Just *perfect* part 2

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The numbers

In the last issue, we defined a perfect number N to be one for which the sum of the divisors d ($1 \le d < N$) is N. We gave the first few perfect numbers, starting with those known by the early Greeks. Here we give an extended list, with some comments about their discovery.

The early Greeks were aware of the first four perfect numbers:

$P_1 = 6$
$P_2 = 28$
$P_3 = 496$
$P_4 = 8128$

Johann Regiomontanus

The Arab mathematician Ibn Fallus (1194–1239) wrote a treatise in which he gave the first seven perfect numbers. So:

$P_5 = 33\ 550\ 336$
$P_6 = 8\ 589\ 869\ 056$
$P_7 = 137\ 438\ 691\ 328$

As the work of Ibn Fallus was not widely known in Europe, these numbers were rediscovered by Regiomontanus (1461), Scheybl (1555) and Cataldi (1603). To the present, some 40 perfect numbers have been discovered.

Structure

Now we might expect to find the early perfect numbers by trial and error, but how did mathematicians obtain the later ones, which clearly get large very rapidly?

Let us review an exercise we did before. You might have found:

$P_1 = 6 \quad = 2 \times 3 \quad = 2^1 \times (2^2 - 1)$
$P_2 = 28 \quad = 4 \times 7 \quad = 2^2 \times (2^3 - 1)$
$P_3 = 496 \quad = 16 \times 31 \quad = 2^4 \times (2^5 - 1)$
$P_4 = 8128 \quad = 64 \times 127 \quad = 2^6 \times (2^7 - 1).$

If you are brave, you can try P_5 and beyond! There is clearly a pretty pattern appearing here, although there are some mysterious gaps. Let us try evaluating the sequence of bracketed terms on the right, including the missing entries:

$(2^2 - 1) = 3$ •
$(2^3 - 1) = 7$ •
$(2^4 - 1) = 15$
$(2^5 - 1) = 31$ •
$(2^6 - 1) = 63$
$(2^7 - 1) = 127$ •

The entries marked with a "•" correspond to our perfect numbers. Before we continue, what do you notice about these entries (that is not true about the remaining two)? Can you make a conjecture?

It looks as though a number of the form $2^n \times (2^{n+1} - 1)$ where the second term is prime will be a perfect number. (A number p is *prime* when its only divisors are 1 and itself. Thus, 3, 7, 31 and 127 are prime numbers. The number 15 is not prime because it is also divisible by 3 and 5.)

Euclid's proof

Perhaps surprisingly, Euclid was able to establish the above conjecture:

Theorem A number $N = 2^n \times (2^{n+1} - 1)$ in which the second term is prime, is a perfect number.

It is not hard to prove this. Let us write

$$M_n = 2^n - 1,$$

and for the purpose of this proof, we set *S(N)* to be the sum of all the divisors of N, including N itself. This means that for N to be perfect, we require $S(N) = 2N$. For example,

$$S(6) = 1 + 2 + 3 + 6 = 12.$$

In fact, if q is a prime number, then $S(q) = q + 1$. Also,

$$S(2^n) = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1.$$

Now what about the number $N = 2^n \times q$, where $q > 2$ is prime? Listing all the divisors, we have:

$$\begin{aligned} S(N) &= 1+2+2^2+\ldots+2^n+q+2q+2^2q+\ldots+2^nq \\ &= 2^{n+1} - 1 + q(2^{n+1} - 1) \\ &= (2^{n+1} - 1)(1 + q) \end{aligned}$$

Finally, suppose that $q = 2^{n+1} - 1$ (where n must be such that q is a prime). Then

$$\begin{aligned} S(N) &= S\{2^n \times (2^{n+1} - 1)\} \\ &= (2^{n+1} - 1)(1 + 2^{n+1} - 1) \\ &= 2^{n+1}(2^{n+1} - 1) \\ &= 2N, \end{aligned}$$

and N is perfect as required.

There are now two questions to be asked:

- *Are there other even perfect numbers which are not of this form?*
  In a posthumous paper in 1849, Euler provided the first proof that Euclid's construction gives all possible even perfect numbers.
- *Are there any odd perfect numbers?*
  It is not known if any odd perfect numbers exist, although by 1991 all numbers up to 10^{300} (a very large number!) had been checked without success.

Mersenne primes

It is clear that our search for perfect numbers has now changed direction. The test for an even perfect number now depends on finding whether the number $M_n = 2^n - 1$ is prime or not.

Numbers of the form M_n are called *Mersenne numbers*; if this number is a prime, it is called a *Mersenne prime*. It is easy to see that M_n will only be prime if n itself is a prime number. For if $n = rs$, then $M_n = 2^{rs} - 1$ will have a factor $2^r - 1$. This is of the binomial form $(x^s - 1) = (x - 1)(x^{s-1} + x^{s-2} + \ldots + 1)$.

Mersenne numbers were named after Marin Mersenne (1588–1648) who was a pioneer in the search for prime numbers.

So which values of prime p generate a Mersenne prime M_p? If you enjoyed writing a little program last time to find the smaller perfect numbers, you might like to write another to find the Mersenne primes. There has been a lot of computing work carried out to determine Mersenne primes. This is because prime numbers are the building blocks of our system of integers, and Mersenne primes give an entry into our understanding of primes in general.

Here is a table with some known results.

Prime p	Mersenne number M_p	M_p prime?	Perfect number $2^{p-1}M_p$
2	3	✓	6
3	7	✓	28
5	31	✓	496
7	127	✓	8128
11	1023	✗	–
13	4095	✗	–

You might like to try extending the table to obtain the next perfect number.

Marin Mersenne

Bibliography

MathWorld. *Perfect number.* http://mathworld.wolfram.com/PerfectNumber.html

The perfect number journey. http://home.pacific.net.sg/~novelway/MEW2/lesson1.html

Perfect numbers. http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Perfect_numbers.html

MathWorld. *Mersenne number.* http://mathworld.wolfram.com/MersenneNumber.html

From Helen Prochazka's

Scrapbook

Robert Peirce writing in Lonely Planet Bhutan

In a mountain village named Laya, I was standing in a schoolhouse and staring at a document entitled Manual for Teachers of Mathematics. What caught my eye was something offered as a "first rule".

Always remember that you are a human being as well as a teacher, that your students are human beings, and that you are here because you have something important to give them that they need.

It is not what one would expect as a first rule for maths teachers anywhere else in the world. But this is Bhutan, and in Bhutan, I am learning, people are not the abstract ciphers they can come to be in a more urban environment. They are human beings who, even in official matters, tend to deal with each other as human beings. "I am not as much concerned about the Gross National Product," the king is supposed to have said, "as I am about the Gross National Happiness."