq N_G(U_\alpha)$ since $H = C_G(z)$, so $[C_{Q_H}(Z_\alpha), U_\alpha] \leq Q_H \cap U_\alpha \leq U$. This is impossible, since by 14.5.21.1, $Q_H/C_H(U)$ is dual to $U/C_U(Q_H)$ as an H-module, so U_α centralizes no hyperplane of $Q_H/C_H(U)$.

Therefore $V_1 \not\leq U_\alpha$. Hence $V_1 \not\leq F$, so we can now refine (b)–(d) to:

$$[U, U_\alpha] = U \cap U_\alpha = F \;\text{ and }\; F \times V_1 = U_0V = C_U(U_\alpha) = U \cap Q_\alpha. \tag{f}$$

Suppose that $U_0 = V_1$. Then by (f), F is a hyperplane of $V = C_U(U_\alpha)$, and by symmetry between γ_1 and α, F is a hyperplane of $C_{U_\alpha}(U)$ and $C_{U_\alpha}(U) \in V^G$. Hence by 14.7.67.2, $C_U(U_\alpha) = V = C_{U_\alpha}(U)$, so that $V_1 \leq U_\alpha$, contrary to our assumption.

Therefore $U_0 > V_1$. By I.1.6.2, $m(\tilde{U}_0) \leq 2$, so that $m(U_0) = 2$ or 3.

Suppose first that $m(U_0) = 3$. Then $\tilde{U}$ is the module $\check{U}$ discussed in Remark 14.7.64. In particular the 2-dimensional $\mathbf{F}_4$-subspace $\tilde{F} = \widetilde{C_U(U_\alpha)}$ is partitioned [4] by $\tilde{V}$, $\tilde{U}_0$, and the three 1-dimensional $\mathbf{F}_4$-spaces spanned by the various $[\tilde{u}, s^*]$ for $s^* \in U_\alpha^\#$ and $\tilde{u} \in \tilde{U} - \tilde{U}_0\tilde{V}$. So as $C_U(U_\alpha) = F \times V_1$ by (f), F has the partition

$$F = F_0 \cup F_1 \cup F_V,$$

where $F_V := F \cap V$, $F_0 := F \cap U_0$, and

$$F_1 := \{[x, y] : x \in U_\alpha - Q_H, y \in U - U_0V\}.$$

[4]Following Suzuki, a *partition of a vector space* is a collection of subspaces such that each nonzero element is contained in a unique subspace.

Now F_1 is invariant under the symmetry interchanging γ_1 and α, so by this symmetry there is a similar partition of F given by

$$F = (F \cap V^g) \cup (F \cap U_0^g) \cup F_1,$$

for $g \in \langle LT, H \rangle$ with $V_1^g = Z_\alpha$. By 14.7.68.1, $z^G \cap U_0^g = \{z^g\}$, so as $z^g \notin F$ and $F_V^{\#} \subseteq z^G$, $F_V = F \cap V^g \leq V^g$. Then as F_V is a hyperplane of V, $V = V^g$ by 14.7.67.2, contrary to our earlier reduction $V_1 \not\leq U_\alpha$.

Therefore $m(U_0) = 2$. This time $\tilde{F}$ is partitioned by $\tilde{U}_0$ and $\tilde{F}_1$, so F has the partition $F = F_0 \cup F_1$, and again using the symmetry between γ_1 and α as above, we conclude that $F = (F \cap U_0^g) \cup F_1$ is also a partition, and then that $F_0 \leq U_0^g$. Further $\tilde{F}_0 = \tilde{U}_0 \leq Z(\tilde{H})$, so $U_0 = U_0^g$ by 14.7.68.2, and hence $g \in N_G(U_0) = H$ as $H \in \mathcal{M}$ by 14.7.67.4, contrary to $V_1^g = Z_\alpha \neq V_1$. This contradiction completes the proof of 14.7.69. □

By choice of γ in Notation 14.7.1, $m(U_\alpha^*) \geq m(U/D) > 0$, where $D := U \cap Q_\alpha$; so as $m(U_\alpha^*) = 1$ by 14.7.69, also $m(U/D) = 1$. Thus again we have symmetry between α and γ_1, as discussed in Remark 14.7.39.

LEMMA 14.7.70. *(1) We may choose α so that $Z_\alpha \leq V_2$.*
(2) $m(U_0) \leq 2$.
(3) $U \cap U_\alpha = U_0 V = [U, U_\alpha]V = [U, U_\alpha]U_0$.
(4) $b = 3$ and $U_\alpha \in U^L$.

PROOF. Observe that if (1) holds, then so does (4) by 14.7.3.4. Thus it suffices to establish (1)–(3).

Let $F := [U, U_\alpha]$. By 14.7.66 and 14.7.69, U_α^* is a subgroup of $Q^* \in Syl_2(K^*)$ of order 2. Then using Remark 14.7.64, $FU_0 = VU_0$, $\tilde{V}\tilde{U}_0 = \tilde{F} \times \tilde{U}_0$, and U_α centralizes no $\mathbf{F}_2$-hyperplane of $\tilde{U}$; so $1 \neq [D, U_\alpha]$, and hence $Z_\alpha = [D, U_\alpha] \leq F \leq U$ using F.9.13.6. By the symmetry between γ_1 and α discussed above, also $V_1 = [D_\alpha, U] \leq F$. By 14.7.68.1, $Z_\alpha \not\leq U_0$.

By Remark 14.7.64, $m(\tilde{U}_0) \leq 2$, so that $m(U_0) \leq 3$. We now make some choices: We may conjugate in $N_H(R_1) = L_1 T$ and preserve the condition $U_\alpha \leq R_1$. As U_α^* is of order 2 in Q^*, conjugating in L_1, we may assume that $U_\alpha^* \leq Z(T^*)$; when $m(U_0) = 3$, we make this choice. When $m(U_0) \leq 2$, we make a more careful choice: As $Z_\alpha \leq F \leq U_0 V$, conjugating in L_1 we may assume that $Z_\alpha U_0 = V_2 U_0$. As $m(U_0) \leq 2$, T centralizes $\tilde{V}_2\tilde{U}_0$ and hence also $\tilde{Z}_\alpha$. Further $[Z_\alpha, Q_H] = V_1$ by 14.7.4.1, so by a Frattini Argument, $T^* = C_T(Z_\alpha)^*$. Now as $H = C_G(z)$, $C_T(Z_\alpha) \leq N_G(U_\alpha)$, so again $U_\alpha^* \leq Z(T^*)$. Thus in either case our choice implies $U_\alpha^* \leq Z(T^*)$.

As $U_\alpha^* \leq Z(T^*)$, T acts on $[\tilde{U}, U_\alpha^*] = \tilde{F}$; hence as $V_1 = [D_\alpha, U] \leq F$, T also acts on F. Recall also that $Z_\alpha \leq F$, so

$$V_1 Z_\alpha \leq F \leq U \cap U_\alpha \leq C_U(U_\alpha) \leq FU_0 = VU_0 = FV. \qquad (*)$$

Suppose first that $V_1 = U_0$. Then (2) holds, and by our choice under this assumption, $Z_\alpha \leq V_2 U_0 = V_2$, so that (1) holds. Further (3) follows from (*), completing the proof of the lemma in this case.

Thus we may suppose that $V_1 < U_0$. Recall $\tilde{F}$ is a complement to $\tilde{U}_0$ in $\tilde{V}\tilde{U}_0$. Further if $m(U_0) = 3$, then from Remark 14.7.64, $\tilde{F} \cap \tilde{V} = 1$, while if $m(U_0) = 2$ then $m(\tilde{F} \cap \tilde{V}) = 1$.

Suppose first that $m(U_0) = 2$. Again (2) holds. Also T acts on V, V_2, and F, and T centralizes $\tilde{Z}_\alpha$ by our choice when $m(U_0) \le 2$; in particular, T^* centralizes $\tilde{F} \cap \tilde{V}$ of rank 1. As $H^* \cong S_5$ by Theorem 14.7.63, $m(C_{\tilde{V}}(T)) = 1 = m(C_{\tilde{F}}(T))$, so as $Z_\alpha \le F$, we conclude that $\tilde{V}_2 = C_{\tilde{V}}(T) = \tilde{V} \cap \tilde{F} = C_{\tilde{F}}(T) = \tilde{Z}_\alpha$, since all these subspaces are of rank 1, and each successive pair is related by inclusion. Thus (1) holds. Then we saw that (4) also holds, so that $U_\alpha \in U^L$, and hence as $V \le U$, also $V \le U_\alpha$, so that $FV \le U \cap U_\alpha$. Then (3) follows from (*), completing the proof of the lemma in this case.

Therefore we may assume $m(U_0) = 3$, and it remains to derive a contradiction. This time as $\tilde{F} \cap \tilde{V} = 1$ and $Z_\alpha \le F$, we have $Z_\alpha \not\le V$. Let $E := V_1 Z_\alpha$, $Y_E := \langle Q_H, Q_\alpha \rangle$, and $Y := O^2(Y_E)$. As H is irreducible on $\tilde{U}/\tilde{U}_0$ and $\tilde{Z}_\alpha \not\le \tilde{U}_0$, it follows from 14.5.15.1 that $[Z_\alpha, Q_H] = V_1$. By the symmetry between γ_1 and α, $[V_1, Q_\alpha] = Z_\alpha$. Then by A.1.14, Y_E induces $GL(E)$ on E, $N_G(E) = Y_E C_G(E)$, and $Y_E \trianglelefteq N_G(E)$. As $Z_\alpha \not\le U_0$ and $H = C_G(z)$, $C_G(E) = C_H(Z_\alpha)$ is a 2-group.

As R_1^* centralizes $\tilde{U}_0 \tilde{V}$ by 14.7.65, R_1 acts on E. We claim $T \le N_G(E)$, so suppose otherwise. Then for $t \in T - R_1$, $F_0 := V_1 Z_\alpha Z_\alpha^t$ is of rank 3, so as T acts on F with $E = V_1 Z_\alpha \le F \le U \cap U_\alpha$, F_0 is contained in $U \cap U_\alpha \cap U_\alpha^t$. Therefore $Y_0 := \langle Y_E, Y_E^t \rangle$ induces $GL(F_0)$ on F_0, since $Aut_{Y_E}(F_0)$ is the stabilizer of E in $GL(F_0)$. But then there is an element of order 3 in $C_{Y_0}(z)$, impossible as $N_H(F_0) \le T$.

Thus $T \le N_G(E)$ as claimed, so T acts on $O^2(Y_E) = Y$, and further $\tilde{Z}_\alpha C_{\tilde{U}_0}(T) = C_{\tilde{U}}(T) = \tilde{V}_2 C_{\tilde{U}_0}(T)$. Therefore $\langle Z_\alpha^{L_1} \rangle = V Z_\alpha$ is of rank 4, as we saw $Z_\alpha \not\le V$.

Let $I := \langle L_1 T, Y \rangle$, $V_I := \langle V_1^I \rangle$, $Q_I := O_2(I)$, and $I^+ := I/Q_I$. Then $(I, L_1 T, YT)$ is a Goldschmidt triple in the sense of Definition F.6.1, so $\alpha := (L_1^+ T^+, T^+, Y^+ T^+)$ is a Goldschmidt amalgam by F.6.5.1, and hence is described in F.6.5.2. Next L_1 has at least five noncentral 2-chief factors, one on $O_2(L_1^*)$ and two each on $\tilde{U}$ and $Q_H/C_H(U)$ using 14.5.21.1. Thus we conclude from F.6.5.2 that $Q_I \ne 1$. In particular I is an SQTK-group and $I \in \mathcal{H}(T) \subseteq \mathcal{H}^e$ by 1.1.4.6, so that $V_I \in \mathcal{R}_2(I)$ by B.2.14. As $E \le V_I$ and $C_G(E)$ is a 2-group, $Q_I = C_I(V_I)$.

We finish much as at the end of the proof of 14.7.32: If Y^+ acts on L_1^+, then as T acts on Y, $I^+ = L_1^+ T^+ Y^+$, so $V_I = \langle V_1^{L_1^+ T^+ Y^+} \rangle = \langle V_1^{Y^+} \rangle = E$, impossible as L_1 does not act on E. Therefore Y^+ does not act on L_1^+, so in particular L_1^+ is not normal in I^+, and so $L_1^+ \ne 1$.

Suppose $Y \le M$. As Y does not act on L_1 but T acts on Y, the projection of Y on $L/O_2(L)$ in $M/O_2(L) = L/O_2(L) \times C_M(L/O_2(L))/O_2(L)$ is the maximal parabolic $L_2 O_2(L)/O_2(L)$. Then $Y = [Y, T \cap L] \le L$, so $E = \langle V_1^Y \rangle \le V$, whereas we saw earlier that $Z_\alpha \not\le V$. Thus $Y \not\le M$.

Assume next that $J(T) \le Q_I$. Then $J(T) = J(Q_I)$ by B.2.3.3, so that $I \le N_G(J(T))$. Then since $M = !\mathcal{M}(LT)$ and $Y \not\le M$, we conclude again using B.2.3.3 that $J(T) \not\le O_2(LT)$. Thus $L_1 = [L_1, J(T)]$ by 14.3.9.2, contradicting $J(T) \le Q_I$ and $L_1^+ \ne 1$.

Therefore $J(I)^+ \ne 1$, so by Theorem B.5.6, either $J(I)^+$ is solvable and the direct product of copies of S_3, or there is $K_I \in \mathcal{C}(J(I))$ with $K_I^+ \ne 1$. In the latter case, $K_I \in \mathcal{L}_f(G,T)$, so by 14.3.4.1, K_I^+ is $L_3(2)$ or A_5.

Let $I^! := I/O_{3'}(I)$. By F.6.11.2, either $I!$ is described in Theorem F.6.18, or $I^! \cong S_3$. But in the latter case, and in case (1) of F.6.18, T^+ is of order 2, so that $T^+ = J(T)^+$, and $I^+ = \langle T^{+I^+} \rangle = J(I^+) \cong S_3$, contrary to L_1^+ not normal in I^+.

Therefore $I^!$ appears in one of the cases (2)–(13) of F.6.18. Further the subcase of case (2) of F.6.18 with $O^2(I^+) \cong 3^{1+2}$ is eliminated, since in that case there is no subnormal subgroup of I^+ isomorphic to S_3. Thus if I^+ is solvable, then by F.6.18, $I^! \cong S_3 \times S_3$, so there is a normal subgroup K_I^+ of I^+ contained in $J(I)^+$ isomorphic to S_3. Then as $Y = [Y, T]$, either $Y^+ = O^2(K_I)^+$ or Y^+ centralizes K_I^+. Similarly either $L_1^+ = O^2(K_I)^+$ or L_1^+ centralizes K_I^+. Therefore as Y^+ does not act on L_1^+, we conclude using F.6.6 that $O^2(I) = \langle Y, L_1 \rangle$ centralizes K_I^+, impossible as $O^2(K_I^+) \not\leq Z(K_I^+)$.

Therefore $I^!$ is nonsolvable, so as I^+ has a subnormal subgroup isomorphic to S_3, $L_3(2)$ or A_5, it follows from F.6.18 that $I^! \cong L_3(2)$. Thus $K_I = O^2(I) = \langle Y, L_1 \rangle$ and $I = K_I T$. But now $E_4 \cong E = [E, Y] \leq [V_I, K_I]$, so as $L_1 T$ centralizes V_1 and $K_I = O^2(I)$, $V_I = [V_I, K_I] = \langle V_1^{K_I} \rangle$ is of rank 3 by H.5.5. This is impossible, since we saw earlier that $\langle Z_\alpha^{L_1} \rangle$ is of rank 4. □

LEMMA 14.7.71. *(1) H has two noncentral 2-chief factors, both isomorphic to $\tilde{U}$, one on U and one on $Q_H/C_H(U)$.*

(2) L_1 has five noncentral 2-chief factors, one in $O_2(\bar{L}_1)$, and four in S.

(3) $[Q, L] \leq S$.

PROOF. The proof is similar to some of the analysis in the second subsection, but is substantially easier. First L_1 has one noncentral chief factor on $O_2(L_1^*)$, two on $\tilde{U}$, and hence also two on $Q_H/C_H(U)$ by the duality in 14.5.21.1. Thus L_1 has at least five noncentral 2-chief factors.

Next as $U_\alpha \leq S$ by 14.7.70.4, using 14.7.66 we have

$$O_2(L_1^*) = \langle U_\alpha^{*L_1} \rangle \leq S^*. \quad (*)$$

Set $Q_K := [Q_H, K] C_H(U)$. As $[Q_K/C_H(U), S] = [Q_K/C_K(U), O_2(L_1^*)]$ is of corank 2 in Q_K, with $[S, Q_K] \leq S$, and as $m(Q_K/C_{Q_K}(V)) = 2$ by the duality in 14.5.21.1, we conclude

$$C_{Q_K}(V) = Q_K \cap Q = (S \cap Q_K) C_H(U). \quad (**)$$

Thus one noncentral 2-chief factor for L_1 in Q lies in S^*, two lie in $U \leq S$, and by (**) a fourth factor also lies in S. Now if (1) holds, then L_1 has four noncentral 2-chief factors in Q_H, so L_1 has exactly five noncentral 2-chief factors by (*). Then as L_1 has at least four noncentral chief factors on S, (2) holds, and of course (3) follows from (2).

Thus it remains to prove (1), so we must show that $[C_H(U), K] \leq U$. But $K = [K, U_\alpha]$, so it suffices to show $[C_H(U_\alpha), U_\alpha] \leq U$.

Now $C_H(U) \leq C_H(Z_\alpha) \leq N_G(U_\alpha)$, so $[U_\alpha, C_H(U)] \leq C_{U_\alpha}(U)$. We will show that $m(C_{U_\alpha}(U)/U \cap U_\alpha) \leq 1$; then as $m([W, U_\alpha]) \geq 2$ for any nontrivial H-chief factor W on $C_H(U)/U$ since $U_\alpha^* \leq K^*$ by 14.7.66, our proof will be complete.

By 14.7.69, $m(U_\alpha^*) = 1$, and by 14.7.70.3, $m(U_\alpha/U \cap U_\alpha) = 2$. So indeed $m(C_{U_\alpha}(U)/U \cap U_\alpha) \leq 1$, as desired. □

LEMMA 14.7.72. *(1) $S = O_2(L) = [O_2(L), L]$.*

(2) S/V is the Steinberg module for L/S.

(3) $U_0 = V_1$.

(4) $V = Z(S) = \Phi(S) = [S, S]$.

PROOF. By 14.7.70.4, $b = 3$. Set $R := \langle U_0^L \rangle$, so that $V \leq R \leq S$, and $\langle (U_0 V)^L \rangle = RV = R$. From Theorem 14.7.63 and 14.7.66, $[U, Q] = V U_0$, and from

14.7.70.3, $VU_0 = U \cap U_\alpha = [U, U_\alpha]V$. Thus U_α centralizes $[U, Q]$, so $R \leq Z(S)$ by 14.7.13.3. Also $\Phi(S) = [S, S] = R$ by 14.7.13.4. In particular $U \not\leq R$ as $S = \langle U^L \rangle$, so as L_1 is irreducible on U/U_0V, $U_0V = R \cap U$. Therefore as $R \leq Z(S) \leq C_H(U)$, we conclude from 14.7.71.1 that $[R, L_1] \leq R \cap U = U_0V$, so $[R, L_1] = [U_0V, L_1] = V$ in view of 14.7.65. Thus $[R, L] \leq V$, so $R = U_0V$. Further $UR/R = [UR/R, L_1] \cong E_4$, so by H.6.5:

(*) S/R is one of: the Steinberg module, the dual of V, the core (denoted $Core$) of the permutation module for LT on LT/L_2T, or the sum of the Steinberg module with either the dual of V or $Core$.

Suppose first that $U_0 = V_1$, so that $R = V$. Then by 14.7.71.2, L_1 has three noncentral chief factors on S/V, so that $S/R = S/V$ must be the Steinberg module, since by 14.7.22.2, the Steinberg module is the only module listed in (*) with this property. It follows that $V = Z(S)$, and then the rest of the lemma holds: For example, $S = [Q, L]$ by 14.7.71.3, and then as $Q_H \cap O_2(L_1) - Q$ contains an involution H-conjugate to an involution in U_α, the double cover of $L_3(2)$ is not involved in L/S, so that $S = [O_2(L), L] = O_2(L)$.

Thus we assume that $V_1 < U_0$, and it remains to derive a contradiction. By 14.7.70.2, $m(U_0) = 2$, so as $R = U_0V$, $m(R/V) = 1$. Therefore as L is irreducible on V, either $R \leq Z(Q)$ or $[R, Q] = V$, and the latter is impossible as $|T : C_T(U_0)| \leq 2$. Thus $R \leq Z(Q)$ and $m(R/V) = 1$, but $|T : C_T(U_0)| \leq 2$ so R is not the extension in B.4.8.3; thus $R = V \oplus C_R(L)$ where $C_R(L) = R \cap Z(L)$ is of rank 1. Hence $C_R(L)V_1 = C_R(T) = C_R(L_1)$, so as $U_0 \leq C_R(L_1)$, there exists $u \in C_{U_0}(LT) - V_1$. But now by 14.7.68.2, $L \leq C_G(u) \leq H$, contrary to $H \not\leq M = !\mathcal{M}(LT)$. □

Recall $M_1 = H \cap M = L_1T = N_H(V)$.

LEMMA 14.7.73. *(1)* $SO_2(K) = O_2(L_1) \in Syl_2(K)$.
(2) $|T \cap L : T \cap K| = 2$.
(3) Let $k \in K - M_1$. *Then* $K = \langle S, S^k \rangle$, $O_2(K) = (S \cap O_2(K))(S^k \cap O_2(K))$ *is of order* 2^{11}, $S \cap S^k = C_{O_2(K)}(U)$, *and* $O_2(K)/U$ *is the* 6*-dimensional indecomposable for* $K/O_2(K)$ *with* $C_{O_2(K)/U}(K) = (S \cap S^k)/U \cong E_4$ *and* $O_2(K)/(S \cap S^k)$ *the* $L_2(4)$*-module.*

PROOF. By 14.7.72.2 and H.6.3.5, $S/V = [S/V, L_1]$. Then as $V = [V, L_1]$, $S = [S, L_1] \leq O_2(L_1) \leq K$. We saw in (*) in the proof of 14.7.71 that $O_2(L_1^*) \leq S^*$, so we conclude that $O_2(L_1^*) = S^* \in Syl_2(K^*)$.

We can now argue much as in the proof of G.2.3: Let $k \in K - M_1$ and set $K_0 := \langle S, S^k \rangle$. Now $K^* = K_0^*$, so $K \leq K_0Q_H$; therefore as $Q_H \leq N_G(S)$, $S^K = S^{K_0}$, so $K \leq \langle S^{K_0} \rangle = K_0$. Then as $S \leq K \trianglelefteq H$, $K = K_0$. Let $P := (S \cap Q_H)(S^k \cap Q_H)$. Then $[P, S] \leq S \cap Q_H \leq P$ and similarly $[P, S^k] \leq P$, so $P \trianglelefteq K$; then as $PS/P \cong S/S \cap P \cong S^* \in Syl_2(K^*)$, $P = O_2(K)$.

Next $U \leq S \cap S^k$, and $[S, S] = \Phi(S) = V \leq U$ by 14.7.72.4, so $(S \cap S^k)/U \leq Z(K/U)$. Further setting $P^+ := P/S \cap S^k$,

$$P^+ = (S \cap P)^+ \oplus (S^k \cap P)^+.$$

For each $s \in S - P$, $[P^+, s] \leq (S \cap P)^+ \leq C_{P^+}(s)$ again since $[S, S] = V \leq S \cap S^k$ using 14.7.72.4. So by G.1.5.3 and Theorem G.1.3, P^+ is the sum of natural modules for $K/O_2(K)$. Hence as $U \leq S \cap S^k$, we conclude from 14.7.71.1 that P^+ is a natural module and $S \cap S^k = C_P(U)$. Therefore $P/(S \cap P) = [P/(S \cap P), L_1]$. Thus as $S \leq O_2(L_1) \leq SP$, $P = [P, L_1](S \cap P) \leq O_2(L_1)$ and $SO_2(K) = SP = O_2(L_1) \in$

$Syl_2(K)$. That is, (1) holds. Further as $S \leq O_2(L_1)$ and $|\bar{T} : O_2(\bar{L}_1)| = 2$, (2) holds.

Let $B \in Syl_3(L_1)$. By 14.7.72.2 and H.6.3.3, $|C_S(B)| = 8$, so as $P/C_P(U) = [P/C_P(U), B]$ and $C_U(B) = V_1$ using 14.7.72.3, $(S \cap S^k)/U = C_S(B)U/U$ is of order 4. As $S = [S, L_1]$, L_1 is indecomposable on P/U, so we conclude (3) holds. □

LEMMA 14.7.74. *(1) $M = L$ and $S = O_2(M)$.*
(2) $H = KT$ and $O_2(H) = O_2(K)$.

PROOF. By 14.7.72.2, S/V is the Steinberg module which is a projective L-module, so $Q/V = Q_C/V \oplus S/V$, where $Q_C/V = C_{Q/V}(L)$. Now $[Q_C, L_1] \leq V$ and L_1^* contains the Sylow 2-subgroup $O_2(L_1^*)$ of K^*, so by Gaschütz's Theorem A.1.39, $[Q_C, K] \leq U$. Then as $Q_C U$ centralizes V, $Q_C U$ centralizes $\langle V^K \rangle = U$, so Q_C centralizes $\langle U^L \rangle = S$.

Let $Q_L := C_Q(L)$, so that $Q_L \leq Q_C$. Then $[Q_L, K] \leq [Q_C, K] \leq U$. Further in the unique nonsplit L-module extension W of V in I.1.6 whose quotient is a trivial L-module, $O_2(L_1)$ does not centralize a vector in $W - V$ (cf. B.4.8.3), so $Q_L V_1 = C_{Q_C U}(L_1)$. Therefore since L_1 contains a Sylow 2-subgroup of K, $\tilde{Q}_L \tilde{U} = \tilde{Q}_L \times \tilde{U}$ with K centralizing $\tilde{Q}_L$ again using Gaschütz's Theorem A.1.39. Then K centralizes Q_L by Coprime Action. So since $K \not\leq M = !\mathcal{M}(LT)$, we conclude $Q_L = 1$.

Let $B \in Syl_3(L_1)$, and set $Q_B := C_{Q_C}(B)$. Then $Q_C = VQ_B$, so $\Phi(Q_B) = \Phi(Q_C) \trianglelefteq LT$. But L is irreducible on V, and $Q_B \cap V = V_1$, so $\Phi(Q_C) \cap V = 1$. Then $[\Phi(Q_C), L] \leq \Phi(Q_C) \cap V = 1$, so that $\Phi(Q_C) \leq C_Q(L) = 1$ by the previous paragraph. Since also $C_{Q_C}(L) = 1$, $m(Q_C/V) \leq \dim H^1(\bar{L}, V) = 1$, with $[Q_C, O_2(L_1)] = V$ in case of equality (again cf. B.4.8.3).

Suppose $V < Q_C$. By 14.7.73.1, $O_2(L_1) = SO_2(K)$, so as we saw that S centralizes Q_C, $[Q_C, O_2(K)] = [Q_C, O_2(L_1)] = V$. Then as $V_1 = [U, O_2(K)]$, $[Q_C U, O_2(K)] = [Q_C, O_2(K)]V_1 = V$. However, K normalizes $Q_C U$, and hence also normalizes $[Q_C U, O_2(K)] = V$, so $H = KT \leq N_G(V) \leq M$, contrary to $H \in \mathcal{H}_z$.

This contradiction shows that $Q_C = V$. Hence $Q = S = O_2(L) \leq O_2(M)$ by 14.7.72.1, so $O_2(L) = O_2(M)$ by A.1.6. By 14.7.72.4, V char S, so that $V \trianglelefteq M$. Thus $M = LT$ by 14.7.67.5, so as $LT = LO_2(LT)$, $O_2(LT) = O_2(M) = O_2(L)$, and hence (1) holds.

Finally using (1) and 14.7.72.2, $4|Q_H| = |R_1| = 4|S| = 2^{13}$, so $|Q_H| = 2^{11} = |O_2(K)|$ by 14.7.73.3, and hence (2) holds. □

Under the hypotheses of this section, we can now identify G as Ru.

THEOREM 14.7.75. *Assume Hypothesis 14.3.1 holds with $L/O_2(L) \cong L_3(2)$ and $\langle V^{G_1} \rangle$ abelian. Then $G \cong Ru$.*

PROOF. We verify that G is of type Ru as defined in section J.1. Then the Theorem follows from Theorem J.1.1.

By 14.7.74.1, $M = L$ and $S = O_2(L)$. Thus as L acts on V and $M \in \mathcal{M}$, $L = N_G(V)$ with $L/S \cong L_3(2)$. By 14.7.72.4, S is special with center V. Of course V is the natural module for L/S, and by 14.7.72.2, S/V is the Steinberg module. Thus hypothesis (Ru1) is satisfied.

As $F^*(L) = O_2(L) = S$ and $V = Z(S)$ by 14.7.72.4, $Z = C_V(T) = V_1$. By Theorem 14.7.63, $H = C_G(Z)$ with $H^* \cong S_5$. By 14.7.72.3, $C_{\tilde{U}}(H) = 1$, so by

Theorem 14.7.63, $\tilde{U}$ is the $L_2(4)$-module for H^*. By 14.7.74.2, $Q_H = O_2(K)$, so by 14.7.73.3, Q_H/U is a 6-dimensional indecomposable for H^*. Thus hypothesis (Ru2) is satisfied. Therefore G is of type Ru, completing the proof of the Theorem. $\square$

14.8. The QTKE-groups with $\mathcal{L}_\mathbf{f}(\mathbf{G},\mathbf{T}) \neq \emptyset$

We now come to a major watershed in this work: We complete the treatment of the case where $\mathcal{L}_f(G,T)$ is nonempty. We begin with the following preliminary result:

THEOREM 14.8.1. *Assume Hypothesis 13.3.1. Then one of the following holds:*
(1) $L/O_2(L) \cong A_6$ and $G \cong Sp_6(2)$ or $U_4(3)$.
(2) $L/O_2(L) \cong A_5$ and $G \cong U_4(2)$ or $L_4(3)$.
(3) $L/O_2(L) \cong L_3(2)$ and $G \cong Sp_6(2)$, $G_2(3)$, HS, or Ru.

PROOF. First by 13.3.2.1, $L/O_2(L) \cong A_5$, $L_3(2)$, A_6, $\hat{A}_6$, or $G_2(2)'$. By Theorem 13.3.16, $L/O_2(L)$ is not $G_2(2)'$. If $L/O_2(L) \cong A_5$, then (2) holds by Theorem 13.6.1. If $L/O_2(L)$ is A_6 or $\hat{A}_6$, then G is $Sp_6(2)$ or $U_4(3)$ by Theorem 13.8.1, so (1) holds. This leaves the case where $L/O_2(L) \cong L_3(2)$. Then G is not $U_4(3)$, as in that case there is no $L \in \mathcal{L}(G,T)$ with $L/O_2(L) \cong L_3(2)$. Further if $G \cong Sp_6(2)$, then (3) holds, so we may assume G is not $Sp_6(2)$. Therefore Hypothesis 14.3.1.1 is satisfied. Let $U := \langle V_1^{G_1} \rangle$. If U is nonabelian then G is $G_2(3)$ or HS by Theorem 14.4.14, so that (3) holds. Thus we may assume U is abelian. Then Theorem 14.7.75 shows that $G \cong Ru$, so that (3) holds, completing the proof. $\square$

We can now easily deduce our main result Theorem D (14.8.2) below from Theorem 14.8.1. Theorem 14.8.1 assumes Hypothesis 13.3.1, and some major reductions are concealed in Hypothesis 13.3.1, so we briefly recapitulate those reductions; they take place in the proof of 13.3.2. In Hypothesis 13.3.1 we assume that $\mathcal{L}_f(G,T) \neq \emptyset$. This rules out the groups in Theorem 2.1.1, so that $|\mathcal{M}(T)| > 1$, and allows us to appeal to the theory based on Theorem 2.1.1. The groups excluded in Hypothesis 13.1.1 are also excluded in Hypothesis 13.3.1, so we are able to apply Theorem 13.1.7 to conclude that $K/O_2(K)$ is quasisimple for each $K \in \mathcal{L}_f(G,T)$. By 1.2.9, $\mathcal{L}_f^*(G,T) \neq \emptyset$, and we pick $L \in \mathcal{L}_f^*(G,T)$. In particular, Hypothesis 12.2.1 is satisfied. The proof of Theorem 12.2.2 discusses how previous work leads to the groups in conclusions (1) and (2) of 12.2.2; Hypothesis 12.2.3 excludes these groups, but they are also excluded in Hypothesis 13.3.1, so Hypothesis 12.2.3 is also satisfied. This allows us to appeal to the work in chapter 12 which restricts the choice for the pair L, V in the Fundamental Setup to those listed in 13.3.2.

THEOREM 14.8.2 (Theorem D). *Assume that G is a simple QTKE-group, with $T \in Syl_2(G)$, and $\mathcal{L}_f(G,T) \neq \emptyset$. Then one of the following holds:*

(1) G is a group of Lie type over $\mathbf{F}_{2^n}$, $n > 1$, of Lie rank 2, but $G \cong U_5(2^n)$ only for $n = 2$.
(2) G is $L_4(2)$, $L_5(2)$, A_9, M_{22}, M_{23}, M_{24}, He, or J_4.
(3) G is $Sp_6(2)$, $U_4(2)$, $L_4^\epsilon(3)$, $G_2(3)$, HS, or Ru.

PROOF. Since the groups excluded in parts (2) and (3) of Hypothesis 13.3.1 appear as conclusions in Theorem D, we may assume that parts (1)–(3) of Hypothesis 13.3.1 are satisfied. Now choose $L \in \mathcal{L}_f^*(G,T)$ with $L/O_2(L)$ not A_5 if possible.

Suppose that $L/O_2(L) \cong A_5$. Then the choice of L above was forced, so that $K/O_2(K) \cong A_5$ for all $K \in \mathcal{L}_f^*(G,T)$. Therefore $J/O_2(J) \cong A_5$ for all $J \in \mathcal{L}_f(G,T)$ by 1.2.4 and A.3.12. Thus part (4) of Hypothesis 13.3.1 is satisfied, completing the verification of Hypothesis 13.3.1.

Now Theorem 14.8.1 completes the proof of Theorem D. $\square$

Part 6

The case $\mathcal{L}_f(G,T)$ empty

CHAPTER 15

The case $\mathcal{L}_f(G, T) = \emptyset$

In this chapter, we complete the treatment of the case $\mathcal{L}_f(G,T)$ empty. Since the previous chapter completed the analysis of the case $\mathcal{L}_f(G,T)$ nonempty, this chapter will complete the proof of our Main Theorem.

Initially we assume Hypothesis 14.1.5, introduced at the start of the previous chapter, with $M := M_f$. Recall that $V(M)$ is defined just before 14.1.2: as mentioned in section A.5, in this chapter we are deviating from our usual meaning of $V(M)$ in definition A.4.7, instead using the meaning in notation A.5.1, namely $V(M) := \langle Z^M \rangle$. In the first two sections of this chapter, we reduce to the case where M and $V := V(M)$ satisfy $m(V) = 4$ and $M/O_2(M) \cong O_4^+(V)$. We treat that final difficult case in the third section. The fourth section then treats the remaining subcase of the case $\mathcal{L}_f(G,T)$ empty when Hypothesis 14.1.5 is not satisfied; this subcase quickly reduces to the situation $\mathcal{L}(G,T)$ empty, or equivalently each member of $\mathcal{H}(T)$ is solvable.

15.1. Initial reductions when $\mathcal{L}_f(G, T)$ is empty

In this section, and indeed until the final section of this chapter, we assume Hypothesis 14.1.5. This Hypothesis isolates the most important subcase of the case $\mathcal{L}_f(G,T)$ empty, and was already introduced at the beginning of the previous chapter. Recall Hypothesis 14.1.5 includes the assumption that $|\mathcal{M}(T)| > 1$, which is appropriate in view of Theorem 2.1.1. Hypothesis 14.1.5 also includes the assumption that there is a unique maximal 2-local M_c containing the centralizer in G of $Z := \Omega_1(Z(T))$; that is,

$$M_c = !\mathcal{M}(C_G(Z)).$$

The case where this condition fails will be treated in the final section of the chapter; in that case Hypothesis 15.4.1.2 of the final section is satisfied.

By 14.1.12.1, there is $M := M_f \in \mathcal{M}(T) - \{M_c\}$, which is maximal under the partial order $\lesssim$ on $\mathcal{M}(T)$ of Definition A.5.2, and M is the unique maximal member of $\mathcal{M}(T) - \{M_c\}$ under $\lesssim$. As in Definition A.5.8, set $V(M) := \langle Z^M \rangle$; as usual $V(M) \in \mathcal{R}_2(M)$ by B.2.14.

The uniqueness theorems in A.5.7, for overgroups of T in M which cover $M/C_M(V(M))$, replace the uniqueness theorems for members of $\mathcal{L}_f^*(G,T)$, used in the treatement of the Fundamental Setup (3.2.1), which are no longer available as $\mathcal{L}_f(G,T)$ is empty.

LEMMA 15.1.1. *Set $V := V(M)$, $R := C_T(V)$, and $\bar{M} := Aut_M(V)$. Then*

(1) Case (II) of Hypothesis 3.1.5 is satisfied with $N_M(R)$ in the role of "M_0", and any $H \in \mathcal{H}_*(T,M)$.

(2) $\hat{q}(\bar{M},V) \leq 2$, and if $q(\bar{M},V) > 2$ then $\hat{q}(\bar{M},V) < 2$.

PROOF. As M is maximal in $\mathcal{M}(T)$ under $\stackrel{<}{\sim}$ and $V = V(M)$, we conclude from A.5.7.2 that $R = O_2(N_M(R))$, $V \in \mathcal{R}_2(N_M(R))$, $\bar{M} = Aut_{N_M(R)}(V) = \overline{N_M(R)}$, and $M = !\mathcal{M}(N_M(R))$. So as $V \trianglelefteq M$, (1) holds. Further for $H \in \mathcal{H}_*(T, M)$, $O_2(\langle H, N_M(R)\rangle) = 1$ as $M = !\mathcal{M}(N_M(R))$, so conclusion (1) of Theorem 3.1.6 does not hold. Then conclusion (2) or (3) of 3.1.6 holds, establishing (2) since $\overline{N_M(R)} = \bar{M}$. □

By 15.1.1, $Aut_M(V(M))$ and its action on $V(M)$ are described in section D.2. Using the fact that $M_c = !\mathcal{M}(C_G(Z))$, we refine that description in the first lemma in this section, which provides the basic list of cases to be treated in the first three sections of this chapter. Recall $\hat{\mathcal{Q}}_*(Aut_M(V(M)), V(M))$ from Definition D.2.1, and set $\hat{J}(Aut_M(V(M)), V(M)) := \langle \hat{\mathcal{Q}}_*(Aut_M(V(M)), V(M))\rangle$.

LEMMA 15.1.2. *Let $V := V(M)$, and set $\bar{M} := M/C_M(V)$ and $\bar{M}_J := \hat{J}(\bar{M}, V)$. Then one of the following holds:*

(1) $\bar{M}_J \cong D_{2p}$ and $m(V) = 2m$, where $(p, m) = (3, 1)$, $(3, 2)$, or $(5, 2)$.

(2) $m(V) = 4$ and $\bar{M} = \bar{M}_J = \Omega_4^+(V) \cong S_3 \times S_3$.

(3) $\bar{M}_J = \bar{M}_1 \times \bar{M}_2$ and $V = V_1 \oplus V_2$, with $\bar{M}_i \cong D_{2p}$, $V_i := [V, M_i]$ of rank $2m$, (p, m) as in (1), and $\bar{M}_1$ and $\bar{M}_2$ interchanged in $\bar{M}$.

(4) $\bar{M}_J = \bar{P}\langle\bar{t}\rangle$ where $\bar{P} := O^2(\bar{M}) \cong 3^{1+2}$, and $\bar{t}$ is an involution inverting $\bar{P}/\Phi(\bar{P})$. Further $m(V) = 6$, and T acts irreducibly on $\bar{P}/\Phi(\bar{P})$.

(5) $\bar{M}_J = \bar{P}\langle\bar{t}\rangle$ where $\bar{P} := O^2(\bar{M}) \cong E_9$ and $\bar{t}$ is an involution inverting $\bar{P}$. Further $m(V) = 4$, and $\bar{T} \cong \mathbf{Z}_4$.

(6) $\bar{M}_J \cong S_3$, $V = [V, M_J] \times C_V(M_J)$ with $m([V, M_J]) = 4$ and $C_V(M_J) \neq 1$, $M/C_M([V, M_J]) = \Omega_4^+([V, M_J])$, and $M \cap M_c = C_M([V, M_J])C_M(C_V(M_J))T$ is of index 3 in M.

PROOF. By 15.1.1.1, $\hat{q}(\bar{M}, V) \leq 2$, while by 14.1.6.1, $\bar{M}$ is solvable. Hence, in the language of the third subsection of section D.2, $(\bar{M}_J, [V, F(\bar{M}_J)])$ is a sum of indecomposables, so there is a partition

$$\hat{\mathcal{Q}}_*(\bar{M}, V) = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_s$$

such that $\bar{M}_J = \bar{M}_1 \times \cdots \times \bar{M}_s$ and $V_0 := [V, F(\bar{M}_J)] = V_1 \oplus \cdots \oplus V_s$, where $\bar{M}_i := \langle \mathcal{Q}_i\rangle$, $V_i := [V, M_i]$, and $(\bar{M}_i, V_i)$ is indecomposable as defined in section D.2. Further by D.2.17, each indecomposable $(\bar{M}_i, V_i)$ satisfies one of the conclusions of D.2.17. Let M_i denote the preimage in M of $\bar{M}_i$. As M permutes the set $\{\mathcal{Q}_i : 1 \leq i \leq s\}$ of orbits of M_J on $\hat{\mathcal{Q}}_*(\bar{M}, V)$, M permutes $\{M_i : 1 \leq i \leq s\}$.

Observe that $F^*(\bar{M}_i) = O_p(\bar{M})$ for some odd prime p (depending on i), so for each nontrivial 2-element $\bar{t}$ in $\bar{M}_i$, $C_{O_p(\bar{M})}(\bar{t})$ is cyclic by A.1.31.1. Thus if $\bar{M}_i$ is not normal in $\bar{M}$, then as the product $\bar{M}_J$ of the $\bar{M}_j$ is direct, $\bar{M}_i^{\bar{M}} = \bar{M}_i^T$ is of order 2, and $m_p(\bar{M}_i) = 1$, so that $\bar{M}_i$ falls into case (1) or (2) of D.2.17. In particular, if $m_p(\bar{M}_i) > 1$, then $\bar{M}_i \trianglelefteq \bar{M}$.

Let $K_1, \ldots, K_a$ be the groups $\langle M_i^M\rangle$, and set $W_i := [V, K_i]$; then $\bar{M}_J = \bar{K}_1 \times \cdots \times \bar{K}_a$ and $V_0 = W_1 \oplus \cdots \oplus W_a$. Further $V = V_0 \oplus C_V(F(\bar{M}_J))$ by Coprime Action.

Assume first that $J(T) \not\leq C_M(V)$. Then as $M \neq M_c$, we conclude from 14.1.7 that either (1) or (3) holds, with $(p, m) = (3, 1)$.

Thus in the remainder of the proof, we may assume that $J(T) \leq C_M(V)$. Therefore since M is maximal in $\mathcal{M}(T)$ under $\stackrel{<}{\sim}$, we may apply 14.1.4 to conclude that $M_c \stackrel{<}{\sim} M$. Then since $M_c \not\leq M$, A.5.6 gives

$$\text{There is no } 1 \neq X \leq V \text{ with } X = \langle (Z \cap X)^{M \cap M_c} \rangle \trianglelefteq M. \quad (*)$$

Suppose first that $(\bar{M}_1, V_1)$ satisfies case (6) of D.2.17; we will derive a contradiction. For then $\bar{P} := O^2(\bar{M}_1) = \bar{P}_1 \times \bar{P}_2 \times \bar{P}_3$, with $\bar{P}_j \cong \mathbf{Z}_3$ for each j, and $V_1 = U_1 \oplus U_2 \oplus U_3$, where $U_j := [V, P_j]$ is of rank 2 for P_j the preimage of $\bar{P}_j$. As $m_3(\bar{M}_1) > 1$, M_1 and V_1 are normal in M by the second paragraph of the proof, so M permutes $\mathcal{X} := \{P_1, P_2, P_3\}$. Then T fixes some member of $\mathcal{X}$, say P_1, so that T acts on $[V, P_1] = U_1$, and on $U := U_2 \oplus U_3$. Thus $1 \neq Z_1 := Z \cap U_1$ and $1 \neq Z_U := Z \cap U$. So $P_1 \leq C_G(Z_1) \leq M_c = !\mathcal{M}(C_G(Z))$, and similarly $P_2P_3 \leq C_M(Z_U) \leq M_c$. Then $V_1 = \langle Z_1^{P_1}, Z_U^{P_2P_3} \rangle = \langle (Z \cap V_1)^{M_c \cap M} \rangle$, contrary to (*). This completes the proof that no $(\bar{M}_i, V_i)$ satisfies conclusion (6) of D.2.17.

Next suppose for the moment that $(\bar{M}_1, V_1)$ satisfies conclusion (3) of D.2.17; in this case, we show that M_1T acts irreducibly on V_1. Again by the second paragraph, M_1 and V_1 are normal in M. As case (3) of D.2.17 holds, $\bar{M}_1 \cong \mathbf{Z}_2/E_9$, with $m(V_1) = 4$ and $O(\bar{M}_1)$ inverted in $\bar{M}_1$. Thus $Aut_M(V_1) \leq N_{GL(V_1)}(\bar{M}_1) \cong O_4^+(V_1)$. Assume now that M_1T acts reducibly on V_1. Then $Aut_T(V_1) \cong \mathbf{Z}_2$ or E_4, and in either case $Z \cap V_1 \cong E_4$ and $M = \langle C_M(z) : z \in Z^\# \cap V_1 \rangle$, so $M \leq M_c$ as $M_c = !\mathcal{M}(C_G(Z))$. This contradiction completes the proof that if $(\bar{M}_i, V_i)$ satisfies case (3) of D.2.17, then M_iT acts irreducibly on V_i.

Next we introduce a basic case division for the proof of the lemma: Let $Z_i := Z \cap W_i$, and suppose that $W_i = \langle Z_i^{K_i} \rangle$ for some i, and that either $C_V(M_J) \neq 1$, or $a > 1$. Then $C_Z(K_i) \neq 1$, so that $K_i \leq C_G(C_Z(K_i)) \leq M_c = !\mathcal{M}(C_G(Z))$. Then W_i is generated by $Z_i^{K_i} \subseteq Z_i^{M \cap M_c}$, so as $W_i \trianglelefteq M$ since $K_i = \langle M_i^M \rangle$, we have a contradiction to (*). Thus we conclude that either

(i) $a = 1$ and $V = V_0$, or

(ii) For each i, $\langle Z_i^{K_i} \rangle < W_i$.

We first assume that case (i) does not hold; then case (ii) holds, and we will show that conclusion (6) is satisfied in this case. Choose notation so that $W_1 = V_1 \oplus \cdots \oplus V_b$; then $b \leq 2$ by paragraph two. As (ii) holds, $\langle Z_1^{K_1} \rangle < W_1$, so K_1T acts reducibly on W_1, and hence $M_1N_T(V_1)$ acts reducibly on V_1. Now in D.2.17, $\bar{M}_1$ acts reducibly only in case (3), and in case (1) when $m(V_1) = 4$. But earlier we showed $M_1N_T(V_1)$ is irreducible on V_1 in case (3), so $\bar{M}_1 \cong S_3$ with $m(V_1) = 4$, and in particular $\hat{q}(\bar{M}_1, V_1) = 2$ and $Aut_M(V_1) \leq N_{GL(V_1)}(\bar{M}_1) = \Omega_4^+(V_1)$.

Since $\langle C_{V_1}(N_T(V_1))^{M_1} \rangle < V_1$, $m(C_{V_1}(N_T(V_1))) = 1$, so $|Aut_T(V_1)| > 2$. Then as $Aut_M(V_1) \leq \Omega_4^+(V_1)$, $Aut_T(V_1) \cong E_4$, so $Aut_M(V_1))$ is either $\Omega_4^+(V_1)$ or $S_3 \times \mathbf{Z}_2$. However in the latter case, $M = K_1C_M(Z_1)$, so as $W_1 > \langle Z_1^{K_1} \rangle$, also $W_1 > \langle Z_1^M \rangle$, contrary to $V = \langle Z^M \rangle$.

Therefore $Aut_M(V_1) = \Omega_4^+(V_1)$. Now as $\bar{M}_1 = Aut(\bar{M}_1)$, $\bar{M}_0 := N_{\bar{M}}(\bar{M}_1) = \bar{M}_1 \times C_{\bar{M}_0}(\bar{M}_1)$, with $m_3(C_{\bar{M}_0}(\bar{M}_1)) \leq 1$ using A.1.31.1. Suppose $s > 1$. Then by symmetry, $\bar{M}_2 \cong S_3$, so $\bar{M}_0 = \bar{M}_1 \times \bar{M}_2 \times C_{\bar{M}_0}(\bar{M}_1\bar{M}_2)$, and then $O(\bar{M}_1)O(\bar{M}_2) = O^{3'}(\bar{M}_0)$ by A.1.31.1. This is a contradiction as $Aut_M(V_1) = \Omega_4^+(V_1)$ and $[V_1, M_2] = 1$. Therefore $s = a = 1$, and hence $K_1 = M_1 = M_J$ and $W_1 = V_1 = V_0$. If $V = V_0$, then case (i) holds, contrary to our assumption, so we may assume that $C_V(M_J) \neq 0$. Now $C_M(V_0)$ and $C_M(C_V(M_J))$ lie in $M_c = !\mathcal{M}(C_G(Z))$, and $|M : C_M(V_0)C_M(C_V(M_J))T|$ divides $|Aut_M(V_0) : \bar{K}_1\bar{T}| = 3$; then as $M \not\leq M_c$, we conclude that $M \cap M_c = C_M(V_0)C_M(C_V(M_J))T$ is of index 3 in M. This completes the proof that conclusion (6) of 15.1.2 holds if case (i) does not hold.

Thus we may assume that case (i) holds, so $\bar{M}_J = \bar{K}_1$ and $V = V_0 = W_1$. Thus $\bar{M}$ is faithful on W_1 and $\bar{M} \leq N_{GL(W_1)}(\bar{K}_1)$.

Assume first that $(\bar{M}_J, V)$ is indecomposable. Then $s = 1$, so $\bar{M}_J = \bar{M}_1$ and $V = V_1$. Recall we showed that conclusion (6) of D.2.17 does not hold for $(\bar{M}_J, V)$. Conclusions (1) and (2) of D.2.17 give conclusion (1) of 15.1.2.

Suppose conclusion (5) of D.2.17 holds. Then $\bar{M}_J = \Omega_4^+(2)$, so $\bar{M} = \bar{M}_J$, for otherwise $\bar{M} = N_{GL(V)}(\bar{M}_J) = O_4^+(V)$ contains transvections, whereas $\hat{q}(\bar{M}, V) = 3/2$ in case (5) of D.2.17. Thus conclusion (2) of 15.1.2 holds in this case.

Suppose conclusion (3) of D.2.17 holds. We showed earlier that $\bar{M} \leq O_4^+(V)$ and M acts irreducibly on V. As conclusion (3) of D.2.17 holds, $\hat{q}(\bar{M}, V) = 2$, so $\bar{T}$ contains no transvections on V. Hence as M is irreducible on V, $\bar{T} \cong \mathbf{Z}_4$, so conclusion (5) of 15.1.2 holds.

Suppose that case (4) of D.2.17 holds. Then $\bar{M}_J = \bar{P}\langle\bar{t}\rangle$, where $\bar{P} = F^*(\bar{M}_J) \cong 3^{1+2}$, $\bar{t}$ inverts $\bar{P}/\Phi(\bar{P})$, and $m(V) = 6$. Hence $\bar{M} \leq N_{GL(V)}(\bar{M}_J) \cong GL_2(3)/3^{1+2}$. If $O^2(\bar{M}) > \bar{P}$, then $m_3(C_{\bar{M}}(\bar{t})) > 1$, contrary to A.1.31.1; thus $O^2(\bar{M}) = \bar{P}$. Therefore if T is irreducible on $\bar{P}/\Phi(\bar{P})$, then conclusion (4) of 15.1.2 holds, so we may assume that T is reducible on $\bar{P}/\Phi(\bar{P})$, and it remains to derive a contradiction. Then $\bar{T} \cong \mathbf{Z}_2$ or E_4, and in either case T acts on subgroups $\bar{P}_1$ and $\bar{P}_2$ of order 3 generating $\bar{P}$. Thus $Z = E_1E_2$, where $1 \neq E_i := C_V(\bar{P}_i\bar{T})$. Therefore the preimages P_i satisfy $P_iT \leq C_G(E_i) \leq M_c = !\mathcal{M}(C_G(Z))$, and hence $M = \langle P_1, P_2\rangle T \leq M_c$, a contradiction.

Finally assume that $(\bar{M}_J, V)$ is decomposable. As (i) holds, $a = 1$. Then from the second paragraph of the proof, $s = 2$ and $(\bar{M}_i, V_i)$ satisfies case (1) or (2) of D.2.17. As $a = 1$, $\bar{M}_1$ and $\bar{M}_2$ are interchanged in M, so that conclusion (3) of 15.1.2 holds.

This completes the proof of 15.1.2. □

15.1.1. Statement of the main theorem, and some preliminaries. Our first goal is to show that case (1) or (3) of 15.1.2 holds, and $V(M)$ is an FF-module for $\bar{M}$. That is, we will prove that either $m(V(M)) = 2$ with $M/C_M(V(M)) = GL(V(M)) \cong L_2(2)$, or $m(V(M)) = 4$ with $M/C_M(V(M)) = O_4^+(V)$.

In the remaining cases there are no quasithin examples; indeed as far as we can tell, there are not even any shadows. But we saw in Theorem 14.6.25 of the previous chapter that quasithin examples do arise in the first case, and many shadows complicate our analysis of the second case, in the third section 15.3 of this chapter.

Thus the remainder of this section is devoted to the first steps in a proof of the following main result:

THEOREM 15.1.3. *Assume Hypothesis 14.1.5, and let $M := M_f$ as in 14.1.12. Then either*

(1) $m(V(M)) = 2$, $M/C_M(V(M)) \cong L_2(2)$, and G is isomorphic to J_2, J_3, ${}^3D_4(2)$, the Tits group ${}^2F_4(2)'$, $G_2(2)' \cong U_3(3)$, or M_{12}; or

(2) $m(V(M)) = 4$, and $M/C_M(V(M)) = O_4^+(V(M))$.

The proof of Theorem 15.1.3 involves a series of reductions, which will not be completed until the end of section 15.2. Thus in the remainder of this section, and throughout section 15.2, we assume G is a counterexample to Theorem 15.1.3. We also adopt the following convention:

NOTATION 15.1.4. Set $M := M_f$. We choose $V := V(M)$ in cases (1)–(5) of 15.1.2, but in case (6) of 15.1.2 we choose $V := [V(M), M_J]$, where M_J is the preimage in M of $\hat{J}(M/C_M(V(M))$. Set $\bar{M} := M/C_M(V)$ and $\bar{M}_0 := \hat{J}(\bar{M}, V)$.

Observe that except in case (6) of 15.1.2, $\bar{M}_0$ coincides with $\bar{M}_J$. We review some elementary but fundamental properties of V:

LEMMA 15.1.5. *(1)* $V = \langle (Z \cap V)^{M_0} \rangle$, *so* $V \in \mathcal{R}_2(M)$.
(2) $C_G(V) \leq M_c$.

PROOF. From the description of V in 15.1.2, $V = \langle (Z \cap V)^{M_0} \rangle$. Then B.2.14 completes the proof of (1). Part (2) follows since $M_c = !\mathcal{M}(C_G(Z))$. □

LEMMA 15.1.6. $O^2(C_M(Z)) \leq C_M(V)$.

PROOF. By 14.1.6.1, $M^\infty \leq C_M(V)$. Let $S := T \cap M^\infty$. By a Frattini Argument, $C_M(Z) = M^\infty K$, where $K := C_M(Z) \cap N_M(S)$. Since $K^\infty \leq N_{M^\infty}(S)$ and $N_{M^\infty}(S)$ is 2-closed, K is solvable. Thus $K = XT$, where X is a Hall $2'$-subgroup of K. Therefore it remains to show $X \leq C_M(V)$, so we may assume $\bar{X} \neq 1$. From the structure of $\bar{M}$ described in 15.1.2, $\bar{M}$ is 2-nilpotent, and hence so is $\bar{X}\bar{T}$. Therefore $\bar{X} = O(\bar{X}\bar{T}) \trianglelefteq \bar{X}\bar{T}$. Then as $\bar{X} \neq 1$, $Z \cap [V, X] = C_{[V,X]}(T) \neq 1$, and $C_{[V,X]}(\bar{X}) = 1$ by Coprime Action, whereas $X \leq K \leq C_M(Z)$. This contradiction completes the proof. □

In the next result we review the cases from 15.1.2 which can occur in our counterexample, except that we reorder them according to the value of $m(V)$:

LEMMA 15.1.7. *One of the following holds:*
(1) $m(V) = 4$, *and* $\bar{M} = \bar{M}_0 \cong S_3$.
(2) $m(V) = 4$, $\bar{M}_0 \cong S_3$, *and* $\bar{M} \cong S_3 \times \mathbf{Z}_3$.
(3) $m(V) = 4$, *and* $\bar{M} = \bar{M}_0 = \Omega_4^+(V)$.
(4) $m(V) = 4$, $\bar{M}_0 = \bar{P}\langle \bar{t} \rangle$ *where* $\bar{P} := O^2(\bar{M}) \cong E_9$ *and* $\bar{t}$ *is an involution inverting* $\bar{P}$, *and* $\bar{T} \cong \mathbf{Z}_4$.
(5) $m(V) = 4$, $\bar{M}_0 \cong D_{10}$, $\bar{T} \cong \mathbf{Z}_2$ *or* $\mathbf{Z}_4$, *and either* $F(\bar{M}) = F(\bar{M}_0)$ *or* $F(\bar{M}) \cong \mathbf{Z}_{15}$.
(6) $m(V) = 8$, $\bar{M}_0 = \bar{M}_1 \times \bar{M}_2$ *where* $\bar{M}_i \cong D_{2p}$ *with* $p = 3$ *or* 5, $M_1^t = M_2$ *for some* $t \in T$, *and* $V = V_1 \oplus V_2$, *where* $V_i := [V, M_i]$.
(7) $m(V) = 6$, $\bar{M}_0 = \bar{P}\langle \bar{t} \rangle$ *where* $P := O^2(\bar{M}) \cong 3^{1+2}$, $\bar{t}$ *is an involution inverting* $\bar{P}/\Phi(\bar{P})$, *and* T *acts irreducibly on* $\bar{P}/\Phi(\bar{P})$.
Furthermore if $V < V(M)$, *then case (3) holds.*

PROOF. Suppose first that $V < V(M)$. Then by definition of V in 15.1.4, case (6) of 15.1.2 holds and $V = [V, M_J]$; it follows that conclusion (3) holds. Thus in the remainder of the proof we may assume that $V = V(M)$, and hence that one of cases (1)–(5) of 15.1.2 holds.

Assume first that $m(V) = 2$. Then case (1) of 15.1.2 holds with $(p, m) = (3, 1)$ and $\bar{M} \cong S_3$. Then as we observed at the start of section 14.2, 14.1.18 shows that Hypothesis 14.2.1 is satisfied. Therefore we may apply Theorem 14.6.25 to conclude that G is one of the groups listed in conclusion (1) of Theorem 15.1.3, contrary to the choice of G as a counterexample.

Thus $m(V) > 2$. Also since G is a counterexample, conclusion (2) of Theorem 15.1.3 does not hold. Thus if case (3) of 15.1.2 holds, then $m(V_i) = 4$ for each i, so that conclusion (6) holds. In case (2) of 15.1.2, conclusion (3) holds. Cases (4) and (5) of 15.1.2 are conclusions (7) and (4).

It remains to treat case (1) of 15.1.2. In this case, $m(V)=4$ as $m(V)>2$, so $\bar{M}_0$ is S_3 or D_{10}. If $\bar{P}:=O^2(\bar{M}_0)=F^*(\bar{M})$, then $\bar{M}\leq Aut(\bar{P})\cong S_3$ or $Sz(2)$, respectively, so that conclusion (1) or (5) holds. Thus we may assume that $\bar{P}<F^*(\bar{M})$. Now $\bar{M}\leq N_{GL(V)}(\bar{M}_0)$, and $N_{GL(V)}(\bar{M}_0)$ is $\Omega_4^+(V)$ or $\mathbf{Z}_4/\mathbf{Z}_{15}$, respectively. Since $\bar{P}<F^*(\bar{M})$, and $O_2(\bar{M})=1$ by 15.1.5.1, one of conclusions (2)–(5) of the lemma holds. This completes the proof. □

Our assumption that G is a counterexample to Theorem 15.1.3 has ruled out the subcases of 15.1.2 in which $\bar{M}$ contains an FF*-offender on V; that is we are left with thoses cases where $q(\bar{M},V)>1$. Indeed:

LEMMA 15.1.8. *One of the following holds:*

(1) $\hat{q}(\bar{M},V)=q(\bar{M},V)=2$.

(2) Case (3) of 15.1.7 holds, where $\hat{q}(\bar{M},V)=3/2$ *and* $q(\bar{M},V)=2$.

PROOF. The proof of 15.1.2 showed that one of the following holds:

(i) $V=V(M)$, and $(\bar{M}_J,V)$ is an indecomposable appearing in one of cases (1)–(5) of D.2.17.

(ii) $V=V(M)$, and the $(\bar{M}_i,V_i)$ are indecomposable and appear in case (1) or (2) of D.2.17; hence case (6) of 15.1.7 holds.

(iii) $V<V(M)$, and hence case (3) of 15.1.7 holds.

In cases (i) and (ii), $\bar{M}_J=\bar{M}_0$ by Notation 15.1.4, so we conclude from the values listed in the corresponding cases of D.2.17 that $\hat{q}(\bar{M}_0,V)=q(\bar{M}_0,V)=2$— unless case (3) of 15.1.7 holds, where (M,V) appears in case (5) of D.2.17, $\bar{M}_0=\bar{M}$, $\hat{q}(\bar{M}_0,V)=3/2$, and $q(\bar{M}_0,V)=2$. However by definition of $\hat{\mathcal{Q}}_*(\bar{M},V)$, if $\hat{q}(\bar{M},V)\leq 2$, then $\hat{q}(\bar{M},V)=\hat{q}(\bar{M}_0,V)$ and $\hat{q}(\bar{M},V)\leq q(\bar{M},V)$. Thus the lemma holds in cases (i) and (ii). In case (iii), conclusion (2) of the lemma holds, again as $(\bar{M},V)$ appears in case (5) of D.2.17. □

Recall that $|\mathcal{M}(T)|>1$ by Hypothesis 14.1.5.3, so that $\mathcal{H}_*(T,M)$ is nonempty. The next few results study properties of members of $\mathcal{H}_*(T,M)$.

LEMMA 15.1.9. *Set* $R:=C_T(V)$. *Then*

(1) $[V,J(T)]=1=[V(M),J(T)]$, *so that* $Baum(T)=Baum(R)$ *and further* $C(G,Baum(T))\leq M$.

(2) M *is the unique maximal member of* $\mathcal{M}(T)$ *under* $\lesssim$.

(3) $\mathcal{H}_*(T,M)\subseteq C_G(Z)\leq M_c$.

(4) For each $H\in\mathcal{H}_*(T,M)$, $O^2(H\cap M)\leq C_M(V)$.

(5) $M=!\mathcal{M}(N_M(R))$.

(6) $N_M(R)\in\mathcal{H}^e$, $V\in\mathcal{R}_2(N_M(R))$, $R=O_2(N_M(R))$, *and* $\overline{N_M(R)}=\bar{M}$; *and case (II) of Hypothesis 3.1.5 is satisfied with* $N_M(R)$ *in the role of "*M_0*" for any* $H\in\mathcal{H}_*(T,M)$.

(7) $N_G(T)\leq M$, *and each* $H\in\mathcal{H}_*(T,M)$ *is a minimal parabolic described in B.6.8, and in E.2.2 if* H *is nonsolvable.*

PROOF. If $J(T)$ does not centralize $V(M)$, then as $m(V)>2$ by 15.1.7, 14.1.7 shows that conclusion (2) of Theorem 15.1.3 holds, contrary to the choice of G as a counterexample. Therefore $J(T)$ centralizes $V(M)$, and hence also centralizes V.

Since M is maximal in $\mathcal{M}(T)$ under $\stackrel{<}{\sim}$, we may now apply 14.1.4 to conclude that (2) holds; and apply 15.1.5.1, (2), and 14.1.2 to complete the proof of (1). Observe that $N_M(R) \in \mathcal{H}^e$ by 1.1.3.2. Using case (b) of the hypothesis of A.5.7.2 rather than case (a), the proof of 15.1.1.1 shows that (5) and (6) hold, and case (II) of Hypothesis 3.1.5 is satisfied with $N_M(R)$ in the role of "M_0" for any $H \in \mathcal{H}_*(T,M)$. Further $M = !\mathcal{M}(N_M(R))$ by (5), so (3) follows from 3.1.7. Then (4) follows from (3) and 15.1.6. Finally $N_G(T) \leq M$ by (1), so (7) follows from 3.1.3.2. □

LEMMA 15.1.10. *If case (6) of 15.1.7 holds with $p = 3$, then $\bar{M} \cong S_3 \;\; wr \;\; \mathbf{Z}_2$.*

PROOF. Since $C_{O_3(\bar{M})}(\bar{T} \cap \bar{M}_i)$ is cyclic by A.1.31.1, $O^2(\bar{M}_0) = O^2(\bar{M})$. Then as $O^2(\bar{M})$ acts on $\bar{M}_i$ and V_i for $i = 1, 2$, $C_{GL(V_i)}(\bar{M}_i) \cong L_2(2)$, and $O_2(\bar{M}) = 1$ by 15.1.5.1, the result follows. □

LEMMA 15.1.11. *For $H \in \mathcal{H}_*(T,M)$:*
(1) $V \leq O_2(H)$.
(2) $U_H := \langle V^H \rangle$ is elementary abelian.

PROOF. Set $R := C_T(V)$. By 15.1.9.5, $O_2(\langle N_M(R), H\rangle) = 1$, so Hypothesis F.7.1 is satisfied with $N_M(R)$, H in the roles of "G_1, G_2". Further as $R = O_2(N_M(R))$ by 15.1.9.6, $R = O_2(C_{N_M(R)}(V))$, so that Hypothesis F.7.6 is also satisfied. Now V is not an FF-module for $Aut_{N_M(R)}(V)$ by 15.1.8, so if (1) holds, we may apply F.7.11.8 to obtain (2).

So we may assume that $V \not\leq O_2(H)$, and it remains to derive a contradiction. By 3.1.3.1, $H \cap M$ is the unique maximal subgroup of H containing T, and by 15.1.9.7, H is described in B.6.8. Then our assumption $V \not\leq O_2(H)$ implies $V \not\leq \ker_{H\cap M}(H)$ by B.6.8.5. Thus Hypothesis E.2.8 is satisfied with $H \cap M$ in the role of "M". Then by E.2.15, $r := \hat{q}(\bar{M}, V) < 2$, so that by 15.1.8, $m(V) = 4$, $\bar{M} = \Omega_4^+(2)$, and $r = 3/2$. Also by 15.1.9.4, $O^2(H \cap M) \leq C_H(V)$. Hence by E.2.17, $Y = \langle V^H \rangle$ is isomorphic to S_3/Q_8^2, $L_3(2)/D_8^3$, or $(\mathbf{Z}_2 \times L_3(2))/D_8^3$. However in the last two cases, $|Aut_T(V)| \geq 8$ by E.2.17, contrary to $|\bar{M}|_2 = 4$. Therefore $Y \cong S_3/Q_8^2$. Set $P := O_2(Y)$, $X_0 := O^2(N_M(R))$, and $X := O^2([X_0, P])$. As $Aut_P(V) \cong E_4 \cong \bar{T}$ and $R = C_T(V)$, $T = PR$, so $\bar{X} = \bar{X}_0 \cong E_9$. Next

$$[R,P] \leq C_P(V) = V \cap P, \tag{$*$}$$

so P centralizes R/V, and hence $X \leq [X_0, P]$ centralizes R/V. Then $V = [R, X]$ so as $F^*(M) = O_2(M) \leq R$, $C_X(V)$ is a 2-group by Coprime Action. Then as $\bar{X} \cong E_9$ and $X = O^2(X)$, it follows that $X \cong A_4 \times A_4$ and $R = V \times C_R(X)$. By (*), $[C_R(X), P] \leq C_V(X) = 1$, so $TX = PRX = PX \times C_R(X)$ with $PX \cong S_4 \times S_4$. But now $[V, J(T)] \neq 1$, contrary to 15.1.9.1. □

LEMMA 15.1.12. *Let $H \in \mathcal{H}_*(T,M)$ and $U_H := \langle V^H \rangle$. Then*

(1) H has exactly two noncentral chief factors U_1 and U_2 on U_H.

(2) There exists $A \in \mathcal{A}(T) - \mathcal{A}(O_2(H))$, and for each such A chosen with $AO_2(H)/O_2(H)$ minimal, A is quadratic on U_H, and setting $B := A \cap O_2(H)$, we have:

$$2m(A/B) = m(U_H/C_{U_H}(A)) = 2m(B/C_B(U_H));$$

$$2m(B/C_B(V^h)) = m(V^h/C_{V^h}(B))$$

for each $h \in H$ with $[B, V^h] \neq 1$; $m(A/B) = m(U_i/C_{U_i}(A))$; and $C_{U_H}(A) = C_{U_H}(B)$.

(3) $H/C_H(U_i) \cong S_3$, S_5, S_3 wr $\mathbf{Z}_2$, or S_5 wr $\mathbf{Z}_2$, with U_i the direct sum of the natural modules $[U_i, F]$, as F varies over the S_3-factors or S_5-factors of $H/C_H(U_i)$. Further $J(H)C_H(U_i)/C_H(U_i) \cong S_3$, S_5, $S_3 \times S_3$, or $S_5 \times S_5$, respectively.

(4) $[\Omega_1(Z(J_1(T))), O^2(H)] = 1$.

PROOF. Let $R := C_T(V)$. By 15.1.9.6, $\bar{M} = \overline{N_M(R)}$. We check that the hypothesis of 3.1.9 holds, with $N_M(R)$ in the role of "M_0": First case (II) of Hypothesis 3.1.5 is satisfied by 15.1.9.6. By 15.1.11, $V \leq O_2(H)$, giving (c). By 15.1.7 and B.1.8, V is not a dual FF-module for $\bar{M} = \overline{N_M(R)}$, giving (d). By 15.1.8, $q(\bar{M}, V) = 2$, giving (a). By 15.1.9.5, $M = !\mathcal{M}(N_M(R))$, giving (b). Finally by 15.1.9.4, $O^2(H \cap M) \leq C_G(V)$, so the hypotheses of part (5) of 3.1.9 are satisfied. Therefore by 3.1.9, (1)–(3) hold.

As A^* is an FF*-offender on U_i, it follows from (3) that there is a subgroup X of Y with $A^* \in Syl_2(X^*)$, $O_2(H) \leq X$, $X/O_2(X) \cong S_3$, and $H = \langle O^2(X), T\rangle$. Now we chose $A \in \mathcal{A}(T)$, and U_H is elementary abelian by 15.1.11.2, with $A \cap U_H \leq A \cap O_2(H) = B$, so $C_{U_H}(A) = A \cap U_H = B \cap U_H \leq C_B(U_H)$. Next by (2),

$$m(A/C_B(U_H)) = m(A/B) + m(B/C_B(U_H)) = 2m(A/B) = m(U_H/C_{U_H}(A)),$$

so $m(U_H C_B(U_H)) \geq m(A)$. Hence $U_H C_B(U_H) \in \mathcal{A}(T)$, so as $U_H C_B(U_H) \leq O_2(H) \leq O_2(X)$, also $U_H C_B(U_H) \in \mathcal{A}(O_2(X))$. Therefore by B.2.3.7, $\Omega_1(Z(J(T)))$ and $\Omega_1(Z(J(O_2(X))))$ are contained in $U_H C_B(U_H)$, so by B.2.3.2, $\Omega_1(Z(J_1(T))) =: E$ and $\Omega_1(Z(J_1(O_2(X)))) =: D$ are also contained in $U_H C_B(U_H)$. In particular, $E \leq O_2(X)$, so $E \leq D$.

If $[E, O^2(X)] = 1$, then $H = \langle O^2(X), T\rangle \leq N_G(E)$, and hence $K \leq \langle O^2(X)^H\rangle \leq C_G(E)$, so that (4) holds. Thus we may assume that $[E, O^2(X)] \neq 1$, and it remains to derive a contradiction. We saw $E \leq D$, so also $[D, O^2(X)] \neq 1$. Then as $O^2(X) = [O^2(X), A]$ by construction, $[D, A] \neq 1$, so in particular $D \not\leq A \cap O_2(X) =: B_X$. Observe that $B_X \geq A \cap O_2(H) = B$. On the other hand, $B_X \in \mathcal{A}_1(O_2(X))$ as $X/O_2(X) \cong S_3$, so D centralizes B_X, and then as $D \not\leq B_X$, $m(DB_X) > m(B_X) = m(A) - 1$, so $DB_X \in \mathcal{A}(T)$. Then as $D \leq U_H C_B(U_H) \leq O_2(H)$, by minimality of $AO_2(H)/O_2(H)$, $B_X \leq O_2(H) \cap A = B$, so that $B_X = B$. But by (2), $C_{U_H}(B) = C_{U_H}(A)$, so

$$D \leq C_B(U_H)U_H \cap C_G(B) = C_B(U_H)C_{U_H}(B) = C_B(U_H)C_{U_H}(A) \leq C_G(A),$$

contrary to an earlier observation. This contradiction completes the proof of 15.1.12. □

LEMMA 15.1.13. *Let $E_1 := \Omega_1(Z(J_1(T)))$. Then*

(1) $C_G(E_1) \not\leq M$.

(2) $[V, J_1(T)] \neq 1$.

(3) Either

(i) for all $A \in \mathcal{A}_1(T)$ with $\bar{A} \neq 1$, $|\bar{A}| = 2$ and $\bar{A} \in \hat{\mathcal{Q}}_*(\bar{M}, V)$, or

(ii) case (3) of 15.1.7 holds.

(4) Either

(a) $\overline{J_1(T)} = \bar{T} \cap \bar{M}_0$ and $\overline{J_1(M)} = \bar{M}_0$, or

(b) Case (3) of 15.1.7 holds, and $\overline{J_1(T)}$ is of order 2.

PROOF. As $\mathcal{H}_*(T,M) \neq \emptyset$, (1) follows from 15.1.12.4. Next if $[V, J_1(T)] = 1$, then by B.2.3.5, $N_M(C_T(V))$ normalizes $J_1(T)$ and hence also normalizes E_1, so that $N_G(E_1) \leq M$ by 15.1.9.5, contrary to (1). This establishes (2).

By (2), there is $A \in \mathcal{A}_1(T)$ with $\bar{A} \neq 1$. Now $m(\bar{A}) \leq m_2(\bar{M})$ and $m_2(\bar{M}) \leq 2$ from 15.1.7. As $A \in \mathcal{A}_1(T)$, $m(V/C_V(A)) \leq m(\bar{A}) + 1$, while $q(\bar{M}, V) > 1$ by 15.1.8, so that $m(V/C_V(A)) = m(\bar{A}) + 1$. Hence for $m(\bar{A}) = 1$ or 2,

$$r_{\bar{A},V} = \frac{m(V/C_V(A))}{m(\bar{A})} = 2 \text{ or } 3/2,$$

respectively. Assume that case (3) of 15.1.7 does not hold. Then by 15.1.8, $\hat{q}(\bar{M}, V) = q(\bar{M}, V) = 2$, and the calculation above shows that $m(\bar{A}) = 1$ and $r_{\bar{A},V} = 2$ for each $A \in \mathcal{A}_1(T)$ with $\bar{A} \neq 1$, so that $\bar{A} \in \hat{\mathcal{Q}}_*(\bar{M}, V)$. This establishes (3).

It remains to prove (4). Suppose first that case (3) of 15.1.7 holds. Then $\bar{M} = \bar{M}_0$ and $\bar{T} \cong E_4$, and $\overline{J_1(T)} \neq 1$ by (2). Therefore either $\overline{J_1(T)} = \bar{T}$, and hence conclusion (a) of (4) holds, or $\overline{J_1(T)}$ is of order 2, and conclusion (b) holds. Thus we may assume that case (3) of 15.1.7 does not hold. Then by (3), $\overline{J_1(T)} \leq \bar{T} \cap \bar{M}_0 =: \bar{T}_0$. As case (3) of 15.1.7 does not hold, either case (6) of 15.1.7 holds or $|\bar{T}_0| = 2$. In the latter case, $\overline{J_1(T)} = \bar{T}_0$ so that $\overline{J_1(M)} = \bar{M}_0$, giving conclusion (a). In the former case, $\bar{A} \leq \bar{M}_i$ for $i = 1$ or 2 since $r_{\bar{A},V} = 2$, and then $\overline{J_1(T)} = \langle \bar{A}, \bar{A}^t \rangle = \bar{T}_0$, so again conclusion (a) holds. This completes the proof of (4). □

LEMMA 15.1.14. *Let $V_E := C_V(J_1(T))$. Then*
(1) $O^2(C_G(Z)) \leq C_G(V_E)$.
(2) $N_M(J_1(T)) \leq N_G(V_E) \leq M_c$.
(3) $N_G(J_1(T)) \leq M \cap M_c$.

PROOF. By 15.1.6, $O^2(C_M(Z)) \leq C_M(V) \leq C_M(V_E)$. Thus if (1) fails, then

$$O^2(C_G(Z)) \not\leq \langle M \cap O^2(C_G(Z))T, O^2(C_G(Z)) \cap C_G(V_E)\rangle,$$

so there exists $H \in \mathcal{H}_*(T,M)$ with $H \leq C_G(Z)$ but $O^2(H) \not\leq O^2(C_G(V_E))$. However since $V_E \leq \Omega_1(Z(J_1(T)))$, this contradicts 15.1.12.4, so (1) is established. Then (1) implies (2) since $M_c = !\mathcal{M}(C_G(Z))$. Finally as $J(J_1(T)) = J(T)$ by (1) and (3) of B.2.3,

$$N_G(J_1(T)) = N_G(J(T)) \cap N_G(J_1(T)) \leq N_M(J_1(T))$$

by 15.1.9.1, so (2) implies (3). □

15.1.2. Eliminating some larger possibilities from 15.1.7.

Our proof of Theorem 15.1.3 now divides into two cases:

Case I. $M = \langle C_M(Z_1), T\rangle$ for some nontrivial subgroup Z_1 of $C_V(J_1(T))$.

Case II. There exists a subgroup X of M containing T with $M = !\mathcal{M}(X)$ and $X/O_2(X) \cong S_3$, D_{10}, or $Sz(2)$.

Case II will be treated in the following section. Cases (1)–(3) and (5) of 15.1.7 appear in Case II, although this fact is not established until lemma 15.2.6 in that section. In the remainder of this section, we treat the three cases from 15.1.7 which appear in Case I. Namely we prove the following theorem:

THEOREM 15.1.15. *None of cases (4), (6) or (7) of 15.1.7 can hold.*

Until the proof of Theorem 15.1.15 is complete, assume G is a counterexample. Thus we are in case (4), (6), or (7) of 15.1.7. As in 15.1.13, let $E_1 := \Omega_1(Z(J_1(T)))$. By 15.1.13.1, $C_G(E_1) \not\leq M$.

As case (3) of 15.1.7 does not hold, $\overline{J_1(T)} = \bar{T} \cap \bar{M}_0$ and $\overline{J_1(M)} = \bar{M}_0$ by 15.1.13.4. Also $V = V(M)$ by 15.1.7.

We begin by determining $V_E := C_V(J_1(T))$ in each of our three cases, and defining some notation:

NOTATION 15.1.16. (a) In case (6) of 15.1.7, $V = V_1 \oplus V_2$ for V_i defined there, and $V_E = Z_1 \oplus Z_2$, where $Z_i := C_{V_i}(T \cap M_0) \cong E_4$.

(b) In case (4) of 15.1.7, $V = V_1 \oplus V_2$, where V_1 and V_2 are the two 4-subgroups of V such that $\bar{M}_i := N_{\bar{M}}(V_i)$ is not a 2-group; in this case $V_E = Z_1 \oplus Z_2$ where $Z_i := V_E \cap V_i$ is of order 2.

(c) Finally in case (7) of 15.1.7, $V_E \cong E_{16}$. In this last case, $\bar{P} := O_3(\bar{M}) \cong 3^{1+2}$. Let $\bar{P}_Z := Z(\bar{P})$, pick $\bar{P}_i$ of order 3 in $\bar{M}_0$ inverted by $T \cap M_0$ for $i = 1, 2$ with $\bar{P} = \bar{P}_Z \bar{P}_1 \bar{P}_2$, and set $Z_i := C_V(\bar{P}_i)$ and $V_2 := [V, P_1]$, so that $V = Z_1 \oplus V_2$, $Z_i \cong E_4$, and $V_2 \cong E_{16}$. In this case $V_E = Z_1 \oplus Z_2$.

In each case, set $S := C_T(Z_1)$, $G_1 := C_G(Z_1)$, $M_Z := G_1 \cap M_c$, and $Q_1 := O_2(M_Z)$.

Observe that in each of the cases in Notation 15.1.16, Case I holds by construction: Namely $Z_1 \leq V_E = C_V(J_1(T))$, and $M = \langle C_M(Z_1), T\rangle$. Also:

LEMMA 15.1.17. *$S \in Syl_2(G_1 \cap M)$, $J(S) = J(T)$, $Baum(S) = Baum(T)$, and $C(G, Baum(S)) \leq M$.*

PROOF. By construction in Notation 15.1.16, S is Sylow in $G_1 \cap M$ and $C_T(V) \leq C_T(Z_1) = S$. So as $J(T) \leq C_T(V)$ by 15.1.9.1, $J(S) = J(T)$ and $\mathrm{Baum}(S) = \mathrm{Baum}(T)$ by (3) and (5) of B.2.3. Then 15.1.9.1 completes the proof. □

LEMMA 15.1.18. *(1) $Z_1 \leq V_E$, and $O^2(C_G(Z)) \leq M_Z$.*

(2) $O^2(M \cap M_c \cap G_1) = O^2(C_M(V))$ and $C_M(Z_1) \not\leq M_Z$.

(3) S, M_Z, and Q_1 are T-invariant.

(4) $S \in Syl_2(G_1)$.

(5) $C(G_1, Q_1) = M_Z = N_{G_1}(Q_1)$, so Hypothesis C.2.3 is satisfied with G_1, Q_1, M_Z in the roles of "H, R, M_H".

(6) Hypothesis 1.1.5 is satisfied with G_1, M_c in the roles of "H, M", for any $1 \neq z \in Z$.

(7) $M_c = !\mathcal{M}(M_Z T)$ and $C(G, Q_1) \leq M_c$.

PROOF. We observed earlier that Case I holds, so in particular, $Z_1 \leq V_E$ and $M = \langle C_M(Z_1), T\rangle$. Then as $O^2(C_G(Z)) \leq C_G(V_E)$ by 15.1.14.1, and $M_c = !\mathcal{M}(C_G(Z))$, (1) follows; and as $M \not\leq M_c$, $C_M(Z_1) \not\leq M_Z$. By 15.1.17, $S \in Syl_2(G_1 \cap M)$ and $N_G(S) \leq M$, so (4) holds. Hence S is also Sylow in M_Z and in $M \cap M_Z = G_1 \cap M \cap M_c$. Since $Z_1 \leq V$, 15.1.5.2 says that $C_M(V) \leq G_1 \cap M \cap M_c$. By construction in 15.1.16, $O^2(\overline{C_M(Z_1)})$ is of prime order, so as $C_M(Z_1) \not\leq M_Z$ and $S \in Syl_2(M_Z)$, it follows that $O^2(M \cap M_c \cap G_1) = O^2(C_M(V))$, completing the proof of (2). We check in each case in 15.1.16 that $\bar{S} = C_{\bar{T}}(V_E)$, so that $S \trianglelefteq T$. By 15.1.9.2, $M_c \stackrel{<}{\sim} M$, so

$$M_c = N_M(V(M_c))C_{M_c}(V(M_c)). \qquad (*)$$

By (1),

$$O^2(C_G(V(M_c))) \le O^2(C_G(Z)) \le O^2(M_Z), \qquad (**)$$

and then by (*) and (**),

$$O^2(M_Z) = O^2(N_{M\cap M_Z}(V(M_c)))O^2(C_{M_c}(V(M_c)).$$

Next from (2), $O^2(M \cap M_Z) = O^2(C_M(V))$, so

$$O^2(M_Z) = O^2(C_M(V))O^2(C_{M_c}(V(M_c)),$$

and hence $O^2(M_Z)$ is T-invariant. Therefore as S is Sylow in M_Z and normal in T, both $O^2(M_Z)S = M_Z$ and $O_2(M_Z) = Q_1$ are also T-invariant, completing the proof of (3). Then as $O^2(C_G(Z)) \le M_Z$ and $M_c = !\mathcal{M}(C_G(Z))$, (7) holds. Since $C(G, Q_1) \le M_c$, $C(G_1, Q_1) = M_Z = N_{G_1}(Q_1)$ and $Q_1 \in \mathcal{B}_2(G_1)$. Then we easily verify Hypothesis C.2.3 with G_1, Q_1, M_Z in the roles of "H, R, M_H", so that (5) holds. Finally for any $1 \ne z \in Z$, $M_c \in \mathcal{M}(C_G(z))$, so (6) follows from 1.1.6 applied to G_1, M_c in the roles of "H, M". □

LEMMA 15.1.19. *(1) $O(G_1) = 1$.*

(2) If $K = O_{2,2'}(K)$ is an M_Z-invariant subgroup of G_1 with $F^*(K) = O_2(K)$, then $K \le M_Z$.

(3) $O_{2,F}(G_1) \le M_Z$.

(4) If $M_Z \le H \le G_1$ with $O_{2,F^*}(H) \le M_Z$, then $H = M_Z$.

(5) $O_\infty(G_1) \le M_Z$.

(6) There exists $L \in \mathcal{C}(G_1)$ with $L/O_2(L)$ quasisimple and $L \not\le M_Z$.

(7) For L as in (6), $O^2(N_M(Z_1))V_2$ acts on L and $[L, V_2] \ne 1$.

PROOF. Observe that $V_2 = [V_2, O^2(C_M(Z_1))]$ by construction in Notation 15.1.16, so V_2 centralizes $O(G_1)$ by A.1.26.1. Also by construction, $1 \ne Z \cap Z_1Z_2 = Z \cap Z_1V_2 =: Z_+$, so that Z_+ centralizes $O(G_1)$. Now by 15.1.18.6, we may apply 1.1.5.2 with any involution of $Z_+^\#$ in the role of "z", so (1) follows.

Assume K_Z is a counterexample to (2). Then K is M_Z-invariant and $S \in Syl_2(G_1)$ by 15.1.18.4, so $O_2(K) \le O_2(KM_Z) \le S \le M_Z$, and hence $O_2(K) \le O_2(M_Z) = Q_1$, so that $Q_1 \in Syl_2(KQ_1)$. Then by 15.1.18.5 and C.2.5, there is an A_3-block X of K with $X \not\le M_Z$. Let $Y := O^2(M_Z)$; then $[Y, X] \le O_2(K) \le N_G(Y)$, so X normalizes $O^2(YO_2(K)) = Y$. However as $M_c = !\mathcal{M}(M_ZT)$ by 15.1.18.7, $X \le N_G(Y) \le M_c$, contrary to the choice of X. This contradiction establishes (2). By (1), $F^*(O_{2,F}(G_1)) = O_2(O_{2,F}(G_1))$, so (2) implies (3).

Assume the hypotheses of (4). Then $Q_1 = O_2(H)$ by A.4.4.1 with M_Z in the role of "K", so $H \le N_G(Q_1) \le M_c$ by 15.1.18.5, establishing (4). By (3), we may apply (4) with $O_\infty(G_1)M_Z$ in the role of "H", to obtain (5). Similarly if (6) fails, then by (3), $O_{2,F^*}(G_1) \le M_Z$, so $G_1 = M_Z$ by (4), contrary to 15.1.18.2.

Finally by 1.2.1.3, $O^2(C_M(Z_1))$ acts on each L satisfying (6), and hence so does $V_2 = [V_2, O^2(C_M(Z_1))]$. Further if V_2 centralizes L, then so does $Z \cap Z_1V_2 \ne 1$, so that $L \le M_c = !\mathcal{M}(C_G(Z))$, contrary to $L \not\le M_Z$. So V_2 is nontrivial on L, establishing (7). □

Recall $J(S) = J(T)$ by 15.1.17, and $S \in Syl_2(G_1)$ by 15.1.18.4. Further 15.1.19.6 shows that there is $L \in \mathcal{C}(G_1)$ with $L/O_2(L)$ quasisimple and $L \not\le M_Z$, so the collection of subgroups studied in the following result is nonempty:

LEMMA 15.1.20. *Let $L\in\mathcal{L}(G_1,S)$ with $L/O_2(L)$ quasisimple and $L\not\leq M_Z$. Set $S_L:=S\cap L$ and $M_L:=M_Z\cap L$. Then $S_L\in Syl_2(L)$ and*

(1) $L\not\leq M$.

(2) Assume $F^*(L)=O_2(L)$. Then $L=[L,J(S)]$, and one of the following holds:

(a) L is a block of type A_5 or $L_2(2^n)$, and M_L is a Borel subgroup of L.

(b) $L/O_{2,Z}(L)\cong A_7$, $L_3(2)$, A_6, or $G_2(2)'$. Further if $L\in\mathcal{C}(G_1)$ then L is a block of type A_7, $L_3(2)$, A_6, or $G_2(2)$. In the last three cases, $M_L=C_L(Z)$ is the maximal parabolic subgroup of L centralizing $C_{U(L)}(S_L)$, and in the first case M_L is the stabilizer of the vector of $U(L)$ of weight 2 centralized by S_L.

(c) $L/O_2(L)\cong L_4(2)$ or $L_5(2)$, and M_L is a proper parabolic subgroup of L.

(3) Assume L is a component of G_1. Then $Z(L)$ is a 2-group, and one of the following holds:

(a) L is a Bender group or $L/O_2(L)\cong Sz(8)$, and M_L is a Borel subgroup of L.

(b) $L\cong L_3(2^n)$ or $Sp_4(2^n)$, $n>1$, or $L/O_2(L)\cong L_3(4)$, and M_L is a Borel subgroup or a maximal parabolic of L.

(c) $L\cong G_2(2)'$, ${}^2F_4(2)'$, or ${}^3D_4(2)$, and $M_L=C_L(Z(S_L))$.

(d) $L/O_2(L)$ is a Mathieu group, J_2, HS, He, or Ru, and $M_L=C_L(Z(S_L))$.

(e) $L\cong L_4(2)$ or $L_5(2)$, and M_L is a proper parabolic subgroup containing $C_L(Z(S_L))$.

PROOF. If $L\leq M$, then by 14.1.6.1 and 15.1.5.2, $L\leq C_M(V)\leq M_c$, contrary to the choice of L; so (1) holds.

Since $S\in Syl_2(G_1)$ by 15.1.18.4, and $L\in\mathcal{L}(G_1,S)$ by hypothesis, $S_L=S\cap L$ is Sylow in the subnormal subgroup L of $\langle L,S\rangle$.

Suppose that $L/O_2(L)$ is a simple Bender group; we claim that M_L is the Borel subgroup B_L of L over S_L. For B_L is the unique maximal overgroup of S_L in L, so as $S_L\leq M_L<L$, it follows that $M_L\leq B_L$; then $N_{M_Z}(L)$ acts on $N_L(O_2(M_L))=B_L$, and hence M_Z normalizes the Borel subgroup $B_0:=\langle B^S\rangle$ of $L_0:=\langle L^S\rangle$. As $L/O_2(L)$ is a Bender group, $F^*(B_0)=O_2(B_0)$, so $B_0\leq M_Z$ by 15.1.19.2, and hence $M_L=B_L$, as claimed.

We now begin the proof of (2), so assume that $F^*(L)=O_2(L)$. Set $H:=LS_H$, where $S_H:=N_S(L)$. Then $S_H\in Syl_2(H)$ and as $F^*(L)=O_2(L)$, also $F^*(H)=O_2(H)$. Let $U:=\langle Z^H\rangle$ and $H^*:=H/C_H(U)$, so that $O_2(H^*)=1$ by B.2.14. As $C_H(U)\leq C_H(Z)\leq M_Z$ and $L\not\leq M_Z$, $L^*\neq 1$; thus as $L/O_2(L)$ is quasisimple, L^* is quasisimple. As $N_G(J(S))\leq M$ by 15.1.17, and $L\not\leq M$ by (1), $J(S)\not\leq O_2(\langle L,S\rangle)$ using B.2.3.3. Now by B.2.5, we may apply B.1.5.4 to conclude that $J(S)^*$ normalizes L^*, so that $L=[L,J(S)]$. Thus U is an FF-module for H^* by B.2.7. Therefore by Theorem B.4.2, L^* is one of $L_2(2^n)$, $SL_3(2^n)$, $Sp_4(2^n)'$, $G_2(2^n)'$, $L_n(2)$, for suitable n, or $\hat{A}_6$, or A_7.

Suppose $L^*\cong L_2(2^n)$. If L is a block then $L/O_2(L)\cong L_2(2^n)$, so M_L^* is a Borel subgroup of L^* by paragraph three, and hence conclusion (a) of (2) holds. So we assume L is not a block, and it remains to derive a contradiction. Now as $L/O_2(L)$ is quasisimple, H is a minimal parabolic, so we may apply C.1.26 to conclude that either $C_1(S_H)$ centralizes L, or $C_2(S_H)\trianglelefteq H$. By 15.1.17, $\mathrm{Baum}(T)=\mathrm{Baum}(S)$, and by C.1.16.3, $\mathrm{Baum}(S)$ acts on L, and hence $\mathrm{Baum}(S)=\mathrm{Baum}(S_H)$ by B.2.3.4.

Now by Remark C.1.19, we may take $C_2(S_H) = C_2(T)$ and $C_1(T) \leq C_1(S_H)$. Then $N_G(C_2(S_H)) \leq M$ by 15.1.17, so $C_2(S_H)$ is not normal in H by (1). Therefore $L \leq C_G(C_1(S_H)) \leq C_G(C_1(T)) \leq M_c$ since $C_1(T) \leq Z$ and $M_c = !\mathcal{M}(C_G(Z))$. However this contradicts our hypothesis that $L \not\leq M_Z$.

Suppose next that $L^* \cong SL_3(2^n)$, $Sp_4(2^n)$, or $G_2(2^n)$ with $n > 1$. Let P_i, $i = 1, 2$, be the maximal parabolics of H over S_H, and $L_i := O^2(P_i)$. Then $L_i \in \mathcal{L}(G_1, S)$ with $L_i/O_2(L_i) \cong L_2(2^n)$, but L_i is not a block since $O_2(L_i^*) \neq 1$ and $[U, L_i] \neq 1$, so by the previous paragraph, $L_i \leq M_Z$. Thus $L = \langle L_1, L_2 \rangle \leq M_Z$, contrary to hypothesis.

If $L^* \cong L_4(2)$ or $L_5(2)$, then as $S_L \leq M_L < L$ and $S_L \in Syl_2(L)$, M_L is a proper parabolic subgroup, so conclusion (c) of (2) holds. If L^* is one of the remaining possibilities, then $L/O_{2,Z}(L)$ is listed in conclusion (b) of (2). Thus to complete the proof of (2), it remains to assume that $L \in \mathcal{C}(G_1)$ with $L^* \cong L_3(2)$, A_6, A_7, $\hat{A}_6$, or $G_2(2)'$, and to verify the final two sentences of (2b).

As L is not a χ_0-block, by 15.1.18.5 we may apply C.2.4 to conclude that Q_1 acts on L. So since $L \in \mathcal{C}(G_1)$, the hypotheses of C.2.7 are satisfied, and hence L is listed in C.2.7.3. Set $Z_S := C_U(S_H)$ and $Z_U := C_{[U,L]}(S_H)$; then $Z \leq U \cap Z(S_H) = Z_S$. By B.2.14, $U = C_{Z_S}(L)[U, L]$, so $Z_S = C_{Z_S}(L)Z_U$. Since $M_c = !\mathcal{M}(C_G(Z))$,

$$C_L(Z_U) = C_L(Z_S) \leq C_L(Z) \leq M_L. \qquad (*)$$

Suppose either that $L/O_{2,Z}(L)$ is of rank 2 over $\mathbf{F}_2$, or that L is an exceptional A_7-block. The module $[U, L]/C_{U,L}(L)$ is described in case (i) or (ii) of Theorem B.5.1.1, and in each module U, $C_L(Z_U)$ is a maximal subgroup of L. Therefore as $M_L < L$, the inequalities in (*) are equalities, so that $M_L = C_L(Z_U) = C_L(Z(S_L))$. Suppose instead that L is an ordinary A_7-block; then Z_U contains vectors z_w of weights $w = 2, 4, 6$, and there is $z \in Z \cap C_{Z_S}(L)z_w$ for some w. Then $C_L(z_w) \leq C_L(z) \leq M_L$. But unless $w = 2$, $Aut_{O_2(C_{LS_H}(z_w))}([U, L])$ contains no FF*-offenders by B.3.2.4, contrary to C.2.7.2. Thus $w = 2$ and as $C_L(z_2)$ is maximal in L, $C_L(z_2) = M_L$.

We have shown that if L is one of the four blocks listed in (2b), then (2b) holds, so we may assume L is not one of these blocks. If L^* is A_6 or $G_2(2)'$, then by C.2.7.3, L is an A_6-block or $G_2(2)$-block, contrary to this assumption. If $L^* \cong L_3(2)$, then by C.2.7.3, L is described in C.1.34. By the previous paragraph, M_Z is the parabolic centralizing Z_S, so case (1) or (5) of C.1.34 holds as the other cases exclude Q_1 normal in that parabolic; thus L is an $L_3(2)$-block, again contrary to assumption. If $L/O_{2,Z}(L) \cong A_7$, then by C.2.7.3, L is either an A_7-block or an exceptional A_7-block. Again the first case contradicts our assumption, and in the second case, we showed in the previous paragraph that M_L^* is the maximal subgroup of index 15 fixing Z_U, rather than the subgroup of index 35 in L^* appearing in case (d) of C.2.7.3.

Thus it only remains to eliminate case (c) of C.2.7.3, where L is a block of type $\hat{A}_6$: Here by B.4.2, the only parabolic P of L such that $O_2(P)$ contains an FF-offender is *not* the parabolic $C_L(Z) = M_L$, contrary to C.2.7.2. This completes the proof of (2).

Finally we prove (3), so assume L is a component of G_1. By 15.1.18.6 we can apply 1.1.5, and in particular L is described in 1.1.5.3. Since $O(G_1) = 1$ by 15.1.19.1, $Z(L) = O_2(L)$ is a 2-group. By 1.1.5.1, $M_Z \in \mathcal{H}^e$, so $M_L \in \mathcal{H}^e$ by 1.1.3.1. By 1.1.5.3, Z is faithfully represented on L with $Aut_Z(L) \leq Z(Aut_S(L))$. Thus if some $z \in Z^{\#}$ induces an inner automorphism on L, then as $M_c = !\mathcal{M}(C_G(Z))$, we have $C_L(Z(S_L)) \leq C_L(z) \leq M_L$.

We first treat cases (a)–(c) of 1.1.5.3, where $L/O_2(L)$ is of Lie type in characteristic 2, and hence described in case (3) or (4) of Theorem C (A.2.3). As M_L is contained in a proper overgroup of $S_L \in Syl_2(L)$, M_L is contained in a proper parabolic subgroup P_L of L (cf. 47.7 in [**Asc86a**]). In cases (a)–(c) of 1.1.5.3, either $L \cong A_6$, or Z induces inner automorphisms on L. In the latter case, $C_L(Z(S_L)) \leq P_L$ by the previous paragraph, and in the former $C_L(S_L) = S_L \leq P_L$ trivially.

Next if $L/O_2(L)$ is of Lie rank 1, then by 1.1.5.3, $L/O_2(L)$ is a simple Bender group, and conclusion (a) of (3) holds by paragraph three. Thus we may assume that $L/O_2(L)$ is of Lie rank at least 2. Then by 1.1.5.3, either L is simple, or $L/O_2(L) \cong L_3(4)$ or $G_2(4)$.

Assume first that $L/O_2(L)$ is defined over $\mathbf{F}_{2^n}$ with $n > 1$. This rules out case (4) of Theorem C, so that $L/O_2(L)$ is in one of the five families of groups of Lie rank 2 in case (3) of Theorem C. Further $L \trianglelefteq G_1$ by 1.2.1.3. If S is nontrivial on the Dynkin diagram of L, then either $L \cong L_3(2^n)$ or $Sp_4(2^n)$ with $n > 1$, or $L/O_2(L) \cong L_3(4)$; further the Borel subgroup B of L over S_L is the unique S-invariant proper parabolic subgroup of L containing S_L, so arguing as in paragraph three, $M_L = B$, and then conclusion (b) of (3) holds.

Thus we may assume that S normalizes both maximal parabolics P_i, $i = 1, 2$, of L over S_L. Then $L_i := P_i^\infty \in \mathcal{L}(G_1, S)$ with $F^*(L_i) = O_2(L_i)$, and $L_i/O_2(L_i)$ is either $L_2(2^m)$ (with m a multiple of n) or $Sz(2^n)$. By a Frattini Argument, $M_Z = M_L N_{M_Z}(S_L)$, and $N_{M_Z}(S_L) = O^2(N_{M_Z}(S_L))S$ acts on the two maximal overgroups P_1 and P_2 of S_L in L. Thus M_Z acts on each parabolic P containing M_L, so $O_{2,2'}(P) \leq M_Z$ by 15.1.19.2. Then if $L_i \leq M_Z$, $P_i \leq M_Z$ by 15.1.19.4, so $P_i = M_L$ by maximality of P_i.

If L_i is not a block, then $L_i \leq M_Z$ by (2). Thus if neither L_1 nor L_2 is a block, then $L = \langle L_1, L_2 \rangle \leq M_Z$, contrary to hypothesis. Therefore L_i is a block for $i = 1$ or 2, so either L is $L_3(2^n)$ or $Sp_4(2^n)$, or $L/O_2(L) \cong L_3(4)$. So if $L_1 \leq M_L$, then $M_Z = P_1$ by the previous paragraph, so that conclusion (b) of (3) holds. Thus we may assume that neither L_1 nor L_2 is contained in M_L. Then (cf. 47.7 in [**Asc86a**]), M_L is contained in the Borel subgroup $P_1 \cap P_2 = P$ over S_L, so $M_L = P$ by the previous paragraph, and again conclusion (b) of (3) holds.

Thus we may assume that $L/O_2(L)$ is defined over $\mathbf{F}_2$. Then from 1.1.5.3, and recalling $Z(L)$ is a 2-group, L is simple. So from Theorem C, L is $G_2(2)'$, ${}^2F_4(2)'$, ${}^3D_4(2)$, $Sp_4(2)'$, $L_3(2)$, $L_4(2)$ or $L_5(2)$. Recall from earlier discussion that $P_c := C_L(Z(S_L)) \leq M_L \leq P_L$ for some proper parabolic P_L of L. However in the first three cases, P_c is a maximal parabolic, so $M_L = P_c$, and hence conclusion (c) of (3) holds. Thus we may assume one of the remaining four cases holds. In those cases, *all* overgroups of S_L are parabolics, so M_L is a parabolic. Thus conclusion (e) of (3) holds if $L \cong L_4(2)$ or $L_5(2)$.

In cases (a)–(c) of 1.1.5.3, we have reduced to $L \cong L_3(2)$ or A_6. We now eliminate these cases, along with case (d) of 1.1.5.3. Since $Z(L) = O_2(L)$, $L \cong A_7$ in the last case. In each case $S_L \cong D_8$, and $Z(S_L)$ is of order 2.

We claim that Z contains a nontrivial subgroup Z_L inducing inner automorphisms on L. If $L \cong L_3(2)$, this follows from earlier discussion. In the other two cases, L is normal in G_1 by 1.2.1.3, so as $Out(L/O_2(L))$ is an elementary abelian

2-group, $\Phi(S)$ induces inner automorphisms on L. Then as $S \trianglelefteq T$ by 15.1.18.3, $1 \neq Z \cap \Phi(S)$ induces inner automorphisms, completing the proof of the claim.

Notice the claim eliminates case (d) of 1.1.5.3 where $L \cong A_7$, as there each $z \in Z^\#$ induces outer automorphisms on L. Thus $L \cong L_3(2)$ or A_6.

Next $Y := O^2(C_G(Z)) \leq M_Z$ by 15.1.18.1, and so Y acts on L by 1.2.1.3. However as L is $L_3(2)$ or A_6, $C_{Aut(L)}(Z(S_L))$ is a 2-group, so as Z_L induces $Z(S_L)$ on L, $O^2(C_{Aut(L)}(Z)) \leq O^2(C_{Aut(L)}(Z(S_L))) = 1$, and hence $Y \leq C_{G_1}(L)$. Then as $M_c = !\mathcal{M}(C_G(Z))$, $L \leq N_G(Y) \leq M_c$, contrary to our choice of L. This completes the treatment of cases (a)–(d) of 1.1.5.3.

Next suppose case (f) of 1.1.5.3 holds. Then $L/O_2(L)$ is sporadic, Z induces inner automorphisms on L, and $Z(S_L O_2(L)/O_2(L))$ is of order 2. Thus by paragraph one, Z induces $Z(S_L)$ on L and $C_L(Z(S_L)) \leq M_L$. Indeed if $L/Z(L)$ is not M_{22}, M_{23}, or M_{24}, then $C_L(Z(S_L))$ is a maximal subgroup, so that $C_L(Z(S_L)) = M_L$. Hence in these cases either conclusion (d) of (3) holds, or $L \cong J_4$, a case we postpone temporarily. Next assume $L/Z(L) \cong M_{22}$, M_{23}, or M_{24}. To complete our treatment of case (f) in these cases, we assume that $C_L(Z(S_L)) < M_L$, and derive a contradiction. Here since $F^*(M_L) = O_2(M_L)$ from paragraph one, the subgroup structure of L determines M_L uniquely as a block of type A_6, exceptional A_7, or $L_4(2)$, respectively. Therefore $[M_L, Z] \neq 1$ as $C_L(Z(S_L)) = C_L(Z_L) = C_L(Z)$. Then $M_L \leq M_c^\infty$, so 1.2.1.1 says M_c^∞ contains a member of $\mathcal{L}_f(G,T)$, contrary to Hypothesis 14.1.5.1.

To complete our treatment of case (f), we may assume $L \cong J_4$. Then there is $K \in \mathcal{L}(G_1, S)$ with $K \leq L$, $F^*(K) = O_2(K)$, and $K \cong M_{24}/E_{2^{11}}$. Now $K \leq M_L$ by (2). But then $L = \langle K, C_L(Z(S_L))\rangle \leq M_L$, contrary to our choice of $L \not\leq M_Z$.

Finally suppose case (e) of 1.1.5.3 holds. We have already treated the cases where $L \cong L_2(4) \cong L_2(5)$ and $L \cong L_3(2) \cong L_2(7)$. Thus L is either $L_3(3)$, or $L_2(p)$ for $p > 7$ a Fermat or Mersenne prime. By 15.1.19.7, $X := O^2(C_M(Z_1))$ acts on L, and V_2 acts nontrivially on L. Thus X is nontrivial on L since $V_2 = [V_2, X]$ by construction of V_2 in 15.1.16. This is impossible since S_L acts on X, whereas neither $L_3(3)$ nor $L_2(p)$ has a subgroup of odd index in which an element of odd order acts nontrivially on a normal elementary abelian 2-subgroup. (Cf. Dickson's Theorem A.1.3 in the case of $L_2(p)$). □

In the remainder of this section, we will eliminate cases from 15.1.20 until we have reduced to case (2c), at which point we will derive our final contradiction.

We begin by eliminating cases (2a) and (3ab) of 15.1.20:

LEMMA 15.1.21. *Assume $L \in \mathcal{C}(G_1)$. Assume further that either $F^*(L) = O_2(L)$ with L an $L_2(2^n)$-block or an A_5-block, or L is a component of G_1 with $L/O_2(L)$ a Bender group, $L_3(2^n)$ or $Sp_4(2^n)$. Then $L \leq M_Z$.*

PROOF. Set $S_L := S \cap L$, $M_L := L \cap M_Z$, and assume that $L \not\leq M_Z$. By 15.1.20, M_L is a either the Borel subgroup B_L of L over S_L, or a maximal parabolic of L. Set $L_0 := \langle L^S\rangle$; then $M_Z \cap L_0$ is either the Borel subgroup $B := \langle B_L^S\rangle$ of L_0 over $S \cap L_0$, or a maximal parabolic of L_0. So in any case, $B \leq M_Z$, and S normalizes B.

When L is a block, $L = [L, J(S)]$ by 15.1.20.2, so the action of $J(S)$ on L is described in Theorem B.4.2. When L is a component, $n > 1$ by 15.1.20.3. Then we conclude that one of the following holds:

(i) $J(S) \trianglelefteq SB$, and L is not an A_5-block.

(ii) $O^2(B) = [O^2(B), J(S)]$, and either $L \cong L_2(4)$ or L is an A_5-block.

(iii) $L \cong U_3(2^n)$, some $A \in \mathcal{A}(S)$ does not induce inner automorphisms on L, and $J(S) \trianglelefteq DS$, where D is the subgroup of B generated by all elements of order dividing $2^n - 1$.

Suppose that case (i) or (iii) holds. Let $B_0 := B$ in case (i), and $B_0 := D$ in case (iii). By 15.1.17, $J(T) = J(S)$ and $B_0 \leq N_G(J(S)) \leq M$. As $B \leq M_Z$, $B_0 \leq M \cap M_Z = M \cap M_c \cap G_1$, so by 15.1.18.2, $B_0 = O^2(B_0)$ centralizes V. Thus $V_2 \leq C_S(B_0) \leq C_S(L)$ from the structure of $Aut(L)$ in (i) or (iii), contrary to 15.1.19.7.

Therefore case (ii) holds. Now $J(T) = J(C_T(V))$ by 15.1.9.1, so that $M = C_M(V)N_M(J(T))$ by a Frattini Argument. Hence by construction of Z_1 and V_2 in 15.1.16, there is a p-subgroup Y of $N_M(J(T)) \cap G_1$ where $p := 3$ or 5, satisfying $SY = YS$ and $V_2 = [V_2, Y]$. Now $YS = SY$, Y acts on $J(T) = J(S)$, and $O^2(B) = [O^2(B), J(S)]$, so it follows from the structure of $Aut(L_0)$ that $[L, Y] = 1$. But then $V_2 = [V_2, Y]$ centralizes L_0, contrary to 15.1.19.7. This contradiction completes the proof. □

We now define notation in force for the remainder of this section: By 15.1.19.6, we may choose $L \in \mathcal{C}(G_1)$ with $L/O_2(L)$ quasisimple and $L \not\leq M_Z$. Set $M_L := M_Z \cap L$ and $S_L := S \cap L$. Then L is described in 15.1.20. In the next lemma, we refine that description.

LEMMA 15.1.22. *One of the following holds:*

(1) L is an A_7-block. Further $L = \langle M \cap L, M_L \rangle$, $M \cap L$ is a proper subgroup of L containing the stabilizer of the partition of type $2^3, 1$ stabilized by S_L, and M_L is the stabilizer of the vector of weight 2 in $C_{U(L)}(S_L)$.

(2) L is a block of type $L_3(2)$, A_6, or $G_2(2)$, M_L is the maximal parabolic subgroup of L centralizing Z, $M \cap L$ is the remaining maximal parabolic over S_L, and $C_S(O^{3'}(M \cap L)) = C_S(L)$.

(3) $F^*(L) = O_2(L)$ with $L/O_2(L) \cong L_4(2)$ or $L_5(2)$, and M_L and $M \cap L$ are proper parabolic subgroups of L which generate L.

(4) $L \cong G_2(2)'$, ${}^2F_4(2)'$, or ${}^3D_4(2)$, $M_L = C_L(Z(S_L))$, $M \cap L$ is the remaining maximal parabolic over S_L, and $C_S(O^{3'}(M \cap L)) = C_S(L)$.

(5) $L/Z(L)$ is J_2, He, or a Mathieu group other than M_{11}, $M_L = C_L(Z(S_L))$, and $C_S(O^{3'}(M \cap L)) = C_S(L)$.

(6) $L \cong M_{11}$, $M_L = C_L(Z(S_L))$, and $O^2(N_{G_1}(J(S))) \leq C_{G_1}(L)$.

(7) $L \cong L_4(2)$ or $L_5(2)$, $L = \langle M_L, M \cap L \rangle$, where M_L is a proper parabolic containing $C_L(Z(S_L))$, $M \cap L$ is a proper parabolic, and $C_S(O^{3'}(M \cap L)) = C_S(L)$.

PROOF. Observe that $M \cap L < L$ by 15.1.20.1, and $M_L < L$ since $L \not\leq M_Z$ by the choice of L. By 15.1.19.1, $Z(L)$ is a 2-group.

We first establish some preliminary technical results. The first is on overgroups of S_L in L. Let $\mathcal{P}$ be the set of $N_S(L)$-invariant subgroups P of L such that $F^*(P) = O_2(P)$, $PN_S(L)/O_2(PN_S(L)) \cong S_3$ or S_3 wr $\mathbf{Z}_2$, and $O^2(P)$ is not a product of A_3-blocks. We show:

(*) For $P \in \mathcal{P}$, either $P \leq M \cap L$ or $P \leq M_L$.

For let $P_0 := \langle P, S \rangle$; then P_0 is a minimal parabolic in the sense of Definition B.6.1, so as $O^2(P)$ is not a product of χ_0-blocks, we conclude from C.1.26 that either $C_1(S)$ centralizes P, or $C_2(S)$ is normalized by P. But $\mathrm{Baum}(T) = \mathrm{Baum}(S)$ by

15.1.17, so by Remark C.1.19 we may take $C_2(S) = C_2(T)$ and $C_1(T) \leq C_1(S)$. Thus $N_G(C_2(S)) \leq M$ by 15.1.17, while $C_G(C_1(S)) \leq M_c$ since $C_1(T) \leq Z$ and $M_c = !\mathcal{M}(C_G(Z))$. This establishes (*).

Next let $G_L := N_{G_1}(L)$, $G_L^+ := G_L/C_{G_1}(L)$, and $Z_L^+ := \Omega_1(Z(N_S(L)^+)))$. We establish:

(**) If $|Z_L^+| = 2$, then $Z^+ = Z_L^+$ and $C_{L^+}(Z^+) = C_L(Z)^+ \leq (M_Z \cap L)^+$. Further if $O^{3'}(M \cap L) \not\leq M_L$, then $C_S(O^{3'}(M \cap L)) = C_S(L)$.

For assume the hypotheses of (**). As $L \not\leq M_Z$ but $M_c = !\mathcal{M}(C_G(Z))$, L does not centralize Z, and hence $Z^+ \neq 1$. Therefore since $|Z_L^+| = 2$, $Z^+ = Z_L^+$, so $O^2(C_{L^+}(Z_L^+)) = O^2(C_L(Z)^+)$ by Coprime Action. Then as the Sylow 2-group S of G_1 centralizes Z, $C_{L^+}(Z_L^+) = C_L(Z)^+ \leq (M_Z \cap G_L)^+$. Therefore if $O^{3'}(M \cap L) \not\leq M_L$, then $O^{3'}(M \cap L)^+$ does not centralize Z_L^+. However, $N_S(L)$ acts on $D := C_S(O^{3'}(M \cap L))$, so if $D^+ \neq 1$, then $1 \neq Z_L^+ \cap D^+$. Then as $|Z_L^+| = 2$, Z_L^+ lies in D^+, and so centralizes $O^{3'}(M \cap L)^+$, contrary to the previous remark. This completes the proof of (**).

Our final preliminary result says:

(!) If $\mathcal{P}_0 \subseteq \mathcal{P}$ with $\langle \mathcal{P}_0 \rangle \not\leq M_L$, then $P \leq M$ for each $P \in \mathcal{P}_0$ with $P \not\leq M_L$. If in addition $|Z_L^+| = 2$, then $C_S(O^{3'}(M \cap L)) = C_S(L)$.

Under the hypothesis of (!), the first statement follows from (*), and so in particular $O^{3'}(M \cap L) \not\leq M_L$. Then the second statement follows from (**).

We now begin to show that one of the conclusions of the lemma must hold. By 15.1.21, cases (2a), (3a), and (3b) of 15.1.20 do not hold.

Suppose that case (2b) or (3c) of 15.1.20 holds, but L is not an A_7-block. Therefore either L is a block of type $L_3(2)$, A_6, or $G_2(2)$, or $L \cong G_2(2)'$, ${}^2F_4(2)$, or ${}^3D_4(2)$. In each case $N_S(L)$ is trivial on the Dynkin diagram of $L/O_2(L)$; when L is a block, this follows since $U(L)/C_{U(L)}(L)$ is the natural module. Thus each minimal parabolic over S_L is $N_S(L)$-invariant. Further in each case, $C_L(Z_L^+)$ is one of these minimal parabolics, with $M_L = C_L(Z_L^+)$ by 15.1.20. Let P denote the other minimal parabolic over S_L, and set $\mathcal{P}_0 := \{P\}$. As $O^2(P)$ is not an A_3-block, $\mathcal{P}_0 \subseteq \mathcal{P}$. Thus if we can show that $|Z_L^+| = 2$, then conclusion (2) or (4) of 15.1.22 will hold by (!). When L^+ is simple, this is a well-known fact (cf. 16.1.4 and 16.1.5) about the structure of $Aut(L)$, so we may assume L is a block. Here $F^*(L^+) = O_2(L)^+ = O_2(L)^+$ by A.1.8, so also $F^*(L^+N_S(L)^+) = O_2(L^+N_S(L)^+) =: Q_L^+$, and hence $Z_L^+ \leq Q_L^+$. But then $Z_L^+ = C_{U(L)^+}(N_S(L)^+)$ by Gaschütz's Theorem A.1.39. Thus $|Z_L^+| = 2$ from the action of L on $U(L)$, completing the proof that the lemma holds in this case.

Next we consider the remaining case in (2b) of 15.1.20, where L is an A_7-block. Here we adopt the notation of section B.3, let P denote the preimage of the stabilizer of the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}$, and set $\mathcal{P}_0 := \{P\}$. Again $\mathcal{P}_0 \subseteq \mathcal{P}$. Further by 15.1.20, M_L is the stabilizer of the vector $e_{5,6}$ of $U(L)$, and hence $P \not\leq M_L$, so $P \leq M$ by (!), completing the proof that conclusion (1) holds in this case.

Now assume that case (2c) or (3e) of 15.1.20 holds, so that $L/O_2(L) \cong L_4(2)$ or $L_5(2)$. Then $S = N_S(L)$ by 1.2.1.3. Let P_c denote the parabolic generated by the minimal parabolics for the interior nodes in the diagram for $L/O_2(L)$. In case (3e), $|Z_L^+| = 2$, and by 15.1.20, M_L is a proper parabolic containing $P_c = C_L(Z_L^+)$.

In case (2c), M_L is some proper parabolic. In any case, let

$$\mathcal{P}_0 := \{\langle P^S\rangle : \quad P \text{ is a minimal parabolic and } \quad P \not\leq M_L\}.$$

Now either $L^+S^+ \cong Aut(L_5(2))$ and $F^*(L) = O_2(L)$, or $\mathcal{P}_0 \subseteq \mathcal{P}$. In the latter case, conclusion (3) or (7) of 15.1.22 holds by (*) and (!), so we may assume the former case holds. Here $L = \langle P_c, P_e\rangle$, where P_e is the parabolic generated by the two end-node minimal parabolics, $P_e \in \mathcal{P}$, and $P_cS/O_2(P_cS) \cong Aut(L_3(2))$. As $P_e \in \mathcal{P}$, P_e is contained in $M_e \in \{M_L, M \cap L\}$ by (*). Then as P_eS is a maximal subgroup of LS, $M_e = P_e$.

If P_c centralizes Z, then $P_c \leq M_c$ as $C_G(Z) \leq M_c$, so $P_c = M_L$ by maximality of P_cS in LS. Then $P_e = M \cap L$ by the previous paragraph, so that conclusion (3) of 15.1.22 holds. Thus we may assume that $[Z, P_c] \neq 1$. Now $W := \langle Z^{P_c}\rangle \in \mathcal{R}_2(P_c)$ by B.2.14, so as $P_cS/O_2(P_cS) \cong Aut(L_3(2))$, and the latter group has no FF-module by Theorem B.5.1, we conclude that $J(S) \leq C_{P_cS}(W) = O_2(P_cS)$. Therefore $J(S) = J(O_2(P_cS))$ by B.2.3.3, and hence $P_c \leq N_G(J(S)) \leq M$ by 15.1.17. As $M \cap L < L$ and P_c is a maximal S-invariant subgroup of L, we conclude that $P_c = M \cap L$, and then $P_e = M_L$ by the previous paragraph, so again conclusion (3) of 15.1.22 holds.

Finally we assume that case (d) of 15.1.20.3 holds, so that L is a component of G_1 with $L/Z(L)$ sporadic, and $Z(L) = O_2(L)$.

Suppose first that $L/Z(L)$ is HS or Ru. Then there is $K \in \mathcal{L}(LS, S) \cap L$ with $F^*(K) = O_2(K)$ and $K/O_2(K) \cong L_3(2)$. Further $O_2(K) \cong \mathbf{Z}_4^3$ or 2^{3+8}, respectively, so KS is not among the conclusions of C.1.34. Hence by C.1.34, there is a nontrivial characteristic subgroup C of S normal in K. Then as $S \trianglelefteq T$ by 15.1.18.3, $\langle K, T\rangle \leq N := N_G(C)$. Then $K \leq N^\infty \leq M_c$ by 14.1.6.3; but this is impossible, as $K \not\leq C_L(Z(S_L))$, whereas $C_L(Z(S_L)) = M_L$ by 15.1.20.3.

Therefore $L/Z(L)$ is a Mathieu group, J_2, or He, and $M_L = C_L(Z(S_L))$ by 15.1.20.3. Assume first that $L/Z(L)$ is not M_{11}, and set $K := \langle M_L, \mathcal{P}\rangle$. Then from the structure of $Aut(L)$, either $K = L$, or $L/Z(L) \cong M_{22}$ and $K/Z(L) \cong A_6/E_{16}$. Moreover in the latter case, $K > M_L$ as we saw in our treatment of M_{22} during the proof of 15.1.20. Thus in any case there is $P \in \mathcal{P}$ with $P \not\leq M_L$, and as $|Z_L^+| = 2$ in these groups, (!) completes the proof that conclusion (5) holds.

It remains to treat the case $L/Z(L) \cong M_{11}$, where $L \cong M_{11}$ by I.1.3, and $L \trianglelefteq G_1$ by 1.2.1.3. Then $Out(L) = 1$, so that $G_1 = L \times C_{G_1}(L)$; in particular $J(S) = J(C_S(L)) \times J(S_L)$, and hence $N_{G_1}(J(S)) \leq N_{G_1}(J(S_L))$. Further $J(S_L) \cong D_8$, so that $N_L(J(S_L)) = S_L$, and hence $O^2(N_{G_1}(J(S))$ centralizes L, so that conclusion (6) holds.

This completes the proof of 15.1.22. □

LEMMA 15.1.23. *If case (6) of 15.1.7 holds, then* $p = 3$, *so* $\bar{M} \cong S_3$ *wr* $\mathbf{Z}_2$.

PROOF. Assume case (6) of 15.1.7 holds. If $p = 3$, then $\bar{M} \cong S_3$ wr $\mathbf{Z}_2$ by 15.1.10. So we may assume that $p = 5$, and it remains to derive a contradiction. Then $\bar{M}_0 \cong D_{10} \times D_{10}$. Hence there is a 5-group $Y \leq C_M(Z_1)$ with $SY = YS$ and $V_2 = [V_2, Y]$. Set $G_0 := N_{G_1}(L)$ and $G_0^* := G_0/C_{G_0}(L)$. By 15.1.19.7, YV_2 acts on L, and V_2 acts nontrivially on L. Thus as Y is faithful and irreducible on V_2, YV_2 is faithful on L. Thus comparing the list in 15.1.22 to the possibilities for $L/O_2(L)$ in A.3.15, we conclude $L \cong {}^2F_4(2)'$, and $Aut_Y(L) \leq Aut_P(L)$, where $P := C_L(Z(S_L))$ with $P/O_2(P) \cong Sz(2)$. This is impossible, as $Aut_{YS}(L)$ does not act irreducibly on an E_{16}-subgroup $Aut_{V_2}(L)$ of P. □

In the remainder of the section, let $Y := O^{3'}(G_1 \cap M)$. As G is a counterexample to Theorem 15.1.15, 15.1.23 says we are in case (4), case (6) with $p = 3$, or case (7) of 15.1.7, so that $\bar{M}_0$ is a $\{2,3\}$-group. In particular $Y \not\leq M_Z$ by 15.1.18.2.

LEMMA 15.1.24. *(1) Either case (4) of 15.1.7 holds, or case (6) of 15.1.7 holds with $p = 3$. In particular, $Z_1 \leq V_1$.*

(2) L is not an $L_3(2)$-block.

(3) For each $Y_0 = O^2(Y_0) \leq Y$ with $Y_0 \not\leq M_Z$, $V_2 = [V_2, Y_0]$ and $|Y_0 : C_{Y_0}(V_2)| = 3$. In particular $V_2 = [V_2, Y]$.

(4) $V_1 \leq C_S(Y)$.

PROOF. By construction of V_2 in 15.1.16, and since $p = 3$ when case (6) of 15.1.7 occurs, $\bar{Y}$ is of order 3 and $V_2 = [V_2, Y]$. Thus (3) follows from 15.1.18.2. Further from 15.1.16, in cases (4) and (6) of 15.1.7, $V_1 \geq Z_1$ and $\bar{Y}$ centralizes V_1, so (4) will follow once we prove (1).

To establish (1), we may assume that case (7) of 15.1.7 holds, and we must produce a contradiction. Let X_0 be the preimage in M of $Z(O^2(\bar{M}))$, $R := O_2(M \cap M_c)$, and $X_1 := O^2(\langle R^{X_0} \rangle)$. We apply 14.1.17 to M_c, X_0 in the roles of "M_1, Y_0" to conclude $\bar{X}_1 = \bar{X}_0 = [\bar{X}_0, R]$ and $[R, C_{X_1}(V)] \leq O_2(M)$. As $\bar{X}_1 = \bar{X}_0$, $Z_1 = [Z_1, X_0]$. Then as $\bar{X}_1 = [\bar{X}_1, R]$ and $[R, C_{X_1}(V)] \leq O_2(M)$, there is a subgroup X_2 of X_1 of order 3 with $Z_1 = [Z_1, X_2]$. So as $m_3(N_G(Z_1)) \leq 2$, A.3.18 eliminates the possibilities in 15.1.22 of 3-rank 2, leaving the case where L is an $L_3(2)$-block. Then by 1.2.1.3, $X_2 = O^2(X_2)$ normalizes L and $O^{3'}(G_1) = LO^{3'}(C_{G_1}(L/O_2(L)))$. Therefore as $X_2 \not\leq G_1$ and $m_3(N_{G_1}(Z_1)) \leq 2$, $L = O^{3'}(G_1)$.

Thus to establish both (1) and (2), it suffices to assume L is an $L_3(2)$-block. As usual let $U(L) := [O_2(L), L]$. By 15.1.22, M_L is the parabolic of L centralizing $Z_S := \Omega_1(Z(S_L))$, and $M \cap L$ is the remaining maximal parabolic of L over S_L. Let $Y_0 := O^2(M \cap L)$, so that as $L \leq G_1$, $Y_0 \leq Y$ but $Y_0 \not\leq M_L$; hence $V_2 = [V_2, Y_0]$ and $C_{Y_0}(V_2) = O_2(Y_0)$ by (3). Thus $V_2 \leq Z(O_2(Y_0)) \leq U(L)$, so $V_2 \leq [C_{U(L)}(O_2(Y_0)), Y_0] =: U_2$. As L is an $L_3(2)$-block, U_2 is of rank 2, so $V_2 = U_2$ is of rank 2. This eliminates cases (6) and (7) of 15.1.7, and in particular completes the proof of conclusions (1) and (4) as mentioned earlier, though not yet of (2). Thus case (4) of 15.1.7 holds. Further $V_1 \leq C_S(Y_0)$ by (4), and $C_S(Y_0) = C_S(L)$ as L is an $L_3(2)$-block, so L centralizes V_1. Since $Z_1 \leq V_1$ by (1), $C_G(V_1) \leq C_G(Z_1) = G_1$, and hence $L \in \mathcal{C}(C_G(V_1))$. Let $t \in T - S$ and $X \in Syl_3(Y_0^t)$; then X is of order 3, and by our construction of V_1 and V_2 in 15.1.16, $V_1 = [V_1, X]$ and $[V_2, X] = 1$. As $L \in \mathcal{C}(C_G(V_1))$, $Y_0^t = O^2(Y_0^t)$ acts on L by 1.2.1.3, and as X centralizes $V_2 = [U(L), Y_0]$, X centralizes L, since L is an $L_3(2)$-block. Then as $m_3(N_G(V_1)) \leq 2$ and $X \not\leq C_G(V_1)$, arguing as above, we conclude that $L = O^{3'}(C_G(V_1))$. Indeed as X centralizes L, so does $\langle X^{Y_0^t} \rangle = Y_0^t$. Then $U(L) \leq C_S(Y_0^t) = C_S(L^t)$, and by symmetry, $U(L)^t$ centralizes L. Thus $\langle L, T \rangle$ acts on $W := U(L)U(L)^t$. Then setting $N := N_G(W)$, $L \leq N^\infty \leq M_c$ by 14.1.6.3, contrary to the choice of L. This contradiction completes the proof of (2), and hence of the lemma. □

LEMMA 15.1.25. *(1) $L = O^{3'}(G_1)$.*

(2) L is not M_{11}.

(3) V_1 does not centralize L.

(4) $|S : C_S(V_1)| = 2$.

(5) L is not an A_7-block.

(6) $|T : S| = 2$.

PROOF. By 15.1.24.2, L is not an $L_3(2)$-block, while in all other cases of 15.1.22, $m_3(L)=2$; then we obtain (1) from A.3.18.

Let $G_1^* := G_1/C_{G_1}(L/O_2(L))$. By (1), $Y \leq L$, so $Y = O^{3'}(L\cap M)$. By 15.1.17, $J(S)=J(T)$ and $N_G(J(S)) \leq M$. By 15.1.9.1, $J(T)$ centralizes V, so by a Frattini Argument, $Y = C_Y(V)Y_0$, where $Y_0 := O^2(N_Y(J(S)))$. Now $C_Y(V) \leq M_c$ by 15.1.5.2, but we saw $Y \not\leq M_Z$, so $Y_0 \neq 1$. However if $L \cong M_{11}$, then as $Y_0 \leq O^2(N_{G_1}(J(S)))$, Y_0 centralizes L as case (6) of 15.1.22 holds, contradicting $1 \neq Y_0 \leq L$. Hence (2) is established.

Next recall by 15.1.24.1, that we are in case (4) or (6) of 15.1.7, so that $|T:S|=2$ from our construction in 15.1.16; thus (6) holds.

We turn to the proof of (3). Let $t \in T-S$; then $V_2^t = V_1$ since we are in case (4) or (6) of 15.1.7. By 15.1.24.3, $V_2=[V_2,Y]$, and so $V_1=[V_1,Y^t]$. Further a Sylow 3-group of L, and hence also of Y, is of exponent 3, so there is X of order 3 in Y^t faithful on V_1. However if V_1 centralizes L, then as $Z_1 \leq V_1$, $L = O^{3'}(C_G(V_1))$, while $X \not\leq L$ as X is faithful on V_1. This is a contradiction as $L = O^{3'}(N_G(V_1))$ by A.3.18, so (3) is established.

Part (4) follows from the action of $\bar{M}$ on V in cases (4) and (6) of 15.1.7; use 15.1.23 in case (6).

Finally suppose L is an A_7-block. Represent LS on $\Omega := \{1,\ldots,7\}$, and adopt the notation of section B.3. By 15.1.22, Y^*S^* contains the stabilizer P^* of the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}$. Let P be the preimage of P^* and $Y_1 := O^2(P)$. By 15.1.24.4, $V_1 \leq C_S(Y_1)$, and from the representation of LS on $U(L)$, $C_S(Y_1) \leq C_S(L)\langle u,s\rangle$, where $u := e_\theta$, $\theta := \{1,\ldots,6\}$, and $s^* := (1,2)(3,4)(5,6)$. Therefore as s^* does not induce a transvection on $U(L)$, we conclude from (4) that $V_1 \leq C_{LS}(U(L)) = O_2(LS)$. So as $V_1 \leq C_S(L)\langle u,s\rangle$, $V_1 \leq C_S(L)\langle u\rangle$, so V_1 centralizes $K := L_7^\infty$ with $K/O_2(K) \cong A_6$. As $Z_1 \leq V_1$, $K = O^{3'}(C_G(V_1))$ by (1), and then it follows from A.3.18 that $O^{3'}(N_G(V_1)) = K \leq C_G(V_1)$. This is a contradiction, as the subgroup X of order 3 defined earlier acts nontrivially on V_1. Hence the proof of (5) and of 15.1.25 is complete. □

LEMMA 15.1.26. *$F^*(L)=O_2(L)$ and $L/O_2(L) \cong L_4(2)$ or $L_5(2)$.*

PROOF. Assume otherwise. In cases (2), (4), (5), and (7) of 15.1.22, $C_S(O^{3'}(M\cap L)) = C_S(L)$. Thus by 15.1.24.4, $V_1 \leq C_S(Y) \leq C_S(L)$, contrary to 15.1.25.3. Further cases (1) and (6) of 15.1.22 were eliminated in parts (5) and (2) of 15.1.25, leaving only case (3) of 15.1.22, where the lemma holds. □

By 15.1.26, $F^*(LS)=O_2(LS)$. Let $U := \langle Z_2^L\rangle$ and $U_L := [U,L]$. By 15.1.24.1, case (4) or (6) of 15.1.7 holds, where by construction in 15.1.16, Z is a full diagonal subgroup of $Z_1 \oplus Z_2$, so $C_{G_1}(Z) = C_{G_1}(Z_2)$ and $S = C_T(Z_1) = C_T(Z_2)$. Thus $U \in \mathcal{R}_2(LS)$ by B.2.14. Set $(LS)^* := LS/C_{LS}(U)$, and recall $C_{LS}(U) = O_2(LS)$ since $L/O_2(L)$ is simple and $U \in \mathcal{R}_2(LS)$. From 15.1.20.2, $L=[L,J(S)]$, so that U is an FF-module for L^*S^*; then we conclude from Theorem B.5.1 and B.4.2 using I.1.6 that:

LEMMA 15.1.27. *One of the following holds:*

(1) U_L is the orthogonal module or its 7-dimensional cover for $L^* \cong L_4(2)$.
(2) U_L is a 10-dimensional irreducible for $L^* \cong L_5(2)$.
(3) U_L is the sum of the natural module and its dual.

(4) U_L is a sum of at most $n-1$ isomorphic natural modules for $L^* \cong L_n(2)$, where $n = 4$ or 5.

By B.2.14, $U = U_L C_U(L)$. Let Z_U be the projection of Z_2 on U_L with respect to this decomposition.

LEMMA 15.1.28. *U_L is a natural module and $M_L = C_L(Z)$ is the stabilizer of the point Z_U in U_L.*

PROOF. First by 15.1.18.5, $C(G_1, Q_1) = M_Z$, and Hypothesis C.2.3 is satisfied with G_1, Q_1, M_Z in the roles of "H, R, M_H". Thus by C.2.1.2, $O_2(LS) \leq Q_1$. Further L is normal in G_1 by 15.1.25.1, so we may apply C.2.7.2 to conclude that Q_1 contains an FF-offender on U.

As $C_{G_1}(Z) = C_{G_1}(Z_2)$, $C_{LS}(Z_U) = C_{LS}(Z) \leq M_Z$. That is, M_L is an S-invariant proper parabolic containing $C_L(Z_U)$.

Suppose case (1), (2), or (4) of 15.1.27 holds. Then $C_L(Z_U)$ is a maximal parabolic, acting irreducibly on $O_2(C_L(Z_U)^*)$, so by the previous paragraph $M_L = C_L(Z_U)$ and $Q_1^* = O_2(M_L^*)$. Therefore as Q_1^* contains an FF*-offender, we conclude from B.3.2 or B.4.2 that case (1) holds with U_L the natural module for L, so that the lemma holds in this case.

Thus we may assume case that (3) of 15.1.27 holds. Therefore $U = U_1 \oplus U_2$, where U_1 is a natural module for L^*, and U_2 is its dual. Let $Z_0 := C_{U_L}(S)$, so that $Z_U \leq Z_0$. Then either S acts on U_1 and U_2 with $Z_0 = Z_{U,1} \oplus Z_{U,2}$, where $Z_{U,i}$ is the point of U_i fixed by S_L, or else S is nontrivial on the Dynkin diagram of L^*, with $Z_0 = \langle z_1 z_2 \rangle$ where $Z_{U,i} := \langle z_i \rangle$. In either case, $C_L(Z_0)$ contains the parabolic P determined by the interior node(s) of the diagram for L^*. Thus as $C_L(Z_0) \leq C_L(Z_U) \leq M_L$, $Q_1^* \leq O_2(P^*)$. But then by B.4.9.2, Q_1^* contains no FF*-offenders, contrary to an earlier remark. □

LEMMA 15.1.29. *U_L is not a natural module.*

PROOF. By 15.1.24.1, we are in case (4) or (6) of 15.1.7. Adopt the notation of the proof of 15.1.28. By 15.1.28, the projection Z_U of Z_2 on U_L is of order 2. We saw $U = U_L C_U(L)$ and Z is a full diagonal subgroup of $Z_1 Z_2$ with $[Z_1, L] = 1$, $L \not\leq M_c$, and $C_G(z) \leq M_c$ for each $z \in Z^\#$. Thus Z projects faithfully on U_L, so $|Z| = 2$. Therefore case (4) of 15.1.7 holds, rather than case (6) with $p = 3$, so V_2 is of rank 2. As S acts on V_2, and $V_2 = [V_2, Y] \leq [U, L] \leq U_L$, it follows that V_2 is the line in U_L stabilized by S. Thus $N_L(V_2)$ contains the minimal parabolic P_0 of LS over S which is not contained in the maximal parabolic $M_L S = C_L(Z)S$. By 15.1.9.1, $J(T) \leq C_T(V_2)$, and by 15.1.17, $J(T) = J(S)$ and $N_G(J(S)) \leq M$. Hence $J(S) = J(C_S(V_2)) = J(O_2(P_0))$ using B.2.3.3, so that $P_0 \leq M$, and hence $Y_0 := O^2(P_0) \leq Y$ with $V_2 = [V_2, Y_0]$. Let P_2 be the minimal parabolic adjacent to P_0 with respect to the Dynkin diagram of L, let $Y_2 := O^2(P_2)$, and let $K := \langle Y_0, Y_2 \rangle$. Thus $KS/O_2(KS) \cong L_3(2)$ with KS the rank-2 parabolic corresponding to an end node and its neighbor, and $KS \cap M_L S = P_2$. Let $Q := C_T(V)$ and $t \in T - S$. Let P_3 be the remaining end-node minimal parabolic of L, and set $L_0 := O^{3'}(M_L)$. Thus $L_0 S/O_2(L_0 S) \cong L_{n-1}(2)$, and P_2 and P_3 are the end-node minimal parabolics of $L_0 S$. By 15.1.25.1 and 15.1.28, $L_0 = O^{3'}(C_{G_1}(Z))$, so as $O^2(C_G(Z)) \leq C_G(V_E)$ by 15.1.14.1, and $Z_1 \leq V_E$ by 15.1.18.1, $L_0 = O^{3'}(C_G(Z))$. Thus T acts on L_0. Hence as P_2 and P_3 are the end-node minimal parabolics of L_0, Y_2^t is either Y_2 or $Y_3 := O^2(P_3)$.

Next $O_2(P_0) = C_S(V_2)$, and as case (4) of 15.1.7 holds, $C_S(V_2) = C_T(V)$, so $Q = O_2(P_0)$. Thus from the structure of the rank-2 parabolics of LS: $O_2(Y_3) \leq Q$ so that $Q \in Syl_2(QY_3)$, as the nodes determining P_0 and P_3 are not adjacent in the diagram of L; but $Q \notin Syl_2(P_2)$, as the nodes determining P_0 and P_2 are adjacent. Therefore as t acts on Q, $Y_2^t \neq Y_3$, so $Y_2^t = Y_2$.

Let $Q_2 := O_2(P_2)$; as T acts on Y_2 and on S, T acts on Q_2. But $P_2 = C_{KS}(Z)$, and K has noncentral 2-chief factors on both $O_2(L)$ and $O_2(K)O_2(L)/O_2(L)$, so that K is not an $L_3(2)$-block. We conclude from C.1.34 that there is a nontrivial characteristic subgroup C of Q_2 with $C \trianglelefteq KQ_2$, and hence $C \trianglelefteq KT$. Then $H := \langle T, K\rangle \leq N_G(C)$. On the other hand, Y_0 centralizes Z_1 but $V_2 = [V_2, Y_0]$, so from our construction in 15.1.16, $M = \langle T, Y_0\rangle C_M(V)$. Then by A.5.7.1, $M = !\mathcal{M}(\langle T, Y_0\rangle)$. Since $Y_0 \leq K$, we conclude $H \leq N_G(C) \leq M$. But then $K = K^\infty \leq C_M(V)$ by 14.1.6.1, contrary to $V_2 = [V_2, Y_0]$. This contradiction completes the proof of 15.1.29. □

Observe that 15.1.28 and 15.1.29 supply a contradiction which establishes Theorem 15.1.15.

15.2. Finishing the reduction to $\mathbf{M_f/C_{M_f}(V(M_f)) \simeq O_4^+(2)}$

In this section, we complete the proof of Theorem 15.1.3, begun in section 15.1. Thus we assume G is a counterexample to Theorem 15.1.3.

15.2.1. Preliminary reductions. Recall we are assuming Hypothesis 14.1.5; in particular by 14.1.5.2,

$$M_c = !\mathcal{M}(C_G(Z)).$$

We continue Notation 15.1.4: namely we set $M := M_f$, and set $V := V(M)$ unless case (6) of 15.1.2 holds, where we set $V := [V(M), M_J]$. Also M_0 is the preimage in M of $\hat{J}(\bar{M}, V)$.

Since Theorem 15.1.15 eliminated cases (4), (6), and (7) of 15.1.7, we have reduced to the remaining cases in 15.1.7, which we summarize below for convenience:

LEMMA 15.2.1. *$m(V) = 4$, and one of the following holds:*

(1) $\bar{M} = \bar{M}_0 \cong S_3$.

(2) $\bar{M}_0 \cong S_3$ and $\bar{M} \cong S_3 \times \mathbf{Z}_3$.

(3) $\bar{M} = \bar{M}_0 = \Omega_4^+(V)$.

(4) $\bar{M}_0 \cong D_{10}$, $\bar{T} \cong \mathbf{Z}_2$ or $\mathbf{Z}_4$, and either $F(\bar{M}) = F(\bar{M}_0)$ or $F(\bar{M}) \cong \mathbf{Z}_{15}$. Furthermore if $V < V(M)$, then case (3) holds.

LEMMA 15.2.2. *If $T \leq X \leq M$ with $M_0 \leq XC_M(V)$ or $M_0 \leq XN_M(Z \cap V)$, then $M = !\mathcal{M}(X)$.*

PROOF. Let $M_1 \in \mathcal{M}(X)$. By 15.1.5.1, $V = \langle (Z \cap V)^X\rangle$, and by 15.1.9.2, $M_1 \lesssim M$, so $M_1 \leq M$ by A.5.6. □

LEMMA 15.2.3. *Let $R_c := O_2(M \cap M_c)$, $Y := O^2(\langle R_c^{O^2(M_0)T}\rangle)$, and $M^* := M/O_2(M)$. Then*

(1) $1 \neq \bar{Y} = [\bar{Y}, \bar{R}_c]$, and one of the following holds:

(i) $\bar{Y} = O^2(\bar{M}_0) \cong Y^*$. Further if case (3) of 15.2.1 holds, then $C_M(V)$ is a $3'$-group.

(ii) Case (3) of 15.2.1 holds, $\bar{R}_c \cong \mathbf{Z}_2$ *inverts* $O^2(\bar{M}_0) = O^2(\bar{M}) = \bar{Y}$, $Y^* \cong 3^{1+2}$, *and* $O^2(O_{2,Z}(Y))$ *is the subgroup* $\theta(C_M(V))$ *generated by all elements of* $C_M(V)$ *of order* 3.

(iii) Case (3) of 15.2.1 holds, $\bar{R}_c \cong \mathbf{Z}_2$ *inverts* $\bar{Y} \cong Y^* \cong \mathbf{Z}_3$, *and a Sylow* 3*-subgroup of* M *is isomorphic to* $\mathbf{Z}_3 \times \mathbf{Z}_{3^n}$ *for some* $n \geq 1$.

(2) $Y \trianglelefteq M$.

(3) If $\bar{Y} = O^2(\bar{M}_0)$, *then* $M = !\mathcal{M}(YT)$.

(4) $M = (M \cap M_c)O^2(M_0)$.

(5) $R_c^* Y^*$ *centralizes* $C_M(V)^*$.

PROOF. Part (3) follows from 15.2.2. Set $Y_0 := O^2(M_0)$. To establish the remaining parts, we apply case (b) of 14.1.17 with M_c in the role of "M_1". By 14.1.17: $\bar{R}_c \neq 1$, $\bar{Y} = [\bar{Y}_0, \bar{R}_c]$, and $R_c^* Y^*$ centralizes $C_M(V)^*$, so that (5) holds. As $\bar{R}_c \neq 1 = O_2(\bar{M})$ using 15.1.5.1, $\bar{Y} \neq 1$.

Assume for the moment that $\bar{Y}$ is cyclic. Then $\bar{Y}$ is of prime order from 15.2.1, and $\bar{Y}$ is inverted in $\bar{R}_c$. Hence as R_c^* centralizes $C_Y(V)^*$ by (5), we conclude $C_Y(V)^* = 1$, so that $\bar{Y} \cong Y^*$.

We next prove (1). If case (3) of 15.2.1 does not hold, then $\bar{Y}_0$ is of prime order, so $\bar{Y} = \bar{Y}_0$, and then conclusion (i) of (1) holds by the previous paragraph. Thus we may assume that case (3) of 15.2.1 holds. Since R_c^* is faithful on $F^*(M^*) \leq Y_0^* C_M(V)^*$, but R_c^* centralizes $C_M(V)^*$, R_c^* is faithful on $O_3(M^*)$, so that $m_3(C_M(V)) \leq 1$ by 14.1.17.4.

Suppose first that $\bar{Y} = \bar{Y}_0$, so that $\bar{Y} = O^2(\bar{M})$. Since $C_Y(V)^* \leq Z(Y^*)$ by (5), we conclude from A.1.21 and A.1.24 that either $Y^* \cong \bar{Y} \cong E_9$ or $Y^* \cong 3^{1+2}$. In the former case, conclusion (i) of (1) holds: for $C_M(V)^*$ is centralized by $Y^* \cong E_9$ by (5), so that $C_M(V)$ is a $3'$-group since $m_3(M) = 2$. In the latter case as $\bar{T} \cong E_4$, we conclude from (5) that $\bar{R}_c$ is the subgroup of order 2 in $\bar{T}$ which centralizes $C_Y(V)^*$, and hence inverts $\bar{Y}$; then since $m_3(C_M(V)) \leq 1$, conclusion (ii) of (1) holds.

Thus we may suppose that $\bar{Y} < \bar{Y}_0$. Therefore $\bar{Y}$ is of order 3, so $\bar{R}_c$ is of order 2, and $\bar{Y} \cong Y^*$ by the second paragraph of the proof. Since Y^* centralizes $C_M(V)^*$ and $m_3(C_M(V)) \leq 1$, conclusion (iii) of (1) holds, completing the proof of (1).

We next prove (4). First assume the subcase of case (4) of 15.2.1 where $\bar{T} \cong \mathbf{Z}_4$ and $O(\bar{M}) \cong \mathbf{Z}_{15}$ does not hold. In the remaining cases, $\bar{M} = \bar{Y}_0 N_{\bar{M}}(\bar{T})$, and $N_{\bar{M}}(\bar{T}) = \overline{N_M(T)}$ by a Frattini Argument, so as $N_M(T) \leq N_M(Z) \leq M_c = !\mathcal{M}(C_G(Z))$, (4) holds. Now consider the excluded subcase. By 15.1.13.4, $\overline{J_1(T)} = \bar{T} \cap \bar{M}_0$, so $\overline{J_1(T)}$ centralizes $O_3(\bar{M})$. Then $O^{3'}(M)$ acts on $V_E := C_V(J_1(T))$, so that $O^{3'}(M) \leq M_c$ by 15.1.14.2. Hence $M = YTO^{3'}(M) = Y_0(M \cap M_c)$, completing the proof of (4).

As $M \cap M_c$ acts on R_c and Y_0, $M \cap M_c$ and Y_0 act on $[Y_0, R_c] = Y$. Then as $M = (M \cap M_c)Y_0$ by (4), (2) holds. □

Recall that 15.1.12 describes the possible structures for $H \in \mathcal{H}_*(T, M)$. We next eliminate one subcase of 15.1.12.3:

LEMMA 15.2.4. *If* $H \in \mathcal{H}_*(T, M)$, *then* $H/O_2(H)$ *is not* S_5 *wr* $\mathbf{Z}_2$.

PROOF. Assume otherwise, define R_c and Y as in 15.2.3, set $M^* := M/O_2(M)$ and $X := O^2(H \cap M)$. By 15.1.9.7 we may apply E.2.2 to conclude that $X^* \cong E_9$. Further $X \leq C_M(V) \leq M \cap M_c$ by 15.1.9.4 and 15.1.5.2. Therefore case (3) of 15.2.1

does not hold, as in that case $m_3(C_M(V)) \leq 1$ by 15.2.3.1. Thus $\bar{Y} = O^2(\bar{M}_0) \cong Y^*$ by 15.2.3.1, so $Y^*X^* = Y^* \times X^*$ as Y^* centralizes $C_M(V)^*$ by 15.2.3.5. Then as M is an SQTK-group, $O^2(\bar{M}_0) \cong Y^*$ is a $3'$-group, so case (4) of 15.2.1 holds.

Next $K := O^2(H) = K_1K_1^t$ for $t \in T - N_T(K_1)$ and $K_1 \in \mathcal{C}(H)$ with $K_1/O_2(K_1) \cong A_5$ and $K_1 \not\leq M$. Observe that $F^*(K_1) = O_2(K_1)$ by 1.1.3.1. Let $X_i := X \cap K_i$ and $S := N_T(K_1)$, and observe that $O_2(XT) \leq S$, while $J(T) \leq S$ by 15.1.12.3. By 15.2.3.2, $O_2(Y) \leq O_2(M) \leq R_c$, and hence $R_c \in Syl_2(YR_c)$. Then as $X \leq M \cap M_c$, $R_c \leq O_2(XT) \leq S$, so $S \in Syl_2(G_1)$, where $G_1 := YX_1S$. Also $S \in Syl_2(G_2)$, where $G_2 := K_1S$. Let $G_0 := \langle G_1, G_2 \rangle$.

Suppose first that $O_2(G_0) = 1$. This assumption gives part (e) of Hypothesis F.1.1 with YR_c, K_1 in the roles of "L_1, L_2"; most other parts are straightforward, but we mention: As X^* centralizes $Y^*R_c^*$, $N_{K_1}(S \cap K_1) = X_1(S \cap K_1) \leq XS \leq N_G(YR_c)$. Recall that $Y = [Y, R_c]$ by construction in 15.2.3, so that $YR_c/O_2(YR_c) \cong D_{10}$ or $Sz(2)$. Thus the amalgam $\alpha := (G_1, X_1S, G_2)$ is a weak BN-pair of rank 2 by F.1.9. Furthermore as $S = N_{YS}(R_c)$, α is described in F.1.12. This is a contradiction, as $YR_c/O_2(YR_c) \cong D_{10}$ or $Sz(2)$ while $K_1/O_2(K_1)) \cong A_5$, and no such pair appears in F.1.12.

Therefore $O_2(G_0) \neq 1$. Let $S \leq T_0 \in Syl_2(G_0)$. We saw $J(T) \leq S$, so that $J(T) = J(S)$ by B.2.3.3. As $|T : S| = 2$, $|T_0 : S| \leq 2$, so that T_0 normalizes S. We conclude from 15.1.9.1 that $T_0 \leq N_G(J(T)) \leq M$, so that either $T_0 = S$ or $T_0 \in Syl_2(M)$. But in the latter case, $M = !\mathcal{M}(YT_0)$ by 15.2.3.3, so $K_1 \leq G_0 \leq M$, whereas we saw $K_1 \not\leq M$. Thus $S \in Syl_2(G_0)$. Let $\hat{G}_0 := G_0/O_2(G_0)$ and $M_1 := G_0 \cap M$. If $F^*(\hat{G}_i) = O_2(\hat{G}_i)$ for $i = 1$ and 2, then as above $(\hat{G}_1, \hat{X}_1\hat{S}, \hat{G}_2)$ is a weak BN-pair described in F.1.12, for the same contradiction as before. Thus either $\hat{Y} \cong \mathbf{Z}_5$ or $\hat{K}_1 \cong A_5$.

As $K_1 \in \mathcal{L}(G_0, S)$ and $S \in Syl_2(G_0)$, $K_1 \leq L_1 \in \mathcal{C}(G_0)$ by 1.2.4. As $K_1 \not\leq M$, $L_1 \not\leq M$. As S normalizes K_1, $L_1 \trianglelefteq G_0$ by 1.2.1.3. Indeed since $K_1 \leq L_1$ and $G_1 \leq M_1$, $G_0 = \langle G_1, G_2 \rangle = L_1M_1$. As $|T : S| = 2$ and $S \leq M_1$, $F^*(M_1) = O_2(M_1)$ by 1.1.4.7.

We claim that $G_0 \in \mathcal{H}^e$: If $\hat{Y} \cong \mathbf{Z}_5$, then $V = [V, Y] \leq O_2(Y) \leq O_2(G_0)$, so that $G_0 \in \mathcal{H}^e$ by 1.1.4.3. Suppose on the other hand that $\hat{K} \cong A_5$. We have seen that $G_0 = L_1M_1$ and $F^*(M_1) = O_2(M_1)$. Further $N_G(O_2(Y)) = M$ since $Y \trianglelefteq M$ and $M \in \mathcal{M}$. So it suffices by A.1.10 to show that $F^*(L_1) = O_2(L_1)$. Now as $K_1 \leq L_1$ and $\hat{K}_1 \cong A_5$, $O_2(K_1) \leq\leq O_2(L_1)$. Therefore $\hat{L}_1 \cong L_1/O_2(L_1)$ is quasisimple by 1.2.1.4. Then as $F^*(K_1) = O_2(K_1)$, L_1 does not centralize $O_2(L_1)$, so that $F^*(L_1) = O_2(L_1)$. This completes the proof of the claim that $G_0 \in \mathcal{H}^e$.

Let $R := O_2(YS)$. As Y and S are T-invariant, so is R; so as $M = !\mathcal{M}(YT)$ by 15.2.3.3, $C(G, R) \leq M$, and hence $C(G_0, R) \leq M_1$. Further as $Y \trianglelefteq M_1$, C.1.2.4 says $R \in \mathcal{B}_2(M_1)$ and $R \in Syl_2(\langle R^{M_1} \rangle$, so $R \in \mathcal{B}_2(G_0)$. Thus Hypothesis C.2.3 is satisfied with G_0, M_1 in the roles of "H, M_H".

Suppose first that $R \in Syl_2(RL_1)$. Then L_1 is a χ_0-block by C.2.5, so as $K_1 \in \mathcal{L}(L_1, S)$, we conclude from A.3.14 that $L_1 = K_1$. As $\hat{Y} = O^{5'}(\hat{Y})$ normalizes $\hat{R}$, it centralizes the Sylow group $\hat{R} \cap \hat{K}_1$ of $\hat{K}_1 = \hat{L}_1$, and hence centralizes $\hat{L}_1$. Therefore K_1 normalizes $O^2(YO_2(G_0)) = Y$, and hence $K_1 \leq N_G(Y) \leq M = !\mathcal{M}(YT)$, a contradiction. Thus R is not Sylow in RL_1. However if $Y \not\leq L_1$, then Y normalizes $YS \cap L_1 = S \cap L_1$, so $S \cap L_1 \leq O_2(YS) = R$, contradicting $R \notin Syl_2(RL_1)$. Therefore $Y \leq L_1$.

Suppose $\hat{L}_1$ is not quasisimple. Then by 1.2.1.4, $\hat{F}_1 := F(\hat{L}_1) = F^*(\hat{L}_1)$ is a $3'$-group, so the preimage F_1 of $\hat{F}_1$ lies in M_1 by C.2.6.2. Then $Y \leq F_1$: for otherwise $[F_1, Y] \leq F_1 \cap Y \leq F_1 \cap O_2(Y) \leq O_2(F_1)$, and then $L_1 = [L_1, Y]$ centralizes $\hat{F}_1$, contradicting $F^*(\hat{L}_1) = \hat{F}_1$. Therefore $\hat{Y} \leq O_5(\hat{L}_1)$, so $\hat{L}_1 \cong SL_2(5)/E_{25}$ by 1.2.1.4. In particular S is irreducible on $\hat{F}_1$, impossible as S acts on Y and $Y < F_1$.

Therefore $\hat{L}_1$ is quasisimple, so as $L_1 \trianglelefteq G_0$, L_1 is described in C.2.7.3. Further $\hat{Y}$ is an $\hat{S}$-invariant subgroup of $\hat{L}_1$ with $|\hat{Y} : O_2(\hat{Y})| = 5$, so we may apply A.3.15; comparing the list of A.3.15 with the list of C.2.7.3, we conclude $L_1/O_2(L_1) \cong L_2(2^n)$ or $SL_3(2^n)$ with $n \equiv 0 \mod 4$. This is impossible, as $K_1 \in \mathcal{L}(L_1S, S)$ with $K_1/O_2(K_1) \cong A_5$. This contradiction completes the proof of 15.2.4. □

We are now able to obtain the analogue of 14.2.2.5:

LEMMA 15.2.5. $\mathcal{M}(T) = \{M, M_c\}$.

PROOF. We assume $M_1 \in \mathcal{M}(T) - \{M, M_c\}$, and derive a contradiction. Set $H := M \cap M_1$ and $Z_V := Z \cap V$. As $M_c = !\mathcal{M}(C_G(Z))$, $C_{M_1}(V(M_1)) \leq C_G(Z) \leq M_c \geq N_G(Z_V)$. By 15.1.9.2, $M_1 \stackrel{<}{\sim} M$, so that $M_1 = HC_{M_1}(V(M_1))$, and hence as $C_{M_1}(V(M_1)) \leq M_c$ but $M_1 \not\leq M_c$, also $H \not\leq M_c$. Thus $H \not\leq N_M(Z_V)$, so as $C_M(V) \leq M_c$ by 15.1.5.2, it follows that $\bar{H} \not\leq \bar{M}_c$, so that $\bar{T} < \bar{H}$. On the other hand if $O^2(M_0) \leq HN_M(Z_V)$, then $M = !\mathcal{M}(H)$ by 15.2.2, contrary to $H \leq M_1 \neq M$. Thus $O^2(M_0) \not\leq HC_M(V)$, so that $\bar{H} < \bar{M}$.

Now if either case (1) or (2) of 15.2.1 holds, then $|M : N_M(Z_V)| = 3$ is prime, so as $H \not\leq N_M(Z_V)$, $M = N_M(Z_V)H$, which is contrary to the previous paragraph. Similarly in case (4) of 15.2.1, as $O^2(\bar{M}_0) \not\leq \bar{H}$ and $\bar{M} > \bar{H} > \bar{T}$, $F^*(\bar{M})$ has order 15, and $\bar{H} = O_3(\bar{M})\bar{T}$. But $\bar{M} = \overline{M \cap M_c}O^2(\bar{M}_0)$ by 15.2.3.4, so $\bar{H} = O_3(\bar{M})\bar{T} = \overline{M \cap M_c}$, contrary to the previous paragraph.

Thus case (3) of 15.2.1 holds, so as $\bar{M} > \bar{H} > \bar{T}$, $\bar{H} \cong \mathbf{Z}_2 \times S_3$. Set $M^* := M/O_2(M)$ and $R := O_2(H)$.

Suppose for the moment that $V < V(M)$. Then from Notation 15.1.4, case (6) of 15.1.2 holds. Let M_J denote the preimage in M of $\hat{J}(Aut_M(V(M)), V(M))$, and $V_J := C_{V(M)}(M_J)$; by 15.1.4, $V = [V(M), M_J]$, and by 15.1.2.6, $V(M) = V \times V_J$ with $V_J \neq 1$, $C_M(V)C_M(V_J)T = M_c$, and $|M : M \cap M_c| = 3$ is prime. So as $H \not\leq M_c$, $M = H(M \cap M_c) = HC_M(V)C_M(V_J)$ in this case. We now drop the assumption that $V < V(M)$.

Suppose that $\bar{R} = 1$. Observe that hypothesis (a) of 14.1.17 is satisfied with $V(M)$, $O^2(M)$ in the roles of "V, Y_0" so as $\bar{R} = 1$, we conclude $V < V(M)$ from 14.1.17.1, and we adopt the notation of the previous paragraph. As $\bar{R} = 1$, $R \leq C_M(V)$, so $[C_M(V_J), R] \leq C_M(V(M))$. Also $R = O_2(RC_M(V(M)))$ by 14.1.17.5 applied to $V(M)$ in the role of "V", so $C_M(V_J) \leq N_M(R)$. From the previous paragraph, $M = HC_M(V)C_M(V_J)$, so $M = C_M(V)N_M(R)$, and hence $M = !\mathcal{M}(N_M(R))$ by 15.2.2. Therefore $C(G, R) \leq M$, so $C(M_1, R) = H$, and hence $M_1 = !\mathcal{M}(H)$ by 14.1.16, contrary to $H \leq M \neq M_1$.

Therefore $\bar{R} \neq 1$. Since $\bar{R} \leq O_2(\bar{H})$ with $\bar{H} \cong S_3 \times \mathbf{Z}_2$, $\bar{R} = O_2(\bar{H})$ is of order 2. Let Y_0 denote the preimage in M of $O(\bar{H})$; then $\bar{H} = \bar{Y}_0\bar{T}$. As $O^2(\bar{M})$ is abelian and $T \leq H$, $\bar{Y}_0$ of order 3 is normal in $\bar{M}$.

Set $R_1 := O_2(M_1 \cap M_c)$, $V_1 := V(M_1)$, $\hat{M}_1 := M_1/C_{M_1}(V_1)$, and $M_1^+ := M_1/O_2(M_1)$. Recall $\hat{M}_1 = \hat{H}$ and $C_{M_1}(V_1) \leq M_c$, so $M_1 = (M_1 \cap M_c)H$, and $O^2(\hat{H}) \neq 1$ as $M_1 \not\leq M_c$.

We next construct a subgroup Y_1 of H with $\hat{Y}_1$ of order 3, $Y_1 \trianglelefteq M_1$, and $M_1 = (M_1 \cap M_c)Y_1$.

Suppose first that $V = V(M)$. Then by A.5.3.3, $C_M(V) \le C_{M_1}(V_1) \cap M \le H$, so as $\bar{H} = \bar{Y}_0\bar{T}$, $Y_0 \le H$ and $H = Y_0T$. Therefore $\hat{Y}_0 \trianglelefteq \hat{Y}_0\hat{T} = \hat{H} = \hat{M}_1$. Further $\hat{Y}_0 \neq 1$ as $O^2(\hat{H}) \neq 1$, so as $\bar{Y}_0$ has order 3 and $C_M(V) \le C_{M_1}(V_1)$, we conclude that $\hat{Y}_0$ has order 3. In this case let Y_1 be the preimage in M_1 of $\hat{Y}_0$, so that $\hat{Y}_1 = \hat{Y}_0$ has order 3 and $\hat{M}_1 = \hat{Y}_1\hat{T}$, so that $M_1 = Y_1(M_1 \cap M_c)$ and $Y_1 \trianglelefteq M_1$.

Suppose instead that $V < V(M)$. Then by our earlier discussion, $M = H(M \cap M_c)$ with $|M : M \cap M_c| = 3$. Thus $M = Y_0(M \cap M_c)$, and $\bar{Y}_0$ is the unique $\bar{T}$-invariant subgroup of $\bar{M}$ of order 3 not contained in $\overline{M \cap M_c}$. Let $R_c := O_2(M \cap M_c)$, and define Y as in 15.2.3. As $|M : M \cap M_c| = 3$, $\bar{Y} = [O^2(\bar{M}_0), \bar{R}_c] < O^2(\bar{M}_0) \cong E_9$, so case (iii) of 15.2.3.1 holds, and $Y^* \cong \bar{Y} = [O^2(\bar{M}_0), \bar{R}_c]$. By the uniqueness of $\bar{Y}_0$ mentioned above, $\bar{Y} = \bar{Y}_0$. Therefore $YC_M(V) = (Y_0 \cap H)C_M(V)$. Now R_c acts on $Y_0 \cap H$, $Y^*R_c^*C_M(V)^* = Y^*R_c^* \times C_M(V)^*$ by 15.2.3.5, and $Y^* = [Y^*, R_c^*]$ since $Y^* \cong \bar{Y} = [\bar{Y}, R_c]$. Thus $Y \le H$, so as $|M : M \cap M_c| = 3$ is prime, $H = Y(H \cap M_c)$. Then as we saw $M_1 = H(M_1 \cap M_c)$, $M_1 = Y(M_1 \cap M_c)$. As $Y \trianglelefteq M$ and $\hat{M}_1 = \hat{H}$, $\hat{Y} \trianglelefteq \hat{M}_1$. As $Y \not\le M_c \ge C_{M_1}(V_1) \ge C_M(V_1)$, $\hat{Y} \neq 1$, so as Y^* has order 3 and $C_Y(V) \le C_{M_1}(V_1)$, we conclude $\hat{Y}$ has order 3. In this case, let Y_1 be the preimage in M_1 of $\hat{Y}$, so that $\hat{Y}_1 = \hat{Y}$ has order 3, and $M_1 = Y_1(M_1 \cap M_c)$. This completes the definition of Y_1 in our second case.

Now in either case we have the hypotheses of case (b) of 14.1.17, with M_1, M_c, R_1, Y_1 in the roles of "M, M_1, R, Y_0". We claim $\hat{Y}_1 = [\hat{Y}_1, R_1]$: For otherwise $\hat{R}_1$ is normal in $\hat{Y}_1\widehat{M_1 \cap M_c} = \hat{M}_1$, whereas $O_2(\hat{M}_1) = 1$ by B.2.14. Set $Y_2 := O^2(\langle R_1^{Y_1}\rangle) = O^2(\langle R_1^{Y_1T}\rangle)$, so that Y_2 plays the role of "Y" in 14.1.17. Since R_1 is normal in $M_1 \cap M_c$ and $M_1 = Y_1(M_1 \cap M_c)$, we have $Y_2 = O^2(\langle R_1^{M_1}\rangle)$ normal in M_1, so that $M_1 = N_G(Y_2)$ as $M_1 \in \mathcal{M}$. To complete the proof, we will show that $Y_2 \trianglelefteq M$, so that $M = N_G(Y_2) = M_1$, contrary to our choice of $M_1 \neq M$.

Since $\hat{Y}_1 = [\hat{Y}_1, \hat{R}_1]$ is of order 3, $\hat{Y}_2 = \hat{Y}_1 \cong \mathbf{Z}_3$. But by 14.1.17.3, $C_{Y_2}(V_1)^+$ centralizes R_1^+, so $Y_2^+ \cong \hat{Y}_2 \cong \mathbf{Z}_3$. Moreover $Y_2C_{M_1}(V_1) = Y_1C_{M_1}(V_1)$, so arguing as above when we showed $Y \le H$, we conclude $Y_2 \le H$. Further $Y_2^+ = [Y_1^+, R_1^+]$.

Suppose first that $V < V(M)$. Then by construction $Y \le Y_1$ and $\hat{Y} = \hat{Y}_1 = [\hat{Y}_1, R_1]$, so that $O_2(Y) = C_Y(V_1)$ as $Y^* \cong \mathbf{Z}_3$. Then as R_1 acts on Y, $Y = [Y, R_1]$. Thus $\mathbf{Z}_3 \cong Y_2^+ = [Y_1^+, R_1] \ge [Y^+, R_1] = Y^+ \cong \mathbf{Z}_3$, so $Y_2^+ = Y^+$. Then $Y_2 = O^2(Y_2O_2(M_1)) = O^2(YO_2(M_1)) = Y \trianglelefteq M$ by 15.2.3, completing the proof in this case.

Thus we may assume that $V = V(M)$. Here we saw that $C_M(V) \le H$, so $C_M(V) \le M_1 = N_G(Y_2)$, and $[R, C_M(V)] \le O_2(H) \cap C_M(V) \le O_2(C_M(V)) \le O_2(M)$, so that R^* centralizes $C_M(V)^*$. Further $\bar{R}$ centralizes $O(\bar{H})$, so case (ii) of 15.2.3.1 does not hold, since there $Y^* \cong 3^{1+2}$, in which case involutions not inverting $O^2(\bar{M})$ do not centralize $C_Y(V)^*$. Therefore a Sylow 3-subgroup P of M is abelian by 15.2.3.1. Choose P with $X := P \cap Y_2 \in Syl_3(Y_2)$. Then P centralizes X and normalizes $C_M(V)$. Now we saw $C_M(V) \le C_{M_1}(V_1)$ and $Y_2 \trianglelefteq M_1$, so $\langle X^{C_M(V)}\rangle = Y_2$. Hence P acts on Y_2 so $HP = M$ acts on Y_2, completing the proof. □

Lemma 15.2.6. *Define R_c and Y as in 15.2.3. Then there exists a T-invariant subgroup $Y_1 := O^2(Y_1)$ of Y such that*

(1) $Y_1R_c/O_2(Y_1R_c) \cong S_3$*,* D_{10}*, or* $Sz(2)$*.*
(2) $C_{Y_1}(V) = O_2(Y_1)$*.*
(3) $M = !\mathcal{M}(Y_1T)$*.*

PROOF. Assume case (3) of 15.2.1 does not hold. Here we take $Y_1 := Y$. Then by 15.2.3.1, $\bar{Y} = [\bar{Y}, \bar{R}_c] = O^2(\bar{M}_0)$ and $C_Y(V) = O_2(Y)$, so (2) holds, and (3) follows from 15.2.3.3. Conclusion (1) follows from the structure of $\bar{M}_0$ described in 15.2.1.

So assume that case (3) of 15.2.1 holds. In case (iii) of 15.2.3.1, we again choose $Y_1 := Y$, so that $\bar{Y} = [\bar{Y}, \bar{R}_c]$ is of order 3. In cases (i) and (ii) of 15.2.3.1, we choose Y_0 to be the preimage of a T-invariant subgroup Y_0^* of Y^* of order 3 with $\bar{Y}_0 = [\bar{Y}_0, \bar{R}_c]$ of order 3, and set $Y_1 := O^2(Y_0)$. In each case Y_1 satisfies (1) by construction. In case (iii) of 15.2.3.1, $\bar{Y}_1 = \bar{Y} \cong Y^*$, so (2) holds; in the remaining cases we chose Y_1 with $\bar{Y}_1 \cong Y^*$, so again (2) holds. Finally as $Y_1 = [Y_1, R_c]$, $Y_1 \not\leq M_c$, so (3) follows from 15.2.5. □

LEMMA 15.2.7. *Define* Y *as in 15.2.3 and* Y_1 *as in 15.2.6. Then*
(1) $M = !\mathcal{M}(YT)$*.*
(2) If $1 \neq X = O^2(X) \leq C_M(V)$ *is* T*-invariant, then*
(i) $N_G(X) \leq M$*, and*
(ii) if $|X : O_2(X)| = 3$*, then* X *acts on* Y_1*.*

PROOF. Part (1) follows from 15.2.6.3 as $Y_1 \leq Y$. Assume X satisfies the hypotheses of (2); to prove (2), it suffices by (1) to show that Y acts on X, and that X acts on Y_1 if $|X : O_2(X)| = 3$. Let $M^* := M/O_2(M)$. As T acts on $X = O^2(X)$ and $Y_1 = O^2(Y_1)$, it suffices to show that Y^* acts on X^*, and that X^* acts on Y_1^* if $|X : O_2(X)| = 3$. But as $Y \trianglelefteq M$ by 15.2.3.2, $[X, Y] \leq C_Y(V)$, so if $C_Y(V) = O_2(Y)$, then $[X^*, Y^*] = 1$, and the lemma holds. Thus by 15.2.3.1, we may assume that case (ii) of 15.2.3.1 holds. Then $[X^*, Y^*] \leq Z(Y^*)$ with $Z(Y^*)$ of order 3. Thus if X^* is a $3'$-group, then $X^* = O^2(X^*)$ centralizes Y^* by Coprime Action, and as before the lemma holds. Finally if X^* is not a $3'$-group, then $Z(Y^*) \leq X^*$ as case (ii) of 15.2.3.1 holds, so $[X^*, Y^*] \leq Z(Y^*) \leq X^*$, and once again Y^* acts on X^*. Also if $|X : O_2(X)| = 3$, then $X^* = Z(Y^*)$ acts on Y_1^*, completing the proof. □

LEMMA 15.2.8. *If* $H \in \mathcal{H}_*(T, M)$*, then* $H/O_2(H) \cong S_3$ *wr* $\mathbf{Z}_2$*.*

PROOF. First $H/C_H(U_1)$ is described in 15.1.12.3, where U_1 is a noncentral chief factor for H on $U_H := \langle V^H \rangle$. In particular $O_2(H/C_H(U_1)) = 1$, so $O_2(H) \leq C_H(U_1)$. Recall by 15.1.9.7 that H is a minimal parabolic described by B.6.8; thus $H \cap M = N_H(T \cap H)$ by 3.1.3.1, and $C_H(U_1) \leq H \cap M$ by B.6.8.6a. If $C_H(U_1) > O_2(H)$, then $X := O^2(C_H(U_1)) \neq 1$, so that $X \leq C_M(V)$ by 15.1.9.4; hence by 15.2.7.2, $H \leq N_G(X) \leq M$, contrary to $H \in \mathcal{H}_*(T, M)$. Therefore $O_2(H) = C_H(U_1)$.

Thus $H/C_H(U_1) = H/O_2(H)$ is described in 15.1.12.3. By 15.2.4, $H/O_2(H)$ is not S_5 wr $\mathbf{Z}_2$, and the lemma holds if $H/O_2(H)$ is S_3 wr $\mathbf{Z}_2$, so we may assume that $H/O_2(H) \cong S_3$ or S_5, and it remains to derive a contradiction.

Define R_c and Y as in 15.2.3, and Y_1 as in 15.2.6. We will verify that Hypothesis F.1.1 is satisfied with Y_1R_c, H, T in the roles of "L_1, L_2, S". Most parts are straightforward, but we give a few details: First $Y_1R_c/O_2(Y_1R_c) \cong S_3$, D_{10}, or $Sz(2)$ by 15.2.6.1, while we saw $H/O_2(H) \cong S_3$ or S_5, so that part (c) holds. Next

$M = !\mathcal{M}(Y_1T)$ by 15.2.6.3, so that $O_2(\langle Y_1T, H\rangle) = 1$, and hence part (e) holds. To verify part (d), we must show that $H \cap M$ normalizes Y_1R_c. By 15.1.9.3, $H \leq M_c$, so $H \cap M$ acts on R_c. By 15.1.9.4, $O^2(H \cap M) \leq C_M(V)$, so $O^2(H \cap M)$ acts on Y_1 by 15.2.7.2.

Hence $\alpha := (Y_1(H \cap M), H \cap M, H)$ is a weak BN-pair of rank 2 by F.1.9, and since $N_{Y_1R_c}(T) = T$, α appears in the list of F.1.12. Now U_H is abelian by 15.1.11.2, and H has two noncentral 2-chief factors on U_H by 15.1.12.1, which are natural modules for $H/O_2(H) \cong S_n$ for $n = 3$ or 5 by 15.1.12.3. But these conditions are not satisfied by any member of F.1.12. □

We now define notation which will be in force for the remainder of the section:

NOTATION 15.2.9. Pick $H \in \mathcal{H}_*(T, M)$, and let $Q_H := O_2(H)$, $U_H := \langle V^H\rangle$, and $H^* := H/O_2(H)$. Recall in particular by 15.1.9.3 that

$$H \leq M_c.$$

By 15.1.12.1, H has exactly two noncentral chief factors U_1 and U_2 on U_H. By 15.2.8, $H^* \cong S_3$ wr $\mathbf{Z}_2$. Thus by 15.1.12.4, $m(U_i) = 4$ and $H^* = O_4^+(U_i)$, so $U_i = U_{i,1} \oplus U_{i,2}$ with $U_{i,j} \cong E_4$, $j = 1, 2$, the two definite 2-dimensional subspaces of the othogonal space U_i. Also $H^* = (H_1^* \times H_2^*)\langle t^*\rangle$, where t^* is an involution with $H_1^t = H_2$ and $H_i^* \cong S_3$. This choice for H_1 and H_2 is not unique, but 15.1.12 supplies us with a distinguished choice: Pick $H_i := C_H(U_{1,3-i})$. In particular the subgroups H_i^*, $i = 1, 2$ contain the transvections in H^* on U_1. Let $K_i := O_2(H_i)$ and $K := O^2(H)$.

Next let Δ consist of those $A \in \mathcal{A}(H)$ such that A^* is minimal subject to $A \not\leq Q_H$. By 15.1.12.2, for each $A \in \Delta$, A^* is an FF*-offender on U_1 and U_2. From B.2.9.1 and the description of FF*-offenders in B.1.8.4, A^* is of order 2 by minimality of A^*, so A^* induces transvections on both U_1 and U_2. Thus A lies in either H_1 or H_2, and we can choose notation so that also $H_i = C_H(U_{2,3-i})$. Then $U_{j,i} = [U_j, H_i]$ and $U_{j,1}^t = U_{j,2}$.

For $A \in \Delta$, let $B(A) := A \cap Q_H$; thus $|A : B(A)| = |A^*| = 2$. Let $\Sigma := \{B(A) : A \in \Delta\}$.

Observe by 15.2.8 that $T = M \cap H = N_H(V)$, so $|V^H| = 9$. For $h \in H$, let $\Delta(V^h) := \Delta \cap T^h$, $\Delta'(V^h) := \Delta - \Delta(V^h)$; $\Sigma(V^h) := \{B(A) : A \in \Delta(V^h)\}$, and $\Sigma'(V^h) := \Sigma - \Sigma(V^h)$.

LEMMA 15.2.10. *Let $\bar{D} \in \mathcal{Q}(\bar{T}, V)$. Then*

(1) $\bar{D} \leq \bar{M}_0$ and $m(\bar{D}) = 1$.
(2) $m([V, D]) = 2$.
(3) $[V, D] \trianglelefteq T$.

PROOF. By 15.2.1, $\mathcal{Q}(\bar{T}, V) \subseteq \Omega_1(\bar{T}) \leq \bar{M}_0$. Then the lemma follows easily from 15.2.1: For example (3) follows as $\bar{T}$ is abelian, and in case (3) of 15.2.1, $m(\bar{D}) = 1$ since $\bar{D}$ acts quadratically on V. □

LEMMA 15.2.11. *Let $B \in \Sigma'(V)$ and $Z_S := [V, B]$. Then*

(1) $\bar{B} \in \mathcal{Q}(\bar{M}, V)$.
(2) $Z_S \leq Z(K)$.
(3) $E_4 \cong Z_S \trianglelefteq T$.

PROOF. Recall $B = B(A)$ for some $A \in \Delta'(V)$, and we may assume without loss that $A \leq H_1$. Let $X := \langle \Delta \cap T \cap H_1 \rangle$ and $I_1 := \langle X^{H_1} \rangle$; then $X \trianglelefteq N_T(H_1)$. As $A \leq H_1$, $K_1 = [K_1, A] \leq I_1$. Also $H_1 = Q_H \langle X, A \rangle$, so $I_1 = \langle X, A \rangle$.

We saw $T = N_H(V)$, so A does not normalize V, and hence $[V, A] \neq 1$. But $B \leq O_2(H)$, so B does normalize V, and by 15.1.12.2, $C_{U_H}(A) = C_{U_H}(B)$; so $[V, B] \neq 1$. Therefore (1) holds by 15.1.12.2, and then (3) follows from 15.2.10. Recall A^* is of order 2; thus $1 = m(A/B) = m(B/C_B(U_H))$ by 15.1.12.2. Also 15.1.12.2 shows A is quadratic on U_H, so $Z_S = [V, B]$ is centralized by A. Further as $\Delta \subseteq \mathcal{A}(H)$, $X \leq J(T) \leq C_G(V)$ by 15.1.9.1, so Z_S is centralized by $\langle X, A \rangle = I_1$. Thus by (3), $Z_S \trianglelefteq \langle I_1, T \rangle = H$, so $K = \langle K_1^H \rangle \leq C_G(Z_S)$, and hence (2) holds. □

Recall from Notation 15.2.9 that $H \leq M_c$. Set $K_c := \langle K^{M_c} \rangle$.

LEMMA 15.2.12. *Let $B \in \Sigma'(V)$ and $Z_S := [V, B]$. Then*

(1) $K \in \Xi(G, T)$.

(2) One of the following holds:

(i) $K_c = K$.

(ii) $K_c \in \mathcal{L}^*(G, T) = \mathcal{C}(M_c)$ and $K_cT/O_2(K_cT) \cong Aut(L_n(2))$, $n = 4$ or 5.

(iii) $K_c = LL^t$ with $L \in \mathcal{L}^*(G, T) = \mathcal{C}(M_c)$ and $L/O_2(L) \cong L_2(2^n)$, n even, or $L_2(p)$ for some odd prime p.

(3) $M_c = !\mathcal{M}(H)$ and $Z_S \trianglelefteq H$, so $N_G(Z_S) \leq M_c$.

(4) $K_c = O^{3'}(M_c) \leq C_G(Z)$.

(5) Case (6) of 15.1.2 does not hold, so $V = V(M)$.

PROOF. Part (1) is immediate from 15.2.8. By 1.3.4, either $K = K_c$, or $K_c = \langle L^T \rangle$ for some $L \in \mathcal{C}(M_c)$ described in (1)–(4) of 1.3.4. Suppose the latter holds. By 14.1.6.2, $L \in \mathcal{L}^*(G, T)$. As $Aut_T(K/O_2(K)) \cong D_8$, we conclude from 1.3.4 that either $LT/O_2(LT) \cong Aut(L_n(2))$, $n = 4$ or 5, or $L < K_c$ and $L/O_2(L) \cong L_2(2^n)$ or $L_2(p)$. Therefore (2) is established.

By 15.2.5, $M_c = !\mathcal{M}(H)$, while by (2) and (3) of 15.2.11, H normalizes Z_S, so (3) holds. In case (i) of (2), as $Aut_T(K/O_2(K)) \cong D_8$ and $Aut(K/O_2(K)) = GL_2(3)$, it follows as $T \in Syl_2(N_G(K))$ that $Aut_G(K/O_2(K)) = Aut_T(K/O_2(K))$, so $O^{3'}(M_c) = KO^{3'}(C_{M_c}(K/O_2(K)))$. Therefore as $K/O_2(K) \cong E_9$ and $m_3(M_c) \leq 2$, we conclude $K_c = K = O^{3'}(M_c)$. In cases (ii) and (iii) of (2), we obtain $K_c = O^{3'}(M_c)$ using A.3.18 and 1.2.2.a. If $K < K_c$, then $K_c = K_c^\infty$ centralizes Z by 14.1.6.3. If $K = K_c$, this follows from 15.1.9.3. This completes the proof of (4).

By (4), $O^{3'}(M \cap M_c) \leq C_M(Z)$. However in case (6) of 15.1.2, $|M : M \cap M_c| = 3$ and $O^2(\bar{M}) \cong E_9$ with $\bar{T} = C_{\bar{M}}(Z \cap V)$. This contradiction establishes (5). □

LEMMA 15.2.13. *(1) Either*

(i) case (1) or (4) of 15.2.1 holds, with $\bar{M} \cong S_3$, D_{10}, or $Sz(2)$, or

(ii) case (3) of 15.2.1 holds, and $O(\bar{M}) = [O(\bar{M}), B]$ for each $B \in \Sigma'(V)$.

(2) Let $\bar{B}_0$ be the unique subgroup of $\bar{T}$ of order 2 with $O(\bar{M}) = [O(\bar{M}), \bar{B}_0]$. Then for each $B \in \Sigma'(V)$, $\bar{B} = \bar{B}_0$, and $C_V(B) = [V, B] = [V, \bar{B}_0] = C_V(\bar{B}_0)$.

PROOF. Assume conclusion (i) of (1) does not hold. Then either one of cases (2) or (3) of 15.2.1 holds, or case (4) of 15.2.1 holds with $F^*(\bar{M}) \cong \mathbf{Z}_{15}$. Pick $B \in \Sigma'(V)$. By 15.2.11.1 and 15.2.10.1, $\bar{B}$ is of order 2. Then either $\bar{X} := C_{O(\bar{M})}(\bar{B}) \cong \mathbf{Z}_3$, or case (3) of 15.2.1 holds with $O(\bar{M}) = [O(\bar{M}), \bar{B}]$. Assume the former case holds. Then we compute that $\bar{X}$ acts faithfully on $[V,B] =: Z_S$, so $X \le N_G(Z_S) \le M_c$ by 15.2.12.3. Hence $X \le C_G(Z)$ by 15.2.12.4, impossible as X is nontrivial on Z_S, and $1 \neq Z \cap Z_S$. Therefore the latter case holds for each $B \in \Sigma'(V)$, and hence conclusion (ii) of (1) holds. This completes the proof of (1). Then (1) implies (2). □

For the remainder of the section, we define $\bar{B}_0$ and $Z_S := [V,\bar{B}_0]$ as in 15.2.13.2. Thus $Z_S = [V,B]$ for each $B \in \Sigma'(V)$ by 15.2.13.2. Let $S := C_T(Z_S)$.

Lemma 15.2.14. *(1)* $M_c = C_G(Z)$.
(2) $Z \le Z_S$, *and either*
(a) $S = T$ *and* $Z = Z_S \cong E_4$, *or*
(b) $\bar{M} \cong Sz(2)$ *or* $\Omega_4^+(V)$, Z *is of order* 2, *and* $|T:S| = 2$.
(3) $Baum(T) \le S$.
(4) $Z_S = \Omega_1(Z(S))$.
(5) $M \cap M_c = C_M(Z) = C_M(V)T$.
(6) $m(\bar{M},V) = 2$ *and* $a(\bar{M},V) = 1$.

Proof. As $\bar{M}$ is solvable, $a(\bar{M},V) = 1$ by E.4.1. Then by inspection of the cases in 15.2.13.1, and recalling $V = V(M)$ by 15.2.12.5 so that $Z \le V$, (2) and (6) hold, and $C_{\bar{M}}(Z) = \bar{T}$. Recall $C_M(V) \le M_c$ by 15.1.5.2. In case (i) of 15.2.13.1, $\bar{T}$ is maximal in $\bar{M}$, so $C_M(V)T = M \cap M_c$ as $M_c \not\le M$. In case (ii) of 15.2.13.1, this holds as $O^{3'}(M \cap M_c) \le C_M(V)$ by 15.2.12.4. Thus (5) is established. Further $M_c = (M \cap M_c)C_{M_c}(V(M_c))$ since $M_c \lesssim M$ by 15.1.9.2, so M_c centralizes Z by (5), and hence (1) holds.

If $Z_S = Z$, then $S = T$ so (3) and (4) are trivial; thus we may assume that $\bar{M} \cong Sz(2)$ or $\Omega_4^+(2)$ with Z of order 2. Then $\mathrm{Baum}(T) \le C_T(V) \le S$ by 15.1.9.1, completing the proof of (3). Finally $Z_S \le \Omega_1(Z(S)) =: Z_0$ and $\mathbf{Z}_2 \cong Z = C_{Z_0}(T)$; so as T/S is of order 2, $m(Z_0) \le 2m(Z) = 2 = m(Z_S)$ using 15.2.11.3, and hence $Z_0 = Z_S$, establishing (4). □

15.2.2. A uniqueness theorem. This subsection is devoted to establishing the following uniqueness theorem:

Theorem 15.2.15. $M_c = !\mathcal{M}(C_{M_c}(Z_S))$.

The proof of Theorem 15.2.15 involves a series of reductions. Until it is complete, we assume $I \in \mathcal{H}(C_{M_c}(Z_S))$ with $I \not\le M_c$, and work toward a contradiction. Set $M_I := M_c \cap I$ and $N_I := M \cap I$. In particular

$$M_I < I.$$

Since $Z \le Z_S$ by 15.2.14.2, and $M_c = C_G(Z)$ by 15.2.14.1, while we chose $C_{M_c}(Z_S) \le I$:

Lemma 15.2.16. $C_G(Z_S) = C_{M_c}(Z_S) \le M_I$.

Recall $H \le M_c$ by Notation 15.2.9; so as $Z_S \le Z(K)$ by 15.2.11.2, $KS \le M_I$. As $C_{M_c}(Z_S) \le I \not\le M_c$ and $M_c = C_G(Z)$, $Z < Z_S$. Hence case (b) of 15.2.14.2 holds, so by that result:

Lemma 15.2.17. *(1)* $\bar{M} \cong Sz(2)$ *or* $\Omega_4^+(2)$.
(2) $|Z| = 2$.
(3) $|T:S| = 2$.

LEMMA 15.2.18. *(1)* $S \in Syl_2(I)$.

(2) $B := Baum(T) = Baum(S)$ *and* $C(I, B) \leq N_I$.

(3) Either

(i) $N_I \leq M_I$, *or*

(ii) $N_I \not\leq M_I$, *case (ii) of 15.2.13.1 holds, with* $\bar{M} = \Omega_4^+(V)$, *and* $N_I = C_M(Z_1)$ *is of index* 6 *in* M, *for some complement* Z_1 *to* Z *in* Z_S.

PROOF. Recall Z_S is of order 4 by 15.2.11.3. Let $S \leq T_I \in Syl_2(I)$. By 15.2.14.3 and B.2.3.5, $B = \mathrm{Baum}(S) = \mathrm{Baum}(T_I)$, and $C(G, B) \leq M$ by 15.1.9.1. Thus (2) holds and also $T_I \leq N_I(B) \leq M$, so as $N_{\bar{M}}(\bar{S}) = \bar{T}$ and $|T : S| = 2$ by 15.2.17, T_I is either T or S. But if $T_I = T$, then $I \leq M$ by 15.2.5 since $I \not\leq M_c$, and we saw $K \leq M_I$; but this is contrary to $H \not\leq M$. Thus (1) is established.

Next using 15.2.16, $C_M(V) \leq C_M(Z_S) \leq M_I$, so if $O^2(N_I) \leq C_M(V)$, then conclusion (i) of (3) holds. Thus we may assume $X := O^2(N_I) \not\leq C_M(V)$.

Suppose $\bar{X} \trianglelefteq \bar{M}$. Then T acts on $SXC_M(V)$ and K, so T acts on $G_0 := \langle SXC_M(V), K\rangle$. Now $O_2(I) \leq S \leq G_0 \leq I$, so $TG_0 \in \mathcal{H}(T)$. But by 15.2.14.5, $M \cap M_c = C_M(V)T$, so as $X \not\leq C_M(V)$, $TG_0 \not\leq M_c$. Hence $M = !\mathcal{M}(TG_0)$ by 15.2.5, so $K \leq G_0 \leq M$, contrary to $H \not\leq M$.

Therefore $\bar{X}$ is not normal in $\bar{M}$, so case (ii) of 15.2.13.1 holds, and $\bar{X}$ is one of the two subgroups of $O(\bar{M})$ of order 3 not normal in $\bar{M}$. Thus conclusion (ii) of (3) holds, completing the proof. □

Recall $C_1(S)$ from Definition C.1.18.

LEMMA 15.2.19. *Define* $B := Baum(S)$.

(1) If conclusion (i) of 15.2.18.3 holds, then $C(I, B) \leq M_I \geq C_I(C_1(S))$.

(2) If conclusion (ii) of 15.2.18.3 holds, then Hypothesis C.2.3 is satisfied with I, $O_2(N_I)$, N_I *in the roles of "*H, R, M_H*".*

PROOF. Assume first that conclusion (i) of 15.2.18.3 holds. Then $N_I \leq M_I$, so $C(I, B) \leq N_I \leq M_I$ by 15.2.18.2. Further $C_1(S) \trianglelefteq T$ as $S \trianglelefteq T$ by 15.2.17.3, so $1 \neq Z \cap C_1(S)$ and hence $C_I(C_1(S)) \leq C_G(Z \cap C_1(S)) \leq M_c = !\mathcal{M}(C_G(Z))$, completing the proof of (1).

Next assume conclusion (ii) of 15.2.18.3 holds, and let $R := O_2(N_I)$. Then $\bar{N}_I \cong S_3$, so $R \leq C_M(V)$. But $C_M(V) \leq C_M(Z_S) \leq N_I$, so $R = O_2(C_M(V)) = O_2(M)$, and hence $C(I, R) \leq N_I$. Then as $R = O_2(N_I)$, the remaining two conditions of Hypothesis C.2.3 are trivially satisfied, so (2) holds. □

LEMMA 15.2.20. *(1) The hypotheses of 1.1.5 are satisfied with* I, M_c *in the roles of "*H, M*" for each* $z \in Z^\#$.

(2) $F^*(M_I) = O_2(M_I)$.

(3) $O(I) = 1$.

PROOF. Let $I_0 \in \mathcal{M}(I)$; then part (1) holds for I_0 in the role of "I" by 1.1.6. Then by 15.2.18.1, S is Sylow in I and I_0. In particular, $O_2(I_0 \cap M_c) \leq O_2(I \cap M_c) \leq S$ by A.1.6. Hence as I_0 satisfies the hypotheses of 1.1.5,

$$C_{O_2(M_c)}(O_2(I \cap M_c)) \leq C_{O_2(M_c)}(O_2(I_0 \cap M_c)) \leq T \cap I_0 = S \leq I,$$

and so (1) holds. Then (2) follows from (1) and 1.1.5.1. As usual $1 \neq U_K := [U_H, K]$ centralizes $O(I)$ by A.1.26.1, and $Z \leq U_K$ since $Z = \Omega_1(Z(T))$ has order 2 by 15.2.17.2, so (3) follows from 1.1.5.2. □

Lemma 15.2.21. *(1) If S is not irreducible on $K/O_2(K)$ then $KS = H_1H_2$, $K_1^t = K_2$ for $t \in T - S$, and K_c centralizes Z_S, so that $K_c = O^{3'}(M_I)$.*

(2) If $K = K_c$ then $C_M(V)$ is a $3'$-group.

(3) Assume $K_c/O_2(K_c) \cong L_4(2)$, conclusion (ii) of 15.2.18.3 holds, and there is an S-invariant subgroup $Y_1 = O^2(Y_1)$ of N_I with $Y_1S/O_2(Y_1S) \cong S_3$ and $Y_1 \not\leq M_I$. Then S is irreducible on $K/O_2(K)$.

Proof. Assume S is not irreducible on $K/O_2(K)$. From Notation 15.2.9, $K^*T^* \cong O_4^+(2)$, so T^* is irreducible on K^* and hence $S^* < T^*$. But $|T : S| = 2$ by 15.2.17.3, so $O_2(KT) \leq S$. As $J(T) \leq S$ by 15.2.18.2, and $|T^* : J(T)^*| = 2$ by 15.1.12.3, $J(T)^* = S^*$. Further $J(T)^* \in Syl_2(H_1^*H_2^*)$ by 15.1.12.3, so $KS = H_1H_2$, and hence $K_1^t = K_2$ for $t \in T - S$.

Recall $K_c = \langle K^{M_c}\rangle = O^{3'}(M_c)$ is described in 15.2.12.2. If $K = K_c$, then (1) holds by 15.2.11.2, so we may assume $K < K_c$. Then $K_c = \langle L^T\rangle$ for some $L \in \mathcal{C}(M_c)$ described in 15.2.12.2. Now using A.1.6, $O_2(K_cT) \leq O_2(KT) \leq S = C_T(Z_S)$, so as $L \notin \mathcal{L}_f(G,T)$ by Hypothesis 14.1.5.1, $K_c \leq C_G(Z_S) \leq M_I$ by 1.2.10 and 15.2.16, completing the proof of (1).

Assume in addition the hypotheses of (3); we will obtain a contradiction to our assumption that S is not irreducible on $K/O_2(K)$, and hence establish (3). As $|Y_1 : O_2(Y_1)| = 3$ and $Y_1 \leq N_I$ but $Y_1 \not\leq M_I$, $\bar{Y}_1 \cong \mathbf{Z}_3$. By hypothesis, case (ii) of 15.2.18.3 holds, so $N_I = C_M(Z_1)$ is of index 6 in M, for some complement Z_1 to Z in Z_S; in particular, $\bar{Y}_1 = O(\bar{N}_I)$ is not T-invariant. Define Y as in 15.2.3, and set $\hat{M} := M/O_2(M)$. If case (iii) of 15.2.3.1 holds, then $\hat{Y} \cong \mathbf{Z}_3$ is T-invariant, so $YY_1/O_2(YY_1) \cong \bar{Y}\bar{Y}_1 \cong E_9$, and hence $C_M(V)$ is a $3'$-group as $m_3(M) \leq 2$.

Next KS is the maximal parabolic subgroup of K_cS determined by the end nodes of the Dynkin diagram for $K_c/O_2(K_c)$. Let $Y_0 := O^2(P)$, where P is the minimal parabolic determined by the middle node. If $Y_0T \not\leq M$, then $Y_0T \in \mathcal{H}_*(T,M)$, contrary to 15.2.8; hence $Y_0T \leq M$, so $Y_0 \leq C_M(V)$ by 15.2.14.5. Thus $C_M(V)$ is not a $3'$-group, so case (ii) of 15.2.3.1 holds by the previous paragraph. Therefore $\hat{Y} \cong 3^{1+2}$ and $\hat{Y}_0 = Z(\hat{Y})$. Now $\bar{S}$ inverts $\bar{Y}_1$ which is not $\bar{T}$-invariant, so as $|T : S| = 2$, $\bar{S}$ is the subgroup of order 2 of $\bar{T}$ inverting $\bar{Y}$, and so $\hat{S}$ centralizes $\hat{Y}_0$, This is impossible, as $K_c \leq M_I$ by (1), so $S \in Syl_2(K_cS)$ by 15.2.18.2, and then $SY_0/O_2(SY_0) \cong S_3$. So (3) is established.

Finally assume $K = K_c$. Then as $M \cap K = O_2(K)$ by 15.2.8, and $K = O^{3'}(M_c)$ by 15.2.12.4, we conclude $O^{3'}(M \cap M_c) = 1$, so (2) holds. □

Lemma 15.2.22. *Assume conclusion (ii) of 15.2.18.3 holds, $F^*(I) = O_2(I)$, and $C_M(V)$ is a $3'$-group. Assume S is not irreducible on $K/O_2(K)$, and there is $Y_1 = O^2(Y_1) \leq N_I$ which is S-invariant with $Y_1S/O_2(Y_1S) \cong S_3$. Then $[Y_1, K_2] \not\leq Y_1 \cap K_2$.*

Proof. Assume $[Y_1, K_2] \leq Y_1 \cap K_2$. Then for $t \in T - S$, $[Y_1^t, K_1] \leq Y_1^t \cap K_1$ as $K_2^t = K_1$ by 15.2.21.1. Next as conclusion (ii) of 15.2.18.3 holds, $\bar{N}_I = C_{\bar{M}}(Z_1)$ is of index 6 in $\bar{M} \cong \Omega_4^+(V)$ for some complement Z_1 to Z in Z_S. Then as $Y_1 \leq N_I$ and $C_M(V)$ is a $3'$-group, $\bar{Y}_1 = O_3(\bar{N}_I)$, $\bar{Y}_1\bar{Y}_1^t = O(\bar{M}) \cong E_9$, and M has Sylow 3-subgroups isomorphic to E_9. Define Y as in 15.2.3; as $Y \trianglelefteq M$ but Y_1 is not T-invariant, YY_1 contains a Sylow 3-subgroup of M, so by a Frattini Argument, we may take t to act on YY_1, and thus $YY_1 = Y_1Y_1^t$ with $[Y_1, Y_1^t] \leq Y_1 \cap Y_1^t$.

Let $X := \langle Y_1, K_1\rangle$; then $[X, X^t] \leq X \cap X^t$, in view of the commutator relations established in first sentence of the proof, along with the relations $[Y_1, Y_1^t] \leq Y_1 \cap Y_1^t$

and $[K_1, K_2] \le K_1 \cap K_2$. Now S acts on X, $F^*(I) = O_2(I)$ by hypothesis, and $S \in Syl_2(I)$ by 15.2.18.1; thus $F^*(XS) = O_2(XS)$ by 1.1.4.4. Then $F^*(X) = O_2(X)$ by 1.1.3.1, so that $O_2(X) \neq 1$. It follows that $O_2(XX^t) \neq 1$. Then as T acts on XX^t, $XX^tT \in \mathcal{H}(T)$. This is a contradiction, as $Y_1Y_1^tT \le XX^tT$ and $M = !\mathcal{M}(Y_1Y_1^tT)$ by 15.2.2, so that $K = K_1K_2 \le M$, contrary to $H \not\le M$. □

Recall $C_{M_c}(Z_S) = C_G(Z_S)$ by 15.2.16. During the remainder of the proof of Theorem 15.2.15, take I minimal subject to $I \in \mathcal{H}(C_G(Z_S))$ and $I \not\le M_c$. Recall $M_I < I$, so as $M_I = I \cap M_c$, M_I is a maximal subgroup of I.

For $X \le G$, let $\theta(X)$ be the subgroup generated by all elements of X of order 3.

Lemma 15.2.23. *$F^*(I) \neq O_2(I)$.*

Proof. We assume $F^*(I) = O_2(I)$ and derive a contradiction. We begin with some preliminary reductions.

Suppose first that there is $X_0 = O^{3'}(X_0) = X_0^\infty \trianglelefteq I$ with X_0 nontrivial on $Z_0 := \Omega_1(Z(O_2(X_0)))$ and $X_0 \le M_c$. As $X_0 = O^{3'}(X_0)$, $X_0 \le K_c$ by 15.2.12.4. Since $S \in Syl_2(I)$ by 15.2.18.1, $S \cap O_2(K_c) = I \cap O_2(K_c)$, so X_0 acts on $S \cap O_2(K_c)$. Therefore as $|T : S| = 2$ by 15.2.17.3, $|O_2(K_c) : S \cap O_2(K_c)| \le 2$, so $[O_2(K_c), X_0] \le S \cap O_2(K_c) \le N_G(X_0)$, Thus $X_0 = (X_0O_2(K_c))^\infty$ is $O_2(K_c)$-invariant. Hence $[X_0, O_2(K_c)] \le O_2(X_0) \le C_G(Z_0)$, so by the Thompson $A \times B$-lemma, X_0 is nontrivial on $C_{Z_0}(O_2(K_c))$. But since $K_c \in \mathcal{H}^e$ by 1.1.3.1, $C_{Z_0}(O_2(K_c) \le \Omega_1(Z(O_2(K_c))) =: Z_c$. Hence K_c is nontrivial on Z_c, so that $K_c \in \mathcal{L}_f(G,T)$ by 1.2.10, contradicting part (1) of Hypothesis 14.1.5. Thus no such X_0 exists.

Next assume that either X is a χ-block of I, or $X \in \mathcal{C}(I)$ with $X/O_2(X) \cong L_3(2)$ and X is described in C.1.34. Set $X_0 := \langle X^S \rangle$ and $U_0 := \langle Z_S^{X_0} \rangle$. Observe that $X_0 \trianglelefteq I$ either by 1.2.1.3, or when X is an A_3-block since $|X^I| \le m_3(I) \le 2$. If X is an A_7-block, then $X = O^{3'}(I)$ by A.3.18, so $K \le X$. In the remaining cases, $m_3(X) = 1$ and K acts on X, so as $m_3(KX) \le 2$, $K_0 := O^2(K \cap X) \neq 1$.

Consider first the subcase where X is an A_3-block or an $L_2(2^n)$-block. By 15.2.11.2, Z_S centralizes K, so that $Z_S \cap U_0$ centralizes $\langle K_0^S \rangle = X_0$; this is impossible, as $Z_S \cap U_0 \not\le C_{U_0}(X_0)$ in these blocks (cf I.2.3.1).

This leaves the subcases where either X is an A_5-block or an A_7-block, or $X/O_2(X) \cong L_3(2)$. Then $X_0 \not\le M_c$ by paragraph two, so $M_0 := X_0 \cap M_I < X_0$. Notice $C_{X_0}(Z_S) \le M_0$ by 15.2.12.3. If X is an A_5-block, then $C_{X_0}(Z_S)$ is a Borel subgroup of X_0, so M_0 is that Borel subgroup. If X is an A_7-block, then we saw $K \le X$, so $K \le M_0$ by 15.2.11.2, and hence $M_0 = K(X \cap S)$ is the maximal subgroup stabilizing the partition $\{\{1,2,3,4\},\{5,6,7\}\}$, using the notation of section B.3. Finally if $X/O_2(X) \cong L_3(2)$, then since $C_{X_0}(Z_S) \le M_0 < X_0$, case (5) of C.1.34 is eliminated by B.4.8.2, as in that case Z_S centralizes X_0. Thus $C_{X_0}(Z_S)$ is a maximal parabolic of X_0, so M_0 is that maximal parabolic.

Let $Q_I := O_2(KS)$; in each case $M_0 \trianglelefteq KS$, so $Q_I = O_2(M_0Q_I)$. Further M_0 contains a Sylow 2-group of X_0, so $O_2(X_0Q_I) \le Q_I$ by A.1.6. Next $Q_I \trianglelefteq H$ as $|T : S| = 2$, so $C(I, Q_I) \le M_I$ by 15.2.12.3. Then as $M_0 < X_0$, $J(Q_I) \not\le O_2(X_0Q_I))$, so there is an FF*-offender in $Aut_{Q_I}(U_0)$ by B.2.10. Hence by B.3.2.4, X is not an A_5-block or an A_7-block. Further cases (2)–(4) of C.1.34 are eliminated since $C_{X_0}(Z_S) \le M_0$, leaving case (1) of C.1.34 where X is an $L_3(2)$-block. Thus we have shown that I possesses no χ-blocks, and if $X \in \mathcal{C}(I)$ with $X/O_2(X) \cong L_3(2)$ and

X is described in C.1.34, then X is an $L_3(2)$-block. This completes our preliminary reductions.

Suppose first that conclusion (i) of 15.2.18.3 holds. Then by 15.2.19.1 and C.1.28, $I = M_I L_1 \cdots L_s$, where L_i is a χ-block. But then $M_I = I$ by the previous paragraph, a contradiction.

Therefore conclusion (ii) of 15.2.18.3 holds. Set $R := O_2(N_I)$. Then case (2) of 15.2.19 holds, so Hypothesis C.2.3 is satisfied by I, R, N_I in the roles of "H, R, M_H". If $O_{2,F}(I) \not\leq N_I$, then by C.2.6, there is an A_3-block X of I with $X \not\leq N_I$, contrary to an earlier reduction. Thus $O_{2,F}(I) \leq N_I$. On the other hand, if $O_{2,F^*}(I) \leq N_I$, then $R = O_2(I)$ by A.4.4.1, contradicting $N_I = N_I(R)$ and $K \not\leq N_I$. Thus there is $X \in \mathcal{C}(I)$ with $X/O_2(X)$ quasisimple and $X \not\leq N_I$. Further $K = O^2(K)$ normalizes X by 1.2.1.3.

If R does not act on X, then X is a χ-block by C.2.4, contrary to an earlier reduction. Thus R acts on X, so X is described in C.2.7.3. Let $M_X := M \cap X$, and this time set $M_0 := M_I \cap X$, so that $M_1 := C_X(Z_S) \leq M_0$ by 15.2.12.3.

Suppose that case (g) of C.2.7.3 holds. Then $X/O_2(X) \cong SL_3(2^n)$, M_X is a maximal parabolic of X, and (XR, R) is an MS-pair described in C.1.34. Assume first that $n > 1$. Then $M_1 = P^\infty$, where P is the maximal parabolic of XS over S other than M_X. As $m_3(KC_X(Z_S)) = 2$, $K_0 := O^2(K \cap M_1) \neq 1$; then as S acts on K_0, n is even. Then $O_{2,Z}(X) > O_2(X)$, so as $m_3(KO_{2,Z}(X)) = 2$, $O_{2,Z}(X) \leq K$. This is impossible, as $O_{2,Z}(X) \leq M_X$ while $K \cap M = O_2(K)$. Therefore $n = 1$, so by an earlier reduction, X is an $L_3(2)$-block and $M_0 = M_1$ is the maximal parabolic of X over $S \cap X$ other than M_X. If $X < \langle X^S \rangle =: X_0$, then $M_X S/O_2(M_X S) \cong O_4^+(2)$. This is impossible, as $\bar{M}_X \bar{S} \cong S_3$ since conclusion (ii) of 15.2.18.3 holds, while $O_4^+(2)$ has no such quotient group. Thus $X \trianglelefteq I$ by 1.2.1.3, so by minimality of I, $I = M_I X$. Then S is not irreducible on $K/O_2(K)$ since $m_3(K \cap X) = 1$, so $K_c \leq M_I$ by 15.2.21.1. Further $KS = H_1 H_2$ by 15.2.21.1, so

(*) K_1 and K_2 are the S-invariant subgroups K_+ of K with $|K_+ : O_2(K_+)| = 3$.

As $m_3(KX) = 2 = m_3(K)$, by (*) we may assume $K_1 = O^2(K \cap X)$. Then by another application of (*), $[X, K_2] \leq O_2(X)$. Further $K_1 = O^2(M_0) \trianglelefteq M_I$, so that $K = K_c$ by 15.2.12.2. Thus $C_M(V)$ is a $3'$-group by 15.2.21.2, so $O^{3'}(N_I) =: Y_1 = O^{3'}(M_X)$, and Y_1 is S-invariant with $Y_1 S/O_2(Y_1 S) \cong S_3$. As $[X, K_2] \leq O_2(X)$, $[Y_1, K_2] \leq K_2 \cap Y_1$, contrary to 15.2.22.

Thus case (g) of C.2.7.3 is eliminated. An earlier reduction showed that X is not a χ-block; this eliminates case (a) of C.2.7.3, and the subcases of (b) where X is a χ-block. In the remaining cases, $m_3(X) = 2$, and then $X = \theta(I)$ by A.3.18; so as $KS \leq C_G(Z_S)$ by 15.2.11.2, $KS \leq M_1 \leq M_0$. In particular $m_3(M_0) = 2$, with $KS/O_2(KS) \cong S_3 \times S_3$; so by inspection of the list in C.2.7.3 (recalling that $Out(Sp_4(4))$ is cyclic; cf. 16.1.4 and its underlying reference), either X is an A_7-block, or $X/O_2(X) \cong L_4(2)$ or $L_5(2)$. The former case was eliminated earlier, so the latter holds. Now M_1 is a proper parabolic of X containing K, so either $M_1 S = KS$ is determined by a pair of non-adjacent nodes, or $X/O_2(X) \cong L_5(2)$ and $M_1 S$ is a maximal parabolic determined by all the nodes except one interior node. Let $U := [\langle Z_S^X \rangle, X]$. By B.2.14, $\langle Z_S^X \rangle = UC_{\langle Z_S^X \rangle}(X)$, so that $C_X(U \cap Z_S) = C_X(Z_S)$. Now by C.2.7.2, U is an FF-module for $(XS)^+ := XS/O_2(XS)$, and hence is described in Theorem B.5.1. In particular one of the following holds:

(a) U is the sum of isomorphic natural modules, and M_1 is an end-node maximal parabolic.

(b) U is the sum of a natural module and its dual, and M_1 is the parabolic determined by the interior nodes.

(c) $U/C_U(X)$ is the 6-dimensional orthogonal module for $X^+ \cong L_4(2)$.

(d) U is a 10-dimensional module for $X^+ \cong L_5(2)$.

As $K \leq M_1$, case (b) is eliminated. Let $K_I := O^{3'}(M_1)$. Assume case (d) occurs. Then $K_I/O_2(K_I) \cong \mathbf{Z}_3 \times L_3(2)$. Now $X = O^{3'}(I)$ by A.3.18; so as $C_G(Z_S) \leq I$ and $M_1 = C_X(Z_S)$, $K_I = O^{3'}(C_G(Z_S))$ is T-invariant, and so $O^{3'}(O_{2,3}(K_I))$ is a T-invariant subgroup of 3-rank 1. But this is impossible as T is irreducible on $K/O_2(K)$. Assume case (a) occurs. Then as $K \leq M_1$, $X/O_2(X) \cong L_5(2)$, and $K_I/O_2(K_I) \cong L_4(2)$. In particular as S acts on M_1, S is trivial on the Dynkin diagram of $X/O_2(X)$, and so S is not irreducible on $K/O_2(K)$. Then by 15.2.21.2, $K_c = O^{3'}(M_I)$, so $K_I = K_c$. As conclusion (ii) of 15.2.18.3 holds, $O^{3'}(N_I) \not\leq M_I$, so the minimal parabolic P of X not contained in K_I is contained in N_I. Thus $Y_1 := O^2(P) \leq N_I$ with $Y_1S/O_2(Y_1S) \cong S_3$, but $Y_1 \not\leq M_I$. This contradicts 15.2.21.3.

Thus case (c) holds. In this case, $M_1S = KS$ is the maximal parabolic determined by the end nodes. We apply an argument made in an earlier reduction, with M_1, U in the roles of "M_0, U_0", to conclude that for $Q_I = O_2(KS)$, $Aut_{Q_I}(U)$ contains an FF*-offender. But this is not the case for this parabolic and representation by B.3.2.6.

This contradiction finally completes the proof of 15.2.23. □

By 15.2.20.3, $O(I) = 1$, so as $F^*(I) \neq O_2(I)$ by 15.2.23, $E(I) \neq 1$. By 15.2.20.2, $F^*(M_I) = O_2(M_I)$, so there is a component L of I with $L \not\leq M_I$, and by 15.2.20.1, L is described in 1.1.5.3. Further as $O(I) = 1$, $Z(L)$ is a 2-group. Let $L_0 := \langle L^S \rangle$, $S_L := S \cap L_0$, and $M_L := L_0 \cap M_I$. As usual $L_0 \trianglelefteq I$ by 1.2.1.3. Recall by our minimal choice of I that M_I is a maximal subgroup of I; hence $I = L_0M_I$. By 1.1.5.3, Z is faithful on L.

LEMMA 15.2.24. *$L/Z(L)$ is not of Lie type and characteristic* 2.

PROOF. Suppose otherwise. Then we are in one of cases (a)–(c) of 1.1.5.3, and $L \cong A_6$ in case (c), since $Z(L)$ is a 2-group.

Now $S_L \in Syl_2(L_0)$ and $S_L \leq M_L$. Further $M_L < L_0$ since M_I is a maximal subgroup of $I = L_0M_I$. So since the maximal S-invariant overgroups of S_L in L_0 are parabolics over S_L, M_L is such a parabolic. Also $Z \leq C_I(M_L)$ by 15.2.14.1, and Z is faithful on L. We conclude from the list in (a)–(c) of 1.1.5.3 that L is defined over $\mathbf{F}_2$, and if $L \cong L_3(2)$, then $M_L = S_L$, so that $N_S(L)$ is nontrival on the Dynkin diagram of L. Now if $m_3(L_0) = 2$ then $L_0 = O^{3'}(I)$ by A.3.18 or 1.2.2.a, so $K \leq C_L(Z) \leq M_L$, and hence $m_3(C_L(Z)) \geq 2$; but this is not the case for the groups of 3-rank 2 defined over $\mathbf{F}_2$ in Theorem C (A.2.3). Therefore $m_3(L_0) = 1$, so $L_0 = L \cong L_3(2)$. But now $Aut_{M_I}(L)$ is a 2-group, so K centralizes L and hence $m_3(KL) = 3$, contrary to I an SQTK-group. □

We are now in a position to complete the proof of Theorem 15.2.15. By 15.2.20.3, L is described 1.1.5.3, and indeed appears in one cases (d)–(f) by 15.2.24, and Z is faithful on L. Further in case (d), $L \cong A_7$ since $Z(L)$ is a 2-group.

If $L \cong L_2(p)$ for p a Mersenne or Fermat prime, then $p > 7$ by 15.2.24, and $C_{L_0}(Z) = S_L$. Then as K centralizes Z, and $Aut_{M_I}(L)$ is a 2-group, $[K, L_0] = 1$, so $L \leq N_I(K) \leq M_c = !\mathcal{M}(H)$ by 15.2.12.3, contrary to the choice of L.

Thus L is not $L_2(p)$. In the remaining cases $m_3(L) = 2$, so $L = O^{3'}(I)$ by A.3.18. Thus $K \leq C_L(Z)$, so $m_3(C_L(Z)) = 2$. Inspecting the list of groups remaining in 1.1.5.3, we conclude $L \cong J_4$. But then $O^2(C_L(Z))/O_2(O^2(C_L(Z)) \cong \hat{M}_{22}$, whereas $C_G(Z) = M_c$ by 15.2.14.1, $K_c = O^{3'}(M_c)$ by part (4) of 15.2.12, and K_c contains no such section by part (2) of the latter result. This contradiction completes the proof of Theorem 15.2.15.

15.2.3. The final contradiction. For $X \leq G$, write $\Lambda(X)$ for the subgroup generated by all involutions of X.

LEMMA 15.2.25. *(1) Case (1) or (4) of 15.2.1 holds, with $\bar{M} \cong S_3$, D_{10}, or $Sz(2)$.*
(2) $\Lambda(T) \leq S$.

PROOF. If (1) fails, then conclusion (ii) of 15.2.13.1 holds. In this case there is Z_1 of order 2 in Z_S with $C_{\bar{M}}(Z_1) \cong S_3$. In particular as $M \cap M_c = C_M(V)T$ by 15.2.14.5, $C_M(Z_1) \not\leq M_c$, contrary to Theorem 15.2.15. Therefore (1) holds. By (1), $\bar{S} = \Omega_1(\bar{T})$, so (2) holds. □

Recall $U_H = \langle V^H \rangle$ from Notation 15.2.9.

LEMMA 15.2.26. *(1) $r(G,V) > 1$ and hence $s(G,V) > 1$.*
(2) $W_0(T,V) \leq C_T(V)$.
(3) $W_0(Q_H,V) \leq C_H(U_H)$.

PROOF. Let U be a hyperplane of V. Suppose first that $\bar{M}$ is not $Sz(2)$. Then $Z = Z_S$ is of rank 2 by examination of the cases in 15.2.25.1, and hence $Z \cap U \neq 1$. Then as $C_G(Z) = M_c$ by 15.2.14.1, $C_G(Z) = C_G(Z \cap U)$. Similarly for $g \in M - M_c$, $C_G(Z^g) = C_G(Z^g \cap U)$ and $ZZ^g = V$, so that

$$C_G(U) \leq C_G(Z \cap U) \cap C_G(Z^g \cap U)) = C_G(Z) \cap C_G(Z^g) = C_G(V).$$

Thus $r(G,V) > 1$ in this case.

So assume $\bar{M} \cong Sz(2)$. In this case $m(Z_S) = 2$ and $m(Z) = 1$. Here M has two orbits on nonzero vectors of V of lengths 5 and 10, and hence two orbits on hyperplanes of V, which are also of lengths 5 and 10. Notice by 15.1.9.5 that Hypothesis E.6.1 is satisfied, so if U is T-invariant then E.6.13 says $C_G(U) \leq N_G(V)$. If U is not T-invariant, then $|U^M| = 10$, so as $\bar{T}$ is cyclic, we may assume that s normalizes U and hence centralizes a nontrivial 2-subspace of U, so that $Z_S = C_V(s) \leq U$. As $V = \langle Z^M \rangle$, there exists $g \in M - M_c$ with $Z^g \not\leq U$. By Theorem 15.2.15, $C_G(Z_S^g \cap U) \leq M_c^g$, so as $M_c^g = C_G(Z^g)$ by 15.2.14.1, with $Z^g \not\leq Z_S^g \cap U \neq 1$ and Z_S^g of rank 2, we conclude that $C_G(Z_S^g \cap U) = C_G(Z_S^g)$. Thus $C_G(U) = C_G(V)$ as in the previous paragraph.

Therefore $r(G,V) > 1$ in either case. Since $m(\bar{M},V) > 1$ by 15.2.14.6, also $s(G,V) > 1$, so that (1) holds. Furthermore $a(\bar{M},V) = 1$ by 15.2.14.6. Now E.3.21.1 implies (2), and (2) implies (3). □

LEMMA 15.2.27. *(1) $O_{2,F^*}(M_c) \leq K_c(M \cap M_c)$.*
(2) $V \leq O_2(M_c)$.
(3) If $Z_S \cap V^g \neq 1$, then $[V,V^g] = 1$.

PROOF. We claim that $O_{3'}(M_c) \leq M \cap M_c$: for otherwise we may choose a T-invariant $3'$-subgroup K of M_c minimal with respect to $J := KT \not\leq M$; then $J \in \mathcal{H}_*(T, M)$, whereas members of $\mathcal{H}_*(T, M)$ are not $3'$-groups by 15.2.8. So the claim holds, and hence as $O^{3'}(M_c) = K_c$ by 15.2.12.4, (1) holds.

If (2) fails, then $[K_c, V] \neq 1$ by (1), so $K < K_c$ as $V \leq O_2(H)$ by 15.1.11.1. Thus case (ii) or (iii) of 15.2.12.2 holds. and then by $K_c = [K_c, V]$ by (1). Let $Q_c := O_2(M_c)$, $V_c := V \cap Q_c$, $K_c^* T^* := K_c T / O_2(K_c T)$, and $\tilde{M}_c := M_c/Z$. Then $C_{M_c}(\tilde{Q}_c) \leq Q_c$ by A.1.8. Thus as $V^* \neq 1$, V does not centralize $\tilde{Q}_c$, so as $[\tilde{Q}_c, V] \leq \tilde{V}_c$, $m(\tilde{V}_c) \geq 1$. On the other hand since $m(Z) \geq 1 \leq m(V^*)$ and V has rank 4, $m(\tilde{V}_c) \leq 2$ with equality holding only if V^* and Z are of rank 1. Next in the groups in (ii) and (iii) of 15.2.12.2, no normal subgroup of T^* induces a group of $\mathbf{F}_2$-transvections with fixed center on a chief section of Q_c by B.4.2, keeping in mind in (ii) that T^* is nontrivial on the Dynkin diagram of K_c^*. Therefore we conclude that $[V, \tilde{Q}_c] = \tilde{V}_c$ is of rank 2, so that V^* and Z are indeed of order 2. It follows that $V^* = Z(T^*)$, $K_c T$ has just one noncentral 2-chief factor W, and (e.g., by D.3.10, B.4.2, and B.4.5) either

(i) $K_c^* T^* \cong S_8$ or $Aut(L_5(2))$, and either W is the 6-dimensional orthogonal module for S_8, or W is the sum of the natural module for $K_c^* \cong L_n(2)$ ($n = 4$ or 5) and its dual; or

(ii) $K_c^* T^* \cong L_3(2)$ wr $\mathbf{Z}_2$ and $W = W_1 \oplus W_2$, where $W_i := [W, K_{c,i}]$ is the natural module for the direct factor $K_{c,i}^* \cong L_3(2)$ of K_c^*.

Next as Z is of order 2 and Z_S is of order 4, $1 \neq \tilde{Z}_S$. As $\tilde{Z}_S \leq Z(\tilde{T})$, we conclude $\tilde{Z}_S \leq \widetilde{Q_c \cap V} = \tilde{V}_c$ using B.2.14. Thus the projection W_Z of $\tilde{Z}_S$ on W is nontrivial and centralized by H^* by 15.2.11.2. As $H^* \cong S_3$ wr $\mathbf{Z}_2$, it follows that in (i), $K_c^* T^* \cong S_8$ and W is the orthogonal module; and in (ii), H^* is the parabolic of $K_c^* T^*$ over T^* stabilizing a point of W. In either case, there is a parabolic P^* of $K_c^* T^*$ not contained in H^*, minimal subject to being T^*-invariant; further $P^*/O_2(P^*) \cong S_3$ in (i), and $P^* \cong S_3$ wr $\mathbf{Z}_2$ in (ii). By minimality of P^*, if the preimage P is not contained in M, then $P \in \mathcal{H}_*(T, M)$. We conclude from 15.2.8 that $P \leq M$ in (i), while in (ii) we get $P \leq M$ from 15.2.11.2 since $[W_Z, P] \neq 1$ by construction. Then by 15.2.14.5, $O^2(P) \leq C_P(V) \leq C_P(Z_S)$, again contrary to $[W_Z, P] \neq 1$. This contradiction completes the proof of (2).

Now by (2), $V^x \leq Q_c \leq T$ for each $x \in M_c$, so $[V, V^x] = 1$ by 15.2.26.2. Finally assume that $1 \neq Z_S \cap V^g$ for some $g \in G$. As $V \leq Z(J(T))$ and $N_G(J(T)) \leq M$ by 15.1.9.1, we may apply Burnside's Fusion Lemma A.1.35 to conclude that M controls fusion in V. Therefore we may take $g \in N_G(Z_S \cap V^g)$ by A.1.7.1, and hence $g \in M_c$ by Theorem 15.2.15. Then $[V, V^g] = 1$ by the initial remark of the paragraph, so (3) holds. □

LEMMA 15.2.28. $[U_H, \Lambda(Q_H)] \leq Z_S$.

PROOF. Observe that $\Lambda(Q_H) \leq \Lambda(T) \leq S$ by 15.2.25.2, and S centralizes V/Z_S in each case of 15.2.25.1. Thus as $\Lambda(Q_H) \trianglelefteq H$ and $U_H = \langle V^H \rangle$, the lemma holds. □

We are now in a position to complete the proof of Theorem 15.1.3.

Observe that the pair M, H satisfies Hypotheses F.7.1 and F.7.6 in the roles of "G_1, G_2". Form the coset graph Γ on the pair M, H, and adopt the notation of section F.7. In particular γ_0 and γ_1 are the points of Γ stabilized by M and H, respectively. For $\alpha := \gamma_1 g$ and $\beta := \gamma_0 y$, set $U_\alpha := U_H^g$, $Z_\alpha := Z_S^g$, and $V_\beta := V^y$. Let $b := b(\Gamma, V)$. Also we choose a geodesic

$$\gamma_0, \gamma_1, \ldots, \gamma_b =: \gamma$$

with $V \not\leq G_\gamma^{(1)}$. As V is not an FF-module for $\bar{M}$ by 15.1.8, b is odd by F.7.11.7. From 15.2.8, $Q_H = G_{\gamma_1}^{(1)}$. Thus as $V \leq Q_H$ by 15.1.11.2, $b > 1$. As b is odd, G_γ is a conjugate of H, so $G_\gamma^{(1)} = O_2(G_\gamma) =: Q_\gamma$.

While Hypothesis F.8.1 does not hold, we can still make use of arguments in section F.8. As in section F.8, define $D_\gamma := U_\gamma \cap Q_H$ and $D_H := U_H \cap Q_\gamma$. By choice of γ, $V \not\leq Q_\gamma$, so $D_H < U_H$. Indeed V does not centralize U_γ, so there is $g \in G$ with $\gamma_1 g = \gamma$ and $[V, V^g] \neq 1$. But if $D_\gamma = U_\gamma$, then $V^g \leq W_0(Q_H, V) \leq C_H(U_H) \leq C_H(V)$ by 15.2.26.3, a contradiction. Therefore $D_\gamma < U_\gamma$, so we have symmetry between γ_1 and γ (cf. Remark F.9.17).

Next

$$m(U_\gamma/D_\gamma) = m(U_\gamma^*) \leq m_2(H^*) = 2,$$

so by symmetry, $m(U_H/D_H) \leq 2$. Now $[U_H, D_\gamma] \leq Z_S$ by 15.2.28, so by symmetry $[U_\gamma, D_H] \leq Z_\gamma$. Then $[D_H, D_\gamma] \leq Z_S \cap Z_\gamma$, while as $[V, V_\gamma] \neq 1$, we conclude from 15.2.27.3 that $Z_S \cap Z_\gamma = 1$. Thus $[D_H, D_\gamma] = 1$. Next $m(D_\gamma/C_{D_\gamma}(V)) \leq m_2(\bar{M}) = 1$ by 15.2.25.1, and by symmetry $m(D_H/C_{D_H}(V^g)) \leq 1$, so

$$m(U_H/C_{U_H}(V^g)) \leq m(U_H/D_H) + 1 \leq 3.$$

Therefore V^{g*} induces a transvection on each of the 2-chief factors of H on U_H appearing in 15.1.12, so $V^{g^*} \leq H_i^*$ for $i = 1$ or 2. Hence $m(V^g/(V^g \cap Q_H)) = 1$. By symmetry, $m(V/(V \cap Q_\gamma)) = 1$, and

$$[V \cap Q_\gamma, V^g \cap Q_H] \leq [D_\gamma, D_H] = 1.$$

Therefore as $s(G, V) > 1$ by 15.2.26.1, E.3.6 says $V \cap Q_\gamma \leq C_G(V^g)$, and then by another application of those lemmas, $V^g \leq C_G(V \cap Q_\gamma) \leq C_G(V)$, contrary to the choice of V^g.

This contradiction completes the proof of Theorem 15.1.3.

15.3. The elimination of $\mathbf{M_f}/\mathbf{C_{M_f}}(\mathbf{V}(\mathbf{M_f})) = \mathbf{S_3}$ wr $\mathbf{Z_2}$

In this section, we complete our treatment of the groups satisfying Hypothesis 14.1.5, by proving:

THEOREM 15.3.1. *Assume Hypothesis 14.1.5. Then G is isomorphic to J_2, J_3, ${}^3D_4(2)$, the Tits group ${}^2F_4(2)'$, $G_2(2)'$, or M_{12}.*

Observe that the groups in Theorem 15.3.1 have already appeared in Theorem 15.1.3, so that we will be working toward a contradiction. On the other hand, the shadows of the groups $Aut(L_n(2))$, $n = 4$, 5, S_9, A_{10}, $Aut(He)$, and L wr $\mathbf{Z}_2$ for $L \cong S_5$ or L of rank 2 over $\mathbf{F}_2$ arise, and cause difficulties: Each of these groups possesses $M \in \mathcal{M}(T)$ such that $V(M)$ is of rank 4 and $Aut_M(V(M)) = O^+(V(M))$.

For $X \leq G$, we let $\theta(X)$ denote the subgroup generated by all elements of X of order 3.

15.3.1. Preliminary results. Recall by Hypothesis 14.1.5.2 that

$$M_c = !\mathcal{M}(C_G(Z)).$$

Throughout section 15.3 we assume G is a counterexample to Theorem 15.3.1. Let $M := M_f$ be the unique maximal member of $\mathcal{M}(T) - \{M_c\}$ under the partial order $\lesssim$ of Definition A.5.2, supplied by 14.1.12. Recall in particular that $M \in \mathcal{M}(N_G(C_2))$, where $C_2 := C_2(\text{Baum}(T))$ is the characteristic subgroup of Baum(T) from C.1.18. Let $V := V(M)$.

We summarize what has been established in this chapter so far:

LEMMA 15.3.2. *(1) $m(V) = 4$ and $\bar{M} = O_4^+(V)$.*

(2) $Z = C_V(T)$ is of order 2.

(3) $\mathcal{M}(T) = \{M, M_c\}$.

(4) If $T \leq X \leq M$, then either

(i) $O^2(X) \leq C_M(V)$, or

(ii) $\bar{X} = \bar{M}$ and $M = !\mathcal{M}(X)$.

(5) M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.

(6) $N_G(T) \leq M$. In particular, members of $\mathcal{H}_*(T, M)$ are minimal parabolics described in B.6.8, and in E.2.2 when nonsolvable.

PROOF. As G is a counterexample to Theorem 15.3.1, and the groups in 15.3.1 appear as conclusions in Theorem 15.1.3, conclusion (2) of 15.1.3 holds, giving (1). Then (1) implies (2) as $V = V(M) = \langle Z^M \rangle$, and (2) and 14.1.12.1 imply (5). Since $N_G(T)$ preserves the relation $\lesssim$ on $\mathcal{M}(T)$, $N_G(T) \leq M$ by (5); then 3.1.3.2 completes the proof of (6).

Assume the hypotheses of (4). By (1), $\bar{T}$ is maximal in $\bar{M}$, so either $O^2(X) \leq C_M(V)$ so that (i) holds, or $\bar{X} = \bar{M}$. In the latter case, (ii) holds by A.5.7.1. Thus (4) is established.

Finally suppose $M_1 \in \mathcal{M}(T)$. By (5), $M_1 = (M \cap M_1)C_{M_1}(V(M_1))$. As usual $C_{M_1}(V(M_1)) \leq M_c$ since $M_c = !\mathcal{M}(C_G(Z))$. We apply (4) to $M \cap M_1$ in the role of "X": in case (i) of (4), $M \cap M_1 \leq C_M(V) \leq C_G(Z) \leq M_c$, so that $M_1 \leq M_c$; in case (ii) of (4), $M = !\mathcal{M}(M \cap M_1)$ so that $M_1 = M$. Thus (3) holds. □

LEMMA 15.3.3. *(1) $M = !\mathcal{M}(N_M(C_2(\text{Baum}(T))))$.*

(2) If Baum$(T) \leq S \leq T$, then Baum(T) = Baum(S), and further $N_G(S) \leq N_G(\text{Baum}(S)) \leq M$.

PROOF. Set $C_2 := C_2(\text{Baum}(T))$, and recall $M \in \mathcal{M}(N_G(C_2))$. By 15.3.2.2 and 14.1.11, $M = C_M(V)N_M(C_2)$, so that (1) follows from 15.3.2.4. Choose S as in (2). Then Baum(T) = Baum(S) by B.2.3.4, and $N_G(S) \leq N_G(\text{Baum}(S)) \leq N_G(C_2)$, so that (2) follows from (1). □

LEMMA 15.3.4. *$M_c = C_G(Z)$.*

PROOF. Let $U := V(M_c)$, and assume the lemma fails; then $U > Z$. Next $M_c \lesssim M$ by 15.3.2.5, so $M_c = C_M(U)X$ where $X := M \cap M_c$, and hence $U = \langle Z^X \rangle$. Thus as $Z < U$, $O^2(X) \not\leq C_G(V)$, so $M = !\mathcal{M}(X)$ by 15.3.2.4, contrary to $M_c \neq M$. □

By 15.3.2.1, $\bar{M} = O_4^+(V)$ preserves an orthogonal-space structure on V, so $V = V_1 \oplus V_2$, where V_1 and V_2 are the two definite 2-dimensional subspaces of V.

Thus $\bar{M}=(\bar{M}_1\times\bar{M}_2)\langle\bar{t}\rangle$, where $M_i:=C_M(V_{3-i})$, $V_i=[V,M_i]$, $\bar{M}_i\cong O_2^-(V_i)\cong S_3$, and $\bar{t}$ is an involution with $M_1^t=M_2$. Set $S:=N_T(V_1)$ and $Z_S:=C_V(S)$.

Lemma 15.3.5. *Let* $E:=\Omega_1(Z(J(T)))$ *and* $B:=Baum(T)$. *Then either*

(1) $[V,J(T)]=1$, $B=Baum(C_T(V))$, $V\le E$, *and* $C(G,B)\le M$, *or*

(2) $\bar{B}=\overline{J(T)}=\bar{S}\cong E_4$ *and* $E\cap V=Z_S\cong E_4$.

Proof. If $J(T)\le C_T(V)$ then $V\le E$ and $B=\mathrm{Baum}(C_T(V))$ by B.2.3.5; thus $M=C_M(V)N_M(B)$ by a Frattini Argument, and hence $C(G,B)\le M$ by 15.3.2.4. Otherwise $\overline{J(T)}\ne 1$, and then (2) follows from B.1.8. □

Lemma 15.3.6. *(1)* $Baum(T)=Baum(S)$.

(2) $N_G(S)\le N_G(Baum(S))\le M$.

(3) $S\in Syl_2(C_G(C_{V_1}(S)))$.

(4) $S\in Syl_2(N_G(V_1))$.

Proof. By 15.3.5, $\mathrm{Baum}(T)\le S$, so that (1) and (2) follow from 15.3.3.2. As $S\in Syl_2(C_M(C_{V_1}(S)))$, (2) implies (3), and similarly (2) implies (4). □

Lemma 15.3.7. *Let* $R_c:=O_2(M\cap M_c)$ *and* $Y:=O^2(\langle R_c^M\rangle)$. *Then* $\bar{Y}=O^2(\bar{M})$, $M\cap M_c=C_M(V)T$,

$$M=N_G(Y)=!\mathcal{M}(YT),$$

$O_2(YT)=C_T(V)$, *and either*

(1) $O_2(Y)=C_Y(V)$ *with* $Y/O_2(Y)\cong E_9$, *and* $Y=O^{3'}(M)$; *or*

(2) $Y/O_2(Y)\cong 3^{1+2}$, $O_{2,Z}(Y)=C_Y(V)$, $\bar{R}_c$ *is cyclic*, $Y=\theta(M)$, *and* $M\cap M_c$ *has cyclic Sylow* 3*-subgroups.*

Proof. We apply case (b) of 14.1.17 with M_c in the role of "M_1", and $Y_0:=O^2(M)$. By 14.1.17.1, $\bar{R}_c\ne 1$. As $\bar{R}_c\trianglelefteq\bar{T}$, $\bar{R}_c$ contains $Z(\bar{T})=:\langle\bar{r}\rangle$, and $\bar{r}$ inverts $\bar{Y}_0$ by 15.3.2.1, so $\bar{Y}_0=[\bar{Y}_0,\bar{R}_c]$ and $M\cap M_c=C_M(V)T$. Further applying parts (2) and (3) of 14.1.17, we conclude $\bar{Y}=\bar{Y}_0$ and $Y^*R_c^*$ centralizes $C_M(V)^*$, where $M^*:=M/O_2(M)$. In particular, $M=!\mathcal{M}(YT)$ by 15.3.2.4, and of course $M=N_G(Y)$ as $Y\trianglelefteq M\in\mathcal{M}$. Also $V=\langle Z^Y\rangle$, so that $V\in\mathcal{R}_2(YT)$ by B.2.14.

As $\bar{r}$ inverts $\bar{Y}_0=O^2(\bar{M})$ and $[r^*,C_Y(V)^*]=1$, r inverts y of order 3 in each coset of $C_Y(V)$ in Y. Therefore since Y^* centralizes $C_M(V)^*$, $Y^*=\Omega_1(Y^*)$ is a 3-group. As $\Phi(Y^*)\le C_Y(V)^*\le C_{Y^*}(r^*)$, it follows that $\Phi(Y^*)\le Z(Y^*)$, and hence $Y^*\cong Y/O_2(Y)\cong E_9$ or 3^{1+2} by A.1.24. Further $O_2(YT)=C_T(V)$.

If $Y^*\cong E_9$, then as $M=YC_M(Y/O_2(Y))T$ and $m_3(M)\le 2$, $Y=O^{3'}(M)$, so that (1) holds. If $Y^*\cong 3^{1+2}$, $m_3(C_M(V))=1$ by 14.1.17.4, so that $Y=\theta(M)$, and $C_M(V)T$ has cyclic Sylow 3-subgroups. Further $\bar{R}_c$ is embedded in $C_{Aut(Y^*)}(Z(Y^*))\cong Q_8$, while $\bar{T}\cong D_8$, so $\bar{T}$ contains no Q_8-subgroup. Thus $\bar{R}_c$ is cyclic, completing the proof of (2). □

In the remainder of this section, define Y as in lemma 15.3.7.

Lemma 15.3.8. $Z_S=\Omega_1(Z(S))\cong E_4$.

Proof. Let $Z_0:=\Omega_1(Z(S))$. As T/S is of order 2, $Z=C_{Z_0}(T)$ is of rank at least $m(Z_0)/2$, so $m(Z_0)\le 2$ as Z is of order 2. As $Z_S=C_V(S)$ is of rank 2, the lemma follows. □

Finally we eliminate a configuration which appears at various points later, the case $V=O_2(Y)$:

LEMMA 15.3.9. *$V < O_2(Y)$, so that $Y \not\cong A_4 \times A_4$.*

PROOF. Assume $V = O_2(Y)$, or equivalently that $Y \cong A_4 \times A_4$. By 15.3.7, $Y \trianglelefteq M$. As $Aut(A_4) \cong S_4$, $Aut(Y) \cong S_4$ wr $\mathbf{Z_2}$ with $C_{Aut(Y)}(V) = Aut_V(Y)$. Thus $YC_M(V) = Y \times C_M(Y)$. Since Z has order 2 and $C_T(Y) \trianglelefteq T$, $C_M(Y)$ has odd order; thus $C_M(Y) = 1$ as $F^*(M) = O_2(M)$. Therefore $M \leq Aut(Y)$, so as $\bar{M} \cong O_4^+(2)$ by 15.3.2.1, we conclude $M \cong S_4$ wr $\mathbf{Z_2}$. But this is ruled out by Theorem 13.9.1. □

15.3.2. Uniqueness theorems for $\mathbf{Y}$ and $\mathbf{O^2(C_Y(V_i))}$. Our first main goal, in Theorem 15.3.45 in this subsection, is to show that $M = !\mathcal{M}(C_Y(V_i)S)$; to do so, we first show that $M = !\mathcal{M}(YS)$. We prove the two results simultaneously, adopting a suitable hypothesis to cover both cases, and eventually establish the common uniqueness result in 15.3.44.

Thus in this subsection, we assume:

HYPOTHESIS 15.3.10. *Either*

(1) $Y_+ := Y$, or

(2) $M = !\mathcal{M}(YS)$ and $Y_+ := O^2(C_Y(V_1))$.

Let

$$\mathcal{H}_+ := \mathcal{H}(Y_+S, M) = \{I \in \mathcal{H}(Y_+S) : I \not\leq M\},$$

and write $\mathcal{H}_{+,*}$ for the mimimal members of $\mathcal{H}_+$ under inclusion. As our goal is to show that $M = !\mathcal{M}(Y_+S)$, we will assume $\mathcal{H}_+$ is nonempty, and derive a contradiction. Given $I \in \mathcal{H}_+$, define $M_I := M \cap I$, $U_I := \langle Z^I \rangle$, $I^* := I/C_I(U_I)$, and $R := O_2(Y_+S)$.

LEMMA 15.3.11. *Assume $I \in \mathcal{H}_+$. Then*

(1) $S \in Syl_2(I)$.

(2) $C_I(U_I) \leq M_I$.

(3) If case (2) of Hypothesis 15.3.10 holds, then $Y_+ = O^2(Y \cap I)$ and $N_G(V_i) \leq M \geq N_G(Y_+)$ for $i = 1, 2$.

(4) $N_I(Y_+) = M_I$.

(5) Either:

(i) $Y_+/O_2(Y_+)$ is E_9 or 3^{1+2}, $Y_+S/O_{2,\Phi}(Y_+S) \cong S_3 \times S_3$, $R = O_2(YS) \trianglelefteq YT$, and $R = C_T(V)$. Further $R = O_2(N_I(R))$, $R \in Syl_2(\langle R^{M_I} \rangle)$, and $C(G, R) \leq M$. Or:

(ii) Case (2) of Hypothesis 15.3.10 holds, $Y_+S/R \cong S_3$, $Y_+ = O^{3'}(M_I)$, and $R = C_S(V_2)$.

(6) $Y_+ = \theta(M_I)$.

(7) $F^*(M_I) = O_2(M_I)$.

(8) $O(I) = 1$.

(9) $N_{I^*}(Y_+^*) = M_I^* < I^*$, and $Y_+^* \neq 1$.

(10) Either

(i) $Baum(R) = Baum(S)$ and $C(I, Baum(S)) \leq M_I$, or

(ii) $Y_+ = [Y_+, J(S)]$, so $Y_+^* \leq J(I)^*$.

(11) $J(I)^* \not\leq M_I^*$.

(12) If $L \leq I$ with $[V, O^2(Y \cap L)] \neq 1$ and $L \not\leq M$, then no nontrivial characteristic subgroup of S is normal in $\langle L, S \rangle$.

PROOF. Let $S \leq T_I \in Syl_2(I)$. By 15.3.6.2, $N_G(S) \leq M$, so as $|T:S| = 2$, $T_I \leq M$. Thus either (1) holds, or $T_I \in Syl_2(M)$, and in the latter case, 15.3.2.4 supplies a contradiction as $\bar{Y}_+ \neq 1$. Thus (1) is established.

We next prove (2)–(6).

Suppose first that case (1) of Hypothesis 15.3.10 holds. Then (3) holds vacuously, and (4) holds as $M = N_G(Y)$ by 15.3.7. Also $V = \langle Z^Y \rangle \leq U_I$, so (2) holds as $M = N_G(V)$. Further from 15.3.7 and the structure of $\bar{M}$ in 15.3.2.1, (6) and the first three statements of (5i) hold. As $R \trianglelefteq YT$, $M = !\mathcal{M}(N_G(R))$ by 15.3.7, so that $C(G,R) \leq M$. Also $R = C_T(V)$ as $O_2(\bar{Y}\bar{S}) = 1$. As $Y \trianglelefteq M_I$ and $R = O_2(YS)$ with $S \in Syl_2(I)$, $R \in \mathcal{B}_2(M_I)$ and $R \in Syl_2(\langle R^{M_I}\rangle)$ by C.1.2.4. Then as $N_I(R) \leq M_I$, $R = O_2(N_I(R))$, completing the proof that conclusion (i) of (5) holds in this case.

Now suppose that case (2) of Hypothesis 15.3.10 holds, so that $Y_+ = O^2(C_Y(V_1))$. Then $M = !\mathcal{M}(YS)$ by Hypothesis 15.3.10, so that $Y \not\leq I$. Then as $|Y : Y_+O_2(Y)| = 3$, $Y_+ = O^2(Y \cap I)$. Further as V_1, V_2, and Y_+ are normal in YS, the remainder of (3) and also (4) follow as $M = !\mathcal{M}(YS)$. Then as $V_2 \leq \langle Z^{Y_+}\rangle \leq U_I$, (2) follows from (3). We next prove (5) and (6). First suppose that conclusion (1) of 15.3.7 holds. Then $Y = O^{3'}(M)$, so $Y_+ = O^{3'}(M_I)$ as $Y_+ = O^2(Y \cap I)$. Thus (6) holds, and visibly conclusion (ii) of (5) holds. Therefore we may assume that conclusion (2) of 15.3.7 holds. Then $Y^* \cong 3^{1+2}$ and $Y_+/O_2(Y_+) \cong E_9$, with $Y_+S/R \cong S_3 \times S_3$ using the structure of $\bar{M}$ in 15.3.2.1. This time $Y = \theta(M)$, so (6) holds. As $C_T(V) = C_S(Y_+^*)$, $R = C_T(V) = O_2(YS)$, so $C(G,R) \leq M$ as $M = !\mathcal{M}(YT)$. As $Y_+ \trianglelefteq M_I$ by (6) and $R = O_2(Y_+S)$ with $S \in Syl_2(I)$, $R \in \mathcal{B}_2(M_I)$ and $R \in Syl_2(\langle R^{M_I}\rangle)$ by C.1.2.4. Then as $N_I(R) \leq M_I$, $R = O_2(N_I(R))$, so that conclusion (i) of (5) holds.

It remains to prove (7)–(12).

As $|T:S| = 2$ and $F^*(M) = O_2(M)$, 1.1.4.7 implies (7). In case (1) of Hypothesis 15.3.10, $V = [V,Y]$ so that $O(I) \leq C_I(V) \leq M_I$ by A.1.26.1, and hence (8) follows from (7). In case (2) of Hypothesis 15.3.10, $V_2 = [V_2, Y_+]$, and (8) follows similarly from (7) as $C_I(V_2) \leq M$ by (3).

Next $X := Y_+C_I(U_I) \leq M_I$ by (2), so $Y_+ = \theta(Y_+C_I(U_I))$ by (6); then (9) follows from (4) as Y_+ is nontrivial on $1 \neq [V_2, Y_+] \leq U_I$ by construction. By (5), $C_T(V) \leq R \leq S$. So if $[V, J(T)] = 1$, then $\mathrm{Baum}(R) = \mathrm{Baum}(S) = \mathrm{Baum}(C_T(V))$ and $C(I, \mathrm{Baum}(S)) \leq M_I$ by 15.3.5 and B.2.3.5, so that conclusion (i) of (10) holds. Then $I = J(I)N_I(J(S)) = J(I)M_I$ by a Frattini Argument, so as $I \not\leq M$, $J(I) \not\leq M$, and hence $J(I)^* \not\leq M_I^*$ by (2), establishing (11). So suppose instead that $[V, J(T)] \neq 1$. Then case (2) of 15.3.5 holds, so that $\overline{J(T)} = \bar{S}$. But by (5), $Y_+ = [Y_+, S]$ and $C_T(V) \leq R \leq S$, so $Y_+ = [Y_+, J(S)]$, and hence conclusion (ii) of (10) holds. Thus if $J(I)^* \leq M_I^*$, then $Y_+^* = \theta(J(I)^*) \trianglelefteq I^*$ by (6), contrary to (9). This completes the proof of (10) and (11).

Assume the hypotheses of (12), with $1 \neq C$ char S and $C \trianglelefteq \langle L, S\rangle$. Then $\langle O^2(Y \cap L), T\rangle \leq N_G(C)$, so as $O^2(Y \cap L) \not\leq C_M(V)$ by hypothesis, $N_G(C) \leq M$ by 15.3.2.4, a contradiction. □

LEMMA 15.3.12. *Assume $I \in \mathcal{H}_+$ and there is $L \in \mathcal{C}(I)$ with $m_3(L) \geq 1$. Also assume $m_3(Y_+) = 2$. Then*

(1) $Y_L := O^2(Y_+ \cap L) \neq 1$. Further R normalizes L and $O_2(L)O_2(Y_L) \leq R$.

(2) Assume that $m_3(L) = 1$ and Y_+ induces inner automorphisms of $L/O_2(L)$. Then $Y_+ = Y_L Y_C$ where $Y_C := O^2(C_{Y_+}(L/O_2(L)))$, $|Y_L|_3 = 3 = |Y_C|$, and $Y_+/O_2(Y_+) \cong E_9$.

PROOF. First $Y_+ = O^2(Y_+)$ normalizes L by 1.2.1.3. Then $m_3(LY_+) \leq 2$ since I is an SQTK-group, so as $m_3(Y_+) = 2$ and $m_3(L) \geq 1$ by hypothesis, $Y_L \neq 1$. As $[Y_L, R]$ is a 2-group, R normalizes L by 1.2.1.3. Further $O_2(L)O_2(Y_L)$ is normalized by Y_+, and so lies in R, completing the proof of (1).

Assume that $m_3(L) = 1$. As $Y_L \neq 1$ by (1) while Y_+ is of exponent 3, $|Y_L|_3 = 3$. Assume also that Y_+ induces inner automorphisms on $L/O_2(L)$; then as $m_3(L) = 1$ and Y_+ is of exponent 3, $Aut_{Y_+}(L) = Aut_{Y_L}(L)$. Hence $Y_+ = Y_L Y_C$, with $Y_L \cap Y_C$ a 2-group as $Z(L/O_2(L)) = 1$. In particular, $Y_+/O_2(Y_+) \cong E_9$ rather than 3^{1+2}, and $|Y_C|_3 = 3$, completing the proof of (2). □

We now begin our analysis of the case $F^*(I) = O_2(I)$. Observe then that $U_I = \langle Z^I \rangle \in \mathcal{R}_2(I)$ by B.2.14.

We partition the problem into the subcases $m_3(Y_+) = 2$ and $m_3(Y_+) = 1$.

THEOREM 15.3.13. *Assume $I \in \mathcal{H}_+$ and $m_3(Y_+) = 2$. Then $F^*(I) \neq O_2(I)$.*

Until the proof of Theorem 15.3.13 is complete, assume I is a counterexample. Then $F^*(I) = O_2(I)$, so that $U_I \in \mathcal{R}_2(I)$ by B.2.14. As $m_3(Y^+) = 2$, case (i) of 15.3.11.5 holds, so $Y/O_2(Y) \cong E_9$ or 3^{1+2}, and $R = C_T(V)$. If case (2) of Hypothesis 15.3.10 holds, then as $Y_+ < Y$, $Y/O_2(Y) \cong 3^{1+2}$ and $C_Y(V)/O_2(C_Y(V)) \cong \mathbf{Z}_3$. Thus:

LEMMA 15.3.14. *(1) $V \leq Z(R)$.*

(2) If case (2) of Hypothesis 15.3.10 holds, then $Y/O_2(Y) \cong 3^{1+2}$ and $|C_Y(V) : O_2(C_Y(V))| = 3$.

LEMMA 15.3.15. *(1) Hypothesis C.2.3 is satisfied with I, M_I in the roles of "H, M_H".*

(2) There exists $L \in \mathcal{C}(J(I))$ with $L \not\leq M_I$, $m_3(L) \geq 1$, $L = [L, Y_+]$, and L^* and $L/O_2(L)$ quasisimple.

(3) Each solvable Y_+S-invariant subgroup of I is contained in M_I.

PROOF. As case (i) of 15.3.11.5 holds with $R = C_T(V)$, (1) follows. By 15.3.11.11, $J(I)^* \not\leq M_I^*$. In particular $J(I)^* \neq 1$, so that U_I is an FF-module for I by B.2.7, and hence $J(I)^*$ is described in Theorem B.5.6. If L^* is a direct factor of $J(I)^*$ isomorphic to S_3, then there are at most two such factors by Theorem B.5.6, so $Y_+^* = O^2(Y_+^*)$ normalizes and hence centralizes L^*, and then $L^* \leq N_{I^*}(Y_+^*) = M_I^*$ by 15.3.11.9. Thus as $J(I)^* \not\leq M_I^*$, Theorem B.5.6 says there exists $L \in \mathcal{C}(J(I))$ with $L \not\leq M_I$, $m_3(L) \geq 1$, and L^* quasisimple. By 1.2.1.3, $Y_+ = O^2(Y_+) \leq N_I(L)$. By 15.3.11.2, $C_I(U_I) \leq M_I$, so as $L \not\leq M_I$, $L = [L, Y_+]$ by 15.3.11.9. Further as L^* is quasisimple and not isomorphic to $Sz(2^m)$ by Theorem B.5.6, $O_{3'}(L) \leq C_I(U_I) \leq M_I$, so by 15.3.11.6, $[O_{3'}(L), Y_+] \leq Y_+ \cap O_{3'}(L) \leq O_2(L)$, and hence $L/O_2(L)$ is quasisimple by 1.2.1.4. Thus (2) holds. Then by 1.2.1.1, each member of $\mathcal{H}_+$ is nonsolvable, so (3) follows. □

During the remainder of the proof of Theorem 15.3.13, pick L as in 15.3.15.2. Set $Y_L := O^2(Y_+ \cap L)$, $Y_C := O^2(C_{Y_+}(L/O_2(L))$, $S_L := S \cap L$, $R_L := R \cap L$, and $M_L := M \cap L$. Let $W_L := \langle V^L \rangle$ and $(LRY_+)^+ := LRY_+/C_{LRY_+}(W_L)$.

Lemma 15.3.16. *(1)* $W_L \in \mathcal{R}_2(LR) \cap \mathcal{R}_2(LRY_+)$ *and* L^+ *is quasisimple.*
(2) $m_3(L) \geq 1$, $Y_L \neq 1$, R *acts on* L, $L = [L, J(R)]$, *and* L *is described in C.2.7.3.*

Proof. By 15.3.15, $m_3(L) \geq 1$ and we can appeal to the results in section C.2. As $m_3(L) \geq 1$, $Y_L \neq 1$, R acts on L, and $O_2(LR) \leq R$ by 15.3.12.1.

As $L/O_2(L)$ is quasisimple and $O_2(LR) \leq R = C_T(V)$, $W_L \in \mathcal{R}_2(LR) \cap R_2(LRY_+)$. As R acts on L and $L \not\leq M_I$, L is described in C.2.7.3. By C.2.7.2, $L = [L, J(R)]$. As $C_G(V) \leq M$ but $L \not\leq M_I$, $L^+ \neq 1$, so as $L/O_2(L)$ is quasisimple, so is L^+. □

Lemma 15.3.17. *One of the following holds:*
(1) $m_3(L) = 1$ *and* $L/O_2(L) \cong L_2(2^n)$, n *even, or* $L_3(2)$.
(2) $m_3(L) = 2$ *and* $Y_+ \leq \theta(I) = L$.
(3) $m_3(L) = 2$ *and* $L^* \cong SL_3(2^n)$ *with* n *even.*

Proof. By 15.3.16, $m_3(L) \geq 1$, $Y_L \neq 1$, and L is described in C.2.7.3.

Suppose first that $m_3(L) = 1$. Then from the list in C.2.7.3, $L/O_2(L) \cong L_2(2^n)$ or $L_3(2^m)$, with m odd. Now as $S \in Syl_2(I)$, $S_L = S \cap L \in Syl_2(L)$ and S_L acts on Y_L since Y_+ is S-invariant. It follows that n is even if $L/O_2(L) \cong L_2(2^n)$, and that $m = 1$ if $L/O_2(L) \cong L_3(2^m)$. That is, (1) holds in this case.

So assume $m_3(L) = 2$. Then (3) holds if $L^* \cong SL_3(2^n)$ with n even; otherwise $\theta(I) = L$ by A.3.18, so that (2) holds. □

In the remainder of the proof of Theorem 15.3.13, we successively eliminate the various possibilities in C.2.7.3 given by 15.3.16.

Lemma 15.3.18. *L is not an* $L_2(2^n)$*-block.*

Proof. Assume otherwise. Then n is even by 15.3.17, while by 15.3.16, $Y_L \neq 1$ and R normalizes L.

Let $L_0 := \langle L^S \rangle$, so that $S_0 := S \cap L_0 \in Syl_2(L_0)$ and $M_0 := M \cap L_0 \geq Y_L$. Then M_0 is an overgroup of S_0 in L_0, so M_0 is contained in a unique Borel subgroup B_0 of L_0, and hence B_0 is M_I-invariant. Therefore as B_0 is solvable, $B_0 = M_0$ by 15.3.15.3. Then as $Y_+ \trianglelefteq M_I$ by 15.3.11.4, we conclude that Y_+ induces inner automorphisms on $L_0/O_2(L_0)$. By 15.3.12.2, $Y_+ = Y_L Y_C$ with $|Y_L|_3 = 3 = |Y_C|_3$, and $Y_+/O_2(Y_+) \cong E_9$. As S_0 is M_I-invariant, $S_0 \leq O_2(Y_+S) = R$; hence $R_L = R \cap L \in Syl_2(L)$, and $R \in Syl_2(LR)$. Thus as $V \leq Z(R)$ and $U(L) = [W_L, L]$ since L is a block, $W_L = C_{W_L}(R_L)U(L)$ by B.2.14, so that

$$V \leq C_{U(L)}(R) C_{W_L}(L). \qquad (*)$$

Now if $[V, Y_L] = 1$, then by (*), we have $V \leq C_{U(L)C_R(L)}(Y_L) = C_R(L)$, since $U(L)/C_{U(L)}(L)$ is the natural module for $L_2(2^n)$. But then $L \leq C_G(V) \leq M$, contrary to $L \not\leq M$. Therefore $1 \neq [V, Y_L]$ is a B_0-invariant subgroup of $[C_{U(L)}(R_L), Y_L]$, so $[V, Y_L] = C_{U(L)}(R_L)$ by (*) and the structure of coverings of the natural module. in I.2.3. But for $b \in B_0 - R$, $C_{U(L)}(b) = C_{U(L)}(L)$, while $O^{2,3}(M_L) \leq C_G(V) \leq C_G([V, Y_L])$ by 15.3.2.1, so we conclude that $(B_0 \cap L)/R_L$ is a 3-group, and hence $n = 2$. Thus as $[V, Y_L] = C_{U(L)}(R_L)$, $[V, Y_L]/C_{[V,Y_L]}(Y_L) \cong E_4$. As $V = V_1 \oplus V_2$ where V_1 and V_2 are the only Y_+-invariant 4-subgroups of V, $C_{U(L)}(L) = 1$ and we may take $V_2 = [V, Y_L]$, and hence $Y_L \leq C_M(V_1) = M_2$. Then by (*), $V_1 \leq C_{W_L}(Y_L)$, so as $L \not\leq M$, also $N_G(V_1) \not\leq M$. Hence by 15.3.11.3, we are in case (1) of Hypothesis 15.3.10, where $Y_+ = Y$. Then $Y/O_2(Y) \cong E_9$, so as $Y_L \leq M_2$, $O^2(M_2) = Y_L$ is S-invariant. Hence S also acts on L and Y_C, so $Y_C = O^2(M_1)$. Let $t \in T - S$, and recall $|T : S| = 2$ so that t normalizes S. As $M_1^t = M_2$, $Y_L^t = Y_C$. Further Y_C centralizes L by C.1.10, and as Y_L contains a Sylow 2-group of L, $U(L) \leq O_2(Y_L)$. Then $U(L)^t \leq O_2(Y_L)^t = O_2(Y_C) \trianglelefteq LS$. Hence $\langle LS, t\rangle = \langle L, T\rangle$ acts on $U(L)U(L)^t$, so that $\langle L, T\rangle \leq M = !\mathcal{M}(YT)$ by 15.3.7, contrary to $L \not\leq M$. □

LEMMA 15.3.19. *$L/O_2(L)$ is not $L_2(2^n)$.*

PROOF. Assume otherwise. Then by 15.3.16 and C.2.7.3, L is a block of type $L_2(2^n)$ or type A_5, so by 15.3.18, L is an A_5-block. Let $L_0 := \langle L^S\rangle$, $S_0 := S \cap L_0$, and $M_0 := M \cap L_0$. Arguing as in the proof of the previous lemma, we conclude that M_0 is the Borel subgroup of L_0 containing S_0, $Y_+ = Y_L Y_C$ with $|Y_L|_3 = 3 = |Y_C|_3$, $Y_+/O_2(Y_+) \cong E_9$, $R_L := R \cap L \in Syl_2(L)$, and $W_L = U(L) \times C_{W_L}(L)$, so

$$V \leq C_{U(L)}(R_L) \times C_{W_L}(L). \qquad (*)$$

Since $U(L)$ is the A_5-module, Y_L centralizes V by (*), so as $C_Y(V) \neq 1$, case (2) of 15.3.7 holds, with $Y/O_2(Y) \cong 3^{1+2}$. In particular $|C_Y(V) : O_2(C_Y(V))| = 3$, so $Y_L = O^2(C_Y(V))$. Further as $Y_+/O_2(Y_+) \cong E_9$, $Y_+ < Y$, so that case (2) of Hypothesis 15.3.10 holds, with $V_2 = [V, Y_+]$, and $N_G(V_2) \leq M$ by 15.3.11.3. Now $\mathrm{End}_{L/O_2(L)}(U(L)) \cong \mathbf{F}_2$ so that Y_C centralizes $U(L)$. Thus $V_2 = [V_2, Y_+] \leq [U(L)C_R(L), Y_C] \leq C_R(L)$. Then $L \leq N_G(V_2) \leq M$, contrary to the choice of L. □

LEMMA 15.3.20. *$L/O_2(L)$ is not $SL_3(2^n)$ with $n > 1$ or $Sp_4(4)$.*

PROOF. Assume otherwise. By 15.3.16, L is described in C.2.7.3, and in particular as W_L is an FF-module for L^+R^+, S is trivial on the Dynkin diagram of L^+ by Theorem B.4.2. Further as S normalizes Y, $S_L Y_+ = Y_+ S_L$, so as each solvable overgroup of S_L in LY_+ is 2-closed, Y_+ acts on S_L. Thus Y_+S acts on both maximal parabolics P_i of L. If $X_i := P_i Y_+ S \not\leq M$, then $X_i \in \mathcal{H}_+$, contrary to 15.3.19. Thus $L = \langle P_1, P_2\rangle \leq M$, contrary to the choice of L. □

LEMMA 15.3.21. *L is not a block of type A_6, $G_2(2)$, $\hat{A}_6$, or A_7, and L is not an exceptional A_7-block.*

PROOF. Assume L is one of the blocks appearing in 15.3.21. By 15.3.17, $Y_+ \leq L$, so that $Y_+ = Y_L$. As $Y_+ S_L = S_L Y_+$, L is not of type A_6 or $G_2(2)$, since no proper parabolic in these groups has 3-rank 2. Similarly if $L/O_2(L) \cong \hat{A}_6$, the preimage of a proper parabolic does not contain 3^{1+2}, and if $L/O_2(L) \cong A_7$, then L has abelian Sylow 3-groups; thus $Y_+S/R \cong S_3 \times S_3$ in these two cases. Hence L is not an an exceptional A_7-block, since in that case $LS/O_2(LS) \cong A_7$ rather than S_7. Further L is not an ordinary A_7-block, since in that case M_L^+ has no normal E_9-subgroup by C.2.7.3. This leaves the case where L is an $\hat{A}_6$-block, where by C.2.7.3, $S^+Y_+^+$ is the stabilizer of a 2-dimensional $\mathbf{F}_4$-subspace U of $[W_L, L]$. Now $[W_L, L]$ has the structure of an $\mathbf{F}_4 L$-module on which $Z(L^+)$ induces scalars in $\mathbf{F}_4$, and $U = U_1 \oplus U_2$ is the sum of two Y_+-invariant $\mathbf{F}_4$-points, so $U_i = V_i$. Thus is impossible as S interchanges the two $\mathbf{F}_4$-points in an $\hat{A}_6$-block, but S acts on V_1 and V_2 by definition. □

LEMMA 15.3.22. *$L/O_2(L) \cong L_n(2)$ for $n = 3$, 4, or 5.*

Proof. Observe that 15.3.19, 15.3.20, and 15.3.21 have eliminated all other possibilities for L^* from the list of C.2.7.3 provided by 15.3.16. Then as $L/O_2(L)$ is quasisimple by 15.3.15.2, and the Schur multiplier of L^* is a 2-group by I.1.3, $O_{2,Z}(L) = O_2(L)$ so that $L/O_2(L) \cong L^*$ is simple. □

Lemma 15.3.23. *$L/O_2(L)$ is not $L_3(2)$.*

Proof. Assume $L/O_2(L) \cong L_3(2)$, and set $U_L := [W_L, L]$. By 15.3.12.2, $Y_+ = Y_LY_C$ with $|Y_L|_3 = 3 = |Y_C|$ and $Y_+/O_2(Y_+) \cong E_9$. By 15.3.16, R acts on L, and by C.2.7.3, $M_L = S_LY_L$ is a minimal parabolic of L and $R = O_2(Y_+S) = O_2(LR)O_2(Y_L)$.

By C.2.7.3, (LR, R) is described in Theorem C.1.34. In particular L has $k := 1$, 2, 3, or 6 noncentral 2-chief factors. If $k = 6$, then case (4) of C.1.34 holds so that $m(\Omega_1(Z(S))) = 3$, contrary to 15.3.8. Thus $1 \le k \le 3$.

From C.1.34, either U_L is the direct sum of $s \le 2$ isomorphic natural modules, or U_L is a 4-dimensional indecomposable. Thus either $[U_L, Y_C] = 1$, or $s = 2$ and $U_L = [U_L, Y_C]$.

Assume first that $[V, Y_L] = 1$. Then as $Y_L \ne 1$ and Y is faithful on V in case (1) of 15.3.7, case (2) of 15.3.7 holds with $Y/O_2(Y) \cong 3^{1+2}$, and $Y_L = O^2(C_Y(V))$. We saw $Y_+/O_2(Y_+) \cong E_9$, so $Y_+ < Y$, and hence case (2) of Hypothesis 15.3.10 holds. Then $N_G(V_i) \le M$ by 15.3.11.3, Y_+ centralizes V_1, and $V_2 = [V_2, Y_+] = [V_2, Y_C] \le C_{W_L}(Y_L)$. As $N_G(V_i) \le M$ but $L \not\le M$, L centralizes neither V_1 nor V_2. From the previous paragraph, U_L is either a sum of isomorphic natural modules, or a 4-dimensional indecomposable with a natural quotient, while M_L is a minimal parabolic of L with $R = O_2(M_LR)$. Thus $V \le Z(R)$ while the fixed points of the unipotent radical R on any extension in B.4.8 of U_L with trivial quotient lie in U_L, so we conclude that

$$V \le U_LC_{W_L}(L).$$

By the previous paragraph, either $[U_L, Y_C] = 1$, or $s = 2$ and $U_L = [U_L, Y_C]$. In the first case,

$$V_2 = [V_2, Y_C] \le [U_LC_{W_L}(L), Y_C] = [C_{W_L}(L), Y_C] \le C_{W_L}(L),$$

contrary to an earlier remark. In the second case,

$$V_1 = C_V(Y_C) \le C_{U_L}(Y_C)C_{W_L}(LY_C) = C_{W_L}(LY_C),$$

contrary to the same remark.

Therefore $[V, Y_L] \ne 1$. Now $Y_L = [Y_L, S_L]$ while $[Y_C, S_L] \le O_2(Y_C)$, so from the action of S on Y_+, Y_L and Y_C are normal in Y_+S, and $\{Y_L, Y_C\}$ is the set $\mathcal{Y}$ of S-invariant subgroups of Y_+ with Sylow 3-group of order 3. In particular, S acts on Y_L and hence on L.

Suppose that case (2) of Hypothesis 15.3.10 holds. Then by 15.3.14.2, $Y/O_2(Y) \cong 3^{1+2}$ with $C_Y(V) > O_2(Y)$. As $\mathcal{Y} = \{Y_L, Y_C\}$ while $[V, Y_L] \ne 1$, it follows that $Y_C = O^2(C_Y(V))$, so $N_G(Y_C) \le M$ as $M = !\mathcal{M}(N_G(Y_C))$ by 15.3.7.2. But L normalizes $O^2(Y_CO_2(L)) = Y_C$, contradicting $L \not\le M$.

Thus we are in case (1) of Hypothesis 15.3.10, so $Y = Y_+$, and hence from earlier discussion, $Y = Y_CY_L$ and $Y/O_2(Y) \cong E_9$. Therefore we may take $Y_C = O^2(M_1)$ and $Y_L = O^2(M_2)$, since $\{Y_L, Y_C\} = \mathcal{Y}$. Thus $Y_L^t = Y_C$, and $V_2 = [V, Y_L]$. As $V_2 = [V, Y_L]$ is S-invariant, Y_LS_L is the parabolic of L stabilizing the line V_2 in $Z(O_2(L))$. Hence case (5) of C.1.34 does not hold, as no such line exists in that case.

Set $Q := [O_2(L), L]$ as in C.1.34, and observe that $[Z, L] \leq U_L \leq Q$. We will complete the proof by showing that for $t \in T - S$, $W := QQ^t$ is normalized by LS, and hence also by T as $|T : S| = 2$. Then as $Y = Y_L Y_C \leq \langle Y_L, t\rangle$, $\langle L, T\rangle \leq N_G(W) \leq M = !\mathcal{M}(YT)$ by 15.3.7, contrary to the choice of L.

Assume that $k = 3$, so that case (3) of C.1.34 holds. Then as Y_L stabilizes the line V_2 in the natural module $Z(Q)$, $Q = [Q, Y_L]$, so $Q \leq O_2(Y_L)$. Further Y_C centralizes the natural module $Z(Q)$ since $\mathrm{End}_{L/O_2(L)}(Z(Q) = \mathbf{F}_2$. As $Q/Z(Q)$ is the direct sum of two natural modules, either Y_C centralizes Q, or $Q = [Q, Y_C]$. In the latter case $Q = O_2(Y_C) \cap O_2(Y_L)$, so Q is t-invariant, whereas Y_C and Y_L have three and two nontrivial 2-chief factors, on $Q/Z(Q)$, respectively. Therefore $[Q, Y_C] = 1$, so $Y_C = O^2(Y_C)$ centralizes L by Coprime Action. Then $Q^t \leq O_2(Y_L)^t = O_2(Y_C) \leq C_S(L)$, so that $W = QQ^t \trianglelefteq LS$, which suffices as mentioned above.

Suppose finally that $k = 1$ or 2, so that case (1) or (2) of C.1.34 holds. In each case $Q = [Q, Y_L]C_Q(P_L)$, for $P_L \in Syl_3(Y_L)$, and as Y_C centralizes the line V_2 stabilized by Y_L in a natural submodule in Q, Y_C centralizes L from the structure of $Aut(L)$. Thus $Q = O_2(Y_L)C_Q(P) \leq O_2(Y_L)C_S(P)$, for $P \in Syl_3(Y)$, and by a Frattini Argument we may assume $t \in T - S$ normalizes P, and hence also $C_S(P)$. Therefore $Q^t \leq O_2(Y_C)C_S(P) \leq O_2(LS)$, so $[Q^t, LS] \leq [O_2(LS), L] = Q$, and hence $W = QQ^t \trianglelefteq LS$, which again suffices as mentioned earlier. □

LEMMA 15.3.24. *$L/O_2(L)$ is not $L_4(2)$ or $L_5(2)$.*

PROOF. Assume $L/O_2(L) \cong L_n(2)$ for $n := 4$ or 5. Then Y_+ is solvable and S-invariant of 3-rank 2, $Y_+ \leq L$ by 15.3.17, and $S \cap L \in Syl_2(L)$ as $S \in Syl_2(I)$. Thus $LS \in \mathcal{H}_+$, so we may take $I = LS$, and hence $U_I = \langle Z^L\rangle$. As Sylow 3-subgroups of L are isomorphic to E_9, $Y_+/O_2(Y_+) \cong E_9$ rather than 3^{1+2}. Then $Y_+S/R \cong S_3 \times S_3$ from the action of S on Y, so S is trivial on the Dynkin diagram of $L/O_2(L)$.

Suppose first that $n = 4$. Then Y_+S is the maximal parabolic of LS over S determined by the end nodes, so $Y_+S_L = L \cap M$ as $L \not\leq M$. This parabolic has unipotent radical $R_L/O_2(L) \cong E_{2^4}$.

Set $U_L := [W_L, L]$. By 15.3.16, $L = [L, J(R)]$, so there are FF-offenders on U_L with respect to R, and in particular U_L is an FF-module for $L/O_2(L)$. As $1 \neq [Z, Y_+] \leq U_L$, $U_L/C_{U_L}(L)$ is not the orthogonal module, so we conclude from Theorem B.5.1 that U_L is either the sum of a natural module and its dual, or the sum of at most $n-1$ isomorphic natural modules. Now by B.2.14, $U_L Z = U_L C_{U_L Z}(L)$, and we let Z_L denote the projection of Z on U_L with respect to this decomposition. Then $Z \leq Z_L C_{W_L}(L)$, so that $C_L(Z_L) = C_L(Z)$.

Assume first that U_L is a sum of isomorphic natural modules. Then $W_L = U_L C_{W_L}(L)$ by I.1.6.6. Also $1 \neq O^2(C_{Y_+}(Z_L)) = O^2(C_{Y_+}(Z)) \leq O^2(C_{Y_+}(V))$, so case (2) of 15.3.7 holds. Hence $Y_+ < Y$, so that case (2) of Hypothesis 15.3.10 holds, and thus $N_G(V_1) \leq M$ by 15.3.11.3. But now $V_1 \leq C_{W_L}(Y_+) = C_{W_L}(L)$, so $L \leq C_G(V_1) \leq M$, contrary to the choice of L.

Therefore U_L is the sum of a natural module and its dual. Since $Y_+S_L = L \cap M$ is the maximal parabolic over S_L determined by the end nodes, each $L_3(2)$-parabolic P over S_L satisfies $P \not\leq M$ and $[Z_L, O^2(Y_+ \cap P)] \neq 1$. Then applying 15.3.11.12 to P in the role of "L", no nontrivial characteristic subgroup of S is normal in PS. Thus $(O^2(P)S, S)$ is an MS-pair, and hence is described in C.1.34. But P has two noncentral chief factors on U_L and one on $O_2(P)/O_2(L)$, so case (3) or (4) of C.1.34

holds, since in the other cases in C.1.34 there are at most two such factors. Case (4) is eliminated as $m(\Omega_1(Z(S))) \geq 3$, contrary to 15.3.8. Suppose case (3) holds, set $Q_P := [O_2(P), P]$, and let $W_L = W_1 \oplus W_2$ with $W_i \in Irr_+(L, W_L)$; we may choose notation so that $[W_1, P]$ is of rank 3 and $W_2 = [W_2, P]$. Then $Z(Q_P)$ is a natural module for P/Q_P and $Q_P/Z(Q_P)$ is the sum of two copies of the dual of $Z(Q_P)$, impossible as $[W_1, P]C_{W_2}(P) \leq Z(Q_P)$.

So $n = 5$. Let P be a maximal parabolic of L over S_L containing Y_+. Since $Y_+S/R \cong S_3 \times S_3$, L is generated by such parabolics, so we may assume $P \not\leq M$. Thus $PS \in \mathcal{H}_+$, and we obtain a contradiction from earlier reductions as $P/O_2(P) \cong S_3 \times L_3(2)$ or $L_4(2)$. □

Observe that 15.3.22, 15.3.23, and 15.3.24 establish Theorem 15.3.13.

We now complete the elimination of the case $F^*(I) = O_2(I)$ under Hypothesis 15.3.10, by treating the remaining subcase where $m_3(Y_+) = 1$ in the following result:

THEOREM 15.3.25. *If $I \in \mathcal{H}_+$ then $F^*(I) \neq O_2(I)$.*

Until the proof of Theorem 15.3.25 is complete, assume I is a counterexample. The proof will be largely parallel to that of Theorem 15.3.13, except this time our list of possibilities for L^* will come from Theorem B.5.6 rather than C.2.7.3, and the elimination of those cases will be somewhat simpler. As $F^*(I) = O_2(I)$, $U_I = \langle Z^I \rangle \in \mathcal{R}_2(I)$ by B.2.14.

LEMMA 15.3.26. *(1) Case (2) of Hypothesis 15.3.10 holds, $Y_+S/R \cong S_3$, $Y_+ = O^{3'}(M_I)$, and $R = C_S(V_2)$.*

(2) There is $L \in \mathcal{C}(J(I))$ with $L \not\leq M_I$, $L = [L, Y_+]$, and L^* and $L/O_2(L)$ quasisimple.

(3) Each solvable Y_+S-invariant subgroup of I is contained in M_I.

PROOF. By Theorem 15.3.13, $m_3(Y_+) = 1$. Then (1) follows from 15.3.11.5. The proofs of (2) and (3) are the same as those in 15.3.15. □

During the remainder of the proof of Theorem 15.3.25, pick L as in 15.3.26.2. As L^* is a component of $J(I)^*$, $L^* = [L^*, J(S)^*]$ and $U_L := [U_I, L]$ is an FF-module for $Aut_{LJ(S)}(U_L)$ by B.2.7, so L^* is described in Theorem B.5.6.

Recall $\mathcal{H}_{+,*}$ denotes the set of members of of $\mathcal{H}_+$ minimal under inclusion.

LEMMA 15.3.27. *(1) $L \trianglelefteq I$.*

(2) $Y_+ \leq L$.

(3) L^* is not $SL_3(2^n)$, n even, or $\hat{A}_6$.

(4) If $I \in \mathcal{H}_{+,*}$, then $I = LS$, and M_I is the unique maximal subgroup of I containing Y_+S.

PROOF. Suppose first that L is not normal in I, and let $L_0 := \langle L^S \rangle$. By 1.2.1.3 and Theorem B.5.6, $L^* \cong L_2(2^n)$ or $L_3(2)$. By 1.2.2, $L_0 = O^{3'}(I)$, so $Y_+ \leq L_0$. Then as $Y_+S/R \cong S_3$ by 15.3.26.1, and $S \in Syl_2(I)$ by 15.3.11.1, $L^* \cong L_2(2^n)$—since when $L^* \cong L_3(2)$, there is no S-invariant subgroup of L_0 with Sylow 3-group of order 3. Let B_0 be the Borel subgroup of L_0 containing $S \cap L_0$; then $M_0 := M \cap L_0 \leq B_0$, and B_0 is M_I-invariant and solvable, so $M_0 = B_0$ by 15.3.26.3. Then as $Y_+ \leq B_0$, n is even. But now $m_3(M_I) \geq m_3(B_0) > 1$, contrary to 15.3.26.1. This contradiction establishes (1).

By 15.3.26.2, $L^* = [L^*, Y_+^*]$. Comparing the list of Theorem B.5.6 with that in A.3.18, we conclude that one of the following holds:

(i) $m_3(L) = 1$ and L^* is $L_2(2^n)$ or $L_3(2^m)$, m odd.
(ii) $L = \theta(I)$.
(iii) $L^* \cong SL_3(2^n)$, n even.

In case (ii), (2) holds. In case (i), as $Y_+S/R \cong S_3$ and $Out(L/O_2(L))$ is abelian, Y_+^* induces inner automorphisms on L^*. Then the projection Y_L^* of Y_+^* on L^* is contained in M_I^* by 15.3.26.3, so the preimage Y_L is contained in $O^{3'}(M_I) = Y_+$ by 15.3.26.1, so that (2) holds again. Finally if (iii) holds, then $Z(L)^* \leq M_I^*$ by 15.3.26.3, so $O^{3'}(M_I)^* = Y_+^* = Z(L^*)$ by 15.3.26.1, contradicting $L^* = [L^*, Y_+^*]$. This completes the proof of (2), and the same argument shows that L^* is not $\hat{A}_6$, completing the proof of (3) also.

Finally assume that $I \in \mathcal{H}_{+,*}$. By (1), S acts on L, and by (2), $Y_+ \leq L$. Thus as $L \not\leq M$, $LS \in \mathcal{H}_+$, so $I = LS$ by minimality of I. Similarly M_I is the unique maximal subgroup of I containing Y_+S, so (4) holds. □

Until the proof of Theorem 15.3.25 is complete, assume $I \in \mathcal{H}_{+,*}$. Thus $I = LS$ by 15.3.27.4.

LEMMA 15.3.28. *Let $M_L := M \cap L$; then one of the following holds:*

(1) $L^* \cong L_2(2^n)$, n even, and M_L^* is a Borel subgroup of L^*.

(2) $L^* \cong L_3(2)$, $Sp_4(2)'$, or $G_2(2)'$, and M_L^* is a minimal parabolic of L^*, so that S is trivial on the Dynkin diagram of L^*.

(3) $I^* \cong S_8$ and M_L^* is the middle-node minimal parabolic of L^*S^*.

PROOF. Suppose first that $L^* \cong A_7$. Then as $Y_+S/R \cong S_3$ by 15.3.26.1, $Y_+^*S^*$ is either the stabilizer of a partition of type $2^3, 1$, or is contained in the stabilizer $I_{4,3}^*$ of a partition of type $4, 3$. In the latter case, the preimage $I_{4,3}$ is contained in M_I by 15.3.27.4, whereas $m_3(M_I) = 1$ by 15.3.26.1. In the former, $Y_+^*S^* \leq I_1^* \cong A_6$ or S_6, and this time $I_1 \leq M_I$ for the same contradiction.

Then by 15.3.27.3 and Theorem B.5.6, L^* is of Lie type and characteristic 2, so as $S \in Syl_2(I)$, M_L^* is a maximal S-invariant parabolic of L^* by 15.3.27.4. As $O^{3'}(M_L^*) = Y_+^*$ with $Y_+S/R \cong S_3$ by 15.3.26.1, we conclude by inspection of the list of Theorem B.5.6 and appeals to parts (3) and (4) of 15.3.27 that one of cases (1)–(3) of the lemma holds. □

LEMMA 15.3.29. *(1) No nontrivial characteristic subgroup of S is normal in I.*

(2) $N_G(V_1) \leq M$ and V_1 centralizes Y_+.

PROOF. As $Y_+ \leq L$ by 15.3.27.2 and $I = LS$, we may apply 15.3.11.12 to obtain (1). By 15.3.26.1, case (2) of Hypothesis 15.3.10 holds, so that V_1 centralizes Y_+, and $N_G(V_1) \leq M$ by 15.3.11.3, so (2) holds. □

LEMMA 15.3.30. *L^* is not $L_2(2^n)$.*

PROOF. Assume L^* is $L_2(2^n)$. By 15.3.28, M_L^* is a Borel subgroup of L^*, so as $R = O_2(Y_+S)$, $R \in Syl_2(LR)$. Then by 15.3.27.4, LR is a minimal parabolic in the sense of Definition B.6.1, so we conclude from 15.3.29.1 and C.1.26 that L is a block of type $L_2(2^n)$ or A_5. Next M_I acts on $[V, Y_+] = V_2 \cong E_4$, so $V_2 \leq U(L)$ is an M_L-invariant line. Thus L is not an A_5-block, so L is an $L_2(2^n)$-block and in particular

$C_{Aut(L)}(Y_+) = 1$. Then by 15.3.29.2, $V_1 \leq C_S(Y_+) \leq C_S(L)$, so $L \leq C_G(V_1) \leq M$ by 15.3.29.2, contrary to the choice of L. □

LEMMA 15.3.31. *$I^* \cong S_8$.*

PROOF. Assume otherwise; by 15.3.30 and 15.3.28, we may assume that case (2) of 15.3.28 holds; that is $L^* \cong L_3(2)$, $Sp_4(2)'$, or $G_2(2)$. As $V_2 = [V_2, Y_+] \cong E_4$ is a Y_+S-invariant line in U_L, it follows from Theorem B.5.6 that M_L^* is the parabolic stabilizing the line V_2 in some $W \in Irr_+(L^*, U_L)$, and hence for each such W when $[\Omega_1(Z(S)), L]$ is a sum of two isomorphic natural modules for $L^* \cong L_3(2)$. By 15.3.29.1, (LS, S) is an MS-pair in the sense of Definition C.1.31, and by 15.3.27.3, L is not a $\hat{A}_6$-block. Therefore by C.1.32, either L is a block of type A_6 or $G_2(2)$, or $L^* \cong L_3(2)$ and L is described in C.1.34. In particular, $L/O_2(L)$ is simple in each case, so that $L/O_2(L) = L^*$. Set $Q := [O_2(L), L]$. As $L \trianglelefteq I = LS$, $Q = [O_2(I), L]$.

In case (4) of C.1.34, $m(\Omega_1(Z(S))) = 3$, contrary to 15.3.8, and case (5) of C.1.34 does not hold, as M_L^* stabilizes the line V_2. Thus only cases (1)–(3) of C.1.34 can arise when $L^* \cong L_3(2)$.

Next by 15.3.29.2, $V_1 \leq C_I(Y_+) =: D$. It will suffice to show that $D = C_I(L)$: for then $L \leq C_G(V_1) \leq M$ by 15.3.29.2, contrary to the choice of L. Set $I^+ := I/C_I(L)$; it remains to show that $D^+ = 1$.

Suppose that D centralizes Q. Then $[D, Q] \leq C_L(Q) \leq O_2(L)$, so as L/Q is quasisimple, $[D, L] \leq Q$. Thus $[D, L] \leq C_Q(Q) = Z(Q)$. Therefore $O^2(D^+) = 1$ by Coprime Action. Further as $\Phi(Z(Q)) = 1$, $\Phi(D^+) = 1$ (cf. the argument in the proof of C.1.13); so as Y_+ centralizes D, $D^+ = 1$ from the structure of the the covering of the L^*-module $Z(Q)/C_{Z(Q)}(L)$ in I.2.3.

Therefore we may assume $[D, Q] \neq 1$. Thus $Q \not\leq Y_+$. In case (3) of C.1.34, $Z(Q)$ is a natural module for L^* and $Q/Z(Q)$ is a sum of two modules dual to $Z(Q)$. In this case, and when L is a block of type A_6 or $G_2(2)$, since M_L^* is the parabolic stabilizing the line V_2 in $Z(Q)$, $Q = [Q, Y_+] \leq O_2(Y_+)$. Therefore case (1) or (2) of C.1.34 holds. Then $C_{I^*}(Y_+^*) = 1$, so $[D, L] \leq Q$. Further the intersection of Y_+ with each $W \in Irr_+(L, Q)$ is a hyperplane W_0 of W, so as $DQ \trianglelefteq DL$ and Q is abelian, DQ centralizes $\langle W_0^L \rangle = W$. Therefore DQ centralizes Q, a contradiction completing the proof. □

Now $L/O_2(L)$ is quasisimple by 15.3.26.2, $L^* \cong A_8$ by 15.3.31, and the Schur multiplier of A_8 is a 2-group by I.1.3. Then as $I = LS$, $O_2(I) = C_I(U_L) = C_S(U_L)$.

LEMMA 15.3.32. *(1) U_L is the 6-dimensional orthogonal module for I^*.*

(2) $C_{Z_S}(L) = Z_S \cap V_1 =: Z_1$ is of order 2.

(3) $L = O^{3'}(C_G(Z_1))$.

(4) $X := O^{3'}(C_G(Z_S)) = O^{3'}(K)$, where K is the maximal parabolic of L over $S \cap L$ determined by the end nodes of the diagram of L^*.

(5) $W := \langle U_L, U_L^t \rangle = U_L \times U_L^t$ for $t \in T - S$, and XS normalizes W.

(6) Let $(XS)^+ := XS/C_S(W)$ and $P := O_2(XS)$. Then $P^+ = C_S(U_L)^+ \times C_S(U_L^t)^+$.

PROOF. By 15.3.31, $I^* \cong S_8$, and then by 15.3.28.3, M_L^* is the middle-node minimal parabolic. Therefore as U_L is an FF-module for I^*, B.5.1 says that either $U_L/C_{U_L}(L)$ is the orthogonal module, or U_L is the sum of a natural module and its dual. Then as $V_2 = [V_2, Y_+]$ is an S-invariant line of U_L, the former case holds with $C_{U_L}(L) = 1$, giving (1). Recall that $Z_S \cong E_4$ by 15.3.8, and that $Z_i := Z_S \cap V_i$ is of order 2 for $i = 1, 2$. Further $Z_1V_2 = ZV_2 = \langle Z^{Y_+}\rangle \leq \langle Z^L\rangle = U_I$, so $U_I = Z_1U_L$. Then as $Z_1 \leq Z(S)$, $U_I = U_LC_{U_I}(LS)$ by B.2.14, and $C_{U_I}(LS) = C_{Z_S}(L) = C_{Z_S}(Y_+) = Z_1$, so (2) holds.

Next $I_1 := C_G(Z_1) \in \mathcal{H}_+$, so $S \in Syl_2(I_1)$ by 15.3.11.1. Thus $L \leq L_1 \in \mathcal{C}(I_1)$ by 1.2.4, and A.3.12 says that either $L = L_1$ or $L_1/O_2(L_1) \cong L_5(2)$, M_{24}, or J_4. As S is nontrivial on the Dynkin diagram of L^*, it follows that $L = L_1$, and then (3) folllows from A.3.18.

Let K be the S-invariant maximal parabolic of L, and set $X := O^2(K)$ and $P := O_2(XS)$. Thus $XS/P \cong S_3$ wr $\mathbf{Z}_2$ is determined by the end nodes of the Dynkin diagram of L^*. By (1), $XS = C_I(Z_S)$, so (3) implies (4). Then as $Z_S \trianglelefteq T$, T acts on $O^2(C_I(Z_S)) = X$ and P. Let $t \in T - S$. Then T acts on $U_L \cap U_L^t$, so if $U_L \cap U_L^t \neq 1$, then $Z \leq U_L \cap U_L^t$, whereas $Z_S \cap U_L = Z_S \cap V_2 = Z_2$. Thus $U_L \cap U_L^t = 1$, so as $U_L \trianglelefteq XS$ and T acts on XS, (5) holds.

Adopt the notation of (6) and let $P_I := O_2(I)$. As XS is irreducible on P^*, either $P_I^t \leq P_I$, or $P = P_IP_I^t$ and (6) follows from (5). But in the former case $P_I^t = P_I$, so that $\langle T, L\rangle$ acts on P_I; then as $Y_+ \not\leq C_M(V)$, $L \leq M$ by 15.3.2.4, contrary to the choice of L. Thus (6) holds. $\square$

We can now complete the proof of Theorem 15.3.25. Let X, W, and P be defined as in 15.3.32.

Let $\mathcal{B}$ be the set of $A \in \mathcal{A}(S)$ such that $A^* \neq 1$, and A^* is minimal subject to this property. Choose some $A \in \mathcal{B}$. By B.2.5, $A^* \in \mathcal{P}^*(I^*, U_L)$. Now B.3.2.6 describes the possible FF-offenders, and the only strong FF-offender is generated by four transvections; so from B.2.9 we conclude that one of the following holds:

(i) $A^* \cong E_8$ is regular on $\Omega := \{1, \ldots, 8\}$.
(ii) A^* is generated by a transposition.
(iii) $A^* = D := \langle (1, 2), (3, 4), (5, 6), (7, 8)\rangle$.
(iv) $A^* = D \cap L^*$.

In particular in each case, $A \not\leq P$. Further $m(A^*) = m(U_L/C_{U_L}(A))$ except in case (iii), where $m(A^*) = 4$ and $m(U_L/C_{U_L}(A)) = 3$.

Let $\mathcal{C} := \mathcal{B} \cap \mathcal{B}^t$ for $t \in T - S$. As $A \not\leq P$, $Aut_A(U_L^t) \neq 1$ by 15.3.32.6, so there is $A_+ \leq A$ such that $Aut_{A_+}(U_L^t) \in \mathcal{P}^*(Aut_{I^t}(U_L^t), U_L^t)$ by B.1.4.4. Then $A_+ \not\leq P$ by the previous paragraph applied to U_L^t in place of U_L, so $A_+^* \neq 1$ again using 15.3.32.6. Hence by minimality of A^*, $A_+^* = A^*$. Thus $A \in \mathcal{C}$.

Let $\widehat{XT} := XT/O_2(XT)$, so that $\hat{S} \cong D_8$, and set $S_0 := S \cap LO_2(LS)$. Observe:

(I) In (i), $|\hat{A}| = 2$ and $\hat{S}_0 = \hat{A} \times Z(\hat{S})$.
(II) In (ii), $|\hat{A}| = 2$ and $\hat{A} \not\leq \hat{S}_0$.
(III) In (iii), $\hat{A}$ is the 4-subgroup of $\hat{S}$ distinct from $\hat{S}_0$.
(IV) In (iv), $\hat{A} = Z(\hat{S})$.

Let $B := C_A(W)$, $m_0 := m(\hat{A})$, $m_1 := m(A^* \cap P^*)$, $m_2 := m(Aut_{A\cap P}(U_L^t))$, $m_3 := m(U_L/C_{U_L}(A))$, and $m_4 := m(U_L^t/C_{U_L^t}(A))$. Then $m(A) \leq m_0 + m_1 + m_2 + m(B)$. Also $m(BW) = m(B) + m_3 + m_4$. Therefore as $m(A) \geq m(BW)$ since $A \in \mathcal{A}(S)$,

$$m_0 + m_1 + m_2 \geq m_3 + m_4. \qquad (!)$$

Suppose first that $\hat{S} < \hat{T}$. Then $\hat{T}$ is Sylow in $GL_2(3)$, so $\hat{T} \cong SD_{16}$, and hence $\hat{S}_0^t$ is the 4-subgroup in $\hat{S}$ distinct from $\hat{S}_0$. As $A \in \mathcal{C}$, it satisfies one of conclusions

(I)–(IV), and also one of the analogous conclusions on U_L^t. Then inspecting (I)–(IV), we conclude that either:

(a) A^* is in case (ii) and $Aut_A(U_L^t)$ is in case (i), or vice versa; or:
(b) A^* and $Aut_A(U_L^t)$ are in case (iv).

However in case (a), the tuple of parameters $(m_0, m_1, m_2, m_3, m_4)$ is $(1,0,2,1,3)$, contrary to (!). Similarly in case (b), the tuple is $(1,2,2,3,3)$, again contrary to (!).

Thus $\hat{T}=\hat{S}$. So as $\hat{S}_0$ and $Z(\hat{S})$ are normal in $\hat{S}$, their preimages are normal in T. Then inspecting (I)–(IV), we conclude that A^* and $Aut_A(U_L^t)$ always appear in the same case of (i)–(iv). In cases (i), (ii), and (iv), we calculate the tuple of parameters to be $(1,2,2,3,3)$, $(1,0,0,1,1)$, and $(1,2,2,3,3)$, again contrary to (!). We conclude A^* is in case (iii). In particular, $A \not\leq S_0$; so since A is an arbitary member of $\mathcal{B}$, it follows that $J(S_0) \leq C_I(U_L)$. Thus $J(S_0) = J(C_I(U_L)) \trianglelefteq \langle T, L\rangle$, again contrary to $L \not\leq M = !\mathcal{M}(YT)$ by 15.3.2.4 since $Y_+ \not\leq C_M(V)$.

This completes the proof of Theorem 15.3.25.

Theorem 15.3.25 has reduced the treatment of Hypothesis 15.3.10 to the case $F^*(I) \neq O_2(I)$. As $O(I)=1$ by 15.3.11.8, there is a component L of I, and $Z(L)=O_2(L)$. As $F^*(M_I)=O_2(M_I)$ by 15.3.11.7, $L \not\leq M$. Thus to complete our proof that $M = !\mathcal{M}(Y_+S)$, it remains to determine the possibilities for L, and then to eliminate each possibility.

Set $L_0 := \langle L^S\rangle$, $S_L := S \cap L$, and $M_L := M \cap L$. Let z denote a generator of Z.

LEMMA 15.3.33. *(1) If L_1 is a Y_+S-invariant subgroup of L_0 with $F^*(L_1) = O_2(L_1)$, then $L_1 \leq M_I$.*

(2) The hypotheses of 1.1.5 are satisfied with I, M_c is the roles of "H, M".

(3) If $K \in \mathcal{C}(I)$ then $K \not\leq M$, $K=[K,z]$ is described in 1.1.5.3, $O(K)=1$, and

$$F^*(C_K(z)) = O_2(C_K(z)).$$

PROOF. Choose L_1 as in (1); if $L_1 \not\leq M$, then $L_1Y_+S \in \mathcal{H}_+$, contrary to Theorem 15.3.25, so (1) holds. Next let $H \in \mathcal{M}(I)$, so in particular $H = N_G(O_2(H))$. Then $H \in \mathcal{H}_+$, so $S \in Syl_2(H)$ by 15.3.11.1. Thus $S = T \cap H$ and $O_2(H) \leq O_2(I \cap M_c)$ by A.1.6, so

$$C_{O_2(M_c)}(O_2(I\cap M_c)) \leq C_T(O_2(H)) \leq T\cap H = S \leq I,$$

and hence (2) follows since $C_G(z) = M_c$ by 15.3.4. Then (2) and 1.1.5 imply (3), since we saw $O(I)=1$. □

Observe that 15.3.33.3 applies to L in the role of "K". We now begin to eliminate the various possibilities for L in 1.1.5.3.

LEMMA 15.3.34. *$L/Z(L)$ is not $Sz(2^n)$. Hence $m_3(L) \geq 1$.*

PROOF. If $L/Z(L) \cong Sz(2^n)$, then Y_+S acts on a Borel subgroup B of L_0, so $B = M_I \cap L_0$ by 15.3.33.1, since B is a maximal S-invariant subgroup of L_0. By 15.3.11.4, $M_I = N_I(Y_+)$, so as automorphisms of $L_0/O_2(L_0)$ of order 3 acting on B are nontrivial on $B/O_2(L_0)$, we conclude $Y_+ \leq C_I(B) = C_I(L_0)$. Thus $L_0 \leq N_I(Y_+) = M_I$ by 15.3.11.4, contrary to our choice of L. □

Again we will divide the proof into two cases: $m_3(Y_+) = 2$ and $m_3(Y_+) = 1$. We eliminate the first case in the next theorem:

THEOREM 15.3.35. *Case (2) of Hypothesis 15.3.10 holds, $Y_+S/R \cong S_3$, $Y_+ = O^{3'}(M_I)$, $Y_+ < Y$, and $R = C_S(V_2)$.*

Until the proof of Theorem 15.3.35 is complete, assume I is a counterexample. Therefore case (i) of 15.3.11.5 holds, so:

LEMMA 15.3.36. *(1) $Y_+S/O_{2,\Phi}(Y_+S) \cong S_3 \times S_3$.*
(2) $R = C_T(V)$.

The next lemma eliminates the shadow of $G \cong A_{10}$, where $L \cong A_6$. It also eliminates the shadows of $G \cong L$ wr $\mathbf{Z}_2$, for various groups L of Lie rank 2 over $\mathbf{F}_2$.

LEMMA 15.3.37. *One of the following holds:*

(1) $Y_+ \leq L_0$.

(2) $L = L_0 \cong L_2(2^n)$ or $U_3(2^n)$ with n even, or $L_3(2)$. Further $Y_+ = Y_LY_C$ where $Y_L := O^2(Y_+ \cap L)$, $Y_C := O^2(C_{Y_+}(L))$, $|Y_L|_3 = 3 = |Y_C|$, and $Y_+/O_2(Y_+) \cong E_9$.

(3) $L = L_0$, with $L \cong L_3(2^n)$, n even, or $L/O_2(L) \cong L_3(4)$. Further $Y_L := O^2(Y_+ \cap L) \neq 1$ and $Y_+ = Y_L\langle y\rangle$ with y of order 3 inducing a diagonal outer automorphism on L.

PROOF. By 15.3.34, $m_3(L) \geq 1$. Thus if $L < L_0$, then $L_0 = O^{3'}(I)$ by 1.2.2, so (1) holds. Therefore we may assume $L = L_0$. By 15.3.12.1, $Y_L \neq 1$.

Suppose first that $m_3(L) = 1$. Then $|Y_L|_3 = 3$. Further by 15.3.33.3 and 1.1.5.3, one of the following holds: L is $L_2(2^n)$, L is $L_3^\delta(2^m)$ with $2^m \equiv -\delta \mod 3$, or L is $L_2(p)$ for some Fermat or Mersenne prime p. Then as $Y_L \neq 1$ and S_L acts on Y_L, n is even in the first case; in the second case, either $L \cong L_3(2)$, or m is even and $L \cong U_3(2^m)$; and in the third case, $p = 5$ or 7, so that L is $L_2(4)$ or $L_3(2)$ and so appears in previous cases. Now if L is $L_2(2^n)$ or $U_3(2^m)$, then M_I acts on the Borel subgroup B of L containing S_L, so $B = M_L$ by 15.3.33.1 and maximality of B in L. Thus B acts on Y_+ by 15.3.11.4. Hence Y_+ induces inner automorphisms on L. This also holds if L is $L_3(2)$ since there $Out(L)$ is a 2-group. Then by 15.3.12.2, $Y_+ = Y_LY_C$ with $|Y_L|_3 = 3 = |Y_C|_3$ and $Y_+/O_2(Y_+) \cong E_9$. Then (2) holds.

Finally suppose $m_3(L) = 2$. Again by 15.3.33.3, L is described in 1.1.5.3 with $O(L) = 1$, and then by A.3.18, either

(i) $L = \theta(I)$, or

(ii) $L \cong L_3^\epsilon(2^n)$ with $2^n \equiv \epsilon \mod 3$, or $L/O_2(L) \cong L_3(4)$. Further some y of order 3 in Y_+ induces a diagonal outer automorphism on L.

In case (i), $Y_+ \leq L$, so that (1) holds. In case (ii), $Y_+ = Y_L\langle y\rangle O_2(Y)$ is S-invariant of 3-rank 2, so $\epsilon = +1$ and hence (3) holds, completing the proof of the lemma. □

The next lemma rules out conclusion (3) of 15.3.37, and eliminates the first appearance of a shadow of $Aut(He)$, where $L/Z(L) \cong L_3(4)$.

LEMMA 15.3.38. *If $m_3(L) = 2$, then $Y_+ \leq L$.*

PROOF. Assume otherwise, and let $I^+ := I/C_I(L)$. Then case (3) of 15.3.37 holds, and in particular some element y of order 3 in Y_+ induces a diagonal outer automorphism on L. If $L/O_2(L) \cong L_3(4)$, let $n := 2$; in the remaining cases $L \cong$

$L_3(2^n)$ with n even. As $S \in Syl_2(I)$ acts on Y_+, Y_+S acts on the Borel subgroup B of L over $S \cap L = S_L$, and then $Y_L := O^2(Y_+ \cap L) \leq B \leq M_I$ by 15.3.33.1. Hence by 15.3.11.6, $M_I = N_I(B)$, n is coprime to 3, and $Y_L/O_2(Y_L) \cong \mathbf{Z}_3$. Also $[V, Y_L] \neq 1$: for otherwise $V \leq C_S(Y_L) = C_S(L)$, so $L \leq N_G(V) = M$, contrary to the choice of L. Since Y_L is S-invariant with $Y_L/O_2(Y_L)$ of order 3, $Y_L \leq M_i$ for $i = 1$ or 2, and we may choose notation so that $i = 2$. Thus $E_4 \cong V_2 = [V, Y_L] \trianglelefteq B$, so $n = 2$, and V_2^+ is the root group $Z(S_L^+)$ of L^+. Further $V_1 = C_V(Y_L) \leq C_G(L)$, so $N_G(V_1) \not\leq M$, and hence by 15.3.11.3, case (1) of Hypothesis 15.3.10 holds, so $Y = Y_+$ and $Y_L = O^2(Y \cap M_2)$. Hence $Y/O_2(Y) \cong E_9$. Let $Y_D := O^2(Y \cap M_1)$ and $t \in T - S$. Then $Y_L^t = O^2(Y \cap M_2)^t = O^2(Y \cap M_1) = Y_D$, and $Y = Y_L Y_D$. As case (3) of 15.3.37 holds, an element of order 3 in Y_D induces a diagonal outer automorphism on L^+. Also $Y_D \leq M_1 \leq C_G(V_2)$, so from the structure of $PGL_3(4)$, $V_2^+ = C_{S_L^+}(Y_D)$ and $S_L = [S_L, Y_D]$. Of course $[S_L, Y_D] \leq O_2(Y_D)$, so as $O_2(Y_L) = S_L$ and $Y_L^t = Y_D$, $S_L = O_2(Y_D)$ by an order argument. Thus t acts on S_L, and hence also on $Z(S_L^+) = V_2^+$. Since t interchanges V_1 and V_2, $V_1 \leq V_2 Z(L)$.

Now $C_{Z(L)}(Y_D) = 1$, and we showed $V_2^+ = C_{S_L^+}(Y_D)$. Further $Z(L) = C_{S_L}(Y_L)$. Thus $V_2 = C_{S_L}(Y_D)$ and $Z(L)^t = C_{S_L}(Y_L^t) = C_{S_L}(Y_D) = V_2 \cong E_4$, so $E_4 \cong V_1 = V_2^t = Z(L)$.

Next $C_S(Y_L) = C_S(L)$, so conjugating by t, $|C_S(Y_D)| = |C_S(L)|$, and as $Y = Y_L Y_D$, $C_S(Y) = C_S(L) \cap C_S(Y_D)$. Then as $C_S(Y_D) \leq V_2 C_S(L)$ and $|V_2 C_S(L) : C_S(L)| = |V_2| = 4$, $|V_2 C_S(L) : C_S(Y_D)| = 4$. Therefore $|C_S(L) : C_S(Y)| = |C_S(L) : C_S(L) \cap C_S(Y_D)| \leq 4$, so as $C_{V_1}(Y_D) = 1$, $C_S(L) = V_1 C_S(Y)$. But by 15.3.8, $\Omega_1(Z(S)) = Z_S \leq V$, and $C_V(Y) = 1$, so we conclude that $C_S(Y) = 1$; hence $V_1 = C_S(L)$. Thus $C_T(V) = O_2(YS) = O_2(Y) = S_L$. As $YS/C_T(V) = \bar{Y}\bar{S} \cong S_3 \times S_3$, we conclude that $L^+ Y_D^+ S^+ = Aut(L^+)$. As $O(I) = 1$, $V_1 = C_S(L) = C_I(L)$, so $L = F^*(I)$ and hence $I = LY_D S$.

Let $X \in Syl_3(Y)$; then $C_{S_L}(X) = 1$, and by a Frattini Argument, $N_{YS}(X) \cong S_3 \times S_3$. Let $E := N_S(X)$; then $E = \langle \tau, f \rangle$, where τ and f are involutions inducing a graph and a field automorphism on L^+, respectively. Further $X = X_L \times X_D$, where $X_A := X \cap Y_A$ for $A := L, D$, f inverts X, and τf centralizes V_1. By a Frattini Argument, we may choose $t \in N_T(X)$, so $\langle t, \tau \rangle \cong \bar{T} \cong D_8$. Thus we may choose t to be an involution and $\tau^t = \tau f$.

Let w be of order 4 in $\langle \tau, t \rangle$, and set $W := \langle w, S_L \rangle$; then $|T : W| = 2$, so as $G = O^2(G)$, $\tau^G \cap W \neq \emptyset$ by Thompson Transfer. As $W/S_L = \bar{W} \cong \mathbf{Z}_4$ and $w^2 = f$, τ is fused to a member of $W_0 := \langle f \rangle S_L$.

Next $V_1 = Z(L) \trianglelefteq I$. Then $I_1 := N_G(V_1) \in \mathcal{H}_+$, so $S \in Syl_2(I_1)$ by 15.3.11.1, and thus $L \leq L_1 \in \mathcal{C}(I_1)$ by 1.2.4. Then $m_3(L_1) \geq m_3(L) = 2$, so applying our reductions so far to I_1, L_1 in the roles of "I, L", we conclude that $I_1/O_2(I_1) \cong Aut(L_3(4))$, and hence $L = L_1$ and $I = I_1$. That is, $I = N_G(V_1)$.

Let $\langle v \rangle = Z_S \cap V_1$, $G_v := C_G(v)$, and $\dot{G}_v := G_v/\langle v \rangle$. Then $\mathbf{Z}_2 \cong \dot{V}_1$ and $\dot{L}\dot{S} = C_{\dot{G}_v}(\dot{V}_1)$ as $I = N_G(V_1)$. By I.3.2, $L \leq O_{2',E}(G_v)$, so $F^*(G_v) \neq O_2(G_v)$, and hence $|G_v|_2 < |T|$ as G is of even characteristic. Thus as $S \leq G_v$ and $|T : S| = 2$, $S \in Syl_2(G_v)$. Therefore $L \leq L_v \in \mathcal{C}(G_v)$ by 1.2.4, with the embedding described in A.3.12. As $L \leq O_{2',E}(G_v)$, L_v is quasisimple. Indeed as $\dot{L}$ is a component of $C_{\dot{L}_v}(\dot{V}_1)$, while the only embedding of $L_3(4)$ appearing in A.3.12 is in M_{23}, and M_{23} has trivial Schur multiplier by I.1.3, we conclude $L_v = L$. Thus $G_v \leq N_G(L) \leq$

$N_G(V_1) = I$, so as M_1 is transitive on $V_1^{\#}$, V_1 is a TI-subgroup of G by I.6.1.1. Then as $V_1 \cong E_4$:

(*) V_1 is faithful on any subgroup $F = O^2(F)$ on which it acts nontrivially.

Recall τ is fused to an element of W_0. But L is transitive on the involutions in fL, and each involution in L is fused into V under L and hence is in $z^G \cup v^G$. Thus to obtain a contradiction and complete the proof, it remains to show that τ is not fused to f, v, or z.

Recall that τf centralizes V_1. Suppose $\tau f = v^g$ for some $g \in G$. Then V_1 normalizes V_1^g since V_1 is a TI-subgroup of G, and hence by I.6.2.1, $[V_1, V_1^g] = 1$. Thus $V_1^g \leq C_G(V_1) = C_I(V_1) = L\langle \tau f \rangle$, so V_1^g centralizes $F := O^2(C_L(\tau f)) \cong E_9$ by (*). Then as $|C_{Aut(L)}(F)| = 2$, $1 \neq V_1^g \cap V_1$, so $V_1^g = V_1$ as V_1 is a TI-subgroup, contrary to $v^g = \tau f \notin V_1$. Thus τf is not fused to v, so as $\tau^t = \tau f$, τ is not fused to v. Similarly using (*) and Generation by Centralizers of Hyperplanes A.1.17, $O(C_G(\tau f))$ centralizes V_1, so $O(C_G(\tau f)) \leq O(C_I(\tau f)) \cong E_9$.

Next $O^2(C_L(\tau)) \cong L_2(4)$, so applying I.3.2 as above, we conclude $F^*(C_G(\tau)) \neq O_2(C_G(\tau))$, and hence τ is not fused to z in G. Therefore τ is fused to f in G, so τf is also.

Let $L_f := O^2(C_L(f))$ and $G_f := C_G(f)$; then $L_f \cong L_3(2)$, and again using I.3.2, $L_f \leq O_{2',E}(G_f)$. We saw earlier that $O(C_G(\tau f))$ is an elementary abelian 3-group of rank at most 2, and f is fused to τf; so $O_{2',E}(G_f)^\infty = E(G_f)$, and hence $L_f \leq E(G_f)$.

Suppose that there is a component L_1 of G_f of 3-rank 1. As f is fused to τf, and a Sylow 3-subgroup of $C_I(\tau f)$ is isomorphic to 3^{1+2}, there is $3^{1+2} \cong B \leq G_f$. But now $m_3(L_1 B) = 3$ from the structure of $Aut(L_1)$ with L_1 of 3-rank 1 in Theorem C (A.2.3), contrary to G_f an SQTK-group. Thus no such component exists. But $L_f \leq E(G_f)$ so there is a component L_2 of G_f which is not a $3'$-group, and then as $m_3(G_f) \leq 2$, L_2 is of 3-rank 2 and $L_2 = O^{3'}(E(G_f))$, so $L_f \leq L_2$. Indeed $L_3(2) \cong L_f$ is a component of $C_{L_2}(v)$, so we conclude using Theorem C that $L_2/Z(L_2) \cong L_3(4)$, J_2, or $L_3(7)$. By A.3.18, either $C_{G_f}(L_2)$ is a $3'$-group or $L_2/O_2(L_2)$ is $SL_3(4)$ or $SL_3(7)$, with $O(Z(L_2))$ the unique subgroup of order 3 in $C_{G_f}(L_2)$.

As τf is fused to f, there is a conjugate U_1 of V_1 with $L_3 := O^2(C_G(\langle U_1, f \rangle)) = O(C_{G_f}(U_1)) \cong 3^{1+2}$. In particular, U_1 acts nontrivially on L_2, and hence U_1 acts faithfully on L_2 by (*). By the previous paragraph, either L_3 is faithful on L_2 or $Z(L_2) = Z(L_3)$. We conclude from the structure of centralizers of involutions in $Aut(L_2)$ in our three cases that $L_3 = O(C_{G_f}(U_1)) \leq C_{G_f}(L_2)$, contrary to the previous paragraph. □

Lemma 15.3.39. $L = L_0$.

Proof. Assume $L < L_0$. Then $L_0 := LL^s$ for some $s \in S - N_S(L)$, with L described in 1.2.1.3. Then $m_3(L) = 1$ by 15.3.34, and by 15.3.33.3, L is also described in 1.1.5.3 with $O(L) = 1$, so L is simple and $L_0 = L \times L^s$. Therefore as $m_3(Y_+) = 2$ and $Y_+ \leq L_0$ by 15.3.37, $Y_+ = XX^s$ where $X := O^2(Y_+ \cap L)$. Next there is $Y_2 \leq Y_+ \cap M_2$, with $V_2 = [V, Y_2]$ and Y_2 is S-invariant. As Y_2 and V_2 are s-invariant, they are diagonally embedded in L_0, so X is the projection of Y_2 on L, and $V_L = [V_L, X]$, where V_L is the projection of V_2 on L. Similarly s acts on $O^2(C_{Y_+}(V_2)) =: Y_1$, so Y_1 is also diagonally embedded in L_0 with projection X on

L. Now Y_1 centralizes V_2 and hence also V_L, whereas $[V_L, Y_1] = [V_L, X] \neq 1$, a contradiction. □

The next lemma rules out conclusion (1) of 15.3.37, and eliminates the shadow of $G \cong S_9$ where $L \cong A_5$, and also those of $G \cong L$ wr $\mathbf{Z}_2$ for $L \cong L_3(2)$ and A_5.

LEMMA 15.3.40. *$Y_+ \leq L$.*

PROOF. Assume $Y_+ \not\leq L$. Then $m_3(L) = 1$ by 15.3.38, and $L = L_0$ by 15.3.39, so that case (2) of 15.3.37 holds, and in that notation of the lemma, $Y_+ = Y_L Y_C$ with $|Y_L|_3 = 3 = |Y_C|_3$, and $Y_+/O_2(Y_+) \cong E_9$. As L is S-invariant, the subgroups Y_L and Y_C are S-invariant. By 15.3.36.1, $Y_+S/R \cong S_3 \times S_3$, so from the structure of $\bar{M}$ in 15.3.2.1, $\{Y_C, Y_L\} = \{Y_1, Y_2\}$, where $Y_2 \leq Y_+ \cap M_2$ with $V_2 = [V, Y_2]$, and $Y_1 \leq Y_+ \cap M_1$.

If $Y_C = Y_2$ then $V_2 = [V_2, Y_C] \leq C_G(L)$, so as $L \not\leq M$, $C_G(V_2) \not\leq M$, and hence $Y = Y_+$ by 15.3.11.3. Then since $V_1 = [V_1, Y_1]$, interchanging the roles of V_1 and V_2 if necessary, we may assume instead that $Y_L = Y_2$. As $Y_L = Y_2$, $V_2 = [V_2, Y_L] \leq L$.

Suppose first that $L \cong L_2(2^n)$ or $U_3(2^n)$. Then M_I acts on the Borel subgroup over S_L, so M_L is that Borel subgroup by 15.3.33.1. In particular M_L acts on $V_2 \cong E_4$, so we conclude $n = 2$. Then as $Aut_{M_I}(V_2) \cong S_3$, $L \cong L_2(4)$.

Therefore $L \cong L_2(4)$ or $L_3(2)$ as case (2) of 15.3.37 holds, so $Y_2 = Y_L \cong A_4$. In particular $Y/O_2(Y)$ is E_9 rather than 3^{1+2} as Y_2 has one noncentral 2-chief factor, so $Y = Y_+ = Y_2 \times Y_2^t \cong A_4 \times A_4$ for $t \in T - S$, contrary to 15.3.9. □

Assume for the remainder of the proof of Theorem 15.3.35 that $I \in \mathcal{H}_{+,*}$. By 15.3.39, $L \trianglelefteq I$, by 15.3.40, $Y_+ \leq L$, and by 15.3.33.3, $L \not\leq M$. Thus $LS \in \mathcal{H}_+$, so $I = LS$ by minimality of I. Let $I^+ := I/C_I(L)$.

LEMMA 15.3.41. *(1) $F^*(I^+) = L^+$ is simple and described in 1.1.5.3.*

(2) M_I^+ is a 2-local of I^+ containing a Sylow 2-subgroup S^+ of I^+ with $Y_+^+ \trianglelefteq M_L^+$, $Y_+^+S^+/O_{2,\Phi}(Y_+^+S^+) \cong S_3 \times S_3$, and M_I^+ is maximal in I^+ subject to $F^*(M_I^+) = O_2(M_I^+)$.

PROOF. Part (1) follows from 15.3.33.3. By 15.3.11.4 and Coprime Action, $M_I^+ = N_{I^+}(Y^+)$. The remaining two assertions follow from 15.3.36.1 and Theorem 15.3.25. □

LEMMA 15.3.42. *L^+ is of Lie type and characteristic 2.*

PROOF. Assume otherwise. If $L^+ \cong A_7$, then as $Y_+ \leq L$ and Y_+ is S-invariant, $Y_+ \cong A_4 \times \mathbf{Z}_3$. Thus M_I is the stabilizer in I of a partition of type $4, 3$, as that stabilizer is the unique maximal subgroup of I containing Y_+S. This contradicts $F^*(M_I) = O_2(M_I)$ in 15.3.11.7.

By 15.3.11.8, $O(L) = 1$. Thus by the previous paragraph, L must appear in case (e) or (f) of 1.1.5.3. Inspecting the 2-local subgroups of the groups in those cases for subgroups satisfying the conclusions of 15.3.41.2, we conclude that $I^+ \cong Aut(J_2)$. Then as $V \leq Z(R)$ by 15.3.36.2, and $[V, Y_+]$ is normal in M_I, we conclude $[V, Y_+] \cong E_4$, and hence case (2) of Hypothesis 15.3.10 holds, so $V_2 = [V, Y_+]$. But now $V_1 \leq C_I(Y_+) \leq C_I(L)$ from the structure of $Aut(J_2)$, so $L \leq C_G(V_1) \leq M$ by 15.3.11.3, contrary to the choice of L. □

We are now in a position to complete the proof of Theorem 15.3.35.

By 15.3.42, L^+ is of Lie type and characteristic 2, and hence is described in cases (a)–(c) of 1.1.5.3. By 15.3.41.2, M_L^+ is a maximal S-invariant parabolic of I^+ and S^+ acts on Y_+^+ with $Y_+^+S^+/O_{2,\Phi}(Y_+^+S^+) \cong S_3 \times S_3$. Therefore either $L^+ \cong L_4(2)$ or $L_5(2)$, or L^+ is of Lie rank 2 and defined over $\mathbf{F}_{2^n}$ for $n > 1$ with Y_+ contained in the Borel subgroup B of L over S_L.

Assume the latter case holds. Then as $Y_+S/O_{2,\Phi}(Y_+S) \cong S_3 \times S_3$, we conclude from the structure of $Aut(L^+)$ that $L^+ \cong L_3(2^n)$ with n even. As $O^{2,3}(B) \leq C_M(V)$ by 15.3.2.1, and $[V, Y_+] \trianglelefteq B$, we conclude $n = 2$. Since $O(L) = 1$ by 15.3.33.3, $L \cong L_3(4)$, so that $m_3(B) = 1$, a contradiction.

Therefore $L \cong L_4(2)$ or $L_5(2)$. As $Y_+S/R \cong S_3 \times S_3$, we conclude that S is trivial on the Dynkin diagram of L. By 15.3.41.2, M_I^+ is maximal in I^+ subject to $F^*(M_I^+) = O_2(M_I^+)$, so $L^+ \cong L_4(2)$ and $M_I^+ = Y_+^+S^+$ is the maximal parabolic determined by the two end nodes. Therefore $Y_+^+S^+$ is irreducible on $O_2(Y_+^+S^+)$ of order 16, impossible as $E_4 \cong V_2 = [V_2, Y_+] \trianglelefteq Y_+S$.

This contradiction completes the proof of Theorem 15.3.35.

Finally we complete our analysis of Hypothesis 15.3.10 by eliminating the only possiblity left in Theorem 15.3.35:

THEOREM 15.3.43. *Y_+S/R is not S_3.*

PROOF. Assume otherwise; then case (ii) of 15.3.11.5 holds, so

(a) $Y_+ = O^{3'}(M_I)$ with $|Y_+ : O_2(Y_+)| = 3$.

In particular, case (2) of Hypothesis 15.3.10 holds, so that

(b) $[V_1, Y_+] = 1$ and $V_2 = [V_2, Y_+] \cong E_4$.

Further $N_G(V_i) \leq M$ by 15.3.11.3, so as $L \not\leq M$, $[V_1, L] \neq 1$. Therefore as $V_1 \leq C_S(Y_+)$ by (b):

(c) $[V_1, L] \neq 1$ and $C_S(Y_+) \neq C_S(L)$.

Suppose $Y_+ \cong A_4$. As $|Y_+|_3 = 3$, case (1) of 15.3.7 holds and $Y_+ = O^{3'}(M_2)$. Thus $Y = Y_+ \times Y_+^t \cong A_4 \times A_4$ for $t \in T - S$, contrary to 15.3.9. Therefore:

(d) Y_+ is not isomorphic to A_4.

Arguing as in the the proof of 15.3.37, one of the following holds:

(i) $Y_+ \leq \theta(I) = L_0$.

(ii) $L = L_0$ is of 3-rank 1 with $L/O_2(L) \cong L_2(2^n)$, $L_3^\delta(2^m)$ with $2^m \equiv -\delta$ mod 3, or $L_2(p)$ for some Fermat or Mersenne prime p.

(iii) $L = L_0 \cong L_3^\epsilon(2^n)$, $2^n \equiv \epsilon \mod 3$, or $L/O_2(L) \cong L_3(4)$. Further some y of order 3 in Y_+ induces a diagonal outer automorphism on L.

Suppose that Y_+ does not induce inner automorphisms on L. Then as $Y_+S/R \cong S_3$, conclusion (iii) holds. As $Y_+S = SY_+$, Y_+S acts on a Borel subgroup B of L over $S \cap L$, so by 15.3.33.1, $B \leq M_L$. But then $m_3(M_I) > 1$, contrary to (a).

Therefore Y_+ induces inner automorphisms on L, and hence case (i) or (ii) holds. As $L \not\leq M$, $[L, Y_+] \neq 1$ by 15.3.11.4, so the projection Y_L of Y_+ on L is nontrivial. Now $N_S(L)$ acts on Y_L, so $S \cap Y_L \in Syl_2(Y_L)$ and hence Y_L normalizes $O^2(Y_+O_2(S \cap Y_L)) = Y_+$, so that $Y_L \leq M_I$ by 15.3.11.4. Thus $Y_L \leq O^{3'}(M_I) = Y_+$, so $Y_L = Y_+$ since $|Y_+|_3 = 3$.

As S normalizes Y_+ and $Y_+ \leq L$, S normalizes L, and hence $L \trianglelefteq I$. As $Y_+ \leq L$ and $I \in \mathcal{H}_{+,*}$, $I = LS$. Let $I^+ := I/C_I(L)$. Now we obtain the following analogue of 15.3.41, using the same proof, but replacing the appeal to 15.3.36.1 by an appeal to (a):

(e) $F^*(I^+) = L^+$ is simple, and is described in 1.1.5.3. Further M_I^+ is a 2-local of I^+ containing a Sylow 2-subgroup S^+ of I^+ with $Y_+^+ = O^{3'}(M_I^+)$, $S^+Y_+^+/R^+ \cong S_3$, and M_I^+ is maximal subject to $F^*(M_I^+) = O_2(M_I^+)$.

We now eliminate the various possibilities for L^+ arising in 1.1.5.3 and satisfying condition (e).

Suppose first that L^+ is of Lie type over $\mathbf{F}_{2^n}$, and hence is described in cases (a)–(c) of 1.1.5.3. Then M_L^+ is a maximal S-invariant parabolic by (e).

Assume that $n > 1$. Then as $Y_+ = O^{3'}(M_L)$, we conclude $L^+ \cong L_3(2^n)$ or $L_2(2^n)$ with n even, or $U_3(2^n)$, and M_L^+ is a Borel subgroup of L^+. Then $E_4 \cong V_2 = [V, Y_+] \trianglelefteq M_L$ by (b), so we conclude that $n = 2$. But from the structure of $Aut(L)$, $C_S(Y_+) = C_S(L)$, contrary to (c).

So $n = 1$. As $Y_+ = O^{3'}(M_L)$ and M_L is a maximal S-invariant parabolic, either L^+ is of Lie rank 2, or $I^+ \cong S_8$ and M_L^+ is the middle-node minimal parabolic isomorphic to S_3/Q_8^2. As $E_4 \cong V_2 \trianglelefteq M_I$, the last case is eliminated. Now by (b), $V_1 \leq C_S(Y_+)$, but $[V_1, L] \neq 1$ by (c), and again $V_2 \trianglelefteq M_I$, so we conclude that $I^+ \cong S_6$. However in this case $Y_+ \cong A_4$, contrary to (d).

Suppose next that L^+ is sporadic. We inspect the list of possible sporadics in case (f) of 1.1.5.3 for subgroups I^+ of $Aut(L)$ such that there is a 2-local M_I^+ satisfying (e) and with $E_4 \cong V_2 = [V_2, Y_+] \trianglelefteq M_I$. We conclude $L^+ \cong M_{12}$. But then $C_S(Y_+) = C_S(L)$, contrary to (c).

If $L^+ \cong A_7$, then arguing as in the sporadic case, M_I is the stabilizer of a partition of type $2^3, 1$, so $Y_+ \cong A_4$, again contrary to (d).

From the list of 1.1.5.3, this leaves the case where L is $L_3(3)$ or $L_2(p)$, p a Fermat or Mersenne prime; we may take $p > 7$ as $L_2(5) \cong L_2(4)$ and $L_2(7) \cong L_3(2)$ were eliminated earlier. However in each case, there is no candidate for M_I satisfying (e). This completes the proof of 15.3.43. □

By Theorems 15.3.35 and 15.3.43:

THEOREM 15.3.44. *Assume Hypothesis 15.3.10. Then* $M = !\mathcal{M}(Y_+S)$.

In the remainder of this subsection we deduce information about the structure of M and of members of $\mathcal{H}(T, M)$ from these uniqueness results.

THEOREM 15.3.45. *For* $i = 1, 2$*:*

(1) $M = !\mathcal{M}(C_Y(V_i)S)$.
(2) $N_G(V_i) \leq M$.
(3) $C_G(C_{V_i}(S)) \leq M$.

PROOF. First Hypothesis 15.3.10.1 is satisfied with $Y_+ := Y$, so by 15.3.44, $M = !\mathcal{M}(YS)$. Therefore Hypothesis 15.3.10.2 also holds with $Y_+ := O^2(C_Y(V_1))$, so (1) holds since V_1 and V_2 are conjugate in M. Then as $V_i \trianglelefteq YS$ and $C_Y(V_i)S \leq C_G(C_{V_i}(S))$, (2) and (3) follow from (1). □

Recall that we view V as a 4-dimensional orthogonal space of sign $+1$ over $\mathbf{F}_2$, and $\bar{M}$ as the isometry group of this space. In particular, there are two M-classes of involutions in V: the 9 singular involutions fused to z under M, and the 6 nonsingular involutions in $V_1^\# \cup V_2^\#$. We will show next that these classes are not fused in G. Recall weak closure parameters $r(G,V)$ and $w(G,V)$ from Definitions E.3.3 and E.3.23.

LEMMA 15.3.46. *(1) M controls G-fusion of involutions in V.*

(2) For $g \in G - M$, $V \cap V^g$ is totally singular. In particular if $1 < U < V$ with $N_G(U) \not\leq M$, then U is totally singular.

(3) $r(G,V) > 1$.

(4) $W_0(T,V)$ centralizes V, so that $w(G,V) > 0$, $V^G \cap M \subseteq C_G(V)$, and $N_G(W_0(T,V)) \leq M$.

(5) $N_G(Z_S) \leq M \geq C_G(v)$ for $v \in V$ nonsingular.

PROOF. Let $\langle v \rangle = Z_S \cap V_1$; as we just observed, v and z are representatives for the orbits of M on $V^\#$. Now $S \in Syl_2(C_M(v))$, and $C_G(v) \leq M$ by 15.3.45.3, so v is not 2-central in G, and hence is not fused to the 2-central involution z. Thus (1) holds. As $C_G(v) \leq M = N_G(V)$, (1) and A.1.7.2 say that V is the unique member of V^G containing v, so (2) holds. As no hyperplane of V is totally singular, (2) implies (3). Similarly Z_S is not totally singular, so (5) holds.

It remains to prove (4), so suppose $g \in G - M$ and $A := V^g \leq T$. We must show $[V,A] = 1$, so assume otherwise.

Assume first that $V \leq N_G(A)$. Then we have symmetry between V and A, $1 \neq [V,A] \leq V \cap A$, and $[V,A]$ is totally singular by (2). As $[V,A]$ is totally singular, $m(\bar{A}) = 1$ and $\bar{A} = \langle \bar{a} \rangle$ with $V_1^a = V_2$. But as $m(\bar{A}) = 1$, V centralizes the hyperplane $C_A(V)$ of A, so that V induces a group of transvections on A, contrary to $V_1^a = V_2$ and symmetry.

Therefore $V \not\leq N_G(A)$. In the notation of Definition F.4.41, by (3), $U := \Gamma_{1,\bar{A}}(V) \leq N_V(A)$, so $U < V$. Hence $m(\bar{A}) > 1$ so $m(\bar{A}) = m_2(\bar{M}) = 2$, and so $\bar{A}$ is one of the two 4-subgroups of $\bar{T}$. As $V = \Gamma_{1,\bar{S}}(V)$, $\bar{A}$ is the 4-subgroup distinct from $\bar{S}$, so $U = Z^\perp$ and $C_U(A) = Z$. Let $B := C_A(V)$; then $m(B) = 2 = m(Aut_U(A))$. As $V \not\leq N_G(A)$, $C_G(B) \not\leq N_G(A)$, so B is totally singular in A by (2). This is impossible, as U centralizes B and $m(Aut_U(A)) = 2$, whereas the centralizer of a totally singular line is of 2-rank 1.

Therefore $W_0(T,V)$ centralizes V, and hence $w(G,V) > 0$ and $V^G \cap N_G(V) \subseteq C_G(V)$. Then by a Frattini Argument, $M = C_M(V)N_M(W_0(T,V))$, and it follows that $N_G(W_0(T,V)) \leq M$ by 15.3.2.4. □

LEMMA 15.3.47. *If $x \in C_G(Z_S)$, then either $[V,x] = 1$ or $V^G \cap N_G([V,x]) \subseteq C_G(V)$.*

PROOF. Assume $[V,x] \neq 1$. As $x \in C_G(Z_S)$, $x \in M$ by 15.3.46.5. Therefore as $[V,x] \neq 1$ and x centralizes Z_S, $[V,x]$ is not totally singular, so $N_G([V,x]) \leq M$ by 15.3.46.2. But then $V^G \cap N_G([V,x]) \subseteq C_G(V)$ by part (4) of 15.3.46. □

Observe that $M_c \in \mathcal{H}(T,M)$, and in particular $\mathcal{H}(T,M)$ is nonempty.

In the remainder of the section, H denotes a member of $\mathcal{H}(T,M)$.

Let $M_H := M \cap H$, $V_H := \langle V^H \rangle$, $U_H = \langle Z_S^H \rangle$, $Q_H := O_2(H)$, and $H^* := H/Q_H$. By 15.3.2.3 and 15.3.4, $H \leq M_c = C_G(Z)$, so we can form $\tilde{H} := H/Z$.

LEMMA 15.3.48. *If case (2) of 15.3.7 holds, assume that $C_Y(V) \leq H$. Then*

(1) Hypothesis F.9.1 is satisfied with Y, Z_S, Z in the roles of "L, V_+, V_1".

(2) $\tilde{Z}_S \leq Z(\tilde{T})$, $\tilde{U}_H \leq \Omega_1(Z(\tilde{Q}_H))$, and $\Phi(U_H) \leq Z$.

(3) $Q_H = C_H(\tilde{U}_H)$.

(4) $O_2(H^*) = 1$.

PROOF. By 15.3.46.5, $N_G(Z_S) \leq M = N_G(V)$, so that part (c) of Hypothesis F.9.1 holds. Let $L_1 := O^2(C_Y(Z))$. By 15.3.7, $C_M(Z) = TC_M(V)$, so $L_1 \leq C_M(V)$. Now part (b) of Hypothesis F.9.1 holds as $Z_S \trianglelefteq T$ and $\tilde{Z}_S$ is of order 2. Further $\tilde{Z}_S \leq Z(\tilde{T})$. Part (d) holds as $M = !\mathcal{M}(YT)$ by 15.3.7.

We next establish part (a) of F.9.1. As $C_G(Z_S) \leq M$ by 15.3.46.5, and $C_M(Z) = TC_M(V)$, $C_G(Z_S) = C_M(V)S$, so that using Coprime Action,

$$X := O^2(\ker_{C_H(\tilde{Z}_S)}(H)) \leq C_M(V),$$

and hence $[X, Y] \leq C_Y(V)$. In case (1) of 15.3.7, $C_Y(V) = O_2(Y)$; thus $[Y, X] \leq O_2(Y)$ and $L_1 = 1$ so $L_1T \leq H$. In case (2) of 15.3.7, $C_Y(V) = O_{2,Z}(Y)$, and $L_1 \leq C_Y(V) \leq H$ by hypothesis. If X is a $3'$-group then again $[Y, X] \leq O_2(Y)$ as $Aut(Y/O_2(Y))$ is a $\{2,3\}$-group. If X is not a $3'$-group then as $O^2(C_Y(V)) = \theta(C_M(V))$ by 15.3.7, $[Y, X] \leq C_Y(V) \leq XO_2(Y)$. Thus in any case $[Y, X] \leq XO_2(Y)$, so as $X \trianglelefteq XT$, YT normalizes $O^2(XO_2(Y)) = X$. It follows that $X = 1$, as otherwise $H \leq N_G(X) \leq M = !\mathcal{M}(YT)$ by 15.3.7, contrary to $H \in \mathcal{H}(T, M)$. Thus $\ker_{C_H(\tilde{Z}_S)}(H)$ is a 2-group, and hence lies in Q_H. This completes the verification of part (a) of F.9.1.

Finally under the hypothesis of part (e) of F.9.1, $V^g \leq W_0(N_G(V)) \leq C_G(V)$ by 15.3.46.4, so part (e) holds. This completes the verification of (1). By (1), we may apply F.9.2 to obtain the remaining conclusions of 15.3.48. □

The next result eliminates case (2) of 15.3.7; in particular Lemma 15.3.48 applies thereafter to all members of $\mathcal{H}(T, M)$.

LEMMA 15.3.49. *(1)* $O^2(M_H) \leq C_{M_H}(V)$.

(2) $Z = [Z_S, O_2(M_c)]$.

(3) Case (1) of 15.3.7 holds, so $Y/O_2(Y) \cong E_9$, $O_2(Y) = C_Y(V) = C_Y(Z)$, *and* $Y = O^{3'}(M)$.

(4) $C_M(V)$ *and* M_H *are* $3'$*-groups.*

(5) $N_G(Z_S) = N_M(Z_S)$ *is a* $3'$*-group.*

PROOF. Part (1) follows since $M \cap M_c = C_M(V)T$ by 15.3.7.

Since $C_Y(V) \leq C_G(Z) = M_c \in \mathcal{H}(T, M)$, enlarging H ir necessary, we may assume when case (2) of 15.3.7 holds that H contains $C_Y(V)$, so that 15.3.48 applies to H.

Let $U_C := C_{U_H}(Q_H)$; we claim:

(a) $O_{2,F^*}(H)$ centralizes U_C.

For $U_C \leq Z(Q_H)$, so as $\mathcal{L}_f(G, T) = \emptyset$ by Hypothesis 14.1.5, each member of $\mathcal{C}(H)$ centralizes U_C by A.4.11, and hence $O_{2,E}(H)$ centralizes U_C. Also by Coprime Action, $U_C = C_{U_C}(O_{2,F}(H)) \oplus [U_C, O_{2,F}(H)]$, so as $Z \leq C_{U_C}(M_c)$ by 15.3.4, and $H \leq M_c$, it follows that $[U_C, O_{2,F}(H)] = 1$, completing the proof of (a).

Set $\hat{U}_H := U_H/U_C$. By 15.3.48, H^* is faithful on $\tilde{U}_H$ and $O_2(H^*) = 1$, while $F^*(H^*)$ centralizes $\tilde{U}_C$ by (a); thus $F^*(H^*)$ is faithful on $\hat{U}_H$, and then also H^* is faithful on $\hat{U}_H$. In particular $U_H \not\leq Z(Q_H)$, so $[Z_S, Q_H] = Z$ by 15.3.48.2, and hence (2) holds since we may take M_c in the role of "H".

By 15.3.7, (3) holds iff $C_M(V)$ is a $3'$-group, in which case $M \cap M_c = C_M(V)T$ is a $3'$-group, and hence M_H is also a $3'$-group as $H \leq M_c$. That is, (3) and (4) are equivalent. Further $N_G(Z_S) = N_M(Z_S)$ by 15.3.46.5, so as $N_{\bar{M}}(Z_S) = \bar{T}$ is a 2-group, (4) implies (5).

Thus we may assume that (3) fails, and it remains to derive a contradiction. Hence case (2) of 15.3.7 holds, so that $Y/O_2(Y) \cong 3^{1+2}$, $Y = \theta(M)$, $C_Y(V) = O_{2,Z}(Y)$, and $Y_0 := O^2(C_Y(V)) \neq 1$. Hence:

(b) $Y_0 = \theta(M_H)$.

Let $\Omega := \Omega_1(Q_H)$. Now $[Q_H, Y_0] \leq Q_H \cap Y_0 \leq O_2(Y_0)$, so $\bar{\Omega} \leq \Omega_1(\overline{C_T(Y_0/O_2(Y_0))}) = Z(\bar{T})$. Thus $\Omega \leq S$, so as $U_H = \langle Z_S^H \rangle$:

(c) $U_H \leq Z(\Omega)$.

(d) V_H is elementary abelian.

For $[V, Q_H] \leq V \cap Q_H \leq \Omega \leq C_H(U_H)$ by (c), so V_H centralizes $Q_H/C_H(U_H)$. Now Hypothesis F.9.1 holds by 15.3.48.1, and U_H is abelian by (c), so we may apply F.9.7 to conclude that $Q_H/C_H(U_H)$ is H-isomorphic to the dual of $\hat{U}_H$. So as H^* is faithful on $\hat{U}_H$, $V_H \leq Q_H$. In particular V_H normalizes V, so V commutes with each H-conjugate of V by 15.3.46.4, and hence V_H is abelian, establishing (d).

We next extend Hypothesis F.9.1 to:

(e) Hypothesis F.9.8 holds.

For suppose $Z \leq V \cap V^g$ for some $g \in G$. As $M_c = C_G(Z)$, and M controls G-fusion in V by 15.3.46.1, we conclude from A.1.7.1 that M_c is transitive on $\{U \in V^G : Z \leq U\}$. Thus we may take $g \in M_c$, and then $[V, V^g] = 1$ by (d) applied to M_c in the role of "H". Thus condition (f) of Hypothesis F.9.8 holds. Further case (ii) of condition (g) of Hypothesis F.9.8 holds by 15.3.47.

We now adopt the notation of the latter part of section F.9 and obtain:

(f) $[E_H, V_\gamma] = 1 = [E_\gamma, V_H]$. In particular, $C_{V_H}(U_\gamma/Z_\gamma) = C_{V_H}(U_\gamma)$.

For as V_H is elementary abelian by (d), $E_\gamma = V_\gamma \cap Q_H \leq \Omega$, and so $[E_\gamma, U_H] = 1$ by (c). Thus as $Z_S \leq U_H$, $E_\gamma \leq C_T(Z_S)$. Suppose E_γ does not centralize V. Then $1 \neq [V, E_\gamma] \leq V_\gamma$, so as V_γ is abelian, $V_\gamma \in V^G \cap C_G([V, E_\gamma]) \subseteq C_G(V)$ by 15.3.47, contradicting our assumption that $[V, E_\gamma] \neq 1$. Thus E_γ centralizes V. But as $E_\gamma \leq Q_H$, E_γ normalizes each H-conjugate of V, so this argument gives the second equality in (f). Before completing the proof of (f), we recall $[V, U_\gamma] \neq 1$ since $V \not\leq G_\gamma^{(1)}$, so as $[D_\gamma, V] \leq [E_\gamma, V] = 1$:

(g) $D_\gamma < U_\gamma$.

By (g) we have symmetry between γ_1 and γ as discussed in the first paragraph of Remark F.9.17, so that the remaining equality in (f) follows from that symmetry. Further by F.9.16.4, we can choose γ so that $0 < m(U_\gamma^*) \geq m(U_H/D_H)$, and hence by (f):

(h) U_γ^* and V_γ^* are quadratic FF*-offenders on $\tilde{U}_H$.

Choose $h \in H$ with $\gamma_0 = \gamma_2 h$, set $\alpha := \gamma h$, and observe $V_\alpha^* \leq O_2(Y_0^* T^*)$—since from the proof of 15.3.48, Y_0 plays the role of "L_1". Let $J_H := \langle V_\alpha^H \rangle$. We show:

(i) J_H is the product of $\mathcal{C}$-components L of J_H with $L = [L, Y_0]$.

For if J_H is not the product of members of $\mathcal{C}(J_H)$, then by (h) and Theorem B.5.6, there is L^* subnormal in J_H^* with $L^* \cong S_3$, $O_3(L^*) = [O_3(L^*), V_\alpha^*]$, and $[\tilde{U}_H, L]$ of rank 2. Further Y_0 acts on L^* as there are at most two H-conjugates of L^* in Theorem B.5.6 and $Y_0 = O^2(Y_0)$. As $O_3(L^*) = [O_3(L^*), V_\alpha^*]$ and $V_\alpha^* \leq O_2(Y_0^* T^*)$, $O_3(L^*) \neq Y_0^*$. Hence Y_0 centralizes $L/O_2(L)$ so that L normalizes $O^2(Y_0 O_2(L)) =$

Y_0. Thus $L \leq N_G(Y_0) = M$, contrary to (b) since we just saw $O_3(L^*) \neq Y_0^*$. This contradiction shows that J_H is the product of members of $\mathcal{C}(J_H)$. Similarly $L = [L, Y_0]$ for each $L \in \mathcal{C}(J_H)$, completing the proof of (i).

Applying (i) to any overgroup of Y_0T in H we conclude

(j) Each solvable overgroup of Y_0T in H is contained in M_H.

Pick $L \in \mathcal{C}(J_H)$ and let $L_0 := \langle L^T \rangle$ and $U_0 := [U_H, L_0]$. Then $L_0Y_0T \in \mathcal{H}(Y_0T, M)$, so replacing H by L_0Y_0T, we may assume $H = L_0Y_0T$. As $\tilde{Z}_S \leq Z(\tilde{T})$ by 15.3.48, $\tilde{U}_H = \tilde{U}_0 C_{\tilde{U}_H}(H)$ by B.2.14. Let $\tilde{Z}_0$ be the projection of $\tilde{Z}_S$ on $\tilde{U}_0$ with respect to this decomposition; thus $C_{H^*}(\tilde{Z}_0) \leq N_H(Z_S)^* \leq M_H^*$ by 15.3.46.5.

By Theorems B.5.1 and B.5.6, L^* is A_7, $\hat{A}_6$, $L_n(2)$ for $n := 4$ or 5, or a group of Lie type of Lie rank 1 or 2 over some $\mathbf{F}_{2^e}$. Set $T_0 := T \cap L_0$. When L^* is of Lie type, let B denote the Borel subgroup of L_0 containing T_0.

Assume first that $Y_0 \not\leq L_0$. Then L^* is not A_7 or $\hat{A}_6$ by A.3.18. Further $T_0 = Y_0T \cap L_0$ is Y_0T-invariant, so Y_0T acts on B. Then $B \leq M_H$ by (j), so as we are assuming $Y_0 \not\leq L_0$, we conclude from (b) that B is a $3'$-group acting on Y_0. As $L_0 = [L_0, Y_0]$ by (i), we conclude from the structure of $Aut(L^*)$ for L^* of Lie type that $B = T_0$, and so L is defined over $\mathbf{F}_2$. Then $Out(L_0^*)$ is a $3'$-group from the list of possibilities in Theorem B.4.2, so Y_0 induces inner automorphisms on L_0^*, and this time we obtain a contradiction from (j) and (b) since the projection of Y_0^* on L_0^* is Y_0T-invariant and nontrivial. Thus we have shown:

(k) $Y_0 \leq L_0$.

Suppose that L^* is of Lie type, and defined over $\mathbf{F}_{2^e}$ with $e > 1$; then from Theorem B.4.2, L^* is $L_2(2^e)$, $SL_3(2^e)$, $Sp_4(2^e)$, or $G_2(2^e)$. Further as T_0 acts on Y_0, Y_0 is contained in B, and e is even. Then by (b), $\theta(N_{L_0^*}(Y_0)) = Y_0^*$ is of 3-rank 1, so we conclude $L^* = L_0^* \cong L_2(2^e)$. As V_α^* is an FF*-offender contained in $O_2(Y_0^*T^*)$, we conclude from Theorem B.4.2 that $\tilde{U}_0/C_{\tilde{U}_0}(L)$ is the natural module for L^*. But then $\tilde{Z}_0 \leq C_{\tilde{U}_0}(Y_0) \leq C_{\tilde{U}_0}(L)$, contrary to $U_H = \langle Z_S^H \rangle$. Therefore L^* is not of Lie type of $\mathbf{F}_{2^e}$ with $e > 1$.

Applying (j) and (b) as at the end of the proof of (k), we conclude that $L = L_0$ if $L^* \cong L_3(2)$; so using 1.2.1.3, we have reduced to:

(l) $L_0 = L$, L^* is $L_n(2)$, $3 \leq n \leq 5$, A_6, $\hat{A}_6$, A_7, or $G_2(2)'$, and either $Y_0^*T_0^*$ is a minimal parabolic of L^* of Lie type, or L^* is A_7 or $\hat{A}_6$.

(m) $V_H > U_HV$.

For suppose that $V_H = U_HV$; then because $[U_H, Q_H] \leq V_1$ by 15.3.48.2, $[V_H, Q_H] = [U_H, Q_H][V, Q_H] \leq V_1V = V$. Further $[V, Q_H] \neq 1$ by (2), so $Z(\bar{T}) \leq \bar{Q}_H$ as $Z(\bar{T})$ is of order 2. Thus $Z_S \leq [V, Q_H]$, and hence $Z_S \leq [V_H, Q_H] \leq V$. Therefore $[V_H, Q_H]$ is not totally singular in V, so $H \leq N_G([V_H, Q_H]) \leq M$ by 15.3.46.2, contrary to $H \in \mathcal{H}(T, M)$.

(n) V_γ^* is a strong FF*-offender on $\tilde{U}_H$.

Suppose otherwise. By the choice of γ, $m(U_\gamma^*) \geq m(U_H/D_H)$, and $U_\gamma \leq V_\gamma \leq C_H(D_H)$ by (f), so as V_γ^* is not a strong offender, we conclude that $\tilde{D}_H = C_{\tilde{U}_H}(V_\gamma)$, $U_\gamma^* = V_\gamma^*$, and $m(U_\gamma^*) = m(U_H/D_H)$. By the last equality we have symmetry between γ and γ_1 (as discussed in the second paragraph of Remark F.9.17) so also $V_H = U_H C_{V_H}(U_\gamma/Z_\gamma)$ by that symmetry. Further $C_{V_H}(U_\gamma/Z_\gamma) = C_{V_H}(U_\gamma)$ by (f), so U_γ centralizes V_H/U_H. Hence as $L = [L, U_\gamma]$, L centralizes V_H/U_H, so as $H = LY_0T$, $V_H = \langle V^H \rangle = U_HV$, contrary to (m). Thus (n) holds.

Observe that L^* is A_6 or $L_n(2)$, $3 \leq n \leq 5$, since in the remaining cases in (l), L^*T^* has no strong FF*-offenders by Theorem B.4.2, contrary to (n).

Suppose that $L^* \cong L_3(2)$. As $V_\alpha^* \leq O_2(Y_0^*T^*)$ and $Y_0^*T^*$ is the stabilizer of the point $\tilde{Z}_0$ in $\tilde{U}_0$, V_γ^* is not a strong offender on $\tilde{U}_H$ by Theorem B.5.1, contrary to (n). Thus L^* is not $L_3(2)$.

Suppose next that $L^* \cong L_n(2)$ for $n = 4$ or 5. As $Y_0^*T_0^*$ is a T-invariant minimal parabolic by (l), either LT is generated by overgroups H_1 of Y_0T with $H_1/O_2(H_1) \cong S_3 \times S_3$ or $L_3(2)$, or $H^* \cong S_8$ with $Y_0^*T_0^*$ the middle-node minimal parabolic of L^*. In the first case, $L \leq M$ by our previous reductions, contrary to $H \not\leq M$. In the second case, $Y_0T = C_H(\tilde{Z}_0)$, so by Theorem B.5.1, $\tilde{U}_0$ is the sum of the natural module and its dual; hence $O_2(Y_0^*T^*)$ contains no FF*-offender by B.4.9.2iii, whereas V_α^* is such an offender by (h).

Thus $L^* \cong A_6$. But then as $V_\alpha^* \leq O_2(Y_0^*T^*)$ with $Y_0^*T^*$ the stabilizer of the point $\tilde{Z}_0$, V_γ^* is not a strong FF*-offender on $\tilde{U}_H$ by B.3.2, contrary to (n). This contradiction completes the proof of 15.3.49. □

15.3.3. The case $\langle \mathbf{V^{M_c}} \rangle$ nonabelian. Recall from 15.3.49.4 that case (1) of 15.3.7 holds, and in particular 15.3.48 applies to all $H \in \mathcal{H}(T, M)$.

In this subsection, we will assume that $\langle V^{M_c} \rangle$ is nonabelian, and derive a contradiction via an application of the methods in section 12.8; in particular we will use Theorem G.9.3. Thus we will reduce to the following situation, to be treated in the final subsection:

THEOREM 15.3.50. *V_H is abelian for each $H \in \mathcal{H}(T, M)$.*

Until the proof of Theorem 15.3.50 is complete, assume H is a counterexample. Then $\langle V^{M_c} \rangle$ is also nonabelian, so as usual in the nonabelian case of section F.9, we take $H := M_c$. Recall $M_c = C_G(Z)$ by 15.3.4, so $V_H = \langle V^{C_G(Z)} \rangle$. Set $U := U_H = \langle Z_S^{C_G(Z)} \rangle$.

LEMMA 15.3.51. *(1) $V^* \neq 1$.*

(2) Either

(a) U is nonabelian, $\bar{U}$ is the 4-subgroup of $\bar{T}$ distinct from $\bar{S}$, and $\bar{U}$ is a Sylow group of $\Omega_4^+(V)$, or

(b) U is elementary abelian, $U \leq S$, $Z(\bar{T}) \leq \bar{U}$, and $Z_S = V \cap U$.

(3) $Y = [Y, U]$.

(4) $[V, Q_H] \leq V \cap Q_H$ and $[V, U] \leq V \cap U$.

PROOF. If $V \leq Q_H$ then the members of V^H normalize V, so that V_H is abelian by 15.3.46.4, contrary to our choice of H as a counterexample. Thus (1) holds, so $[\tilde{U}, V] \neq 1$ by 15.3.48.3, and hence $\bar{U} \neq 1$. By 15.3.48.2, $\Phi(U) \leq Z$, so $\bar{U}$ is elementary abelian, and as $T \leq H$, $\bar{U} \trianglelefteq \bar{T}$, so $Z(\bar{T}) \leq \bar{U}$ as $Z(\bar{T})$ is of order 2. As $U = \langle Z_S^H \rangle$, U is nonabelian iff $U \not\leq C_T(Z_S) = S$ iff conclusion (a) of (2) holds. Thus if U is abelian then $U \leq S$, so as $Z(\bar{T}) \leq \bar{U}$, $Z_S \leq V \cap U \leq C_V(U) \leq C_V(Z(\bar{T})) = Z_S$, and hence conclusion (b) of (2) holds. As $Z(\bar{T}) \leq \bar{U}$, $\bar{Y} = [\bar{Y}, \bar{U}]$, so (3) holds. Part (4) follows as V normalizes Q_H and U, and vice versa. □

LEMMA 15.3.52. *U is nonabelian.*

PROOF. Assume U is abelian; then case (b) of 15.3.51.2 holds. Thus $Z_S = V \cap U$ with $[U, V] \leq U \cap V$ by 15.3.51.4, so V^* induces a group of transvections on $\tilde{U}$ with center $\tilde{Z}_S$. Then $V_H^* = \langle V^{*H^*}\rangle$ is generated by transvections, $\tilde{U} = \langle \tilde{Z}_S^H \rangle$, and $V^* \trianglelefteq T^*$, so by G.6.4.4, $V_H^* = H^* \cong L_n(2)$, $2 \le n \le 5$, S_6, or S_7, and $\tilde{U}/C_{\tilde{U}}(H^*)$ is the natural module for H^*. As $C_{H^*}(\tilde{Z}_S)$ is a $3'$-group by 15.3.49.5, we conclude that $H^* \cong S_3$. Then $m(U) = 3$ and $Z_S = C_U(V) = U \cap V$. Now as $O_2(Y) = C_Y(V)$ by 15.3.49.3, $[O_2(Y), U] \le C_U(V) \le V$; then in view of 15.3.51.3, Y centralizes $O_2(Y)/V$, so that $V = O_2(Y)$. Thus $Y \cong A_4 \times A_4$, contrary to 15.3.9. □

LEMMA 15.3.53. *(1)* $\bar{U}\bar{Y} = \Omega_4^+(V) = \bar{N}_1 \times \bar{N}_2$ *with* $\bar{N}_i \cong S_3$ *and* $V = [V, N_i]$.
(2) $V \cap Q_H = V \cap U = [U, V] = Z^\perp$ *is the hyperplane of* V *orthogonal to* Z. *Thus* V^* *is of order* 2.

PROOF. By 15.3.52, conclusion (a) of 15.3.51.2 holds, giving (1). Next by (1), $[U, V] = Z^\perp$, so as $V^* \ne 1$ by 15.3.51.1, and $[U, V] \le U \cap V$ by 15.3.51.4, (2) follows. □

For the remainder of this subsection, define N_i as in 15.3.53, and set $Y_i := O^2(Y \cap N_i)$.

LEMMA 15.3.54. *Let* $g \in Y$ *with* Z^g *not orthogonal to* Z *in* V, *and set* $I := \langle U, U^g\rangle$, $P := O_2(I)$, *and* $W := U \cap P$. *Then*

(1) $I = YU$.
(2) $P = WW^g$ *and* $V \le Z(P)$.
(3) $U \cap U^g = W \cap W^g = Z^\perp \cap Z^{g\perp} \cong E_4$.
(4) $P/V = P_1/V \oplus P_2/V$, *where* $P_i/V := [P/V, Y_i] = C_{P/V}(N_{3-i})$, *and* P_i/V *is the sum of* s *natural modules for* $\bar{N}_i$.
(5) $[W^g, U] \le W$ *and* W^g *normalizes* U.

PROOF. We verify the hypotheses of G.2.6, with V, Y, Z, U in the roles of "V_L, L, V_1, U" By 15.3.481, G.2.2 is satisfied by the tuple of groups, and the remaining hypotheses of G.2.6 hold by 15.3.53. Hence the conclusions of G.2.6 hold with $V(U \cap U^g)$ in the role of "S_2". Thus conclusions (1) and (2) of 15.3.54 follow from G.2.6, and conclusion (4) will follow from G.2.6.5 once we show that $U \cap U^g \le V$.

As $W^g \le T$, W^g normalizes U, so $[W^g, U] \le P \cap U = W$, and hence (5) holds. Further $\Phi(U^g) \le Z^g$ by 15.3.48.2, so $[U \cap U^g, W^g] \le W \cap Z^g$. But $Z^g \le V$, so $W \cap Z^g \le U \cap V$, and hence $W \cap Z^g = 1$, since we chose $Z^g \not\le Z^\perp$, and $Z^\perp = U \cap V$ by 15.3.53.2. Thus W^g centralizes $U \cap U^g$, and by symmetry, W centralizes $U \cap U^g$, so using (2), $P_0 := (U \cap U^g)V \le Z(P)$. Further by G.2.6.4, I centralizes P_0/V, so since $P_0 \le Z(P)$, we may apply Coprime Action to conclude $P_0 = V \times C_{P_0}(Y)$. Now T normalizes $YU = I$, and hence normalizes the preimage P_0 of $C_{O_2(I)/V}(Y)$ in I, and then also normalizes $C_{P_0}(Y)$. Therefore as $\Omega_1(Z(T)) = Z \le V$, we conclude $C_{P_0}(Y) = 1$ so that $P_0 = V$, and hence $U \cap U^g \le V$. As mentioned earlier, this completes the proof of (4), and we established (5) earlier, so it remains to complete the proof of (3). But by 15.3.53.2, $U \cap V = Z^\perp$, so as $U \cap U^g \le V$,

$$U \cap U^g = (U \cap V) \cap (U^g \cap V) = Z^\perp \cap Z^{g\perp} = W \cap W^g \cong E_4.$$

□

In the next few lemmas, we use techniques similar to those in section 12.8 to study the action of H on U.

For the remainder of the subsection, define g, W, P, P_i, and s as in 15.3.54.

LEMMA 15.3.55. *U is extraspecial, and* $V = Z(P)$.

PROOF. First U is nonabelian by 15.3.52, so that $Z = \Phi(U)$ by 15.3.48.2; hence $U = U_0 Z_U$, where U_0 is extraspecial and $Z_U := Z(U)$. Thus we must show that $Z_U = Z$. As $U = \langle Z_S^H \rangle$ is nonabelian and Z_S is of order 4, $Z_S \cap Z_U = Z$; then as $C_{\tilde{V}}(T) = \tilde{Z}_S$, $V \cap Z_U = Z$. Therefore $[V, Z_U] \leq V \cap Z_U = Z$, but no member of $\bar{M}$ induces a transvection on V with singular center, so $Z_U \leq C_U(V) = W$. Hence also $Z_U^g \leq W^g$.

As $Z_U \cap V = Z$, $Z_U^g \cap V = Z^g$, so by 15.3.54.3,

$$Z_U^g \cap U = Z_U^g \cap (U \cap U^g) = Z_U^g \cap (Z^\perp \cap Z^{g\perp}) \leq Z^g \cap Z^\perp = 1.$$

Then as W^g normalizes U and W normalizes U^g by 15.3.54.5, $[Z_U^g, W] \leq Z_U^g \cap U = 1$, so as $P = WW^g$ by 15.3.54.2, $Z_U^g \leq Z_P := Z(P)$. Therefore also $Z_U \leq Z_P$. By 15.3.54.4, the irreducibles for Y on P/V are not isomorphic to those on V, so $Z_P = V \oplus Z_0$, where Z_0 is the sum of the Y-irreducibles on Z_P not isomorphic to those on V. Thus T acts on Z_0, so as $Z \leq V$, $Z_0 = 1$. Thus $Z_P = V$, so as $Z_U \leq Z_P$, $Z_U = V \cap Z_U = Z$, completing the proof. □

LEMMA 15.3.56. *Let $y \in Y_1 - O_2(Y_1)$, $V_0 := \langle Z^{Y_1} \rangle$, $F := U \cap H^y$, $X := F^y$, $E := F \cap F^y$, and $t \in T - UC_T(V)$. Then*

(1) The power map and commutator map make $\tilde{U}$ into an orthogonal space with $H^* \leq O(\tilde{U})$.

(2) $m(\tilde{U}) = 2(s+2)$.

(3) $X \cap Q_H = E$, $[X, F] \leq E$, $V_0 = ZZ^y$, and $\tilde{E}$ is totally singular of rank $s+2$ in the orthogonal space $\tilde{U}$.

(4) $X^* \cong E_{2^{s+1}}$ induces the full group of tranvections on $\tilde{E}$ with center $\tilde{V}_0$.

(5) $\tilde{U} = \tilde{E} \oplus \tilde{E}^t$ and X^* induces the full group of transvections on $\tilde{E}^t$ with axis $\widetilde{C_{E^t}(V_0)}$.

(6) $X^* \cap X^{*t} = V^*$ is of order 2, and $X^* X^{*t} \cong D_8^s$.

(7) $\tilde{Z}_S = C_{\tilde{U}}(\langle X^*, t^* \rangle)$.

PROOF. Part (1) follows from 15.3.55. By 15.3.54.4, $|P| = 2^{4s+4}$, while by parts (2) and (3) of 15.3.54, $|P| = 2^{2(m(W)-1)}$. Thus $m(W) = 2s+3$. By 15.3.53.1 and 15.3.54.1, $m(U/W) = 2$, so (2) follows.

As $y \in Y_1 - O_2(Y_1)$, $z^y \in Z^\perp - Z$, so $z^y \in U - Z$ by 15.3.53.2. Thus as U is extraspecial, $|U : F| = 2$; and the argument in 8.14 of [**Asc94**], which is essentially repeated in the proof of G.2.3, gives us the structure of $J := \langle U, U^y \rangle$: $J/O_2(J) \cong S_3$, $ZZ^y = V_0 \cong E_4$, $O_2(J) = FF^y = FX = C_J(V_0)$, $[E, J] \leq V_0$, and for some r, $O_2(J)/E$ is the direct sum of r natural modules for $J/O_2(J)$ with $[O_2(J)/E, U] = F/E$. Thus

$$J \text{ has } r+1 \text{ noncentral 2-chief factors.} \qquad (*)$$

Moreover J and E are normal in $N_G(V_0)$.

As $O_2(J)/E$ is abelian and $O_2(J) = XF$, $[X, F] \leq E$. Similarly as $[XF/E, U] = F/E$ and $|U : F| = 2$, for $u \in U - F$ the map $\varphi : X/E \to F/E$ defined by $\varphi(xE) := [u, x]E$ is a bijection. Therefore as $[U, Q_H] = Z \leq E$ by 15.3.52 and 15.3.48.2, $X \cap Q_H = E$. Finally $\Phi(E) \leq \Phi(U) \cap \Phi(U^y) = Z \cap Z^y = 1$, so by (1), $\tilde{E}$ is totally singular in the orthogonal space $\tilde{U}$.

Next $\bar{J} = \bar{Y}_1 \bar{U} \cong S_3 \times \mathbf{Z}_2$, with $\bar{F} = \bar{X} = Z(\bar{J}) = \bar{U} \cap \bar{N}_2$; in particular $[Z^\perp, X] = Z$. By 15.3.54.4, Y_1 has s noncentral chief factors on P/V, and by 15.3.53.1, Y_1 has two noncentral chief factors on V. Thus J has $s+2$ noncentral

2-chief factors, so $s+2=r+1$ by (*). Further $m(U/X)=1$ and $m(X/E)=r$, so using (2),

$$m(\tilde{E})=m(\tilde{U})-(r+1)=2(s+2)-(s+2)=s+2,$$

completing the proof of (3). Further $\bar{F}\neq\bar{F}^t$, so $F\cap F^t\leq P$. Also $[E\cap P,Y_1]\leq[E,J]\leq V_0\leq V$, so $(E\cap P)/V\leq C_{P/V}(Y_1)=P_2/V$ by definition of P_2, and then $E\cap E^t\leq P_2\cap P_2^t=V$.

Now (4) holds, using an argument in the proof of 12.8.11.3; indeed the argument is easier here since U is extraspecial. Next using 15.3.53.2, $\widetilde{V\cap U}=Z^\perp=V_0V_0^t$, and we saw earlier that $[Z^\perp,X]=Z$, so X centralizes $\widetilde{V\cap U}$. Therefore X acts on V_0 and V_0^t, so since E and E^t are normal in $N_G(V_0)$ and $N_G(V_0^t)$, respectively, X acts on E and E^t. We saw earlier that $E\cap E^t\leq V$, so

$$E\cap E^t\leq U\cap U^y\cap U^{yt}\cap V=Z^\perp\cap Z^{y\perp}\cap Z^{yt\perp}=Z.$$

Then as $m(\tilde{U})=2m(\tilde{E})$, $\tilde{U}=\tilde{E}\oplus\tilde{E}^t$. Since the action of H^* on $\tilde{U}$ is self-dual, the action of X^* on $\tilde{E}^t$ is dual to its action on $\tilde{E}$, so (5) holds. By 15.3.53.2, V^* is of order 2, so (4) and (5) imply (6) and (7). □

In the remainder of the proof of Theorem 15.3.50, define V_0, X, E, and y as in lemma 15.3.56.

Lemma 15.3.57. *(1) H is irreducible on $\tilde{U}$.*

(2) If $1\neq K^*=O^2(K^*)\trianglelefteq H^*$ and the irreducibles of K^* on $\tilde{U}$ are of rank at least 3, then $[K^*,V^*]\neq 1$, and either

(a) K^* is irreducible on $\tilde{U}$, or

(b) $\tilde{U}=\tilde{U}_1\oplus\tilde{U}_2$, where the $\tilde{U}_i$ are irreducible K^*-modules of rank $s+2$, and V^* induces a transvection on each $\tilde{U}_i$.

Proof. Let $\tilde{U}_0$ be a nonzero H-submodule of $\tilde{U}$. Then $C_{\tilde{U}_0}(T)\neq 0$, so by 15.3.56.7, $\tilde{Z}_S\leq\tilde{U}_0$. Thus $\tilde{U}=\langle\tilde{Z}_S^H\rangle\leq\tilde{U}_0$, so (1) holds.

Assume the hypothesis of (2). By (1) and Clifford's Theorem, $\tilde{U}$ is a semisimple K^*-module, and by hypothesis, each $\tilde{J}\in Irr_+(K^*,\tilde{U})$ is of rank at least 3. If $[K^*,V^*]=1$ then K^* acts on $[\tilde{U},V^*]$; this is impossible as $[\tilde{U},V^*]=\widetilde{V\cap U}$ is of rank 2 by 15.3.53.2, contradicting $m(\tilde{J})>2$ for $\tilde{J}\in Irr_+(K^*,[\tilde{U},V^*])$. Thus $[K^*,V^*]\neq 1$.

Similarly if V^* does not normalize some $\tilde{J}$, then $m([\tilde{U},V^*])\geq m(\tilde{J})>2$ by hypothesis, again contrary to 15.3.53.2. Thus we can write $\tilde{U}=\tilde{J}_1\oplus\cdots\oplus\tilde{J}_k$ where $\tilde{J}_i\in Irr_+(K^*,\tilde{U})$ and $\tilde{J}_i$ is V^*-invariant. Again using 15.3.53.2,

$$2=m([\tilde{U},V^*])=\sum_{i=1}^k m([\tilde{J}_i,V^*])\geq k,$$

so that (2) holds. □

The next lemma eliminates the shadow of $Aut(L_4(2))$, and begins to zero in on the shadows of $Aut(L_5(2))$ and $Aut(He)$.

Lemma 15.3.58. *(1) $H^*\cong Aut(L_3(2))$.*

(2) $s=1$ and $U\cong D_8^3$.

(3) $\tilde{U}=\tilde{U}_1\oplus\tilde{U}_1^t$, for $t\in T-UC_T(V)$, and some natural submodule $\tilde{U}_1$ for $O^2(H^*)\cong L_3(2)$, such that $\tilde{U}_1^t$ is dual to $\tilde{U}_1$.

PROOF. Observe first that $s > 0$: For if $s = 0$, then by 15.3.54.4, $Y \cong A_4 \times A_4$, contrary to 15.3.9.

Next let $V_U := V \cap U$; as $V_U = Z^{\perp}$ by 15.3.53.2, $N_G(V_U) \leq M$ by 15.3.46.2; so as $\tilde{V}_U = [\tilde{U}, V^*]$ by 15.3.53.2, $C_{H^*}(V^*) \leq M_H^*$. Now using (4) and (5) of 15.3.49, $N_{H^*}(\tilde{V}_U)$, $C_{H^*}(\tilde{Z}_S)$, and $C_{H^*}(V^*)$ are $3'$-groups.

Let K^* be a minimal normal subgroup of H^*. As H^* is faithful and irreducible on $\tilde{U}$ using 15.3.57.1, $\tilde{U} = [\tilde{U}, K^*]$. If K^* is a 3-group, then as $C_{H^*}(V^*)$ is a $3'$-group, V^* inverts K^*; so by 15.3.56.2 and 15.3.53.2,

$$2(s+2) = m(\tilde{U}) = 2m([\tilde{U}, V^*]) = 4,$$

contradicting $s > 0$.

Therefore K^* is not a 3-group, so each irreducible for K^* on $\tilde{U}$ has rank at least 3. Thus by 15.3.57.2, $[K^*, V^*] \neq 1$, and K^* satisfies one of the two conclusions of 15.3.57.2. Suppose K^* is solvable. Then K^* is a p-group for some prime $p > 3$. As $m([\tilde{U}, V^*]) = 2$, it follows (cf. D.2.13.2) that $[K^*, V^*] \cong \mathbf{Z}_5$. However as $s > 0$, there is a D_8-subgroup D^* of H^* with center V^* by 15.3.56.5. As V^*, and hence also D^*, is faithful on $[K^*, V^*]$, this is a contradiction.

Therefore K^* is not solvable, so as K is an SQTK-group, K^* is the direct product of at most two isomorphic nonabelian simple groups.

Suppose first that conclusion (b) of 15.3.57.2 holds. Then $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_2$ is the sum of two K^*-irreducibles $\tilde{U}_i$ of rank $s+2$ with V^* inducing a transvection on each $\tilde{U}_i$. By G.6.4.4, $K^*V^* \cong L_n(2)$, $3 \leq n \leq 5$, S_6, or S_7, and $\tilde{U}_i$ is a natural module for K^*. Let $\langle \tilde{u}_i \rangle = [\tilde{U}_i, V^*]$; then $\tilde{V}_U = \langle \tilde{u}_1, \tilde{u}_2 \rangle$, so as $N_{K^*}(\tilde{V}_U)$ is a $3'$-group, we conclude $K^* \cong L_3(2)$, and so $s = 1$.

Next as K^* is irreducible on $\tilde{U}_i$, and $\tilde{U}_i$ is not self-dual, $\tilde{U}_i$ is totally singular, and $\tilde{U}_2$ is dual to $\tilde{U}_1$. Thus $Irr_+(K^*, \tilde{U}) = \{\tilde{U}_1, \tilde{U}_2\}$ is permuted by H^*, and as H^* is irreducible on $\tilde{U}$ by 15.3.57.1, H^* is transitive on $\{\tilde{U}_1, \tilde{U}_2\}$. Further as $End_{K^*}(\tilde{U}_i) \cong \mathbf{F}_2$, $C_{H^*}(K^*) = 1$, so $H^* \cong Aut(L_3(2))$. completing the proof of the lemma in this case.

Thus we may assume that K^* is irreducible on $\tilde{U}$. By 15.3.56.6, $m_2(H^*) \geq s+1$, so using 15.3.56.2, $m(\tilde{U}) = 2(s+2) \leq 2(m_2(H^*)+1)$. As $[K^*, V^*] \neq 1$ by 15.3.57.2, the hypotheses of Theorem G.9.3 are satisfied with K^*, $\tilde{U}$, X^* in the roles of "H, V, A", so H^* and its action on $\tilde{U}$ are described in Theorem G.9.3. As $m([\tilde{U}, V^*]) = 2$ with $V^* \leq Z(T^*)$, we conclude: cases (0)–(2) and (15)–(17) do not hold (see e.g. chapter H of Volume I for the Mathieu groups); in cases (6)–(10), $n \leq 2$; and in case (13), $\tilde{U}$ is a natural module rather than a 10-dimensional module. As $s > 0$, $m(\tilde{U}) \geq 6$; therefore case (3) does not hold, nor does (6) or (7) when $n \leq 2$, completing the elimination of those cases. As $\tilde{Z}_S = C_{\tilde{U}}(T)$ is of order 2, and $C_{H^*}(\tilde{Z}_S)$ is a $3'$-group, the remaining cases are eliminated. □

Let $K := O^2(H)$ and $T_K := T \cap K$, so that $K^* \cong L_3(2)$ by 15.3.58.1.

LEMMA 15.3.59. *(1)* $U = Q_H$.
(2) $T_K \in Syl_2(YU)$.
(3) $|T : T_K| = 2$.
(4) $M = YT$.

PROOF. Let $Q_C := C_T(U)$. As $\tilde{Q}_H = C_{\tilde{T}}(\tilde{U})$ by 15.3.48, and U is extraspecial, $Q_H = UQ_C$. Now $[Q_C, V] \leq C_V(U) = Z$, so $[\tilde{Q}_C, V^*] = 1$. Then as $K = [K, V]$

and $K = O^2(K)$, we conclude using Coprime Action that K centralizes Q_C. Thus $Q_C = C_T(T_K)$. By 15.3.58, $U \leq K$ and $K^* \cong L_3(2)$, so $\hat{K} := K/U \cong L_3(2)$ or $SL_2(7)$ and hence $|T_K| \geq 2^{10}$ and $\hat{Q}_C \leq \Phi(\hat{T}_K)$.

Next $X^* = [X^*, O^2(N_K(V_0))]$, so as $X \cap Q_H = E \leq U$ by 15.3.56.3, $X \leq K$. Thus as $\hat{T}_K = \hat{X}\hat{X}^t\hat{Q}_C$ and $\hat{Q}_C \leq \Phi(\hat{T}_K)$, $T_K = \langle X, X^t\rangle U$. Further $X \leq \langle U^Y\rangle = YU$, so $T_K \leq YU$. Then (2) holds as $|T_K| \geq 2^{10}$ and $|YU|_2 = 2^{10}$ by 15.3.54.4 since $s = 1$.

Since $F^*(YU) = O_2(YU)$, since $V = Z(P)$ by 15.3.55, and since Q_C centralizes T_K, $Q_CV = C_{YUQ_C}(P)$. Thus by Coprime Action, $Q_CV = Q_Y \times V$, where $Q_Y := C_{Q_CV}(Y)$. Then as T acts on Q_Y, and $\Omega_1(Z(T)) = Z \leq V$, $Q_Y = 1$, so $Q_C \leq Z$, establishing (1) and (3). Then $O_2(YT) = O_2(Y)$, so as $Y \trianglelefteq M$, $F^*(M) = O_2(M) = O_2(Y)$ using A.1.6. By 15.3.58.2, $s = 1$, so from 15.3.53.1 and 15.3.54.4, $Aut_Y(B) = O^2(N_{GL(B)}(Aut_Y(B)))$ for $B \in \{V, O_2(Y)/V\}$. Therefore $Y = O^2(M)$ by Coprime Action, so (4) holds. □

Let $D_M \in Syl_3(C_M(V_1))$ and $D_H \in Syl_3(H)$; observe D_M and D_H both have order 3. Let $\langle v\rangle = Z_S \cap V_1$ and $Z = \langle z\rangle$. By 15.3.46.5, $C_G(v) \leq M$.

By 15.3.59.4, $M = YT$, and $s = 1$ by 15.3.58.2, so by 15.3.54.4, $C_M(D_M) = D_M \times J_M$, where $J_M \cong S_4$ and $V_1 = O_2(J_M)$. By construction, an involution $t \in J_M - V_1$ induces a transvection on V, and hence $t \notin UC_T(V)$.

Next a Sylow 2-group of $C_M(D_M)$ is dihedral of order 8 with center $\langle v\rangle$, and as $C_G(v) \leq M$, $|C_G(D_M)|_2 = 8$. On the other hand, from the structure of H described in 15.3.58 and 15.3.59, $|C_H(D_H)|_2 = 2^4$, so $D_M \notin D_H^G$. Thus as $D_H \in Syl_3(C_G(z))$, $t \notin z^G$. Summarizing:

LEMMA 15.3.60. *(1)* $D_M \notin D_H^G$.
(2) An involution t *in* $T \cap J_M - V_1$ *is not in* $UC_T(V)$, *and* $t \notin z^G$.

LEMMA 15.3.61. *(1)* $t \notin v^G$.
(2) All involutions in K *are in* z^G *or* v^G.

PROOF. As $\tilde{U} = \tilde{U}_1 \oplus \tilde{U}_1^t$ by 15.3.58.3, $m(\tilde{U}) = 2m([\tilde{U}, t])$, so $\tilde{U}$ is transitive on involutions in $\tilde{t}\tilde{U}$. Thus $O^2(C_{H^*}(t^*)) = O^2(C_H(t)^*)$, and hence t centralizes a conjugate of D_H. But by 15.3.46.5, $C_M(v) = C_G(v)$, so D_M is Sylow in $C_G(v)$ by construction. Thus (1) follows from 15.3.60.1.

From the action of H on U described in 15.3.58, H has two orbits on involutions in $U - Z$: $(U_1 - Z) \cup (U_1^t - Z) \subseteq z^G$ and v^H. Let $a \in V - U$ with U_a the preimage in U of $C_{\tilde{U}}(a)$. Then all involutions in $K - U$ are fused into aU_a under H, so it remains to show that each such involution is in $z^G \cup v^G$. Now $|U : U_a| = 4 = |\bar{U}|$ by 15.3.53, so $U_a = C_U(V) = C_U(U \cap V)$. Thus $U_a \cong E_4 \times D_8$, and all involutions in U_aV are in the two E_{16} subgroups A_1 and A_2 of U_aV.

Next $VU_a \leq P$; let $P^+ := P/V$. From the description of I in 15.3.54, $U_a^+ = [P^+, U]$ is an isotropic line in the orthogonal space P^+ with one singular point, and I is transitive on singular and nonsingular points of P^+. Thus A_i^+, $i = 1, 2$, are the nonsingular points in U_a^+. Therefore there is D_i of order 3 in I centralizing A_i^+ and $[Z, D_i]$ is a singular line in the orthogonal space V, so $[D_i, Z] \leq V^\perp = U \cap V$. Let a_i generate $C_{A_i}(D_i)$. If $a_i \in U$, then each member of A_i is fused into U under D_i, so that (2) holds. Thus we may assume $a_i \notin U$. Here each member of $A_i - \langle a_i\rangle[a_i, P]$ is fused into U, and P is transitive on $a_i[a_i, P]$, so it remains to show the a_i is fused to z or v.

Let $B_i := C_P(D_i)V$. Then $B_i \cong E_{64}$ and I has four orbits on $B_i^{\#}$: z^I and v^I, and orbits of length 12 and 36 on $B_i - V$. Further $B_i \leq T_K$, and $E_i := B_i \cap U$ is of rank at most $m(U) = 4$, while $m(B^*) \leq m(H^*) = 2$, so B_i^* is a 4-group in T_K^* and $\tilde{E}_i = C_{\tilde{U}}(B_i^*)$. Then $B_i = C_{T_K}(E_i)$ is invariant under $N_H(E_i) = N_H(B_i^*) \cong S_4$, so as $N_H(B_i^*)$ does not act on V^*, $N_G(B_i)$ does not act on V. Thus as $v \notin z^G$, the two orbits of I on $V^{\#}$ are fused to its two orbits on $B_i - V$, so all involutions in B_i are fused to z or v, completing the proof of (2). □

We now eliminate the shadows of $Aut(L_5(2))$ and $Aut(He)$, and establish Theorem 15.3.50.

First the involution t of 15.3.60.2 is in $T - T_K$, since $T_K = UC_T(V)$ by 15.3.59.2. By 15.3.59.3, $|T : T_K| = 2$, so as G is simple, $t^G \cap T_K \neq \emptyset$ by Thompson Transfer. Thus $t \in z^G \cup v^G$ by 15.3.61.2. However this contradicts 15.3.60.2 and 15.3.61.1. This contradiction completes the proof of Theorem 15.3.50.

15.3.4. The case $\langle \mathbf{V^{M_c}} \rangle$ abelian. By Theorem 15.3.50, V_H is abelian for each $H \in \mathcal{H}(T, M)$. This will allow us to use weak closure in 15.3.63, and to verify Hypothesis F.9.8. Then Hypothesis F.9.8 eventually leads to a contradiction.

LEMMA 15.3.62. *(1) M_c is transitive on $\{V^g : g \in G$ and $Z \leq V^g\}$.*
(2) If $V \cap V^g \neq 1$, then $[V, V^g] = 1$.

PROOF. Part (1) folllows from 15.3.46.1 using A.1.7.1, since $M_c = C_G(Z)$ by 15.3.4. If $g \in G - M$ and $V \cap V^g \neq 1$, then as $V \cap V^g$ is totally singular by 15.3.46.2 and M is transitive on singular vectors, we may take $Z \leq V \cap V^g$. Therefore $V^g \leq V_{M_c} \leq C_G(V)$ by (1) since V_{M_c} is abelian by Theorem 15.3.50. □

LEMMA 15.3.63. *Assume $r(G, V) \geq 3$. Then*
(1) $W_1(T, V) \leq C_T(V)$, so $w(G, V) > 1$.
(2) $n(H) > 1$ for each $H \in \mathcal{H}(T, M)$.

PROOF. Assume $W_1(T, V)$ does not centralize V, and let A be a hyperplane of V^g with $A \leq T$ and $\bar{A} \neq 1$. In particular $V \not\leq M^g$ by 15.3.46.4, so as $r(G, V) \geq 3$ by hypothesis, $m(V^g/C_{V^g}(V)) = m(\bar{A}) + 1 > 2$, and hence $m(\bar{A}) = 2 = m_2(\bar{M})$. As $\bar{M}$ is solvable, $a(\bar{M}, V) = 1$ by E.4.1, so there is a hyperplane B of A with $C_A(V) \leq B$ such that $1 \neq [C_V(B), A] =: V_B$. As $r(G, V) \geq 3$ and $m(V^g/B) = 2$, $C_V(B) \leq M^g$, so $1 \neq V_B \leq V \cap V^g$, contrary to 15.3.62.2. Thus $[V, W_1(T, V)] = 1$, establishing (1).

By A.5.7.2, $M = !\mathcal{M}(N_M(C_T(V)))$, while $r(G, V) > 1 < w(G, V)$ by our hypotheses and (1). Thus (2) follows from E.3.35.1. □

Recall that Hypothesis F.9.1 holds by 15.3.48.1 and 15.3.49.3.. Further 15.3.62.2 gives part (f) of Hypothesis F.9.8, while case (ii) of part (g) of Hypothesis F.9.8 holds by 15.3.47. Thus Hypothesis F.9.8 holds, so we conclude from F.9.16.3 that:

LEMMA 15.3.64. *$q(H^*, \tilde{U}_H) \leq 2$.*

LEMMA 15.3.65. *(1) If $H \in \mathcal{H}_*(T, M)$, then $n(H) = 1$.*
(2) $r(G, V) = 2$.

PROOF. By 15.3.46.3, $r(G, V) \geq 2$, so if (2) fails then $r(G, V) \geq 3$, and hence $n(H) > 1$ for $H \in \mathcal{H}(T, M)$ by 15.3.63.2. Thus (1) implies (2), so it remains to establish (1).

Assume $H \in \mathcal{H}_*(T, M)$ with $n(H) > 1$. Then in view of 15.3.2.6, H is described in E.2.2. In particular $K_0 := O^2(H) = \langle K^T \rangle$ for some $K \in \mathcal{C}(H)$, $K_0/O_2(K_0)$ is of Lie type over $\mathbf{F}_{2^n}$ for some $n > 1$, and setting $M_0 := M \cap K_0$, M_0 is a Borel subgroup of K_0. As M_H is a $3'$-group by 15.3.49.4, n is odd.

By A.1.42.2, we may pick $\tilde{I} \in Irr_+(K_0, \tilde{U}_H, T)$; set $\tilde{I}_T := \langle \tilde{I}^T \rangle$. We apply parts (4) and (5) of F.9.18 to the list of possibilities in E.2.2 defined over $\mathbf{F}_{2^n}$ with n odd. In view of 15.3.64, we may also appeal to Theorems B.4.2 and B.4.5; this determines the modules from the restrictions given in F.9.18. In particular as n is odd, there is no orthogonal module for $L_2(2^n)$. We conclude that one of the following holds:

(i) $K/O_2(K)$ is a Bender group, and $\tilde{I}/C_{\tilde{I}}(K)$ is the natural module for $K/O_2(K)$. Further either $K = K_0$ and $I = I_T$; or $K < K_0$, $K/O_2(K) \cong L_2(2^n)$ or $Sz(2^n)$, and for $t \in T - N_T(K)$, $I_T = I + I^t$ and $[I, K^t] = 0$.

(ii) $K/O_2(K) \cong SL_3(2^n)$ or $Sp_4(2^n)$, T is nontrivial on the Dynkin diagram of $K/O_2(K)$, and $\tilde{I}_T/C_{\tilde{I}_T}(K)$ is the sum of a natural module and its conjugate by an outer automorphism nontrivial on the diagram.

(iii) $K_0/O_2(K_0) \cong \Omega_4^+(2^n)$, and $\tilde{I}_T$ is the orthogonal module.

Now by Theorems B.5.1 and B.4.2, $K_0T/O_2(K_0T)$ has no FF-modules, except in (i) with $K/O_2(K) \cong L_2(2^n)$, where $K_0T/O_2(K_0T)$ has no strong FF-modules. We conclude from F.9.18.6 that either $I_T = [U_H, K_0]$, or case (i) holds with $K/O_2(K) \cong L_2(2^n)$, and $[U_H, K]/I$ is an extension of the natural module for $K/O_2(K)$ over a submodule centralized by K. (Recall that $n > 1$ is odd).

As T centralizes $\tilde{Z}_S$ and $H = K_0T$, $\tilde{U}_H = [\tilde{U}_H, K_0]C_{\tilde{U}_H}(H)$ by B.2.14. By 15.3.49.1, $O^2(M_0)$ centralizes V, and hence M_0 centralizes $\tilde{Z}_S$. It follows from the structure of the modules described in (i)–(iii), that H centralizes $\tilde{Z}_S$. But then K centralizes Z_S by Coprime Action, and so K centralizes $\tilde{U}_H$, contrary to $K^* \neq 1$. This contradiction completes the proof of 15.3.65. □

As $r(G, V) = 2$ by 15.3.65.2, there is $E_4 \cong E \leq V$ with and $G_E := N_G(E) \not\leq M$. Further E is totally singular by 15.3.46.2. Pick E so that $T_E := N_T(E) \in Syl_2(M_E)$, where $M_E := N_M(E)$. Let $Y_E := O^2(N_Y(E))$, $Q_E := O_2(G_E)$, and $V_E := \langle V^{G_E} \rangle$.

LEMMA 15.3.66. *(1)* $\bar{T}_E = \bar{T} \cap \Omega_4^+(V)$.
(2) $|T : T_E| = 2$.
(3) $\bar{T}_E$ *is the* 4*-subgroup of* $\bar{T}$ *distinct from* $\bar{S}$.
(4) $Z \leq E$.
(5) $Y_ET_E/O_2(Y_ET_E) \cong S_3$, $V = [V, Y_E]$, $Q_E \leq C_G(E)$, *and* $O_2(Y_ET_E) = C_{Y_ET_E}(E)$.
(6) $G_E = Y_ET_EC_G(E)$.
(7) $C_G(E) \leq M_c$.

PROOF. As E is a totally singular line in V, $Aut_M(E) = GL(E)$, so that $Q_E \leq C_G(E)$ and (1) and (5) hold. Then (1) implies (2)–(4), and as Y_ET_E induces $GL(E)$ on E, (6) holds. Finally $C_G(E) \leq C_G(Z) = M_c$ by (4) and 15.3.4. □

LEMMA 15.3.67. *(1)* $R := C_T(V) = C_G(V)$ *and* $M = YT$.
(2) $T_E \in Syl_2(G_E)$ *and* $B_E := Baum(T_E) \leq R$, *so that* $C(G, B_E) \leq M$.
(3) $G_E = Y_EX_ET_E$, *where* $X_E := O^2(C_G(E)) \not\leq M$, *with* $X_E/O_2(X_E) \cong Y_E/O_2(Y_E) \cong \mathbf{Z}_3$, *and* Y_E *and* X_E *are normal in* G_E.

(4) $G_E/Q_E \cong S_3 \times S_3$ *and* $X_E = [X_E, J(R)]$.

PROOF. By 15.3.5.2, if $A \in \mathcal{A}(T)$ with $A \not\leq C_T(V) = R$, then either $\bar{A} \leq \bar{M}_i$ or $\bar{A} = \bar{S}$. Therefore by 15.3.66.3, $J(T_E) = J(R)$, so that $B_E = \mathrm{Baum}(R)$ by B.2.3.5. Hence $C(G, B_E) \leq M$ as $M = !\mathcal{M}(N_M(R))$ by A.5.7.2. In particular $N_G(T_E) \leq M$, so as $T_E \in Syl_2(M_E)$, $T_E \in Syl_2(G_E)$, and hence (2) holds.

Let $Q_c := O_2(M_c)$ and $P_c := Q_c \cap G_E$. By 15.3.49.2, $\bar{Q}_c \neq 1$, so as $\bar{Q}_c \trianglelefteq \bar{T}$ while $Z(\bar{T})$ is of order 2 and lies in $\bar{T}_E$ by 15.3.66.3, $Z(\bar{T}) \leq \bar{P}_c$ and $Y_E = [Y_E, P_c]$. As P_c is $(M_c \cap G_E)$-invariant, $P_c^{G_E} = P_c^{Y_E}$ since $G_E = Y_E T_E C_G(E)$ by 15.3.66.6; thus as $Y_E = [Y_E, P_c]$, $Y_E = O^2(\langle P_c^{G_E}\rangle) \trianglelefteq G_E$. So as $V = [V, Y_E]$ by 15.3.66.5, $V_E = [V_E, Y_E]$. Further $V^{G_E} = V^{Y_E T_E C_G(E)} = V^{C_G(E)} \subseteq V^{M_c}$, so that $V_E \leq V_{M_c}$. Thus V_E is abelian since V_{M_c} is abelian by Theorem 15.3.50.

Let $S_E := O_2(Y_E T_E) = C_{Y_E T_E}(E)$ and $C_E := C_G(E)$. Then $S_E = C_T(E) \in Syl_2(C_E)$ by (2). Let $\dot{G}_E := G_E/Q_E$. Then $\dot{G}_E = \dot{Y}_E\langle\dot{\tau}\rangle \times \dot{C}_E$, where $\tau \in P_c - S_E$ and $\dot{Y}_E\langle\dot{\tau}\rangle \cong S_3$. As $\tau \notin S_E$ and $E \cong E_4$, $C_E(\tau) = Z$. As $G_E = Y_E\langle\tau\rangle C_E$ and $Y_E\langle\tau\rangle$ acts on V, $V_E = \langle V^{G_E}\rangle = \langle V^{C_E}\rangle$.

Let $\check{G}_E := G_E/E$. Now $[V, S_E] = E$ from 15.3.66.1, so Q_E centralizes $\check{V}_E$ as $Q_E \leq S_E$. Then as we saw $\dot{G}_E = \dot{Y}_E\langle\dot{\tau}\rangle \times \dot{C}_E$ and $\dot{Y}_E\langle\dot{\tau}\rangle \cong S_3$,

$$\check{V}_E = \check{V}_{E,1} \oplus \check{V}_{E,2}$$

is a C_E-invariant decomposition, where $V_{E,1} := C_{V_E}(\tau)$, and $V_{E,2} := C_{V_E}(\tau^y)$ for $1 \neq \dot{y} \in \dot{Y}_E$. Thus $V_{E,1}^y = V_{E,2}$.

Let $I := J(C_E)$. By (2), $J(T_E) \leq C_T(V) \leq S_E$, so that $J(T_E) = J(S_E)$ by B.2.3.3. Then $G_E := IN_{G_E}(J(T_E)) = IM_E$ by a Frattini Argument and (2), so as $G_E \not\leq M$, we conclude $I \not\leq M$. Thus $\dot{I} \neq 1$.

Let $I_0 := N_I(C_T(\check{V}_E))$. In order to determine the structure of I_0, temporarily replacing G_E by $N_{G_E}(C_T(\tilde{V}_E))$ if necessary, we may assume that $Q_E = C_T(\tilde{V}_E)$. We will drop this assumption later, once we have determined I_0, and then complete the proof of (1) and (3).

Let $\hat{G}_E := G_E/C_{G_E}(V_E)$. Now $\langle\tau, Q_E\rangle \leq T_E$, so $\Phi(\langle\bar{\tau}, \bar{Q}_E\rangle) \leq \Phi(\bar{T}_E) = 1$, and hence $\Phi(\langle\tau, Q_E\rangle \leq C_G(V)$. We saw earlier that $\dot{\tau}$ centralizes $\dot{C}_E$, so C_E acts on $\langle\tau, Q_E\rangle$. Then as $V_E = \langle V^{C_E}\rangle$, we conclude that $\Phi(\langle\tau, Q_E\rangle) \leq C_G(V_E)$, and hence $\Phi(\langle\hat{\tau}, \hat{Q}_E\rangle) = 1$. Therefore as Q_E centralizes $\check{V}_E$, $\hat{Q}_E$ induces a group of transvections on $V_{E,1}$ with center $C_E(\tau) = Z$. Next $[Y_E, Q_E] \leq O_2(Y_E) = C_{Y_E}(V)$, so as $[Y_E, Q_E] \trianglelefteq G_E$, $[Y_E, Q_E] \leq C_{G_E}(V_E)$, and hence $[\hat{Y}_E, \hat{Q}_E] = 1$. Then as $Y_{E,1}^y = Y_{E,2}$, $C_{Q_E}(V_{E_1}) = C_{Q_E}(V_E)$. Hence as Q_E induces a group of transvections on $V_{E,1}$ with center Z, we conclude $m(V_E/C_{V_E}(\hat{W})) = 2m(\hat{W})$ for each $\hat{W} \leq \hat{Q}_E$.

As $\dot{I} \neq 1$, there is $A \in \mathcal{A}(T_E)$ with $\dot{A} \neq 1$. Let $B := A \cap Q_E$ and $D := C_A(V_E)$. Then since $C_{V_E}(A) = A \cap V_E$ as $A \in \mathcal{A}(T_E)$,

$$m(\dot{A}) + m(\hat{B}) + m(D) = m(A) \geq m(DV_E)$$

$$\geq m(D) + m(V_E/(A \cap V_E)) = m(D) + m(V_E/C_{V_E}(A))$$

so that

$$m(\dot{A}) + m(\hat{B}) \geq m(V_E/C_{V_E}(A)).$$

Further using an earlier remark with $\hat{B}$ in the role of "$\hat{W}$",

$$m(V_E/C_{V_E}(A)) = m(V_E/C_{V_E}(B)) + N = 2m(\hat{B}) + N,$$

where $N := m(C_{V_E}(B)/C_{V_E}(A))$. Therefore $m(\dot{A}) \geq m(\hat{B}) + N$ and hence

$$2m(\dot{A}) \geq 2m(\hat{B}) + 2N = m(V_E/C_{V_E}(A)) + N.$$

Further

$$m(V_E/C_{V_E}(A)) \geq m(\check{V}_E/C_{\check{V}_E}(A)) = 2m(\check{V}_{E,1}/C_{\check{V}_{E,1}}(A)),$$

so $m(\dot{A}) \geq m(\check{V}_{E,1}/C_{\check{V}_{E,1}}(A)) + N/2$ with $N \geq 0$, and hence $\dot{A}$ is an FF*-offender for the FF-module $\check{V}_{E,1}$.

Next $m_3(G_E) \leq 2$ since G_E is an SQTK-group, and $\dot{G}_E \cong S_3 \times \dot{C}_E$, so $m_3(C_E) \leq 1$. Therefore by Theorem B.5.1, $\dot{I} \cong L_2(2^n)$, $L_3(2^m)$, m odd, or S_5, with $\check{V}_{E,1}/C_{\check{V}_{E,1}}(I)$ the natural module or the sum of two natural modules for $L_3(2^m)$. As $G_E = IM_E$, $V_E = \langle V^I \rangle$, and as $\check{V} \leq Z(\check{S}_E)$, $\check{V}_E = [\check{V}_E, I]\check{V}$ and $C_{\dot{I}}(\check{V}) = C_{\dot{I}}(\check{V}_\#)$, where $\check{V}_\# \neq 1$ is the projection of $\check{V}$ on $\check{V}_{E,1}$ in the decomposition of $\check{V}_E$. Also $\check{V}_\# \leq C_{\check{V}_{E,1}}(\check{S}_E)$, and by (2), $N_{\dot{I}}(J(S_E)) \leq \dot{C}_E \cap \dot{M}_E \leq C_{\dot{G}_E}(\check{V})$, so $N_{\dot{I}}(J(S_E)) \leq C_{\dot{I}}(\check{V}_\#)$. Using the structure of $J(S_E)$ from Theorem B.4.2 we conclude that $\dot{I} \cong S_3$, S_5, or $L_3(2)$. But in the last two cases, as $\check{V}_\# \leq Z(\check{S}_E)$, $O^{3'}(C_I(\check{V})) \neq 1$, contradicting 15.3.49.5.

Therefore $\dot{I}_0 \cong S_3$ and $m([\check{V}_{E,1}, I]) = 2$. At this point, we drop the temporary assumption that $Q_E = C_T(\check{V}_E)$.

By a Frattini Argument, $I = I_0 C_I(\check{V}_E)$, while $C_I(\check{V}_E) \leq N_I(V) \leq M_E$. Thus as $G_E = M_E I$, $G_E = M_E I_0$, so $|G_E : M_E| = 3$. Therefore $O^{2,3}(G_E) = O^{2,3}(C_G(V))$ is normal in M and G_E, so $O^{2,3}(G_E) = O^{2,3}(C_M(V)) = 1$ as $G_E \not\leq M \in \mathcal{M}$. Then as $C_M(V)$ is a $3'$-group by 15.3.49.4, (1) holds. Hence $M_E = Y_E T_E$, so as $|G_E : M_E| = 3$, (3) holds. Further $X_E = [X_E, J(R)]$ since $J(T_E) = J(R)$ by (2), so $G_E/Q_E \cong S_3 \times S_3$ since $Y_E T_E/O_2(Y_E T_E) \cong S_3$ and $R \leq O_2(M_E)$. This completes the proof of 15.3.67. □

Next Z is contained in exactly two totally singular 4-subgroups E and $F := E^s$ of V, where $s \in S - T_E$. Observe $T_E = T_E^s = T_F$ acts on $Y_F := Y_E^s$ with $T_E \in Syl_2(Y_F T_E)$, and $Y = Y_E Y_F O_2(Y)$. Let $G_1 := Y_F T_E$, $G_2 := X_E T_E$, and $G_0 := \langle G_1, G_2 \rangle$. Set $L_i := O^2(G_i)$ and $Q_i := O_2(G_i)$ for $i = 1, 2$. Thus $G_i/Q_i \cong S_3$ and $T_E = G_1 \cap G_2 \in Syl_2(G_i)$.

LEMMA 15.3.68. *(1)* $G_0 \leq N_G(Y_E)$.
(2) $V \leq Z(Q_0)$*, where* $Q_0 := O_2(G_0)$.
(3) $T_E \in Syl_2(M_0)$ *for each* $M_0 \in \mathcal{M}(G_0)$.
(4) (G_0, G_1, G_2) *is a Goldschmidt triple.*
(5) $Q_0 = O_{3'}(G_0)$.

PROOF. By construction, $Y_E \trianglelefteq YT_E$, so G_1 acts on Y_E. By 15.3.67.3, G_2 acts on Y_E. Thus $G_0 = \langle G_1, G_2 \rangle$ acts on Y_E, establishing (1). By 15.3.66.5, $V = [V, Y_E]$, so $V \leq O_2(Y_E)$ and hence $V \leq Q_0$ by (1). Set $R := C_T(V)$ as in 15.3.67.1. Then $Q_0 \leq Q_1 \cap Q_2 = R = C_{T_E}(V)$, so $V \leq Z(Q_0)$. Hence (2) holds. Next let $M_0 \in \mathcal{M}(G_0)$. As $N_G(T_E) \leq M$ by 15.3.67.2, if $T_E \notin Syl_2(M_0)$ then we may take $T \leq M_0$. But then $YT = \langle Y_F, T \rangle \leq M_0$, so $X_E \leq M_0 = M = !\mathcal{M}(YT)$ by 15.3.7, contrary to 15.3.67.3. Hence (3) holds and $T_E \in Syl_2(G_0)$, so (4) holds.

Let $P := O_{3'}(G_0))$. By F.6.11.1, P is 2-closed with $T_E \cap P = Q_0$, so P is solvable. By (2), $V \leq O_2(P)$, so $O(P) \leq C_G(V)$, and hence $O(P) = 1$ as $C_G(V) = R$ by 15.3.67.1. Thus $F^*(P) = Q_0$. Let $X := J(T_E)P$, $T_0 := T_E \cap X$, and $Z_0 := R_2(X)$. Then $T_0 \in Syl_2(X)$ as Q_0 is Sylow in P, and $F^*(X) = O_2(X)$.

By 15.3.67.2, $J(T_E) \leq R$, so that $J(T_E) \trianglelefteq Y_F T_E$, and so Y_F acts on X. As X is a $3'$-group with $F^*(X) = O_2(X)$, $X = C_X(Z_0)N_X(J(T_E))$ by Solvable Thompson Factorization B.2.16. As Y_F acts on X, $F = \langle Z^{Y_F} \rangle \leq Z_0$ by B.2.14, so $P = C_P(F)N_P(J(T_E))$. But by 15.3.67, $C_G(F)$ and $M = YT$ are $\{2,3\}$-groups, so $P = Q_0$ as $N_G(J(T_E)) \leq M$ by 15.3.67.2. Thus (5) holds, completing the proof of 15.3.68. □

We are now in a position to complete the proof of Theorem 15.3.1.

Let $Q_0 := O_2(G_0)$ and $\dot{G}_0 := G_0/Q_0$. By 15.3.68.3 and F.6.5.1, $(\dot{G}_1, \dot{T}_E, \dot{G}_2)$ is a Goldschmidt amalgam. Since $G_1 \cap G_2 = T_E$, and $O_{3'}(G_0) = Q_0 \leq T_E$ by 15.3.68.5 case (i) of F.6.11.2 holds, so $\dot{G}_0$ is described in Theorem F.6.18.

Let $V_0 := \langle V^{G_0} \rangle$. By 15.3.68.2, $V_0 \leq \Omega_1(Z(Q_0))$. Also $C_{G_0}(V_0) \leq C_{G_0}(V) = R$ is a 2-group by 15.3.67.1, so $Q_0 = C_{G_0}(V_0)$. By 15.3.67.4, $X_E = [X_E, J(T_E)] \leq J(G_0) =: X$, so V_0 is an FF-module for $\dot{G}_0$. Thus examining the list of Theorem F.6.18 for groups appearing in Theorem B.5.6, and recalling that $J(T_E) \trianglelefteq G_1$ by 15.3.67.2, we conclude that $\dot{X} \cong S_3$, $L_3(2)$, A_6, S_6, A_7, S_7, $\hat{A}_6$, or $G_2(2)$.

Assume first that $\dot{X} \cong S_3$. Then $X_E = O^2(X)$, so $Z \leq C_{G_0}(X)$, and hence $F = \langle Z^{Y_F} \rangle \leq C_{G_0}(X)$. But then X_E acts on $EF = Z^{\perp}$, so $X_E \leq M$ by 15.3.46.2, contrary to 15.3.67.3.

In the remaining cases, $O^2(X) = O^2(G_0)$ by Theorem F.6.18, so $Y_F \leq X$. However $Q_0 O_2(Y_F) \leq C_{T_E}(V) = R$, so $O_2(\dot{Y}_F)$ centralizes V, while $[V, Q_1] = F$, so $\dot{Q}_1 > O_2(\dot{Y}_F)$. This eliminates the cases $\dot{G}_0 \cong L_3(2)$, A_6, A_7, or $\hat{A}_6$, so that $\dot{G}_0$ is S_6, $\hat{S}_6$, S_7, or $G_2(2)$. As $V = [V, Y_F] \leq [V_0, X]$, $V_0 = [V_0, X]$. Thus $O_2(\dot{Y}_F)$ centralizes the 4-dimensional subspace V of the FF-module $V_0 = [V_0, X]$ for $\dot{X}$, so we conclude using Theorem B.5.1 that $\dot{G}_0$ is $\hat{S}_6$ and $m(V_0) = 6$. But now $N_{\dot{X}}(V_1)$ has a quotient A_5, whereas $N_G(V_1) \leq M$ by 15.3.45.2, and M is solvable by 15.3.67.1.

This contradiction completes the proof of Theorem 15.3.1.

15.4. Completing the proof of the Main Theorem

In this section, we complete the treatment of the case $\mathcal{L}_f(G,T)$ empty, and hence also the proof of the Main Theorem. Our efforts so far have in effect reduced us to the case $\mathcal{L}(G,T)$ empty (cf. 15.4.2.1 below).

More precisely, since we have been assuming that $|\mathcal{M}(T)| > 1$, and since Theorem 15.3.1 completed the treatment of groups satisfying Hypothesis 14.1.5, we may assume that condition (2) of Hypothesis 14.1.5 fails. Thus in this section, we assume instead:

HYPOTHESIS 15.4.1. *G is a simple QTKE-group, $T \in Syl_2(G)$, and*

(1) $\mathcal{L}_f(G,T) = \emptyset$.

(2) Let $Z := \Omega_1(Z(T))$. Then $|\mathcal{M}(C_G(Z))| > 1$.

The section culminates in Theorem 15.4.24, where we see that $L_3(2)$ and A_6 are the only groups which satisfy Hypothesis 15.4.1.

We now define a collection of subgroups similar to the set $\Xi(G,T)$ of chapter 1: Let $\xi(G,T)$ consist of those T-invariant subgroups $X = O^2(X)$ of G such that $XT \in \mathcal{H}(T)$ and $|X : O_2(X)|$ is an odd prime. Let $\xi^*(G,T)$ consist of those $X \in \xi(G,T)$ such that $\exists! \mathcal{M}(XT)$.

Because of Hypothesis 15.4.1, members of $\xi(G,T)$ have few overgroups, so that $\xi^*(G,T)$ is nonempty in many situations—cf. 15.4.3.1 and 15.4.12.

15.4.1. Preliminary results, and the reduction to $\mathbf{C_G(Z) = T}$. Recall $\Xi_+(G,T)$ is defined before 3.2.13.

LEMMA 15.4.2. *(1) $\mathcal{L}(G,T) = \emptyset$.*

(2) Each member of $\mathcal{M}(T)$ is solvable.

(3) If $X \in \Xi(G,T)$ then $[Z,X] \neq 1$ and $X \in \Xi^*_f(G,T)$ but $X \notin \Xi_+(G,T)$. In particular $X/O_2(X)$ is a 3-group or a 5-group.

(4) Assume $M_0 \in \mathcal{H}(T)$ with $M = !\mathcal{M}(M_0)$, and $V \in \mathcal{R}_2(M_0)$ with $R := C_T(V) = O_2(C_{M_0}(V))$ and $V \leq O_2(M)$. Then $\hat{q}(M_0/C_{M_0}(V), V) \leq 2$.

PROOF. Assume $\mathcal{L}(G,T) \neq \emptyset$. Then there is $L \in \mathcal{L}^*(G,T)$. Setting $L_0 := \langle L^T \rangle$, $N_G(L_0) = !\mathcal{M}(\langle L,T \rangle)$ by 1.2.7.3. But by Hypothesis 15.4.1.1, $\mathcal{L}_f(G,T) = \emptyset$, so $\langle L,T \rangle \leq C_G(Z)$ and hence $N_G(L_0) = !\mathcal{M}(C_G(Z))$, contrary to Hypothesis 15.4.1.2. Thus (1) holds, and since $\mathcal{C}(M) \subseteq \mathcal{L}(G,T)$ for $M \in \mathcal{M}(T)$, (1) and 1.2.1.1 imply (2).

Supppose $X \in \Xi(G,T)$. By (1), $X \in \Xi^*(G,T)$, so by 1.3.7, $N_G(X) = !\mathcal{M}(XT)$. Thus $[Z,X] \neq 1$ by Hypothesis 15.4.1.2, so $X \in \Xi^*_f(G,T)$. Hence $X \notin \Xi_+(G,T)$ by 3.2.13, completing the proof of (3).

Assume the hypotheses of (4), and let $\hat{q} := \hat{q}(M_0/C_{M_0}(V), V)$. Pick $H \in \mathcal{H}_*(T,M)$, and let $Q_H := O_2(H)$. Observe that Hypothesis D.1.1 is satisfied with M_0, H in the roles of "G_1, G_2": First, by hypothesis $M = !\mathcal{M}(M_0)$, so that $O_2(\langle M_0, H \rangle) = 1$, and hence part (3) of Hypothesis D.1.1 holds. Second, $C_T(V) = O_2(C_{M_0}(V))$, with $O_2(C_{M_0}(V)) = O_2(M_0)$ since $V \in \mathcal{R}_2(M_0)$, so that part (2) of D.1.1 holds. Finally by 3.1.3.1, $H \cap M$ is the unique maximal subgroup of H containing T, so that part (1) of D.1.1 holds. Now recall by B.5.13 that if the dual V^* is an FF-module for $M_0/C_{M_0}(V)$, then $q(M_0/C_{M_0}(V), V) \leq 2$. Thus combining conclusions (2), (3), and (4) of the *qrc*-lemma D.1.5 into case (ii) below, one of the following holds:

(i) $V \not\leq Q_H$.

(ii) $q(M_0/C_{M_0}(V), V) \leq 2$.

(iii) $V \leq R \cap Q_H \trianglelefteq H$, the dual V^* is not an FF-module for $M_0/C_{M_0}(V)$, $U := \langle V^H \rangle$ is abelian, and H has a unique noncentral chief factor on U.

If (ii) holds, then the conclusion of (4) holds and we are done.

Assume that (i) holds. We verify Hypothesis E.2.8 with $H \cap M$ in the role of "M": As $V \not\leq Q_H$, $T \not\leq Q_H$, so $H \not\leq N_G(T)$; hence by 3.1.3.2, H is a minimal parabolic in the sense of Definition B.6.1, and so is described in B.6.8. Therefore by B.6.8.5, $\ker_{H \cap M}(H)$ is 2-closed with Sylow group Q_H, so $V \not\leq \ker_{H \cap M}(H)$. Finally $V \leq O_2(M)$ by hypothesis, so that $V \leq O_2(H \cap M)$. Now $\hat{q}(Aut_H(V), V) \leq 2$ by E.2.13.2, and again (4) holds.

This leaves case (iii), so as H is solvable by (2), $R \in Syl_2(O^2(H)R)$ by D.1.4.4.2. But now $O_2(\langle M_0, H \rangle) \neq 1$ by Theorem 3.1.1, contrary to an earlier observation. This completes the proof of (4), and hence of 15.4.2. □

Following the notational convention of chapter 1, set $\xi_f(G,T) := \xi(G,T) \cap \mathcal{X}_f$.

LEMMA 15.4.3. *Let $X \in \xi(G,T)$, with $|X : O_2(X)| = p$ where p is chosen maximal among such X, and suppose $p > 3$. Then*

(1) $\exists!\mathcal{M}(XT)$*, so that* $X \in \xi^*(G,T)$*.*
(2) $X \leq O_{2,p}(M)$ *for* $M \in \mathcal{M}(XT)$*. In particular,* $O_2(X) \leq O_2(M)$*.*
(3) $[Z,X] \neq 1$*, so* $X \in \xi_f(G,T)$*.*
(4) $p = 5$*.*

PROOF. Let $M \in \mathcal{M}(XT)$, and set $M^* := M/O_2(M)$. If $q \neq p$ is an odd prime such that $[O_q(M^*), X^*] \neq 1$, then $Aut_X(O_q(M^*))$ is embedded in $GL_2(q)$ by A.1.25.2, so p divides $q - \epsilon$ for $\epsilon := \pm 1$. This is impossible as $p \geq q$ by maximality of p, with p and q odd. Therefore $X^* \leq C_{M^*}(O^p(F(M^*))) =: H^*$, and as M is solvable by 15.4.2.2, $F^*(H^*) = O_p(H^*) \times O^p(Z(H^*))$. As $p > 3$, the solvable subgroup $Aut_H(O_p(H^*))$ of $GL_2(p)$ is p-closed using Dickson's Theorem A.1.3, so that $X^* = O^{p'}(X^*) \leq O_p(H^*) \leq O_p(M^*)$, establishing (2) for each $M \in \mathcal{M}(XT)$.

Let $N_G(X) \leq M_X \in \mathcal{M}$. We will show that $M = M_X$; then since M is an arbitrary member of $\mathcal{M}(XT)$, (1) will be established. Let $K := O_{2,F}(M)$ and $Y := O^2(N_K(X))$; by (2), $X \leq Y \leq M \cap M_X =: I$. Let X_0 be the preimage in M of X^*; as X is T-invariant, $X = O^2(X_0)$, so $N_M(X^*) = N_M(X)$. Thus $Y^* = O^2(N_{K^*}(X^*))$. Next $C_{K^*}(Y^*) \leq C_{K^*}(X^*) \leq Y^*$ as K^* is of odd order. Thus the hypotheses of case (b) of A.4.4 are satisfied with M, M_X, $YO_2(M)$, in the roles of "H, K, X". Therefore $R := O_2(I) = O_2(M)$ by A.4.4.1, and $C(M_X, R) \leq I$ by A.4.4.2, so Hypothesis C.2.3 is satisfied by M_X, I in the roles of "H, M_H". Further C.2.6.2 says that either $O_{2,F}(M_X) \leq I$, or M_X has an A_3-block L with $L \not\leq I$. In the first case $M = M_X$ by A.4.4.3, as desired. In the second $[L,Y] \leq O_2(L)$, so taking Y_Z to be the preimage in M of $Z(O_p(M^*)) \leq Y^*$ and $Y_0 := O^2(Y_Z)$, L normalizes $O^2(Y_0 O_2(L)) = Y_0$. Then $L \leq N_G(Y_0) = M$ as $M \in \mathcal{M}$, contrary to $L \not\leq I$. This contradiction completes the proof of (1).

If $[Z,X] = 1$ then $XT \leq C_G(Z)$ and $M = !\mathcal{M}(XT)$ by (1), so that also $M = !\mathcal{M}(C_G(Z))$, contrary to Hypothesis 15.4.1.2. Thus (3) holds.

Next $V := \langle Z^X \rangle \in \mathcal{R}_2(XT)$ and $V \leq O_2(M)$ by B.2.14, and as $[Z,X] \neq 1$ and $X/O_2(X)$ has prime order, $C_{XT}(V) = O_2(C_{XT}(V)) = C_T(V)$. Then by (1) we may apply 15.4.2.4 with XT in the role of "M_0", to conclude $\hat{q}(XT/C_{XT}(V), V) \leq 2$. Then as $p > 3$ by hypothesis, D.2.13.1 shows $p = 5$, so (4) holds. □

LEMMA 15.4.4. *Each member of* $\mathcal{M}(T)$ *is a* $\{2,3,5\}$*-group.*

PROOF. Suppose some $M \in \mathcal{M}(T)$ has order divisible by $p > 5$, and choose p maximal subject to this constraint. As M is solvable by 15.4.2.2, there is a Hall $\{2,p\}$-subgroup H of M containing T. Let P denote a Sylow p-subgroup of the preimage in H of $\Omega_1(Z(H/O_2(H)))$; then $TP \in \mathcal{H}(T)$ with P elementary abelian, and $m_p(P) \leq 2$ since H is an SQTK-group. If $m_p(P) = 2$ and T is irreducible on P, then $H \in \Xi(G,T)$, contrary to 15.4.2.3. Thus there is $X \leq TP$ with $X \in \xi(G,T)$, contrary to 15.4.3.4. □

LEMMA 15.4.5. $C_G(Z)$ *is a* $\{2,3\}$*-group.*

PROOF. If not, arguing as in the proof of 15.4.4, there is a nontrivial elementary abelian 5-subgroup P of $C_G(Z)$ with $PT \in \mathcal{H}(T)$, and there is $X \leq PT$ with $X \in \Xi(G,T)$ or $\xi(G,T)$. Since $X \leq C_G(Z)$, the former is impossible by 15.4.2.3, and the latter by 15.4.3.3. □

Recall from 14.1.4 that for $V(M) := \langle Z^M \rangle$:

LEMMA 15.4.6. *If M is maximal in $\mathcal{M}(T)$ with respect to $\lesssim$ and $[V(M), J(T)] = 1$, then M is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.*

The two groups which satisfy Hypothesis 15.4.1 appear in conclusion (2) of the next result. The second subsection completes the treatment of Hypothesis 15.4.1, by eliminating the case where $|Z| > 2$, and the case where $|Z| = 2$ but conclusion (1) of the result fails.

LEMMA 15.4.7. *Assume Z is of order 2. Then either*

(1) There exists a nontrivial characteristic subgroup $C_2 := C_2(T)$ of Baum(T) such that

$$M_f = !\mathcal{M}(N_G(C_2)),$$

and M_f is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$, or

(2) $G \cong L_3(2)$ or A_6.

PROOF. Let $S := \mathrm{Baum}(T)$ and choose $C_i := C_i(T)$ for $i = 1, 2$ as in the Glauberman-Niles/Campbell Theorem C.1.18. Thus $1 \neq C_2$ char S, and $1 \neq C_1 \leq Z$, so as Z is of order 2 by hypothesis, $C_1 = Z$.

Assume (2) fails; we claim that:

(*) For each $M \in \mathcal{M}(T)$, $M = N_M(C_2)C_M(V(M))$.

Assume (*) fails and set $V := V(M)$. If $[V, J(T)] = 1$, then $S = \mathrm{Baum}(C_T(V))$ by B.2.3.5, so (*) holds by a Frattini Argument, contrary to our assumption. Thus $[V, J(T)] \neq 1$.

By 15.4.2.2, M is solvable, so by Solvable Thompson Factorization B.2.16, $\overline{J(M)} = \bar{Y} = \bar{Y}_1 \times \cdots \times \bar{Y}_r$ with $\bar{Y}_i \cong S_3$ and $V = V_1 \times \cdots \times V_r \times C_V(J(M))$, where $V_i := [V, Y_i] \cong E_4$, and Y and Y_i denote the preimages in M of $\bar{Y}$ and $\bar{Y}_i$. As M is an SQTK-group, $r \leq 2$ by A.1.31.1. As $|Z| = 2$, $C_V(J(M)) = 1$ and T is transitive on $\{Y_1, \ldots, Y_r\}$. Thus if $r = 1$, then $m(V) = 2$ and $\bar{M} = \bar{Y} = GL(V)$, while if $r = 2$ then $m(V) = 4$ and $\bar{M}$ is the normalizer $O_4^+(V)$ of $\bar{Y}$ in $GL(V)$. In either case as $C_1 = Z$, $C_M(C_1) = C_M(Z) = C_M(V)T$. Thus as (*) fails, $M > N_M(C_2)C_M(V) = N_M(C_2)C_M(C_1) = \langle N_M(C_2), C_M(C_1)\rangle$. Therefore as M is solvable, we conclude from C.1.28 that there is an A_3-block $A_4 \cong X \trianglelefteq\trianglelefteq M$ such that $X = [X, J(T)]$.

Let $X_0 := \langle X^M \rangle$. By the previous paragraph, either $r = 1$ and $X_0 = X$, or $r = 2$ and $X_0 = X_1 \times X_2$ with $X = X_1$ and $X_2 = X^t$ for suitable $t \in T - N_M(X)$. Let $H \in \mathcal{M}(X_0T)$. As H is solvable by 15.4.2.2, applying C.1.27 to H, X in the roles of "G, K", we conclude that $X_0 \trianglelefteq H$, so $H = N_G(X_0)$ as $H \in \mathcal{M}$. Thus $H = !\mathcal{M}(X_0T)$, so $M = H = N_G(X_0) = !\mathcal{M}(X_0T)$ as $X_0T \leq M \in \mathcal{M}$. Next $X_0T/C_T(X_0) \cong S_4$ or S_4 wr $\mathbf{Z}_2$. As $|Z| = 2$, $Z \leq O_2(X_0)$, so $C_T(X_0) = 1$. Then as $M \in \mathcal{H}^e$, $C_M(X_0) = 1$, so $M = X_0T \cong S_4$ or S_4 wr $\mathbf{Z}_2$. In the second case, Theorem 13.9.1 supplies a contradiction, so suppose the first case holds. Then $T \cong D_8$, so as $F^*(C_G(Z)) = O_2(C_G(Z))$ by 1.1.3.2, we conclude $T = C_G(Z)$. Further Thompson Transfer shows that each noncentral involution of T is fused into $Z(T)$, so that G has one conjugacy class of involutions. Thus (2) holds by I.4.1.2, contrary to our assumption; this completes the proof of the claim (*).

Pick $M_f \in \mathcal{M}(N_G(C_2))$. By (*), for each $M \in \mathcal{M}(T)$, $M \lesssim M_f$. Thus M_f is the unique maximal member of $\mathcal{M}(T)$ under $\lesssim$—in particular M_f is uniquely determined since $\lesssim$ is a partial order (cf. A.5.5, and in particular A.5.4). Thus (1) holds. □

We come to the main result of this subsection:

THEOREM 15.4.8. *$C_G(Z) = T$.*

The proof of Theorem 15.4.8 involves a short series of reductions. Until the proof is complete, assume G is a counterexample. Let $\mathcal{X}$ consist of those $X \in \xi(G,T)$ such that $X \trianglelefteq C_G(Z)$. Recall by 15.4.5 that $X/O_2(X) \cong \mathbf{Z}_3$.

LEMMA 15.4.9. *$\mathcal{X} \neq \emptyset$.*

PROOF. Set $H := C_G(Z)$ and $\hat{H} := H/O_2(H)$. By 15.4.5, H is a $\{2,3\}$-group, so as G is a counterexample to Theorem 15.4.8, $O_3(\hat{H}) \neq 1$. Let $\hat{P} := \Omega_1(Z(O_3(\hat{H})))$. If $\hat{P}$ is of order 3, then $X := O^2(P) \in \mathcal{X}$, so we may assume that $|\hat{P}| > 3$. Then as $m_3(H) \leq 2$, $E_9 \cong \hat{P} = \Omega_1(O_3(\hat{H}))$, so that $C_{\hat{H}}(\hat{P}) = O_3(\hat{H})$ by Coprime Action, and $\hat{H}/O_3(\hat{H})$ is a subgroup of $GL_2(3)$. As H centralizes Z, $PT \notin \Xi(G,T)$ by 15.4.2.3, so $\hat{T}$ is not irreducible on $\hat{P}$. Therefore there is a normal subgroup $\hat{P}_1$ of $\hat{H}$ of order 3, and so $X := O^2(P_1) \in \mathcal{X}$. □

LEMMA 15.4.10. *For each $X \in \mathcal{X}$ and each $M \in \mathcal{M}(XT)$, $X \leq C_M(V(M))$.*

PROOF. Assume X, M is a counterexample, and let $V := V(M)$ and $\bar{M} := M/C_M(V)$. In particular $\bar{X} \neq 1$. If $O_2(\bar{X}) = 1$, then $V = [V,X] \oplus C_V(X)$ by Coprime Action, and $Z \cap [V,X] \neq 1$ as X is T-invariant, contrary to $X \leq C_G(Z)$. Therefore $O_2(\bar{X}) \neq 1$, so $O_2(X) \not\leq O_2(M)$.

Let $M^* := M/O_2(M)$. Thus $1 \neq O_2(X)^* \leq O_2(X^*)$. We claim next that $O_2(X^*)$ centralizes $O_5(M^*)$, so suppose not. Then by A.1.25, $O_2(X^*)$ acts nontrivially on a supercritical subgroup P^* of $O_5(M^*)$, $P^* \cong \mathbf{Z}_5$, E_{25} or 5^{1+2}, and $Aut_X(P^*)$ is a subgroup of $Aut(P^*)/O_5(Aut(P^*)) \cong GL_2(5)$. As $O_2(X^*)$ does not centralize P^* and $X^* = O^2(X^*)$, we conclude that P^* is not of order 5 and $Aut_{X^*}(P^*) \cong \mathbf{Z}_3/Q_8$. Thus $O_2(X^*)$ is irreducible on $P^*/\Phi(P^*)$, and so the preimage P contains a member of $\Xi_+(G,T)$, contrary to 15.4.2.3. This establishes the claim.

As M is a solvable $\{2,3,5\}$-group by 15.4.4, $F^*(M^*) = O_3(M^*)O_5(M^*)$, so $O_2(X^*)$ is faithful on $O_3(M^*)$ by the claim. Again by A.1.25, $O_2(X^*)$ acts nontrivially on a supercritical subgroup P^* of $O_3(M^*)$, $P^* \cong E_9$ or 3^{1+2}, and $O_2(X^*) \cong Q_8$ is irreducible on $P^*/\Phi(P^*)$. Let $Y := O^2(P)$, so that $Y \in \Xi(G,T)$ and $Aut_X(P^*) \cong SL_2(3)$. If $P^* \cong 3^{1+2}$, then as $Aut_{X^*}(P^*) \cong SL_2(3)$, $m_3(XP) = 3$, contrary to M an SQTK-group; thus $P^* \cong E_9$.

Let $H := YXT$, $W := \langle Z^H \rangle$, and $H^+ := H/C_H(W)$; then $W = \langle Z^H \rangle \in \mathcal{R}_2(H)$ and $W \leq O_2(M)$ by B.2.14. As $[Z,Y] \neq 1$ by 15.4.2.3, and $O_2(X^*)$ is irreducible on P^*, $C_Y(W) = O_2(Y)$. Therefore H^+ is the split extension of $P^+ \cong E_9$ by either $SL_2(3)$ or $GL_2(3)$, so W contains an 8-dimensional faithful irreducible H-submodule I. Thus $\hat{q}(H^+,W) \geq \hat{q}(H^+,I) > 2$.

By 15.4.2.3, $Y \in \Xi^*(G,T)$, so that $N := N_G(Y) = !\mathcal{M}(YT)$ by 1.3.7, and as $YT \leq M$, $M = N$. Of course $YT \leq H$, so $M = !\mathcal{M}(H)$. Further $O_2(C_H(W)) = C_T(W)$ since $C_Y(V) = O_2(Y)$. Thus we may apply 15.4.2.4 to conclude $\hat{q}(H^+,W) \leq 2$, contrary to the previous paragraph. □

We are now ready to complete the proof of Theorem 15.4.8. By Hypothesis 15.4.1.2, there exist distinct members M_1 and M_2 of $\mathcal{M}(C_G(Z))$. By 15.4.9, there is $X \in \mathcal{X}$. Now X is not normal in both M_1 and M_2, so we may assume X is not normal in M_1. Let $Y_1 := \langle X^{M_1} \rangle$, and set $M_i^i := M_i/O_2(M_i)$ for $i = 1,2$. By

15.4.10, $X \leq C_{M_i}(V(M_i)) =: H_i$, so as $H_i \leq C_G(Z)$ and $X \trianglelefteq C_G(Z)$, $X \trianglelefteq H_i$. Thus $O_2(X) \leq O_2(H_i)$, so X^i is of order 3 for each i, and $Y_1^1 \cong E_{3^e}$ for some $e \geq 1$. Notice $e \leq m_3(M_1) \leq 2$, so as X is not normal in M_1, $e = 2$. Now $T \leq C_G(Z) \leq N_{M_1}(X)$, so as $Aut_{M_1}(Y_1^1) \leq GL_2(3)$ and $Y_1 = \langle X^{M_1}\rangle$, it follows that $O^2(Aut_{M_1}(Y_1^1)) \cong \mathbf{Z}_3$. Therefore $Y_1 = XX_1$, where $X_1 := O^2(X_0)$ and X_0 is the preimage in Y_1 of $C_{Y_1^1}(P^1)$ for $P \in Syl_3(M_1)$. Thus $X_1 \in \mathcal{X}$, so by 15.4.10, $X_1 \leq H_2$. As $H_2 \leq C_G(Z)$, X^2 and X_1^2 are normal in H_2^2. Therefore $O^2(H_2^2) \leq C_{H_2^2}(Y_1^2)$, so as $m_3(H_2) \leq 2$, $Y_1^2 = \Omega_1(O_3(H_2^2))$. Hence Y_1 is normal in M_2, and Y_1 is normal in M_1 by definition, contrary to the simplicity of G. This contradiction completes the proof of Theorem 15.4.8.

In the remainder of the subsection, we collect some useful consequences of Theorem 15.4.8.

LEMMA 15.4.11. *For each $M \in \mathcal{M}(T)$:*

(1) $O_2(M) = C_M(V(M))$.

(2) M is maximal with respect to $\lesssim$. In particular, there is no unique maximal member of $\mathcal{M}(T)$ under $\lesssim$.

(3) $[V(M), J(T)] \neq 1$.

PROOF. First $C_M(V(M)) \leq C_G(Z) = T$ by Theorem 15.4.8, so as $V(M)$ is 2-reduced, (1) holds. Now if $M \lesssim M_1 \in \mathcal{M}(T)$, then

$$M = C_M(V(M))(M \cap M_1) = O_2(M)(M \cap M_1) \leq M_1$$

by (1), so $M = M_1$. Thus M is maximal in $\mathcal{M}(T)$ under $\lesssim$, so since $|\mathcal{M}(T)| > 1$ by Hypothesis 15.4.1.2, (2) holds. Then (3) follows from (2) and 15.4.6. □

Define $\mathcal{Y}$ to consist of those groups Y in $\Xi(G,T) \cup \xi(G,T)$ such that $Y = [Y, J(T)]$; we show $\mathcal{Y}$ is nonempty in the next lemma. Set $S := \mathrm{Baum}(T)$ and $E := \Omega_1(Z(J(T)))$.

LEMMA 15.4.12. *Let $M \in \mathcal{M}(T)$. Then $\mathcal{Y} \cap M \neq \emptyset$. Further for each $Y \in \mathcal{Y} \cap M$:*

(1) $Y \trianglelefteq M$.

(2) $M = N_G(Y) = !\mathcal{M}(YT)$.

(3) For suitable $s(Y) = 1$ or 2, $Y/O_2(Y) \cong E_{3^{s(Y)}}$ and $m([V(M), Y]) = 2s(Y)$.

(4) $S \in Syl_2(YS)$.

(5) $R_2(YT) = [V(M), Y] \oplus E_Y$, where $E_Y := C_{\Omega_1(Z(O_2(YT)))}(Y)$ and $E = E_Y \oplus C_{[V(M),Y]}(S)$.

(6) Either

(i) $s(Y) = 1$, $YT/O_2(YT) \cong S_3$, and $|E : E_Y| = 2$, or

(ii) $s(Y) = 2$, $YT/O_2(YT) \cong O_4^+(2)$, $Y = Y_1Y_2$ with $Y_i/O_2(Y_i) \cong \mathbf{Z}_3$, $[V(M), Y] = V_1 \oplus V_2$, where $V_i := [V(M), Y_i] \cong E_4$ for $i = 1, 2$, $Y_i = [Y_i, J(T)]$ is S-invariant, and $|E : E_Y| = 4$.

PROOF. Set $\bar{M} := M/C_M(V(M))$. By 15.4.11.3, $[V(M), J(T)] \neq 1$, so as M is solvable, we conclude from Solvable Thompson Factorization B.2.16 and A.1.31.1 that $\bar{X} := [O_3(\bar{M}), J(T)]$ and its action on $V(M)$ are described in (3). Let X be the preimage in M of $\bar{X}$ and $Y_0 := O^2(X)$.

By 15.4.11.1, $O_2(Y_0) = C_{Y_0}(V(M))$. If T acts irreducibly on $\bar{X}$, then Y_0 lies in $\Xi(G,T)$ or $\xi(G,T)$, and hence also in $\mathcal{Y} \cap M$, and Y_0 satisfies (1) and (3). Otherwise, $\bar{X} = \bar{X}_1 \times \bar{X}_2$ where $\bar{X}_i$ is T-invariant; setting $Y_i := O^2(X_i)$ for X_i the preimage in M of $\bar{X}_i$, Y_i lies in $\xi(G,T)$ and hence also in $\mathcal{Y} \cap M$, and Y_i satisfies (1) and (3). In particular $\mathcal{Y} \cap M \neq \emptyset$. By E.2.3, Y_i satisfies (4)–(6) for $i = 0$ or $i = 1, 2$ in our two cases.

Now consider any $Y \in \mathcal{Y} \cap M$. By 15.4.11.1, $O_2(Y) = C_Y(V(M))$. Since $Y = [Y, J(T)]$, $\bar{Y} \leq [O_3(\bar{M}), J(T)]$ by Solvable Thompson Factorization B.2.16. Hence as $Y = O^2(Y)$ is T-invariant, either $Y = Y_0$, or $Y = Y_1$ or Y_2, in our two cases. In particular $Y \trianglelefteq M$, so $M = N_G(Y)$ as $M \in \mathcal{M}$; similarly Y is normal in each member of $\mathcal{M}(YT)$, so (2) holds. □

LEMMA 15.4.13. *For $Y \in \mathcal{Y}$, YT is not isomorphic to $\mathbf{Z}_2 \times S_4$.*

PROOF. Assume otherwise. Then $Z \cong E_4$ and $T \cong \mathbf{Z}_2 \times D_8$ with $\Phi(T)$ of order 2, so $O^2(Aut(T))$ centralizes Z. Thus as $N_G(T)$ controls fusion in Z by Burnside's Fusion Lemma A.1.35, the three involutions in Z are in distinct G-conjugacy classes. Pick an involution $t \in T - O_2(YT)$, and let $R := \langle t \rangle O_2(Y)$. Then $R \cong D_8$ has two YT-classes of involutions t^R and z^Y for $1 \neq z \in Z \cap R = \Phi(T)$. As the three involutions in Z are in distinct G-classes, at most one of the two involutions in $Z - R$ can be G-conjugate to t, so that some $i \in Z^{\#}$ satisfies $i \notin t^G \cup z^G$. Thus $i^G \cap R = \emptyset$, so by Thompson Transfer, $i \notin O^2(G)$, contrary to the simplicity of G. □

15.4.2. The final contradiction.

In this subsection, we assume that G is not $L_3(2)$ or A_6.

LEMMA 15.4.14. *(1) For each $Y \in \mathcal{Y}$, $C_Z(Y)$ is a hyperplane of Z.*
(2) $m(Z) = 2$. Thus $C_Z(Y) \neq 1$.

PROOF. Part (1) follows from 15.4.12.6. By the hypothesis of this subsection, conclusion (2) of 15.4.7 does not hold, and by 15.4.11.2, conclusion (1) of 15.4.7 does not hold. Thus $m(Z) > 1$ by 15.4.7. Indeed $|M(T)| > 1$ by Hypothesis 15.4.1.2, so by 15.4.12 there exist distinct $Y, X \in \mathcal{Y}$ with $!\mathcal{M}(XT) \neq !\mathcal{M}(YT)$, and hence $C_Z(Y) \cap C_Z(X) = 1$. Thus $m(Z) \leq 2$ by (1), so (2) holds. □

LEMMA 15.4.15. *There exists at most one $M \in \mathcal{M}(T)$ such that $s(Y) = 1$ for some $Y \in \mathcal{Y} \cap M$.*

PROOF. Assume $M_i \in \mathcal{M}(T)$, $i = 1, 2$, are distinct, with $Y_i \in \mathcal{Y} \cap M_i$ such that $s(Y_i) = 1$. Let $G_i := Y_i T$ for $i = 1, 2$, and $G_0 := \langle G_1, G_2 \rangle$; then (G_0, G_1, G_2) is a Goldschmidt triple. By 15.4.12.2, $O_2(G_0) = 1$, so by F.6.5.1, $\alpha := (G_1, T, G_2)$ is a Goldschmidt amalgam, and hence α is described in F.6.5.2. As $G_i \in \mathcal{H}(T)$, $G_i \in \mathcal{H}^e$, so α appears in case (vi) of F.6.5.2—namely in one of cases (1), (2), (3), (8), (12), or (13) of F.1.12. By 15.4.14, $Z \not\leq Z(G_i)$ for $i = 1$ and 2, and $m(Z) = 2$. Thus by inspection of the possibilities for α, case (2) of F.1.12 holds; that is $G_1 \cong G_2 \cong \mathbf{Z}_2 \times S_4$. However this contradicts 15.4.13. □

Recall that $S = \text{Baum}(T)$.

LEMMA 15.4.16. *$N_G(S) \leq M$ for each $M \in \mathcal{M}(T)$.*

PROOF. Pick $Y \in \mathcal{Y} \cap M$, and apply Theorem 3.1.1 to S, $N_G(S)$, YT in the roles of "R, M_0, H". Note that by 15.4.12.4, $S \in Syl_2(YS)$, and by 15.4.12.6, YS is a minimal parabolic in the sense of Definition B.6.1, so the hypotheses of 3.1.1 are indeed satisfied. Therefore by Theorem 3.1.1, $O_2(\langle N_G(S), YT\rangle) \neq 1$, so $N_G(S) \leq M$ by 15.4.12.2. □

LEMMA 15.4.17. *Assume $BC_G(B) \leq H \leq N_G(B)$ for some $1 \neq B \leq T$ such that $T_H := T \cap H \in Syl_2(H)$. Assume $T_H \leq I \leq H$, and for $z \in Z^\#$, let $M_z \in \mathcal{M}(C_G(z))$. Then*

(1) The hypotheses of 1.1.5 are satisfied with I, M_z, z in the roles of "H, M, z".

(2) If $Z \cap O_2(I) \neq 1$, then $F^*(I) = O_2(I)$.

PROOF. As $T_H \in Syl_2(H)$, and $Z \leq C_T(B) \leq T_H$ by hypothesis, 1.1.6 says that the hypotheses of 1.1.5 are satisfied with H, M_z, z in the roles of "H, M, z". Then as T_H is Sylow in H and I, $O_2(H \cap C_G(z)) \leq O_2(I \cap C_G(z))$ by A.1.6, so that (1) holds. Assume $Z \cap O_2(I) \neq 1$. Then $N := N_G(O_2(I)) \in \mathcal{H}^e$ by 1.1.4.3, so as $BC_N(B) \leq H \cap N \leq N_N(B)$, $H \cap N \in \mathcal{H}^e$ by 1.1.3.2. Hence as $T_H \leq I \leq H \cap N$, and T_H is Sylow in H, we conclude $I \in \mathcal{H}^e$ from 1.1.4.4. □

LEMMA 15.4.18. *$s(Y) = 2$ for each $Y \in \mathcal{Y}$.*

PROOF. Assume $Y \in \mathcal{Y}$ with $s(Y) = 1$, and let $M_1 := N_G(Y)$; then $M_1 = !\mathcal{M}(YT)$ by 15.4.12.2. Pick $M_2 \in \mathcal{M}(T) - \{M_1\}$; by 15.4.12, we may choose $X \in \mathcal{Y} \cap M_2$, and again $M_2 = N_G(X) = !\mathcal{M}(XT)$. By 15.4.15, $s(X) = 2$, so by 15.4.12, $X = Y_2 Y_2^t$ where $Y_2 = O^2(Y_2) = [Y_2, J(T)]$ is S-invariant with $Y_2/O_2(Y_2) \cong \mathbf{Z}_3$ and $t \in T - N_T(Y_2)$. Set $T_I := N_T(Y_2)$, $Y_1 := Y$, $G_i := Y_i T_I$, and $I := \langle G_1, G_2\rangle$. By 15.4.12.4, $S \in Syl_2(Y_i S)$, so as $S \leq T_I$, $T_I \in Syl_2(G_i)$. Notice $|T : T_I| = 2$.

As $S \in Syl_2(Y_i S)$, $E = \Omega_1(Z(J(T))) = E_i \times F_i$, where $E_i := C_E(Y_i)$, and $F_i := [E, Y_i] \cap E$ is of order 2. In particular E_i is a hyperplane of E. Similarly $E = E_X \times F_X$, where $E_X := C_E(X)$ and $F_X := [E, X] \cap E \cong E_4$. As T acts on $E_X \cap E_1$, if $E_X \cap E_1 \neq 1$, then $C_Z(X) \cap C_Z(Y) \neq 1$, then $M_1 = M_2$ by 15.4.12.2, contrary to the choice of M_2. Thus $E_X \cap E_1 = 1$, so as E_1 is a hyperplane of E, $m(E_X) \leq 1$. By 15.4.14, $1 \neq C_Z(X) \leq E_X$, so $C_Z(X) = E_X$ is of rank 1, and $m(E) = 3$. Thus as E_i is a hyperplane of E, $E_1 \cap E_2 =: E_0 \neq 1$; and G_i centralizes E_0, so $I \leq C_G(E_0)$. In particular, $IE_0 \in \mathcal{H}$, so that I is an SQTK-group.

Next $S = \mathrm{Baum}(T) \leq T_I$, so that $S = \mathrm{Baum}(T_I)$ by B.2.3.5. Thus $N_G(T_I) \leq N_G(S) \leq M_i$ by 15.4.16, so as $T_I = N_T(Y_2) \in Syl_2(C_{M_2}(E_0))$, we conclude that T_I is Sylow in $G_E := C_G(E_0)$, and hence also $T_I \in Syl_2(I)$. Thus (I, G_1, G_2) is a Goldschmidt triple. Let $I^* := I/O_{3'}(I)$. As $T \leq M_1$, $Y_2 \not\leq M_1$ since $M_2 = !\mathcal{M}(XT)$. Then as $M_1 = !\mathcal{M}(YT)$ and $O_2(G_1) \trianglelefteq YT$, we conclude $O_2(G_1) \neq O_2(G_2)$. Thus $\alpha := (G_1^*, T_I^*, G_2^*)$ is a Goldschmidt amalgam by F.6.11.2, so α and I^* are described in Theorem F.6.18.

As $Z \leq T_I \in Syl_2(G_E)$, we may apply 15.4.17 with I, G_E, E_0 in the roles of "I, H, B". We conclude that for $z \in Z^\#$ and $M_z \in \mathcal{M}(C_G(z))$, the hypotheses of 1.1.5 are satisfied with I, M_z, z in the roles of "H, M, z", and $F^*(I) = O_2(I)$ if $Z \cap O_2(I) \neq 1$. By 1.1.5.1, $F^*(I \cap M_z) = O_2(I \cap M_z)$, so by 1.1.3.2, $F^*(C_I(z)) = O_2(C_I(z))$. As $[Z, Y_i] \leq C_I(O(I))$ by A.1.26.1, and $1 \neq Z \cap [Z, Y_1]$, $O(I) = 1$ by 1.1.5.2.

Suppose first that $F^*(I) \neq O_2(I)$. Then there is a component L of I, and Z is faithful on L by 1.1.5.3. Now L is described in one of cases (3)–(13) of F.6.18, so as

$F^*(C_I(z)) = O_2(C_I(z))$ for each $z \in Z^\#$, and $m(Z) = 2$ by 15.4.14.2, we conclude $L \cong A_6$ and $L^*Z^* \cong S_6$. In particular $Y_i \le O^{3'}(I) = L$ for $i = 1, 2$, so $L = O^2(I)$ using F.6.6. Now T acts on $C_{T_I}(Y) = C_{T_I}(L) \times (Z \cap L)$, but T acts on no nontrivial subgroup of $C_{T_I}(L)$ as $C_Z(X) \cap C_Z(Y) = 1$. Therefore as $|T : T_I| = 2$, $C_{T_I}(L)$ is of order 2, and hence $C_{T_I}(L) = E_0$, so $G_i \cong E_4 \times S_4$. Thus $Y_2 \cong A_4$, so $X \cong A_4 \times A_4$. Then as $C_Z(X) = E_X$ is of order 2, $m_2(T_I) \ge 5$, contrary to $G_i \cong E_4 \times S_4$.

Therefore $F^*(I) = O_2(I)$. Let $V_I := \langle Z^I \rangle$, so that $V_I \in \mathcal{R}_2(I)$ by B.2.14. But $C_I(V_I) \le C_I(Z) = T_I$ using Theorem 15.4.8, so $C_I(V_I) = O_2(I)$. Let $\hat{I} := I/O_2(I)$. Now $Y_i = [Y_i, J(T)]$ as $Y_i \in \mathcal{Y}$, and $[Z, Y_i] \ne 1$ by 15.4.12.6, so V_I is an FF-module for $\hat{I}$. Then using Theorem B.5.6 to determine the FF-modules for the possible groups in Theorem F.6.18, it follows as $C_I(Z) = T_I$, that $\hat{I} \cong S_3 \times S_3$. But then Y_2 normalizes $O^2(Y_1O_2(I)) = Y_1$, so that $Y_2 \le N_G(Y) = M_1$, contrary to an earlier remark. □

We now define notation in force for the remainder of the subsection. By Hypothesis 15.4.1.2, we can pick distinct members M_1 and M_2 of $\mathcal{M}(T)$, and by 15.4.12, we can choose $X \in \mathcal{Y} \cap M_1$ and $Y \in \mathcal{Y} \cap M_2$. Thus $M_1 = N_G(X) = !\mathcal{M}(XT)$ and $M_2 = N_G(Y) = !\mathcal{M}(YT)$ by 15.4.12.2. Further $s(X) = s(Y) = 2$ by 15.4.18, so that $Y = Y_1Y_2$ and $X = X_1X_2$ as in 15.4.12.6. Let $T_0 := N_T(Y_1) \cap N_T(X_1)$. By 15.4.12.4, S is Sylow in XS and YS, so as $S \le T_0$ by 15.4.12.6, T_0 is Sylow in XT_0 and YT_0. Let $L_1 := X_1$ or X_2, and $L_2 := Y_1$ or Y_2. Set $G_i := L_iT_0$, and $I := \langle G_1, G_2 \rangle$. Let $V_i := [V(M_i), L_i]$, so that $V_i \cong E_4$ by 15.4.12.6. Observe $|T : T_0| \le 4$ since $|T : N_T(Y_i)| = 2 = |T : N_T(X_i)|$.

Lemma 15.4.19. *(1)* $1 \ne C_E(I) \le Z(I)$. *In particular* $I \in \mathcal{H}$.
(2) $L_{3-i} \not\le M_i$.
(3) $Z \cap Z(I)V_1 \ne 1$.

Proof. If $L_2 \le M_1$ then $YT = \langle L_2, T \rangle \le M_1$, contrary to $M_2 = !\mathcal{M}(YT)$ and the choice of $M_1 \ne M_2$. Thus (2) holds. Similarly $Z \cap Z(I) = 1$.

Let $E_I := C_E(T_0)$. Arguing as in the second paragraph of the proof of 15.4.18, $E_I = E_i \times F_i$, where $E_i := C_E(G_i)$ and $F_i := C_{V_i}(T_0) \cong \mathbf{Z}_2$. Thus $E_0 := C_E(I)$ is of corank at most 2 in E_I. As $C_Z(X) \ne 1$ by 15.4.14, and $C_{E \cap [Z,X]}(T_0) \cong E_4$ by 15.4.12.6, $m(E_I) \ge 3$, so (1) holds. Further as $Z \cap Z(I) = 1$ and $m(Z) = 2$ by 15.4.14.2, $E_I = E_0 \times Z$, so $1 \ne Z \cap E_0F_1 \le Z(I)V_1$, and hence (3) holds. □

By 15.4.19, $I \in \mathcal{H}$, so that $\mathcal{H}(I)$ is nonempty.

Lemma 15.4.20. $T_0 \in Syl_2(I_0)$ *for each* $I_0 \in \mathcal{H}(I)$.

Proof. Assume otherwise, and let $T_0 < T_I \in Syl_2(I_0)$, and $T_1 := T_I \cap M_1 \cap M_2$. As $S \le T_0 \le T_1$, $S = \mathrm{Baum}(T_1)$ by B.2.3.4, and hence $N_{T_I}(T_1) \le N_{T_I}(S) \le T_I \cap M_1 \cap M_2 = T_1$ by 15.4.16. Thus $T_I = T_1$, so we may take $T_I \le T$. Of course $T_I < T$, as otherwise I contains XT and YT, contrary to $!\mathcal{M}(XT) = M_1 \ne M_2 = !\mathcal{M}(YT)$. Therefore $|T : T_I| = 2$ since $|T : T_0| \le 4$. Also $T_I \in Syl_2(J)$ for any $J \in \mathcal{H}(I_0)$, and in particular, $T_0 \in Syl_2(N_G(O_2(I_0)))$.

As $T_I > T_0$, T_I does not normalize at least one of L_1 or L_2, so we may assume T_I does not normalize L_1. Then $X = \langle L_1^{T_I} \rangle \le I \le I_0$ and $R := O_2(XT_I) \trianglelefteq XT$, so as $M_1 = !\mathcal{M}(XT)$,

$$C(G, R) \le M_1.$$

Set $K := \langle X^{I_0}\rangle$ and recall $K \in \Xi(G,T)$. As $M_1 = N_G(X)$, X is not normal in I_0 since $I_0 \not\leq M_1$ by 15.4.19.2.

Observe that either:

(i) $K \in \mathcal{C}(I_0)$ with $KT_I/O_2(KT_I) \cong Aut(L_n(2))$, $n = 4$ or 5, or

(ii) $K = K_1K_1^r$ for some $K_1 \in \mathcal{C}(I_0)$ and $r \in T_I - N_{T_I}(K_1)$, with $K_1/O_2(K_1) \cong L_2(2^n)$, n even, or $L_2(p)$ for some odd prime p.

This follows from 1.3.4, since $K/O_2(K)$ is not $L_3(3)$, M_{11}, or $Sp_4(2^n)$ because T_I induces D_8 on $X/O_2(X)$ since $XS/O_2(XS) \cong S_3 \times S_3$ by 15.4.12.6, while T_I does not act on L_1. Also $K = O^{3'}(I_0)$ by A.3.18 or 1.2.2, so $I \leq KT_I$, and hence without loss $I_0 = KT_I$.

Suppose first that $F^*(K) \neq O_2(K)$, so that K is a product of components of I_0. By an earlier remark, T_0 is also Sylow in $N_G(O_2(I_0))$, so we may apply 15.4.17 with I_0, $N_G(O_2(I_0))$, $O_2(I_0)$ in the roles of "I, H, B" to conclude that for $z \in Z^\#$ and $M_z \in \mathcal{M}(C_G(z))$, the hypotheses of 1.1.5 are satisfied with I_0, M_z, z in the roles of "H, M, z". Thus K is described in 1.1.5.3, and Z is faithful on K. Suppose first that case (ii) holds. As Z is noncyclic and in the center of T_I, while T_I induces D_8 on $X/O_2(X)$, K_1 is not $L_2(p)$ for p odd. Thus $K_1 \cong L_2(2^n)$, so as $L_2(4) \cong L_2(5)$ and n is even, $n \geq 4$. Further as $C(G,R) \leq M_1$, a Borel subgroup B of K is contained in M_1, and hence $B = O^2(B)$ acts on the 4-subgroup V_1 of 15.4.12.6; this is impossible, as when $n \geq 4$, B does not act on a 4-subgroup of $O_2(B)$. Thus (i) holds, in which case we again have a contradiction to Z 2-central, noncyclic, and faithful on K.

Therefore $F^*(K) = O_2(K)$. Let $I_X := I_0 \cap M_1$. Recall $C(G,R) \leq M_1$, so that Hypothesis C.2.3 is satisfied with I_X, I_0 in the roles of "M_H, H". If case (ii) holds, then as R centralizes $X/O_2(X)$, R normalizes K_1, so it follows from C.2.7.3 that either K_1 is a block of type $L_2(2^n)$ or A_5, or $K_1/O_2(K_1) \cong L_3(2)$.

Let $V_I := \langle Z^K\rangle$ so that $V_I \in \mathcal{R}_2(I_0)$ by B.2.14, and set $I_0^* := I_0/O_2(I_0)$. As $K/O_2(K)$ is semisimple, $O_2(I_0) = C_{I_0}(V_I)$. As $L_i = [L_i, J(T)]$, V_I is an FF-module for I_0^*. Then as $C_{I_0}(Z)$ is a 2-group by Theorem 15.4.8, it follows from the description of the modules in C.2.7.3 and C.1.34 for the groups in cases (i) and (ii), that K_1 is an $L_2(2^n)$-block. But once again a Borel subgroup B of K is contained in M_1, and hence $B = O^2(B)$ acts on the 4-subgroup V_1 of case (ii) of 15.4.12.6, so we conclude that K_1 is an $L_2(4)$-block. Finally by 15.4.14, $C_Z(X) \neq 1$, so $C_Z(X) \leq Z(I_0)$ from the structure of K. Thus $I_0 \leq C_G(C_Z(X)) \leq M_1 = !\mathcal{M}(XT)$, contrary to 15.4.19.2. □

Recall $I \in \mathcal{H}$ by 15.4.19.1, so $T_0 \in Syl_2(I)$ by 15.4.20. Then (I, G_1, G_2) is a Goldschmidt triple. Set $Q_i := O_2(G_i)$ and $I^+ := I/O_{3'}(I)$.

LEMMA 15.4.21. *If $F^*(I) = O_2(I)$, then $I = L_1L_2T_0$ with $L_i \trianglelefteq I$.*

PROOF. Assume $F^*(I) = O_2(I)$ and let $V_I := \langle Z^I\rangle$. As $C_I(V_I) \leq C_I(Z) = T_0$ by Theorem 15.4.8, and $V_I \in \mathcal{R}_2(I)$ by B.2.14, we conclude that $C_I(V_I) = O_2(I)$. Let $I^* := I/O_2(I)$, so that I^+ is a quotient of I^*. As $L_i = [L_i, J(T)]$, V_I is an FF-module for I^* and L_i centralizes $O^3(F(I^*))$ by B.1.9.

We claim $\alpha := (G_1^+, T_0^+, G_2^+)$ is a Goldschmidt amalgam. For if not, by F.6.11.2, $Q_1 = Q_2$ and $I^+ \cong S_3$ with $O_{3'}(I)^* \neq 1$. Then $Q_1 = O_2(I)$ and I is solvable by F.6.11.1, so as L_i^* centralizes $O^3(F(I^*))$, I is a $\{2,3\}$-group by F.6.9, contradicting $O_{3'}(I^*) \neq 1$. Thus the claim is established.

By the claim, I^+ is described in Theorem F.6.18. Therefore as V_I is an FF-module for I^*, either the lemma holds, or comparing the list of Theorem F.6.18 with that of Theorem B.5.1, we conclude that I^* is an extension of $L_3(2)$, A_6, A_7, $\hat{A}_6$, or $G_2(2)$, so that $C_I(Z)$ is not a 2-group. The latter case contradicts Theorem 15.4.8. □

LEMMA 15.4.22. *If $F^*(I) \neq O_2(I)$ then $I = KT_0$, $K \cong A_6$, and $I/C_T(K) \cong S_6$.*

PROOF. Assume $F^*(I) \neq O_2(I)$. By 15.4.20, $T_0 \in Syl_2(N_G(O_2(I)))$, so we may apply 15.4.17 to conclude that the hypotheses of 1.1.5 are satisfied and $Z \cap O_2(I) = 1$. Since $V_1 = [V_1, L_1] \cong E_4$, V_1 centralizes $O(I)$ by A.1.26.1, so $1 \neq Z \cap V_1 Z(I)$ centralizes $O(I)$ by 15.4.19.3. Thus $O(I) = 1$ by 1.1.5.2.

If $Q_1 = Q_2$ then $Q_1 = O_2(I)$; but $Z \leq Q_1$ by B.2.14, contradicting $Z \cap O_2(I) = 1$. Thus $Q_1 \neq Q_2$, so (G_1^+, T_0^+, G_2^+) is a Goldschmidt amalgam by F.6.11.2, and I^+ is described in Theorem F.6.18. By F.6.11.1, $O_{3'}(I)$ is 2-closed, so as $Z \cap O_2(I) = 1$, $Z \cap O_{3'}(I) = 1$ and hence $Z \cong Z^+$ is noncyclic; then we conclude from Theorem F.6.18 that I^+ is either $L_2(p^2)$ extended by a field automorphism, or S_7. However $F^*(I \cap M_z) = O_2(I \cap M_z)$ for each $z \in Z^\#$ by 1.1.5.1, so that $F^*(C_I(C_Z(W))) = O_2(C_I(C_Z(W)))$ for $W := X, Y$ by 1.1.3.2. We conclude that $I^+ \cong S_6$. As $O_{3'}(I)$ is 2-closed and $F^*(I) \neq O_2(I)$, it follows that I has a component K with $K/O_2(K) \cong A_6$ and then that $K = O^{3'}(I)$ by A.3.18. Thus $K = O^2(I)$ by F.6.6, so $I = KT_0$. As $E_4 \cong Z$ is faithful on K, $Z(K) = 1$, so the lemma holds. □

LEMMA 15.4.23. *$F^*(I) \neq O_2(I)$.*

PROOF. Assume $F^*(I) = O_2(I)$. Then by 15.4.21, $[L_2, L_1] \leq O_2(L_1)$. We may choose notation so that $L_1 := X_1$, and set $L_1' := X_2$ and $I' := \langle L_1' T_0, L_2 \rangle$. As $[L_1, L_1'] \leq O_2(L_1)$ and $[L_1, L_2] \leq O_2(L_1)$, we conclude $[O^2(I'), L_1] \leq O_2(L_1)$ from F.6.6. However by 15.4.21 and 15.4.22, I' contains an E_9-subgroup P, with $P \cap L_1 = 1$, since $I' \not\leq M_1$ by 15.4.19.2. Therefore as $[O^2(I'), L_1] \leq O_2(L_1)$, $m_3(L_1 P) = 3$, contrary to $N_G(I)$ an SQTK-group. □

We are now ready to establish the main result of this section. By 15.4.23, we may apply 15.4.22, to conclude that $L_i \cong A_4$, $O_2(L_1)O_2(L_2) \cong D_8$, and $O_2(L_1) \cap O_2(L_2) \neq 1$. We may choose $L_1 := X_1$, and set $L_1' := X_2$ and $I' := \langle L_1', L_2 \rangle$. By symmetry, $O_2(L_1') \cap O_2(L_2) \neq 1$, so as $O_2(L_1) \cap O_2(L_1') = 1$,

$$O_2(L_2) = (O_2(L_1) \cap O_2(L_2))(O_2(L_1') \cap O_2(L_2)) \leq O_2(X) \cong E_{16}.$$

This is impossible as $O_2(L_1)O_2(L_2) \cong D_8$ and $O_2(L_1) \leq O_2(X)$.

Since we assume in this subsection that G is not $L_3(2)$ or A_6, this contradiction establishes:

THEOREM 15.4.24. *Assume Hypothesis 15.4.1. Then $G \cong L_3(2)$ or A_6.*

Then combining the main results of this chapter:

THEOREM 15.4.25 (Theorem E). *Assume G is a simple QTKE-group, $T \in Syl_2(G)$, $|\mathcal{M}(T)| > 1$, and $\mathcal{L}_f(G, T) = \emptyset$. Then G is isomorphic to J_2, J_3, ${}^3D_4(2)$, the Tits group ${}^2F_4(2)'$, $G_2(2)'$, M_{12}, $L_3(2)$, or A_6.*

PROOF. If Hypothesis 15.4.1 holds, the groups in Theorem 15.4.24 appear in the list of Theorem E. On the other hand if Hypothesis 15.4.1 fails, then there is a unique member M_c of $\mathcal{M}(C_G(Z))$, so that Hypothesis 14.1.5 holds, and the groups in Theorem 15.3.1 appear as conclusions in Theorem E. □

Although by this point it may feel like something of an anticlimax, we have also completed the proof of the Main Theorem: For suppose G is a simple QTKE-group, with $T \in Syl_2(G)$. If $|\mathcal{M}(T)| = 1$, the groups appearing as conclusions in Theorem 2.1.1 of chapter 2 appear as conclusions in the Main Theorem. So assume that $|\mathcal{M}(T)| > 1$. If $\mathcal{L}_f(G, T) = \emptyset$, then the groups in the conclusion of Theorem E are among those in the conclusion of the Main Theorem. Finally if $\mathcal{L}_f(G, T) \neq \emptyset$, then the groups in the conclusion of Theorem D (14.8.2) appear as conclusions in the Main Theorem.

Thus, after a hiatus of roughly twenty years, there is at last a classification of the quasithin groups of even characteristic. In particular, this result fills that gap in the literature classifying the finite simple groups.

Part 7

The Even Type Theorem

CHAPTER 16

Quasithin groups of even type but not even characteristic

The original proof of the classification of the finite simple groups (CFSG) requires the classification of simple QTK-groups G of characteristic 2-type. (Recall G is of *characteristic 2-type* if $F^*(M) = O_2(M)$ for all 2-local subgroups M of G.) Mason produced a preprint [**Mas**] which goes a long way toward such a classification, but that preprint is incomplete. Our Main Theorem fills this gap in the "first generation" proof of CFSG, since we determine all simple groups in the larger class of QTK-groups of even characteristic. (Recall G is of *even characteristic* if $F^*(M) = O_2(M)$ only for those 2-locals M containing a Sylow 2-subgroup T of G.)

The "revisionism" project (see [**GLS94**]) of Gorenstein-Lyons-Solomon (GLS) aims to produce a "second-generation" proof of CFSG. In GLS, the notion of characteristic 2-type from the first-generation proof is replaced by the notion of *even type* (see p. 55 in [**GLS94**]). In a group of even type, centralizers of involutions are allowed to contain certain components (primarily of Lie type in characteristic 2). In particular, if the centralizer of a 2-central involution has a component, then G is not of even characteristic, and so does not satisfy the hypothesis of our Main Theorem.

To bridge the gap between these two notions of "characteristic 2", this final chapter of our work classifies the simple QTK-groups of even type. More precisely, our main result Theorem 16.5.14 (the Even Type Theorem) shows that J_1 is the only simple QTK-group which is of even type but not of even characteristic. Thus the simple QTK-groups of even type are the groups in our Main Theorem, of even type, along with J_1.

To prove Theorem 16.5.14, we will utilize a small subset of the machinery on standard components from the first generation proof of CFSG. In sections I.7 and I.8 of Volume I, we give proofs of all but one of the results we use; that result is Lemma 3.4 from [**Asc75**], which is a fairly easy consequence of Theorem ZD on page 21 in [**GLS99**].

We are grateful to Richard Lyons and Ronald Solomon for their careful reading of this chapter, and suggestions resulting in a number of improvements.

16.1. Even type groups, and components in centralizers

In this chapter, we assume the following hypothesis:

HYPOTHESIS 16.1.1. *G is a quasithin simple group, all of whose proper subgroups are $\mathcal{K}$-groups, but G is* not *of even characteristic. On the other hand, G is of even type in the sense of GLS.*

The definition of even type is given on p.55 of [**GLS94**]. We will not need to assume that $m_2(G) \geq 3$ as in part (3) of that definition. Part (2) of that definition says that if K is a component of the centralizer of an involution, then $K/Z(K)$ is in the set $\mathcal{C}_2$ of simple groups listed in Definition 12.1 on page 100 in [**GLS94**]; and also that $Z(K)$ satisfies further restrictions given in the final sentences of that definition. We will not reproduce that full list here, since as G is a QTK-group, $K/O_2(K)$ also appears in Theorem C (A.2.3). Instead, intersecting the list of possibilities from Theorem C (A.2.3) with the list of possibilities in Definition 12.1 on page 100 of [**GLS94**], it follows that when G is a QTK-group, G is of even type if and only if:

(E1) $O(C_G(t)) = 1$ for each involution $t \in G$.

(E2) If L is a component of $C_G(t)$ for some involution $t \in G$, then one of the following holds:

(i) $L/O_2(L)$ is of Lie type and characteristic 2 appearing in case (3) or (4) of Theorem C; but L is not $SL_2(q)$, $q = 5, 7, 9$ or $A_8/\mathbf{Z}_2$. Further if $L/O_2(L) \cong L_3(4)$, then $\Phi(O_2(L)) = 1$.

(ii) $L \cong L_3(3)$ or $L_2(p)$, p a Fermat or Mersenne prime.

(iii) $L/O_2(L)$ is M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_4, HS, or Ru.

Observe that from Theorem C, in case (i) either $L/O_2(L)$ is of Lie rank 1, and so of Lie type $A_1 = L_2$, ${}^2B_2 = Sz$, or ${}^2A_2 = U_3$; or $L/O_2(L)$ is of Lie rank 2 and of Lie type A_2, B_2, G_2, ${}^2F_4$, or ${}^3D_4$; or L is $L_4(2)$ or $L_5(2)$.

In the remainder of this introductory section assume that L is a component of the centralizer of some involution of G, and set $\bar{L} := L/Z(L)$. Thus by Hypothesis 16.1.1, L is one of the quasisimple groups listed above. To provide a more self-contained treatment, in this introductory section we collect some facts about L which we use frequently.

First, inspecting the list of Schur multipliers in I.1.3 for the groups L in (E2), and recalling that $O(L) = 1$ by (E1), we conclude:

LEMMA 16.1.2. *(1) If L is not simple, then $Z(L) = O_2(L)$ and $\bar{L} \cong Sz(8)$, $L_3(4)$, $G_2(4)$, M_{12}, M_{22}, J_2, HS, or Ru.*

(2) Either $|Z(L)| \leq 2$, or $\bar{L} \cong Sz(8)$ or $L_3(4)$ with $Z(L) \cong E_4$, or $\bar{L} \cong M_{22}$ with $Z(L) \cong \mathbf{Z}_4$.

Occasionally we need more specialized information about the quasithin groups appearing in 16.1.2, which can be obtained from knowledge of the covering groups L of $\bar{L}$. Such facts are collected in I.2.2.

In the next two lemmas, we list the involutory automorphisms of L and their centralizers in $\bar{L}$. Notice that we write L rather than $\bar{L}$ in those cases where $Z(L) = 1$ by 16.1.2.1.

We begin with the groups of Lie type and characteristic 2 in case (i) of (E2), that is, in case (3) or (4) of Theorem C. Recall that the involutions in classical groups of characteristic 2 are determined up to conjugacy by their Suzuki type: In orthogonal and symplectic groups, the types are denoted a_k, b_k, c_k, as discussed in Definition E.2.6; in linear and unitary groups, the types are denoted j_k, as discussed in Aschbacher-Seitz [**AS76a**]. In each case, k is the dimension of the commutator space for the involution on the natural module for the classical group.

NOTATION 16.1.3. Recall that the types of twisted groups in Theorem C are $^2A_2 = U_3$, $^2B_2 = Sz$, $^3D_4$, and $^2F_4$. We adopt the convention of [**GLS98**] for labeling involutory outer automorphisms of these groups. We emphasize that this convention differs from that of Steinberg which is widely used in the literature (eg. in the Atlas [**C$^+$85**]) in which there are no graph automorphisms of twisted groups. Instead the convention in Definition 2.5.3 in [**GLS98**] is that all involutory automorphisms of groups of type $^2A_2 = U_3$ which are not inner-diagonal are called graph automorphisms, but involutory outer automorphisms of groups of type $^3D_4$ are called field automorphisms. All involutory automorphisms of groups of types $^2B_2 = Sz$ and $^2F_4$ are inner.

LEMMA 16.1.4. *Assume that $\bar{L} \cong X(2^n)$ is of Lie type X and characteristic 2. Let r be an involution in $Aut(L)$, and set $L_r := O^2(C_L(r))$. Then one of the following holds:*

(1) r induces an automorphism on L corresponding to a root involution of $\bar{L}$ (or in $Sp_4(2)$ or $G_2(2)$, if $L = Sp_4(2)'$ or $G_2(2)'$), and $L_r = O^2(C_P(r))$ for the proper parabolic P containing $C_L(r)$.

(2) $L \cong Sp_4(2^n)$, r induces an automorphism of type c_2, and $L_r = 1$.

(3) $L \cong L_4(2)$ or $L_5(2)$, r induces an automorphism of type j_2, and $L_r \cong A_4$ or $\mathbf{Z}_3/(E_4 \times E_4)$, respectively.

(4) r induces a field automorphism on $\bar{L}$ and $L_r \cong X(2^{n/2})'$. Further $Sz(2^n)$ and $^2F_4(2^n)$ have no involutory non-inner automorphisms, and $U_3(2^n)$ has no involutory field automorphisms.

(5) $\bar{L} \cong L_3(2^n)$, n even, r induces a graph-field automorphism on L, and $L_r \cong U_3(2^{n/2})$—unless $n = 2$, where $L_r \cong E_9$.

(6) $\bar{L} \cong L_3^\epsilon(2^n)$, r induces a graph automorphism on $\bar{L}$, and $L_r \cong L_2(2^n)'$.

(7) $L \cong Sp_4(2^n)$, n odd, r induces a graph-field automorphism on L, and $L_r \cong Sz(2^n)'$.

(8) $L \cong L_4(2)$ or $L_5(2)$, r induces a graph automorphism on L, and $L_r \cong A_6$.

(9) $L\langle r\rangle \cong S_8$, r is of type $2^3, 1^2$, and $L_r \cong A_4$.

PROOF. This follows from list of possibilities for L in (E2), and the 2-local structure of $Aut(L)$ (cf. Aschbacher-Seitz [**AS76a**]). □

We turn to the cases in parts (ii) and (iii) of (E2):

LEMMA 16.1.5. *Assume that $\bar{L}$ is not of Lie type and characteristic 2, and r is an involution in $Aut(L)$. Then one of the following holds:*

(1) $L \cong L_3(3)$ and either r is inner with $C_L(r) \cong GL_2(3)$, or r is outer with $C_L(r) \cong S_4$.

(2) $L \cong L_2(q)$ for $q > 7$ a Fermat or Mersenne prime, and either r is inner with $C_L(r) \in Syl_2(L)$, or r is an outer automorphism in $PGL_2(q)$ with $C_L(r) \cong D_{q+\epsilon}$, where $q \equiv \epsilon \mod 4$.

(3) $L \cong M_{11}$, r is inner, and $C_L(r) \cong GL_2(3)$.

(4) $\bar{L} \cong M_{12}$ and either r is inner with $C_{\bar{L}}(r) \cong S_3/Q_8^2$ or $\mathbf{Z}_2 \times S_5$, or r is outer and $C_{\bar{L}}(r) \cong \mathbf{Z}_2 \times A_5$.

(5) $\bar{L} \cong M_{22}$ and either r in inner with $C_{\bar{L}}(r) \cong S_4/E_{16}$, or r is outer with $C_{\bar{L}}(r) \cong L_3(2)/E_8$ or $Sz(2)/E_{16}$.

(6) $L \cong M_{23}$, r is inner, and $C_L(r) \cong L_3(2)/E_{16}$.

(7) $L \cong M_{24}$, r is inner, and $C_L(r) \cong L_3(2)/D_8^3$ or S_5/E_{64}.

(8) $\bar{L} \cong J_2$ *and either* r *is inner with* $C_{\bar{L}}(r) \cong A_5/Q_8 * D_8$ *or* $E_4 \times A_5$, *or* r *is outer and* $C_{\bar{L}}(r) \cong Aut(L_3(2))$.

(9) $L \cong J_4$, r *is inner, and* $C_L(r) \cong Aut(\hat{M}_{22})/D_8^6$ *or* $Aut(M_{22})/E_{2^{11}}$.

(10) $\bar{L} \cong HS$, *and either* r *is inner with* $C_{\bar{L}}(r) \cong S_5/(Q_8^2 * \mathbf{Z}_4)$ *or* $\mathbf{Z}_2 \times Aut(A_6)$, *or* r *is outer with* $C_{\bar{L}}(r) \cong S_8$ *or* S_5/E_{16}.

(11) $\bar{L} \cong Ru$, r *is inner, and* $C_{\bar{L}}(r) \cong S_5/E_{64}/E_{32}$ *or* $E_4 \times Sz(8)$.

PROOF. The Atlas [**C$^+$85**] contains a list of centralizers, as does [**GLS98**]. Neither reference includes proofs for the sporadic groups, but there are proofs in section 5 of chapter 4 of [**GLS98**] when $\bar{L}$ is of Lie type and odd characteristic. Proofs for M_{24}, He, and J_2 appear in [**Asc94**], for M_{11} and M_{12} in [**Asc03b**], and for HS in [**Asc03a**]. Proofs or references to proofs for the remaining groups can be found in [**AS76b**]. □

LEMMA 16.1.6. *Assume* r *is a* 2*-element of* $Aut(L)$ *centralizing a Sylow* 2*-subgroup of* L. *Then either*

(1) $r \in Inn(L)$, *and if* L *appears in case (i) of (E2) and* r *is an involution, then either* $\bar{r}$ *is a long-root involution or* $\bar{L} \cong Sp_4(2^n)$; *or*

(2) $L \cong A_6$ *and* r *induces an automorphism in* S_6.

PROOF. This is well known; it follows from 16.1.4 and 16.1.5 when r is of order 2. □

Our final preliminary results describe the possible embeddings among components of involution centralizers.

LEMMA 16.1.7. *Assume* t *is an involution in* G, L *is a component of* $C_G(t)$, *and* i *is an involution in* $C_G(\langle t, L\rangle)$. *Set* $K := \langle L^{E(C_G(i))}\rangle$. *Then one of the following holds:*

(1) $K = L$.

(2) $L/O_2(L) \cong L_2(2^n)$, $Sz(2^n)$, *or* $L_2(p)$, p *prime,* $K = K_1K_1^t$ *with* K_1 *a component of* $C_G(i)$ *and* $K_1 \neq K_1^t$, $K_1/O_2(K_1) \cong L/O_2(L)$, *and* $L = C_K(t)^\infty$.

(3) K *is a component of* $C_G(i)$, $K = [K, t]$, L *is a component of* $C_K(t)$, *and one of the following holds:*

(a) $K/O_2(K) \cong X(2^{2n})$, *where* X *is a Lie type of Lie rank at most* 2, *but not* $Sz(2^n)$, $U_3(2^n)$, *or* ${}^2F_4(2^n)$, *and* t *induces a field automorphism on* $K/O_2(K)$ *with* $L/O_2(L) \cong X(2^n)'$.

(b) $K \cong L_3(2^{2n})$ *for* $n > 1$, t *induces a graph-field automorphism on* K, *and* $L \cong U_3(2^n)$.

(c) $K/O_2(K) \cong L_3^\epsilon(2^n)$ *for* $n > 1$, t *induces a graph automorphism on* $K/O_2(K)$, *and* $L \cong L_2(2^n)$.

(d) $K \cong Sp_4(2^n)$, $n > 1$ *odd,* t *induces a graph-field automorphism on* K, *and* $L \cong Sz(2^n)$.

(e) $K \cong L_4(2)$ *or* $L_5(2)$, t *induces a graph automorphism on* K, *and* $L \cong A_6$.

(f) $K/O_2(K) \cong M_{12}$ *or* J_2 *and* $L \cong A_5$.

(g) $K/O_2(K) \cong J_2$ *and* $L \cong L_3(2)$.

(h) $K/O_2(K) \cong HS$ *and* $L \cong A_6$ *or* A_8.

(i) $K/O_2(K) \cong Ru$ *and* $L \cong Sz(8)$.

PROOF. Let $\bar{K} := K/O_2(K)$. Since L is a component of $C_G(t)$ and i centralizes L by hypothesis, L is also a component of $C_{C_G(i)}(t)$. We apply I.3.2 with $C_G(i)$ in the role of "H": As $O(C_G(i)) = 1$ by (E1), $O_{2',E}(C_G(i)) = E(C_G(i))$ and the 2-components in that result are components. Then either

(i) $K = K_1K_1^t$ for some component $K_1 \neq K_1^t$ of $C_G(i)$, $L/O_2(L) \cong \bar{K}_1$, and $L = C_K(t)^\infty$, or

(ii) K is a t-invariant component of $C_G(i)$, and L is a component of $C_K(t)$.

In case (i), since G is quasithin, the possibilities in (2) are obtained by intersecting the list of A.3.8.3 with that of (E2). Therefore we may assume that case (ii) holds, and $K > L$ since otherwise conclusion (1) holds. The simple group $\bar{K}$ is described in (E2). The cases in (3) arise by inspecting 16.1.4 and 16.1.5 for involutions $i \in Aut(K)$ such that $C_K(i)$ has a component. We use 16.1.2.1 to conclude that $O_2(K) = 1$ or $O_2(L) = 1$ when appropriate; in case (i) of (3), $Z(L) = 1$ from the structure of the covering group K of $\bar{K} \cong Ru$ in I.2.2.7a. □

LEMMA 16.1.8. *Assume t is an involution in G, L is a component of $C_G(t)$, i is an involution in $C_G(\langle t, L\rangle)$, and $S \in Syl_2(C_G(i))$ with $|S : C_S(t)| \leq 2$. Then L is a component of $C_G(i)$.*

PROOF. Assume otherwise, and set $K := \langle L^{E(C_G(i))}\rangle$; then K is described in case (2) or (3) of 16.1.7, and it remains to derive a contradiction. As $S \in Syl_2(C_G(i))$ and K is subnormal in $C_G(i)$, $S_K := S \cap K \in Syl_2(K)$. Further $|S_K : C_{S_K}(t)| \leq |S : C_S(t)| \leq 2$. However in case (2) of 16.1.7,

$$|S_K : C_{S_K}(t)| \geq |K_1/O_2(K_1)|_2 > 2,$$

so case (3) must hold. But in each subcase of (3), $|S_K : C_{S_K}(t)| > 2$, a contradiction establishing the lemma. □

16.2. Normality and other properties of components

Let $\mathcal{P}$ denote the set of pairs (z, L) such that z is a 2-central involution in G and L is a component of $C_G(z)$.

LEMMA 16.2.1. $\mathcal{P} \neq \emptyset$.

PROOF. Let $T \in Syl_2(G)$. By Hypothesis 16.1.1, G is not of even characteristic, so there is $M \in \mathcal{M}(T)$ such that $O^2(F^*(M)) \neq 1$ and there is $1 \neq z \in \Omega_1(Z(T)) \cap O_2(M)$. Then $O^2(F^*(M))) \leq O^2(F^*(C_M(z)))$, so that $F^*(C_M(z)) \neq O_2(C_M(z))$. Then as $M = N_G(O_2(M))$ since $M \in \mathcal{M}$, $F^*(C_G(z)) \neq O_2(C_G(z))$ by 1.1.3.2. On the other hand by Hypothesis 16.1.1, G is of even type, so by (E1), $O(C_G(z)) = 1$. Therefore $E(C_G(z)) \neq 1$, so there is a component L of $C_G(z)$, and then $(z, L) \in \mathcal{P}$. □

In view of 16.2.1, we assume for the remainder of the chapter:

NOTATION 16.2.2. $T \in Syl_2(G)$, z is an involution in $Z(T)$, $(z, L) \in \mathcal{P}$, $G_z := C_G(z)$, $T_L := T \cap L$, and $T_C := C_T(L)$.

LEMMA 16.2.3. *If t is an involution in T_C with $|T : C_T(t)| \leq 2$, then L is a component of $C_G(t)$.*

PROOF. Let $C_T(t) \leq S \in Syl_2(C_G(t))$, so that $C_T(t) \leq C_S(z)$. Since $T \in Syl_2(G)$,

$$|S : C_S(z)| \leq |S : C_T(t)| \leq |T : C_T(t)| \leq 2,$$

and hence the lemma follows from 16.1.8 with z, t in the roles of "t, i". □

Of course the component L is subnormal in G_z; the main result in this section is 16.2.4 below, showing that in fact L is normal in G_z.

Our eventual goal will be to show that L is *standard* in G, as defined in the next section. As Ronald Solomon has observed, rather than proving that L is normal in G_z, we might instead prove that L is "terminal" in the sense of [**GLS99**] (ie. for each $t \in C_G(L)$, L is a component of $C_G(t)$), and then appeal to Corollary PU_4 in chapter 3 of [**GLS99**] to prove that L is standard. Instead we show directly that L is standard, later in 16.3.2. This allows us to keep our treatment self-contained, and avoid an appeal to a fairly deep result such as Corollary PU_4 of [**GLS99**], with a minimal amount of extra effort.

THEOREM 16.2.4. *$L \trianglelefteq G_z$.*

Until the proof of Theorem 16.2.4 is complete, assume (z, L) is a counterexample. Set $L_0 := \langle L^{G_z} \rangle$ and $H := N_G(L_0)$. By A.3.8, $|T : N_T(L)| = 2$ and $L_0 = LL^u$ for $u \in T - N_T(L)$, so that $T \leq H$ and $[L, L^u] = 1$. The possibilities for L are obtained by intersecting the lists of A.3.8.3 and (E2); 16.1.2.1 allows only one case with $O_2(L) \neq 1$:

LEMMA 16.2.5. *Either $L \cong L_2(2^n)$, $Sz(2^n)$, or $L_2(p)$ with p odd, or $L/O_2(L) \cong Sz(8)$ with $O_2(L) \neq 1$.*

In the remainder of this section we will eliminate the possibilities in the list of 16.2.5.

LEMMA 16.2.6. *(1) L is a component of $C_G(t)$ for each involution $t \in \langle z \rangle L^u$.*

(2) If $L/O_2(L) \cong Sz(8)$ and $O_2(L) \neq 1$, then L is a component of $C_G(s)$ for each involution $s \in O_2(L)$.

PROOF. Let t be an involution in $\langle z \rangle L^u$. From our list in 16.2.5, either

(I) L is simple and has one conjugacy class of involutions, or

(II) $L/O_2(L) \cong Sz(8)$, and $O_2(L) \neq 1$.

If (I) holds, then conjugating in L, we may take $t \in Z(N_T(L))$; then as $|T : N_T(L)| = 2$, L is a component of $C_G(t)$ by 16.2.3, establishing (1) in this case.

Therefore we may assume that (II) holds. Let s be an involution in $O_2(L)$; then the same argument also establishes (2), since $O_2(L) = Z(L) \leq Z(N_T(L))$ as $Out(L/Z(L))$ is of odd order. Thus it remains to establish (1) in case (ii).

By (2), L is a component of $C_G(s)$. Thus we can apply 16.1.7 to s, t in the roles of "t, i". As $s \in L \leq O^2(C_G(t))$, s acts on each component of $C_G(t)$ by A.3.8.1, so that case (2) of 16.1.7 does not occur. Also the only subcase of case (3) of 16.1.7 in which $L/Z(L) \cong Sz(8)$ is subcase (i), and in that subcase L is simple, whereas here $O_2(L) \neq 1$. Thus case (1) of 16.1.7 holds, completing the proof that conclusion (1) holds. □

LEMMA 16.2.7. *$\langle N_G(L), N_G(L^u) \rangle \leq H$.*

PROOF. Let $g \in N_G(L^u)$, and let t be an involution in L^u. By 16.2.6.1, L is a component of $C_G(t)$, so as $C_G(L^u) \leq C_G(t)$, L is a component of $C_G(L^u)$. Since $g \in N_G(L^u)$, L^g is also a component of $C_G(L^u)$. If $L \neq L^g$, then $C_G(L^u)$ contains three isomorphic components L^u, L, and L^g, contrary to A.1.34.2. Thus $L = L^g$, so g normalizes $LL^u = L_0$. Therefore $N_G(L^u) \leq N_G(L_0) = H$, so the lemma holds as $u \in H$. □

In the next few lemmas, we will show that L is tightly embedded in G. Recall that a subgroup K of a finite group G is *tightly embedded* in G if K has even order, but $K \cap K^g$ is of odd order whenever $g \in G - N_G(K)$.

LEMMA 16.2.8. *(1) If $g \in G$ such that $\langle z\rangle L^u \cap (\langle z\rangle L^u)^g$ has even order, then $g \in H$.*

(2) Either L is tightly embedded in G, or $L/O_2(L) \cong Sz(8)$ and $O_2(L) \neq 1$.

(3) If X is a nontrivial 2-subgroup of $\langle z\rangle L^u$, then $N_G(X) \leq H$.

PROOF. Observe that (3) is a special case of (1). Assume the hypotheses of (1). Then there is an involution $t \in \langle z\rangle L^u \cap (\langle z\rangle L^u)^g$, so by 16.2.6, L and L^g are both components of $C_G(t)$. Then L^g normalizes L so that $L^g \leq H$ by 16.2.7, and hence L^g is a component of $C_H(t)$. Applying I.3.2, L^g lies in a 2-component of H, which is a member of $\mathcal{C}(H)$, so that by A.3.7, either $L^g \in \{L, L^u\}$ or $[LL^u, L^g] = 1$. The latter case is impossible, for since $L/O_2(L)$ is not $U_3(8)$, case (1.a) of A.1.34 holds, so that $O^{r'}(H) = LL^u$ for a suitable odd prime r; while in the former, either g or gu^{-1} lies in $N_G(L)$, so $g \in H$ by 16.2.7. Thus (1) holds.

Now if $L^u \cap L^{ug}$ has even order, then $g \in H$ by (1). Hence if $g \notin N_G(L)$, then $L^{ug} = L$, so that $1 \neq L^u \cap L \leq O_2(L)$. Then we conclude from 16.2.5 that $L/O_2(L) \cong Sz(8)$, so that (2) holds. □

LEMMA 16.2.9. *(1) Let p be a prime divisor of $2^n - 1$ if $L/O_2(L) \cong Sz(2^n)$ or $L_2(2^n)$, and let $p := 3$ if $L \cong L_2(r)$ for odd r. Then $L_0 = O^{p'}(H)$.*

(2) $L_0 \not\leq H^g$ for $g \in G - H$.

PROOF. Observe if $L \cong L_2(r)$ for r odd that 3 divides the order of some 2-local subgroup of L. Then part (1) follows as case (a) of A.1.34.1 holds. If $L_0 \leq H^g$ then $L_0 = O^{p'}(L_0) \leq O^{p'}(H^g) = L_0^g$ by (1), so that $g \in N_G(L_0) = H$, and (2) holds. □

When analyzing a tightly embedded subgroup K of a group G, one focuses on the conjugates K^g such that $N_{K^g}(K)$ is of even order. (See e.g. the definition of $\Delta(K)$ in Section 4.) In our present setup, we need a slightly stronger condition, which we establish in the next lemma:

LEMMA 16.2.10. *(1) The strong closure of T_L in $N_T(L)$ with respect to G properly contains $T_L \cup T_L^u$.*

(2) There is $g \in G - H$ such that $|L^g \cap N_H(L)|_2 > 1$.

PROOF. Set $A_1 := T_L$, $A_2 := T_L^u$, and assume $A_1 \cup A_2$ is strongly closed in $N_T(L)$ with respect to G; we check that the hypotheses of Lemma 3.4 of [**Asc75**] are satisfied. First if $A_i^g \cap A_j \neq 1$ for some i, j, then $A_2^{wg} \cap A_2^v \neq 1$ for some choice of $v, w \in \{1, u^{-1}\}$; therefore $wgv^{-1} \in N_G(A_2) \leq H$ by 16.2.8.1, and hence also $g \in H$ as $u \in H$. Thus the subgroup H_0 of H generated by all such elements g plays the role of the group "H" in 3.4 of [**Asc75**]. Next as H permutes $\{L, L^u\}$, $A_1 \cup A_2$ is strongly closed in T with respect to H. Of course $N_T(A_i) = N_T(L)$, so hypothesis (*) of 3.4 of [**Asc75**] is satisfied.

Then since $H_0 \leq H < G$ and $T_L \not\leq T_L^u$, conclusion (3) of 3.4 in [**Asc75**] holds: namely $A_1 \cap A_2 \neq 1$, and A_1 is dihedral or semidihedral. But as $1 \neq A_1 \cap A_2 \leq L \cap L^u \leq O_2(L)$, $L/O_2(L) \cong Sz(8)$ by 16.2.5, so that A_1 is of 2-rank at least 3 and hence not dihedral or semidihedral. This contradiction completes the proof of (1).

As $T_L \in Syl_2(L)$ and H permutes $\{L, L^u\}$, (1) implies (2). □

LEMMA 16.2.11. $O_2(L) = 1$*, so* L *is tightly embedded in* G.

PROOF. If L is not tightly embedded in G, then $1 \neq O_2(L) = Z(L)$ by 16.2.8.2, so to prove both assertions we may assume $Z(L) \neq 1$, and it remains to derive a contradiction. By 16.2.5, $L/Z(L) \cong Sz(8)$.

Set $Z_L := \Omega_1(T_L)$. From I.2.2.4, involutions of $T_L Z(L)/Z(L)$ lift to involutions of T_L, and these involutions are the nontrivial elements of Z_L, so Z_L is elementary abelian. Further $Out(L)$ is of odd order. From these remarks we deduce:

(*) If A is an elementary abelian 2-subgroup of $N_T(L)$ then $A \leq Z_L T_C$ and A centralizes Z_L.

Further as $Z(L) \neq 1$, $m(Z_L) > m(Z_L/Z(L)) = 3$.

By 16.2.10.2, there is $g \in G - H$ such that $L^g \cap N_H(L)$ contains an involution i, and as $T \in Syl_2(H)$ we may take $i \in T$. Then by (*), i centralizes Z_L, so as $C_G(i) \leq H^g$ by 16.2.8.3, $Z_L \leq H^g$; then conjugating in H^g, we may take $Z_L \leq T^g$. Hence by (*), $X := N_{Z_L}(L^g)$ centralizes $V := Z_L^g$. Further $|Z_L : X| \leq |H : N_H(L)| = 2 < |Z_L : Z(L)|$, and hence $X \not\leq Z(L)$. In particular $1 \neq X$, so $V \leq C_G(X) \leq H$ by 16.2.8.3. As $X \leq Z_L$ but $X \not\leq Z(L)$, $N_H(X) \leq N_H(L)$, so $V \leq N_H(L)$. Then as $m(V) > 3 = m_2(Aut(L))$, $C_V(L) \neq 1$, so $L \leq C_G(C_V(L)) \leq H^g$ by 16.2.8.3. Similarly $C_V(L^u) \neq 1$ so that $L^u \leq H^g$, and then $L_0 \leq H^g$, contrary to 16.2.9.2. □

LEMMA 16.2.12. $T_L^G \cap T = \{T_L, T_L^u\}$.

PROOF. Assume otherwise. Then there is $g \in G - H$ with $S := T_L^g \leq T$ but S is not equal to T_L or T_L^u. Now as $|T_L| > 2 = |T : N_T(L)|$, $1 \neq N_S(L) \leq N_S(T_L)$; so as L is tightly embedded in G by 16.2.11, S centralizes T_L (and similarly T_L^u) by I.7.6 with G, L, T_L, T in the roles of "H, K, Q, S ". Then $R := T_L\langle z\rangle \leq C_G(S) \leq H^g$ using 16.2.8.3. As R centralizes $S \in Syl_2(L^g)$, we conclude from 16.1.6 that R induces inner automorphisms on L^g. Then as $|R| = 2|S|$, $1 \neq C_R(L^g)$, so $L^g \leq C_G(C_R(L^g)) \leq H$ by 16.2.8.3. Similarly $L^{ug} \leq H$, so $L_0^g \leq H$, contrary to 16.2.9.2. □

We are now in a position to complete the proof of Theorem 16.2.4.

By 16.2.10.2, there is $g \in G - H$ such that $L^g \cap N_T(L)$ contains an involution i. If i centralizes a Sylow 2-subgroup of L, we may assume by conjugating in L that i centralizes T_L. Then by 16.2.8.3, $T_L \leq C_G(i) \leq H^g$, and conjugating in H^g we may assume $T_L \leq T^g$. But now by 16.2.12, $T_L \in \{T_L^g, T_L^{ug}\}$, contrary to L tightly embedded in G by 16.2.11 since $g \notin H$. Thus i does not centralize any Sylow 2-subgroup of L. But as $O_2(L) = 1$ by 16.2.11, 16.2.5 says that L has one conjugacy class of involutions, so we conclude i induces an outer automorphism on L. Therefore by 16.1.4 and 16.1.5 applied to the list in 16.2.5, either

(i) $L \cong L_2(2^{2n})$, and i induces a field automorphism on L, or

(ii) $L\langle i\rangle \cong PGL_2(p)$.

In case (ii), $C_{L_0}(i) \cong D_{p+\epsilon} \times D_{p+\epsilon}$, where $p \equiv \epsilon \mod 4$ and $\epsilon = \pm 1$. But 3 divides $p+\epsilon$ as p is a Fermat or Mersenne prime, so $C_{L_0}(i)$ contains $E \cong E_9$. This is impossible, since $i \in T_L^g$, so using 16.2.9.1,

$$E \leq O^{3'}(C_{H^g}(i)) \leq C_{L_0^g}(i) \leq C_{L^g}(i) \times L^{ug} \cong D_{p-\epsilon} \times L_2(p).$$

Similarly in case (i), $C_{L_0}(i) \cong L_2(2^n) \times L_2(2^n)$, and by 16.2.8.3 and 16.2.9.1,

$$1 \neq O^2(C_{L_0}(i)) \leq O^{q'}(H^g) = L_0^g,$$

for any prime divisor q of $2^n - 1$. Since $L_2(4) \cong L_2(5)$, we may assume $n > 1$, so such primes q exist and $m_q(C_{L_0}(i)) = 2$ while $m_q(C_{L_0^g}(i)) = 1$. This contradiction completes the proof of Theorem 16.2.4.

16.3. Showing L is standard in G

In Theorem 16.3.7 of this section, we will show that the component L is in *standard form* in G, in the sense of [**Asc75**]: that is $C_G(L)$ is tightly embedded in G, $N_G(L) = N_G(C_G(L))$, and L commutes with none of its conjugates.

To show that L is standard, we show that L is "terminal" in the sense of [**GLS99**], as defined earlier. The next two lemmas show that if L is terminal then L is standard. The proof of the first lemma makes use of the normality of L in G_z which we established in Theorem 16.2.4.

LEMMA 16.3.1. *$C_G(L)$ contains at most one component isomorphic to L, and no component G-conjugate to L.*

PROOF. The first assertion follows from A.1.34.2 with $N_G(L)$ in the role of "H". Assume that L^g is a component of $C_G(L)$. Then by the first assertion,

$$\Theta(L) := \{L^x : x \in G \text{ and } L^x \text{ is a component of } N_G(L)\} = \{L, L^g\}.$$

Since L is not $SU_3(8)$, case (1.a) of A.1.34 holds, so that $LL^g = O^{r'}(N_G(L))$ for a suitable odd prime r, and hence $L^g = O^{r'}(C_G(L))$. It follows that $L = O^{r'}(C_G(L^g))$, so that $\Theta(L) = \Theta(L^g) = \Theta(L)^g$. Thus $g \in N_G(\Theta(L)) =: N$, and hence a Sylow 2-subgroup of N is transitive on $\Theta(L)$ of order 2. This is impossible, as by Theorem 16.2.4, $T \leq N_G(L) \leq N_G(\Theta(L)) = N$ so that T fixes $\Theta(L)$ pointwise but is also Sylow in N. □

LEMMA 16.3.2. *Assume that L is a component of $C_G(t)$ for each involution $t \in C_G(L)$. Then L is standard in G.*

PROOF. Assume the hypothesis of the lemma. We first observe that $C_G(L)$ contains no conjugate of L, verifying the third condition in the definition of "standard form". For if $L^g \leq C_G(L)$, then L^g is a component of $C_G(i)$ for each involution $i \in L$ by hypothesis, so as $C_G(L) \leq C_G(i)$, L^g is a component of $C_G(i)$, contrary to 16.3.1.

Set $H := N_G(L)$, $X := C_G(L)$, and assume that $X \cap X^g$ is of even order for some $g \in G$. We will show that $g \in H$, which will suffice: For then since $C_G(L)$ is of even order and $H \leq N_G(C_G(L))$, $C_G(L)$ is tightly embedded in G and $N_G(C_G(L)) = H$, verifying the remaining conditions for L to be in standard form.

Finally assume that $g \notin H$. Thus $L^g \neq L$, while as $X \cap X^g$ is of even order, there is an involution $t \in X \cap X^g$. By hypothesis, L and L^g are components of $C_G(t)$, so as $L^g \neq L$, $L^g \leq C_G(L)$, contrary to our first observation. □

Observe also (cf. I.7.2.5):

REMARK 16.3.3. If L is standard in G, then for each nontrivial 2-subgroup X of $C_G(L)$,

$$N_G(X) \leq N_G(C_G(L)) = N_G(L).$$

To show that L is terminal, we need to eliminate the proper inclusions of L in K in parts (2) and (3) of 16.1.7. The first elimination makes use of an approach suggested by Richard Lyons. Although the method could be applied without appeal to Theorem 16.2.4, it goes more smoothly with such appeal. The method could also be used to eliminate other proper containments in 16.1.7, but it is easier to use other arguments like those in 16.3.9.

LEMMA 16.3.4. *Assume t is an involution in T_C, and L is a component of $C_K(i)$ for some component K of $C_G(t)$ and some involution $i \in C_G(L\langle t\rangle)$ with $K = [K, i]$. Then K is not $U_3(2^n)$, M_{12}, J_2, HS, or Ru.*

PROOF. Assume K is one of the components we wish to eliminate. Inspecting the list of possibilities for the pair L, K in 16.1.7, we find that L is simple, so $T_LT_C = T_L \times T_C$. By Theorem 16.2.4, T acts on L, so $|T : T_C| \leq |Aut(L)|_2$, while by inspection of the pairs on our list, $|K|_2 > |Aut(L)|_2$, so $|K|_2|T_C| > |T|$.

When $K \cong U_3(2^n)$ set $V := T_L$, and in the remaining cases choose V of order 2 in $T_L \cap Z(T)$. Thus in any case $T_C \cap V = 1$ and $T \leq N_G(V)$, so as $T \in Syl_2(G)$, $T \in Syl_2(N_G(V))$. Thus for $S \in Syl_2(N_G(V))$, $S = T^g$ for some $g \in N_G(V)$, and setting $S_A := T_A^g$ for $A \in \{C, L\}$, $S_C \trianglelefteq S$, $|K|_2|S_C| > |S|$ and $S_C \cap V = 1$.

Next we claim that there exists $S_K \in Syl_2(K)$ such that $V = Z(S_K)$: This is clear from the embedding of L in K when K is $U_3(2^n)$, while in the remaining cases we will show that 2-central involutions in L are 2-central in K, so the claim holds there too. Choose S_K so that $S_i := C_{S_K}(i) \in Syl_2(C_K(i))$. Then S_i contains an involution z in $Z(S_K)$, and by inspection of $C_K(i)$ for the pairs on our list, this forces $z \in L$: This is evident if $Z(S_i) \leq L$, while in the remaining cases all involutions in $C_{S_i}(L)$ are not 2-central and all involutions in $Z(S_i) - L$ are fused into $C_{S_i}(L)$ under $N_K(S_i)$.

By the claim, $S_K \leq N_K(V)$; let $S_K \leq S \in Syl_2(N_G(V))$. Then $|S_K||S_C| = |K|_2|S_C| > |S|$, so $S_K \cap S_C \neq 1$. By the claim, $Z(S_K) = V$, so as $1 \neq S_K \cap S_C \trianglelefteq S_K$, $V \cap (S_K \cap S_C) \neq 1$, contrary to $S_C \cap V = 1$. □

LEMMA 16.3.5. *Assume E is a subgroup of G of order 4, and K is a component of $C_G(e)$ for each $e \in E^\#$. Let i be an involution in $C_G(EK)$. Then one of the following holds:*

(1) K is a component of $C_G(i)$.

(2) $K < I$, where I is a component of $C_G(i)$ such that E is faithful on I, $O_2(I) \neq 1$, $E \cong C_{Aut(I)}(Aut_K(I)) \cong E_4$, and either

(a) $K \cong A_5$ and $I/O_2(I) \cong M_{12}$ or J_2, or

(b) $K \cong Sz(8)$ and $I/O_2(I) \cong Ru$.

PROOF. Assume that conclusion (1) fails, and set $I := \langle K^{E(C_G(i))}\rangle$. Then I and the action of an involution $t \in E^\#$ on I are described in conclusion (2) or (3) of 16.1.7, with K, I in the roles of "L, K ". Observe that $C_E(I) = 1$ since $K < I$ and K is a component of $C_G(e)$ for each $e \in E^\#$, so E is faithful on I.

Suppose the pair (t, I) satisfies case (2) of 16.1.7. Then $I = I_1I_1^t$ with $I_1 \neq I_1^t$, for some component I_1 of $C_G(i)$ such that $I_1/Z(I_1) \cong K/Z(K)$. Thus $N_E(I_1) \neq 1$

as E is of order 4, so by hypothesis K is a component of the centralizer of an involution $e \in N_E(I_1)$. Thus as e centralizes K, which is a full diagonal subgroup of $I_1 I_1^t = I$, e centralizes I, contrary to E faithful on I.

Therefore (t, I) is described in case (3) of 16.1.7. Then as K centralizes the subgroup E of order 4 faithful on I, we conclude from 16.1.4 and 16.1.5 (applied to I described in 16.1.7.3) that $E \cong E_4$, and $I/O_2(I) \cong M_{12}$, J_2, HS, or Ru. By 16.3.4, $O_2(I) \neq 1$. We may assume that conclusion (2) fails, so $I/O_2(I) \cong HS$, with $E = \langle s, t\rangle$ a 4-group such that $E(C_I(s)) \cong A_8$ and $E(C_I(t)) \cong A_6$. But this contradicts the hypothesis that K is a component of $C_I(e)$ for each $e \in E^\#$. □

LEMMA 16.3.6. *Assume $E \leq T_C$ is of order 4, with L a component of $C_G(e)$ for each $e \in E^\#$. Then L is a component of $C_G(i)$ for each involution $i \in C_G(EL)$.*

PROOF. Assume otherwise. Let i be a counterexample to the lemma, set $G_i := C_G(i)$, and take $E\langle i\rangle \leq T_i \in Syl_2(N_{G_i}(L))$. As $T \in Syl_2(N_G(L))$ by Theorem 16.2.4, we may assume $T_i \leq T$, so that $T_i = C_T(i)$. Then $i \in C_T(L) = T_C$.

As L is a component of $C_G(e)$ for each $e \in E^\#$, and we are assuming the lemma fails, $L < I := \langle L^{E(G_i)}\rangle$, where I, E, and L are described in 16.3.5.2 with L, I in the roles of "K, I". In particular $O_2(I) \neq 1$. Set $R := C_{T_C}(i)$; as $T_i = C_T(i)$, $R = C_{T_i}(L)$, so $z \in R$. Also $T_L \leq T_i$ since $i \in C_G(L)$, so $C_{T_L T_C}(i) = T_L R$.

Let $R_0 := C_R(I)$. By 16.3.5.2, $E \cong E_4$ is faithful on I and $Aut_E(I) = C_{Aut(I)}(Aut_L(I))$, so $R = R_0 E$ with $E \cap R_0 = 1$. Next $R < T_C$: for otherwise $T_L T_C \leq T_i$, so that $|T : T_i| \leq |T : T_L T_C| \leq |Out(L)|_2 \leq 2$ by inspection of the cases in 16.3.5.2, contrary to 16.1.8 with z in the role of "t".

Now pick the counterexample i so that R is maximal. As $R < T_C$, there is $y \in N_{T_C}(R) - R$ with $y^2 \in R$. Suppose $X := R_0 \cap R_0^y \neq 1$. Then as R normalizes R_0 and y normalizes R, R also normalizes R_0^y, and hence normalizes X. Therefore there is an involution i_1 in X central in $R\langle y\rangle$, contrary to the maximality of R.

Therefore $R_0 \cap R_0^y = 1$, so R_0 is isomorphic to a subgroup of $R/R_0 = R_0E/R_0 \cong E_4$, and in particular $\Phi(R_0) = 1$. As $O_2(I) \neq 1$, from (5b) and (7b) of I.2.2, non-2-central involutions of $I/Z(I)$ lift to 4-elements of I, so either $C_I(L) \cong Q_8$, or $I/O_2(I) \cong M_{12}$ and $C_I(L) \cong \mathbf{Z}_4$. In either case there is $e \in E^\#$ inducing an inner automorphism on I, so that $e = r_0 f$ with $r_0 \in R_0$ and $f \in C_I(L)$; then f is of order 4 with $f^2 \in Z(I)$, so r_0 is also of order 4, contradicting $\Phi(R_0) = 1$. □

We are now ready to state the main result of this section:

THEOREM 16.3.7. *L is standard in G.*

Until the proof of Theorem 16.3.7 is complete, assume L is a counterexample. Thus by 16.3.2, there is an involution $t \in C_G(L)$ such that L is not a component of $G_t := C_G(t)$. Recall $T_C = C_T(L) \in Syl_2(C_G(L))$, so we may assume $t \in T_C$.

LEMMA 16.3.8. *T_C is dihedral or semidihedral of order at least 8. In particular $C_{T_C}(t) = \langle z, t\rangle$.*

PROOF. As L is not a component of G_t, $t \neq z$, so $|T_C| > 2$. Since T normalizes T_C by Theorem 16.2.4, we may choose $E \leq T_C$ of order 4 with $E \trianglelefteq T$, and set $S := C_{T_C}(E)$. Then $|T : C_T(e)| \leq 2$ for each $e \in E^\#$, and hence L is a component of $C_G(e)$ by 16.1.8. Hence L is a component of $C_G(s)$ for each $s \in S^\#$ by 16.3.6. Therefore $t \in T_C - S$, and if $C_S(t) > \langle z\rangle$, then applying 16.3.6 to a subgroup of $C_S(t)$ of order 4 in the role of "E", we contradict our assumption that

L is not a component of G_t. Therefore $C_S(t) = \langle z \rangle$, so as S is of index 2 in T_C, $C_{T_C}(t) = \langle t, z \rangle \cong E_4$. Then by a lemma of Suzuki (cf. Exercise 8.6 in [**Asc86a**]), T_C is dihedral or semidihedral. As $T_C > S \geq E$ and $|E| = 4$, $|T_C| > 4$. □

Let $K := \langle L^{E(G_t)} \rangle$. By assumption, $K > L$, so K, L, and the action of z on K are described in case (2) or (3) of 16.1.7. To prove Theorem 16.3.7, we will successively eliminate those possibilities, beginning with the reduction:

LEMMA 16.3.9. *One of the following holds, with $O_2(K) \neq 1$ in cases (2)–(4):*
(1) $K \cong L_4(2)$, $L \cong A_6$, and z induces a graph automorphism on K.
(2) $K/O_2(K) \cong M_{12}$ and $L \cong A_5$.
(3) $K/O_2(K) \cong J_2$ and $L \cong A_5$ or $L_3(2)$.
(4) $K/O_2(K) \cong HS$ and $L \cong A_6$ or A_8.

PROOF. We observe first that either
(i) $t \notin K$, so that $m_2(K\langle t \rangle) \geq m_2(K) + 1$, or
(ii) $t \in K$, so that $t \in Z(K)$ and hence $Z(K) \neq 1$.

On the other hand, T normalizes L by Theorem 16.2.4, and $m_2(T_C) = 2$ by 16.3.8, so

$$m_2(K\langle t \rangle) \leq m_2(T) = m_2(LT) \leq m_2(Aut(L)) + m_2(T_C) = m_2(Aut(L)) + 2. \quad (!)$$

We will eliminate those cases in 16.1.7 not appearing in the lemma, primarily by appeals to (!). Set $\bar{K} := K/Z(K)$, $L^* := L/Z(L)$, and $m := m_2(L^*)$.

Suppose first that K is not quasisimple, so that case (2) of 16.1.7 holds. By 16.1.2, either L is simple, or $L^* \cong Sz(8)$ with $m(Z(L)) \leq 2$. Furthermore as $Inn(L^*) \leq Aut(L) \leq Aut(L^*)$, using 16.1.4 and 16.1.5, $m_2(Aut(L)) = m$; Thus (!) says that

$$m_2(K\langle t \rangle) \leq m + 2. \quad (*)$$

Further $m_2(K) \geq 2m$. Thus if $t \notin K$, then $2m + 1 \leq m + 2$ by paragraph one and (*), so that $m \leq 1$, contrary to L^* simple. On the other hand if $t \in K$, then $Z(K) \neq 1$ and hence $L^* \cong \bar{K} \cong Sz(8)$. Thus $2m \leq m + 2$ by (*), contrary to $m = m_2(Sz(8)) = 3$.

Therefore K is quasisimple, and so K is described in one of the subcases of part (3) of 16.1.7.

Suppose first that one of subcases (a)–(d) holds, but that $\bar{K}$ is neither $Sp_4(4)$ nor $G_2(4)$. By 16.3.4, K is not $U_3(2^m)$. Further either K is simple, and then L is also simple in each case; or $Z(K) \neq 1$, and then by 16.1.2, $\bar{K} \cong L_3(4)$ and $\Phi(Z(K)) = 1$. Hence when $Z(K) \neq 1$, involutions in $\bar{K}$ lift to involutions in K, and so as $\Phi(Z(K)) = 1$, $m_2(K) = m_2(\bar{K}) + m(Z(K))$. Therefore in any case, $m_2(T) \geq m_2(\bar{K}) + 1$ from paragraph one. By inspection, $m_2(\bar{K}) \geq 2m$, and $m_2(Aut(L)) = m$. Thus from (!), $2m + 1 \leq m + 2$, contradicting $m > 1$.

The lemma holds if K is $L_4(2)$, or if K appears in case (f)–(h) of 16.1.7, using 16.3.4 to conclude that $O_2(K) \neq 1$ in the latter cases. So we may assume that $\bar{K}$ is $Sp_4(4)$, $G_2(4)$, $L_5(2)$, or Ru. Now by 16.1.2, either K is simple, or $\bar{K} \cong G_2(4)$ or Ru with $Z(K)$ of order 2. By inspection, $m_3(Aut(L)) = 3$ in each case, so (!) says that

$$m_2(K\langle t \rangle) \leq 5. \quad (!!)$$

However by inspection, $m_2(\bar{K}) \geq 6$, so if K is simple, then (!!) supplies a contradiction. Thus $\bar{K} \cong G_2(4)$ or Ru and $|Z(K)| = 2$. In the latter case $m_2(K) \geq 6$

by I.2.2.7b, contrary to (!!). Thus $\bar{K} \cong G_2(4)$. By I.2.2.5a, 2-central involutions of $\bar{K}$ lift to involutions of K; so since the unipotent radical of the stabilizer in $\bar{K}$ of a point in the natural representation contains a product of two long roots groups with elements permuted transtively by a subgroup $L_2(4)$ of a Levi complement, we conclude that $m_2(K) \geq 5$. Therefore $Z(K) = \langle t \rangle$ by (!!).

Next from I.2.2.5b, short-root involutions in $\bar{K}$ lift to elements of order 4 in K squaring to t. We may choose such a u of order 4 to normalize L. But now as T is Sylow in $N_G(L)$, if necessary replacing T by a Sylow 2-subgroup of $N_G(L)$ containing $\langle u, z \rangle$, we may assume that u lies in T and so normalizes T_C. But by 16.3.8, T_C is dihedral or semidihedral of order at least 8 and $C_{T_C}(t) = \langle z, t \rangle$, while as $u^2 = t$, t centralizes the characteristic subgroup of T_C isomorphic to $\mathbf{Z}_4$. □

LEMMA 16.3.10. *Neither t nor tz is in z^G.*

PROOF. Suppose $z^g = t$. Then as L^g and K are distinct components of G_t described in 16.3.9, $m_3(KL^g) > 2$, contrary to G quasithin.

Therefore $t \notin z^G$. But by 16.3.8, $tz \in t^{T_C}$, so also $tz \notin z^G$. □

LEMMA 16.3.11. *(1) $\langle t, z \rangle \in Syl_2(C_{G_t}(L))$.*
(2) $\langle t \rangle \in Syl_2(C_{G_t}(K))$.
(3) $K \cong A_8$, $L \cong A_6$, and z induces a transposition on K.

PROOF. By 16.3.8, $\langle t, z \rangle =: E \in Syl_2(C_{G_t}(L\langle z \rangle)$, and by 16.3.10, $tz \notin z^G$, so z is weakly closed in E with respect to G_t. Hence (1) holds, and of course (1) implies (2) since z does not centralize K.

Assume (3) fails. Then K appears in one of cases (2)–(4) of 16.3.9. Thus $1 \neq O_2(K)$, so by (2), $O_2(K) = \langle t \rangle$, and if z induces an inner automorphism on K, then $z \in K$. Let $\bar{K} := K/\langle t \rangle$.

Suppose z induces an inner automorphism on K. Then $z \in K$ by the previous paragraph, so as z centralizes L, we conclude from 16.1.5 that $\bar{z}$ is a non-2-central involution of $\bar{K}$. Then from I.2.2.5b, the lift in K of $\bar{z}$ is of order 4, a contradiction.

Thus z induces an outer automorphism on K. Again using 16.1.5, z centralizes a non-2-central involution $\bar{u}$ in $\bar{K}$. Thus a preimage u of $\bar{u}$ in K is of order 4, and $\bar{u}$ acts on $\bar{L} = O^2(C_{\bar{K}}(z))$, so u acts on L. Now the argument in the last paragraph of the proof of 16.3.9 supplies a contradiction. □

Let $T_t := C_T(t)$; as $T \in Syl_2(G_z)$ and $L \trianglelefteq G_z$, we may choose t so that $T_t \in Syl_2(C_{G_t}(z))$. Let $T_t \leq P \in Syl_2(G_t)$.

As $K\langle z \rangle \cong S_8$ by 16.3.11.3, we can represent $K\langle z \rangle$ as the symmetric group on $\Omega := \{1, \ldots, 8\}$ with $z := (1, 2)$. Then there is an involution $u \in C_K(z)$ acting as $(1, 2)(3, 4)$ on Ω and inducing a transposition on $L \cong A_6$. Let y denote a generator for the characteristic cyclic subgroup Y of index 2 in T_C provided by 16.3.8. Choose $w \in \{u, tu\}$ with $|C_Y(w)|$ maximal.

LEMMA 16.3.12. *$J(T) = R \times T_L \times \langle w \rangle$, where either*

(1) w centralizes T_C, $R := T_C$ if T_C is dihedral, and R is the dihedral subgroup of T_C of index 2 if T_C is semidihedral; or

(2) $|Y| > 4$, $y^w = yz$, and $R = \langle y^2, t \rangle$.

PROOF. First $\langle u, t \rangle$ acts on Y with $y^t = y^{-1}$ or $y^{-1}z$ for T_C dihedral or semidihedral, respectively. Further $L\langle u \rangle \cong S_6$ by construction, so that $m_2(T/T_C) = 3$, while $m_2(T_C) = 2$ by 16.3.8; hence

$$m_2(T) \leq m_2(T/T_C) + m_2(T_C) = 3 + 2 = 5,$$

so as $m_2(T) \geq m_2(P) \geq m_2(S_8) + 1 = 5$, all inequalities are equalities. Hence $m_2(T) = 5$, and for each $A \in \mathcal{A}(T)$, $m(A/A \cap T_C) = 3$ and $m(A \cap T_C) = 2$. Thus $ALT_C/T_C \cong S_6$, so $A \leq T_C L\langle u\rangle$ and hence $J(T) = J(T_0)$, where $T_0 := T_C T_L\langle u\rangle$.

If $\langle u, t\rangle$ is not faithful on Y, then w centralizes T_C since we chose $|C_Y(w)|$ maximal; therefore $T_0 = T_C \times T_L\langle w\rangle$, and (1) follows. Thus we may assume that $\langle u, t\rangle$ is faithful on Y, so $|Y| > 4$ and $y^w = yz$. Then we calculate that $\Omega_1(T_0) = \langle y^2, t\rangle \times T_L\langle w\rangle$, and then that (2) holds. □

We now complete the proof of Theorem 16.3.7.

As $L \cong A_6$, $T_L \cong D_8$. It follows from 16.3.12 and 16.3.8 that $\Omega_1(\Phi(J(T))) = \langle z, v\rangle$, where $\langle v\rangle = Z(T_L)$, and $\langle z, v\rangle \leq Z(T)$. On the other hand, by 16.3.11.2, $\langle t\rangle = C_P(K)$, so $P = \langle t\rangle \times Q$, where $Q := (P \cap K)\langle z\rangle \cong D_8$ wr $\mathbf{Z}_2$. Thus $J(P) = \langle t\rangle \times S_1 \times S_2$, with $S_i \cong D_8$, and $\Omega_1(\Phi(J(P)) = \langle s_1, s_2\rangle$, where $\langle s_i\rangle = Z(S_i)$, s_i has cycle structure 2^2 on Ω, and $s_1 s_2$ has cycle structure 2^4.

Now by 16.3.12, $J(T) = R \times T_L \times \langle w\rangle$, where R and T_L are dihedral of order at least 8. Then by the Krull-Schmidt Theorem A.1.15, either $|R| > |T_L|$ and $N_G(J(T))$ normalizes $RZ(J(T))$ and $T_L Z(J(T))$, or $|R| = |T_L| = 8$ and $N_G(J(T))$ permutes the pair. Thus $N_G(J(T))$ permutes $\{z, v\}$ since $\langle z\rangle = \Omega_1(\Phi(RZ(J(T)))$ and $\langle v\rangle = \Omega_1(\Phi(T_L Z(J(T))))$. Next $J(P) \leq T^g$ for some $g \in G$, and $m_2(P) = 5 = m_2(T)$, so $J(P) \leq J(T^g)$. Then $\langle s_1, s_2\rangle = \Phi(J(P)) \leq \Phi(J(T^g)) = \langle z, v\rangle^g$. This is impossible as $\langle z, v\rangle \leq Z(T)$, whereas $\langle s_1, s_2\rangle \not\leq Z(P)$.

This contradiction completes the proof of Theorem 16.3.7.

LEMMA 16.3.13. *(1)* $L^G \cap C_G(L) = \emptyset$.

(2) L is standard in G.

(3) If $C_G(L) \cap N_G(L^g)$ is of even order for some $g \in G - N_G(L)$, then $L \not\leq N_G(L^g)$.

PROOF. Observe that (2) is just a restatement of Theorem 16.3.7, and (1) is a restatement of the condition in the definition of standard form that L commutes with none of its conjugates.

Assume the hypothesis of (3) and $L \leq N := N_G(L^g)$. Thus $L^g \neq L$, and there is an involution $i \in C_N(L)$. By Remark 16.3.3, L is a component of $C_N(i)$, so we may apply I.3.1 with N, $\langle i\rangle$ in the roles of "H, P", to conclude that $L \leq KK^i$, where K and K^i are (not necessarily distinct) 2-components of N. If $L^g \leq KK^i$, then $L^g \in \{K, K^i\}$, so as $i \in N = N_G(L^g)$, $L \leq KK^i = L^g$, contrary to $L \neq L^g$. Therefore $[L^g, KK^i] = 1$ by 31.4 in [**Asc86a**], so $L \leq C_G(L^g)$, contrary to (1). □

16.4. Intersections of $\mathbf{N_G(L)}$ with conjugates of $\mathbf{C_G(L)}$

Recall that in Notation 16.2.2, z is an involution in the center of T, and L is a component of $G_z = C_G(z)$. By Theorem 16.3.7, L is standard in G.

With this setup, we could now finish quickly by quoting some of the machinery on standard subgroups and tightly embedded subgroups in the Component Paper [**Asc75**] and the Tightly Embedded Subgroup Paper [**Asc76**], and some of the classification theorems in the literature based on that theory. But since GLS do not use this machinery, we will only use some comparatively elementary results from that theory, which we have reproduced in section I.7.

In this section we develop some technical tools, which we apply in the final section to show that J_1 is the only group satisfying Hypothesis 16.1.1.

Set $K := C_G(L)$, $H := N_G(L)$, and $H^* := H/K$. As L is standard in G, $H = N_G(K)$. Thus for each nontrivial 2-subgroup X of K, $N_G(X) \leq H$ by Remark 16.3.3. In particular,

$$G_z \leq H.$$

For $K' \in K^G$, define $L(K') := L^g$, where $g \in G$ with $K^g = K'$; as $N_G(K) = N_G(L) = H$, this definition is independent of the choice of g, and the set of such elements is a coset of H in G.

Recall $T_L = T \cap L$ and $T_C = T \cap K$.

Our discussion in this section will be based on an analysis of the set

$$\Delta = \Delta(K) := \{K' \in K^G - \{K\} : |N_{K'}(K)|_2 > 1\}.$$

To see that Δ is nonempty under our hypotheses, we appeal to I.8.2: Since G is simple, K is not normal in $G = \langle K^G \rangle$. Therefore if Δ is empty, then H is strongly embedded in G by I.8.2. Then by the Bender-Suzuki classification (see Theorem SE on p. 20 of [**GLS99**]) of simple groups with strongly embedded subgroups, $G = O^{2'}(G)$ is a Bender group, contrary to our assumption in Hypothesis 16.1.1 that G is not of even characteristic. Thus we conclude that

$$\Delta \text{ is nonempty.}$$

Recall that in Notation 16.2.2, $T \in Syl_2(G)$, and then $T \in Syl_2(H)$ using Theorem 16.2.4. Then as Δ is nonempty we can extend that earlier Notation by adopting:

NOTATION 16.4.1. $K' \in \Delta$, $L' := L(K')$, $H' := N_G(K')$, and $R \in Syl_2(N_{K'}(K))$ with $R \leq T$. For each involution r in R, set $L_r := O^2(C_L(r))$. Also set $H^* := H/K$.

Since $R \leq T$ in Notation 16.4.1, R normalizes T_L and T_C by Theorem 16.2.4.

Our next result lists elementary properties of the members of $\Delta(K)$:

LEMMA 16.4.2. *(1) $R \cong N_{T_C}(R) = C_{T_C}(R) = N_{T_C}(K') \in Syl_2(N_K(K'))$, with $R \cap K = 1$. In particular R is faithful on L and $|N_{K'}(K)|_2 = |N_K(K')|_2$.*

(2) $L \not\leq H'$.

(3) $R = K' \cap T$.

(4) There exists $g \in G$ with $K' = K^g$ and $N_T(K') \leq T^g$.

(5) For each $1 \neq X \leq R$, $N_G(X) \leq H'$.

(6) If $N_T(R) \in Syl_2(N_H(R))$, then $C_T(R\langle z \rangle) \in Syl_2(C_G(R\langle z \rangle))$.

PROOF. Part (5) is a restatement of Remark 16.3.3. We apply parts (1) and (2) of I.7.7 with K', K, $T_C R$ in the roles of "K, K^g, S" to obtain $N_{T_C}(R) = C_{T_C}(R) \cong R$. By I.7.7.3, $N_{T_C}(R)$ is Sylow in $N_K(K')$, completing the proof of (1).

By (1), $|N_K(K')|$ is even, so (2) follows from 16.3.13.3. Part (3) holds since $R \in Syl_2(N_{K'}(K))$ and $R \leq T$. Let $g \in G$ with $K' = K^g$ and $N_T(K') \leq T' \in Syl_2(H')$. As $T^g \in Syl_2(H')$ there is $y \in H'$ with $T^{gy} = T'$, so replacing g by gy, we may take $N_T(K') \leq T^g$. Thus (4) holds.

If $N_T(R) \in Syl_2(N_H(R))$ then $C_T(R) \in Syl_2(C_H(R))$, so as $z \in Z(T)$, $C_T(R\langle z \rangle) \in Syl_2(C_H(R\langle z \rangle))$. Thus (6) follows as $G_z \leq H$. □

The next result says that Δ defines a symmetric relation on K^G.

LEMMA 16.4.3. *(1)* $K \in \Delta(K')$.
(2) $L' = [L', z]$.

PROOF. Part (1) is a consequence of 16.4.2.1. By 16.4.2.1, $z \in Z(T) \cap T_C \leq N_{T_C}(K')$. Thus (2) follows from 16.4.2.1 and the fact that Δ is symmetric. □

LEMMA 16.4.4. *(1) Assume* R *is of order* 2. *Then* $C_{T_C}(R) = \langle z \rangle$ *is also of order* 2, $m_2(RT_C) = 2$, *and* RT_C *is dihedral or semidihedral. If furthermore* $Z(L) \neq 1$, *then* $z \in Z(L)$.
(2) If T_C *is cyclic, then* $K = T_C O(K)$ *and* $C_K(z) = T_C$.

PROOF. Assume R is of order 2. Then by 16.4.2.1, $C_{T_C}(R)$ is also of order 2, so that $C_{T_C}(R) = \langle z \rangle$. Hence by Suzuki's lemma (cf. Exercise 8.6 in [**Asc86a**]), RT_C is dihedral or semidihedral, so that (1) holds.

Next assume T_C is cyclic. Then by Cyclic Sylow 2-Subgroups A.1.38, $K = T_C O(K)$. Further as $G_z \leq H = N_G(K)$, $C_{O(K)}(z) \leq O(G_z) = 1$ by (E1), so that (2) holds. □

Now we begin the process of obtaining restrictions on H, and in particular on the Sylow 2-subgroup R of $N_{K'}(K)$.

LEMMA 16.4.5. *Assume* p *is an odd prime with* $m_p(L) > 1$, *and* i *is an involution in* K. *Then either*
(1) $L = O^{p'}(C_G(i))$, *or*
(2) $p = 3$, $O^{3'}(C_G(i)^*) \cong PGL_3^{\epsilon}(2^n)$ *or* $L_3^{\epsilon,\circ}(2^n)$, *with* $2^n \equiv \epsilon \mod 3$, *and* $L = O^{3'}(LC_K(i))$. *In particular,* $O^{3'}(H^*) \cong PGL_3^{\epsilon}(2^n)$ *or* $L_3^{\epsilon,\circ}(2^n)$.

PROOF. Recall $L \trianglelefteq C_G(i)$ by Remark 16.3.3, and $O(L) = 1$ by (E1). Thus as L is in the list of (E2), $C_G(i)$ satisfies conclusion (1) or (2) of A.3.18, so the lemma holds. □

The next lemma eliminates the shadow of $L_2(p^2)$ (p a Fermat or Mersenne prime) extended by a field automorphism, and the shadow of S_7. These groups are quasithin, and have a 2-central involution with centralizer $\mathbf{Z}_2 \times PGL_2(p)$, but the groups are neither simple nor of even type.

LEMMA 16.4.6. *If* $L \cong L_2(q)$, q *odd, then no involution in* R *induces an outer automorphism in* $PGL_2(q)$ *on* L.

PROOF. Let r denote an involution in R with $L\langle r \rangle \cong PGL_2(q)$. Recall q is a Fermat or Mersenne prime or 9 by (E2). Further if $q \neq 9$, $Aut(L) \cong PGL_2(q)$, so either $H^* \cong PGL_2(q)$, or $q = 9$ and $H^* \cong Aut(PGL_2(9)) \cong Aut(A_6)$.

If $q \neq 9$, let $R_0 := R \cap LK$; while if $q = 9$, let R_0 be the subgroup of R inducing automorphisms in S_6. Then $R = R_0\langle r \rangle$. If $R_0 \neq 1$ there is an involution r_0 in R_0, and $L = \langle C_L(r), C_L(r_0) \rangle \leq H'$ by 16.4.2.5, contrary to 16.4.2.2. Thus $R_0 = 1$, so $R = \langle r \rangle$ is of order 2. Hence by 16.4.4.1, $\langle z \rangle = C_{T_C}(R)$ is also of order 2. Choose T so that $N_T(R) \in Syl_2(N_H(R))$.

Let $E := \langle r, z \rangle$ and $T_E := C_T(E)$. As RT_L is dihedral, $C_{T_L}(r) =: \langle v \rangle$ is of order 2. Therefore as $C_{T_C}(R) = \langle z \rangle$, $T_E \cap LKR =: V = \langle v, z, r \rangle \cong E_8$, and either $H^* \cong PGL_2(q)$ and $T_E = V$, or $H^* \cong Aut(A_6)$ and $T_E \cong E_4 \times \mathbf{Z}_4$. Further $T_E \in Syl_2(C_G(E))$ by 16.4.2.6. As RT_L is dihedral, $rv \in r^{N_{T_L}(T_E)}$. From the structure of $Aut(L)$, $Z(T^*) = \langle v^* \rangle$, so $Z(T) \leq \langle v \rangle T_C \cap T_E = \langle v, z \rangle$, and hence $Z(T) = \langle z, v \rangle$.

We claim that z is weakly closed in $Z(T)$ with respect to G; the proof will require several paragraphs. Suppose the claim fails. Then using Burnside's Fusion Lemma A.1.35, $N_G(T)$ induces $\mathbf{Z}_3$ on $Z(T)$, and in particular is transitive on $Z(T)^{\#}$. Thus there are $h, k \in N_G(T)$ such that $v = z^h$ and $vz = z^k$; and in particular, $N_G(T)$ transitively permutes $\{T_C, T_C^h, T_C^k\} =: \mathcal{T}$. As K is tightly embedded in G, distinct members of $\mathcal{T}$ intersect trivially.

Since T_L and T_C^k are normal in T, $T_L \cap T_C^k$ is normal in T. Then as $Z(T) \cap T_L = \langle v \rangle$ is of order 2, v lies in $T_L \cap T_C^k$ if this group is nontrivial; but this is impossible as $v = z^h \in T_C^h$ and $T_C^h \cap T_C^k = 1$ by the previous paragraph. Therefore $[T_L, T_C^k] \leq T_L \cap T_C^k = 1$, so that $T_C^k \cong T_C^{k*} \leq C_{T^*}(T_L^*)$. Now by 16.1.6, either $C_{T^*}(T_L^*) = \langle v^* \rangle$ or $H^* \cong Aut(A_6)$ and $C_{T^*}(T_L^*) = \langle v^*, x^* \rangle$, where x induces a transposition on L and $L = \langle C_L(v), C_L(x) \rangle$. Thus by 16.4.2.2, $T_C^{k*} \neq \langle v^*, x^* \rangle$ when $H^* \cong Aut(A_6)$, so $|T_C^k| = 2$. Therefore $T_C = \langle z \rangle$ is of order 2. Then $z^h = v \in [T, T]$ since $L\langle r \rangle \cong PGL_2(q)$, so as $h \in N_G(T)$, $z \in [T, T]$. Thus $|T : T_L| > 4$, so that $q = 9$ and $H^* \cong Aut(A_6)$. Then $[T^*, T^*] = Y^*$, where $\mathbf{Z}_4 \cong Y \leq T_L$, so $[T, T] \leq T_C Y$ and hence $\langle v \rangle = \Phi([T, T]) \trianglelefteq N_G(T)$. This contradiction completes the proof of the claim that z is weakly closed in $Z(T)$ with respect to G. In particular, $z \notin v^G$.

By 16.4.2.4 there exists $g \in G$ with $K' = K^g$ and $N_T(K') \leq T^g$. By 16.4.3.2, $L' = [L', z]$.

We next establish symmetry between L, r and L', z by showing that $L'\langle z \rangle \cong PGL_2(q)$. Assume otherwise and recall $E = \langle r, z \rangle$ and $T_E \in Syl_2(C_G(E))$ is abelian. Thus if $q = 9$, z does not induce an automorphism of L' contained in S_6. Therefore we may assume that z induces an inner automorphism on L'. By 16.4.2.1, R is Sylow in $C_{K'}(z)$, so as $z \in K'L'$ and R is of order 2, R is Sylow in K'. Hence $z^{L'} \cap Z(T^g) \neq \emptyset$, impossible as r is weakly closed in $Z(T^g)$ by the claim. Therefore $L'\langle z \rangle \cong PGL_2(q)$.

Recall that $rv \in r^L$, so if $v \in L'$, then by symmetry $zv \in z^{L'}$, contrary to the claim. Hence $v \notin L'$. Recall that $\langle z, r, v \rangle = V = \Omega_1(T_E) \cong E_8$, and there is $t \in N_{T_L}(T_E)$ with $[t, r] = v$. As $v \notin L'$,

$$V \cap L' =: \langle u \rangle \neq \langle v \rangle,$$

and by symmetry there is $s \in N_{T^g}(T_E)$ with $[s, z] = u$. Set $X := N_G(V)$ and $X^+ := X/C_X(V)$, so that $X^+ \leq GL(V) \cong L_3(2)$, and t^+ and s^+ are transvections in X^+ on V, with centers v, u, and axes $Z(T) = \langle z, v \rangle$, $Z(T^g) = \langle r, u \rangle$, respectively. As $Z(T) \neq Z(T^g)$ and $v \neq u$, $\langle t^+, s^+ \rangle$ is either D_8 or S_3 from the structure of $L_3(2)$. In the first case the unique hyperplane W of V normalized by $\langle t^+, s^+ \rangle \cong D_8$ is centralized by either t or s, say t; but then $W = Z(T)$ is not centralized by s, so that $z \neq z^s \in Z(T)$, contrary to the claim. Hence $\langle t^+, s^+ \rangle \cong S_3$, and so $V = V_1 \oplus V_2$, where

$$V_1 := \langle u, v \rangle = [V, \langle s, t \rangle] \cong E_4,$$

and

$$V_2 := C_V(\langle s, t \rangle) = Z(T) \cap Z(T^g) = \langle vz \rangle = \langle ur \rangle.$$

In particular as u is fused to v and all involutions in $D := T_L\langle u \rangle$ are in v^G. As $z \notin v^G$ by the claim, $z^G \cap D = \emptyset$.

Next if $T_C > \langle z \rangle$, then $N_{T_C}(V)^+$ is a transvection on V with axis $Z(T)$ and center $\langle z \rangle$, so $\langle N_{T_C}(V)^+, t^+, s^+ \rangle \cong S_4$ is the stabilizer in $GL(V)$ of vz, and hence is transitive on $V - \langle vz \rangle$, which is impossible since $v \notin z^G$ by the claim. Therefore $T_C = \langle z \rangle$, so $T_C T_L R = T_C \times D$ as $vz = ur$.

Suppose that $H^* \cong PGL_2(q)$. Then $|T : D| = 2$, so since we saw that $z^G \cap D = \emptyset$, $z \notin O^2(G)$ by Thompson Transfer, contrary to the simplicity of G.

Therefore $H^* \cong Aut(A_6)$, so $T_E \cong E_4 \times \mathbf{Z}_4$. Now $\langle s, t \rangle$ acts on T_E and $C_V(\langle t, s \rangle) = \langle vz \rangle$, so we conclude that $\langle vz \rangle = \Phi(T_E)$. Next $D = T_L \langle u \rangle$ and $T = T_E T_L$ with T_E abelian, so that $D \trianglelefteq T$. As $vz \notin D$ and $vz \in \Phi(T_E)$, $T/D \cong \mathbf{Z}_4$. Then since we saw that $z^G \cap D = \emptyset$, $z \notin O^2(G)$ by Generalized Thompson Transfer A.1.37.2, contrary to the simplicity of G, completing the proof of 16.4.6. □

In the next lemma, we deal with the only case (other than those eliminated in 16.4.6 and the case $L \cong L_2(2^n)$) where for some involution r in R, L_r is not generated by its p-elements as p varies over those odd primes such that $m_p(L) > 1$.

LEMMA 16.4.7. *If $L^* \cong M_{22}$, then no involution in R induces an outer automorphism on L with $C_{L^*}(r) \cong Sz(2)/E_{16}$.*

PROOF. Assume r is a counterexample, and set $L_R := O^2(N_L(O_2(L_r)))$. Recall $L_r = O^2(C_L(r))$. Then $R = R_0\langle r \rangle$, where $R_0 := R \cap LK$. From the structure of the extension of L^* by a 2-group in I.2.2.6a, L_r is isomorphic to $\mathbf{Z}_5/E_{16}$ if $|Z(L)| < 4$, but isomorphic to $\mathbf{Z}_5/Q_8D_8$ if $Z(L) \cong \mathbf{Z}_4$.

We first show that $R_0 = 1$, so we assume that $R_0 \neq 1$ and derive a contradiction. Then as $L_r \leq H'$, $[L_r, C_{R_0}(r)] \leq L_r \cap K' =: Y \trianglelefteq L_r$, and $Y^* \neq 1$ as $C_{H^*}(L_r^*) = 1$. Thus Y^* contains the unique minimal normal subgroup $O_2(L_r^*)$ of L_r^*, so $O_2(L_r) \leq Y \leq K'$. It follows from 16.4.2.5 that $L_R \leq H'$. Further as $O(L') = 1$ by (E1), $L' = O^{3'}(H')$ by A.3.18, and hence $L_R \leq L'$. But then $O_2(L_r) \leq K' \cap L' \leq Z(L')$, impossible as $m_2(O_2(L_r)) \geq 3$ by the first paragraph, while $Z(L')$ is cyclic by 16.1.2.2.

Thus $R_0 = 1$, so $R = \langle r \rangle$ is of order 2. So by 16.4.4.1, $C_{T_C}(R) = \langle z \rangle$ is of order 2, and $T_C R$ dihedral or semidihedral. Thus if $Z(L) \neq 1$, $\langle z \rangle = \Omega_1(Z(L))$. Conversely if $z \in L$, then $Z(L) \neq 1$ and $\langle z \rangle = \Omega_1(Z(L))$. By 16.4.3.2, $L' = [L', z]$.

Again we establish symmetry between L, z, r and L', r, z, by showing that the action of z on L' is the same as that of r on L: First RT_C is dihedral or semidihedral and isomorphic to a Sylow group of H/L, so as $L_r \leq H'$ and L_r is irreducible on $O_2(L_r)/\Phi(O_2(L_r)) \cong E_{16}$, we conclude that $O_2(L_r) \leq L'$ and $O_2(L_r) \cap K' \leq \Phi(O_2(L_r))$. As z centralizes L_r, we conclude from 16.1.5.5 that z induces an outer automorphism of L' with $C_{L'/O_2(L')}(z) \cong Sz(2)/E_{16}$, establishing the symmetry.

In particular as $|Out(L')| = 2$, $z \notin [H', H']$. But if $Z(L) \cong \mathbf{Z}_4$, then we saw that $z \in [O_2(L_r)), O_2(L_r]$; so $|Z(L)| \leq 2$, and hence $O_2(L_r) \cong E_{16}$ from our earlier discusion. Further we saw $O_2(L_r) \leq L'$.

Set $E := \langle r, z \rangle$, $V := O_2(L_r)E$, and choose $g \in G$ with $K^g = K'$ and $N_T(K') \leq T^g$. Set $M := \langle N_H(V), N_{H'}(V) \rangle$ and $M^+ := M/C_M(V)$; then $N_L(V)^+ \cong S_5$.

Assume for the moment that $Z(L) \neq 1$, so that $Z(L) = \langle z \rangle$ is of order 2. Then V is a 6-dimensional indecomposable for $N_L(V)^+$, so by symmetry between r and z, also $N_{L'}(V)^+$ is indecomposable on V, and hence M is irreducible on V when $Z(L) \neq 1$. Now assume that $Z(L) = 1$. Then $V = \langle z \rangle \oplus O_2(L_r)\langle r \rangle$ as an $N_L(V)^+$-module, and $O_2(L_r)\langle r \rangle$ is a 5-dimensional indecomposable with trivial quotient.

Since we saw $O_2(L_r) \le L'$, we conclude by symmetry that $O_2(L_r) \trianglelefteq M$ when $Z(L) = 1$.

Suppose that $T_C > \langle z \rangle$ and set $S_C := N_{T_C}(V)$. Then as $[V, S_C] = [R, S_C] \le V \cap T_C = \langle z \rangle = C_{T_C}(R)$, S_C is of order 4 and $S_C^+ = \langle t^+ \rangle$, where t^+ induces a transvection on V with center $\langle z \rangle$ and axis $O_2(L_r)\langle z \rangle$. By symmetry there is $t_0 \in M \cap T_C^g - R$ which induces a transvection on V with center $\langle r \rangle$ and axis not containing z. As $\langle z \rangle$ is the center of t^+, $C_{M^+}(t^+) \le C_M(z)^+ = S_C^+ \times N_L(V)^+$. In particular $S_C^+ = O_2(C_{M^+}(t^+))$, so either $O_2(M^+) = 1$ or $O_2(M^+) = S_C^+$. But in the latter case, M^+ acts on z, whereas $[t_0, z] \ne 1$. Thus $O_2(M^+) = 1$. Let $X^+ := \langle t^+, t_0^+ \rangle$; then $X^+ \cong S_3$ centralizes $O_2(L_r)$ from the structure of subgroups of the general linear group generated by a pair of transvections.

Assume first that $Z(L) = 1$. Then $O_2(L_r) \trianglelefteq M$ and $N_L(V)$ centralizes $V/O_2(L_r)$, so

$$[X^+, N_L(V)^+] \le C_{M^+}(O_2(L_r)) \cap C_{M^+}(V/O_2(L_r)) \le O_2(M^+) = 1$$

by Coprime Action, and hence $N_L(V)$ normalizes $[V, X] = \langle z, r \rangle$. This is a contradiction as r does not centralize L_V. Therefore $Z(L) \ne 1$, so M is irreducible on V by earlier remarks. As M^+ contains the transvection t^+, with $C_{M^+}(t^+) \le C_M(z)^+ \cong \mathbf{Z}_2 \times S_5$, it follows from G.6.4 that $M^+ \cong S_7$ and V is the natural module. This is a contradiction as the noncentral chief factor for $N_L(V)^+$ on V is the natural module for $L_2(4)$, whereas the centralizer in A_7 of a transvection has the A_5-module as its noncentral chief factor. This contradiction completes the elimination of the case $T_C > \langle z \rangle$.

So $T_C = \langle z \rangle$. Then $T = T_L T_C R$ acts on V, so $T^+ = T_L^+ \cong D_8$ is Sylow in M^+. We may apply A.3.12 to conclude that $L_V \le N \in \mathcal{C}(M)$, and the embedding of L_V^+ in N^+ is described in A.3.14. As D_8 is Sylow in N^+T^+ and there is a nontrivial $\mathbf{F}_2 N^+T^+$-representation of dimension 6, we conclude that either $N^+ = L_V^+$ or $N^+ \cong A_7$. In the first case, M acts on $C_V(L_V) = \langle z \rangle$, and then by symmetry, M also centralizes r, a contradiction as L_V does not centralize r. In the second case, as $m(V) = 6$ and $S_5 \cong N_L(V)^+ = C_{M^+}(z)$, we conclude that V is the core of the 7-dimensional permutation module for M^+, and that z is of weight 2 in that module. This is impossible, as we saw at the end of the previous paragraph. This contradiction completes the proof of 16.4.7. $\square$

LEMMA 16.4.8. *Let r be an involution of R. Then either*

(1) $L_r \le L'$, or

(2) $O^{3'}(H^*) \cong PGL_3^\epsilon(2^n)$ or $L_3^{\epsilon,\circ}(2^n)$, $2^n \equiv \epsilon \mod 3$, r induces an inner automorphism on L, $n \ne 3$, and $O^3(L_r) \le L'$.

PROOF. As usual recall that $L_r \le C_G(r) \le H'$. Let $\bar{H}' := H'/K'$, and define

$$\Lambda(L_r) := \langle O^{p'}(L_r) : \ p \text{ is an odd prime such that } \ m_p(L) > 1 \rangle.$$

Applying 16.4.5 with L', r in the roles of "L, i", either $\Lambda(L_r) \le L^g$ or $O^{3'}(\bar{H}') \cong PGL_3^\epsilon(2^n)$ or $L_3^{\epsilon,\circ}(2^n)$.

Assume first that the latter case holds. In particular, $n \ge 2$.

We will first treat the subcase where r induces an outer automorphism on L. Then by 16.1.4, either L_r is $PSL_3(2^{n/2})$, $U_3(2^{n/2})$ with $n > 2$, or $L_2(2^n)$; or r induces a graph-field automorphism on $L^* \cong L_3(4)$ and $L_r \cong E_9$. As $m_3(L') = 2$, $Z(L')$ is a 2-group, and $L'K'$ is an SQTK-group, $C_{K'}(r)$ is a $3'$-group; so as

$O_{3'}(L_r) = 1$, L_r is faithful on L' and $\bar{L}_r \trianglelefteq O^2(C_{\bar{H}'}(\bar{z}))$. We conclude from 16.1.4 first that z induces an outer automorphism on L', and second as $O^{3'}(\bar{H}')$ is $PGL_3^{\epsilon}(2^n)$ or $L_3^{\epsilon,\circ}(2^n)$ that either $\bar{L}_r \leq \bar{L}'$, or $O^2(\bar{H}') \cong PGL_3(4)$ and z induces a graph-field automorphism on L' with $O^2(C_{L'}(z)) \cong E_9 \cong L_r$. In the former case conclusion (1) holds, so we may assume the latter. Then $|C_{L^*}(r^*) : C_L(r)^*| \leq |Z(L)| =: m$, and by 16.1.2.2, $m \leq 4$. On the other hand, $C_{L^*}(r^*)$ contains a Q_8-subgroup faithful on L_r^*, so if $m < 4$, then r centralizes $x \in L$ with x^2 inverting L_r; but then $\bar{x}^2 \in \bar{L}'$ by 16.4.2.5, so that $L_r = [L_r, x^2] \leq L'$, and conclusion (1) holds again. Finally if $m = 4$, then r centralizes $Z(L)$ by I.2.2.3b, so that $Z(L) \leq C_G(r) \leq H'$ using 16.4.2.5. Then as $m_2(C_{Aut(L')}(L_r)) = 1$, $Z(L) \cap K' \neq 1$, contradicting K tightly embedded in G.

Next we treat the subcase where r induces an inner automorphism on L. Recall here that $|C_L(r)/O_2(C_L(r))| = (2^n - \epsilon)/3$. Assume that $n > 3$. Then there are prime divisors $p > 3$ of $|C_L(r)|$, and $m_p(L) > 1$ for each such p; hence $L = O^{p'}(H)$ by 16.4.5, so that $O^{p'}(L_r) \leq O^{p'}(H') = L'$. Thus $O^3(L_r) \leq L'$, so that conclusion (2) holds. Finally suppose that $n = 3$, so that $L \cong U_3(8)$, $|L_r : T_L| = 3$, and $O^2(L_r)$ centralizes $Z(T_L)$. Let x be of order 3 in L_r; we may assume (1) fails, so $x \notin L'$. By 16.4.5, $O^{3'}(\bar{H}') \cong PGU_3(8)$ or $U_3^{\circ}(8)$, and $C_{K'}(r)$ is a $3'$-group, so as $x \notin L'$, $\bar{x} \notin \bar{L}'$. As $1 \neq \bar{z} \in C_{\bar{H}'}(x)$, $C_{\bar{H}'}(x)$ is of even order, and $\bar{x}$ is of order 3; so from the structure of $Aut(U_3(8))$, and as $O^{3'}(\bar{H}') \cong PGU_3(8)$ or $U_3^{\circ}(8)$, $\bar{x} \in \bar{L}'$, a contradiction.

This completes the treatment of the case where $O^{3'}(\bar{H}')$ is described in case (2) of 16.4.5, so we may assume that $\Lambda(L_r) \leq L^g$. In particular if $L_r = \Lambda(L_r)$, then conclusion (1) holds, so we may assume that $\Lambda(L_r) < L_r$; thus $L_r \neq 1$.

Suppose first that $L \cong L_2(q)$, q odd. Then q is a Fermat or Mersenne prime or 9 by (E2), and r does not induce an outer automorphism in $PGL_2(q)$ on L by 16.4.6. If $q = 9$ and r induces an outer automorphism in S_6, then $L_r \cong A_4$, contrary to our assumption that $\Lambda(L_r) < L_r$. Thus we conclude from 16.1.4 that r induces an inner automorphism of L. But then $C_L(r)$ is a 2-group, so $L_r = 1$, again contrary to assumption.

Next suppose $L^* = X(2^n)$ is of Lie type. We can assume by the previous paragraph that L is not $L_2(4) \cong L_2(5)$, $L_3(2) \cong L_2(7)$, or $Sp_4(2)' \cong L_2(9)$. Hence as $\Lambda(L_r) < L_r$, we conclude from 16.1.4 and 16.1.2.1 that $L \cong L_2(2^n)$ for $n > 2$ even, and r induces a field automorphism on L. For each involution $i \in C_{T_C}(r)$, $L_r \cong L_2(2^{n/2})$ is a component of $C_{H'}(\langle i, r\rangle)$ using 16.4.2.5. Therefore either $L_r \leq L'$ so that conclusion (1) holds, or $L_r \leq K'$, and we may assume the latter. Then using 16.4.2.5,

$$L = \langle C_L(j) : \ j \text{ is an involution in } \ L_r\rangle \leq H',$$

contrary to 16.4.2.2.

It remains only to consider the cases in (E2) where $L \cong L_3(3)$ or L^* is sporadic. Inspecting centralizers of involutory automorphisms of L using 16.1.5, we conclude that $L_r = \Lambda(L_r)$, except in the situation which we already eliminated in 16.4.7. This completes the proof of 16.4.8. □

We now focus on those members of Δ with the property that involutions of R induce inner automorphisms on L. Set

$$\Delta_0 = \Delta_0(K) := \{K' \in \Delta : \Omega_1(R) \leq KL \ \text{ for } \ R \in Syl_2(N_{K'}(K))\}.$$

We will first show in 16.4.9.2 that Δ_0 is nonempty. The shadow of S_{10} is eliminated toward the end of the proof: that is, a transposition in S_{10} is a 2-central involution with centralizer $\mathbf{Z}_2 \times S_8$, such that $\Delta_0 = \emptyset$; of course S_{10} is neither simple nor quasithin.

LEMMA 16.4.9. *(1) If $K' \notin \Delta_0$, then $|R| = 2$.*

(2) $\Delta_0 \neq \emptyset$.

(3) Assume $J \in K^G$, and some involution i in J induces a nontrivial inner automorphism on L. Then $J \in \Delta_0$.

PROOF. First assume (1) and the hypothesis of (3). Then $i \in LK - K$, so $J \neq K$ and $|N_J(K)|_2 > 1$, and hence $J \in \Delta$. Now if $|N_J(K)|_2 > 2$ then $J \in \Delta_0$ since we are assuming (1), while if $|N_J(K)|_2 = 2$, then $\langle i \rangle \in Syl_2(N_J(K))$ with $i \in LK$, so that $J \in \Delta_0$ by definition. Thus (1) implies (3), so it remains to establish (1) and (2).

If $\Delta = \Delta_0$ then (1) is vacuous and (2) holds as Δ is nonempty, so we may assume $K' \in \Delta - \Delta_0$ and pick some involution $r \in R$ inducing an outer automorphism on L. Then by 16.4.8, $L_r \leq L' \leq C_G(R)$.

We now prove (1). By inspection of the centralizers of involutory outer automorphisms of L^* listed in 16.1.4 and 16.1.5, one of the following holds:

(I) $C_{H^*}(L_r^*) = \langle r^* \rangle$.

(II) $L^*T^* \cong S_8$ and r^* is of type $2^3, 1^2$.

(III) $L^* \cong M_{12}$ and $L_r^* \cong A_5$.

In case (I), as R^* centralizes L_r^* and $R \cong R^*$, $R = \langle r \rangle$ is of order 2, and hence (1) holds in this case. Thus we may assume that (II) or (III) holds. In either case, $C_{H^*}(L_r^*) \cong E_4$, so either R is of order 2 and (1) holds, or $R^* = C_{H^*}(L_r^*) \cong E_4$, and we may assume the latter.

Suppose case (II) holds. Then there is $s \in R^\#$ with s^* of type $2, 1^6$. But then $[R^*, L_s^*] \neq 1$, a contradiction since $L_s \leq L' \leq C_G(R)$ by 16.4.8.

Therefore case (III) holds. Then there is $s \in R^\#$ with $s^* \in L^*$ but s^* not 2-central in L^*. Let s_L denote the projection of s on L. If $O_2(L) \neq 1$, then from I.2.2.5b, s_L is of order 4, so $s = s_L s_C$ with $s_C \in N_{T_C}(K')$ of order 4, impossible as $N_{T_C}(K') \cong R \cong R^* \cong E_4$ by 16.4.2.1. Therefore $O_2(L) = 1$, and hence $C_{L^*}(s^*) = C_L(s)^*$, with $C_L(s) \leq N_G(K')$ by 16.4.2.5. Thus $\langle s_L \rangle = [C_{T_L}(s), r] \leq T \cap K' = R$ by 16.4.2.3, and hence $s = s_L \in L$. Then

$$T_C = C_{T_C}(s) \leq N_{T_C}(K') = C_{T_C}(R) \cong R \cong E_4,$$

so $T_C \cong E_4$ centralizes R. Then as $L^*\langle r^* \rangle = Aut(L^*)$, $T = T_C T_L R \leq C_G(T_C)$ so $T_C \leq Z(T)$. Hence R is in the center of each Sylow 2-subgroup of H' containing R. As $C_T(s) \leq H'$ and $[R, C_{T_L}(s)] \neq 1$, this is a contradiction. Thus (1) is established.

We may assume that (2) fails, and it remains to derive a contradiction. Now $R = \langle r \rangle$ is of order 2 by (1). By 16.4.4.1, $C_{T_C}(r) = \langle z \rangle$ is of order 2, and $T_C R$ is dihedral or semidihedral. Set $E := \langle r, z \rangle$. By (1) and (3):

$$\text{If } J \in \Delta, \text{ then } |N_J(K)|_2 = 2 \text{ and } J \cap KL \text{ is of odd order.} \qquad (*)$$

By 16.4.3.1, $K \in \Delta(K')$, so we have symmetry between K and K'. Thus applying (*) with the roles of K and K' reversed, we conclude that z induces an outer automorphism on L'.

Next we show:

(+) If $r^l = rv$ for some $l \in L$ and $1 \neq v \in C_L(r)$, then r^l acts on K' with $\langle r \rangle \in Syl_2(C_{K'}(r^l))$ and $v \notin L'$.

For assume the hypotheses of (+), and let $J := (K')^l$. Then $r^l \in C_J(r) \leq N_J(K')$, and as R has order 2, $r^l \notin K'$, so that $J \neq K'$. Thus $J \in \Delta(K')$, and then by (*), $R \in Syl_2(N_{K'}(J))$, and also $J \cap K'L'$ is of odd order so that $v \notin L'$. As $\langle r \rangle = R \in Syl_2(N_{K'}(J))$ and $C_G(r^l) \leq N_G(J)$, also $\langle r \rangle \in Syl_2(C_{K'}(r^l))$, completing the proof of (+).

As r induces an outer automorphism on L, by 16.1.6, r does not centralize T_L unless L is A_6. In the first case, there is $l \in T_L$ with $r \neq r^l \in C_G(r)$, and in the second by inspection there is $l \in L$ with this property; thus in any case there is $l \in L$ with $r \neq r^l \in C_G(r)$. Let $J := (K')^l$.

Suppose that $T_C > \langle z \rangle$. Then r^l normalizes some subgroup $X = \langle x, r \rangle$ of order 4 in K', and $\langle r \rangle \in Syl_2(C_{K'}(r^l))$ by (+), so that $\langle r^l, X \rangle \cong D_8$, and hence $v := rr^l = r^{lx} \in J^x \cap L$. Thus $J^x \notin \Delta$ by (*), so $J^x = K$ and hence $v \in K \cap L = Z(L)$, so $z = v \in L$ as $\langle z \rangle = C_{T_C}(r)$. Further $r^L \cap C_G(r) = \{r, rz\}$, so $|C_{L^*}(r^*) : C_L(r)^*| = 2$ and $r^* C_L(r)^* \cap r^{*L^*} = \{r^*\}$, so there are involutions in $C_{L^*}(r^*) - C_L(r)^*$. Suppose $L^* \cong L_3(4)$ or M_{22}. Examining 16.1.4 and 16.1.5 for outer automorphisms r with $C_{L^*}(r)$ not perfect, we conclude that either $L^* \cong L_3(4)$, r is a graph-field automorphism, and $C_{L^*}(r^*) \cong Q_8/E_9$; or $L^* \cong M_{22}$ and $C_{L^*}(r^*) \cong \mathbf{Z}_4/\mathbf{Z}_5/E_{2^4}$. However in both cases each subgroup of $C_{L^*}(r^*)$ of index 2 contains all involutions in $C_{L^*}(r^*)$, contrary to an earlier remark.

We have shown that either $\langle z \rangle = T_C \in Syl_2(K)$, or $z \in L$ and L^* is not $L_3(4)$ or M_{22}.

As r induces an outer automorphism on L, $L_r \leq L'$ by 16.4.8. So if $rv = r^l$ for some $l \in L$ and $1 \neq v \in L_r$, then $v \in L'$ which is contrary to (+). Thus:

$$r^L \cap rL_r = \{r\}. \tag{!}$$

It is now fairly easy to eliminate most possibilities for the involutory outer automorphism r on L in 16.1.4 and 16.1.5; indeed the next few paragraphs will be devoted to the reduction to the following cases:

(i) $L \cong A_6$ or A_8.
(ii) $L^* \cong L_3(4)$, and r^* induces a graph-field automorphism on L^*.

We may assume that neither (i) nor (ii) holds, and will derive a contradiction. Since r induces an outer automorphism on L, L is not $L_3(2)$ by 16.4.6. Then as (ii) does not hold, by inspection of the outer automorphisms in the remaining cases in 16.1.4 and 16.1.5, L_r is of even order. Choose notation so that $C_T(r) \in Syl_2(C_H(r))$.

Suppose first that $L \cong M_{22}$ or HS, and let $T_r := T \cap RC_L(r))$; then $Z(T_r) = R \times Z_r$, where $\mathbf{Z}_2 \cong Z_r = \langle v \rangle \leq L_r$. So as $T_r < RT_L \in Syl_2(RL)$, $rv \in r^{N_{T_L}(T_r)}$, contrary to (!). Thus if $L^* \cong M_{22}$ or HS, we may assume that $O_2(L) \neq 1$.

Now assume that $O_2(L) = 1$, and recall L is not A_6, A_8, M_{22}, or HS by assumption. Therefore by inspection of the outer automorphisms in the possibilities remaining in 16.1.4 and 16.1.5, L is transitive on the involutions in rL. Then since L_r is of even order (recalling $L^* \ncong L_3(4)$ as we are assuming that (ii) fails), (!) supplies a contradiction.

Thus to establish our reduction, it remains to treat the case $1 \neq O_2(L) = Z(L)$. Since L^* admits the outer automorphism r^* of order 2, we conclude from 16.1.2.1 that L^* is $L_3(4)$, $G_2(4)$, M_{12}, M_{22}, J_2, or HS. If $|T_C| = 2$, then $\langle z \rangle = Z(L)$. If

$|T_C| > 2$, we showed $z \in L$ and L^* is not $L_3(4)$ or M_{22}. So in any case $z \in Z(L)$; it then follows by 16.1.2.2 that $Z(L) = \langle z \rangle$ is of order 2.

Suppose first that $L^* \cong L_3(4)$, and recall we are assuming that r^* does not induce a graph-field automorphism on L^*. Assume that r^* induces a field automorphism on L^*, so that $L_r \cong L_3(2)$ by 16.1.4.4. Let $L_{r,1}$ be a maximal parabolic of L_r; then there is an r-invariant maximal parabolic L_1 of L with $L_{r,1} \leq L_1$ and $O_2(L_{r,1}) \leq O_2(L_1)$. Then $O_2(L_1)$ is transitive on $rO_2(L_{r,1})$, contrary to (!). Therefore by 16.1.4, r^* induces a graph automorphism on L^*, so $L_r \cong L_2(4)$ by 16.1.4.6. Let $U := T \cap L_r \cong E_4$ and $V := EU$; thus $V \cong E_{16}$ and as $Z(L) = \langle z \rangle$, $N_{T_L}(V)/V \cong E_4$ induces a group of $\mathbf{F}_2$-tranvections on V with axis $V_0 := \langle z, U \rangle \cong E_8$. Now $R \leq T \leq N_G(T_L)$, so that $[N_{T_L}(V), r] =: W \leq U$ is a hyperplane of V_0, and hence $W = U$. Then for $1 \neq v \in W$, $rv \in r^L$, contrary to (!). This establishes the reduction to (ii) when $L^* \cong L_3(4)$.

Next suppose that $L^* \cong G_2(4)$, M_{12}, J_2, or HS. Then $|Z(L)| = 2$, so $Z(L) = \langle z \rangle$; and from I.2.2.5b, r inverts y of order 4 in L with $y^2 = z$, so $r^y = rz$. Hence the involutions in $rZ(L)$ are fused under L: for example, from the description of $C_{L^*}(r^*)$ in 16.1.4 or 16.1.5, and observing that $C_{L^*}(r^*) \cong G_2(2)$ when $L^* \cong G_2(4)$, choose $y^* \in C_{L^*}(r^*) - L_r^*$, and observe $y^*r^* \in r^{*L}$, so r inverts y as required. But we showed earlier that $r^*v^* = r^{*l}$ for some involution $v^* \in L_r^*$ and $l \in L$, and now from I.2.2.5a, we may choose a preimage v to be an involution. So as $r^l \in rvZ(L)$ and the involutions in $rZ(L)$ are fused, $rv \in r^L$, again contrary to (!).

This leaves the case $L^* \cong M_{22}$, where in view of 16.1.5 and 16.4.7, $C_{L^*}(r^*) = L_r^* \cong L_3(2)/E_8$. Choose an involution s^* inducing an outer automorphism of the type in 16.4.7, with $r^* \in s^*O_2(L_s)$; then as we saw in the proof of 16.4.7, since $Z(L) = \langle z \rangle$ is of order 2, the group $O_2(L_s^*) \cong E_{16}$ splits over $Z(L)$, so $J(C_T(r))$ is the group V of rank 6 in the proof of 16.4.7, now with s in the role of "r". Now the argument in the final paragraph of that proof again supplies a contradiction. This finally completes the proof of the reduction to (i) or (ii).

Thus to prove (2), it remains to eliminate the groups in parts (i) and (ii) of the claim. Choose T so that $N_T(R) \in Syl_2(N_H(R))$. As in 16.4.2.4, choose $g \in G$ with $K^g = K'$ and $N_T(K') \leq T^g$.

Suppose first that (ii) holds. Then $L_r \cong E_9$ by 16.1.4.5, and $N_{LE}(E) = L_rQ$, where $Q/E \cong Q_8$ and $\langle z, r \rangle = E = C_Q(L_r)$.

Assume that $O_2(L) \neq 1$. Then since $T_C = \langle z \rangle$ is of order 2, $\langle z \rangle = Z(L)$. Also $C_Q(E)/E \cong \mathbf{Z}_4$ is irreducible on L_r, and Q induces a transvection on E with center $\langle z \rangle$. By symmetry, Q^g induces a transvection on E with center $\langle r \rangle$, so $X := \langle Q, Q^g \rangle$ induces S_3 on E. Since $N_T(R) \in Syl_2(N_H(R))$, $C_T(E) \in Syl_2(C_G(E))$ by 16.4.2.6, so by a Frattini Argument, $Y := O^2(N_G(E) \cap N_G(C_T(E)))$ is transitive on $E^\#$. Hence as $L_r \leq L'$,

$$L_r = O^2\Big(\bigcap_{y \in N_G(E)} L^y\Big) \trianglelefteq N_G(E),$$

and then as $C_T(E)$ is irreducible on L_r, either $[L_r, Y] = 1$ or Y induces $SL_2(3)$ on L_r. In the former case as $C_T(E)$ is irreducible on L_r, there is an element of order 3 in $Y - L_r$ centralizing L_r, contradicting G quasithin. In the latter, $N_T(E)Y/E \cong GL_2(3)$ has a unique Q_8-subgroup, so that $Q \trianglelefteq N_T(E)Y$, impossible as $Aut_Q(E)$ is not normal in $Aut_{QY}(E)$.

Therefore $O_2(L) = 1$, so $L \cong L_3(4)$. Then Q centralizes E, and there is a complement P to E in Q. Let $\bar{H}' := H'/K'$. Each Q_8-subgroup of $C_{\bar{H}'}(\bar{z})$ is contained in $\bar{L}'\langle\bar{z}\rangle$, so that $P \leq T^g \cap K'L'E \leq L'E$. Hence $1 \neq \Phi(P) \leq L'$. This contradicts (+) as $\Phi(P) = \langle v\rangle$ where v is an involution in $C_L(r)$ with $rv \in r^L$.

Thus case (i) holds, so $L \cong A_6$ or A_8. Since $R = \langle r\rangle$, $T_C = \langle z\rangle$, and $N_T(K') \leq T^g$, $r = z^g$. Further r induces an outer automorphism on L, and RL is not $PGL_2(9)$ by 16.4.6, so we conclude that $z^G \cap LT \subseteq LE$ and $LE = LR \times \langle z\rangle \cong S_n \times \mathbf{Z}_2$, with $n := 6$ or 8. Represent LE on $\Omega := \{1, \ldots, n\}$ with kernel $T_C = \langle z\rangle$. Observe that either $LT = LE$ and $T = T_LE$, or $H^* \cong Aut(A_6)$ and $|LT : LE| = |T : T_LE| = 2$.

By (*), $z^G \cap L\langle z\rangle = \{z\}$, so since $z^G \cap LT \subseteq LE$, we conclude $z^G \cap LT \subseteq \{z\} \cup rL \cup rzL$. If $n = 6$, we may choose notation so that r induces a transposition on Ω, and if $n = 8$, r is either a transposition, or of type $2^3, 1^2$. In any case, setting $m := n/2 + 1$, there is a subgroup A_0 of T_LR generated by a set of $m - 1$ commuting transpositions, and by the $m - 1$ conjugates of r under L in A_0. Thus there is $E_{2^m} \cong A \leq LE$ with $\alpha := \{z\} \cup (r^L \cap A)$ of order m.

Let a, b, c be a triple from α. Then $c := z^x$ for suitable $x \in G$; set $X := L^x\langle z^x\rangle$. By the previous paragraph,

$$z^G \cap X = \{z^x\}. \qquad (**).$$

Then $a, b \in L^xA - X$, so as $z^G \cap L^xT^x \subseteq L^xE^x$ and $|(LE)^x : X| = 2$, $ab \in X$. Hence by (**), neither ab nor abz^x is in z^G. Thus no product of two or three members of α is in z^G.

Now assume $n = 8$, let $\alpha = \{a_1, \ldots, a_5\}$, and take $a_5 = z^x$. By the previous paragraph, a_1a_2 and a_3a_4 are in X, so $a_1a_2a_3a_4 \in X$. Hence by (**), neither $a_1a_2a_3a_4$ nor $a_1a_2a_3a_4z^x = a_1a_2a_3a_4a_5$ is in z^G. Thus no product of four or five members of α is in z^G, so $z^G \cap A = \alpha$. But each involution in T is fused into A under L, so we conclude that $z^G \cap \langle rz\rangle L = \emptyset$, and hence $z \notin O^2(G)$ by Thompson Transfer, [1] contrary to the simplicity of G.

Therefore $n = 6$. Set $\alpha = \{a_1, \ldots, a_4\}$ and $z^G \cap T =: \beta$. First assume $H^* \ncong Aut(A_6)$. Then $LT = LE = LA$ and $T = T_LR \times \langle z\rangle$. If $t := a_1a_2a_3a_4 \notin z^G$, then the argument of the previous paragraph supplies a contradiction, so $t \in z^G$. Hence $\beta = \alpha \cup \{t\}$ is of order 5. But $\mathcal{A}(T)$ is of order 2, while the member A of $\mathcal{A}(T)$ is normal in T, so A is weakly closed in T with respect to G. Hence by Burnside's Fusion Lemma A.1.35, $N_G(A)$ is transitive on β. Then as $N_{G_z}(A)/A$ induces S_3 on β, $Aut_G(A)$ is a subgroup of S_5 of order 30, a contradiction.

Therefore $H^* \cong Aut(A_6)$. Thus $\mathcal{A}(T) = \{A, A^s\}$ for $s \in T - AT_L$, so $C_G(z)$ is transitive on $\mathcal{A}(C_G(z))$; hence by A.1.7.1, $N_G(A)$ is transitive on β. In particular as $|N_T(A) : A| = 2$, $|\beta| \neq 4$, so $t \in \beta$ and $|\beta| = 5$, for the same contradiction as at the end of the previous paragraph.

This finally completes the proof of 16.4.9. □

In view of 16.4.9:

In the remainder of this chapter, we choose $K' \in \Delta_0$. Thus $\Omega_1(R) \leq KL$, where $R \in Syl_2(N_{K'}(K))$.

LEMMA 16.4.10. *$R \in Syl_2(K')$.*

PROOF. This is more or less the argument on page 101 of [**Asc76**]: By 16.4.2.1, $R \cong N_{T_C}(K') = C_{T_C}(R) =: S$. Assume $R \notin Syl_2(K')$; then also $S < T_C$, so in

[1] Here we are in particular eliminating the shadow of S_{10}.

particular T_C does not centralize R. For $1 \neq r \in R$, observe by 16.4.2.5 that $C_{T_C}(r) \leq N_{T_C}(K') = C_{T_C}(R) = S$, so that $C_{T_C}(r) = S$. By parts (4) and (6) of I.7.7, $S \trianglelefteq T_C$ and R is abelian.

Set $R_0 := R \cap LK$, $U := \Omega_1(R)$, and recall $K' \in \Delta_0$ so that $U \leq R_0$. In particular $R_0 \neq 1$. For $r \in R_0$, we can write $r = s(r)l(r)$ with $s(r) \in T_C$ and $l(r) \in L$. Observe that $l(r)$ is determined up to an element of $L \cap K = Z(L)$, and then $s(r)$ is determined by $l(r)$. Also $s(r)Z(L) \subseteq C_{T_C}(s(r)) \leq C_{T_C}(r)$. Then as $C_{T_C}(r) = S < T_C$, $s(r)Z(L) \subseteq S$, but $s(r) \notin Z(T_C)$. Indeed $\hat{s}(r) := s(r)Z(L)$ is a uniquely determined element of $\hat{S} := S/Z(L)$, and $\hat{s} : R_0 \to \hat{S}$ is a group homomorphism. Also $R \cap L = 1$ as $C_{T_C}(r) = S < T_C$ for $r \in R^{\#}$, so $\ker(\hat{s}) = R_0 \cap Z(L) = 1$ and hence $\hat{s}$ is injective. For $R_1 \leq R_0$, let $s(R_1)$ be the preimage in S of $\hat{s}(R_1)$. As $s(r) \notin Z(T_C)$ for each $r \in R^{\#}$, $s(U) \cap Z(T_C) = 1$.

As R is abelian, U is elementary abelian. Suppose $s(U)$ is elementary abelian. Then since $s(U)/Z(L) \cong \hat{s}(U) \cong U \cong \Omega_1(S) \geq s(U)$, we conclude that $Z(L) = 1$ and the map $s : U \to \Omega_1(S)$ is an isomorphism. So as $S \trianglelefteq T_C$, we have a contradiction to $s(U) \cap Z(T_C) = 1$.

Thus for some $u \in U^{\#}$, we may choose $s(u)$, and hence also $l(u)$, of order at least 4. Thus $1 \neq l(u)^2 \in Z(L)$, so $Z(L) \neq 1$. Therefore L^* is not $L_3(4)$, since $Z(L)$ is elementary abelian, and from I.2.2.3b, involutions in L^* lift to involutions in L. Hence by inspection of the remaining groups in 16.1.2.1, $Out(L)$ has Sylow 2-groups of order at most 2, so $|R : R_0| \leq 2$. Therefore

$$|R| = |S| \geq |\hat{s}(R_0)||Z(L)| = |R_0||Z(L)| \geq |R||Z(L)|/2.$$

So as $|Z(L)| \geq 2$, all inequalities are equalities, and hence $S = s(R_0)$ with $Z(L) = \langle l(u)^2 \rangle$ of order 2. Thus there is $1 \neq v \in U$ with $E := \langle s(v) \rangle Z(L)$ a normal subgroup of T_C of order 4. Then $|T_C : C_{T_C}(s(v))| = |T_C : C_{T_C}(v)| = 2$, so $S = C_{T_C}(s(v))$ is of index 2 in T_C. Thus $RS \trianglelefteq RT_C$, so as $N_{T_C}(R) = S \cong R$ is abelian, while R is a TI-set in $T_C R$ under $N_G(T_C R)$ by I.7.2.3, $SR = R \times R^t$ for $t \in T_C - S$. Therefore $\Omega_1(S) = C_{UU^t}(t) = [U, t]$, and hence $s(w) \in Z(T_C)$ for each $w \in U$, contrary to $s(U) \cap Z(T_C) = 1$. This contradiction completes the proof. □

The next lemma summarizes some fundamental properties of members of Δ_0; in particular it shows that Δ_0 defines a symmetric relation on K^G.

LEMMA 16.4.11. *(1) R centralizes $T_C \cong R$. In particular, $T_C \leq H'$.*
(2) $K \in \Delta_0(K')$.
(3) If we choose g as in 16.4.2.4, then $R = T_C^g$.

PROOF. By 16.4.10, $R \in Syl_2(K')$, so that $R \cong T_C$. Then since $R \cong C_{T_C}(R)$ by 16.4.2.1, T_C centralizes R, and so (1) holds using 16.4.2.5.

By 16.4.3.1, $K \in \Delta(K')$. Suppose that $K \notin \Delta_0(K')$. Then by 16.4.9.1, $|T_C| = 2$, and by 16.4.9.1, z induces an outer automorphism on L'. Applying 16.4.8 with the roles of K and K' reversed, we conclude that $L_z := O^2(C_{L'}(z)) \leq L$. As $C_G(r) \leq N_G(L')$, we conclude that $L_z \trianglelefteq L_r$. Comparing the fixed points of outer and inner automorphisms of order 2 in 16.1.4 and 16.1.5, we conclude $L^* \cong M_{12}$ and $L_r^* = L_z^* \cong A_5$. As r induces an inner automorphism on L, if $Z(L) \neq 1$, then I.2.2.5b says that the projection r_L of r on L is of order 4, so $r = r_C r_L$ with $r_C \in T_C$ of order 4, contradicting $|T_C| = 2$. Thus $Z(L) = 1$, so $C_L(r) \cong \mathbf{Z}_2 \times S_5$. However $C_L(r) \leq C_{H'}(z)$ by 16.4.2.5, and as z induces an outer automorphism on L', no element of $C_{H'}(z)$ induces an outer automorphism on L_z. Therefore (2) holds.

Choose g as in 16.4.2.4; that is, so that $N_T(K') \leq T^g$. Then $R \leq T_C^g \leq K'$ and $R \in Syl_2(K')$, so $R = T_C^g$, establishing (3). □

16.5. Identifying $\mathbf{J_1}$, and obtaining the final contradiction

In this final section, we first see that $G \cong J_1$ when $L/Z(L)$ is a Bender group. Then we eliminate all other possiblities for L appearing in (E2), to establish our main result Theorem 16.5.14.

Recall R is faithful on L by 16.4.2.1 and $H^* = H/K$. Set $U := \Omega_1(R)$; as $K' \in \Delta_0$,

$$U \leq KL.$$

Let u denote an element of $U^\#$, and set $U_C := \Omega_1(T_C)$. By 16.4.11.3, if we choose g as in 16.4.2.4, then $R = T_C^g$ and hence $U = U_C^g$; in particular, $U_C \in U^G$.

Proposition 16.5.1. *If L^* is a Bender group, then $G \cong J_1$.*

Proof. By hypothesis, $L^* \cong L_2(2^n)$, $U_3(2^n)$, or $Sz(2^n)$. Set $U_L := \Omega_1(T_L)$. Then there is $X \leq N_L(T_L)$ with $X \cong \mathbf{Z}_{2^n-1}$ and X regular on $U_L^{*\#}$. Now either $Z(L) = 1$, or from 16.1.2, $L^* \cong Sz(8)$, with $U_L^* = \Omega_1(T_L^*)$ from I.2.2.4. Thus as $U \leq KL$,

$$U \leq U_C U_L =: V = U_C \times [V, X],$$

with X regular on $[V, X]^\#$ and $U_C = C_V(X)$.

We claim that there is $g \in M := N_G(V)$ with $K^g = K'$. Suppose first that $L^* \cong L_2(2^n)$ or $Sz(2^n)$. Recall by 16.4.6 that if $L \cong L_2(4) \cong L_2(5)$, then no involution in U induces an outer automorphism on L, so that $V = \langle U^G \cap T\rangle$. In the remaining cases, no outer automorphism of L is an FF^*-offender on U_L, so that $V = J(T)$. Thus in each case, V is weakly closed in T with respect to G, so by Burnside's Fusion Lemma A.1.35, M controls fusion in V, and hence there is $g \in M$ with $U_C^g = U$ and hence $K^g = K'$, as claimed. So we may assume $L \cong U_3(2^n)$. Choose g as in 16.4.2.4, so that $N_T(K') \leq T^g$, and as observed earlier, $R = T_C^g$ and $U = U_C^g$. Fix $u \in U^\#$. Set $Y_u := O^3(L_u)$ if $n \neq 3$ and $Y_u := L_u$ if $n = 3$; then $Y_u \leq L'$ by 16.4.8. But in either case $T_L \leq Y_u$, so $T_L \leq L'$ and hence $T_L = T_L^g$ as $N_T(K') \leq T^g$. Thus $U_L = U_L^g$, so as $U = U_C^g$,

$$V^g = (U_L U_C)^g = U_L U_C^g = U_L U = V,$$

completing the proof of the claim.

Pick g as in the claim, and set $M^+ := M/C_G(V)$. As $U_C = V \cap K$ and K is tightly embedded in G, U_C is a TI-set in V under the action of M by I.7.2.3. Further $X^+ = N_L(V)^+ \trianglelefteq (M \cap H)^+ = N_{M^+}(U_C)$ since $N_G(T_C) \leq H$ by 16.4.2.5, and X^+ is regular on $[V, X]^\#$, while $T^+ \in Syl_2(M^+)$ acts on X^+. So we have a Goldschmidt-O'Nan pair in the sense of Definition 14.1 of [**GLS96**]; hence we may apply O'Nan's lemma Proposition 14.2 in [**GLS96**] with M^+, X^+, V in the roles of "X, Y, V": Observe that conclusion (i) of that result does not hold, since there M normalizes U_C—whereas here $g \in M - N_G(U_C)$. Similarly conclusions (ii) and (iii) of that result do not hold, since here T normalizes U_C, but does not normalize U_C there. Thus conclusion (iv) of that result holds: $m(U_C) = 1$, $m(V) = 3$, and $M^+ \cong Frob_{21}$. In particular, $n = 2$, so that $L \cong L_2(4)$ or $U_3(4)$. But the latter case is impossible, since we saw in the unitary case that g normalizes T_L, so that $M^+ = \langle X^+, X^{g+}\rangle$ acts on $\Phi(T_L) = U_L$, whereas M^+ is irreducible on V.

Thus $L \cong L_2(4)$. Then $H = LK$ by 16.4.6, so R induces inner automorphisms on L. Recall R is faithful on L, so R is elementary abelian; hence as $|U_C| = 2$, $U_C = T_C$ is of order 2. Therefore $C_K(z) = T_C$ by 16.4.4.2, so that $G_z = L \times T_C \cong L_2(4) \times \mathbf{Z}_2$, and hence G is of type J_1 in the sense of I.4.9. Then we conclude from that result that $G \cong J_1$. □

In the remainder of the chapter, assume G is not J_1; therefore by Proposition 16.5.1, L^* is not a Bender group. To complete the proof of our main result Theorem 16.5.14, we must eliminate each remaining possibility for L in (E2).

Recall from 16.4.3.2 that $L' = [L', z]$, and that z induces an inner automorphism on L' since $K \in \Delta_0(K')$ by the symmetry in 16.4.11.2.

LEMMA 16.5.2. *(1) If* $R \cap Z(T) \neq 1$ *and* g *is chosen as in 16.4.2.4, then* $g \in N_G(T)$.

(2) Assume u, z *are involutions in* R, T_C *whose projections on* L, L^g *are 2-central, and that* $|T : T_C T_L| \leq 2$. *Then either*

(a) $R \cap Z(T_1) \neq 1$ *for some* $T_1 \in Syl_2(H)$, *and we may choose* T_1 *so that* $R \trianglelefteq T_1 \in Syl_2(H \cap H')$, *or*

(b) $T_C T_L^l =: T_0 \in Syl_2(H \cap H')$ *for some* $l \in L$, *with* $R \trianglelefteq T_0$, $|T_0| = |T|/2$, *and there exists* $g \in N_G(T_0)$ *with* $K^g = K'$.

PROOF. Assume that $R \cap Z(T) \neq 1$, and that g is chosen as in 16.4.2.4. Now $T \leq C_G(R \cap Z(T)) \leq N_T(K')$ using 16.4.2.5, so $T^g = T$ as $N_T(K') \leq T^g$ by the choice of g. Thus (1) holds.

Assume the hypotheses of (2). Then u centralizes T_L^l for some $l \in L$, so $T_L^l \leq H'$ by 16.4.2.5. Also $T_C \leq H'$ by 16.4.11.1, so $T_0 := T_C T_L^l$ acts on some $R_1 \in Syl_2(K' \cap H)$. But by 16.4.10, $R \in Syl_2(K')$, so by Sylow's Theorem there is $x \in K' \cap H$ with $R_1^x = R$, and thus $T_2 := T_0^x$ acts on R. Let $T_2 \leq T_1 \in Syl_2(H)$. By hypothesis $|T_1 : T_2| \leq 2$, so either $R \trianglelefteq T_1$, or $T_2 = N_{T_1}(R) \in Syl_2(H \cap H')$. In the former case, $R \cap Z(T_1) \neq 1$ and conclusion (a) of (2) holds, so we may assume the latter. Thus $T_0 = T_2^{x^{-1}} \in Syl_2(H \cap H')$. By 16.4.11.2 we have symmetry between K and K', so there is $S \in Syl_2(K'L')$ with S Sylow in $H \cap H'$. Thus there is $h \in H \cap H'$ with $T_0^h = S$, so as h acts on $K'L'$, T_0 is Sylow in $K'L'$. Let $y \in G$ with $K^y = K'$; then T_0 and T_0^y are Sylow in $K'L'$, so there is $w \in K'L'$ with $T_0^{yw} = T_0$, and hence conclusion (b) of (2) holds with $g := yw$. □

We now begin the process of eliminating the possibilities for L remaining in (E2). Let u denote an involution in U, and recall $z \in T_C \cap Z(T)$. Also $R \cap K = 1$ by 16.4.2.1, so that by 16.4.11.1,

$$R^* \cong R \cong T_C.$$

In particular as $U \leq LK$,

$$U \cong U^* \leq T_L^*.$$

LEMMA 16.5.3. *L is not* A_6.

PROOF. Assume otherwise. Then $U \cong U^* \leq T_L^* \cong D_8$, and hence $U \cong \mathbf{Z}_2$, E_4 or D_8. Now in the notation of Definition F.4.41, $X := \Gamma_{1,U}(L) \leq H'$ using 16.4.2.5. But if $U^* \cong D_8$, then $X = L$, contrary to 16.4.2.2. Assume $U^* \cong E_4$. Then $X \cong S_4$ with $O_2(X^*) = U^*$; so as X acts on K' while $U = \Omega_1(R)$, X acts on $K' \cap O_2(XU) = U$, and hence $3 \in \pi(Aut_{H'}(U))$. Then as $Out(L')$ is a $3'$-group, $3 \in \pi(N_{K'}(U))$, so that $m_{2,3}(H) > 2$, contradicting G quasithin. Hence

$U = \Omega_1(R)$ is of order 2, so as $R \cong T_C$, $m_2(R) = 1 = m_2(T_C)$. Recall u denotes the involution in U and z the involution in T_C. The projection v of u on L is 2-central in LT, so conjugating in L if necessary, without loss $v \in Z(T)$, and then $u \in \langle z, v\rangle =: E \leq Z(T)$. Now we may choose g as in 16.4.2.4, so that $u = z^g$ by 16.4.11.3, and $g \in N_G(T)$ by 16.5.2.1. Now

$$[T, T] = Y_L \times Y_C, \qquad (*)$$

where Y_C is the preimage in T_C of $[T/T_L, T/T_L]$, and Y_L is of index at most 2 in the cyclic subgroup Y of order 4 in T_L.

Assume that $Y_C = 1$; that is, that T/T_L is abelian. Set $Z := \Omega_1(Z(T))$. Then either $Z = E$, or $Z = E\langle t\rangle$ where t induces a transposition on L. Further $N_G(T)$ centralizes $[T, T] \cap Z = \langle v\rangle$ by (*). However g does not centralize Z since $u = z^g$, so $g \notin T$, and hence we may assume g has odd order. As $g \in N_G(T)$, g centralizes v, so $Z = E\langle t\rangle = \langle v\rangle \times [Z, N_G(T)]$ where $[Z, N_G(T)]$ is of rank 2, and g induces an element of order 3 on Z. But then either z or $u = zv$ lies in $[Z, N_G(T)]$, so as $u = z^g$, we conclude $z \in [Z, N_G(T)]$, and then $zz^g = z(zv) = v \in [Z, N_G(T)]$, whereas we saw $N_G(T)$ centralizes v.

Therefore $Y_C \neq 1$. Then as $m_2(T_C) = 1$, $\langle z\rangle = \Omega_1(Y_C)$, so by (*), $\Omega_1([T, T]) = E \leq \Omega_1(Z(T))$. Therefore as $g \in N_G(T)$, g induces an element of order 3 on E, and $N_G(T)$ is transitive on $E^{\#}$. Thus as Y_L is cyclic, so is Y_C by the Krull-Schmidt Theorem A.1.15, and then $Y_C \cong Y_L$, so that Y_C is cyclic of order at most 4.

Next $Y_C \trianglelefteq T$, so $Y_C^g \trianglelefteq T$. Now $s \in T_L - Y$ inverts Y and centralizes Y_C, so if $|Y_C| = 4$, then s does not act on Y_C^g, a contradiction.

Thus $Y_C = \langle z\rangle$, so $Y_L = \langle v\rangle$ as $Y_L \cong Y_C$. As $Y_L = \langle v\rangle$, L^*T^* is A_6 or S_6. Assume $L^* \cong A_6$. Then $T = T_L \times T_C$, so $T_C \cong Q_8$ since $m_3(T_C) = 1$ and $\langle z\rangle = Y_C \cong [T/T_L, T/T_L] \cong [T_C, T_C]$. But $R = T_C^g$, so $Q_8 \cong T_C \cong R \cong R^* \leq T^* = T_L^*$, impossible as $T_L^* \cong D_8$. Therefore $L^*T^* \cong S_6$, so $T^* \cong \mathbf{Z}_2 \times D_8$. Then as $T_C \cong R \cong R^* \leq T^*$, while $m_2(T_C) = 1$ and $[T, T_C] \neq 1$, we conclude that $T_C \cong \mathbf{Z}_4$ and $t \in T - T_C T_L$ inverts T_C. As $T^* \cong \mathbf{Z}_2 \times D_8$, we may pick t so that t centralizes T_L and $t^2 \in T_C$. Let $T_1 := T_C\langle t\rangle$; then $T = T_1 \times T_L$, with $T_1 \cong D_8$ or Q_8, and $T_L \cong D_8$. Now $g \in N_G(T)$ is transitive on $E^{\#}$; but this is impossible, as by the Krull-Schmidt Theorem A.1.15, $N_G(T)$ permutes $\{\Phi(T_1 Z(T)), \Phi(T_L Z(T))\}$. □

LEMMA 16.5.4. *L^* is not $L_3(4)$.*

PROOF. Assume otherwise. As $U^* \leq T_L^*$ and all involutions in L are 2-central in L from I.2.2.3b, u centralizes a Sylow 2-group of L. Then as R centralizes T_C, we may take $u \in Z(T_L T_C)$.

Suppose first that $U^* \not\leq Z(T_L^*)$. Then $Y := \Gamma_{1,U}(L)$ contains a maximal parabolic P of L, and $Y \leq H'$ by 16.4.2.5. If $P \leq L'$, then $P \leq C_G(U)$, so $U^* \leq C_{L^*}(P^*) = 1$, a contradiction. Thus $P \not\leq L'$, so K' has an $L_2(4)$-section, and hence $m_{2,3}(H') > 2$, contradicting G quasithin.

Therefore $U^* \leq Z(T_L^*)$, so $U \cong \mathbf{Z}_2$ or E_4. Now $J(T) = J(T_L T_C) = T_L J(T_C) = T_L U_C$, where $U_C = \Omega_1(T_C) \cong U$. As $u \in Z(T_C T_L)$, $u \in Z(J(T))$. Recall $U_C \in U^G$ so there is $g \in G$ with $U^g = U_C$. Thus $u^g \in U_C \leq Z(J(T))$, and by Burnside's Fusion Lemma A.1.35, we may take $g \in M := N_G(J(T))$. Hence g acts on $Z(J(T)) =: V$, where $V = U_C \times V_L$, with $V_L := [V, X] \cong E_4$ for X of order 3 in $N_L(T_L)$. Now we argue, just as in the proof of Proposition 16.5.1, that X is regular in $V_L^{\#}$, and U_C is a TI-set under M, so again by Proposition 14.2 in [**GLS96**], conclusion (iv) of that result holds: namely, $m(V) = 3$ and $Aut_M(V) \cong Frob_{21}$. But then M acts irreducibly on V, impossible as M acts on $\Phi(J(T)) = V_L$. □

NOTATION 16.5.5. For $u \in U^{\#}$, define $X_u := O^3(L_u)$ if $L \cong L_3(2^n)$, n even, and $X_u := L_u$ otherwise. In the former case, $n > 2$ by 16.5.4. Thus in any event, $X_u \leq L^g \leq C_G(R)$ by 16.4.8, so $R^* \leq C_{H^*}(X_u^*)$. Further for i an involution in T_C, we can define $X_i \leq L^g$ analogously. Then $X_u \leq X_i$ as $X_u \leq C_{L^g}(i)$, so by symmetry between L, u and L^g, i, $X_i \leq X_u$. Thus we may define $X := X_u = X_i$.

Inspecting the possibilities for L^* remaining in (E2) after 16.5.1, 16.5.3, and 16.5.4, we conclude from 16.1.4 and 16.1.5 that for each involution j^* in L^*, $X_j \neq 1$ except when $L \cong Sp_4(2^n)$ with $n > 1$, and j^* is of type c_2.

Observe that the fourth part of the next lemma supplies another assertion about the symmetry between K and K'.

LEMMA 16.5.6. *Let $\bar{H}' := H'/K'$.*

(1) $X = X_u = X_i$ for all involutions $i \in T_C$ and $u \in R$.

(2) $R^* \leq C_{L^*T^*}(X^*)$.

(3) If we choose choose g as in 16.4.2.4, then $g \in N_G(X)$.

(4) The following are equivalent:

(a) Some involution in R^* is 2-central in L^*.

(b) Each involution in $\bar{T}_C$ is 2-central in $\bar{L}'$.

(c) Some involution in $\bar{T}_C$ is 2-central in $\bar{L}'$.

(d) Each involution in R^* is 2-central in L^*.

(5) Assume that $Z(L) = 1$, and for each $J \in \Delta_0$ and each involution i in $N_J(K)$, that i^* is not 2-central in L^*. Let v be the projection of u on L, and suppose there is $l \in L$ with vv^l an involution of X. Then $\bar{v}\bar{v}^l$ is not 2-central in $\bar{L}'$.

PROOF. We already observed that (1) and (2) hold. We saw in 16.4.11.3 that if we choose g as in 16.4.2.4, then $T_C^g = R$, so (1) implies (3).

Suppose u^* is 2-central in L^*. By (1), for each involution $i \in T_C$, $X_u = X_i$, so by inspection of the centralizers of involutions of H^* listed in 16.1.4 and 16.1.5 remaining after 16.5.1, 16.5.3, and 16.5.4, we conclude that $\bar{i}$ is also 2-central in $\bar{L}'$. Thus (4a) implies (4b). Then as $K \in \Delta_0(K')$ by 16.4.11.2, by symmetry (4c) implies (4d). Of course (4b) implies (4c), and (4d) implies (4a), so (4) holds.

Assume the hypotheses of (5). As $Z(L) = 1$, $u = jv$ for some $j \in T_C$ with $j^2 = 1$, so $uu^l = (jv)(jv^l) = vv^l \in X \leq L'$ by hypothesis and (1), and $\bar{u}^l = \bar{v}\bar{v}^l$. Let $i := u^{lg^{-1}}$ and $J := (K')^{lg^{-1}}$. As $uu^l \in X$ while $g \in N_G(X)$ by (3), $u^{g^{-1}}u^{lg^{-1}} \in X$; thus as $u^{g^{-1}} \in K$, $i = u^{lg^{-1}} \in J \cap XK$, so $J \in \Delta_0$ by 16.4.9.3. By hypothesis i^* is not 2-central in L^*, so conjugating by g, $\bar{v}\bar{v}^l = \bar{u}^l$ is not 2-central in $\bar{L}'$, establishing (5). □

LEMMA 16.5.7. *Assume $Z(L) = 1$, $|C_{T^*}(T_L^*)| = 2$, and $|T^* : T_L^*| \leq 2$. Then*

(1) R^* contains no 2-central involution of L^*.

(2) L has more than one class of involutions.

PROOF. As $U \leq LK$ while $Z(L) = 1$ by hypothesis, (1) implies (2). Hence we may assume that (1) fails, and it remains to derive a contradiction.

By 16.5.6.4, the hypotheses of 16.5.2.2 are satisfied. If case (a) of 16.5.2.2 holds, then replacing T by the subgroup "T_1" defined there, we may assume $R \trianglelefteq T$; further by 16.5.2.1, we may choose $g \in N_G(T)$ with $K^g = K'$. Otherwise case

(b) of 16.5.2.2 holds, and replacing T by the subgroup T^l defined there, we may assume $T_0 := T_C T_L = N_T(R)$ is of index 2 in T, and take $g \in N_G(T_0)$ with $K^g = K'$. Set $T_1 := T$ or T_0 in the respective cases, and set $Z_C := Z(T_1) \cap T_C$ and $Z_L := Z(T_1) \cap T_L$. Thus $g \in N_G(T_1)$, so g acts on $Z(T_1)$ and $T_1 = N_T(R)$. By hypothesis, $T_L T_C = T_L \times T_C$ and $C_T(T_L) = T_C Z(T_L)$ with $|Z(T_L)| = 2$, so $Z_L = Z(T_L) = C_{T_L}(T)$, and as $Z(T_1) \leq C_T(T_L)$,

$$Z(T_1) = Z_L \times Z_C. \tag{$*$}$$

By 16.4.2.1, $Z_C \cap Z_C^g = 1$, so as g acts on $Z(T_1)$ and $|Z(T_1) : Z_C| = |Z_L| = 2$ by (*), we conclude Z_C is also of order 2. Hence T centralizes $Z_L \times Z_C$. Then since R is normal in T_1, $1 \neq R \cap Z(T_1)$ is central in T by (*), so that $R \cap Z(T) \neq 1$; thus case (a) of 16.5.2.2 holds, and hence $T_1 = T$.

As $T_1 = T$, $Z(T) = Z_L \times Z_C$ is of rank 2 and $g \in N_G(T)$. In particular $R = T \cap K^g = (T \cap K)^g = T_C^g$. Also as $Z_C^g \cap Z_C = 1$, g induces an element of order 3 on $Z(T)$ so either Z_C^g or $Z_C^{g^2}$ is not equal to Z_L, and replacing g^2 by g of necessary, we may assume $Z_C^g \neq Z_L$. As $T_C \trianglelefteq T$, also $R = T_C^g \trianglelefteq T$, so $R \cap L \trianglelefteq T$. Hence as $Z(T) \cap R = Z_C^g$ is of order 2 and does not lie in L, $R \cap L = 1$. Thus $[T_L, R] \leq T_L \cap R = 1$, so $R^* \leq C_{T^*}(T_L^*) = Z_L^*$. Therefore as $|Z_L^*| = 2$ and $R \cong R^*$, R is of order 2, and hence $T_C = Z_C$. Then as $|T^* : T_L^*| \leq 2$, $|T : T_L| \leq 4$, so that $[T, T] \leq T_L$. By Proposition 16.5.1 and our assumption that G is not J_1, L is not $L_2(2^n)$, so by inspection of the groups in (E2), T_L is nonabelian. Therefore as Z_L is of order 2, $Z_L = Z(T) \cap [T, T] \trianglelefteq N_G(T)$. This is impossible, as $\langle g \rangle$ is irreducible on $Z(T) \cong E_4$. □

LEMMA 16.5.8. *(1) L is not $L_2(p)$, p an odd prime, or $L_3(3)$.*
(2) L is not M_{11}, M_{22}, M_{23}.

PROOF. If L is a counterexample to (1) or (2), then L has one class of involutions, $Z(L) = 1$, $Out(L)$ is of order at most 2, and $C_{T^*}(T_L^*)$ is of order 2; hence the lemma follows from 16.5.7. □

LEMMA 16.5.9. *L is not $U_3(3)$.*

PROOF. Assume otherwise. Then L has one class of involutions, $X \cong SL_2(3)$, and $C_{H^*}(X^*) \cong \mathbf{Z}_4$ or Q_8. Thus $R^* \cong R$ is of 2-rank 1 by 16.5.6.2, and $T_C \cong R$ by 16.4.11.1. Then $Z := \Omega_1(Z(T)) \cong E_4$, with $u \in Z := \langle z, v \rangle$, where $z \in T_C$ and $v \in T_L$. Choose g as in 16.4.2.4; then $T_C^g = R$ by 16.4.11.3. As $R \cap Z(T) \neq 1$, $g \in N_G(T)$ by 16.5.2.1, and as $T_C^g = R$, g is nontrivial on Z. Then as Z is of rank 2 we conclude that g induces an element of order 3 on Z. This is impossible, as $g \in N_G(X)$ by 16.5.6.3, so g acts on $Z(X) = \langle v \rangle$. □

LEMMA 16.5.10. *Assume L^* is not of Lie type of Lie rank 2 over $\mathbf{F}_{2^n}$ for some $n > 1$. Then*

(1) If L^* is of Lie type in characteristic 2, then L is ${}^2F_4(2)'$, ${}^3D_4(2)$, $L_4(2)$, or $L_5(2)$.

(2) If L^* is not of Lie type and characteristic 2, then $L^* \cong M_{12}$, M_{22}, M_{24}, J_2, J_4, HS, or Ru; and if $L^* \cong M_{22}$, then $Z(L) \neq 1$.

(3) $|T^* : T_L^*| \leq 2$.

(4) $|C_{T^*}(T_L^*)| \leq 2$.

(5) Either $Z(L) \neq 1$, or R^* contains no 2-central involution of L^*.

(6) Assume that $Z(L) = 1$, v is the projection of u on L, and there is $l \in L$ with vv^l an involution in X. Then vv^l is not 2-central in L'.

PROOF. Part (2) follows from 16.5.8 and inspection of the groups in (E2).

Suppose that L^* is of Lie type in characteristic 2. By Proposition 16.5.1 and our assumption that G is not J_1, L is not of Lie rank 1, and by hypothesis L^* is not of Lie rank 2 over $\mathbf{F}_{2^n}$ for some $n > 1$. Thus from Theorem C, either L^* is of Lie rank 2 over $\mathbf{F}_2$, or $L^* \cong L_4(2)$ or $L_5(2)$. By 16.5.8 L is not $L_3(2) \cong L_2(7)$, by 16.5.9 L is not $U_3(3) \cong G_2(2)'$, and by 16.5.3 L is not $A_6 \cong Sp_4(2)'$. Thus (1) holds since L is simple by 16.1.2.1.

Next by inspection of $Aut(L^*)$ for L^* listed in (1) and (2), $|Out(L^*)|_2 \leq 2$, so (3) holds. Similarly (4) follows from inspection of $Aut(L^*)$. Then (5) follows from (3), (4), and 16.5.7.1. Finally assume the hypotheses of (6). Then the hypotheses of 16.5.6.5 are satisfied using (5), so (6) follows from that result. □

LEMMA 16.5.11. *L^* is of Lie type in characteristic 2.*

PROOF. Assume otherwise; then L^* is in the list of 16.5.10.2.

Suppose first that u^* is 2-central in L^*. Then by 16.5.10.5, $Z(L) \neq 1$, so applying 16.1.2.1 to the list in 16.5.10.2, L^* is M_{12}, M_{22}, J_2, HS, or Ru. Furthermore using 16.1.5, we find that either $C_{H^*}(X^*)$ is of order 2, or L^* is HS and $C_{H^*}(X^*) \cong \mathbf{Z}_4$. Thus by 16.5.6.2, either $|R| = 2$, or L^* is HS and $R \cong \mathbf{Z}_4$. In any case $\langle u \rangle = \Omega_1(R)$ and $\langle z \rangle = \Omega_1(T_C)$. Further if L^* is M_{22}, then as $T_C \cong R \cong \mathbf{Z}_2$, $T_C = \langle z \rangle$ and $Z(L) \cong \mathbf{Z}_2$. Hence we conclude from 16.1.2.2 that in each case $\langle z \rangle = \Omega_1(T_C) = Z(L) \cong \mathbf{Z}_2$. Then from the structure of the covering group L of L^* in parts (5)–(7) of I.2.2, either:

(a) There is a unique $v \in uZ(L)$ such that there exists $x \in O_2(X)$ with $x^2 = v$.
(b) L^* is Ru, and setting $Y_1 := C_{O_2(X)}(\Phi(O_2(X)))$, $Y := [Y_1, Y_1]$ is of of order 2, and $Y^* = \langle u^* \rangle$.

In case (a) set $Y := \langle u \rangle$. Thus in any case $Y^* = \langle u^* \rangle$. Further T normalizes X, and hence centralizes Y, so T centralizes $Z(L)Y = \langle z, u \rangle$. Therefore $1 \neq u \in R \cap Z(T)$. Choose g as in 16.4.2.4; then $g \in N_G(T)$ by 16.5.2.1, and $g \in N_G(X)$ by 16.5.6.3. Next $|C_{T^*}(T_L^*)| = 2$ in each case, so $Z(T_L^*) = \langle u^* \rangle$ and hence $Z := \Omega_1(Z(T)) = \langle z, u \rangle \cong E_4$. By our choice of g and 16.4.11.3, $R = T_C^g$ and hence $u = z^g$. Then as g acts on T, g induces an element of order 3 on Z, and in particular $\langle g \rangle$ acts irreducibly on Z. This is impossible since g acts on X and hence on $Y < Z$.

Therefore u^* is not 2-central in L^*. Thus as M_{22} has one class of involutions, L^* is not M_{22}.

Inspecting the list of centralizers of non-2-central involutions in 16.1.5 for the remaining groups in 16.5.10.2, either $C_{H^*}(X^*)$ is of order 2, or L^* is M_{12}, M_{24}, J_2, HS, or Ru and $C_{H^*}(X^*) \cong E_4$. Arguing as in the previous paragraph, either $|R| = 2 = |T_C|$, or one of the exceptional cases holds with $R \cong E_4 \cong T_C$. In any case, $\Phi(R) = 1 = \Phi(T_C)$, so $R = U$.

Assume $L \cong J_4$ or M_{24}. Then $Out(L) = 1$, so $T = T_L \times T_C$ with $\Phi(T_C) = 1$, and hence $z \notin \Phi(T)$ for $T \in Syl_2(C_G(z))$, and T_C is in the center of $O^{2'}(C_G(z))$. Thus if $|T_C| = 2$, then $z^G \cap T_L \neq \emptyset$ by Thompson Transfer, whereas for each involution $a \in T_L$, $a \in \Phi(C_{T_L}(a))$ by 16.1.5.9. Thus $|T_C| > 2$, so $L \cong M_{24}$, and then U is not centralized by $O^{2'}(C_L(u))$; so as $U = R = T_C^g$, this is contrary to T_C in the center of $O^{2'}(C_G(z))$.

Therefore L^* is M_{12}, J_2, HS, or Ru. If $Z(L) \neq 1$, then from (5b) and (7b) of I.2.2, the projection v of u on L is of order 4, so $u = vt$ with $t \in T_C$ of order 4, contrary to $\Phi(T_C) = 1$. Hence $Z(L) = 1$, so if v is the projection of u on L and there is $l \in L$ with vv^l an involution of X, then by 16.5.10.6, vv^l is not a 2-central involution of L'. However $X \cong A_5$, A_5, A_6, or $Sz(8)$, respectively, with all involutions in X 2-central in L, and the involutions in vX are in v^L, so we have a contradiction which completes the proof of 16.5.11. □

LEMMA 16.5.12. *L^* is of Lie type in characteristic 2 of Lie rank 2.*

PROOF. Assume otherwise. By 16.5.11 and 16.5.10.1, L is $L_n(2)$ for $n := 4$ or 5. Thus H^* is either L^* or $Aut(L^*)$, so $H = LKT$. Recall z is an involution in $T_C \cap Z(T)$. If T_C is cyclic, then by 16.4.4.2, $T_C = C_K(z)$, so that $G_z = LT$.

Next L has two classes j_1 and j_2 of involutions, where j_1 is the class of transvections and the 2-central class. Hence $u^* \in j_2$ by 16.5.10.5, so by 16.1.4.3, $X \cong A_4$ or $\mathbf{Z}_3/2^{4+4}$, for $n = 4$ or 5, respectively. Also $C_{T^*}(X^*) \cong E_4$, unless $L^*T^* \cong S_8$, in which case $C_{T^*}(X^*) \cong D_8$. Thus by 16.5.6.2, either R is a subgroup of E_4, or $H^* = L^*R^* \cong S_8$ and $R \cong \mathbf{Z}_4$; R is not D_8 as $\Omega_1(R^*) = U^* \leq T_L^*$.

Suppose $H^* \cong S_8$ with $R \cong \mathbf{Z}_4$. Then $T_C \cong R \cong \mathbf{Z}_4$, so $G_z = LRT_C$ by paragraph one. Hence $T = T_L T_C R$ centralizes T_C, so $T_C \leq Z(G_z)$. Then $R \leq Z(C_G(u))$, whereas $R^* \cong \mathbf{Z}_4$ is not central in a subgroup D_8 of $C_{L^*}(u^*)$.

Therefore $R \cong \mathbf{Z}_2$ or E_4, so that $R = U$ and hence $R^* \leq T_L^*$. Let v denote the projection of u on L.

Assume first that $n = 5$. Then

$$\Phi(O_2(X)) =: V = [V, X] \oplus C_V(X),$$

with the involutions in the 4-groups $[V, X]$ and $C_V(X)$ of type j_2, and the diagonal involutions of type j_1. Then $v \in C_V(X)$, and there is $l \in L$ with $v^l \in [V, X]$, so that vv^l is 2-central in L', contrary to 16.5.10.6.

Hence $n = 4$, so that $L \cong L_4(2)$. Assume $R \cong E_4$. Then for $r \in R - \langle u \rangle$, the projection of r on L is in $O_2(C_L(X)) - \langle v \rangle$, so as $C_G(u) \leq H'$ by 16.4.2.5, $v \in [r, C_{T_L}(u)] \leq K'$, so $u = v \in L$. Similarly $r \in L$, so that $R = O_2(C_L(X))$. Then there is $y \in L$ of order 3 faithful on R. But as $m_3(N_L(O_2(X))) = 2$ and G is quasithin, K is a $3'$-group, so $y \in O^{3'}(H') = L' \leq C_G(R)$, a contradiction.

Therefore $R = \langle u \rangle$ is of order 2, so as $T_C \cong R$, $T_C = \langle z \rangle$ is of order 2, and $G_z = LT$ by paragraph one. Set $A := O_2(C_L(u))T_C$. Then $A \cong E_{32}$ and $A = J(C_{T_L T_C}(u))$. If $H^* \cong L_4(2)$ then $T = T_L \times \langle z \rangle$, so that $z \notin \Phi(T)$ with $T \in Syl_2(C_G(z))$. However $z^G \cap T_L \neq \emptyset$ by Thompson Transfer, and each involution $a \in L$ satisfies $a \in \Phi(C_L(a))$ (cf. parts (1) and (3) of 16.1.4). Therefore $LT \cong S_8 \times \mathbf{Z}_2$ with $D_8 \times E_8 \cong O_2(C_{LT}(u)) = J(O_2(C_{LT}(u)))$. Choose g with $T \cap H' = N_T(K') \leq T^g$ as in 16.4.2.4, so that $R = T_C^g$ by 16.4.11.3, and hence $u = z^g \in z^G \cap A - \{z\}$ and $A \leq T^g$.

Suppose first that $A \leq L'K'$. Then $A \leq T_L^g T_C^g$, so $A = J(C_{(T_L T_C)^g}(z)) = O_2(C_{L'}(z)) \times R$, and hence A plays the same role for the pairs L', T_C and L, R. Next $A \cap j_2$ is an orbit of length 6 on $A^\# \cap L$ under $N_H(A)$, with $Aut_H(A) \cong O_4^+(2)$. Further if $y \in G$ such that $z^y \in LK$ projects on a member of the class j_1, then $K^y \in \Delta_0$ by 16.4.9.3, contrary to 16.5.10.5. Thus no member of $z^G \cap LK$ projects on j_1, so as $N_H(A)$ has two orbits of length 6 on the elements of A projecting on members of j_1 and u is such a member, we conclude $z^G \cap A =: \alpha$ is of order 7 or 13. Set $M := N_G(A)$ and $M^+ := M/C_M(A)$. Since A plays the same role for both pairs

L', T_C and L, R, $C_M(u)$ moves z, so z^M is of order 7 or 13. Further $A = \langle z^M \rangle$, so M^+ acts faithfully on z^M. Since $|L_5(2)|$ is not divisible by 13, $|z^M| = 7$, so $M^+ \leq S_7$. As $C_{M^+}(z) \cong O_4^+(2)$, $|M^+| = 2^3 \cdot 3^2 \cdot 7$. But S_7 has no subgroup of index 10.

Therefore $A \not\leq L'K'$. Hence $LT \cong S_8 \times \langle z \rangle$, so from the structure of S_8, $\mathcal{A}(T) = \{A, A_1, A_1^t, B\}$ for suitable B and $t \in T_L$, where $J(O_2(C_{LT}(u))) = \{A, A_1\}$. As $A \not\leq L'K'$, $A_1 = O_2(C_{L'}(z))R = J(C_{T_L^g T_C^g}(u))$, so $A_1 \in A^G$. Observe that each member of $\mathcal{A}(T)$ is normal in $J(T)$, so by Burnside's Fusion Lemma, $I := N_G(J(T))$ is transitive on $A^G \cap J(T)$, and hence $A_1 \in A^I$. As $|T : J(T)| = 2$, I is not transitive on $\mathcal{A}(T)$, so $A^I = \{A, A_1, A_1^t\}$ and I induces S_3 on $\mathcal{A}(T)$. But $J(T) \cong \mathbf{Z}_2 \times D_8 \times D_8$, so by the Krull-Schmidt Theorem A.1.15, I permutes the two involutions generating the Frattini subgroups of the D_8-subgroups, so that $O^2(I)$ centralizes $\Phi(J(T))$. Then as $Z(J(T)) \cong E_8$, by Coprime Action, $O^2(I) \leq O^2(C_G(Z(J(T))) \leq O^2(G_z) = O^2(LT) = L$; then $I = N_L(J(T))T \leq N_G(A)$, contrary to $|A^I| = 3$. This completes the proof of 16.5.12. □

LEMMA 16.5.13. *(1) u^* is not in the center of T^*.*
(2) L^* is not $L_3(2^n)$ or $Sp_4(2^n)$.

PROOF. By 16.5.12, $L^* \cong Y(2^n)'$, where Y is one of the Lie types A_2, C_2, G_2, ${}^2F_4$, or ${}^3D_4$. Further if $n = 1$, then L is the Tits group ${}^2F_4(2)'$ or ${}^3D_4(2)$ by 16.5.10.1, and so (1) holds by 16.5.10.5. Thus we may assume that $n > 1$. Further L^* is not $L_3(4)$ by 16.5.4, so by 16.1.2.1, either $Z(L) = 1$ or L^* is $G_2(4)$.

We first treat the case where u^* is a long-root involution. (When $L \cong Sp_4(2^n)$, either class of root involutions can be regarded as "long", as the classes are interchanged in $Aut(L)$.) Thus u^* is 2-central in L^*. Let Z denote the root group of the projection v of u on L—unless L^* is $G_2(4)$ with $Z(L) \neq 1$, where we let $Z := [N_L(Z_1), Z_1]$ where Z_1 is the preimage in L of the root group of u^*. Set $P := N_L(Z)$ and recall the definition of $X := X_u$ from Notation 16.5.5. As u^* is a long-root involution, either

(a) P is a maximal parabolic of L, and we check (cf. 16.1.4.1) that $X = P^\infty$, or

(b) $L \cong L_3(2^n)$ and $X = C_P(Z)$ if n is odd, while $X = O^3(C_P(Z))$ if n is even.

In case (b), $n > 2$ by 16.5.4. Thus in any case $X \neq 1$ and $Z^* = C_{L^*T^*}(X^*)$, so that $V := ZT_C = C_T(X)$, and $T_C \cong R \cong R^* \leq Z^*$ by 16.5.6.2. In particular $\Phi(R) = \Phi(T_C) = 1$. Choose g as in 16.4.2.4; thus $V = ZT_C \leq T \cap H' = N_T(K') \leq T^g$. Further $X^g = X$ by 16.5.6.3, so

$$V = C_T(X) \leq C_{T^g}(X) = C_{T^g}(X^g) = C_T(X)^g = V^g,$$

and hence $g \in N_G(V) \cap N_G(X) =: M$. Let $T_0 := N_T(X)$ and notice that either $T_0 = T$, or T^* is nontrivial on the Dynkin diagram of $L^* \cong Sp_4(2^n)$ with T_0 of index 2 in T. Let $\bar{M} := M/C_M(V)$. We can finish much as in the proof of Proposition 16.5.1: For $\bar{P} \cong \mathbf{Z}_{2^n-1}$ is regular on $Z^\# = [V, P]^\#$, $T_C = C_V(P)$ is a TI-set in V under the action of M by I.7.2.3, $N_M(T_C) \leq N_M(P)$ by 16.4.2.5, and $\bar{T} \in Syl_2(\bar{M})$ acts on $\bar{P}$. Thus again we have the hypotheses for a Goldschmidt-O'Nan pair in Definition 14.1 of [**GLS96**], so we may apply O'Nan's lemma Proposition 14.2 in [**GLS96**]. Conclusion (i) of that result is eliminated since $g \in M - N_G(T_C)$, so either $m(V) = 3$ and $\bar{M} \cong Frob_{21}$ is irreducible on V, or T_C^M is of order 2^n where $n = m(Z^*) > 1$. The latter case is impossible as $|T : T_0| \leq 2$. In the former as M

is irreducible on V and M normalizes X containing Z, $V \le X \le L$. Thus $T_C \le L$ so $Z(L) \neq 1$, and hence L^* is $G_2(4)$ and $Z(L) = T_C$ is of order 2 by 16.1.2.2. Then $X/V \cong L_2(4)/E_{2^8}$ and the chief factors for X/V on $O_2(X)/V$ are natural modules. However $Y := O^{7'}(M)$ centralizes $X/O_2(X)$ as $Aut(A_5) = S_5$, so since the group of units of $End_{X/V}(O_2(X)/V)$ is $GL_2(4)$, Y centralizes $O_2(X)/V$. Then as $V = \Phi(O_2(X))$, Y centralizes $O_2(X)$ by Coprime Action, a contradiction as Y induces $\mathbf{Z}_7$ on V. Therefore u^* is not a long-root involution.

If $L^* \cong L_3(2^n)$ then all involutions of L^* are long-root involutions, so the lemma is established in this case. Further when L^* is of type G_2, ${}^2F_4$, or ${}^3D_4$, the 2-central involutions are the long-root involutions, so the lemma holds in these cases too. Thus we may assume $L^* \cong Sp_4(2^n)$, so that $Z(L) = 1$ by 16.1.2.1. As u^* is not a root involution, u^* is 2-central of type c_2 in L^* by 16.1.4.2, so we may take $u^* \in Z(T^*)$; thus the projection v of u is in $Z(T)$, so $u \in Z(T_L T_C)$ since R centralizes T_C. Proceeding as in the proof of 16.5.4, $U^* \le Z(T_L^*)$. As $U^* \cong U$ and U^* contains no root elements, $m(U) \le n$. Also as in 16.5.4, $J(T) = T_L J(T_C) = T_L U_C$, where $U_C = \Omega_1(T_C) \cong U$ is elementary abelian. Let $M := N_G(J(T))$. Recall that $U_C \in U^G$, so by Burnside's Fusion Lemma A.1.35, $U_C \in U^M$. Now $V := Z(J(T)) = U_C V_L$ is elementary abelian of order rq^2, where $V_L := Z(T_L) \cong E_{q^2}$, $q := 2^n$, and $r := |U_C|$. Further M acts on $\Phi(J(T)) = V_L$ and on V, so as $U_C \cap V_L = 1$, also $U_C^m \cap V_L = 1$ for each $m \in M$. Let β be the set of involutions in $V - V_L$ either contained in U_C, or projecting on a member of $V_L - (Z_1 \cup Z_2)$, where Z_1 and Z_2 are the two root groups in V_L. Then

$$|\beta| = (q^2 - 1 - 2(q-1) + 1)(r-1) = ((q-1)^2 + 1)(r-1).$$

Let γ be the set of involutions contained in a member of U_C^M. If $y \in G$ such that $z^y \in H$ and z^{y*} is a root involution of L^*, then $K^y \in \Delta_0$ by 16.4.9.3, contrary to 16.5.10.5. It follows that $\gamma \subseteq \beta$. Also $L \cap M$ contains a Cartan subgroup Y of $N_L(T_L)$ of order $(q-1)^2$, and Y acts regularly on U^{*Y} and hence also on U^Y. Therefore as K is tightly embedded in G, and $N_M(K)$ normalizes $V \cap K = U_C$, U_C is a TI-subset of V under the action of M by I.7.2.3, so

$$|\gamma| \ge ((q-1)^2 + 1)(r-1) = |\beta|,$$

and hence as $\gamma \subseteq \beta$, we conclude that $\gamma = \beta$ and $U_C^M = \gamma$ is of order $1 + (q-1)^2$. This is impossible since $1 + (q-1)^2$ is even, while $T \le N_M(U_C)$ and T is Sylow in G. This contradiction completes the proof of 16.5.13. □

We are now in a position to establish our main result Theorem 16.5.14.

By 16.5.12 and 16.5.13.2, we have reduced the possibilities for L in (E2) to the case where $L^* \cong G_2(2^n)'$, ${}^2F_4(2^n)'$, or ${}^3D_4(2^n)$. By 16.5.10.1, $n > 1$ if $L^* \cong G_2(2^n)$, and by 16.5.13.1, u^* is a short-root involution in L^*. By 16.1.2, either $Z(L) = 1$, or L^* is $G_2(4)$ and $Z(L)$ is of order 2. However in the latter case, from I.2.2.5b, u^* lifts to an v element of order 4, so $u = cv$ with $c \in T_C$ of order 4. This is impossible, as $C_{H^*}(X^*) \cong E_4$, so $\Phi(R^*) = 1$ by 16.5.6.2, and hence $R^* \cong R \cong T_C$ is elementary abelian. Thus $Z(L) = 1$.

Let V be the root group of the projection v of u on L—except when L is ${}^3D_4(2^n)$, where we set $V := Z(X)$. Then (cf. 16.1.4 and [**AS76a**] for further details) one of the following holds:

(a) $L \cong G_2(2^n)$, $X \cong L_2(2^n)/E_{2^{2n}}$ is an $L_2(2^n)$-block, and $E_{2^n} \cong V^* = C_{H^*}(X^*)$.

(b) $L \cong {}^2F_4(2^n)'$, $X/O_2(X) \cong L_2(2^n)'$, $Z(O_2(X)) = V \oplus W$, where $W := [Z(O_2(X), X]$ is the natural module for $X/O_2(X)$, and $V = Z(X)$.

(c) $L \cong {}^3D_4(2^n)$, $X/O_2(X) \cong L_2(2^n)'$, $Z(O_2(X)) = V \oplus W$, where $W := [Z(O_2(X)), X]$ is the natural module for $X/O_2(X)$, and $V = Z(X) \cong E_{2^n}$.

In case (a), set $W := O_2(X)$. We finish as in several earlier arguments: In each case, $W^\#$ is the set of long root involutions in $Z(O_2(X))$, and $vv^l \in W^\#$ for suitable $l \in L$, contrary to 16.5.10.6.

This final contradiction establishes:

THEOREM 16.5.14 (Even Type Theorem). *Assume G is a quasithin simple group, all of whose proper subgroups are $\mathcal{K}$-groups. Assume in addition that G is of even type, but not of even characteristic. Then $G \cong J_1$.*

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