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ABSTRACT

This document reports on the 11th annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA). Plenary and response lectures and speakers include: "The Description and Analysis of Mathematical Processes" (Nicolas Herscovics); "To Know Mathematics is to Go Beyond Thinking That 'Fractions Aren't Numbers'" (Rochel Gelman, Melissa Cohen, & Patrice Hartnett); "General Mathematical Processes: Theory, Practice and Research" (Alan Bell); "Exploring the Process Problem Space: Notes on the Description and Analysis of Mathematical Processes" (Alan H. Schoenfeld). Symposia topics include: "Realistic Mathematics Education, Beliefs or Theory"; "Sex Differences in Mathematics Ability"; "Clinical Investigations in Mathematics Teaching Environments: Journeys from Research Questions to Data Collection Strategies"; "Changes in Student Assessment Occasioned by Function Graphing Tools"; and "Alternative Conceptions of Probability: Implications for Research, Teaching, and Curriculum." Includes a listing of author addresses. (MKR)

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Volume 2: Plenary Lectures and Symposia

September 20–23, 1989
New Brunswick, New Jersey, U.S.A.

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EDITORS' NOTE

We are pleased to present the plenary lectures, responses, and symposium papers from the Eleventh Annual Meeting of PME-NA at Rutgers University, September 20-23, 1989. The theme for the 1989 PME-NA Meeting, the description and analysis of mathematical processes, is reflected in these papers in a variety of ways, ranging from the characterization of the mathematical processes involved in the construction of conceptual schemes, to assessment methods for ascertaining early understandings in the domain of fractions. We commend these papers to you for further study.

The editors would like to thank again the New Jersey Department of Higher Education for partial financial support in the preparation and publication of these Proceedings.

Carolyn A. Maher
Gerald A. Goldin
Robert B. Davis

Plenary Lectures and Responses

THE DESCRIPTION AND ANALYSIS OF MATHEMATICAL PROCESSES

Nicolas Herscovics, Concordia University

INTRODUCTION

The Program Committee of this year's meeting has chosen as a theme for the plenary sessions "The Description and Analyses of Mathematical Processes". It seems to me that this choice reflects PME's continuing preoccupation with fundamental questions regarding the nature of mathematics and the topics that should concern us in mathematics education. In 1983 when we met in Montreal, we discussed the need to situate research in mathematics education in a broad epistemological perspective that is, viewing it as the growth of knowledge (Vergnaud,1983). Four years later, this was continued in our discussions of constructivism and its relationship to mathematics education (Sinclair, 1987). Last year in DeKalb, Magdalene Lampert (1988) gave us a very impressive demonstration of how one could actually teach primary school mathematics in the spirit of Lakatos that is, as an intellectual process rather than as a product. In a sense, it might be fair to say that after convincing ourselves of the importance of mathematical processes and the need to make them the topic of instruction, we now want to discuss the means to research these processes.

Questions regarding appropriate research paradigms are difficult and challenging, for the description and analysis of mathematical processes imply finding the means of reporting a subject's mathematical thinking and then interpreting it in terms of the subject's own logical system. Fortunately or unfortunately, we cannot penetrate inside another person's head and thus we are restricted to making inferences based on observed behavior. For instance, how a young child handles addition problems with concrete objects can be detected by direct observation. However, this will not necessarily provide us with information on how the same child will perform mental arithmetic. And of

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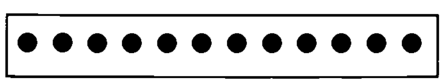
course, restricting oneself to mere observations in studying the performance of mental arithmetic without asking the children to verbalize their thinking is pretty useless. However, even with the most verbal of our students, any analysis based on our observations must take into account the problem of considering a set of responses, which presumably are logical within the subject's cognitive system, and discovering this logic without imposing our own logical system. In other words, we must try to fathom what the students see with their own eyes.

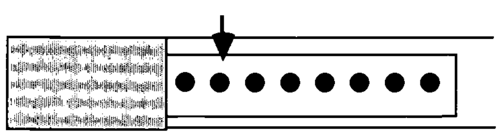
But even when we can observe a given mathematical procedure, its interpretation by the subjects cannot always be inferred and predicted. A recent set of results provides a very nice example.

These last few years, together with my colleague Jacques Bergeron, I have spent some time doing research on the child's construction of the number concept. More specifically, we have dealt with the kindergarten level (Bergeron & Herscovics, 1989a,b; Herscovics & Bergeron,1989). The age range was from 5:2 to 6:7 (average age 5:11) and their enumeration skills ranged from 9 to beyond 76 (averaging 45.7). As everyone knows, the simplest form of enumeration involves establishing a one-to-one correspondence between a set of discrete objects and the number-word sequence. All the 91 children we have interviewed knew how to enumerate a set of chips.However, their numerical skills extended well beyond this. All but 4 out of 91 children could count on from a given number in the number word sequence. We thus expected positive results when we asked children to count on from 6 in a row of chips as in the following task:

Children were presented with a cardboard on which we had glued 12 chips.



The cardboard was then inserted into a partially opaque plastic bag.



The interviewer explained : **Look, there are some hidden chips. When I counted them, I started from here** (pointing to the first hidden chip on the left) **and when I got here** (putting the arrow next to the sixth chip), **this one was the sixth.**

Can you continue counting from here on, from the sixth one?

Not surprisingly, we found that all but 5 of the 91 children could continue counting on in this enumeration task. But what did counting on mean to them? Did they associate this procedure with the cardinality of the set? How could we possibly find this out? This proved rather simple. We asked them :

Can you tell me how many chips are in the whole bag?

Only 33 of the kindergartners (36%) were able to provide the answer. Most of the children either guessed another number or indicated that they would have to count the chips in the bag. To these children, the fact that they had just counted on from 6 and reached 12 did not relate to the cardinality of the set. Counting on was used here as meaningless procedure. Perhaps these children still need to formulate a one-to-one correspondence between the number words and the objects in order to establish the cardinality of a set. Perhaps the difficulty here is related to the visibility of the objects. What would happen if the same task was repeated but without hiding the first four chips? This example shows quite nicely that even when one can identify and describe the procedure being used, one cannot take for granted that it will have the same significance for the learner as it does for the interviewer or the researcher. In analyzing this result, one cannot find a simple explanation.

THE CONSTRUCTION OF CONCEPTUAL SCHEMES

The most serious mistake one could make in a discussion of mathematical processes would be to assume that these processes are universal, that they are the same for the learning child and the professional mathematician. This kind of assumption has led some educators to believe that their own introspection would be sufficient to bring about an understanding of the problems faced by school children. The fallacy with this argument is that young children need construct the very basic mathematical concepts and processes on which the adult or mathematician builds new mathematical knowledge. There is no reason to believe that the construction of the initial fundamental concepts involves the same kind of processes as those used in developing or deriving the more advanced ones. One cannot compare the young child's construction of the number concept with that of the mathematician's construction of the set of all algebraic numbers, that is, the set of all numbers that are roots of polynomial equations.

Another possible error would be to assume that mathematical processes are the same, regardless of the mathematical activity involved. This explains why the early models developed to describe the understanding of mathematics were either so primitive or so impractical. Yet, one can identify three broad areas of mathematical activity that are quite distinct: the construction of conceptual schemes is quite different from problem solving or the elaboration of a mathematical proof. For instance, the activities involved in the construction of the notion of angle are quite different from those used in trying to draw an equilateral triangle with just a pencil and a ruler (without a compass or a protractor); and these activities are very different from trying to prove that the sum of the angles in all triangles is necessarily 180 degrees.

The construction of the notion of angle is an example of the construction of a conceptual scheme. The construction of conceptual schemes requires a long period of time and involves building new cognitive structures. On the other hand, when we give a problem to our students, like drawing an equilateral triangle, we usually assume that they do possess the mathematical knowledge needed to solve it; what is required from them in solving the given problem is a restructuring of their existing knowledge. The third type of activity, that involving the concept of mathematical proof goes beyond problem solving. Fischbein , the first president of PME, has shown that teenagers who knew how to prove a certain mathematical conjecture still felt the need to verify it numerically (Fischbein & Kedem,1982). Few mathematicians would be able to understand this kind of response.

I hope to have convinced you that we should not try to tackle the topic of mathematical processes as a unique, universal process. Instead, I will restrict myself to something with which I am more familiar: the mathematical processes involved in the construction of conceptual schemes. Not too long ago, I would have used the expression 'concept formation' instead of 'the construction of conceptual schemes'. But I recently re-read Richard Mayer's book on Thinking and Problem Solving (1977) to discover that 'concept formation' had been used to describe the identification of a concept from examples and non-examples (pp. 31-33). Clearly, one cannot conceive the construction of fundamental mathematical concepts such as number and the four operations in terms of 'concept formation'. This would be as ludicrous as showing an apple to a child and telling him or her that "it is not a number".

The construction of a fundamental concept in mathematics involves many different ideas that need to be related to each other into some kind of cognitive grid. As opposed to the acquisition of isolated parcels of knowledge, the development of fundamental mathematical concepts involves linking together several different notions into some organic whole forming some kind of cognitive matrix. But all this knowledge needs to be significant and relevant. This can be achieved only when it can be related to problem-situations, that is, situations in which this knowledge provides answers to some perceived problem. For instance, what would be the point in learning the sequence of the number words unless these were used to answer questions about cardinality and rank? Thus, the expression '**conceptual scheme**' is used here to convey both the idea of a cognitive grid or cognitive matrix, as well as the relevant problem-situations.

MODELS OF UNDERSTANDING

One cannot undertake the study of mathematical processes involved in the construction of conceptual schemes from a purely empirical perspective. This inevitably limits such a study to a restricted reductionist aspect of the construction process. But more importantly, any so-called 'empirical study' ignores the fact that whatever is being investigated is always structured consciously or unconsciously by the researcher's own cognition. The need to describe the cognitive processes involved in the construction of knowledge has been recognized by leading psychologists. These processes have been discussed variously in terms of 'thinking' or 'understanding'. Since understanding is the result of some form of thinking, the distinction is not too important.

Bruner (1960) has suggested distinguishing between 'intuitive thinking' (implicit perception of a problem) and 'analytic thinking' (explicit awareness of information and operations involved) while stressing their complementary nature. Regarding the construction of knowledge he has pointed out:

> "It is true that the usual course of intellectual development moves from enactive through iconic to symbolic representation of the world, it is likely that an optimum sequence will progress in the same direction. Obviously, this is a conservative doctrine. For when the learner has a well-developed symbolic system, it may be possible to by-pass the first two stages. But one does so with the risk that the learner may not possess the imagery to fall back on when his symbolic transformations fail to achieve a goal in problem solving." (Bruner, 1966, p.49)

Bruner's emphasis was on the representational aspect of knowledge construction in general. The first model of understanding directly related to the learning of mathematics must be attributed to the British psychologist Richard Skemp who suggested in 1976 two distinct types: **instrumental understanding** referred to blind use of memorized rules ("rules without reason") whereas **relational understanding** was described as "knowing both what to do and why" (Skemp, 1976). About a year later, my colleague Victor Byers and I suggested a more discriminating model that involved four modes of understanding, the first two where those suggested by Skemp, and the other two, **intuitive understanding** and **formal understanding** where based on Bruner's theory. Our model took into account the exceptional importance of mathematical symbolism in mathematics by distinguishing between content (the mathematical ideas) and form (their representations). And this was incorporated into our definition of formal understanding which we described as "the ability to connect mathematical symbolism and notation with relevant ideas and to combine these ideas into chains of logical reasoning" (Byers & Herscovics, 1977). We called it the **tetrahedral model** since we felt that the understanding of a mathematical topic always involved a mixture of the four kinds of understandings and that this idea could be conveyed by representing understanding as a point inside a tetrahedron whose vertices corresponded to the four modes.

The tetrahedral model was very elegant , but when Jacques Bergeron and I tried to apply it to describe the construction of conceptual schemes such as number and the four operations, it proved somewhat inadequate. The model, as well as the preceding ones were more oriented towards problem solving rather than the construction of concepts. That is when we started developing new models more appropriate to our needs (Herscovics & Bergeron, 1983). To a large extent, our work was inspired by Piagetian theory. Piaget's approach was somewhat different since he perceived various levels of understanding. This has been best summarized by Ginsburg and Opper (1979):

> "The first of these levels is *motoric* or *practical* understanding. This is the level of action...Another level of understanding is *conceptualization*. Here the child reconstructs internally the actions that were previously performed directly on objects...A third level of knowledge involves *consciousness* and *verbalizations*. Now the child can deal with concepts on an abstract level, and can express his mental operations in words. The child can reflect on his own thought." (Ginsburg & Opper, p.234)

Of course, we could not use this model literally since it was too developmental and did not quite describe the pupil's construction of school mathematics. Further readings of Piaget indicated that he distinguished between empirical abstraction (dealing with the physical properties of objects) and reflective abstraction (dealing with actions and their coordination) (Piaget,1973). It was reflective abstraction that led the child to the apprehension of the invariance of a mathematical notion such as number in his famous conservation experiments. In fact, Piaget used the terms 'reflective abstraction' and 'logico-mathematical abstraction' interchangeably. These considerations led us to our first model of understanding dealing specifically with the construction of conceptual schemes (Herscovics & Bergeron, 1981). In the next three years, this initial model was constantly improved in the sense of providing clearer criteria for the different levels of understanding (Herscovics & Bergeron,1982, 1983,1984).

In 1987 we had what can best be described as a 'conceptual leap' in our conceptualization of the understanding of mathematics. We presented in

DeKalb our '**Extended Model of Understanding**' since we suggested expanding our previous model into a two-tier one, the first tier dealing with the understanding of the physical pre-concepts and the second tier dealing with the emerging mathematical concepts (Herscovics & Bergeron,1988a). In our initial presentation, we illustrated the usefulness of our model by applying it to the construction of the number concept. This last year, our collaborators have been able to apply the new model to describe an arithmetic operation, that of early multiplication (Nantais & Herscovics,1989), to describe a geometric concept, that of length and its measure (Héraud,1989a), as well as to describe some algebraic concepts such as that of a point in the Cartesian plane, the slope of a straight line and its equation (Dionne & Boukhssimi,1989). The new model is so revealing in the description of the mathematical processes involved in the construction of conceptual schemata that the next section will be devoted to a summary of its structure and its applications.

THE EXTENDED MODEL OF UNDERSTANDING

As mentioned earlier, the extended model consists of two tiers, the first tier dealing with the understanding of the physical pre-concepts and the second tier describing the understanding of the emerging mathematical concept. In this model, the **understanding of preliminary physical concepts** involves three levels of understanding:

intuitive understanding which refers to a global apprehension of the notion at hand; it results from a type of thinking based essentially on visual perception; it provides rough non-numerical approximations;

procedural understanding refers to the acquisition of logico-physical procedures (dealing with physical objects) which the learners can relate to their intuitive knowledge and use appropriately;

logico-physical abstraction refers to the construction of logico-physical invariants, the reversibility and composition of logico-physical transformations and generalizations about them.

The understanding of the **emerging mathematical concept** can be described in terms of three components of understanding:

procedural understanding refers to the acquisition of explicit logico-mathematical procedures which the learner can relate to the underlying preliminary physical concepts and use appropriately;

logico-mathematical abstraction refers to the construction of logico-mathematical invariants together with the relevant logico-physical invariants, the reversibility and composition of logico-mathematical transformations and operations, and their generalization;

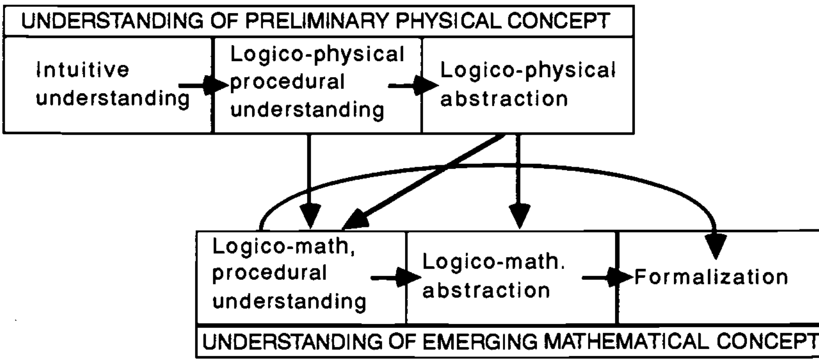
formalization refers to its usual interpretations, that of axiomatization and formal proof which at the elementary level could be viewed as the discovery of axioms and the elaboration of logical mathematical justifications. Two additional meanings are assigned to formalization: that of enclosing a mathematical notion into a formal definition, and that of using mathematical symbolization for notions for which prior procedural understanding or abstraction already exist to some degree.

This model suggests a distinction between on one hand logico-physical understanding which results from thinking about procedures applied to physical objects and about spatio-physical transformations of these objects, and on the other hand logico-mathematical understanding which results from thinking applied to procedures and transformations dealing with mathematical objects. In this framework, one can contend with reflective abstraction of actions operating in the physical realm without necessarily describing it as somehow having to be mathematical. This is not too different from the views Piaget developed in his later years. Piaget has acknowledged this when he suggested two forms of reflective abstraction:

> We will speak in this case of "pseudo-empirical abstractions" since the information is based on the objects; however, the information regarding their properties results from the subject's actions on these objects. And this initial form of reflective abstraction plays a fundamental psycho-genetic role in all logico-mathematical learning, as long as the subject requires concrete manipulations in order to understand certain structures that might be considered too 'abstract'.

Piaget, J. (1974).**Adaptation vitale et psychologie de l'intelligence.**Paris: Hermann, 83-84. (our translation)

The existence of two tiers in our model takes into account the subject's action on his or her physical environment. The two forms of reflective abstraction are comparable to the two aspects of understanding in our model: Piaget's pseudo-empirical abstraction is equivalent to our logico-physical abstraction, his logico-mathematical abstraction is the same as the one we mention in our second tier. The following diagram gives a global view of the model:



It should be noted that the three levels of understanding included in the first tier are essentially linear. Without prior intuitive understanding, the acquisition of concrete procedures could hardly qualify as understanding. Similarly, one cannot expect the child to achieve any logico-physical abstraction without being able to reflect on the procedures used .

Nevertheless, the model as a whole is not linear. The aspects of understanding identified in the second tier need not await the completion of the physical tier. Well before they achieve logico-physical abstraction, children can start acquiring the various relevant arithmetic procedures by the quantification of problems introduced in the first tier. Formalization need not await the completion of logico-mathematical abstraction; for instance, the formalization of the arithmetic procedures will occur much earlier than formalization of the axioms.

Let us remember that the objective here is to develop a model that will adequately describe the construction of a conceptual scheme. We now will look at three such applications.

Application to early multiplication

Table 1a: The understanding of multiplicative situations at the concrete level

Intuitive understanding	Logico-physical procedural understanding	Logico-physical abstraction
-**visual apprehension** discrimination of situations that are multiplicative and others that are not: equal or unequal subsets -recognition of a rectangular array as a mult. situation -comparison between two mult.situations in which one of the factors is changed (number of rows or number of elements in rows)	-**generation** of mult. situations using logico-physical procedures based on the iteration of 1:1 and 1:n correspondences -transforming an additive situation (unequal subsets) into a multiplicative one -covering a rectangle w equal rows or columns -re-interpreting a 1:n multip. situation as a multiple 1:1 correspondence	- **invariance** of the whole wrt irrelevant spatio-physical transformations: -changed configurations of the subsets -'physical' commutativity (90 deg.rotation) -'physical distributivity (4x5 and 4x6 are mult. configurations and so is their composition giving 4x11

If we ask any adult, including teachers, what is the meaning of multiplication of natural numbers, most of them will answer "multiplication is repeated addition". This is quite correct but is limited to a procedural answer, that is, an indication of how the operation has to be carried out. It thus defines multiplication as an extension of addition. But is that the case? Back in 1983 we came to the conclusion that in order to go beyond a procedural perspective, we needed to examine how one could create situations that might be viewed as being somehow 'multiplicative' (Bergeron & Herscovics,1983, Herscovics, Bergeron & Kieran,1983). Piaget & Szeminska had suggested that the iteration of a 1:1 correspondence created a multiplicative situation. We agreed with them but pointed out that the iteration of a one-to-many correspondence was equally successful.

Table 1b: The understanding of the emerging arithmetic multiplication

Logico-math.procedural understanding	Logico-mathematical abstraction	Formalization
-appropriate **explicit arithmetical procedures**: -skip counting on a number line; -adding at least two subsets and counting the others ("3x4 is ...4...8,9,10,11,12") -repeated addition (4+4 is 8 and 4 is 12) -meaningful memorization of number facts	-operation **reversible, invariance** perceived -decomposition into different numerical factors (20=2x10, 20=4x5) -perception of factors as divisors -equivalence of various products:4x6=8x3 -perception of commutativity -perception of distributivity	symbolic representation -interpretation of '4x3' -recognition of 3+3+3+3 as the mult.situation 4x3 -notational represent. of commutativity: ab = ba -of distributivity: a(b + c) = ab + ac

That children can generate quite early various multiplicative situations is not too surprising. But can one claim that by iterating a one-to-one or a one-to-many correspondence they are actually aware of the situation as being multiplicative? Of course not. This claim can only be made when they perceive the whole set as resulting from the iteration of such correspondences. Using this last criterion as a working definition of the pre-concept of multiplication, one is then in a position to perform a conceptual analysis of this notion. Tables 1a and 1b identify the different criteria that can be used to describe the understanding of this notion.

Quite clearly, the knowledge represented in these cognitive matrices is a far cry from the simple procedural definition of multiplication. These tables summarize the analysis presented this summer at the PME meeting in Paris. Using this analysis as a basis, we have developed some very imaginative tasks aimed at evaluating the children's actual knowledge of multiplication (Nantais & Herscovics,1989). The work on the concept of length by Bernard Héraud is equally revealing.

Application to the concept of length

Table 2a:The understanding of physical length

Intuitive understanding	Procedural understand.	Logico-phys.abstraction
visual apprehension -child can distinguish between short and long, a little, a lot (of licorice) - can compare and decide which is longer, shorter, has the same length, based on visual perception	-aligning side by side two objects to see which one extends the other -seriation of a set of rods by comparing them two at a time, by multiple comparisons (as with a fistful of pencils)	**invariance** of length wrt figural transformations: given two straws of equal length, perceives invariance when one is displaced horizontally, vertically, at an angle, when one straw is partially hidden

Table 2b: The understanding of the measure of length

Procedur.understanding	Logico-math. abstraction	Formalization
-aligning units of equal length without any gaps or overlaps -measuring with fewer units be re-using some used before -measuring by iterating one single unit	-invariance of length and the measure of length wrt figur.transformations; -variability of measure wrt the choice of the measuring standard; inverse relationship between numerical measure and unit size	-use of conventional units -symbolic representation -understanding the use of a standard ruler

Again, if we ask an adult what it means to understand the concept of length, the usual answer refers to its measure. In his analysis of this concept, Héraud (1989a) distinguishes between the understanding of length in the physical sense and the measure of length viewed as the emerging mathematical concept. Thus, applying the model to this conceptual scheme we get the cognitive matrices shown in Table 2a and 2b. These cognitive matrices have been used to develop a sequence of corresponding tasks and Bernard Héraud has experimented these in some case studies (Héraud,1989b).

Application to the concept of linear equation

The last conceptual analysis that I wish to present is that of the straight line. Jean Dionne and Driss Boukhssimi (1989) have distinguished between the geometric straight line which they have taken as being in some sense the physical pre-concept, and its equation, which they have taken as the emerging mathematical (algebraic) concept. Their analysis provides the following cognitive matrices:

Table 3a: Understanding the geometric straight line

Intuitive understanding	Procedural understand.	Logico-phys.abstraction
Visual apprehension of the constant direction of a straight line; of inclination; horizontal and vertical;	–angular measure of line; –finding slope of line; –given slope, drawing corresponding lines; –given slope and point, drawing corresp. line	–**invariance** of slope; –parallel lines have equal slopes; –conversely, each slope determines a family of parallel lines.

The above cognitive matrix shows that even for some concepts in secondary school mathematics, our model brings together into some organic whole a vast amount of knowledge we can describe as a 'conceptual scheme'. Of course, the model may not be adequate to describe the construction of more advanced concepts that are formal extensions. For instance, it may be difficult to justify other than formally the extension of the multiplication of natural numbers to the multiplication of negative integers.

Table 3b: Understanding the algebraic straight line

Procedur.understanding	Logico-math. abstraction	Formalization
–covering a line through origin with points and finding a relationship between coordinates (e.g. "y-coord. is twice the value of x-coord.) –using a parallel translation to find pattern (e.g. "y-coord. is twice the x-coord. plus 3") –discovery of constant coord. for horizontals and verticals.	–pointwise perception of straight line as infinite set of points; –representation by (x,y) of a 'variable' point on line; –recognition of y=ax as representing sheaf of lines through origin, y=mx + b for other lines, y=k for horizontals, x=k for verticals; –generalization: each straight line can be represented by an equation	–expressing invariance of slope, m, by using a fixed point (a,b) and a variable point (x,y) in the formula for slope: m=(y-b)/(x-a) –transformations to other forms of the equation

As mentioned earlier, the model is not linear. Nor does it describe how a specific learner progresses in the acquisition of his knowledge. Nevertheless, the model proves to be extremely useful in providing a means of structuring our study of mathematical cognition. It provides a frame of reference enabling us to view learning in a constructivist perspective and to follow the learner among the possible paths. Let us now look at the mathematical processes involved.

PROCESSES INVOLVED IN THE CONSTRUCTION OF CONCEPTUAL SCHEMES

Not so long ago we would have been tempted to consider each of the six cells in our extended model as a distinct mathematical process. The weakness in such an interpretation is that it reduces all logical thinking to mathematical thinking or vice versa, all mathematical thinking to logical thinking, and thus does not discriminate between intellectual processes applied to physical objects and those applied to mathematical objects. However, even if we consider the processes involved at the first tier to be logico-physical, they nevertheless must be included among the processes involved in the construction of mathematical concepts. Ignoring the first tier would bring about a purely formal type of mathematical knowledge.

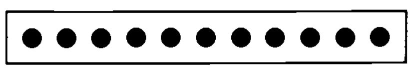
Our model of understanding identifies distinct intellectual processes. Applying the model to describe the understanding of a conceptual scheme results in the gathering of criteria for each component. One can then use these criteria to design corresponding tasks. For instance, we have done this for the concept of natural number by using prior research by Piaget, Karen Fuson, Leslie Steffe, Gelman & Gallistel, James Hiebert, Hermine Sinclair and many more. Nevertheless, the existence of such a model leads one to raise many new questions.

Applying this model to the number concept we have identified the notion of **plurality** , that is, the distinction between one and many, and the notion of **position** of an element in an ordered set, as two preliminary physical concepts (Bergeron & Herscovics,1988, Herscovics & Bergeron, 1988b). Defining number as a **measure of plurality** and also as a **measure of position**, we could identify these as the emerging numerical concepts. Using the above analysis we have developed a sequence of about forty tasks aimed at uncovering the child's numerical knowledge in about three or four interviews lasting on average 30 minutes each. I would like to describe some of the new tasks we have designed to assess the children's understanding of these notions.

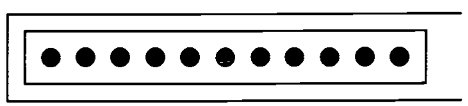
In view of the first part of our definition of number as a <u>measure</u> of <u>plurality</u> , the logico-mathematical abstraction of cardinal number must reflect both the **invariance of plurality** and the **invariance of its measure** with respect to irrelevant spatio-physical transformations. Over twenty years ago, Piaget's collaborator Pierre Gréco (Gréco & Morf,1962) felt the need to distinguish between the children's conception of plurality and the meaning they attach to enumeration. They modified the original conservation task involving two equal rows of chips by asking the children to count one of the rows before stretching the other one; they then asked **how many** chips were in the elongated row while screening it from view. Those who could answer the question were said to **conserve quotity**. Gréco found that many five-year-olds claimed that there were seven chips in each row but that the elongated row had more. Thus, these subjects **conserved quotity without conserving plurality**. For these children, to conserve quotity simply meant that they could maintain the numerical label associated with the elongated row, but their count was not yet a measure of plurality, since they thought that the plurality had changed. It is only when both plurality and quotity are conserved, when both invariances are perceived, that number can become a measure of plurality. At that stage, one can claim that the child has achieved a logico-mathematical abstraction of cardinal number. Of course, the Piaget and the Gréco tasks are not the only ones which can be used to assess abstraction of cardinal number. Let us describe the task we have designed to assess the effect of the visibility of the objects on their sense of cardinality (Herscovics & Bergeron,1989).

Children were given in the first interview a row of 11 chips glued on a piece of cardboard. They were told:

Here is a large cardboard with little chips glued to it.

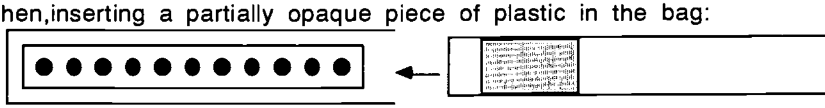


Look, I'm putting the cardboard in a bag (inserting it)

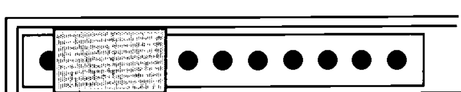


Good, are all the chips in the bag?

Then,inserting a partially opaque piece of plastic in the bag:



Look, I'm putting a plastic strip in the bag



And now, are there more chips in the bag, less chips, or the same number as before?

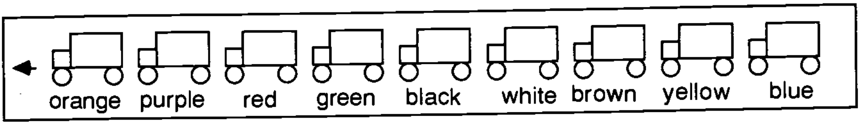
In the second interview, this task was repeated but the children were asked to count up the number of chips before they were inserted in the bag. The following table shows the results obtained:

No of subjects	Invariance of plurality	Invariance of quotity	Invariance of both
n=91	17 (18.7%)	55 (60.4%)	9 (6.3%)

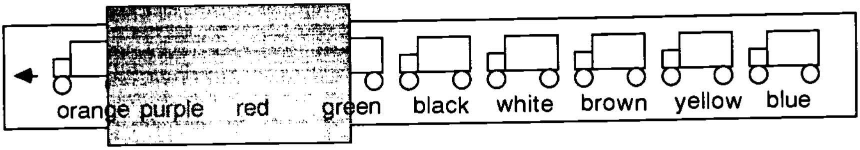
The low results on the invariance of plurality are most astonishing. They indicate that among most kindergartners the visibility of the objects is still primordial. This is not a question of the permanence of the objects, since it is acquired well before the age of five. Nor is it a question of the enumerability of the partially hidden set as evidenced by the invariance of quotity. Visibility of the objects affects these children's apprehension of plurality.

We have somewhat similar results on the invariance of ordinality with respect to the visibility of the objects. Resulting from the second part of our definition of number, number taken as a measure of position, the logico-mathematical abstraction of ordinal number must reflect both the variability and invariance of position as well as the variability and invariance of its measure as determined by enumeration. In designing the tasks on ordinality we have tried to maintain a parallel with some of the tasks designed to assess the understanding of cardinal number. For instance, in the case of cardinality we distinguished between the **invariance of the plurality** of a set under some transformations and the **invariance of quotity**, that is, the invariance of the numerical label associated with a set as a result of enumerating it or enumerating another set in one-to-one correspondence with it. Similarly, we could distinguish between the **variability or invariance of the position** of an element in an ordered set and the **variability or invariance of "ordity"**. "Ordity" refers to the variability or invariance of the numerical label associated with its position, a label that has been obtained by using enumeration to find its rank or the rank of a corresponding element in another set which is in a one-to-one correspondence with the first set. The following task was designed to assess the children's perception of the invariance of ordinality with respect to the visibility of some of the objects (Bergeron & Herscovics,1989a):

A row of 9 little trucks was drawn on a cardboard, each truck colored differently.



The children were told: " Look, here is a parade of trucks. Can you show me the white truck?" (in sixth position). After it was duly pointed out, the interviewer announced "The parade must now go under a tunnel" and then proceeded to slide the cardboard under a 'tunnel' in such a way that part of the first truck was still visible but three trucks were hidden.



The children were then asked:"**Do you think that the white truck has kept the same number in the parade or do you think that it now has a different number**?"

The above task was repeated in the second interview but the children were now asked to <u>find the rank</u> of the white truck. After they had counted to determine its rank (sixth), the parade was inserted into the tunnel and the interviewer asked again "Now, can you tell me the number of the white truck in the parade?". The following table provides the data obtained with this second task assessing the invariance of position and of ordity :

No of subjects	Invariance of position	Invariance of ordity	Invariance of both
n=91	36 (39.6%)	47 (51.6%)	28 (30.8%)

Combining both the invariance of position and the invariance of ordity, we note that the results are somewhat better than those achieved on the comparable cardinality task. But visibility of the objects is not the only thing affecting the kindergartners' thinking. We have designed an ordinal task that might be construed as analogous to the Piaget and Gréco tasks on cardinality involving a comparison of two parallel rows.

The interviewer aligned a row of 9 identical cars, and asked the children "Would you make a parade just like mine and next to it?" while handing over another 9 cars. Then using a blue colored sheet of paper (the river) and a small piece of cardboard to represent a ferry she explained: "The parades must cross the river on a little ferry boat. But the ferry can only carry two cars at a time, one car from each parade. When we are ready, we take one car in my parade" (putting her lead car on the ferry), "and one car from your parade" (asking the children to put their lead car on the ferry). The ferry then crossed the river with the two cars, unloaded them, and came back for two more:

The cars were then put back in their initial position and the subjects were told: "Now I"m putting this little arrow on this car (the seventh car in the interviewer's row). Can you put this other arrow on the car in your parade which has the same number as mine?" Once this was done, the interviewer announced "Now look, the parades move on" while moving the child's parade a small distance and moving her own parade somewhat further by the length of two cars:

The children were then asked : "**Do you think that the two cars with the arrows will cross the river at the same time?**" Following their answer, they were asked to show the interviewer how the two parades were to cross the river in order to verify that they were aware that the cars had to be ferried in pairs. If they did not succeed, the task was repeated the following interview. If they did succeed, they were stopped after two pairs had crossed and they were asked again if they thought that the two cars with the little arrows would cross at the same time. This last question was intended to assess any possible change that might occur since the two indicated cars were now in fourth position.

Following the above task on the invariance of position with respect to translation, the invariance of ordity was immediately assessed. The "river" and "ferryboat" were removed and the two parades of cars we re-aligned next to each other. The interviewer then stuck a little arrow on the sixth car in one of the parades while asking the children "Can you tell me the number of this little car?". After they had counted to determine the rank, they were asked to put another little arrow on the car in the other parade that had the same number. When this was completed, the interviewer moved both parades but made sure that one parade was two cars ahead of the other one.

She then asked: "Now, without counting, can you tell the number of this car?" indicating one with an arrow, and immediately, with her two hands, shielding the parade from the child's view . Whenever the subjects could not provide a correct answer, they were asked if they remembered the number of the car with the arrow in the other parade (which they had counted). If they did not remember it, the task was repeated on the following interview. The following table provides data on the invariance of position (correct in both instances, when cars in sixth and fourth position) and on the invariance of ordity:

No of subjects	Invariance of position	Invariance of ordity	Invariance of both
n=91	12 (13.2%)	60 (65.9%)	10 (11.0%)

The results on the invariance of position are much lower than on the tunnel task (13% vs 39%) but they are better on the ordity question (66% vs 52%).

The three tasks that I have described dealt with the partial visibility of the objects and the effect of translation on the child's apprehension of ordinality. One could assume that it is only at the kindergarten level that children are so affected by their visual apprehension. However, that this is not so is evidenced by the data collected on first graders by Anne Bergeron (1989). Her results obtained in Grade 1 are nearly the same as those we found in kindergarten.

Having discussed the evaluation of the mathematical processes involved in the construction of conceptual schemes, let us now look at the analysis of theses processes.

ANALYSIS OF MATHEMATICAL PROCESSES

Within the present context, the purpose of any analysis of mathematical processes is to establish the different ways that children are likely to construct a conceptual scheme. While each cell in our model represents a distinct intellectual process, one should not consider these as unrelated. The various arrows we have indicated between the cells are indicative of some of the relationships of which we are presently aware. For most of the indicated arrows, examples can be found in the three conceptual analyses we have included. However, some relationships may not be as evident. For instance the arrow relating logico-physical abstraction to logico-mathematical procedural understanding needs some explanation and can be illustrated in the following example.

If we take another look at the data presented in the introductory remarks regarding the kindergartners' ability to integrate the counting on procedure into their notion of cardinality, we may raise some interesting conjectures. For instance, would the conservation of plurality be a necessary pre-requisite? After all, since children's numerical thinking can be affected by a change in configuration, then perhaps they need to have achieved this kind of invariance prior to to handling tasks involving counting on a partially visible set. This would relate the invariance of plurality, which in our analysis of number belongs to the cell dealing with logico-physical abstraction of number, with logico-mathematical procedural understanding exemplified by the counting on procedure.

For this particular conjecture we have some interesting results. In our sample of 91 children 86 could continue counting the row of chips. The following table indicates a possible relationship between the conservation of plurality on one hand and the synthesis of counting on and cardinality on the other hand:

Conserve plurality	Can say how many chips	Cannot say how many
51	21 (41.2%)	30 (58.8%)
Do not conserve plurality		
35	10 (28.6%)	25 (71.4%)

The data indicate that conservation of plurality is not a sufficient condition since 59% of those who perceive this invariance do not know how many chips they have counted. But is it a necessary condition? Well, here is where researchers looking for 'clean' results may feel annoyed. It is a 'mild' necessary condition since its predictive value is only of the order of 71%.

Again, we have here evidence to the effect that despite our attempts at establishing some kind of cognitive hierarchy in the acquisition of knowledge, the children's learning is not linear and we will always discover cognitive hierarchies different from the ones we expect. Nevertheless, these analyses may yield a small number of likely 'construction profiles' that may prove valuable to teachers and educators.

BY WAY OF CONCLUSION

Research on the mathematical processes described in this paper is lengthy and difficult. It usually takes a few years to get anywhere. As a first step, one can do a search of the research literature on a given conceptual scheme and start filling in some cells in the cognitive matrix provided by the model. But then, inevitably, one discovers that some aspects of understanding suggested by the model have never been raised. Finding criteria for the evaluation of these aspects is not usually spontaneous. Translating these criteria into tasks and questions requires a fair amount of exploration. The first few experiments are essentially pilot studies in which one tests the tasks and the wording of the related questions. But then, these pilot studies usually reveal that some important elements are missing in the initial design. Of course, since thinking processes are the focus of our research, we have no choice but to use interviewing methods with a few subjects dealt with as case studies. Eventually, one assembles enough experience to more ambitious enterprises. One ends up with an extensive program for a sequence of interviews. These interviews are semi-standardized, thus enabling the interviewer to explore some unexpected aspect of an individual's thinking. But the purpose of semi-standardization is essentially to warrant the possibility to gather reliable data for larger samples in order to discover various trends and to establish likely construction profiles. Of course, the size of the sample is limited by the time and personel resources available. For instance, our initial research on number was started in 1982. It was only six years later that we were sufficiently satisfied with our planned interviews to handle a sample of 90 children.

The type of research suggested in this paper has some important pedagogical implications. For instance, teachers can be trained to use models of understanding to answer the question "What does it mean to understand (a specific conceptual scheme)?" We have been quite successful in including what we have called 'epistemological analysis' in the training of teachers in various Quebec Universities (Bergeron & Herscovics,1985). It is epistemological in the sense of providing a frame of reference in which one can study the growth of an individual's conceptual knowledge. Two major doctoral dissertations have been produced within this context. Jean Dionne's thesis has shown that this kind of training in conceptual analysis has an impact on the teachers' perception of both mathematics and mathematics education (Dionne, 1986,1988). Another thesis was completed this year by Nicole Nantais (1989) and dealt with the design and experimentation of mini-interviews, that is, short 10 minute semi-standardized interviews designed for use by the teachers in their classroom to evaluate a specific aspect of understanding of some key concept being studied.

Our model of understanding is not an instructional one. Nevertheless it can have an important effect on teaching. The model clearly shows that the learning of any fundamental concept in mathematics is necessarily a complex enterprise and that it cannot be reduced to the mere acquisition of procedures providing in Skemp's words a bit of 'instrumental understanding'. From the broader perspective provided by conceptual analysis, teachers can start designing some lessons aimed specifically at establishing the kind of situations that might lead students to integrate different aspects of understanding of a conceptual scheme in the curriculum. But the design of such lessons might also be part of the research domain. Investigators might plan a series of teaching experiments to assess if the intended understanding has been achieved.

Finally, the research on mathematical processes might have a serious impact on diagnosis and remediation. The conceptual analysis achieved with our model of understanding provides a broad perspective relating various elements into a conceptual whole. One can use it to discover the sources of a student's failure to acquire a given conceptual scheme. Quite often, the difficulties are due to an instrumental type of learning of some mathematical procedure. Designing activities corresponding to the first tier of our model dealing with the understanding of the preliminary physical concepts may provide the necessary foundation on which to build the mathematical knowledge.

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TO KNOW MATHEMATICS IS TO GO BEYOND THINKING THAT "FRACTIONS AREN'T NUMBERS"

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Introduction

The source of our interest in the acquisition of mathematical competencies is our concern with early cognitive development and how environments might nurture concept acquisition at later points. We come to our work on number concepts as researchers of cognitive development and not as mathematics educators. That is, we focus on asking what knowledge systems young children construct -- be these mathematical or otherwise -- on the fly, as they go about interacting with environments that do not yet include those presented in schools.

Findings from a variety of labs researching similar questions converge on the conclusion that young children are busy learning a great deal about some domains. In some cases, the knowledge they acquire before they start school is extensive enough to warrant our granting them something akin to a theory. Of interest is that these implicit or intuitive theories can be much like those held by individuals who have been offered the benefits of formal schooling. That is, early knowledge systems sometimes are robust enough to continue to either co-exist with what is learned later or even resist inputs designed to foster change in them. For example, even students who have taken courses in college physics answer questions about motion as if they hold to a pre-Newtonian theory of motion (McCloskey, 1983). Indeed, the list of misconceptions that some students continue to believe in after they have had formal instruction is ever-growing and includes concepts from algebra, chemistry, electricity, statistics, etc.

We do not mean to conclude that instruction in the foregoing areas is never successful. Indeed, many students do benefit from the

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opportunities offered to them in school and thereby acquire knowledge systems of motion, electricity, and the like, that are closer to the versions held by experts. The fact that learning occurs for some students allows us to contrast these individuals with students who continue to hold on to their informally learned theories. We can ask how this happened, how to characterize the learning that takes part in those who do approach our interpretations of the data we offer them. It is akin to a theory change, a paradigm switch? Or it is more a matter of just learning enough?

In order to study these kinds of questions about conceptual development and change, we have chosen to focus on different possible conceptions of number. We do so for several reasons. First, there have been changes in the definition of what is or is not a number. These changes are well - described in texts on the history of mathematics. As we will see, this offers us clues regarding potential points of difficulty along the road from one conception of number to more advanced (complex?) ones. Second, we can ask to what extent, if any, do alternative formal accounts of the nature of a number relate to different psychological representations of numbers.

In what follows, we first summarize what appears to be an early and robust idea about number, this being that numbers are representations that result from counting things and/or using counting-based algorithms to add and subtract. We then turn to ask how children interpret a variety of novel inputs they encounter when they begin school. Our focus is on those kinds of data that are relevant to the idea that fractions are also numbers. For, we thought Kindergarten, Grade 1 and Grade 2 children might interpret these inputs in ways more consistent with the idea that a number can only be a positive integer. This initial exploration of how children initially interpret the number line, numerical terms, and numerical representations of number leads us to suggest, using the words of one of our articulate second grade subjects, that "fractions aren't numbers" for these young children. Questionnaire results from still older children make it clear that this belief can persist through the elementary school years. They also point to a relationship between being able to talk about the language of mathematics and a move toward an alternative conception of number, one that assigns numberhood status to fractions, be they written in decimal or fraction format. We end with the idea that the shift here represents a first step in a theory change as to what numbers are.

Some Findings About Non-Schooled Notions of Number

It is now apparent that all over the world, both young children and non-schooled adults use counting algorithms to solve arithmetic problems. These are often made up and resemble those that schooled children invent. Whether schooled or unschooled, individuals have strong tendencies to decompose natural numbers into manageable or known components, to count when adding or subtracting, and to use repeated addition (or subtraction) to solve "multiplication" (or "division") problems. These multiplication and division solutions are used both before and after children have been taught more standard multiplication and division algorithms in school. They are also used by un-schooled adults and children in a variety of settings and cultures (e.g., Carraher, Carraher, & Schliemann, 1985; Ginsburg, 1977; Groen & Resnick, 1977; Lave, 1988; Resnick, 1986; Saxe, 1988; Saxe, Guberman & Gearhardt,1987; Starkey & Gelman, 1982)

Some of the algorithms invented by older children and un-schooled adults are more complex than those used by preschool children. Still, as Resnick (1986) notes in her analysis of these, all use principles of counting and addition (or subtraction) with the positive integers. An example from her own work illustrates this. Pitt (7 years-7 mos) said that he solved "two times three" as follows:" *...two threes...one three is three, one more equals six*". Similarly, schooled and unschooled individuals in Africa and Latin America use a combination of number decomposition moves and repeated additions (or subtractions) to solve the multiplication problems presented by investigators. Young children are also less able to work with large numbers; otherwise, it is hard to distinguish their invented, out-of-school, solutions from older childrens' and adults'. (Carraher et al.,1985; Saxe,1988). Lave, Murtaugh, and de la Rocha's (1984) work with shoppers in a California supermarket provide an especially compelling documentation of how everyday "intuitive" solutions for determining unit prices are preferred over any taught in school.

Elsewhere we have suggested that the widespread invention of algorithms that are based on counting and/or repeated addition (subtraction) algorithms provides further support for our conclusion that a universal set of implicit principles governs the acquisition of initial mathematical concepts (Gelman,1982; Gelman & Meck, 1986). Our idea has been that a skeletal set of counting principles, in conjunction with some principles of addition and subtraction, promote the universal uptake of mathematical data that are relevant to these principles. The skeletal principles provide the a priori structure necessary for learners to notice, assimilate, and store relevant data in an organized manner (Gelman, in press; Gelman & Greeno, in press). For example, it looks like young children's implicit knowledge of the counting principles guides their initial attention to the counting sequence(s) in their language, the subsequent learning of which promotes the further development of these principles. Implicit knowledge of counting principles serves to identify the class of behaviors that are potential counting ones as opposed to, say, potential labelling ones. If a child hears a speaker say "one, one, one" while pointing to a set of like items in a display, that child is most likely to assume that "one, one, one" is a string of sounds that include a common label for each object (Markman, 1989). If instead, the same child hears "one, two, three" for the same input, she is more likely to assume that she has heard a string of sounds that can serve as tags for counting. In the former case, the constraint on the use of the string is that exemplars of the same object class share the same name; in the latter case the constraint on the generated sound string is that each and every object in a to-be-counted collection must be uniquely tagged; it matters not whether the items are or are not identical (Gelman & Meck,1986). In other words, implicit principles outline the constraints on the use rules and thereby provide children with clues for keeping separate their interpretations of novel sound sequences. Of course, sorting of the different sound sequences into different classes of words is but a first step. Children still have to master the entries in the identified conventional count list of their linguistic community (Fuson, 1988).

Many have contrasted the discrepancy between the relative ease of learning to count and use counting-based arithmetic solutions in everyday settings with the difficulty of leaming and applying school-taught mathematics (Lave, 1988; Resnick,1987; Saxe, 1988). What should schools do about this discrepancy? Should they make a concerted effort to adopt teaching models that more closely resemble the learning that occurs in everyday situations? This could be a reasonable policy if **both** of the assumptions that are underlie this recommendation hold. These are (1) that schooling should build on what children find relatively easy to learn, and (2) that children bring to school a knowledge of mathematics that is readily expanded if everyday settings are mimicked in schools.

We share the view that instruction should take into account the knowledge base a child brings to school. But it is not clear that the settings that supported mathematical learning in these childrens' everyday interactions are the ones that should be imported into the classroom. We have to ask whether knowledge so acquired provides a ready base upon which to learn more mathematics, let alone achieve the goals of many a mathematics teacher and teach facility with the language and structure of at least some of the body of mathematical knowledge. Indeed, we are beginning to worry that learners' inclinations to apply their initial theory or intuitions about number to the novel inputs offered in schools interferes, for a very long period of time, with mastery of crucial material. Although their initial, pre-instructional assumptions about the nature of numbers serves many well in everyday interactions, we will see that these self-same assumptions may contribute to the difficulty of learning some kinds of mathematics.

Fractions Are Not What You Get When You Count Things

Although there are principles that both guide and structure early learning about counting, this could be a mixed blessing. Learning to count and add without such principles is exceedingly hard, even with the aid of highly structured input and much opportunity for rote learning practice. For example, for Downs Syndrome children, the large majority of whom appear not to be guided by an intuitive understanding of counting, learning to count progresses very slowly or not at all, despite intensive drill (Gelman & Cohen,1988; Irwin,1989). In contrast, normal preschool children are able to count in principled ways, despite more limited and less consistent input data (Gelman, Massey & McManus, in press).

Whatever the mathematical prowess of the young child, it is beginning to look as if further learning requires one transcend the principles that guided their early learning. Much of mathematics involves operations and entities other than counting and the addition and subtraction of the counting numbers (or the positive integers). Might it be that when it comes time to go beyond the early knowledge that is built upon the counting principles that continued adherence to these principles will hinder progress? Insofar as the principles that guide early arithmetic learning do not incorporate more advanced operations with numbers, and insofar as they do not work without modification when applied to numbers that are not count numbers, difficulty in learning modern mathematics might well be the rule as opposed to the exception. To illustrate why, we need not go too deeply into mathematics. The idea that a fraction is a non-counting number but nevertheless a number, will serve our purposes -- especially since it seems to be a watershed in elementary school mathematics learning (e.g. Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980).

If it is correct to characterize the data on unschooled mathematical abilities as consistent with the conclusion that numbers are first thought to be "what one gets when one counts things", then common classroom inputs for learning about fractions may be misinterpreted by young pupils. One cannot count things to get the answer to "Which is more 1/2 or 1/4; 1.5 or 1.0." Similarly, number lines are not simple representations of whole numbers and yet young children might think they are. Or young children might know that one can take apart a circle and a rectangle and get two "halves" on both occasions and still not appreciate why each of these halves can be represented with the written expression 1/2. The interpretation of the latter might be assimilated to the idea that such marks on paper must be about whole numbers.

In the absence of implicit principles for dealing with fractions, young learners might "overgeneralize" their counting principles and produce a distorted assimilation of the instructional data on fractions to an implicit theory of number that cannot handle such data. For example, if they do assimilate their understanding of the language of fractions to the implicit system of mathematical principles available to them, they should "read" fractions, or non-integer numerals, as if these are representations for the counting numbers as opposed to ways consistent with the principles underlying fractions. Thus, for example, they might choose $\frac{1}{4}$ as more than $\frac{1}{2}$; or when asked to place $1\frac{1}{2}$ circles on a number line, they might decide they have "two" things and place the stimuli at the position for 2 on the line.

The study of children's knowledge of fractions serves two goals: It serves as an entry point for assessing limits on the young child's knowledge of mathematical principles, and it provides a way of considering how young children first interpret some of the symbolic tools of mathematics they encounter in their classrooms. As we will see see, these two goals converge on a theoretically and pedagogically important question.

The Main Study

Background

Studies of how school-aged children work with rational numbers have focused on children who are already half way through elementary school or even in high school (e.g. Behr, Wachsmuth, Post & Lesh, 1984; Hiebert & Wearne, 1986; Kerslake, 1986; Nesher & Peled 1986). We know of none that have targeted children in their first few years of school, presumably because early math instruction does not focus on this topic. Since we were as much interested in how children would interpret reasonably novel as well as familiar data in this domain, our first study focused on Kindergarten (N=16, mean age 5 years - 9 months), First Grade (N=12, mean age, 6 years - 9 months), and Second Grade (N=12, mean age 7 years - 9 months) children.

We thought it was especially important to focus on young children. Two conclusions emerge from relevant research with older children. First, even older children have a tendency to extend inappropriately generalizations they learn about integer operations to operations with other numbers. For example, they seem to believe that the product that results when one multiplies two numbers is always larger than either of the original numbers -- even when the numbers are fractions. (e.g. Greer, 1987; Hart,1981). Note that this principle is valid for the *addition* of positive rationals but not for their multiplication. More generally, many secondary school students lack a mathematical understanding of division and multiplication (e.g., Fishbein, Deri, Nello, & Marino,1985; Vergnaud, 1983), a fact that cannot help but make one wonder how they could possibly understand that fractions are numbers, albeit not integers. These findings led us to think that it might be exceedingly hard for teachers to penetrate the child's spontaneous theory of number, despite lessons designed to do so. Before considering such a possibility one has to determine whether the proposed initial theory influences how children deal with their first encounters with fractions. If the young do bring with them the idea that numbers are what one gets when one counts things, they might start building, at an early age, erroneous representations of data meant to exemplify alternative notions of what numbers are about. These representations could, in turn, stand in the way of children's correct interpretation of later lesson plans on fractions.

Children in the early grades receive some instruction about fractions. They also encounter fraction-relevant material. Since our sample was interviewed at the end of their school year, even the Kindergarten children had had some experience with a standard number line and representations of $\frac{1}{2}, \frac{1}{4},$ and $\frac{1}{3}$. Experience with the latter included at least their spoken forms, whole-word written examples of the corresponding phrases, their numerical representations, and appropriately labelled and marked measuring cups. The children in the first and second grades had more experiences like these, both in terms of classroom presentations and testing opportunities. Therefore, although their curricula did not delve into the conceptual and arithmetic characteristics of fractions as numbers, the children were offered some relevant data about the nature of fractions. Our study was designed to provide information on how these children interpret such offerings.

Design and Procedures

Plan of the study : Children who participated in the study attended one of three schools in the Greater Philadelphia area, all of which drew from middle class samples. We had signed permission from the parents of children to interview them. This was done in a quiet room away from their classrooms. For all but a few children, the interviews were conducted on three different days. The first and second days of the interview were separated by at least two days (but not more than a week). The third day of the interview could occur anywhere between one and two months after the second. Exceptions to this timetable were forced by the approaching end of the school year. Several children were presented with their third day items within one to two hours after they finished their second day items because they were about to go on their summer holidays.

The design of the study included a pretest, a five-phase placement interview, and a follow-up battery. The sequence of the placement phases was designed to provide more and more task-relevant information without giving specific answers. Brown, Campione and Bryant (1984) recommend the use of such progressively more explicit "hinting" in order to bring out whatever competence a child might have. Our own work confirms their view that such hints serve to limit misunderstanding about the task and therefore false negative attributions by an investigator. (e.g. Gelman, 1978; Gelman & Meck, 1986).

The items for the first two phases of our placement sequence were presented without hints so as to allow us to obtain some baseline data. Each successive phase after these introduced more and more relevant mathematical information and offered more detailed mathematical descriptions about the props in the task. This meant that we provided successively more hints about the nature of the task as we moved to the use of more and more explicit mathematical language.

We anticipated that the way children would respond to our hints would be theoretically informative. In other studies we have found that hints are differentially effective, depending on how much mathematical competence a child brings to the task to start. For example, in our study of how Downs Syndrome and normal preschool children solved a novel counting problem, the majority of the Downs Syndrome children failed to benefit from hints, even when these were very explicit (Gelman & Cohen, 1988). In contrast, the preschool children improved their solutions, even when simply given the chance to try again. Other findings in the study converged on the conclusion that the preschool children were better able to take advantage of hints because they brought a more principled understanding to the task. This knowledge, albeit incomplete, enabled them to take advantage of subtle hints, to recognize their own errors and to try again. Of interest here is whether a similar pattern will emerge: will some children benefit from hints and others not? If so, we can ask whether this is due to a related difference in fraction-relevant knowledge and/or the ability to apply it. For example, it might be that initial levels of success on the fraction placement task will be related to the ability to interpret the terms and symbols used in later phases.

All sessions were tape recorded and a sample were videotaped for later transcription.

Details about the study : Table 1 presents the organization of the study which included a pretest, a 5-phase fraction placement interview, and a follow-up set of items. The pretest questions and interactions familiarized children with "our special number line". When children came into the room, they saw 'The Count', a puppet from the American television program Sesame Street, sitting in the middle of the table alongside the folded-up number line. They were told that The Count had come to visit them at school because he wanted to learn new things about numbers.

TABLE 1: SEQUENCE OF ITEMS IN THE FRACTION PLACEMENT STUDY

PART OF STUDY	INSTRUCTIONS AND/OR TASKS
	DAY 1 OF INTERVIEW
	PRETESTS
A. SOME RELEVANT VOCABULARY	-E Shows S 1-1/2 (1/2,1-1/4, 1/4,1/3) circles and asks "How many"? -If necessary, provide answers.
B. INTRODUCTION OF NUMBER LINE	- E talks about the "special number line", circles instead of numbers. - S asked to show where 1&3 are. -Then asked where 2 goes; what number would be before 1 and after 3. (E tells S if necessary). -Talk between S and E about order/next,"more/less", "same", e.g. "Is one less than or more than two"? -S given task to place N (1,2,3,&4) circles.
	BEGIN FRACTION PLACEMENT PHASES
PHASE 1: NO DEMONSTRATIONS OR LABELS FOR TEST DISPLAYS	-S is asked to place and explain placement for 1-1/2, 1-1/4, 1-1/3 . - end phase with E repeating 1-1/2 if S wrong on it; now show if necessary.
PHASE 2: NO DEMONSTRATIONS OR LABELS FOR TEST DISPLAYS	- Same as Phase 1, except values 1/2, 1/3, 1/4. - Repeat 1/2 as above if necessary. E asks if fractional parts are "same" or "different".
PHASE 3: BEGIN WITH HINTS (I.E. INFO. THAT COULD BE RELEVANT BUT DOES NOT GIVE ANS)	e.g. hint: *"We can put this (1/2) on a number line like this between 0 and 1 because it is less than 1 and more than 0.", ...etc.*
--PLACEMENT TEST ITEMS;	- S now asked to place 2-1/2, 1-1/4, 1/3. (E does not identify values)
--MORE HINTS	- Ending items: E asks about numbers between 1 & 2 and 0 &1; S counts and solves
--ARITHMETIC & COUNTING	X+0 (where X is an integer), 1/2+1/2, 1/2+0.
	END DAY 1

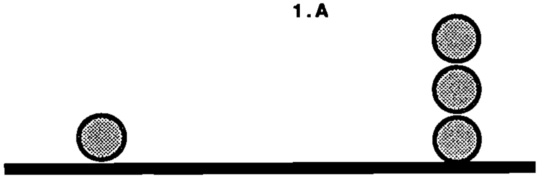
TABLE 1 CONTINUED:

PART OF STUDY	INSTRUCTIONS AND/OR TASKS
	DAY 2 OF INTERVIEW
	(2 - 7 DAYS AFTER DAY 1)
	PLACEMENT PHASES CONTINUED
PHASE 4: BEGIN WITH REVIEW.	-S places 1,2,3 for review.
E NOW LABELS ITEMS WHEN THEY ARE FIRST PRESENTED.END WITH ORDERING QUERIES ABOUT LABELLED VALUES.	-" This is 1-1/4 (1-1/3).Where does it go? -"Which is more, 1-1/4....?"
PHASE 5: ALL ITEMS LABELLED & COMPARED IN SEQUENCED PAIRS. S ASKED WHICH IS MORE OF TWO WRITTEN PAIRS (NOT LABELLED)	-S places and asked to explain placements for 1/2 & 1/3. -More questions with 1/2 vs 1/4; 1/56 vs 1/75
	END DAY 2 OF INTERVIEW
	DAY 3 OF INTERVIEW
	(BETWEEN ONE AND TWO MONTHS LATER)
FOLLOW-UP	-S reads 1/2,1/4 (on cards); - S answers "Which is 1/2?"; "Which is more?" -Talk about measuring cups & their fractional contents; More ordering questions about 1/2 & 1/4.
	END OF INTERVIEW

As can been seen in Table 1, to start the pretest, a child was first shown, one at a time, displays of $1\frac{1}{2}$, $\frac{1}{2}$, $1\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{3}$ circles and asked "how much (or how many since children offered this alternative)" were present. If a child could not name $1\frac{1}{2}$ or $1\frac{1}{4}$, the experimenter pointed to the whole circle and said "This is one circle". Then, while pointing to a part of a circle ($\frac{1}{2}$ or $\frac{1}{4}$), she asked "What's this?" The part was correctly identified for children who could not answer on their own.

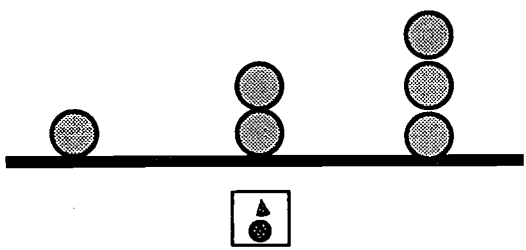
After this introduction to relevant terms, the experimenter unfolded her "special number line" schematized in Fig.1A. It can be seen that our number line did not use numerals to represent the cardinal values for successive integers. Instead, the line had one black circle where the integer 1 should have been, and three black, same-size circles at the place for 3. The number line display was 4'4" long and each circle was 4.75" in diameter. The line was intentionally long so as to given the impression that it went on and on in both directions; it covered the length, and then some, of work tables found in schools.

1.A



The Number Line As First Presented

1.B



A Sample Placement Trial Setup

Figure 1: Schematic Representations Of The "Special Number Line" During Pretesting (1.A) And A Placement Trial (1. B).

Once unfolded the child was told that "our special number line does not have numbers written on it but shows where the numbers should be" and then asked to show us where "1" (and then,"3") were. This done, the child was handed two black circles and asked to show where to put "2" and then do this. Then, while pointing to the left side of "1", the experimenter asked the child what number came before 1; similarly the child was asked what came after 3. This time we did not put anything on the line to represent these values.

The pretest continued with talk about the ordering relations between the whole numbers, e.g. " Is 4 more than, less than, or the same as 3; is 2 more than 1, less than 1 or the same?" If necessary, feedback was provided since we wanted to encourage children to think of ordering numerical values. Then, to end the pretest, children were told "The number line shows the numbers in order". To show us that they knew "how our number line works", they were then asked to place sets of N (1, 2, 3, or 4) small circles (ones much smaller than those on the line) "where they belong".

The pretest phase was followed by a five-phase sequence of fraction placements and related test-items. A sample fraction placement trial is shown in the bottom half of Fig.1. On such trials a child's task was to put each test display below the point on the line "where it belonged" and then explain why it went there.

Throughout the placement phases the color of the whole and fractional parts of circles varied (red, yellow, blue) across, but not within, a display. For example, all items in the $2\frac{1}{2}$ display were red, those in the $1\frac{1}{2}$ display, blue. Size of circle was held constant across and within displays.

No hints were provided for any of the test items used during either of the first two placement phases. In phase 1, test displays contained $1\frac{1}{2}$, $1\frac{1}{3}$, and $1\frac{1}{4}$ circles. The second phase test displays contained $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a circle. Once Phase 2 was done, children were asked how the test items for that phase were the same or different.

As indicated, no hints were provided for the test items used during the first two phases. However, if children erred on $1\frac{1}{2}$ in Phase 1 or $\frac{1}{2}$ in Phase 2, these items were repeated at the end of their respective testing periods. As during the pretest, children now were asked whether the numerical value represented was equal to, more than, or less than a whole number value.

The experimenter started Phase 3 by placing $\frac{1}{2}$ circle between the positions for 0 and 1, and saying "we can put one half between zero and one because it is more than zero and less than one". A similar trial followed with a display with $1\frac{1}{2}$ circles. Finally, the child was told "We can count $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2". Then the child was asked to place the test items ($2\frac{1}{2}$, $1\frac{1}{4}$, $\frac{1}{3}$) and explain their placement. The experimenter did *not* verbalize the corresponding numerical values for these test items.

The experimenter resumed talking about number at the end of the phase-3 part of the placement interview. No props were used for this. A child was simply asked whether there were numbers between 1 and 2 (0 and 1), and if so how many there were. If a child said that there were no numbers between the named integers, the experimenter went on to ask if $1\frac{1}{2}$ (or $\frac{1}{2}$) was such a number. The opportunity to count by rote and solve *verbally* presented arithmetic problems ended the interview for the first day.

Phase 4, which occurred at least 2 days, and as much as a week, after the previous placement phase, started with a brief reminder about the "special number line". For all subsequent test and demonstration items, the experimenter now used the terms that corresponded to the numerical values instantiated by the displays when she presented these. Children were questioned extensively about the ordering relations of the values represented by the described test items as well as the ordered positions the corresponding stimuli should assume on the number line.

Phase 5 was much like Phase 4 and came right after it. These phases differed mainly in terms of the display values presented -- one difference that parallels the difference between Phases 1 and 2. As indicated in Table 1, for these last two placement phases, children had but two test displays per phase. For all of the previous ones they received 3 test displays per placement phase.

Notice that the end of Phase 5 included some items designed to assess the extent to which our subjects interpreted the task as one that had something to do with counting things. The follow-up phase, a month to two later, yielded data on how the children read non-integer numerals when each is presented separately on a card. Finally, as can be seen in

Table 1, the design included some arithmetic items so that we could assess whether there is a relationship between our assessments of arithmetic skill and level of success on the fraction placement task.

Results

By the end of the pretest, all children could place sets of 1, 2, and 3 at the correct whole number positions. Their subsequent responses to the fraction placement target items were coded for each phase of the testing. Inspection of individual patterns of responding across Phases 1 and 2 (i.e. before hints were offered) revealed four patterns of response categories:

(1) *Correct (At Least 50%)* : There were children who placed items so as to integrate a metrically ordered positioning of the fractional parts and whole circles without feedback. Children who did this on at least 3 of their total of 6 Phase 1 and 2 test trials were assigned to this baseline category. We did this even if children were not perfect because those who did not meet the at-least-50%-correct criterion responded differently than those who did. Therefore all other patterns of Phase 1 and 2 responses were assigned to one of the following categories.

(2) *Parts Alone Rank Ordered* : Some children neglected any whole circles in a display and responded as if they simply rank ordered the relative sizes of the parts. For example, one child placed $1\frac{1}{4}$ at "1", $1\frac{1}{3}$ at "2" and $1\frac{1}{3}$ between "2" and "3". Such responses map relative amount of area to relative length without regard to the size of the interval between successive points on the number line.

(3) *Whole Number Placements* : Children in this category used counting strategies. Some placed test displays as if they had counted the number of separable parts. For example, several children placed the display with $1\frac{1}{2}$ circles at "2"; and all displays with $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ circles at "1". Other children responded as if they ignored the fractions of a circle and simply counted the remaining whole circles. Such children also placed the preceding list of stimuli at either "1" or "0".

(4) *Others* : All response patterns that differed from the above three were coded in this category. For phases 1 and 2 these included those where children placed successive test displays from left to right, put each test item at a different position without any concern for order, or generated sequences that we could not decode. Once the experimenter began to show children where to put displays containing one half of a circle (during Phase 3), some started to mimic her. Mimics simply placed nearly all displays that had *any* parts on them halfway between two whole number positions. That is, they even placed displays containing $\frac{1}{3}$, and $\frac{1}{4}$ at half-way points between successive instantiations of whole numbers.

Number Line Placement Results

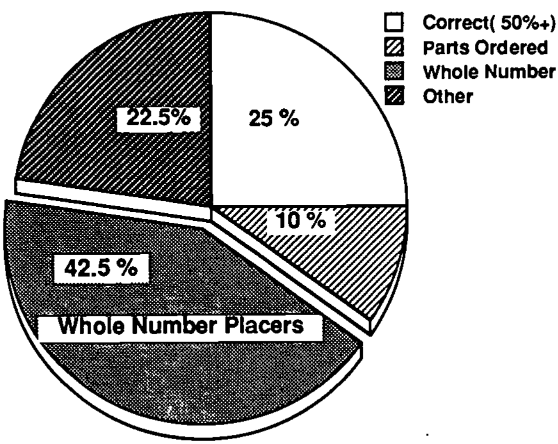


Figure 2. Pattern of Placements During Phase 1 and 2

Since the first two placement sessions were run without feedback or hints, these sessions provide baseline data on how children interpreted the task. As shown in Fig. 2, only 25% of the children were able to use the number line correctly on at least half of their trials. The predominant tendency of those in the study was to place the test displays as if they had counted the items therein. More than 40 percent of the children did this. Another 22.5% produced solutions that were not discernibly task-relevant. Finally, a few children mapped their ordering of the relative size of fractional parts to ordered points on the number line. If they did this, they ignored both distance and the position of the whole numbers as instantiated by the number of circles at a point.

The mean ages of children in the Correct, Only Parts Ordered, Whole Number, and Other groups were: 7 years-6 months, 6 years-0 months, 6 years-3 months, and 7 years-3 months, respectively. More Kindergarten children were placed in the Whole Number and Only Parts Group; children in Grade 1 tended to fall into the Other group; those in Grade 2 were more likely to be in the Correct group. ANOVA's revealed reliable effects of grade ($F_{3,36}=6.12$, $p=.002$) and age ($F_{3,36}=6.62$, $p=.001$). Post-hoc Scheffe tests indicated that children in the Correct group were developmentally more mature than those in either the Order Parts ($F=3.35$ and 3.42, for age and grade, respectively, $p<.05$) and Whole Number (Counters) groups ($F=4.89$ and 4.74, for age and grade, $p<.05$). Differences between other groups were not reliable, including those between the Correct and Other groups. (Comparable inferences follow from wherever similar tests can be performed with the Kruskal - Wallis test, a nonparametric statistic.)

Although there was a correlation between between developmental level and our assignment of children to a baseline placement group, it was not large. The *r*'s between group assignment and age or grade were .26, and .22, respectively. In a factor analysis of the data, both variables loaded exclusively on one of two factors. Group assignment carried the weight of the second factor. What follows considers what else besides a child's developmental status might have contributed to success level on the placement task. In some of these analyses the four fraction placement groups described above are collapsed into two: *Correct Placers* and *Incorrect Placers*. Subjects in the original Correct group make up the former, those in the remaining three groups (Orders Parts Only, Counters, and Other) make up the latter when this is done.

Related Abilities

Initial Responses to the Novel Number Line.

Table 2 summarizes how children assigned to the different fraction placement groups dealt with the special number line. It can be seen that all groups did rather well on these pretest items. Children in the Correct placement group were perfect on all their pretest items and although children in each of the remaining groups had some problems with these items, the differences are not that large. Given our hypothesis that children would assign whole numbers to a correct corresponding relative position on the number line, we would have been surprised had these differences been greater.

TABLE 2: HOW CHILDREN IN EACH BASELINE GROUP RESPONDED TO PRETEST ITEMS

Baseline Group & Pretest Answer Patterns		Test Items and Answers		
		Zero	Fill In 2 & Place N	"Which is more, etc."
		Says "Zero" (or Nothing)	Places (Knows) 2,3,4	Mean Correct (N = 4)
			SCORE	
I	Correct Placements on At least 50% of trials (n=10)	100 %	100 %	4.00
II	Rank Ordering of Size of Parts of Circle along Line (n=4)	50 %	100 %	3.75
III	CountsThings and Uses Whole Number Positions (n=17)	75 %	89 %	3.12
IV	Other (n=9)	78 %	89 %	3.78

Differential Effects of Hinting.

To determine how the different groups of children responded to hints, we looked at who improved as a function of hints. A child was scored as improved if they were in the top group (Correct on at least 50% of their trials) to start and their percentage of correct trials increased. Children who started in another group and then met the criterion for the Correct group, were also scored as improvers.

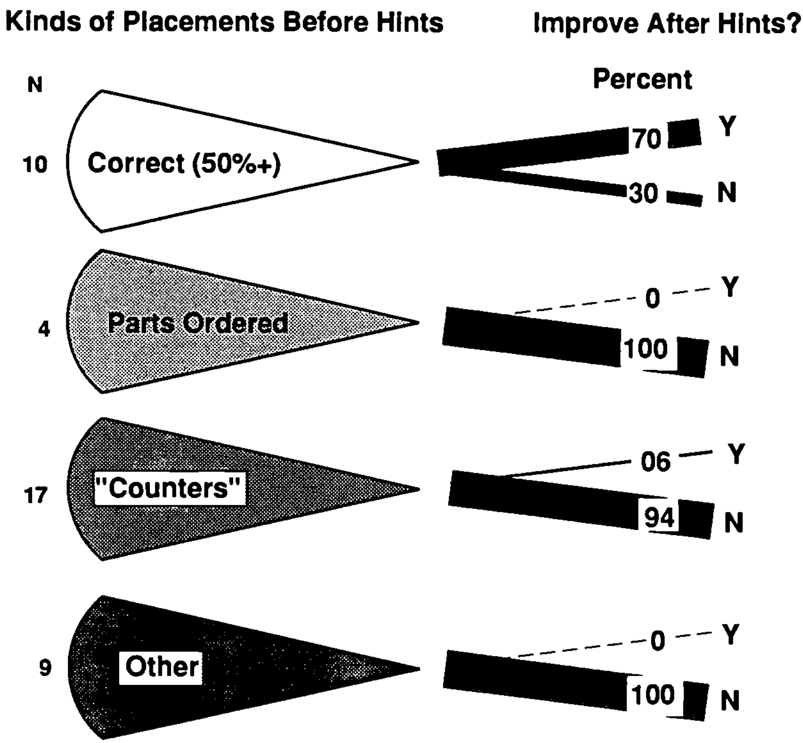


Figure 3: Percent Ss in Pre-Hint Placement Groups Who Improve After Hints

Figure 3 summarizes the effects of hints. It shows that children who did well to start tended to improve when given hints. In contrast, those who thought the task was about counting numbers or the ordering of size of parts resisted our hints. So did children who used other task-irrelevant strategies and therefore started in the Other group. Hence, children who started as non-fraction placers continued to respond as non-fraction placers, continuing to apply their initial solutions or moved to one of the Other solutions, that is, ones that were even less consistent with the present task requirements.

If anything, hinting had a deleterious effect on children who were not in the Correct group to start. The general tendency for the Counters was to stop counting and to mimic the experimenter once hints were introduced. A similar post-hint tendency to place all fraction items at some common point between two whole number positions characterized the children who started out in the "Other" category. In contrast, no child who was classified as Correct on the basis of their Phase 1 and 2 answers, that is before the hints were introduced, simply imitated the experimenter's demonstrations that started in Phase 3. One child who started in the Correct group did switch to a counting solution after the hints were introduced. All others in this group either maintained their initial levels of performance or improved.

In our introduction we considered two possible effects of hints. These simply might serve to clarify the intent of the task. If so, all children who erred to start should have done better after the hinting phases. Alternatively, hints might interact with children's initial level of understanding, helping only those who already had some understanding of the concepts involved in the task. The second of these possibilities best describes the results.

Ability to Read $\frac{1}{2}$ and $\frac{1}{4}$.

Next we consider whether both the differences in initial success levels and tendency to benefit from hints went along with other differences in the ability to deal with both fractions as well as mathematical symbols.

TABLE 3: HOW EACH CHILD IN EACH PLACEMENT GROUP READ CARDS WITH 1/2 AND 1/4

SUBJECT AND BASELINE (PREHINT) GROUP ASSIGNMENT	1/2	1/4
CORRECT		
1A	1 plus 2, 3, 5	1 plus 4
2A	half, one half	one fourth
3A	one half	one fourth
4A	a half	one four
5A	1 and a half	1 and a fourth
6A	a half	one fourth
7A	1 and a half	1 and a fourth
8A	1 and a half	1 and four quarters
9A	one quarter	one fourth
10A	1, 2	1, 4
ORDER PARTS ONLY		
1B	1 and 2	1 and 4
2B	1 line 2	1 line 4
3B	1, 2	1,4
4B	1 plus 2	1 plus 4
WHOLE NUMBER (COUNTERS)		
1C	1 and 2	1 and 4
2C	1 and 2	1 and 4
3C	1, 2	1, 4
4C	twelve	fourteen
5C	1 plus 2	1 plus 4
6C	2,1 "saying backwards"	4,1 "saying backwards"
7C	1 and a 2	1 and a 4
8C	1, 2	1, 4
9C	four	five
10C	1 and 2	4 and 1
11C	1 and a 2	4, 1
12C	1 and 2	1 and 4
13C	1, 2	1, 4
14C	1, 2	1, 4
15C	3	5
16C	1 and 2	1 and 4
17C	1, 2	1, 4
OTHER		
1D	1 and 2	1 and 4
2D	1 and 2	1 and 4
3D	1 and a half	1 and a fourth
4D	1 and 2	1 and 4
5D	one half	one fourth
6D	1 and 2	1 and 4
7D	1, 2	1, 4
8D	1, 2	4, 1
9D	one half	one fourth

As is evident in Table 3, there was a strong relation between pre-hinting placement success and the ability to read the non-integer numergraphs $\frac{1}{2}$ and $\frac{1}{4}$. Almost none of the children in any of the "Incorrect" groups could read the test items appropriately. Instead they had a potent inclination to **misread the fractions as two integers**, e.g. "1,2"; "1 and 2"; etc. In addition, many of these children read the division symbol as either "and", "plus", or "line". Finally, some children behaved as if they had been given an addition problem; when asked to read $\frac{1}{2}$ and $\frac{1}{4}$, they answered "three" and "five", respectively.

To say children failed to read correctly the non-integer numergraphs is to simplify the matter. The results suggest that the children once again revealed an assumption that the task had something to do with counting things and adding whole numbers. Where we assumed we provided inputs about fractions, they behaved as if they were offered further cases to which they could apply their view that numbers are what one gets when one counts and adds the results of counting. What follows is consistent with our interpretation.

More on How Fractions are Treated as Count Number Inputs.
Do Young Children Think $\frac{1}{4}$ is More than $\frac{1}{2}$?

If our test items detected the assimilatory power of the idea that numbers are what one gets when one counts, our subjects might be expected to choose $\frac{1}{4}$ as more than $\frac{1}{2}$. Since we saw that they did not treat these expressions as fractions, they might fall back on comparing only the 4 and 2 and therefore say that the symbol $\frac{1}{4}$ stands for a greater value than does the symbol $\frac{1}{2}$.

Children in the Correct baseline group do somewhat better than those in the Incorrect groups on the "which is more" task, but only because, as a group, they failed to respond in a consistently correctly or wrong way, not because they did well. Subjects who were in the remaining "incorrect" groups had a reliable bias to select the reciprocals with the larger denominators as more for both the $\frac{1}{4}$ vs $\frac{1}{2}$ and $\frac{1}{56}$ vs $\frac{1}{76}$ pairs of numerals. ($x^2_1 = 11.27, p < .001$ and $8.06, p < .01$, respectively).

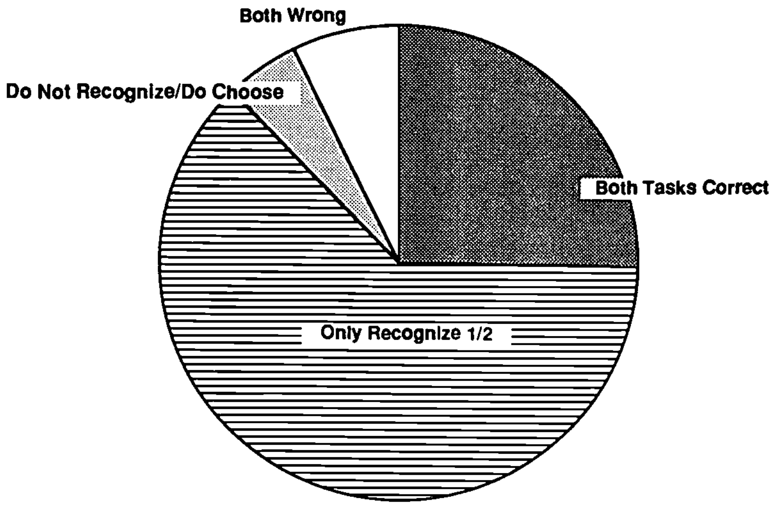


Figure 4. Ability to Recognize 1/2 and Choose it as More Than 1/4.

One possible explanation for the "$\frac{1}{4}$ is more than $\frac{1}{2}$" result is that the children could not recognize $\frac{1}{2}$ when asked to point to one of the two test cards, "the one that shows $\frac{1}{2}$". This might be expected given the way they read these expressions out loud. The relevant data are shown in Fig. 4 which shows that well over 75% of the sample were able to point correctly to $\frac{1}{2}$ when shown both $\frac{1}{2}$ and $\frac{1}{4}$, i.e., most of the children could pass an item like ones on multiple choice tests. The problem is that only 25% of these same children could also interpret these symbols (or their referents) in a manner consistent with the mathematical meaning expressed by them.

Other Arithmetic Tasks.

We included two other problems where we thought children might make a counting-based error, $(\frac{1}{2}+\frac{1}{2})$ and $(0+\frac{1}{2})$. Fig. 5 shows the extent to which children did so. The bias to do this on the $\frac{1}{2}$ vs $\frac{1}{4}$ item is also plotted. Table 4 summarizes error tendencies on these tasks as a function of fraction placement skill.

TABLE 4. PLACEMENT ABILITIES AND ERROR TENDENCIES WHEN CHILDREN CHOOSE THE GREATER OF $\frac{1}{2}$ AND $\frac{1}{4}$, ADD $\frac{1}{2}+\frac{1}{2}$, AND ADD $\frac{1}{2}+0$*

		Baseline Placement Group		
Error Kind/ Tendency	Correct	Only Parts Ordered	Counts Parts/ Wholes	Other
	(n=10)	(n=4)	(n=17)	(n=9)
Percent Errors For All Trials	30	75	78	67
Percent of All Trials with Count-like Errors	30	58	55	33
Percent Ss Who Contribute At Least One Count-Like Error	60	100	94	78

*A count-like error was coded as when S said that $\frac{1}{4}>\frac{1}{2}$; $\frac{1}{2}+\frac{1}{2}$ was "two" circles or "two and a half"; and $\frac{1}{2}+0$ was "one" or "one and a half". These responses contributed to the figures in the bottom two rows. Those in the top row are based on all errors.

Note that answers of the form $(0+\frac{1}{2})=\{1, 1\frac{1}{2}\}$ are relatively infrequent, especially as compared to ones of the form that $\frac{1}{4}>\frac{1}{2}$ or $(\frac{1}{2}+\frac{1}{2})$ =2. Such a pattern of results is consistent with both Evans' (1983) and Wellman and Miller's (1986) conclusion that children quickly master a purely procedural algorithm for dealing with zero addition problems, i.e. repeat the non-zero value whenever it is to be added to zero. To do this one need not know that zero is the identity element under addition and it seems that children as young as ours do not (Wellman & Miller, 1986).

In Table 4 we can see that Correct Placers were less likely to make errors on the arithmetic problems summarized in Fig. 5. Still, if we consider the extent to which children made a count-like error on at least one of the three problems, we see that even in our most skilled placement group, 60% of the children did this. It would seem premature then to say that they had a deep understanding of fractions as a numbers in their own right.

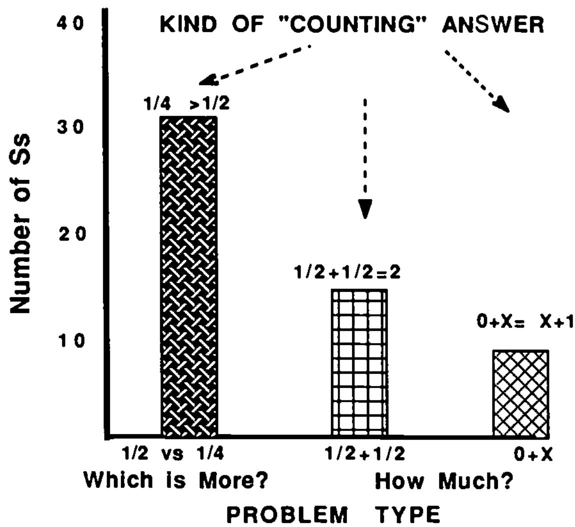


Figure 5. Number Of Children Who Make Count-Like Errors On Different Problem Types

How Much Do Correct Placers Know About Fractions?

How much credit should we give the children who mapped our test items onto the number line and did well on a variety of other tasks? In particular, can we say they understand the isomorphism between points on the number line and numbers. Do they think that that fractions (in either decimal notation or not) serve to make the system of numbers

effectively continuous by filling in the gaps between two integer points on a line ? To grant such understanding[1], we at least need evidence that children know there is a very large (actually infinite), number of non-integer numergraphs between any two integer points on the number line.

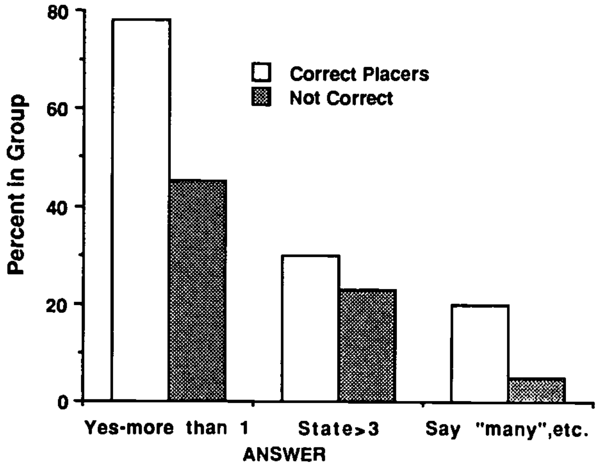


Figure 6. Do Children Think There are Numbers Between the Integers and If so, How Many Do They Say There Are?

Figure 6 shows how children answered questions about whether there are numbers between 0 and 1, 1 and 2 and if so how many such numbers there might be. Those who were scored as Correct on our placement task were more inclined to agree that there are numbers between 0 and 1. Still, the bias of the children was to say that there are about 3 fractions between the integers. This suggests they were simply recalling the three fractions we used during testing. Very few children in any group could tell us that there were "many", "hundreds", or "thousands", or "as many as you want", possible numbers between two integers.

[1] Technically speaking, fractions make the points on the number line dense. We doubt that our subjects would appreciate the reasons for distinguishing between the notions of continuous and dense.

In brief, even our best subjects were not inclined to interpret arithmetic problems correctly; they did not do well when asked to select the greater of two written fractions. Further, they were inclined to answer that there were but a few numbers between two integers.

Of course, we thought it possible that our young subjects would make such errors. They were all too young to have received much instruction about fractions. What is not clear, however, is whether instruction overcomes these initial erroneous assimilations to a theory that numbers are what one gets when one counts. Findings from a variety of studies highlight the possibility that still older children lack a mathematical understanding of why a fraction can be treated as an entity that is a number. For example, Kerslake (1986) notes that very few pupils in her secondary school sample were either familiar with or accepting of the division aspect of a fraction. Instead they thought of fractions in terms of the number of parts in a given whole, a conclusion that is also reached by Silver (1983) on the basis of his interviews with college students in a teacher preparation course. It could well be then that these pupils think that the remainders they get when they do long division can be "thrown" away, this because only the whole numbers count as true numbers.

We too suspect that the tendency to think about fractions in terms of particular examples of part-whole representations is a barrier students must get beyond or be willing to set aside before they can understand why anyone thinks fractions are numbers. It seems that this alone will not suffice. To illustrate why, consider the student in Silver's study who answered that $(\frac{1}{2}+\frac{1}{3}) = \frac{2}{5}$. Such solutions are reminiscent of children in our study who said that $\frac{1}{4} > \frac{1}{2}$ (because 4 is more than 2), and read such representations as lists of whole numbers. If Silver's subjects were inclined to view the marks on paper that stand for fractions as representations of integers, then they might have actually thought that their answer was correct, this because 1+1=2 and 2+3 =5. If so, the belief that numbers (and their numeral representations) are what one gets when one counts might continue to influence pupils' interpretations of arithmetic problems well after students have had instruction that treats of fractions as numbers and not just part-whole descriptors.

A similar conclusion is reached about students conceptions of decimal numbers. To quote Hiebert and Wearne (1986): "To our knowledge, all researchers who report data on ordering decimals ... suggest that many students have trouble judging the relative magnitude of decimal fractions if they have different numbers to the right of the decimal point (e.g., 1.3 and 1.295). Most errors can be accounted for by assuming that students ignore the decimal points and treat the numbers as whole numbers" (p.205). This generalization is supported by Hoz and Gorodetsky's (1983) data on how decimal fractions are read out loud, e.g. "seven" or "zero seven" for 0.7, "thirty-eight meters" for 0.38 m, "3 meters and 8 centimeters" for 3.8 m, etc. Hiebert and Wearne are careful to point out that this is not the only misconception that pupils have about fractions and the interpretation of the convention for representing them as non-integer numergraphs. Others develop as a function of input, for example, that the more numbers there are to the right of the decimal, the smaller the value represented (Sackur-Grisvard & Leonard, 1985; Resnick & Nesher, 1983). But the point still remains: these are hardly correct mathematical understandings of these symbols. More importantly, it appears that even decimal fractions are assimilated to a theory of numbers based on the idea that "numbers are what you get when you count". A few results from a second study lends support to this idea and help make contact between those in the literature and our study with young children. It also introduces our discussion.

The Questionnaire Study

Some Details about the Study

We were able to coordinate our interests in the way children think about fractions with the goals of the mathematics superintendent of a school district in a relatively large MidWestern suburban school district, henceforth assigned the alias The Prairie School District. We were able to present, in written form, a number of items that were selected to (a) relate our finding to those in the literature and (b) assess older students conceptual understanding of fractions. Since the school district offered to include their gifted classrooms, we sampled from two ability levels, regular and gifted. The school system assigned children to these different groups, as early as the second grade, on the basis of overall aptitude and performance levels. Children in the gifted groups were in self-contained classrooms in the regular schools and moved as a group through the grades.

Figure 7 summarizes a sample of our data on what children in different grades and programs did with a battery of "which is more items". It is clear that their responses are consistent with those in the literature on fractions and decimals (See above for a review). The one result that stands out and that is perhaps new has to do with the gifted - regular comparison. Put simply, there is a large effect of this variable, so much so that gifted 4th grade classes outperform regular 7th and 8th grade classes. A similar result is revealed in the final table we present, which reproduces some of the answers children gave to our question "Why are there two numbers in a fraction?". We think the answers speak for themselves. It is only the gifted children who can begin to articulate the mathematical principles that contribute to this notational convention.

TABLE 5: SOME ANSWERS TO WHY ARE THERE TWO NUMBERS IN A FRACTION"

GRADE and SUBJECT NUMBER

Regular 4th and 5th Grade

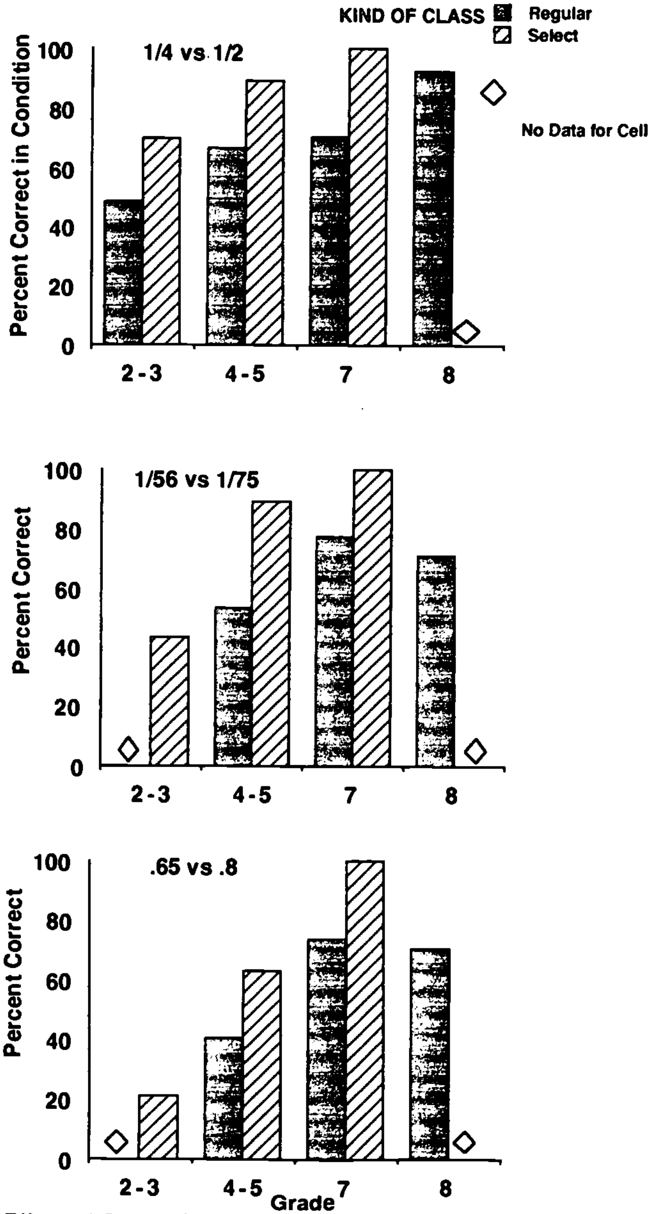
1. "Because if there weren't 2 numbers then you couldn't have a fraction."
2. "It is two numbers"
3 "can't explain"
4. "To be eqivelent."
5. "Because a fraction is a part of something. Not the hole thing."
6. "Because you have to hav a denomonator and a numarator"
7. "1 for one certain color on the thing 4 for all the things in the group"

Gifted 4th and 5th Grade Class

6. "4 is how many piedes and 1 is how many you got"
8. "the bottom one is how many in the whole and the top one is how many are left a in the whole."
9. "The denmunator is the hole, the numirator is how many peices you have of the hole"
10. "...one of the numbers stand for how many thimes something is divided up into and the other how many are taken."

Regular Grade 7

11. "Because they wanted it that way."
12. " 1/4 1 explains how many you have. 4 explains how many there are"
13. "1 shows you how many you have and the other shows you how many are there"
14. "Because 1 number is how many of something you have &how many are in a whole."
16. "Because It Broke in to fractions I GUESS"
18. "because it is a portion of a number"
20. "you need a numberator & denominator"
21. "the top one explain how many of the Bottom one there are"
24. "It is less than one whole"
25. "to show both halfs"26."Because it shows how many items you have out of a given amount of numbers."



Effect of Grade, Ability & Item Pair on % Correct More Choices

Discussion

Young children assimilate data that we think of in terms of fractions to a conceptual scheme appropriate only for the positive integers. This tendency to do so appears as soon as they are introduced to fraction relevant materials and persists at least into the secondary school level. Indeed, judging by Lave's (1988) work, it persists into adulthood in a large part of the population. This assimilatory tendency is manifest early on in the inappropriate placement of fractional parts on the number line and in a persistent misreading of the notation for fractions. It continues to be evident for many years in the misordering of the fractions, in either fractional or decimal form, and an inability to answer questions designed to probe conceptual understanding of principles. Our evidence suggests that preschool conceptual schemes can dominate what many children extract from the curriculum.

It is significant that the above difficulties are much less evident in a selective sample of children -- even as early as the fourth grade. Many of these children order non-integer numerals appropriately; they correctly assimilate what they, like all elementary school children, are taught. Moreover, they give appropriate answers to questions about underlying principles, even when these questions touch on matters that are not an explicit part of the curriculum.

These findings suggest that there are severe limits on what can be expected from purely situational instruction in mathematics. Such instruction may never induce many pupils to exceed the limits of the conceptual scheme they bring to the school environment. Might it be that mastery of fractions is a recognized watershed in the elementary curriculum because it requires students to transcend this intuitive conceptual scheme? They have to learn to apply the relational and combinatorial operations of arithmetic symbols that are not defined by either counting procedures or by language that refers to the intuitively given aspects of arithmetic reasoning. In Skemp's (1971) terms, they have to accommodate their ideas of what numbers are if they are to understand why fractions are numbers.

The correlation between early mastery of the ordering of fractions and the ability to talk coherently about the principles underlying the construction of fractions suggests that learning to apply language to the principles governing arithmetic operations may be crucial. Put differently, learning the mathematics we want to teach today may go hand in hand with learning the language of mathematics. But the language of mathematics does not share the same syntax as does English, or for that matter, any natural language. It has its own notational system, terms *and, plus, or, add,* etc., and entities. Some terms and entities have different meanings in the language of mathematics than do their cognates in a natural language, this because their meaning is embedded in mathematical principles and their operations.

If it is the case that learning to go beyond what we know about mathematics without study requires mastering some of the language and syntax of mathematics, then we are faced with the challenge of how to encourage conceptual development that involves the acquisition of a new theory of number. Our discussion here of why this may not be easy takes off from Gelman and Gallistel's consideration of the difference between the preschooler's and the mathematician's notion of number (Chapter 11, 1978).

Gelman and Gallistel remind their readers that the formal description of arithmetic is a product of late-nineteeth and early-twentieth-century mathematics. Its emergence was accompanied by a profound shift in mathematicians' views about the relation between the laws of arithmetic and the definition of what constitutes a number. One way to characterize this is to suggest that that Kronecker's view as to what is given and what is derived no longer holds; where once the numbers could be taken as given and all else derived, there is a sense in which just the opposite view has taken hold. In the modern view a number is any abstract entity, no matter how bizarre it might seem from the psychological point of view, if it can be shown to behave in accord with the laws of arithmetic. Whatever intellectual discomfort negative numbers generate -- after all one cannot count (-X) real-world things, they are an essential part of modern formal arithmetic. Similarly, no matter how articulately a child denies numberhood status to fractions, they are numbers in modern formal arithmetic. For me, property of closure with respect to division would not apply otherwise.

The idea that the operation of division is related to the definition of a fraction as a number helps introduce a way to characterize the nature of the difference between a conception of number that admits to fractions as numbers and one that does not. Before developing this account, we find it useful to review briefly some terminology we carry forward from Gelman and Gallistel.

These authors distinguished between *numerons* and *numerlogs* to highlight the difference between the conventional spoken counting tags (numerlogs) employed by a culture, and the mental entities (numerons) to which the numerlogs are mapped. Nowhere in their account of the implicit system of early mathematical abilities is there an entity that corresponds to the division of one numerosity by another. Nor could there be. In their system numerons are what one gets when one counts items. In general, the result of dividing one numerosity by another is not a countable numerosity and cannot therefore be represented by a numeron. At least on the basis of current data, there is no justification for attributing an arithmetic or mathematical understanding of division to young children. They might know about cutting things into parts (Miller, 1984), but this is quite a different matter than knowing about dividing one numerical representation by another. Indeed, Kerslake (1986) suggests that many secondary school pupils in England fail to move beyond the idea that a fraction is a part of a whole. She suggests that this is related to Hart's (1981) report that there are secondary level students who avoid fractions and deny that it is possible to divide a number by one that is larger than it. Similar conclusions have been reached about pupils in Canada and the United States (e.g. Behr, Lesh, Post, & Silver, 1983; Chaffe-Stengel & Noddings,1982; Larson, 1980).

It is likely that one cannot accept the idea that there are numbers rendered by the operation of dividing one numerlog by another without the ability to use numerlogs to represent linguistically the cardinal values of sets, an ability in its own right that takes time to develop (Fuson, 1988; Gelman & Greeno, in press). Gelman & Greeno (in press) outline a learning sequence wherein the count words first re-represent the numerons that represent the numerosities of counted sets. Informally, this amounts to saying that the numerlog "five" represents the numeron obtained in counting sets of five. "Six" represents the consequence of counting a set that has one more item than do sets of 5, and so on. Although knowledge of the numerlogs for the positive integers marks an advance, there is no denying that it is far from what one means when one talks about using "five" or 5 as mathematical entities unto themselves, as we do when we say that 5 has the inverse (-5). In fact, such knowledge is at best a prerequisite for learning what Saxe (1988) refers to as a culture's orthography for number.

Learning about fractions involves more than working with numerlogs as cardinal numbers. More generally, learning about fractions seems to depend on being able to use, in mathematically meaningful ways, conventional mathematical terms, tools and symbols, including: the number line; non-integer numerlogs like "one half"; numerals -- the written symbols, or what we will call numergraphs, for corresponding numerlogs; and the notational systems for writing fractions (non-integer numergraphs) as the result of division or in terms of the decimal system. Thus, the mastery of the idea that fractions are indeed numbers requires both he learning of principles that go beyond those implicit foundational ones available to the young child and the use of a language mediated system (See also Hiebert & Wearne, 1986).

Given the foregoing, learning that fractions are numbers should not be all that easy -- never mind the vast majority of the domain that we call mathematics. Since the formal principles that define the relations between multiplication and division include ones that are not included in those for addition -- the distributive law is not a law of addition but a higher-order one -- in what sense, if any, can mathematically meaningful learning about fractions build on what is already intuitively given? Fractions are numbers generated by the division of 2 numerosities, not by counting things or parts. Therefore, we should not presume that the requisite principles for dealing with entities like fractions *as numbers* exist when talk about fractions is first introduced in school. In fact, the young child's tendency to interpret number-like inputs as novel cases of counting data makes sense when considered from her point of view. Since it will a while before the same child has the opportunity to engage division as a mathematical operation, it might not be surprising that her initial tendencies are exceedingly robust.

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GENERAL MATHEMATICAL PROCESSES

Theory, Practice and Research

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The theme proposed for these plenary lectures - Mathematical Processes - has been central to much of my work, and I share the interest expressed by Herscovics in his opening paragraph in 'teaching ... mathematics in the spirit of Lakatos ... as an intellectual process ... in making processes the topic of instruction; and researching these processes'. I shall first describe my own understanding of the nature of mathematics and why the general mathematical processes are such a crucial aspect of it; then comment on the papers of Gelman and Herscovics in relation to this theme; and, finally, examine existing knowledge and needed research on the place of the processes in the mathematical curriculum. In particular, I shall ask how far a curriculum centred on general mathematical processes leads naturally to the acquisition of the body of mathematical concepts and skills generally agreed to be essential.

Applied and Pure Mathematical Processes

Mathematics has two aspects, roughly fitting the traditional labels applied and pure. First, it is a means of gaining insight into some aspect of the environment. For example, the exponential or compound growth function gives us insight into the way in which a population with a given growth rate grows over time - first slowly and then with an increasingly rapid rate of increase. Some of these properties are encapsulated in well known puzzles - such as that of the water lily, doubling its area each day, where will it be the day before it covers the whole pond? - or in the frequent exhortations of our financial salesmen to consider how a modest investment might grow. Another related example is that of the decrease of the rate of inflation - which many people believe means that prices are coming down. In the home environment, a little knowledge of the symmetry group of the rectangular block will tell us when we have turned the mattress on the bed as many ways round as we can; and a modest knowledge of probablility and statistics will help us to interpret advertising claims about what toothpaste seven out of ten movie stars use, and not to be excessively hopeful that our next child will be a boy if we have already produced three girls. These are all 'useful' aspects of mathematics - and note, by the way, that they all depend on the application of conceptual awareness, not on any technical skill; they are useful in the same way as is the knowledge gained in most of the subjects of the curriculum - history, geography, literature, science - that is, deriving from knowledge of some key facts and explanatory concepts.

The second aspect of mathematics is somewhat less loudly commended in public nowadays. It is the pure mathematical aspect which it shares with art and music, the solution and construction of puzzles and problems, and the enjoyment of recognising and making patterns. Mathematical problems in newspapers and magazines still attracts a following, and we might speculate that the capacity to appreciate mathematics as an art to enjoy is initially present in most people, though it often gets suppressed by distasteful school experiences.

These two modes of interaction of people with mathematics, representing the applied and the pure mathematical approaches, have been identifiable throughout history as the mainsprings of mathematical activity.

Freudenthal (1968) also distinguishes applied and pure mathematical processes.

> "Arithmetic and geometry have sprung from mathematising part of reality. But soon, at least from the Greek antiquity onwards, mathematics itself has become the object of mathematising."

> "What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematising reality and if possible even that of mathematising mathematics."

More briefly,

> "Mathematics concerns the properties of the operations by which the individual orders, organises and controls his environment".

(E A Peel, 1971, p. 157)

To approach more closely the question of the nature of mathematical processes, we must unpick somewhat the meaning of 'mathematising reality'; or ask what *are* the operations by which we organise the environment. To do this, it will be helpful to trace the evolution through history of notions of the nature of mathematics, and its content and its processes.

To the Greeks, mathematics was the study of numbers, magnitudes and figures; but even as early as this, deductive proof was equally well established as the characteristic mathematical activity. Plato, in The Republic, says

> "Those who study geometry and arithmetic ... assume the existence of odd and even number, and three kinds of angles: these things they take as known and consider that there is no need to justify them either to themselves or to others, because they are self-evident to everyone; and starting from them, they proceed consistently step to step to the propositions which they set out to examine."

The explicit recognition of a wider subject matter can be attributed to Boole (1847, 1854) who, in his algebra of the Laws of Thought, used the letters x, y for propositions, · and + for "and" and "or" connectives, and 1 and 0 for truth and falsity. He said, "It is not the essence of mathematics to be conversant with the ideas of number and quantity, and ..... it is concerned with operations considered in themselves, independently of the various ways in which they may be applied." This was the culmination of a century or more of puzzlement about the nature of negative and imaginary numbers (e.g. d'Alembert), of infinitesimals, of "imaginary double points at infinity" (Stirling, 1717).

Thus, with Boole, the content of mathematics is being recognised as consisting essentially of the *relations* between objects, and not the properties of the objects themselves. This idea was taken further when the Bourbaki (1968) set about the task of generating the whole of mathematics from the notions of set and element. In this scheme relations are sets of ordered pairs, functions are kinds of relation (mappings of a domain set into a range), algebraic structures are sets with laws of compositon (which are, themselves, functions), topologies are certain kinds of identified sets of subsets, and so on (Choquet, 1962).

Along with this broadening of the content of mathematics, it was becoming recognised that the mathematical process does not only consist of the exposition and demonstration of mathematical truth. The process of discovering the concepts and the generalisations began to receive more attention. Poincaré (1956), Weyl (1940), Polya (1954), Lakatos (1976), among others, have explored both the psychological aspects of the creative process and also the mathematical strategies themselves. The Bourbakiste analysis has contributed important insights into the mathematical process. Choquet affirms that "the axiomatic method is analogous to an automatic production line; the mother-structures to the machine tools". These structures are "those associated with the equivalence relation, the order structures, the algebraic structures, the topological structures, etc. (Choquet, 1962).

Basic Mathematical Actions

To turn the mother structures into machine tools is to say that the way of generating new mathematics is to classify, to compare and order, to combine, to reverse, to transform, to recognise nearness, in the material one is studying. This applies both within mathematics and to non-mathematical material. Fielker (1973) has shown how a rich sequence of geometrical study for a primary school can be built up in practice by the application of the "mother-structures" to simple geometrical elements such as straight lines, circles and a set of wooden shapes, and Gattegno (1973) shows how an extensive mathematics curriculum can be developed by the use of the same basic structural operations on a set of Cuisenaire rods.

Let us view the development of mathematics in this way. Number and space are a collection of concepts constructed by the mind out of its interaction with the world, numbers out of the experience of repetition, geometrical ideas from perceptions of sameness in physcial objects. In each of these fields, further acts of classification take place in which sets are constructed of objects which are agreed to be the same in some way. Next, pairs of objects are compared with other pairs and sometimes the *relationship* is judged to be the same between the two pairs - it may be two pairs of numbers with a common difference, or such that the first is greater than the second, or pairs of objects of the same shape but different size. Considering the transformations which take the member of these pairs into each other leads to the identification, collection and classification of functions, for example: linear, square, reciprocal, enlargement, shear. Thus algebra arises as the set of structures which emerge from the study of number and space. At the same time, the set of functions becomes sufficiently large and variegated to constitute a third field of raw material in which the classifying, relating and transforming process can operate. Thus, to summarise, the content of mathematics is that body of knowledge which is generated by the application to experience of the basic mathematical actions of classifying, relating and transforming.

Representation, Abstraction, Generalisation and Proof

To provide an adequate description of the general processes of mathematics we need to consider, in addition to these basic actions, the broader processes of representation, abstraction, generalisation and proof. I have discussed these at length elsewhere (Bell, 1976, 1979), so will include here only some remarks about representation which are needed for the subsequent discussion.

The representation of a situation by a diagram or a symbolic expression, or by a 'model' is the abstract sense, is so central to mathematics that it is hard to realise that quite a high degree of algebraic sophistication was acheived by the Babylonians and the Greeks without any symbols apart from a crude number system. Dienes (1961) says

> "The structures now being considered by mathematicians are so complex that it would be quite impossible to dispense with symbolism. The symbols remind the mathematician of what it is that he is really supposed to be thinking about; but more than this, the mechanisms of some of the well learnt mathematical techniques make it possible for the mathematician to skip a great number of steps, or in other words he allows the mechanism to do a part of the thinking for him."

Weyl (1940) in a lecture entitled *The Mathematical Way of Thinking* argues that, for an adequate characterisation of mathematics, along-side the axiomatic method must be placed *symbolic construction*, as the method by which mathematics is distilled from the raw material of reality. (He cites as examples the capturing of the infinite set of integers by the positional notation, and of a point of a continuum by an infinite binary decimal; and an extension of this last process to topological schemes.)

Incorporating symbolistion we arrive at the following definition of mathematics.

Mathematics consists of *structures, and their associated models and symbol systems*. By *structure* is meant a *system of relational concepts*. Examples of structures are the group S_3, the group in general, the rational number, the plane quadrilaterials, the functions $y = kx^2$. A *symbol system* is some set of physical objects, usually marks on paper, which has a set of transformation rules determining how these marks may be manipulated, derived from the relationships among the denoted concepts. Thus the transformation $a = b/\sin 15^0 \rightarrow a \sin 15^0 = b$ is a physical movement of the marks (probably perceived as such when performed) which derives its validity from knowledge of the denoted concepts and their relationships. Symbols are visible and movable but concepts are invisible and abstract; hence the tendency to teach symbol-transformations by rote and thus to detach the symbols from their meanings.

An Example

An example which shows the use of the characteristic mathematical processes in a context almost devoid of number and spatial aspects is the following. The *Table of Kindred and Affinity* appears at the back of the Church of England Prayer Book. It defines which relatives one may not marry. Some of these are unsurprising; to marry ones sister or brother is regarded as genetically undesirable. But what about ones brother's wife - assuming she became available, by death or divorce? Some factor other than genetics is being considered here. Questions naturally arise about the basis on which this list is constructed and whether they have got it right! To inquire into these it is helpful first to try to reduce the list to a comprehensible set of principles. If one eliminates the sex differences, and denotes a parent relationship by P, with grandparent P^2, child P^{-1}; a sibling by Sb and a spouse by Sp, the first three items on both lists become P^2, P^2Sp, SpP^2. The next ones (taking the second list so as not to be sexist!) become PSb, (for Father's Brother), PSb again (for Mother's Brother), PSbSp (twice) and so on. When the sex differences are combined in this way, 21 of the 30 relationships on each list remain distinct. One begins to notice some curious irregularities. For example. although the grandparents generation is listed first (Nos. 1-3), the grand children are not the last, but are at Nos. 19-24), and the same

A TABLE
OF
KINDRED AND AFFINITY

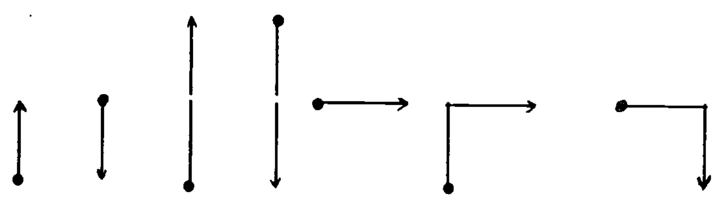
Wherein Whosoever Are Related Are Forbidden In Scripture
And Our Laws to Marry Together

	A Man May Not Marry His		A Woman May Not Marry With Her
1.	Grandmother	1.	Grandfather
2.	Grandfather's Wife	2.	Grandmother's Husband
3.	Wife's Grandmother	3.	Husband's Grandfather
4.	Father's Sister	4.	Father's Brother
5.	Mother's Sister	5.	Mother's Brother
6.	Father's Brother's Wife	6.	Father's Sister's Husband
7.	Mother's Brother's Wife	7.	Mother's Sister's Husband
8.	Wife's Father's Sister	8.	Husband's Father's Brother
9.	Wife's Mother's Sister	9.	Husband's Mother's Brother
10.	Mother	10.	Father
11.	Step-Mother	11.	Step-Father
12.	Wife's Mother	12.	Husband's Father
13.	Daughter	13.	Son
14.	Wife's Daughter	14.	Husband's Son
15.	Son's Wife	15.	Daughter's Husband
16.	Sister	16.	Brother
17.	Wife's Sister	17.	Husband's Brother
18.	Brother's Wife	18.	Sister's Husband
19.	Son's Daughter	19.	Son's Son
20.	Daughter's Daughter	20.	Daughter's Son
21.	Son's Son's Wife	21.	Son's Daughter's Husband
22.	Daughter's Son's Wife	22.	Daughter's Daughter's Husband
23.	Wife's Son's Daughter	23.	Husband's Son's Son
24.	Wife's Daughter's Daughter	24.	Husband's Daughter's Son
25.	Brother's Daughter	25.	Brother's Son
26.	Sister's Daughter	26.	Sister's Son
27.	Brother's Son's Wife	27.	Brother's Daughter's Husband
28.	Sister's Son's Wife	28.	Sister's Daughter's Husband
29.	Wife's Brother's Daughter	29.	Husband's Brother's Son
30.	Wife's Sister's Daughter	30.	Husband's Sister's Son

THE END

Taken from 'The Book of Common Prayer'

generation sibling relationships are not in the middle, at Nos 13-15, as one might expect in a systematic table, but after the children's generation. However, these are simply matters of order in the list. More surprising is the fact that there are six relationships listed in the grandchildren's generation, but only three in the grandparents! On examination, this turns out to be only a terminological asymmetry, since the term grandfather is used to cover mother's father and father's father, whereas son's son and daughter's son are listed separately. (Note also that grandparent is not used, nor uncle/aunt, nephew/niece). To make sense of the list of 21 distinct relationships, we may start by separating out those which do not contain a Spouse relationship (whatever the reasons for excluding these, they cannot be genetic). This leaves 7 relationships, 4 'vertical' ones: P, P^{-1}, P^2, P^{-2}; and 3 involving Sibling: Sb, PSb, SbP^{-1}. (We do not have SbP nor $P^{-1}Sb$ since ones sibling's parent is ones own parent: symbolically, $SbP = P$ and $P^{-1}Sb = P^{-1}$.) This set could be characterised as *all those people who are within one or two blood relationships, (sibling, parent/child), of oneself.* A diagramatic representation of this set may be helpful:



P $\quad P^{-1}$ $\quad P^2$ $\quad P^{-2}$ $\quad Sb$ $\quad PSb$ $\quad SbP^{-1}$

The remaining 14 relationships are those obtained from these seven by including a spouse relationship at the beginning or end of the chain - *but not in the middle!* Thus ones grandmother's husband is forbidden (even if he is not ones own grandfather), but not ones father's wife's father, who is at the same 'distance' in terms of relationships; P^2Sp and SpP^2 are out, but $PSpP$ can be married (if desired).

This seems to be a real anomaly, leading one to wonder whether the church authorites who drew up this list realised there was this gap in their system. Of course, we still have the question why spouse relationships appear in the list at all. This must be left 'for further research'. For our present purposes, the example illustrates how the application of very basic mathematical ideas and symbolism can help to reveal some order in an initially complex situation.

Further Examples

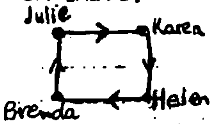
We shall now give a few more examples of mathematical tasks, for the classroom, which illustrate the relevance of the processes. The first, *Dress Mix Ups*, arose when a class of 11-12 year old girls was discussing a recent school show in which they had taken part, and which involved them in a number of changes of costume. Discussion of problems in the dressing room of finding ones own correct costume led to the question of how many ways there might be of picking up the wrong one. It was agreed that a particular rearrangement could be represented by a diagram in which an arrow from J to K meant that Julie's clothes were being worn by Karen. The girls were asked first to work out the possible mix ups for four people; one of them produced the diagrams shown.

This work contains two errors. The fifth type is essentially the same as the first, the only difference being in the order in which the girls take their places in the square. All these changes are 4-cycles, that is they involve *each* girl passing her clothes to another, in a chain. In the same way, the sixth type is the same as the third, since it concerns two pairs, each person changing with the other person in the pair. Otherwise, this work is correct, and the arrow representation brings out clearly the different types of interchange.

Dress mix ups.

The girls could get their clothes mixed up 2 ways in the first type.

Like this.



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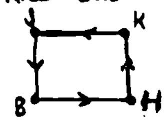
like this.



The dresses could be passed to the wrong girl six ways in the second type.

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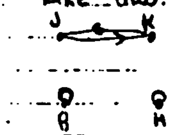

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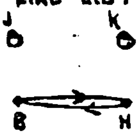

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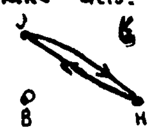

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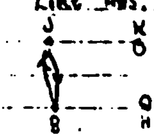

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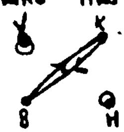

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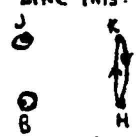
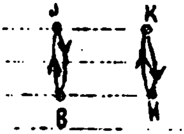


The girls could get their dresses mixed up 2 ways in the third type.

Like this.



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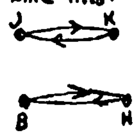
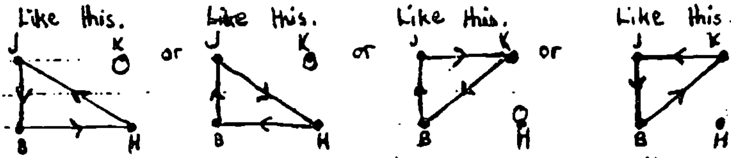
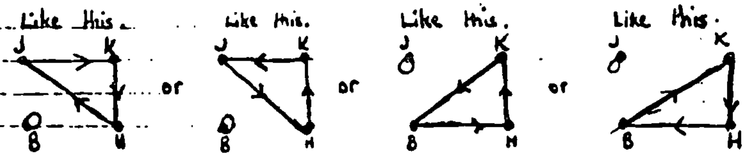
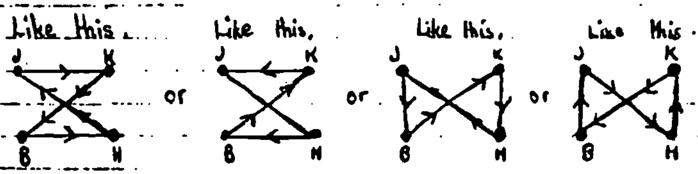


There could be 8 mix-ups in the 4th type.





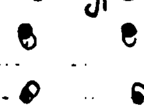
There could be 4 mix-ups in the 5th type.



The girls could get their dresses mixed up one way in the 6th type and one way in the 7th type.

6th type

7th type



Extensions of this task involved considering larger (and smaller) groups of girls; 5 is quite an interesting case. The result of combining two of these 4-person changes is another interesting question. For example, if two changes of the third type are made in succession - or one of the first and one of the second ... do any new types arise? Do two changes of type 4 always produce type 3?

Here, productive questions are being generated by the use of the basic mathematical processes and actions - first, representing, then classifying and combining. Reversing is also worth considering. The activity has also generated some new concepts in the shape of the different types - 3 cycle, double 2-cycle and so on. This is a process of abstraction.

Another example is afforded by the well known three coins puzzle. Given 3 coins showing tails, using any number of moves consisting of turning over 2 coins at a time, obtain 3 heads. Most pupils are reluctant to believe this is impossible, and experiment extensively, generally without success, sometimes thinking they have done it, but then not being able to repeat their 'winning' sequence. The need for recording games arises, and it is often this which leads to the realisation that there is only a small number of essentially different moves, and these can never lead to the desired outcome.

> TTT always → HHT in some order, then turning HT leaves the state unchanged, while turning HH returns to TTT. Nothing else is possible.

(I once asked an 11 year old class, when they had reached frustration point with this game, "Do you think I could do it?" Answer, after some consideration, "No..... but a computer could!")

Extensions of this puzzle lead to some interesting generalisations. 4 coins, turn 2 is trivial, but 4, turn 3 is interesting. An easy sequence is *TTTT* → *THH*H → HTTH; this is symmetrical with respect to T and H, so can be continued using the same moves in the reverse order, with T and H interchanged, to result in HHHH. (This means changing THH in HTTH ....). It is clear that, in general, if n is odd and k is even, then n coins, turn k is impossible. But are all others possible? Also, if n,k and m,k, are possible, so is n + m,k; and if n,k and m, l are possible, is n + m, k + l? In particular: if 4,3 and 5,3 are possible, so is 9,3; and if 4,3 and 5,2 are possible, we can consider whether 9,5 is. (I expect that you found the statements easier to comprehend in the numerical case than in the literal - and that you saw the numerical cases as potentially general; you abstracted easily the implied relationships in the numerical statements. We seem to find abstracting easier than re-embodying an abstract statement.)

In this problem the processes of representation, generalisation and proof are in play; and, particularly in extending the problem, the basic actions of combining and separating are used.

These examples are chosen to highlight aspects of the mathematical process, and they might well be used for this purpose in school. We give now a few examples drawn more obviously from mainstream curriculum topics - taken from our own course, *Journey into Maths*, for 12 year olds (Bell, Wigley and Rooke, 1978). The first of these provides an entry into fractions, ratios and percentages. It gives a list of raw marks in a number of school subjects, and asks the pupils to compare them; this leads to questions of equality.

The following marks were obtained by Jane in her end of term tests:

English	12 out of 20	Geography	34 out of 55
Science	25 out of 40	Religious Education	17 out of 30
Housecraft	24 out of 50	Music	28 out of 45
History	27 out of 60	German	37 out of 65
Handcrafts	24 out of 40	French	4 out of 7
Mathematics	34 out of 50		

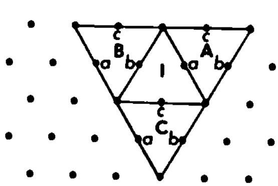
In which subject did she do best? Which was her worst subject?
Make a new list of subjects and marks to show the order from best to worst.
If you find some subjects difficult to compare, work through PC 2 first.

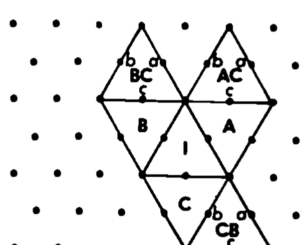
Some comparisons can be made mentally - it is easy to separate those which are below half marks from those that are above; and comparing each more closely with half marks, separating off the difference, gives many further indications of order. The list becomes as shown.

English	+2/20	or 1/10	Geography	+ 6.5/55	>1/10
Science	+5/40	>1/10	RE	+2/30	1/15
Housecraft	-1/50		Music	+5.5/45	>1/9
History	-3/60		German	+4.5/56	<1/10
Handcraft	+4/40	1/10	French	+.5/7	1/14
Maths	+9/50	~2/10			

It is now clear that History is the lowest mark, Housecraft next, and Maths the highest; and that RE and French are probably the third and fourth lowest, though it needs checking just how near to 1/10 is German. To check this, we may note that, since 4.5 is 9 x 0.5, 0.5/7 is 4.5/63, which is greater than 4.5/65; and that (multiplying by 4.5), 1/15 is 4.5/67.5, which is less than 4.5/65, so the order is French>German>RE. Another step of the same type, separating off the differences from 1/10, might well settle the remaining questions. On the other hand, at some point, one might seek a general method, such as converting all to percentages. These could be *estimated* first, then calculated, perhaps with a machine. The point here is to note not only the relevance of generalisation, but also that the basic mathematical actions of comparing, combining, separating, provide a powerful method of attack.

A final example is the task entitled *Triangle Algebra.*





In this task, a triangle is moved around the plane by half turns about the midpoints of its sides. The spaces are labelled by the sequence of letters denoting the movements required to reach them from the central starting position. Thus AB is the space reached by half turns about the midpoints a, b in succession.

Since a given space can be reached by move than one route, certain 'words' are equivalent. In particular, there are identity words denoting movements which bring the triangle back to its starting point. Exploring these interrelations between the algebra and the geometry of the situation is the substance of the task. One such relation is that each AB translates the triangle in a particular direction. BA reverses it, and BC, CB and so on have similar properties. This leads into the generalisation that the results of two successive half turns about different points P,Q is a translation 2PQ. Here, again, the explorations of combinations and equivalences, and of the relation between situation and representation, drive the inquiry.

One last example from a different field will illustrate how the basic mathematical actions generate problems from a situation. The lists shown, of pop videos and records, both offer situations which give experience of positive and negative numbers, for movements up and down the charts.

NAME OF RECORD	Position Jan 14th		Position Jan 21st		Position Jan 28th	CHANGE OVER 2 WEEKS
Pipes Of Peace			1st	Down 1	( )	Down 1
Relax	( )	Up 4	2nd	Up 1	( )	
What Is Love?	(A)	Down 1	3rd			Down 2
That's Living All Right	( )	Up 14	4th	Up 1	(B)	
A Rockin' Good Way	( )	Up 3	5th			Up 8
Bird of Paradise	( )	Up 13	6th	Down 1	( )	
Marguerita Time			7th	Down 8	( )	Down 12
Tell Her About It	(C)	Down 4	8th			Down 14
Running With The Night			9th	Down 7	(D)	Up 6
Islands In The Stream	( )	Down 2	10th			Down 16

Music Video

Show how each video has moved in the charts since the previous week. One has been done as an example.

Week ending January 28, 1984

THIS WEEK	LAST WEEK	
1	1	ZIGGY STARDUST, David Bowie, Thorn EMI
2	3	DURAN DURAN, Duran Duran, PMI
3	4	NOCTURNE, Siouxsie & The Banshees, Polygram
4	2	NOW THAT'S WHAT I CALL MUSIC VIDEO, Various, PMI/Virgin.
5	8	CLIFF VIDEO CONNECTION, Cliff Richard, Thorn EMI
6	10	VIDEO EP, Phil Collins, PMI
7	—	THE SINGLE FILE, Kate Bush, PMI
8	6	VIDEO EP, David Bowie, PMI
9	5	LIVE OVER BRITAIN, Spandau Ballet, Chrysalis
10	12	TRACK RECORD, Joan Armatrading, A&M
11	7	LIVE AT WEMBLEY, Meat Loaf, Videoform
12	17	SHADOW OF LIGHT, Bauhaus, Kace International
13	9	VIDEOWAVES, Various, Polygram
14	15	COMPLETE MADNESS, Madness, Stiff
15	11	VIDEO SNAP, Jam, Polygram
16	13	LIVE, Phil Collins, PMI
17	—	LIVE, Billy Joel, CBS/Fox
18	20	LET'S SPEND THE NIGHT TOGETHER, Rolling Stones, Thorn EMI
19	14	LIVE, UB40, Virgin
20	16	LIVE, Whitesnake, PMI

In the first list, comparing the positions in the two weeks gives rise in the usual way to the changes such as up 2, down 3 (should the sum of these be zero?) and in the second list, leaving gaps is a way of provoking the need to *combine* the *changes*, including cases where, because the unknown is at the beginning, the stated change has to be *reversed* to obtain the answer.

I have tried to show in this section how the general mathematical processes of representation, abstraction, generalisation and proof, and the actions of comparing, classifying,combining, transforming, reversing, generate appropriate activities for a curriculum.

Mathematical Processes and Research

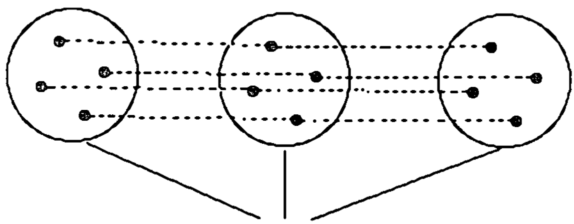
In research, the processes and actions impinge in two ways; first, they can provide a heuristic for generating and elaborating tasks for the study of learners' developing concepts; secondly, they may be themselves the target of research, both to investigate the extent to which they can be developed as abilities and strategies by pupils of different ages and abilities and in different environments, and to study the overall properties of a curriculum in which they are given particular emphasis.

The first of these two aspects - generating tasks for studying pupils' concepts - is relevant to the work presented by Herscovics and by Gelman. Herscovics recognises three mathematical processes, the development of conceptual structures, problem solving and proof. The last two overlap with my description, but I would want to distinguish between general problem solving strategies and the general *mathematical* strategies of abstraction, representation, generalisation and proof. The development of conceptual structures is, I think, somewhat different, as it is a long term process, observable from the outside, as it were, rather than a consciously usable strategy, like the others are.

The extended model of understanding makes an important and helpful distinction betweeen physical or 'applied' and pure mathematical concepts. It is this which has provoked the recognition of the important new distinctions between plurality and quotity and between position and ordity. The reported experiments show a Piagetian ingenuity and the results are striking. Modifications of the experiment in which the ordity question is asked first would be interesting.

I find less helpful the horizontal distinctions in Herscovics' scheme, between intuitive and procedural understanding and logico-physical abstraction - and, I cannot easily relate these titles to the contents of the cells. Another way of generating tasks to investigate aspects of the multiplication concept is by framing a child-relevant definition of the concept, then appplying the basic mathematical actions of comparing, combining, classifying and so on.

My suggested definition is *"the product of two numbers a and b is the total number of objects in a set of a sets, each containing b objects"*.



Such a configuration contains a multiple 1-1 correspondence, connecting the corresponding object in each of the a sets, and a many - one correspondence between the four objects of each set and the set which contains them. Both aspects are inescapably present. This is the physical aspect.

The mathematical aspect is the identification of groups of number-words in the learnt numeral-sequence, one, two, three ...., which occurs with skip or group counting - 1 2 3, 4 5 6, 7 8 9, 10 11 12, ... Here the end points are emphasised and also the number of groups is kept account of; this leads eventually to the table facts such as four 3s are 12. This development may take place independently of the recognition of the physical aspect of multiple sets. (I believe that Steffe has shown that this is so, though I do not have the evidence to hand.)

The mathematical actions applied to this situation - transforming, reversing, combining - give rise to questions such as whether a sets of b contain the same number as b sets of a; whether $b < c \Rightarrow ab < ac$ and $ba < ca$, the physical transformation of the sets of b to the particular case of a rectangular array with a rows, and b columns. All these are aspects of the multiplication concept.

I would take the crucial one to be making the connection between the physical situation and the number sequence, so that the skip or group counted sequence is used to determine the total number of objects, instead of counting all the separate objects. But this is largely a question of to what subset of aspects of the concept does one attribute the label 'understands multiplication' - a kind of demarcation dispute, with no *a priori* answer.

The construction of the concepts of the primary measures such as length, area, capacity, volume, mass/weight, time follows a sequence of *comparison, composition and quantification by unit iteration*. Thus, for length, there is first some means of determining which of two long objects is the longer - by means of juxtaposition. Next, a method of composition is required, in this case, placing the objects end to end and considering them as a single new long object. Taking the first object as a 'unit', the object formed by combining it with a second object, which matches it by juxtaposition, is of length two units, and so on. An object, two of which combine to give a unit, is of length one half, and so on. Thus a set of lengths corresponding to all rational numbers is generated from the two basic acts of comparison and composition (Compare Ellis, 1967). These are the bare bones of the length concept and its measurement. Many details need to be filled in, as has been done by Piaget et al (1960), and now by Héraud (1989), following Hersovics' scheme. The present conceptual analysis begins with the visual appraisal of length comparison, which is essentially an imagined juxtaposition, then actual juxtaposition (is this the correct order?). Comparison may be extended using the notion of transitivity, to non-juxtaposable objects e.g., Piaget's towers on tables of different heights, using a third object as go-between, this object being (a) longer, (b) less long than the objects to be compared. Not all of these aspects appear to be included in the Herscovics/Héraud scheme. Corresponding to the physical/mathematical distinction in the case of multiplication, there is a well-defined distinction here between the comparisons and the establishment of numerical measure by unit iteration. The latter brings with it some additional properties such as the inverse relation between size of unit and measure-number. And, occupying somewhat incongruously the formalisation cell, there is the use of a (numbered or un-numbered) graduated ruler, with its own specific potential misunderstandings concerning the starting point, and counting marks instead of spaces. Here again, I feel the important aspects of the concepts are here, but fit uneasily the cell labels; in particular, the procedural vs. abstraction distinction can sometimes be reversed without altering the essential content, for example, by changing the recognition of the equivalence of A and B into the transformation of A into B.

The geometric and algebraic straight lines, as decribed here, do not, I think, belong to the same conceptual field; the algebraic line is more properly a part of the field of algebraic geometry, in which geometrical problems are solved by co-ordinate methods. Intersections of lines, and of lines and curves, and the gradients of lines are the stuff of algebraic geometry; whereas the notion of gradient is meaningless for the geometric line in the absence of a co-ordinate system.

Fractions

The physical aspect of the fraction concept is labelling by the fraction term (say) one fifth, a quantity, five of which make up some quantity designated as the unit or whole; and (say) three fifths as three one-fifths. The mathematical aspect consists essentially of the recognition of the equivalence of a/b and ka/kb. Some of the various situations in which children may develop these awarenesses are the following:

> *Which is more, half an apple or a quarter of an apple? ... (which would you rather have to eat?)*
>
> *Is a half always bigger than a quarter?*
>
> *If you had half a small apple, could it be smaller than a quarter of a big apple?*

Piaget, Inhelder and Szeminska (1960) gave children shapes of flat clay or paper and asked them to divide them up 'fairly' between 2, 3, or more dolls, either by cutting, folding or drawing. They noted that the particular children in their sample were able to cope successfully with this task at the following ages, on average: $4\text{-}4^1/2$ years for halves of small or regular shapes; 6-7 years for thirds; 7-9 years for sixths, by trial and error; 10 years for sixths, by using a precise plan (e.g. dividing into halves, then each half into thirds). These authors give seven criteria for operational understanding of the spatial part-whole aspect of a fraction including that the parts must exhaust the whole, must be equal in size, and the parts must be considered themselves as wholes, capable of further sub-division.

Freudenthal (1983) notes a number of instances of fraction use in everyday language: "half as much/many/long/heavy/old as ..., half of a cake, half way, half an hour, half of the marbles ..... ", and so on, with increasingly more difficult fractions. It is from such experiences that the notion of half and of other fractions are abstracted.

Kieren (1976) analysed the fraction concept into subconcepts of part/whole, decimal, ratio, quotient, operator and measure, and this analysis was developed and used by the Rational Number Project (Behr et al., 1983).

Payne (1976) reports that the number line model consistently caused difficulty in various teaching experiments with 8 - 12 year old children in Michigan. Novillis (1976) in her investigation of the hierarchical development of the concept of fractions among 10 - 12 year olds confirmed that the number line model was significantly more difficult than either the area part-whole or the subset of a discrete set model. There is a full summary of relevant work in Dickson, Brown & Gibson (1984).

The Gelman et al experiment shows that young children aged 5-7, when faced with tasks which would be regarded by a teacher as embodying fraction concepts, mostly do not recognise the fractional aspects of the material, but respond by ordering the material either according to the number of pieces or to the total size of each display (sometimes ignoring the small pieces and counting only whole circles). The display material consisted of cards showing whole circles and parts; the 'number line' on which they were to be placed showed one, two and three whole circles (just one and three initially) at equidistant points. In this situation it seems a reasonable inference on the children's part that ordering by number or by size is required, particularly if they do not attach the same degree of significance as the teacher does to the special shapes of semicircle and quadrant as denoting half and quarter. As Gelman et al state, these children are assimilating the input to their existing scheme of counting numbers (to which we must surely add *ordering by size*). But I would not take this as an indication that little is to be gained from situational instruction. It is likely that time needs to be spent discussing physical situations which can be described using the fraction concept, and in which the idea of equivalence can be explored, before asking children to order fractions themselves by size - indeed, the question of what that might mean needs discussion. The research quoted above suggests appropriate developmental sequences through which children could eventually reach a mature conception of fractions, including their ordering on a line. In particular, there have been recommendations that the fraction words should be used for some time before introducing the usual symbols. As has been suggested above, failure of communication between teachers and pupils is a particular danger when symbols are used. A good safeguard is to offer the concrete situation and to let the pupils first describe what they see in it; this gives the teacher the possibility of responding in a way which picks up the children's meanings and works with them.

Related to this point is my unease about what children may mean if they say that 'Fractions aren't numbers!'. If, to them, 'number' means what we mean by 'counting number', they are telling us that fractions are not counting numbers but are some other kind of mathematical object. We know that children do recognise the existence of fractions as a way of describing part/whole situations, and, as time goes on, learn that operations can be performed with them. But it is often not until much later that most of them succeed in ordering them on a number line.

Research on Mathematical Processes

I have discussed how the mathematical processes can be used to generate and to extend problems, as well as to solve them. Some relevant questions for research concern the relation between process and content aspects of a curriculum. For example, does a greater emphasis on processes improve, or detract from, the learning of local skillls and concepts? Under what conditions? What are the means by which process learning may be encouraged?

A process enriched mathematics curriculum for pupils aged 11- 13 years was developed and trialled in two Nottinghamshire comprehensive schools, and published as *Journey into Maths - the South Nottinghamshire Project* (Bell, Rooke and Wigley 1978-9). Tests of the main mathematical processes, were designed and used to compare attainments with this curriculum and a standard one; number knowledge was also tested. The project classes did significantly better on *Generating Examples, Recognition of Relationships*, and *Explanations*, and there was no significant difference on a 'control' facet of *Following Instructions*. There was also no significant difference in number achievements. (The question here was whether time devoted to process learning would detract from content learning.) (Bell 1976). A somewhat similar experiment was performed in Calgary with some 20 schools using the same test, with similar results (Brindley, 1980). The tests were further developed and a report of their use was given to PME 5 in Grenoble (Bell and Shiu, 1981; Bell, Shiu and Horton, 1981). Later, a more extensive interview form of test was developed (Fowler, 1984). For a review of other work, see Bell, Costello and Küchemann (1983).

Following this work, my interest has moved towards Diagnostic Teaching, that is the development of effective methods of conceptual learning through conflict and discussion. Experiments have shown very high levels of retention from such teahcing,a nd very low levels from commonly used methods (Bell, 1989). Selection and development of tasks suitable for provoking conflict and discussion of key concepts has highlighted activities, starting with a situation of interest (e.g. Pop Music charts) then generating problems by comparing, combining, transforming, reversing, changing structure, changing context and so on - thus using the characteristic mathematical actions to generate material. Getting the pupils to generate their own problems involves them in using the same actions themselves, and we are now asking to what extent can they do this, consciously and regularly, and thus become capable of directing their own learning to a much greater degree - the metacognitive question. This last is the focus of a current research proposal. The intention is to examine a number of different classroom environments, some normal, some special, to look at strategy development, self direction of learning, and mathematical attainment, and to do this before and after a year-long intervention aimed at enhancing pupils' awareness of their mathematical learning strategies.

We hope that there are other people interested in this area of work too.

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EXPLORING THE PROCESS PROBLEM SPACE:

NOTES ON THE DESCRIPTION AND ANALYSIS OF MATHEMATICAL PROCESSES

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My goal in this reaction paper is to explore the main theme of this conference, the description and analysis of mathematical processes. The paper proceeds inductively. The first section offers an abstraction of the main themes in the two plenary papers, giving an analysis of the theoretical underpinnings of the studies reported by Herscovics (1989) and Gelman, Cohen, & Hartnett (1989), and the ways that their theoretical perspectives play out in the research they report. Those two papers have a great deal in common theoretically and methodologically; moreover, they both deal with rather elementary mathematical notions. For that reason the second section of the paper introduces a third data point, a study of student learning in a much more advanced mathematical arena. On the one hand, that study offers a dramatic contrast with the first two; it differs in its methodological approach and analytical focus as well as in the complexity of the mathematical knowledge it explores. The contrast points to the diversity of processes that need exploration, and the diversity of methodologies appropriate for carrying out those explorations. On the other hand, it will be argued, that study is still in the same qualitative category as the first two: all three focus on conceptual growth and changes in knowledge structures. There are, of course (to say the least!) mathematical processes other than those of knowledge acquisition. The final brief section of this paper sketches the dimensions of the larger problem space, the broad collection of mathematical processes we need to deal with as a research community.

1. An explicit (and reflective) commitment to a constructivist perspective.

Roughly speaking, that commitment is to the idea that humans are not direct perceivers of, but rather *interpreters of,* what they experience. That is, individuals develop interpretive frameworks that shape their perceptions of what they experience in the world, and what sense they make of them.

In Rochel's work[1], for example, one underlying question is: "Will children interpret (particular representations of) fractions in terms of the whole number schemata they possess?" Note that the mechanisms of assimilation and accommodation (in the Piagetian sense) are invoked to help explain the ways that children interpret what they see. Nick's work is also in the Piagetian tradition. A major focus of that work was the search for invariants -- e.g. "conservation of volume" for Piaget, "invariance of ordity" in the tunnel task for Nick.

In both cases, the underlying perspective is similar: "I must try to put aside my own perceptions of the things I am investigating. The issue is, how does the child see them?" [Admittedly, that perspective may seem somewhat obvious now. Let us remember, however, that as little as two decades ago it would have been considered heretical in most educational research circles; a decade ago it would have still been considered out of the mainstream.]

"The" constructivist perspective also plays out in important ways in the methodological aspects of the two bodies of studies. The researchers are aware that what they see in the phenomenology -- their raw data -- is also interpretation. That is, they know they will see what they are biased to see in their data, so they must take precautions against those biases as they try to interpret the data. The result is a questioning stance by the researchers about their own assumptions and data. [In contrast, consider those who believe that intelligence is well defined and is

[1] Given the casual nature of this reaction paper -- a running account of my talk -- I index each plenary presentation by its author's first name. No slight to Rochel's to co-authors, Melissa Cohen and Patrice Hartnett, is intended by the use of "Rochel's work," nor any to Jacques Bergeron by the use of "Nick's work."

measurable by IQ tests, and who then casually refer to those who score above a particular test value as "intelligent people."]

2. An understanding that the cognitive structures that correspond to simple mathematical structures may not be simple at all.

Consider, for example, Nick's counting on task. Children were asked to "count on" from an object identified as the sixth in a row, even though the first few were covered (Figure 1).

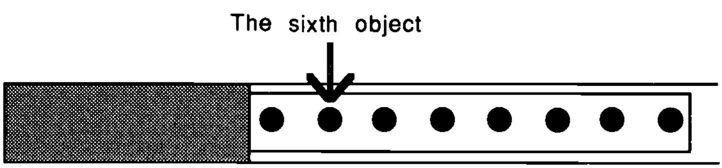

Figure 1. Herscovics' "counting on" apparatus

Most of the children (86 of 91) performed the task without difficulty. Now, for adults, there is a "natural" consequence of the fact that the item Nick identified was the sixth in the row: the number of objects up to and including that object is six. Hence, for example, the number of objects covered is four. [There are six objects up to the identified one; two are visible; hence four remain.] Nick reports that only 33 of the 91 kindergartners in his study were capable of determining how many objects were covered. Thus, the ability to count on from 6 does not necessarily imply that one understands the cardinality principle involved in labeling an object as "sixth."[2] For a case study in "cognition of number is much more complex than number," see Gelman & Gallistel (1986).

[2]To be absolutely certain of this finding, one would have to show that the students could respond appropriately to the following task: "I had six objects on the table. I've covered some of them. [Two remain visible.] How many are covered?" Given that the children can do this, then the missing link is the "the label sixth implies a cardinality of six" link suggested by Herscovics. I assume Nick has data regarding such competencies of his students.

3. A fundamental concern with internal (cognitive) and external (symbolic) representations and their entailments.

When Edith Neimark asked Rochel at the conference whether she was concerned with representations, Rochel's response was "What do you think I've been doing for the past 20 years?" Loosely speaking, both plenary speakers are concerned with what's "in the head" and what's "out there," how each is represented, and what consequences there are to any assumptions one makes about the form and nature of the representations. [Let me also stress the phrase "loosely speaking:" both of the terms "in the head" and "out there" are psychologically and philosophically problematic, and the authors are more careful than I in their use of language.]

4. More importantly, a concern with the *rapprochement* of cognitive and mathematical structures.

That is, how does the set of internal understandings come to be consistent with the mathematical world -- i.e. symbolic entities as constrained by the laws of mathematics?

Here the issues get *really* interesting, and most naive assumptions just don't hold up. The past few decades, especially the last, has seen a re-emphasis on manipulatives and the use of "concrete" materials. But, for example, as Rochel asked, is the number line "concrete" or symbolic? (The only safe answer is "yes.") Or, take Dienes Blocks as a "concrete" manifestation of base 10 arithmetic. A close analysis reveals that some of the physical properties that make Dienes blocks "work" so well for kids -- e.g. the fact that trades (e.g. ten ones for a ten) work because of an inherent fairness in conservation of mass -- may make it harder for kids to see that the property holds independent of the physical objects. And, the arithmetic of Dienes blocks isn't really isomorphic to the arithmetic of base 10. (For example, carries are column-independent in the symbol system: more than 10 in any column results in an addition to the column to the left. In Dienes blocks, the trades are different in each ostensible "column:" ten ones for a "long" in the first "column," and ten "longs" for a "flat" in the 2nd. Trades are optional in the Dienes world, mandatory in the symbol system).

Another interesting issue that emerged in the meeting is that there may be two consistent but independent ways to characterize and present whole numbers: (a) numbers as obtained by the count sequence, and (b) numbers as measure. Given that, what's the best way to have kids make sense of these competing groundings for the notion of number? How should number be "anchored" in kids' experience, how can the entailments of either conception be exploited; what balance in introducing the two systems should be maintained? Here too, whole number represents just the tip of the iceberg. As a more complex example, consider the various ways of conceptualizing fractions. Two of the more elementary ones (see Kieren, 19XX, for more) are "fraction as part-whole" and "fraction as unitary stretching or shrinking operator." Now the part-whole conceptualization is wonderful for some things. With it you can make good sense of fraction equivalence, and have a good basis for representing addition of fractions. On the other hand, part-whole is horrible for making sense of multiplication. The more deeply embedded the notion is, the more difficult it may be to understand what multiplying by a fraction is all about. In contrast, stretching or shrinking is wonderful for multiplication. What's 2/3 of 5/6 of something? Just imagine something being shrinked twice, by the appropriate amounts! Now, however, try to use the stretching or shrinking notion to make sense of fraction addition or subtraction! In short, each conceptualization helps to understand some aspect of what fractions are all about, and may *interfere* with a developing understanding of other aspects of it!

The situation, then, is that the individual encounters a whole bunch of "real world entities" and systems that embody *some* aspects of the mathematical ideas, and which may be in conflict with the partial embodiments in other such systems (compare Dienes Blocks and Unifix Cubes for entailments, for example); they encounter different theoretical and symbolic partial conceptualizations and representations of the mathematical ideas (note that base 10 arithmetic,m even when mastered, is only one aspect of "number."); and sorting through all of this, they're supposed to abstract the underlying mathematical core. Issues regarding the best ways to structure student experiences so that they can navigate through all this territory are subtle indeed. Figure 2 gives a crude illustration of the intellectual thicket. For a more extended and careful discussion, see Schoenfeld, 1986.

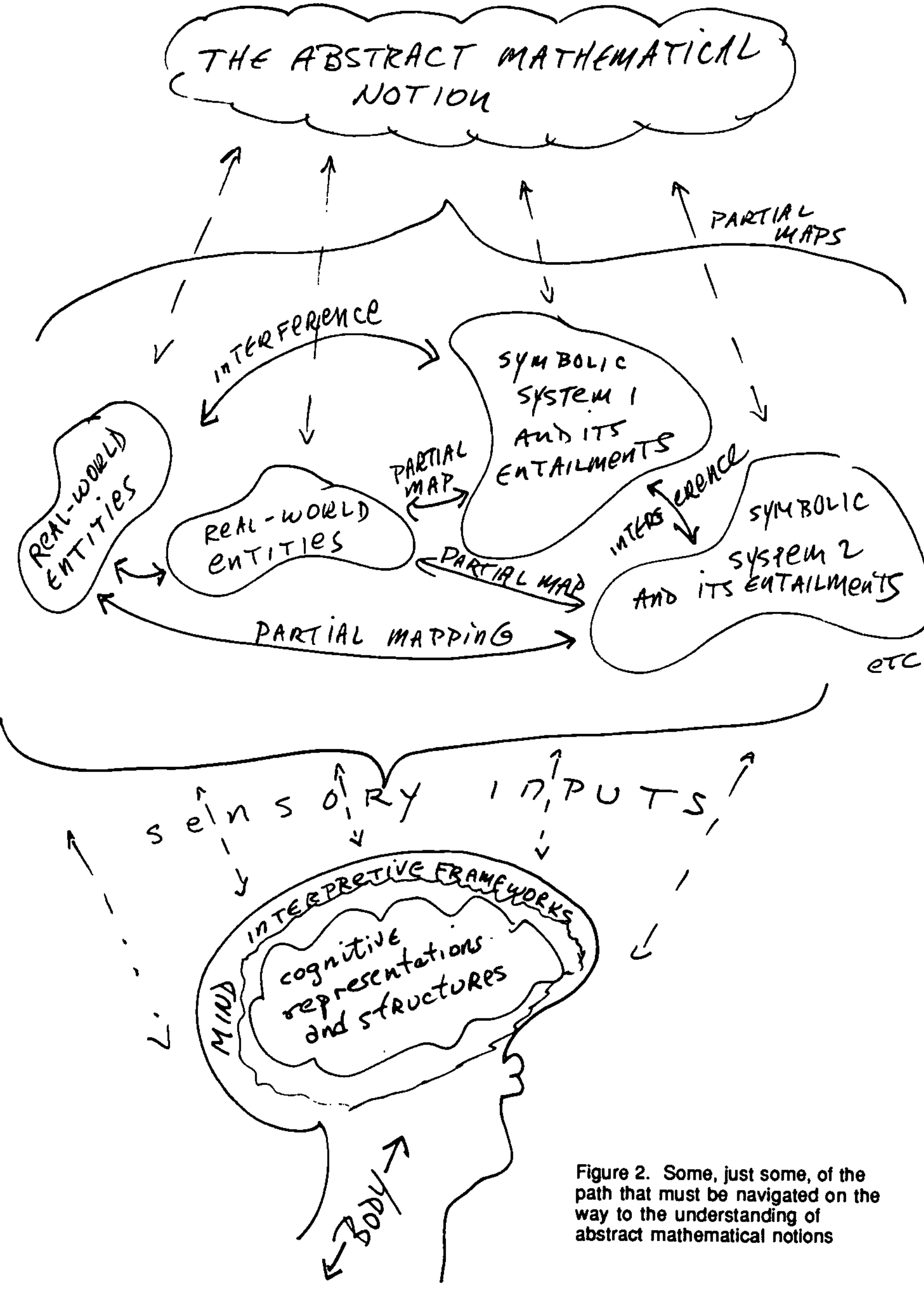

Figure 2. Some, just some, of the path that must be navigated on the way to the understanding of abstract mathematical notions

5. Data analyses are theory-driven.

For Rochel, the current set of research questions lies at the what might be called the "fringe of biological/cognitive constraints." Cross-cultural work indicates that, independent of schooling and culture, there are some universals with regard to children's developing understanding of number -- universals that are strong enough to suggest (as with language studies) that humans are born with biological predispositions to perceive and comprehend number in particular ways. Given this as background, Rochel sought in the current study to push the boundaries. Would there be similar predispositions toward the understanding of (particular representations of) fractions, or does this shift to a new mathematical system transcend the whole number understandings, and require "theory change" (see 6 below)?

Nick's theoretical frame was more explicit. Herscovics is concerned with the acquisition of conceptual schemata, the means by which students come to grips with formal mathematical notions. His framework, reproduced in Figure 3, proposes a development cycle that moves from an intuitive understanding of physical concepts (that is, physical instantiations of the mathematical concept) to the understanding of the emerging mathematical concept.

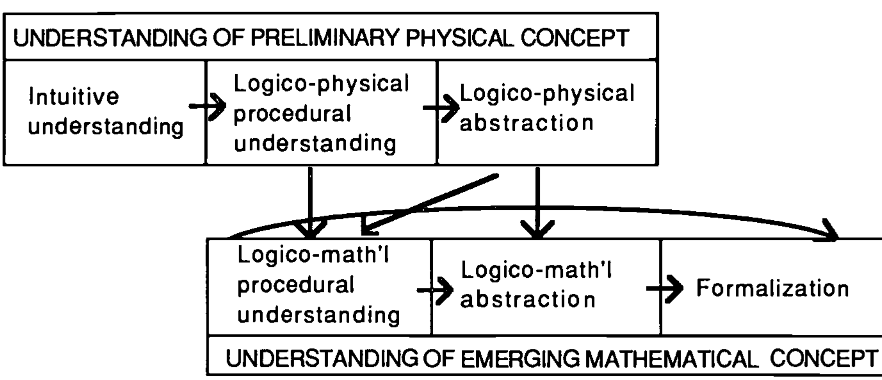

Figure 3. Herscovics' Analytical Frame

While I appreciate the explicitness of both authors' theoretical frames, I do have significant reservations about each. In Rochel's case, I have the feeling that the analytical frames that have served her so well in her investigations of children's understandings of whole number will fall decidedly short as she moves into more complex mathematical domains such as the domain of rational numbers. For one thing, there is a huge increase in the complexity of the mathematical notions students must learn to cope with, and a corresponding increase in the complexity of the corresponding cognitive structures. The various categories in Nick's framework indicate the kinds of things Rochel will have to look for as she moves to examine kids' understandings of more complex mathematical notions. Moreover, as Rochel moves out of the realm of "biologically predisposed" cognitive structures, she is likely to find less regularity of the kinds she has found up to this point. And, as she noted at the conference, the role of language as a mediating factor -- which was minimal in the case of the early acquisition of number -- is going to be central when she studies the acquisition of more complex mathematical concepts.

My reservations about Nick's theoretical framework are different. On the one hand, I do think it has a great deal of heuristic value: It is relatively comprehensive, and in my opinion you run the risk of missing something important if you ignore any of the six categories indicated in the Figure 3. On the other hand, I have a deep distrust of hierarchical frameworks that only have arrows moving in one direction, inexorably progressing from "naive understandings" to "formal knowledge." That distrust is based on two kinds of empirical findings, elaborated in the next main section of this paper: (1) the observation that knowledge structures are far more fragmented, unstable, and localized than any straightforward frame such as Figure 3 would allow, and (2) the observation that there should be lots more arrows, and all of the arrows that do appear in Figure 3 should be bi-directional. For example, I note that when I have represented a concrete situation in formal mathematical language and played with the symbolism for a while, I may well develop new intuitions about the situation itself. Hence there should be an arrow from the last box to the first (and many many others). There is something I tend to call the "structuralist trap," the desire to have real-life phenomena correspond to the artistic representations we draw of them. In general, life is more complex than that, and researchers have to be careful not to be seduced by their pretty pictures. However, let me not come across as being nasty here. As I said above, you ignore any of the boxes in Nick's frame at your peril.

6. Both are interested in theory change -- i.e., learning -- and the mechanisms by which it takes place.

I can't stress enough how important this is. There is, for example, a whole field called developmental psychology -- and very few people in it study development! That is, the vast majority of developmental studies look at cross-sectional attributes of kids examined at ages A_1, A_2, A_3,..., and talk about trends in the ways those attributes change. However, those studies do not look at particular individuals, and ask how the change takes place. That, in my opinion, is where the action is.

Both Rochel and Nick are concerned with the conditions for, and nature of, cognitive change. Nick calls his study an examination of "the construction of conceptual schemes," and Rochel calls hers a study of "theory change." Both ask: How does a student come to grips with new knowledge? And Rochel's data offer clear support for a gradualist, threshold perspective. Those of her students who were in some sense "ready" to make sense of her fraction representation (the ones in her top group) profited greatly from the experimental intervention, while all the other students were did worse after it than before.

7. On methodology.

Note the parallels, derived in spirit from Piaget. Both researchers use a variant of the structured clinical interview. On the one hand, such detailed and structured observations are indeed an appropriate way to focus on process. On the other hand, that's not the only way...

AN NOW FOR SOMETHING (NOT SO) COMPLETELY DIFFERENT

This section briefly describes the main body of work done by my research group over the past few years. As noted in the introduction, the work described here stands in stark contrast to the studies by Herscovics and Gelman *et al.* It deals with fairly advanced mathematical content as opposed to the (mathematically) straightforward notions in the two plenary papers. It deals with a relatively unstructured set of mathematical interactions as opposed to the carefully structured clinical interview formats discussed above. It has a much finer grain size of analysis, and trades off large n (91 for Herscovics, 40 for Gelman *et al.*) for n=1, with a corresponding change in level of detail. In fact, the group is way over on the detailed end of the spectrum. We spent a year and a half analyzing seven hours of videotape of one student learning about functions and graphs.

The work reported here is a distillation taken from a paper affectionately referred to by my research group as "the monster," a 200-page report (Schoenfeld, Smith, and Arcavi, in press) of our findings. I will not try to draw out the explicit contrasts with the two plenary presentations, because they are pretty obvious. However, I note that much of our study resides within the sixth box of Herscovics' Table 3. Thus the contrast is rather pointed. (And, I note, our study is rather incomplete -- it has not yet dealt with much of the territory in Nick's other 5 boxes, but we're working on it.)

Our research over the past four years has focused on students' understandings of the concepts of functions and their graphs in the Cartesian plane. The concept of function is one of the core ideas in mathematics. It is difficult for students to understand, and has been the subject of extensive research in mathematics education (see, e.g., Bell & Janvier, 1981; Buck, 1970; Dreyfus & Eisenberg, 1982; Freudenthal, 1973, 1983; Janvier, 1983, 1987; Tall & Vinner, 1981; Vinner & Dreyfus, in press). A main focus of our research and pedagogical work has been to explore the potential of computer-based learning environments for enhancing instruction in such complex cognitive subject areas. (For general discussions of the potential of such technologies see Hansen, 1984; Nickerson and Zodhiates, 1988; Pea, 1987; Smith, Porter, Leinbach, and Wenger, 1988.)

Graphing is an excellent target domain for cognitive research, and for studies of computer-related instruction. Graphing software can be highly motivational (Dugdale, 1982, 1984). It can present dynamic representations of mathematical objects that students had heretofore experienced only as static objects. Watching the graph of f(t) evolve in real time (ITMA Collaboration, 1984; Mokros & Tinker, 1987) can help students build dynamic interpretations of graphical phenomena, for example the fact that when the graph is horizontal, the quantity f(t) is not changing. In addition, graphing software can -- in stark contrast to students' and teachers' rough sketches -- present essentially veridical representations of graphs on the screen, hypothetically allowing students to "see" what the properties of graphs are, and to compare the properties of different graphs. [Note: what we know to be veridical may not be similarly perceived by students[3] (Goldenberg et al., 1988). There are serious research issues here, discussed below.] Graphing software can simultaneously present multiple representations of mathematical objects (e.g. problem representations, their graphs, their algebraic forms), thereby allowing students to see the mathematical concepts in their different mathematical manifestations (ITMA Collaboration, 1984; Kaput, 1985a,b.). And, such software can allow students to manipulate mathematical objects dynamically. In this way, students can see on the screen what heretofore mathematicians had only been able to imagine for themselves. In addition, in allowing students to analytically and graphically "pick up and move" a complex function like $y = 3x^2 + 5x + 2$, the software helps students to perceive of this complex mathematical object as a single entity (one function in a family of functions), in the way that mathematicians do. In this way it reifies the notion of function as object, helping the student to conceive of it as a "conceptual entity" (Greeno, 1983). Our R&D goals were to develop computational tools to explore the potential of technology to foster learning, and to use the technological environment as a context for detailed studies exploring the nature of thinking and learning in the domain of functions and graphs.

Our technological development included the construction of a software package called GRAPHER (See Schoenfeld, in press), which now runs on Mac II's. GRAPHER consists of three linked micro-worlds: (1) POINT GRAPHER, in which students learn to "build" functions on a point-by-point plotting basis; (2) DYNAMIC GRAPHER, in which students can consider the graphs of families of functions (e.g. $y = \mathbf{a}x^2 + \mathbf{b}x + \mathbf{c}$), changing the parameters and watching the graphs vary in real-time; (3) BLACK BLOBS, a graphing game that is a modification of Sharon Dugdale's (1982, 1984) "Green Globs." When students start the game, a collection of "targets" appears on a Cartesian grid. Students shoot at the targets by entering algebraic equations into the computer. The equations are graphed, and the students score points for each target they hit. See Figure 4.

[3] Indeed, Goldenberg's work and ours indicates that there can be consistent *mis*-perceptions of accurate graphs. Here are two examples. (1) Because of the ways that humans tend to perceive distances, the accurately drawn parabolas $y = x2$ and $y = x2 + 1$ appear <u>not</u> to be parallel, while one is in fact a vertical translation of the other. (2) Mathematically, changing the parameter **b** in the family of functions $\{f(x) + \mathbf{b}\}$ results in the vertical translation of the function $y = f(x)$. Perceptually, however, the family of lines of the form $y = mx + \mathbf{b}$ may appear to move sideways or diagonally as the parameter **b** is changed. Hence the perception belies the mathematics.

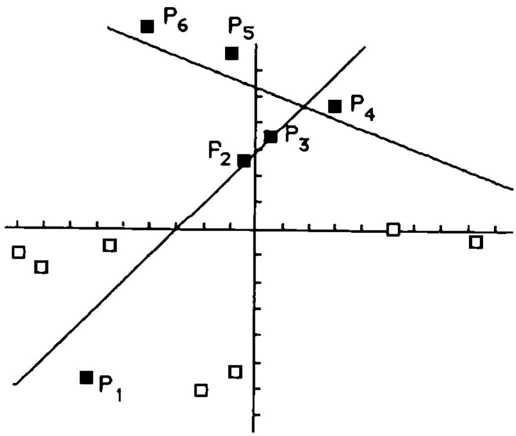

Figure 4. A game situation in which the student's first shot aimed at targets P_1 through P_3, her second shot at targets P_4 through P_6. [Note: she was not constrained to linear functions.

To pilot test our software we brought some students into the lab and asked them to play with it for a while. The videotapes we made of those pilot sessions revealed some interesting student behaviors, which we felt called for close examination. We chose to examine seven hours of tape, comprising one student's complete interactions with the environment (GRAPHER and a graduate student tutor). Little did we know when we began that those seven hours of tape would be the main focus of our work for the next year and a half. The goal of our analyses was to understand, as much as possible, what the mathematics looked like from the student's point of view -- and how her point of view changed by virtue of the interactions with the environment. That is, we wished to get a (reasonably) comprehensive description of what she knew when she entered the lab, and how that knowledge changed.

The subject of our research, IN, was a 16 year-old high school student participating in an honors calculus program on the Berkeley campus in the summer of 1987. At the time she began working with our software -- mostly exploring the properties of straight lines using the "Black Blobs" game, explained below -- she had studied units on graphing in the calculus class, which include the use of derivatives to determine the critical values of functions. The laboratory setting was informal, her interactions with JS (graduate student and her "guide" to the software) casual. There was no fixed agenda. As IN and JS explored the system they determined, mostly extemporaneously, what they would next examine.

Because of the informal nature of these interactions the sessions differ from many laboratory teaching studies, where the subject matter that the student sees is tightly controlled, and the interactions between teacher and student are choreographed in detail. In this case we believe the "messiness" of our data is an advantage rather than a disadvantage. Rather than ignoring social interactions in the data, as one might do in controlled laboratory studies, we were compelled to deal with such interactions in our analyses. Because of this, we feel confident that our methodologies are sufficiently robust to apply to studies of learning in complex educational settings.

Figure 5 represents an overall view of the cognitive territory covered in the IN analyses. The left-hand column identifies four levels of cognitive structure[4] found in our data (reflecting levels of structure that have been characterized in the literature). The middle column illustrates the three top levels -- the standard ones in curricular analyses -- for straight lines. The right-hand column illustrates IN's cognitive structure for the same subject matter, when she began working with us. The task we attempted was to characterize the changes in IN's cognitive structure, as it evolved while she interacted with the environment.

The general structure illustrated in Figure 5 fits comfortably within the cognitive literature. The topmost level reflects the generally accepted observation (Hinsley, Hayes, and Simon, 1977; Rumelhart and Norman, 1983) that our knowledge and our perceptions of the world -- better, our expectations, based on prior experience, of what we will encounter in the world -- are organized at the macro-level in large "chunks" called schemata. The schema illustrated in the middle column is the most familiar one for straight lines: A line has equation $y = \mathbf{m}x + \mathbf{b}$ if and only if its slope is **m** and its y-intercept is **b.** The second level, reflecting knowledge compilation (see, e.g., Anderson, 1983) describes objects in the domain and their familiar properties. For example, if you show a mathematician a near-vertical line and ask if its slope is

[4]The term "levels" refers to the grain size of the objects described. As noted in Figure 1, our use of the term does not imply that these knowledge structures are hierarchically organized, or that knowledge is accessed in hierarchical fashion.

Figure 5

Graphs of Straight Lines: Levels of Analysis and Structure*

Levels:	Traditional View of Subject matter	Our understanding of IN's cognitive structure
1. Macro-organization of knowledge, at the schema level, e.g.:	2-slot schema: L: $y = mx+b$ slope y-intercept	3-slot schema: L: $y = mx+b$ slope x-intercept y-intercept
2. Compiled knowledge: macro-entities and entailments, e.g.:	m is the slope of L m>0: L rises large |m| : L steep (and more...) The point (o,b) is the y- intercept of L... (etc.)	m is the slope of L m>0: L falls large |m| : L flat... the x-intercept has a place in the equation and on the graph... the y-intercept has a place in the equation and on the graph...
3. Fine-grained superstructure supporting domain knowledge: conceptual atoms (nodes) and connections, e.g.:	$m = \frac{y_2 - y_1}{x_2 - x_1}$; $y_2 - y_1$ and $x_2 - x_1$ are directed line segments, so their ratio indicates direction (e.g. + indicating "up, right') and steepness (so much y for so much x).	$m = \frac{y_2 - y_1}{x_2 - x_1}$; but this knowledge is nominal and, while it is used to compute slope, the computation has no graphical entailments. not clear or stable

Figure 5 (Continued)

	when x=0 in y = mx+b, y = 0x+b = b, so the point (0,b) is (a) on the line, and (b) on the y-axis. Hence it is the y-intercept. [We call this level of structure the "Cartesian connection.]	IN knows that "b is the y-intercept," but her understanding is nominal and is not tied to the underlying structure as in the Cartesian connection.
4. The limited applications contexts out of and across which individuals construct the conceptual atoms that are seen at level 3	Void. (In traditional analyses -- and in people who have well developed domain knowledge -- these traces have vanished and the relevant conceptual atoms are the nodes at level 3.)	m is fuzzy, neither stable across contexts nor consistently evoked. Its meanings evolve. The role of the x-intercept evolves over the sessions. In 4 slightly different contexts, IN has 4 different meanings and interpretations for the y-intercept -- ie, the meaning of the term is context- dependent.

*The use of the term "levels" in this context does not presume that the structures discussed are hierarchical, or that they have the customary entailments of hierarchical structures.

large or small, the response will be a nearly automatic "it's large" -- because such properties are familiar. When asked why, however, the mathematician can "unpack" the knowledge produced at level 2, and provide the relevant justifications for it. This unpacking takes place at the third level, the level of fine-grained structure. (See level 3 of the middle column for a partial unpacking of slope and y-intercept.) Our research indicates that for mathematicians the connections illustrated in level 3 are always "in the background" and easily retrievable. The presence of this rich knowledge base, tightly connected, serves as a backdrop for competent performance. It also serves as the link between the manipulations in the algebraic world [in which **m** is simply calculated by the formula $(y_2-y_1)/(x_2-x_1)$] and the graphical world [in which **m** has graphical entailments].

There is an issue of grain size in choosing the "atomic elements" or "nodes" at level 3. One might choose as primitives "the location of points in the plane," for example. Or, one might further decompose them into their Cartesian coordinates, specify the conventions for labeling points in the Cartesian plane, delineate the order principles for the reals, etc. Typically, however, it is assumed that the issues just mentioned -- certainly for mathematicians, and almost certainly for students in calculus classes -- are unproblematic. Consider the notion of y-intercept, for example. For anyone with the slightest degree of sophistication and experience, the statement "the y-intercept of a non-vertical line L is the point where L crosses the y-axis" (see Figure 6) is assumed to be clear, unambiguous, and straightforward. Hence it is reasonable to expect that notions like y-intercept can be the "atoms" or building blocks of cognitive and pedagogical models. With these as the atoms in level 3, the traditional view (middle column of Figure 5) has no fourth level.

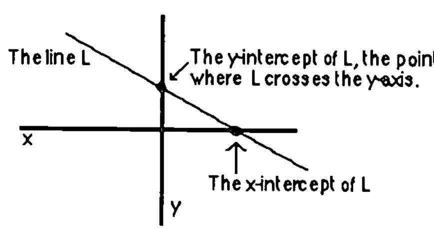

Figure 6. The notion of "intercept" is, ostensibly, a straightforward concept.

Our goal in the research was to capture the nature and evolution of IN's knowledge structures -- how and why they changed -- over her 7 hours of interaction with GRAPHER and JS. The right-hand column of Figure 5 represents IN's initial state. Here is a summary description .

Level 1: Knowledge schemata. IN entered our lab with a non-standard knowledge organization at the schema level, believing that 3 properties of a line (slope, y-intercept, and x-intercept) are necessary and sufficient to characterize a straight line[5]. As a result both of feedback from the system (where entering **m** and **b** suffice to generate a line) and of verbal interactions (the tutor, JS, "squashed" mention of "x-intercept" when IN made it), IN emerged with what appeared to be the standard 2-slot schema. However, her understanding lacked the deterministic underpinnings that are characteristic of mathematical understanding in this domain.

Level 2: entities and entailments: IN entered our laboratory knowing the standard terms (slope, intercepts) but having some misconceptions -- e,g, that lines of positive slope moved down to the right. Largely as a result of receiving visual feedback from the system, but with some direct interventions from JS, IN emerged from the 7 hours with the correct understandings. However...

Level 3: Deep connections. One of the reasons IN could have the misconceptions discussed at level 2 is that when she entered our lab, she lacked the fine-grained structures (the "Cartesian Connection") that support the correct understandings and that contradict the misconceptions. Though IN left our lab with many of the right impressions (e.g. line of positive slope go up to the right), these too were not grounded in the deep structure. Hence they were fragile, and possibly unstable. (Indeed, we have data indicating that old misconceptions die hard; Many of IN's resurfaced in interesting ways, long after they appeared to have been "replaced" with the correct knowledge.)

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[5]We note that such a confusion is "natural" in a novice to the domain: the x- and y-intercepts of a line are both perceptually salient, and there are formulas (namely the slope-intercept formula, $y = \mathbf{m}x + \mathbf{b}$, and the two-intercept formula, $x/\mathbf{a} + y/\mathbf{b} = 1$) in which slope, y-intercept, and x-intercept all play major roles. We note also that IN's mis-interpretation of her mathematical experience fits quite nicely with the literature on science learning, for example the misconceptions literature in physics (diSessa, 1983; McCloskey, 1983).

Level 4: Conceptual "atoms" are not atomic, but are indexed by context. We assert (the "monster" gives the details) that for IN, "y-intercept" was anything but a simple or unitary notion: In each of the game contexts illustrated in Figure 7 (where IN was trying to make the "shot" illustrated with the dashed line), IN had a somewhat *different* meaning for "y-intercept." In context 7a, where she wanted to hit two targets that spanned the y-axis, she had one interpretation and it worked well for her. Her first show was low (see Figure 4) and she compensated for it by making the appropriate change in the y-intercept. When she got to context 7b, however, she faced a new situation: a target was on the y-axis. In this case she conjectured the "b" value in $y = mx+b$ was zero, and was dumbfounded when the line didn't behave the way she wanted it to. Similarly, contexts 7c and 7d represented new situations for her -- and her interpretation of y-intercept was slightly different in each one.

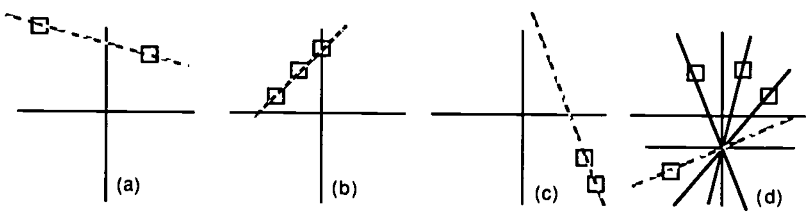

Figure 7. Four game contexts, in which IN had four different interpretations of y-intercept.

Major Findings

Here, in telegraphic form, are some of the main findings of our work.

1. We stress the presence and importance of level 4, and the roles of *indexing by context* and *bridging across contexts* in concept formation. It is clear from the data in the IN study, as well as the half dozen videotapes we have subsequently examined, that knowledge elements during learning are much more fractionated (differentiated by context), stochastic (unreliable in the statistical sense that the same conditions in the same context may not produce the same interpretation with great reliability), and unstable (easily altered if the context is modified even slightly) than suggested by many of the straightforward learning models in the literature (e.g. Siegler & Klahr, 1982). We note that our findings on concept development in this domain, functions and graphs, strongly parallel the results obtained by Confrey (1988) in the domain of exponential functions; also those by VanLehn in his (1989) re-examination

of Anzai and Simon's (1979) strategy acquisition data. We suggest that the process of learning in adult subjects mimics the process of concept development in children (paralleling, for example, the reasons Dienes sought multiple representations for numeration) and propose more fine-grained, microgenetic research of the type we have done to illuminate the mechanisms by which such concept development takes place.

2. IN's learning was slow and organic, with much instability and retrogression. We began our research with the idea that we could identify "learning events," interactions with JS or GRAPHER which resulted in clear and relatively stable changes in IN's cognitive structure (corresponding to the addition of new knowledge, or the deletion of incorrect information). Having gone over the videotapes looking for such learning events with a very fine-toothed comb, we report having found remarkably few. It appears that for learning in a new domain, simple "learning is adding knowledge to the knowledge base" models or straightforward "adding productions to a production system" models of learning (see, e.g. Klahr, 1978) do not do justice to the complex, unstable, and non-monotonic aspects of human learning. (Such findings are consistent with recent research on "local epistemologies" (e.g. diSessa, 1983, 1986) and with the motivations behind connectionist models of cognition.)

3. What we see on the screen is not what the student sees on the screen -- at a number of levels. First, humans consistently mis-perceive veridical representations on the screen (Goldenberg et al., 1988), meaning that "corrective" engagement with the material may be necessary for students to build the correct interpretations of the subject matter. More subtle and equally important, every mathematics teacher who has discussed the software with us has assumed that Black Blobs and its antecedent, Green Globs, are about and in the Cartesian plane (either as a mathematical abstraction or in a concrete realization on the computer screen). However, IN never fully made the mapping from the "game plane" (in which she shot at targets with lines) to the Cartesian plane. For her, lines in the game plane were *fuzzy objects* that were not uniquely determined, but rather members of a class of "good approximations" that nearly went through the centers of the targets. In the game situation illustrated in Figure 3d, IN shot at the target and the point (0,-2) with the line $y = .3x-2$. She was genuinely surprised when her line passed through the point (0,-2) -- despite the fact that the line <u>must</u> pass through that point: when $x = 0$, $y = .3(0)-2 = -2$. This surprise

came from the fact that (for her) the line she shot was a guess, not something mathematically determined. There are clear implications in this finding for software developers and teachers: One simply cannot make facile assumptions about what students see on the screen.

4. A standard assumption in much of the cognitive community [one not shared by many social psychologists or Vygotskeans; see, e.g. Clark, 1985] about analyses of videotapes like ours is that one can factor out the "noise" of social interactions to focus on the cognitive aspects of learning. In contrast, understanding social interactions and the role of language (including mathematics as a language to be learned) are focal points of our analysis. As one example, IN at first completely misunderstood JS's technical use of mathematical terms such as "draw," thinking he meant "produce a sketch" when he meant "derive the graph from the equation." Near the end of the seven hours, they were using terms much the same way. The evolution of IN's mastery of the language of mathematics, and her use of it to communicate with JS, is a central part of our story. As a second example, JS's verbal squashing of IN's use of the term x-intercept (cf. level 1 above) is a social interaction with cognitive consequences. Building on the work of Newman, Griffin, and Cole (1988), we have extended work on "appropriation of problem spaces" to build a theoretical means of discussing how IN and Jack each learn from each other. This work (Moschkovich, 1989) explores IN's growing autonomy and goal-directedness as she becomes familiar with the domain and less dependent on JS to structure her interactions with GRAPHER.

5. We have developed a methodology of competitive argumentation (borrowed in name and spirit from VanLehn, Brown, and Greeno, 1982) for analyzing these data, in which tentative explanations of a subject's current mental state and actions are contrasted with the previous explanations, and used to predict future behavior. As an example, claims that "the student interprets slope this way at time t" are subjected to validity checks for all times prior to time t (were the student's interpretations consistent with this interpretation before, or is there causal evidence of change if not?) and used to predict later behavior (absent causal changes, the student should have the same interpretation attributed to him/her at time t). This methodology allows us, at the microgenetic level, to present detailed, coherent, and reliable analyses of the evolution of students' knowledge structures and behavior.

THE DIMENSIONS OF THE PROBLEM SPACE

There are clear and dramatic contrasts between the studies reported in the first and second parts of this paper. The nature of the studies (large n *versus* small, cross-sectional versus shot-term longitudinal), the level of detail in each, the level of mechanism considered to be explanatory, the methodologies (structured clinical interviews *versus* running microgenetic analyses), the degree of mathematical structure in each of the domains studied, and the characterizations of individual knowledge structures, all differed widely.

Yet, all three studies are actually rather similar, in that they deal with the same theme. Rochel labeled hers a study of "theory change," Nick said his focus was the "construction of conceptual schemes," and mine was a "detailed analysis of evolving knowledge structures." All three of us were concerned with the growth and change of domain knowledge. There's a lot more to mathematical thinking, and mathematical processes, than that.

Consider, for example, a minor modification of Collins, Brown, and Newman's (in press) characterization of different dimensions of learning. They argue that coming to grips with a domain includes:

- Developing the relevant "resources" or domain knowledge;
- Learning the appropriate problem-solving strategies (e.g. mathematical heuristics);
- Working well at the "executive" or "control" level;
- Having the appropriate set of beliefs;
- Becoming acculturated into the community of practitioners in that domain.

Note that the three studies discussed at length in this paper all fall into Collins, Brown, and Newman's first category; hence all the variance described above resides in just one little corner of the problem space. My experience, and my prediction, is that to understand each of the other categories we will need to be concerned with radically different processes, and hence with radically different descriptive tools and analytical methods. Consider what it means to "learn to see the world through the mathematician's eyes," for example, and the role of language and social interactions in the development of a mathematical point of view. My sense of things is that at present we need focus and pluralism, and an occasional step back to look at the big picture. We will make progress by conducting detailed, theory-based studies of the workings of mind, and for the immediate future those studies are likely to have to be as narrow as the ones discussed here. At the same time, there should be broad diversity in what we look at, and the methods we use to do the looking -- I don't believe unified theories or methodologies are around the corner -- and period attempts to pop out of the problem space and reflect on it. We need to work on our descriptions both of the forest and of the trees within.

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SYMPOSIA

I. Realistic Mathematics Education,
Belief or Theory

II. Sex Differences in Mathematics Ability

III. Clinical Investigations in
Mathematics Teaching Environments

IV. Changes in Student Assessment Occasioned
by Function Graphing Tools

V. Alternative Conceptions of Probability

Realistic Mathematics Education (RME)
What does it mean?

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Introduction and overview In this contribution learning mathematics will be considered as *constructing* mathematics.

With respect to this the questions to be raised are: 'This way of learning mathematics, how can it be realised? ' and ' Which features does it have?'

While switching from learning to teaching one also might ask: 'The principles in the background of RME, which are they?' or ' The learning of mathematics as defined by which kind of teaching can it be evoked?'

Examples from early subtraction (< 100) served as a source to illuminate one thing and another. The suggested approach was compared with a competing one. (cf. Thornton e.o. 1983). By way of a brief reflection some general characteristics of the suggested approach and the competing one will be summarized and considered theoretically.

Subtraction (<100) Column subtraction is not a well fitting approach for many applications. Moreover children's strategies often differ from the column one.

Example : *My book counts 53 pages. I have read 26 pages. How many more pages will I have to read to finish the book?*

Children often do not recognise this problem as a subtraction. This concerns three out of four children in the intermediate grades (cf. Treffers a.o. , 1989).

So 53
<u>26 -</u>
... cannot be applied by these pupils. Moreover there exist other obstacles with respect to algorithmic column subtraction:

For half of the pupils from the intermediate grades the algorithm itself is problematic; the order of size of the numbers gets out of sight because of the 'right-to-left method'. The algorithm is artificial, it does not correspond with strategies of mental computation and estimation; it does not fit to methods of complementary counting in both directions and the algorithmic structure deviates from that of many applications.

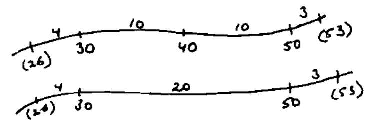
The use of concrete material like blocks also evokes problems as is shown by research. Moreover the lack of ability to apply the algorithm cannot be restored easily. (cf. Resnick & Omanson , 1987). The question is now how to solve this problem from the realistic point of view, that is in a (re)constructive manner?

Well, let us return to the example.Solutions to be expected will - for instance - look like bridging the distance from 26 to 53 by means of counting onwards (complementary counting). .

E.g. a) (26) + 4 (30) + 10 (40) + 10 (50) + 3 (53) -------------> 27

b) (26) + 10 (36) + 10 (46) + 4 (50) + 3 (53) ------------>27 and so on These different methods can be represented on an empty, that is yet unstructured numberline, very well.. Different possibilities to fix key-points on it corresponding to the beforementioned strategies, are available, for example:

a) b)



Solutions vary depending on the choice of the key-points as made.E.g. 4+20+3=27 and so on (1).

The drawbacks of the traditional column-approach of subtraction can be removed by applying the unstructured numberline in the course.Namely, working on it is linked up with contracted methods of counting and complementary counting in two directions. Especially subtractions might be discovered as mirror images of counting onwards strategeies. It also implies the part-whole phenomenon and it conforms to the different methods of pure and applied subtraction. In the program three stages can be distinguished:

1) Working in rows by means of different methods of counting or mental computation complementary as already has been shown.

2) Working in columns by means of different methods of mental computation both in rows and columns, for instance:

a) 53		53
26 -	or	26 -
30 - 3 = 27		30 - 3
		27

Notice that this is conventionalized or stylized mental computation.

b) 53
26 -
33 (a debt of 3 and a possession of 30 makes 27).

3) If required a final algorithm can be aimed at, which meets the preference of children to work from left to right and that corresponds with the historical development of algorithms (2).

As the division in stages shows, the final algorithm, if learned at all, is steered for slowly and gradually.(see also Streefland, 1988)

Theoretical framework of teaching-learning principles. In the examples , although sketched concisely, the fundamental priciples of teaching and learning mathematics according to the didactics of reconstruction can be identified. These are based on Treffers (1987). They will be connected pairwise, as follows:

- evoking construction by giving concrete form by means of contexts (notice the emphasis put on application problems at each stage in the program);
- developing mathematical tools in order to have the pupils moved from concrete to abstract (think of the unstructured numberline as a thinking and working model for subtraction giving room for informal strategies of the pupils);
- giving special assignments like free productions to the pupils in order to evoke reflection;

So the requirements for reflection as the driving energy behind the individual learning processes can be met by selecting special problems in the outline of the course and by having the pupils made individual productions.

- teaching in an interactive way in order to give room for learning as a social activity (this way of teaching focusses on organising the informal approaches of the pupils to serve the learning process - that is making progress in mathematics - of the learning group as a whole); and finally

- intertwining related learning strands in order to have the pupils structured their knowledge. This, however, does not imply exclusion of the existence of other links between the principles as summarized.

Overview and outlook The one-sided approach of column subtraction supported by blocks was critisized. The main objections were:

-children's informal strategies are both blocked and neglected;

-the different levels of proceeding in the learning process as distinguished by cognitive scientists are explained insufficiently, and

-the final level of algorithmization is steered at too rapidly and too rigidly.

A competing approach was presented.Its main characteristics were:

- taking into account children's informal strategies, first formalized in rows and columns;
- providing the pupils with mathematical tools to help them to bridge the gap between the concrete and the formal level (and consequently viewing levels in a different way as usual);
- intertwining the different but related learning strands of counting, mental computation, estimation, column procedures and applications.

The plea for more mental computation in the " NCTM-Standards " was met here. Moreover the suggested approach showed how to connect mental computation with learning (an) algorithm(s). The " Standards " , however, lack suggestions for the envisaged connections. Five theoretical elements were mentioned. They return time and again in general teaching-learning theories. They concern the aspects of (re-)construction, the distinction of levels, the metacognitive aspect and so on (3).

In our approach these aspects or elements acquire a meaning which differs - for instance - from that of the cognitive science approach to mathematics.or the one based on Piaget Our background theoretical framework is a mere synthesis of constructivism and cognitivism.

On the one hand much room is given for children's own contributions to the teaching-learning process and on the other hand the overall goal of our approach is the acquisition of the level of subject-systematics by the pupils, which means the formal level.

Finally attention is drawn for the way our ideas have been presented, that is connecting the general theoretical aspects with a well-elaborated example and contrasting it with a competing approach.

(1): Hassler Whitney described the idea of the unstructured numberline in an unpublished paper which was entitled:" Sane decision making in Mathematics Education ".

See also Grossman (1975), Labinowicz (1985), Madell (1985), Treffers a.o. (1989).

(2): Cf. Streefland (1988) and his contribution on free productions to this conference and also Treffers a.o.(1989).

(3): Cf. Cobb (1987). In piagetian theory for instance the distinguished basic principles can be recognised, albeit that with respect to interaction and learning in a social environment this aspect was recognised by Piaget rather late; see for instance Piagets discussions with the journalist Claude Bringuier.

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The Important Role of Context Problems in Realistic Mathematics Instruction

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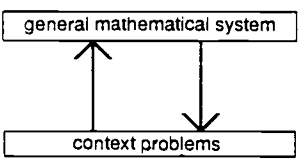
Introduction The fundamental change brought about in mathematics instruction by the realistic approach, is most apparent in the way applications are dealt with. The usual view on mathematics consists of mathematics as a ready-made system with general applicability; and on mathematics instruction as falling apart into learning the mathematical system and learning to apply this system.

Within the realistic approach the stress is on mathematising, so mathematics is viewed as an activity, a way of working. Then learning mathematics coincides with doing mathematics, and solving real life problems forms a essential part of it. Therefore a lot of context problems are integrated into the curriculum from the start (1).

Two models The two fundamentally different views on mathematics and mathematics education imply essentially different processes in learning mathematics.

If we take mathematics as a ready-made system, then applicability is taken care for by the general character of its concepts and procedures, and thus, first of all, one has to adapt this abstract knowledge to solve problems set in the reality. One has to translate real life problems into mathematical problems. We may visualise this in the following way.



This model describes the process of solving a context problem with the help of ready-made mathematics. First the problem is translated; the problem has to be formulated in mathematical terms, as a mathematical problem. Next this mathematical problem is solved with the help of the available mathematical means. And at last the mathematical solution is translated to the original context. On the whole the 'translation' described above boils down to recognising problem types and establishing standard routines.

If we choose the starting-point for our instruction in 'mathematics as an activity', then the whole character of the activity of problem-solving is changed. Then problem-solving is problem-centered, that is, rather then the mathematical tool, the problem is the proper aim.

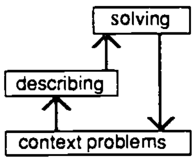
One has to go through the same three stages however:

- translating the context problem into a more formal description;
- solving the problem on this (more or less) formal level;
- translating back the solution.

Nevertheless, the character of these activities differs. One does not aim at fitting the problem in an existing system, but one tries to *describe* the problem in such a way that one is coming to grips with it, and this happens in particular by means of schematising and by means of defining the central relations in the problem situation.

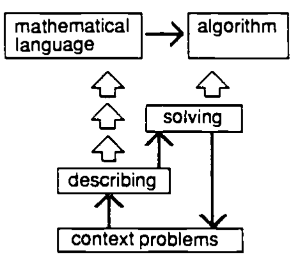
Solving the problem which is stated at this more or less formal level differs greatly from applying a standard procedure. It is a matter of problem solving as well.

Translating the final solution does not differ that much from translating a solution which is produced by a standard procedure. However, now translation and interpretation, are easier since the symbols are meaningful for the problem solver, who is the one who gave them their meaning.

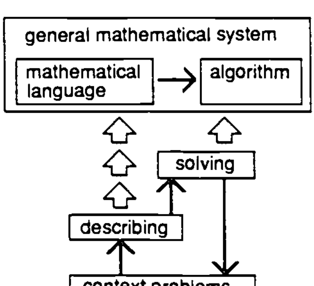


If the instructional programme is full of this kind of problems, it is giving the students the opportunity the learn to mathematise context problems.

Planning a number of similar problems in a sequence will evoke another process. The problem description may develop into an informal language, which in turn may evolve into a more formal language, which looks like the standard language, during a process of simplifying and formalising. This is a process of mathematising too, although a process stretching over a longer period of time. A similar thing happens to the solving procedure. In th long run solving some kind of problem may become routine, that is the procedure may be condensed and formalised in the course of time. In this way real algorithms take shape.



This then is a learning process by which the mathematical system itself can be (re-) constructed.



Treffers (1987) calls the latter process, which is focusing on the mathematisation of mathematical matter, vertical mathematisation. The earlier mentioned process, mathematising context problems, is called horizontal mathematisation.

The two basic models, the 'general model' (the one with the applications afterwards) and the 'integrated model', may be used to compare the American information-processing approach with the Dutch realistic approach.

Applications The ideas presented by information-processing psychologists like Resnick, Greeno and others fit perfectly to the 'general model'. The instruction immediately focuses on the concepts and skills as defined by tradition, that is in their most sophisticated form. A refined form of task analysis is used to divide the learning task into smaller components, aiming at a simplification of the learning process.
Most of the actual instructional designs based on information-processing theory are making use of concrete models. To teach the written algorithms, for instance, one almost always uses Dienes blocks . Since obviously, that these procedures do not reflect all possible real life situations, instruction of applications becomes a necessity. Thus instruction is divided into teaching a fixed procedure and teaching its applications. Instruction of applications then relies on the recognition of semantic structures in word problems. To be sure, one also tries to establish metacognitive skills and strategies. However, most of the students do not know how to apply general heuristics. Lesh (1985) research, for instance, shows that the applicability of heuristics is strongly effected by the availability of domain specific knowledge. Therefore it is not so surprising that Schoenfeld (1987), for instance, abandoned the idea of general heuristics, and changed it for the idea of 'mathematical people'.

One may say that mathematics evolved from applications to a formal system, but now one tries to teach children the formal system first and the applications afterwards. An inversion which Freudenthal rightfully calls anti-didactical. As an alternative for this inverse order Freudenthal (1983) suggest to follow the original order with the help of a didactical phenomenology.
Didactical phenomenology is investigating the historical roots of mathematics by analysing the way mathematical concepts function in reality. Is this manner context problems are found to develop 'intuitive notions', consolidated in mental objects, which in turn form the base for the concepts to be developed. The final concept attainment has to be established by a re-invention process.

In realistic mathematics instruction the domain specific solution procedures function as a base for the learning process. The semantic structures which are discerned by cognitive scientists are perceived here too. However one does not demand recognition of these structures as prerequisite to the application of a standard procedure. One relies on phenomenological structures as a means to simulate domain specific solution procedures. Only after a while students learn that these procedures are interchangeable. First of all the students are stimulated to improve their own solution procedures; to shorten them, to schematise them and to generalise them. In this way the final goal, a standard procedure, is achieved, and the applicability of this knowledge is guaranteed.

However, there is more to it. This approach does not only build a solid base for concept attainment, it also generates a general attitude towards applied problems.While heuristics and metacognition always lead to the application of some algorithm (or a combination of algorithms) in the information-processing approach, the realistic approach consequently capitalises upon the students own ideas. In this way the students will develop the attitude to consider it as self-evident to have a try, to use your head, and to see what can be done with the domain specific knowledge available.

Notes.

1. The examples of a course on long division and multiplication which were presented at the conference had to be omitted in this text.

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Free productions in teaching and learning mathematics

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Introduction and survey The question to be tackled here is: How to influence children toproduce by themselves-albeit under guidance of the teacher -their mathematical abstractions (cp. Cobb, 1987). In order to answer it I will deal with successively :

-children's own - free - production in mathematical instruction - what does it mean?;

-functions of their own production in the teaching/learning process;

- own productions in developmental research after reconstructible instruction;

A few remarks will conclude the exposition .

What is own production? In productive mathematics education children construct and produce their own mathematics.The pupils' mathematical activity expresses itself in their construction and in the production resulting from reflexion on the constructions. Treffers (1987a,p.260) has introduced this distinction, which according to himself is no matter of principle. Free production is rather the most pregnant way in which constructions express themselves.What, however, is own production? In order to answer this question we shall look out for the preconditions and circumstances under which productions emerge or may emerge in instruction. By constructions we mean:

- solving relatively open problems which elicit- in Guilford's terms - divergent production, due to the great variety of solutions they admit, often at various levels of mathematisation (e.g:How to divide three bars of chocolate among four children?), and

-solving incomplete problems, which before being solved require self-supplying of data or references (e.g: A radio message on a 5 km line at Bottleneck Bridge: How many cars may be involved?).

The construction space for free productions might even be wider:

-contriving own problems (easy, moderate, difficult) as a test paper or as a problem book about a theme or for a course, authored to serve the next cohort of pupils. An example, say, for grade one: Think out as many sums as you can with the result five.

Finally there are border problems, that is, of constructive character but with a strong productive component,which require devising symbols, linguistic tools, notations, schemes, or models. In our illustrating problems stress was laid on the various functions own productions can have in the teaching/learning process (as well as in research). In fact, a production problem can involve more than one of these functions. The division according to functions is again a matter of stress rather than of principle.

Functions of own production in the teaching/learning process Preliminary survey If children's learning is to be expressed in their own production, its various functions have to be viewed under the aspect of instruction, that is, according to their didactical value.Without aspiring at completeness the following functions will be distinguished:

-grasping the connection between phenomena in reality and the matching tool of description and organisation (horizontal mathematising);

-seizing the opportunities of continued organising and structuring of mathematical material (vertical mathematising);

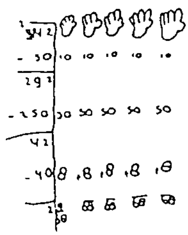
-uncovering learning processes, and reversing wrong trends;

-producing terminology, symbols, notations, schemes, and models serving both horizontal and vertical mathematisation ;

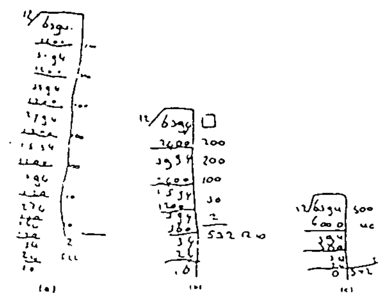
Each of these functions were illustrated by examples and commented on. In all cases it appeared, that being productive in the mathematics lesson provokes both reflection and anticipation on the teaching/learning process.
The various functions will finally be considered within the broader context of course construction and developmental research-.Some remarks will conclude the exposition.

Grasping problems .Example :"The size of The Netherlands" (after Treffers, 1987c), from the domain of calculation and meansuration by estimate:
Somebody affirms that the area of The Netherlands is 36,842 square meters, according to Larousse Encyclopedia, he says. What is your comment?
An impression was given of the course of a lesson with 11 to 12 years olds who have received traditional (rules oriented) instruction, although in their last year (grade six) a few richer problems happened to emerge in the lessons.
Remarks Of course the foregoing example has a merit of its own but in the present context it has been adduced because of its constructive and productive value.Educated estimates and implicit experiential data made explicit, strengthen the grasp on problems, which is one of the functions of construction and production.Solving means tying connections between the real and the arithmetical world by means of mathematical modelling.Growing such connections helps developing a mathematical attitude, in particular horizontal mathematising, that is mathematising real world situations (cf.Treffers,1987a). The spirit of lessons in which this happens is comparable with the direction in which the problem solving courses of Schoenfeld (1987) have been developed.

Seizing opportunities to organise and structure mathematical material. Pupil's own constructions and productions mirror the teaching/learning process, both for the teacher and the educational developer and researcher. Here are two examples. Grossman (1975) reported about unexpected surprises caused by production tasks.She gave a few examples like inventing sums with three or five as their result The *second* example was about long division based on the principles of clever computation and estimating (cf Treffers, 1987) (see also Gravemeijer about contexts) Context problems are at the core of this course, e.g:
'342 stickers are fairly distributed among 5 children; how many does each of them get?'
In such a situation distributing shall be organised. *First* the stickers are handed out piecewise, but soon bigger shares are dispensed. The written report reflects the distributing pattern, which indicates the distribution process. Subsequent steps on the path of mathematisation are predesigned.



In the *second* phase the children are soon satisfied with noting down just one column only - 'all get the same, indeed. Other contexts are also introduced, among which that of grouping. After about 15 lessons the children work on different levels.



In the *third* phase the connection is made to decimals and fractions.Context-dependent answers on divisions with remainder are not neglected for instance interpreting an outcome with remainder.(cf. Gravemeijer). At crucial points in thecourse it is asked to invent problems and to solve them by a slow longwinded manner as well as by a quick and short one.The pupils should learn te reflect on their learning process and to anticipate on even shorter procedures.*Remarks:*

In the first example the teachers' comments showed that the children had amply transgressed the limits of the scholastic domain.The work as published revealed traces showing how they reflected on their activities. One of them - a supposed slow learner - continued intensively, even at home. The other pupil anticipated on sums, three grades higher in the curriculum; up to 10,000-9,995=5! Both pupils reflected on what they had learned within the number system, and consequently they anticipated on the future of the teaching/learning process, the one farther than the other. The teachers were hold up a mirror of their instruction. The *course of long division* as sketched , aims at evoking:

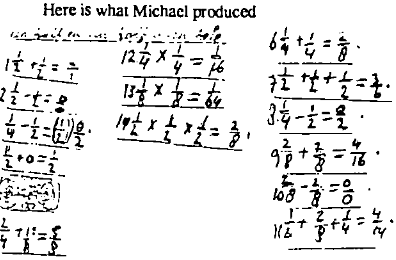
-a process of mental computation and estimating, integrated in context problems;

-a process of progressive mathematising arithmetical methods by means of schematising and shortening. Such an approach of division starts with the informal methods of the children, which are organised and structured. Construction and production play an important part in the process of progressive schematising and shortening, which are aspects of progressive mathematising. During the teaching/learning process the solutions of applied problems are incessantly subjected to inventarisation.Continuously the question of possible shortening is raised.The procedures arising in the process of shortening function in the course to be followed;beacons for those who nearly reached the same level of mathematising. The ultimate standard algorithm of long division is predesigned in this process as the utterly shortened procedure. In a sense this mirrors the historical process of algorithmising long division (cf. Menninger, 1958). Comparative research undertaken in our country has proved that this approach is by far superior to the traditional one. (cf.Rengering,1983;Treffers,1987b).

Uncovering learning processes and reversing wrong trends The previous examples showed the diagnostic value of own constructions andproductions, both illustrating the teaching as well as the learning process. Now we will consider this value for the learning process, in particular cases where constructions and productions reveal wrong ideas and misconceptions.

Example The class had elaborated and described several distribution situations (cp. Streefland,1984 ;1987;1990) e.g: Share 3 chocolate bars among 4 children (each child will get 1/4+1/4+1/4 or 3x1/4 or 1/2+1/4 or 1-1/4 and so on).Then the pupils were challenged to think out such 'number sentences' as had been met in the distribution situations, that is, with halves, fourths and- for the courageous ones - eighths, with 'plus' and 'minus', maybe even with 'times', sums matching distributions.

Here is what Michael produced



Michael's work is typic for - world- wide- mistakes, I called them "N-distractors".(cf. Hart ,1981;Hasemann,1987; Streefland,1984 ; and many others).

Remarks

The diagnosis is clear.The mistakes are the consequence of yielding to the temptation of whole numbers and their rules. It shows that the constitution of the mental object "fraction" has not progressed far enough to resist this temptation. Numerators and denominators were still operated upon separately; their conceptual interdependance was neglected.The task had been set too early, at least for Michael. The concrete sources had been switched off prematurely. Stating this goes to the heart of the function here envisaged. Own constructions and productions unveil the -possibly wrong- personal theoretic basis of reflexion and anticipation in the teaching/learning process.This enhances the diagnostic value of the material. A correct diagnosis promises successful remediation both of learning and teaching.

Producing terminology, symbols, notations, schemes, and models serving both the horizontal and vertical mathematisation Children can contribute to the working apparatus o f mathematics. *Examples*

The *first* example concerned the development of a personal algorithm for subtraction by children based on working with possessions and debts (cf. Madell, 1985 and Labinowicz, 1987). With respect to this pupils can and will invent their own notations (cf. also Van den Heuvel's contribution).

The *second* example dealt with the invention of " pseudonym" to describe non-standard fractions equivalent with standard ones. The quest for a fitting term elicited reflexion. The invented term - which in Dutch sounds less learned than 'pseudonym'- proved to have a long term predictive value.

The *fourth* example also showed the invention of terminology, namely for large numbers (Fynn, 1976) and the *third* example concerned the invention of a symbol for a distribution situation and building schemas with this symbol based on the idea of fair table arrangements in a restautant. The schema proved to be very productive with respect to the equivalence of ratios. The context situation proved to fulfil a model function: the model situation of table service became a situation model, which functions as a cognitive process model.(cp. Greeno, 1976).(cf.Treffers&Goffree,1985; Streefland, 1986 and 1990).

Remarks The part played by the production of terminology, symbols, notations, schemes and models in the shaping of mathematisation, horizontal as well as vertical, was shown by means of the foregoing examples

Education developmental research for the sake of reconstructible instruction Up to now stress was on (re)constructive learning, viewed in the learner's perspective. It started with informal notions and working methods. Reconstruction gradually moved the learner towards more formal mathematical notions, operations and structures.The import of reconstructive learning is often reflected by children's spontaneous mathematical constructions. Curriculum developers and researchers were seldom aware of such signals. Rather than seriously observing children and learning from their constructions and productions, they expect answers on questions and solutions of problems by prematurely theorizing within topical frameworks. Calls for change sounded time and again in the literature on development and research, were not listened to.The results of didactical research in teaching arithmetic are still badly neglected. Fractions is a telling example: fresh starts with all old errors repeated. Nothing is learned from lessons such as taught by didacticians of mathematics like Freudenthal (1973), Hilton (1983) or Usiskin (1979).

A striking illustration of this fact is Brownell & Chazal (1932, p.24), who from the results of drill for the mastery of basic skills conclude: "...the time and accuracy scores on Test B were better than on Test A, ...,..... because the old methods were employed with greater proficiency ".

By " old methods " the authors mean pupils' own informal solutions, which resist instruction against the grain.Wouldn't we have made greater progresses in our knowledge about childrens' mathematical learning if we had built on these telling results of research ?

An important question now is : Does reconstructive learning also apply longitudinally to class instruction ? In order to answer in the affirmative we have to carry on developmental research- which means research in the classroom situation. It aims at developing prototypes of courses and theory-building for teaching and learning the subject involved. Instruction experiments start with provisional material. The teaching/learning process is closely observed. Continual observation and registration of individual learning processes is at the heart of the research. What matters is that pupils' constructions and free productions are*used for building and shaping the teaching course* (cf.Streefland,1990).

In the variety of children's possible proposals to the open problems to be used, one gets a rich choice to find out what is the best fitting, the farthest prospective, and in the long run the most effective. At ours as well as abroad courses have been developed in this way (for science, see Driver (1987); for mathematics, see Treffers (1987 a)).

In this kind of design children, by their learning processes, decisively influence course development- this even extends to supposedly weak learners as some examples proved. It nourishes the source for creating reconstructible instruction. The prototype can serve as a model for establishing and developing derived courses. Such potential instruction is predesigned in textbooks and manuals.Globally the used generative problems with pupils' usable long-term constructions and productions, which emerged in the developmental research, will mark the learning road for fresh pupils' cohorts. In particular the manual will prefigure the material to be expected from the pupils and help to reorganise it with the view on the sequel. In this way the preconditions are fulfilled to have the teachers treated their instruction as free production of teaching as well.

Conclusion The construction principle in education requires a significant part played by children's constructions. What this means for mathematics education has been shown in our introduction on the meaning of realistic mathematics education. Within suchlike education a solid empiric basis is laid for the principle of constructivity by having the children contributed to course development. Horizontal and vertical mathematisation as observed in the historical learning process can be a source of inspiration. In the light of history reconstructive learning is realised on the individual as well as on the classroom level.

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Realistic geometry and instruction theory

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Introduction This contribution deals with a kind of geometry instruction which differs largely from the well known deductive geometry which is being taught at a secondary level in most countries. A plea for 'realistic geometry'.- if not as a replacement than at least as a valuable preparation for more formal geometry.-.will be supported by examples. Further these examples will be used to clarify some aspects of the realistic instruction theory.

Examples of realistic geometry Realistic geometry may be illustrated by some examples.

1. **What do you do when you want to take a photograph of a large group of people and you see that you will not get everybody on the picture?... You will step back.**

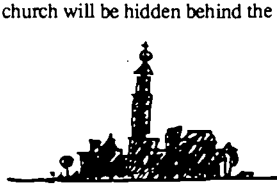
This may seem a rather self-evident solution. But one may wonder about the reason why you get more people on a picture if the photographer stands further away. The answer to that question clearly implies a geometric interpretation of the situation.
The following examples which show the intriguing character of geometrical problems based upon day-by-day experiences.

2. **You're travelling a long way by train. The train is moving through a moonlit landscape. And whatever the speed of the train may be, the moon is catching up with you - it seems. What causes this effect?**

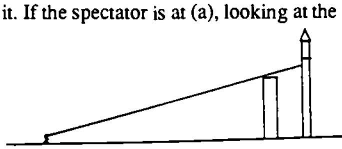
3. **How is it possible that a mirror changes right and left, without changing above and below?**

4. **When you are driving through a long city-street, it is as if the large buildings further away 'sink' behind the smaller ones in the foreground. What is happening?**

Or, to put this last question in another way: In Holland everybody knows the flat polder landscape, a straight horizon, with only a silhouette of a village or a town, where a church rises high above its surroundings. However, when we reach the town the greater part of the church will be hidden behind the other buildings.

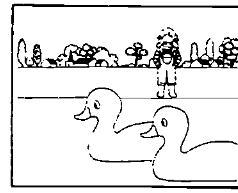


A side-view will clarify what's happening. For instance take a church and a hotel in front of it. If the spectator is at (a), looking at the town, which part of the church does he see?



One may visualise his view with a straight line. So, when he starts moving it appears that the part of the church which is visible diminishes. In other words, it looks as if the church is sinking behind the other building. However, is it sinking or is it shrinking?

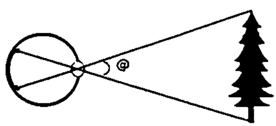
Of course it is growing bigger when you get closer, but the first building is growing faster then the one in the background. Let us elaborate this point a little further.



What is seen as bigger is closer.

What is seen as bigger is closer. That is so self-evident that you do not even think about it. But let us investigate the cause of this effect.

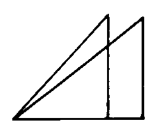
How big an object shows depends on what we may call the image-angle (@).



The size of this angle is determined by the size of the object and the distance between the object and the observer. The bigger the distance, the smaller the angle. So the distance determines the scale at which an object is seen.

If we return to the church and the hotel, the following line of reasoning is possible. If you are far away the scales to which both buildings are seen are almost the same (a). But there is a big difference, if you are closer (b).

a. image-angles with the spectator far away



b. image-angles with the spectator close to the objects

You might say: "Nice problem, but is it geometry, or what's more, is it mathematics?"

We will look at the last question first, then we will get back to the first one. What mathematics consists of depends on what you choose mathematics to be: a ready made system or an activity.

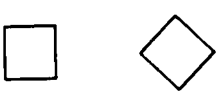
Mathematics and mathematics instruction In realistic mathematics instruction we agree with Freudenthal (1971) who sees mathematics as an activity, like the activity of a mathematician. He explains that it is an activity of solving problems, looking for problems and organising or mathematising a subject matter.

> This can be matter of reality which has to be organised according to mathematical patterns if problems from reality are to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organised according to new ideas, to be better understood, in a broader context, or by an axiomatic approach. (Freudenthal 1971; 413)

Freudenthal opposes the way in which the results of these mathematical activities are being taught. Where the history of mathematics starts with real life problems and evolves to more general and more formal mathematical ideas. The instruction starts with the formal system, presenting the applications afterwards. He calls it an anti-didactical inversion. It deprives the

students of the opportunity to experience mathematics as a mathematical activity.
One may alter this educational tradition by starting with real life problems and promoting mathematising as the main learning principle. Mathematising may a enable the students to re-invent mathematics in stead of absorbing preconstructed mathematics.
Next to mathematising Freudenthal mentions 'looking for problems', with which he refers to a mathematical attitude. Realistic geometry yields a perfect field to develop and practice this reflective attitude. This kind of reflection on practical problems has played a key role in the development of geometry. Geometry started by solving practical problems.
Geometry was a craft knowledge in the first place. Later on one started to study this craft knowledge, not to get profitable applications but out of curiosity. Afterwards it appeared that the results of this activity were applicable as well (1).
This shows us a process in which the solutions to the problems of a earlier period become the tools of the later one. You start out with informal experiential knowledge, which is reflected on and this creates knowledge on a higher level.
So if we take the point of view of mathematics as an activity we may come to the conclusion that reflecting on practical geometrical problems *is* mathematics, which leaves the question 'Is it geometry?' To answer this question we may look at Van Hieles analysis of geometry instruction.

Van Hieles level-theory Van Hiele (1973) analysed geometry instruction, but a completely different kind of geometry than the above mentioned one. He analysed instructional problems in a period when deductive geometry was still on the programme of the Dutch secondary school. Nevertheless his analysis of the problems with this kind of geometry instruction will show us the importance of an informal introduction.
Van Hiele states that there is a communication gap between the teacher and the student. He illustrates this with the interpretation of the geometrical concept 'rhomb'.
He ascertains that the statement 'this figure is a rhomb' does not have the same meaning for teacher and student.



square square seen as a rhomb

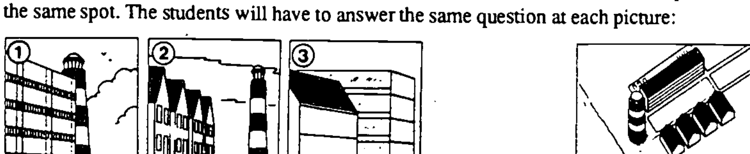
The student might recognise the shape and associate this with the name 'rhomb'.
Students who recognise a rhomb this way may find it hard to see a square as a rhomp,unless this square is in another position.
To a mathematician, and therefore to the teacher too, the label 'rhomb' has quite another meaning. To them a rhomb constitutes of a collection of properties and relations: it is a polygon; all sides are equally long; it is a parallelogram; the sides are parallel two and two; the diagonals are perpendicular, and so on. The teacher will call a square a rhomb because of its properties. He will be willing to accept even a roughly sketched figure as a rhomb if it is agreed that the sides are equally long and parallel two and two.

The difference between teacher and student is a difference in referential framework. These conceptual differences barricade the communication. The teacher and the student use the same words, but these words do not have the same meaning.
The differences in referential framework are seen as differences in conceptual level. And the only way to tackle this problem will be to construct the needed referential framework,

starting at the lowest conceptual level.
Van Hiele distinguishes three conceptual levels in the process of learning mathematics.
At the conceptual level of the teacher words like 'rhomb', side', 'angle', 'square', etc. establish junctions in a framework where each of these concepts constitute of an bunch of properties. This is what Van Hiele calls the *second level.*
There is no such a framework at the *first, or ground-level,* where the labels are still connected to concrete experiences and perceptual objects.
On the *third level,* the relations themselves become object of thinking: the character of relations and the the connection between properties are settled. Which makes it possible to construct a logical system.
The level description shows us that the former Dutch deductive geometry on the secondary school started at the third level. Instruction should start at the first level, with concrete problems and concrete activities. However, what's concrete depends on the students their actual knowledge. In other words, the three levels should not be seen as absolute distinctions, every subject matter content has its own three levels. Concepts like point, line and angle may be concrete for secondary school students, but this knowledge can also be understood as a the third or second level of informal geometry, or orientation-in-space.

Realistic instruction theory The Van Hiele levels nevertheless help to establish a macro structure of a course. We may rephrase the level-description to this goal by the words of Treffers (1987) who speaks of a distinction in an *intuitive phenomenological level,* a *locally-descriptive level,* and a *level of the subject matter systematics.* Next to this macro-structure a micro didactical-structure is needed.
This micro-didactical structure originates from Freudenthals (1983) didactical phenomenology and the re-invention principle (2). The actual micro didactical-structure is given by Freudenthals ideas on the relation between reflection and the transition of micro-levels. In which, as has been said before, *the solutions to the problems of an earlier period become the tools of the later one.*
Here a reflective attitude is the driving force. Such a reflective attitude may be developed by realistic geometry instruction. One may exploit the intriguing aspect of geometry to stimulate this kind of a mathematical attitude. Realistic geometry gives us a splendid opportunity for the development of a mathematical attitude since it can lean upon the vast amount of informal geometrical knowledge possessed by young children.

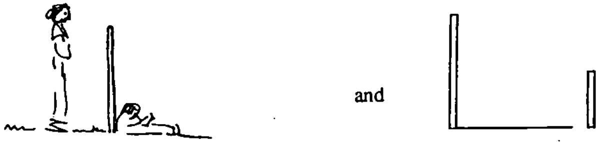
An impression of this kind of geometry instruction may be given by a brief sketch of some activities taken from a realistic textbook series for primary school (Gravemeijer; 1983).
- One may start with some exploration of photographing - where all children are equiped with matchboxes which function as cameras.
- Another time a birds eye view.of a village is presented, together with some pictures of the same spot. The students will have to answer the same question at each picture:



Where stood the photographer?

When reconstructing the point of view of the photographer, the students may implicitly think of lines to represent the way the photographer views the situation.

Later on one can introduce this kind of lines more explicitly with problems like:



Is it possible for uncle Bill to see Marc? Draw the shadow of the second rod.

The shadow can be found by drawing two parallel lines, but one can also reason that a rod twice as small will give a shadow twice as small. The combination of these two insights form an intuitive base for the the understanding of the fixed ratios in similar triangles.

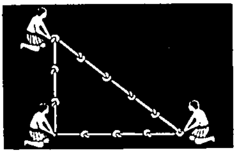
Over some period, this 'shadow-model' will evolve to a more sophisticated triangle-model, where no references to shadows and so on are needed to know that there is a fixed relation between the shape of a triangle and the ratio of its sides.

This brief sketch may give an idea of a possible learning sequence on this topic in realistic geometry. There are some learningstrands: the students make themselfes familiar with the triangle-model, they acquire the habit to visualise geometrical problems and they learn to view a situation from different angles.

Conclusion A realistic geometry programme for primary school may stand as an example for realistic mathematics instruction, which in turn fits into the theoretical framework as given by the Van Hiele levels as a global macro-structure, Freudenthals view on re-invention as a description of the micro-structures, and the didactical phenomenology as an indication of how reality can be used as a starting point for the learning process.

Treffers (1987) a posteriori formulated this theoretical framework of realistic mathematics instruction and he also analysed the main characteristics of the educational programmes and textbookseries developed within this approach, which are described in the contribution of Streefland.

Notes.



(1). We know from the Egyptians for instance that they used the ratios between the sides of similar triangles. They used fixed ratios which could be expressed in whole numbers. They used the ratio of the sides as 3 is to 4 is to 5, to construct a right angle, for instance. In Greece a group of people around Pythagoras got fascinated by these ratios and they started out to investigate these ratios which were not easily to express in whole numbers. Their work founded the trigonometrical knowledge which is so useful in surveying, navigation an astronomy.

(2). See Gravemeijers contribution on context problems to this conference.

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Realistic Mathematics Instruction and Tests

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Introduction Realistic mathematics education does not only imply teaching in a different manner but the usual one, but also testing or evaluating differently. In this respect the realistic approach is - more often than not - associated with observing or having clinical interviews with the pupils. In this section is shown how individual pencil and paper tests can be very well applied here, even to map out the arithmatical and mathematical insigths and skills of children at the beginning of the grade school.

Drawbacks of tests The usual written tests consisting of bare sums suffer from two huge drawbacks. First, the tests reveal only the results and tell nothing about the children's strategies. This lack of information on children's strategies has the consequences that wrong conclusions are likely to be drawn on the children's performance, that too little information can be obtained about the progress of instruction and finally that it is almost impossible to diagnose the children's mathematical difficulties. Second, the tests are too narrow, both with regard to the subject matter and the students. Mostly they are restricted to such subject matter as can easily be tested and futhermore they are not allowing the children to optimally show what they are able to.

Alternatives With regard to secondary education in the Netherlands, the awareness of the drawbacks of the usual written tests from the viewpoint of didactical change and of new curricula has meanwhile led to shaping new testing instruments (De Lange, 1987). In primary education better testing instruments have been a search for in the MORE-project (1). The development of the tests in this project is based on the following requirements: the tests must cover the whole spectrum of the mathematics area concerned, they must offer children the opportunity to show what they are able to, they must provide information about abilities and strategies and they must be easily to handle in a classroom situation. In the next section will be explained briefly how these requirements can be met and how the limits of testing can be extended. A few examples of test items will ilustrate this. The examples to be shown apply to grades 1 to 2.

Easily to be handled in a classroom situation The point of easily to be handled in a classroom situation implies that the tasks must not require extensive oral or written instruction, but on the contrary can be easily presented with a **minimum of explanation**. So one has to look for self-explanatory tasks. Tasks which are selfevident with respect to what the children are expected to do. For example take the item, that is related to a game of darts being thrown at a board (fig.1). The question asked is: "What is the total score?" Although the picture is not quite faithful, together with the question it is sufficient. The children immediately grasp the intention.

Covering the whole spectrum of the mathematics area concerned The requirement of covering the whole spectrum of the mathematics area concerned means that **besides** the usual series of **sums, other chapters** also have to be represented, such as: counting and counting sequence, ratio, measuring and geometry. This not only means that the chapters are tested separately, it is also important that this happens **in their mutual connection**. As an example take the item with the flowers on the roller blind (fig.2). The children are asked whether they can find out the number of flowers in spite of the obstruction of sight by the cats, or not. Obviously this task asks for more than simple resultative counting. It is as well related to measuring, geometry and ratio.

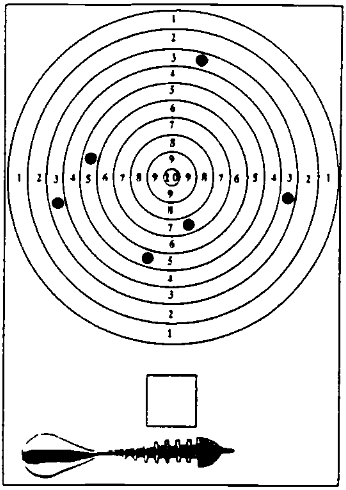

Fig. 1

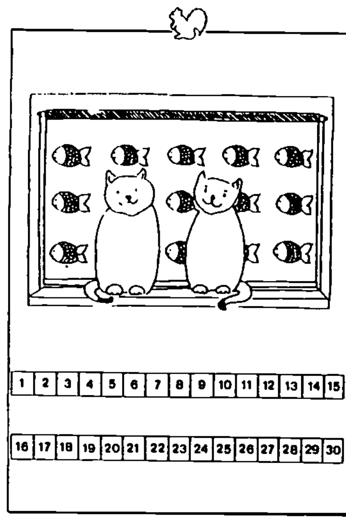

Fig.2

With respect to covering all parts of the spectrum it is also required that the testing instrument allows for **various levels of applying** mathematics in reality. Tasks that don't require much problem analysis as their presentation includes the operation to be carried out as well as all relevant data, are low level under the viewpoint of applying mathematics. High level means large own contributions of the children to the problem analysis. For instance, the item on the birthday treat (fig.3) is of a higher level of application. The question runs: "Suppose there are 30 children in my class, how many bags shall I buy?" First of all, the arithmetical operation is not straightforwardly given. Moreover even correct calculations need not yield the adequate answer (such as 30:9=3 rem 3 in the present case). This example makes quite clear, that applying means more than translating pictures into sums.

Giving children the opportunity to show what they are able to The requirement of giving children the opportunity to show what they are able to, can be realized in several ways: using stimulating and supporting contexts, using tasks with a number of built-in solving levels, creating possibilities for children's own contributions.

The use of **contexts** stimulates and motivates the children; moreover they can imagine what is going on. The more recognizable the task, the more the children can show what they are able to. Except contexts with a general stimulating effect, one can also offer contexts which both link up with informal experiences and function as supporting models. For instance a buying situation where money functions as a model.

Another way to give children the opportunity to optimally show their abilities is to use items with **built-in solving levels**. These items contain supporting information on the test sheet. An example of such a layered test item is the one on 36 sweets which are fairly to be shared by three children. The 36 sweets are pictured on the test sheet. The children may use these at will.

Last but not least, creating possibilities for own contribution offers the children a lot of opportunities to show what they are able to. This can also be done in various ways. One way, is administering test items with **more than one correct result**. These yields the children a latitude to contrive solutions. Another way to grant latitude is **choice tasks** where the choices are built into the task itself. Examples of this are those test items which ask the children to choose something to buy. The different prices imply a sequence of degrees of difficulty. The best way to have children shown what they are able to, is provoking their **own productions**. For instance to ask the children to invent as much sums as possible using a given set of numbers. As fig.4 shows, at the end of the 4th grade this child even produces a sum with a resulting negative number.

Fig. 3

3 8 4 20

3 x 8 = 28

8 - 3 = 5 20 + 4 = 24

4 + 8 = 12

3 - 8 = min 5

Fig.4

Providing information about abilities and strategies The possibilities for tests as mentioned do not only mean enabling children to show what they can, but they also yield information on abilities and strategies. These two requirements are closely connected. Much that contributes to the flexibility of the tests, provides a variety of information as well. Remember the tasks with various **built-in solving levels**. The traces left on the test sheet, reveal the strategies the children have followed. Remember also the possibilities for own contributions, for instance, **choice tasks** and **own productions**. These kinds of test items not only give children the opportunity to show what they are able to, but they can also reveal different manners of tackling a problem. Moreover they can reveal problems and misconceptions of children, which would never have been discovered in ordinary written work where the final result is the only information available.

With respect of the yielding of information on abilities and strategies two more means may be added. First, the presention of certain data in order to find out the ways the students made use of them. Presenting certain data is a means to purposefully look for the insights that children have acquired. For instance by **offering pairs of problems** (14+5=19 and 19-5=) one can diagnose the children's insight in the properties of the operations. This kind of items is particularly informative when the children's answers are being compared with those on numerically similar series of sums without support (18-3=.).

The largest amount of information about children's strategies is provided by stimulating reflection, for instance by the use of **"pieces of scrap paper"**. The pieces are pictured on the test sheet. An example is shown in fig.5a/b. The concerning test item is about two children playing a game: at the end the scores of both of them need to be calculated. In order to do so, the students may use the scrap paper. Some students leave it empty, others write on the test sheet they don't need the scrap paper, but quite a number of pieces show traces of the solving strategies. Take, for instance, the first test sheet (fig.5a). Obviously the student added the numbers one after the other. The student of the second test sheet (fig.5b) took pairs of numbers together. The scrap paper also shows that the additions were made from the left to the right, that is, first the tens were added, and after addition of the units, the tens were corrected.

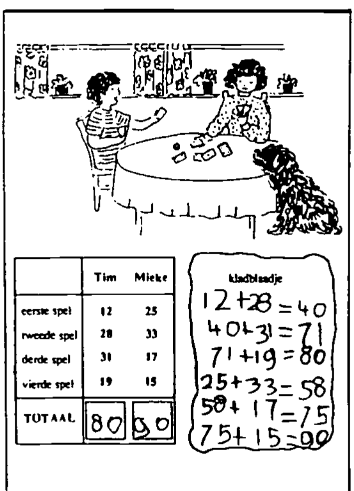

Fig. 5a

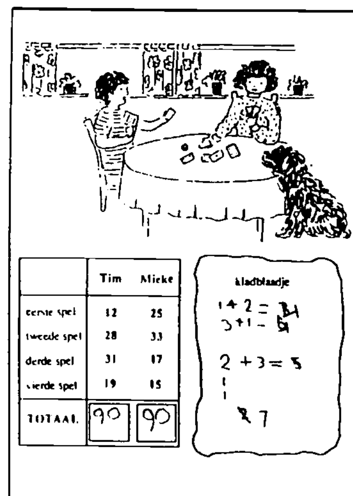

Fig.5b

Some experiences Experiences with tests consisting of items like the forgoing ones, show how revealing this way of testing is and how useful written tests in the classroom environment can be, even for young children. Finally, an example of one of the results will be shown.

It concerns subtraction. The items form part of a test which was administered after three weeks in the first grade. So the children neither got mathematics instruction, nor could they read or write. Moreover they did not have any earlier classroom experience in doing paper and pencil tests. The test was administered to 22 first grade classes; 441 children took the test. The population was quite heterogeneous: rural and city schools, schools with many

foreign children and schools with a great majority of Dutch children. In each class the teacher administered the test according to an instruction form that prescribed all details.
Subtraction was tested in contexts, rather than by formulas. This happened on two levels: tasks where the sheet shows sets of countable objects, and tasks where the cardinalities are given by numerals. The left item of fig.6 shows the "countable" variant. The question is: "How many balloons have been sold?" The item on the right shows the "non-countable" variant. Here the students must indicate the number of florins left after buying the goggles.

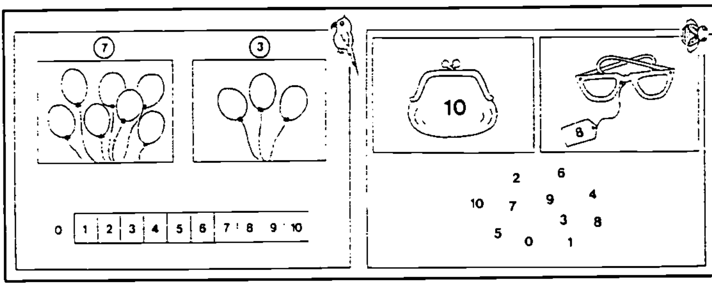

Fig. 6

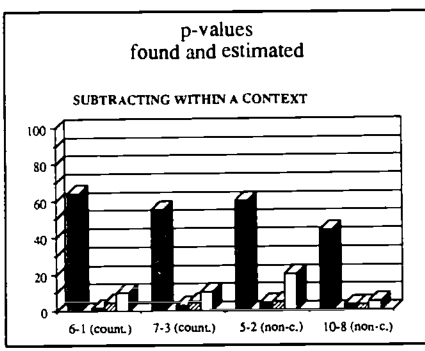

Fig. 7

Fig.7 shows the results. The leftmost column indicates the percentages of subjects that succeeded on the items. The other columns represent the estimated percentages of four small groups of people working as teachers in primary education, as counsellors, or as teacher trainers. It is evident that children under investigation at the start of the first grade possess quite a bit of numerical knowledge and abilities and that they were grossly underestimated, at least by the consulted judges.
However, most revealing was that this was discovered by a written test in a classroom environment. In other words, even in realistic mathematics instruction, tests may have a future.

Note

1. The MORE-project is a collective research project of the Researchgroup OW & OC and the Department of Educational Research, both of the State University of Utrecht. The project is supported by a grant from the Dutch Foundation for Educational Research (SVO-6010). The project is an investigation into the implementation and the effects of realistic didactics compared with the traditional mechanistic brand.

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Sex Differences in Mathematics Ability

The Mathematics Report Card - Are We Measuring Up? Trends and Achievement Based on the 1986 National Assessment

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Gender differences in the performance of 9, 13, and 17 year olds on the 1986 National Assessment in Mathematics will be discussed. The basis for this discussion will be "The Mathematics Report Card--Are We Measuring Up?--Trends and Achievement Based on the 1986 National Assessment" with related statistical data. Comments will focus on five levels of proficiency for 9, 13, and 17 year olds and on a sample of mathematics questions that indicated significant differences in gender performance.

Significant differences in performance by males and females on questions from the College Board's Mathematics Achievement Examination, Levels I and II may also be discussed.

A Cognitive Science Perspective on Mathematical Ability and Sex Differences Therein

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The essence of mathematical ability, as the inherent individual trait that many believe to exist--distinct from the acquired mathematical knowledge and skill achieved as a result of mathematical education--remain undefined and undiscovered. Little evidence exists that can be used to argue that it differs from general intelligence. Yet, too often, the outcomes of standardized tests such as the SAT are taken uncritically as indicators of such an inherent trait, especially when the interpretation of sex differences is at issue. Cognitive science is beginning to provide detailed analyses of the problem solving processes involved in performance on the items that make up tests. These analyses, together with a body of recent research that related performance on tests of various traditional psychometric "abilities" to cognitive models of item performance, can provide an alternative perspective on the interpretation that should be given to such test results. Perhaps new hypotheses about the inherent components of such abilities can be formulated. The points will be illustrated with data about factors that seem to generate sex differences in performance when solving algebra word problems.

THE DEVELOPMENT OF GENDER DIFFERENCES IN MATHEMATICS PERFORMANCE

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Traditional indices of mathematics computation, using standardized test measures, find a developmental gender difference: in elementary school there is no gender difference in scores or if there is, female performance is superior; and beginning in secondary school male performance is superior. These measures emphasize one dimension of computation performance: response accuracy in a timed environment.

The present study examines the process of solution of mathematics to identify stylistic differences in the way tasks are approached and solved by female and male students at different ages. These stylistic differences, which may involve cognitive, motivational, and affective dimensions, may explain the performance sift that has been observed in older students.

This study focuses on cognitive process and also motivational and affective attitudes. This is accomplished through individual interviews with students during which mathematics computations are solved and suitable probes to elucidate solution strategies and problem representations are provided. Systematic interviews probe motivational and attitudinal responses to mathematics in general and to the tasks just completed. Comparisons between male and female responses at two elementary levels and a secondary school level reveal similarities and differences with respect to gender and grade.

Students' Criteria of and Beliefs About the Causes of Success and Mathematics

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and Erna Yackel
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In some perspectives on gender differences in academic motivation, females are seen as having lower expectations of success, valuing success less, and explaining outcomes in different ways. The upshot is a characterization of females as less adaptively motivated than males. We question this perspective with evidence that, in mathematics as well as school in general, students differ (not merely in expectations, explanations, or valuing of success) but in what counts as success.

Gender differences are not large, However, males are more inclined to require evidence of their superiority over others as evidence of success to see success as dependent on superior ability. Females are more inclined to require the experience of insight as a criteria of success and to see success as depending on collaboration with others and attempts to understand. Possible consequences of these trends in conjunction with developmental trends in students' conceptions of ability and intelligence are discussed. Rather than suggesting that females are less well adapted than males, the present evidence raises questions about the adaptiveness and ethical adequacy of schools and societies that make the "male" pattern "adaptive."

The Effects of Single-sex Schooling on Student Achievement and Attitudes in Nigeria

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Women's education in developing countries is of interest to policy makers because of its important contribution to national economic development directly through increased productivity and indirectly through lowered fertility, increased child health and nutrition and increased child school participation. In most developing countries, however, female participation in education lags well behind male participation, with fewer females than males entering, remaining in, or completing every level of schooling. Explanations for this decline emphasize the role that female maturation, marriage and anticipated marriage plays in family decisions to remove girls from (largely coeducational) schooling. Single-sex secondary education, therefore, offers an alternative that could not only facilitate female achievement, but could also promote greater female participation in schooling where the physical safety of adolescents is an issue.

This paper analyzes the effectiveness of single-sex and coeducational schools with respect to student mathematics achievement and gender-stereotypic attitudes toward mathematics, for a sample of 1,012 Form 3 (Grade 9) students controlling for student background, school characteristics and teacher characteristics and practices, single-sex schools have a small positive effect for female students, but a small negative effect for male students, with an effect size of approximately .10 in both cases.

Patterns of Gender Differences on Mathematics Items of the Scholastic Aptitude Test

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For most of the years since the Scholastic Aptitude Test (SAT) was introduced, gender differences in performance on the mathematics section of the SAT have been noted to be about half of a standard deviation in favor of males (Wilder & Powell, 1989). In the early 1980's, these and related findings fueled a heated debate on the nature and causes of gender differences in mathematics and spatial reasoning. Although there is some evidence that in recent years the gender gap in mathematics may be narrowing, the problem is still not clearly defined. As is evident by this symposium and recent publications, researchers continue to explore and debate the factors that may be behind or help account for these differences (e.g., Benbow, 1988; Jacklin, 1989; Wilder & Powell, 1989). Most researchers realize that a sizable proportion of observed gender differences in math performance is experiential in origin: males tend to take more math and related coursed (e.g., physics) and more higher-level math courses. When overall level of math achievement is controlled for, however, there remain both interest in and uncertainty about the various aspects or characteristics of math test items that might cause male/female differences in performance. This investigation adds to the continuing discussion by systematically examining item performance in mathematics on several forms of the SAT. The purpose is to identify <u>relative</u> strengths and weaknesses for males and females associated with the points being tested, the format in which items are presented, and, where applicable, the subject matter in which items are embedded. By studying differences that emerge after matching males and females on overall SAT Mathematics (SAT-M) performance, it is possible to determine whether there are <u>unexpected differences</u> in the ways in which males and females arrive at their total scores. Examination of these differences suggests that there are differences in the ways in which males and females respond to certain kinds of items. Aside from adding to our understanding of the nature of the performance disparity, this information provides data for curriculum developers and teachers to consider in planning mathematics education. Further, the results have implications for test developers and policy-makers as they consider the test specifications and content formulas for future tests.

Method

Six recently administered forms of the SAT-M were selected for study. Only data from high school juniors and seniors for whom English was self-reported to be the best language were included in the sample. In the aggregate, the six SAT forms yielded data for 181,228 males and 198,668 females. Each SAT-M form consisted of 60 items: 40 Problem-Solving items, in a regular mathematics format, and 20 Quantitative Comparison items in a format requiring determinations of whether one quantity is smaller than, larger than, the same as, or indeterminately related to a second quantity. An attempt was made to identify all the ways that math items could differ from one another. Based on findings from previous studies, reviews of test specifications, and suggestions from test development experts, a priori item characteristics were identified and carefully defined. Categories that emerged seemed to fall into one of three main groups: those dealing with points being tested (e.g., Primary content: Arithmetic, Algebra, Geometry, or Miscellaneous); item format (e.g., use of a "Cannot be determined" option); and item content (e.g., use of gender references). In all, over 40 categories were identified. Each Item on each of the six forms was coded accordingly. All 360 items were double-coded; differences in codes assigned to individual items were resolved through discussion.

The Mantel-Haenszel procedure was used to investigate differential item functioning (DIF). This procedure, which has been refined and described by Holland and Thayer (1986), is a noniterative contingency table method for detecting test items that function differently in two groups of examinees. The procedure assumes that if test-takers know approximately the same amount about what is being measured, they should perform in much the same way on an individual test question (item) regardless of their group membership (e.g., gender, ethnicity). For each item, the Mantel-Haenszel procedure provides a single summary measure of the magnitude of DIF. DIF occurs when examinees from two groups who have been matched for overall performance on the test (or some other relevant matching criterian) evidence markedly--and, therefore unexpected--different performance or success on a particular item. One-way analysis of variance techniques were used to identify categories of item characteristics that resulted in significant differences between males and females. The Mantel-Haenszel statistic for each item served as the dependent measure.

It is important to note that results of these analyses cannot be said to reflect absolute differences between males and females. For example, it is inappropriate to say on the basis of these results that females perform better than males on items with a particular characteristic. Rather, significant findings indicate that when males and females have been matched on overall performance, females perform relatively better than males on items with a particular characteristic than they do on items without this characteristic.

Results and Discussion

Points Being Tested. As was mentioned earlier, there are two main item types on the SAT-M: regular math, or problem-solving, and quantitative comparison. No gender differences at this global level were found. Across item types, however, differences were found with regard to the kind of math being tested. When the primary content of the item was Geometry or Arithmetic, males performed relatively better than females, whereas when the item was coded as Miscellaneous, (e.g., Number Sets) females performed relatively better ($F = 8.92, p < .01$). On the related multiple-content category, females, relative to matched males, performed significantly better when Arithmetic/Algebra was required than when Arithmetic/Geometry was required ($F = 7.05$, $p < .01$). These findings support those of Doolittle and Cleary (1987).

Items were coded with regard to the cognitive complexity of the task. There was a consistent and significant trend for females, relative to males, to show a steady decrease in performance from items requiring Factual Knowledge and Math Manipulation to those requiring Higher-Level Mental Processing ($F = 2.79$, $p < .05$). A somewhat contradictory finding dealt with the type of solution that was required by the item. Although one might expect that computed solutions would tend to involve math manipulations or be routine problems (i.e., relatively lower-level mental processing), females performed better than matched males when a General solution was required rather than a Computed Solution ($f = 3.94$, $p < .05$). Further review is necessary to understand this relationship.

Three categories that were investigated dealt with a spatial/visual factor. Only one of the three resulted in significant findings. In this category, an item was coded with regard to the format of the item stimulus. For example, 65 items had figures, 11 involved graphs, 7 had tables, etc. The majority of the items (268) had no such stimuli. With the exception of the one item that involved a Venn diagram, males consistently performed better than matched females on items with figures, graphs, or tables ($f = 2.88$, $p < .01$). Another related category dealt with whether the item had a spatial component. Items were coded on a continuum from those that had a Primary Spatial Component (9 items) or involved Ordinary Geometry (49 items) to those that had No Spatial Component (242 items). Although there was a tendency for males to perform relatively better than females on spatial items, the results were not significant. This finding is not surprising given the research that suggests that gender differences exist on only some kinds of spatial tasks (Linn & Peterson, 1985) and there were insufficient numbers of items with a spatial component to break the category into subcategories in this study. The third spatial-related category dealt with whether the item referred to a figure that was Drawn to Scale (70 items), Not Drawn (7 items), or did not refer to a figure (283 items). No items were identified that involved figures that were provided but not drawn to scale. Results here were not significant.

Several other categories associated with points being tested were investigated. With one exception, either results were not significant or the numbers of items per coding category were too few to be considered reliable. The one exception pertained to a category dealing with special topics. This category identified items dealing with such topics as Money, Time, Fractions, Rate, Linear and Liquid Measure, Percent, and so on. With the possible exceptions of Fractions and Counting, males tended to perform relatively better than matched females on the items that could be considered Special Topics, compared with those items for which this label was not applicable.

Item Content. The second major aspect of items that was considered focused on the kind of language and subject matter of items, that is, the actual words and subject matter used--although not *per se* tested for--in framing questions. Although categories that meet this description have been associated more typically with verbal tests, it was thought useful here to explore whether categories that have been associated with gender differences on verbal tests and could be applied to mathematics items would yield similar findings across fields.

The four main item content categories that yielded significant results were related to each other. The first category compared performance on items that were abstract or not drawn from "real life" with those that were less routine, applied, "real life" problems. Females performed relatively better on unapplied mathematics items, whereas males performed relatively better on the items drawn from "real life"

($\underline{F}$ = 66.04, $\underline{p}$ < .0001). Similarly, females performed relatively better on items that were Very Textbook-Like, while males performed better on items that were Not Textbook-Like ($\underline{F}$ = 6.07, $\underline{p}$ < .001). Not surprisingly, items that were not like the math curriculum tended to be those that dealt with real life problems. These items also tended to be word problems, on which females did consistently worse than males. The remaining two item content categories related to items with gender references or minority stimuli. On verbal tests, females have tended to perform relatively better on items with people--regardless of the sex or ethnic status of these people (Carlton & Harris, 1989). The reverse was true on the SAT-M, presumably because the presence of people on a math item virtually requires the item to be a word problem.

Item Format. A final group of categories that were studied related to the format of test items or the formal characteristics of test items. Based on obvious format differences and on previous studies, many variables were examined. Of these, only a few differentiated male/female performance, while others showed no consistent patterns of differences. Several of the variables that showed gender differences could be associated with word problems. For example, various length categories were considered. Consistent with the previously mentioned performance on word problems, longer items tended to favor males. Similarly, items coded as requiring more difficult reading (measured largely by length of stem) also favored males.

The remaining categories that were considered failed to show consistent differences. For example, although the format used to display the five options (i.e., vertically versus horizontally) has been a significant factor in Analogy items (Carlton & Harris, 1989), it is not associated with gender differences on the SAT-M. Other mathematics item format variables have evinced ethnic differences in performance but were not associated with gender differences. These included, for example, items that required examinees to identify a maximum or minimum value or items for which the responses are Dependent on the Options as opposed to Independent of the Options. The ordering of the options was a significant variable in White/Asian-American, White/Black, and White/Hispanic comparisons (Carlton & Harris, 1989), but this was not the case for matched males and females: like the Asian-Americans, Blacks, and Hispanics, females performed slightly better when there was so sequential ordering of options than when the options were listed from Least to Greatest, but this difference was not significant.

Summary and Implications

As was evident from the findings of this investigation, males and females who achieved the same overall score on a test may not have arrived at that score with the same pattern of responses. There are a multitude of factors that make some items relatively easier or harder for different groups of examinees even after overall test score has been controlled for. One of these factors seemed to be associated with the kind of mathematics skill that was being measured. Consistent with past research, males performed relatively better on Geometry and Geometry/Arithmetic items, while females performed relatively better on Miscellaneous and Arithmetic/Algebra items. Also, males found items with a stimulus format (i.e., figure, graph, or table) relatively easier, while females performed relatively better when there was no stimulus format.

A somewhat contradictory finding was that males performed better when the item called for a computed solution, whereas females performed relatively better when the item called for a general solution: however, females seemed to find routine problems and those calling for math manipulation (lower-level cognitive processing) relatively easier, whereas males seemed to find items requiring higher-level cognitive processing relatively easier. This finding needs further review.

Finally, and perhaps most interesting, was that females performed relatively better on items that were very much like the curriculum and not "real life" problems, whereas males tended to perform relatively better on the less routine, "real life" word problems. This finding suggests that males may use math more than females in everyday life and could lend support to the argument that males perform better in math because they view it as more valuable in their lives (Fennema & Sherman, 1977). It also suggests that females might benefit from greater curricular emphasis on various applications of mathematics.

A further implication of this investigation is that changes in test specifications could be expected to influence gender differences in performance. Test developers and policy-makers need to evaluate findings such as these and take them into account in making decisions about future tests. Although it is important to note that the majority of categories that were con- sidered failed to show gender differences between groups of males and females matched on overall test scores, an examination of those that did show differences yields a better understanding of more global differences in test performance.

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CLINICAL INVESTIGATIONS IN MATHEMATICS TEACHING ENVIRONMENTS: JOURNEYS FROM RESEARCH QUESTIONS TO DATA COLLECTION STRATEGIES

Bob Underhill, Virginia Tech, Organizer and Presenter
Carolyn Maher, Rutgers University, Moderator
Tom Carpenter, University of Wisconsin at Madison, Presenter
Lynn Hart, Georgia State University, Presenter
Pam Schram, Michigan State University, Presenter

Many researchers are exploring the complex domains we call mathematics teaching and learning environments. Coming to know these environments from the perspectives of key participants is at the core of understanding the dynamics of these environments. In this symposium, four researchers will describe the terrains they have traversed in such ways as will high-light the decision-making processes used in translating research questions into clinical research data collection strategies. Following twelve minute presentations, about 40 minutes will be devoted to a group discussion with these researchers serving as a panel.

During the past decade, many researchers both within and outside the mathematics education community have turned their attention to the beliefs, knowledge and actions of teachers. These researchers have sought to understand more fully certain attributes of teachers and teaching and to study the processes which are inherent in the professional actions of teachers. Thanks to their work, we understand much more now than we did even a few short years ago about important aspects of teachers as decision makers and about instructional processes.

The advent of intense interest and research in teaching environments has led researchers to expand their research repertoires beyond quantitative data collection and analyses to encompass important elements of qualitative research methodologies and analyses.

The use of quantitative research methods and data analyses has a long and rich history within the mathematics education research community. A natural by-product of this history, as seen within the context of historically strong interest in methodological issues, has been the evolution of routinized procedures for moving from research questions to data collection strategies, i.e., conventions have been established within the educational research community. On the other hand, since there has not been a long history of qualitative research within the mathematics education community, such conventions or routinized procedures do not exist.

Members of the community have been struggling with this issue, and this symposium seeks to provide a forum for public examination and discussion of these issues. The aim of this symposium is to briefly present the processes by which four research projects have sought to relieve some of the opaqueness and to operationalize the connections between their research questions and their data collection strategies. We invite interested parties to hear of our experiences and to help us raise questions and issues for public exploration. Brief synopses of the four presentations follow.

TOM CARPENTER

The overriding question addressed by the Cognitively Guided Instruction, research project was: How does knowledge about students' thinking influence teachers' beliefs, their instruction, and their students' achievement? In addressing this general question we focused on a clearly specified segment of content, the development of additional and subtraction concepts and skills. Cognitive research in this area provided a principled analysis of the content domain as well as a map of children's cognitive processes. This analysis supplied a framework to analyze teachers' knowledge and beliefs, classroom instruction, and students' achievement.

We employed multiple methodologies and multiple measures in our research. These included correlational and experimental studies as well as case studies. To study classroom processes, we used quantifiable observation scales with 40 classes as well as open ended case study observation methods with a small sample of teachers. To study the knowledge and beliefs of teachers and students, we relied on written scales as well as clinical interviews. We employed both structured and open-ended clinical interview procedures. Through these various measures, we sought to create a coherent picture of how teachers use research based knowledge about children's cognitive processes to understand their own students' thinking and use that knowledge in instruction. The case studies provide a context and meaning for the findings of the large scale studies, and the large scale studies provide a measure of the generalizability of the observations from the case studies. .sp .ce

LYNN HART

The Problem Solving and Thinking Project is in the third year of a three-year study sponsored by the National Science Foundation. Two research questions were raised initially: (1) Can an individual's metacognitive activity and knowledge be changed through an instructional intervention which focuses on metacognitive experience and knowledge?, and (2) Will problem-solving performance be changed as a result of changes in metacognitive activity and knowledge? Basically we were interested in the relationship between metacognition and problem-solving performance. We derived a working definition of metacognition wherein we came to describe metacognitive activity as the monitoring and regulation of cognitive and metacognitive knowledge and activity; metacognitive knowledge as beliefs about person, self or task and metacognitive experience as an awareness of metacognitive activity. From this we were able to develop data collection strategies.

To study metacognitive activity we needed to observe individuals in problem-solving situations. To assure confidence in our observations we needed triangulation. Our decision was to videotape subjects problem-solving alone in a think-aloud format, problem-solving in a small group, and teaching problem solving. We were then able to use the videotapes to study metacognitive knowledge, but additional data were gained from a structured videotaped interview with one of the researchers and from self-reported logs which focused on person, strategy and task beliefs. To study problem-solving performance, once again we used the videotaped sessions of each subject in addition to paper-and-pencil problem-solving tasks. Finally, to study change, data were collected before, during and after an instructional intervention.

PAM SCHRAM

The MSU longitudinal study examines the implementation of a sequence of innovative mathematics and methods courses for undergraduate elementary education majors. The main question informing this study is, "What is the nature and extent of changes in the beliefs and knowledge about mathematics, mathematics teaching and mathematical learning among preservice teachers as a result of the intervention?"

Data sources include questionnaires, course and classroom observations, field notes and clinical interviews. A questionnaire was administered at various points throughout the study and is the one instrument that has remained constant. Interviews with participants in the intensive sample evolved as the study progressed. For example, student responses to the questionnaire indicated that students had adopted a particular "language" from program coursework. An interview was developed to explore the meaning attached to this language.

Preliminary analyses of data provided insights that have influenced both the intervention and the content of the interviews. An analytical framework was developed to examine students' orientations to the teaching and learning of mathematics.

BOB UNDERHILL

The project "Learning to Teach Mathematics: The Evolution of Teachers' Instructional Decisions and Actions" (NSF #MDR 8652476) is beginning its second year of a two-year study designed to address three areas: (1) individuals' experiences, (2) the university experience, and (3) the school experience. Social organization and cultures of schools, the teacher education program, and the mathematics methods course constitute one major domain of investigation as the project seeks to describe and document the influence of these sociocultural areas on the claims (assertions on a knowledge/beliefs continuum) and thinking of novices during their final year of pre-service education. Because of the complexity of the project and the limited amount of time available in the symposium, this domain will be used to discuss a portion of our journey from research questions to data collection strategies.

The major sociocultural research questions of year one were: (1) What are the characteristics of the various components of the sociocultural environment? and (2) How do the components of the sociocultural environment effect the participants' claims during their year-long field-based learning-to-teach program? To collect data which would help us answer these questions, the research team of four mathematics educators, a cognitive psychologist and an educational anthropologist relied heavily upon the expertise of the anthropologist. The journey will be described and a hand-out will be distributed which will help elucidate the operational work of the research team. The talk and handout will attempt to clarify the evolution of interview protocols, observation field notes, and lists of artifacts which eventually became the data sources from which two major areas (university and schools) and five sub-areas (math methods and teacher education program; school division, building, and classroom) were investigated.

Changes in Student Assessment Occasioned by Function Graphing Tools

ASSESSING PRECALCULUS STUDENT ACHIEVEMENT IN A COURSE REQUIRING USE OF A GRAPHING UTILITY: BACKGROUND AND PRELIMINARY RESULTS

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University of Wisconsin-Madison
Bert K. Waits
The Ohio State University

Scientific, graphing, and symbol manipulating calculators are being incorporated into the school mathematics curriculum. As a result the problem solving strategies, techniques, and procedures taught and learned are changing. These changes create a need for new tests, test items, and assessment procedures that will accurately measure student mathematics achievement. This paper briefly describes the Ohio State University Calculator and Computer Precalculus Curriculum (C^2PC) Project; the materials developed by that project require regular student use of a graphing utility. Preliminary data from the field test of the C^2PC materials are given.

In 1975 the National Advisory Committee on Mathematical Education (NACOME) recommended that calculators become an integral part of mathematics instruction (NACOME, 1975, p. 42). Subsequently, the College Board (1983, 1985), the Commission on Standards for School Mathematics (CSSM) (1989), the Conference Board of the Mathematical Sciences (1983), the National Colloquium *Calculus for a New Century* (Steen, 1987), the National Council of Teachers of Mathematics (NCTM) (1980, 1986), and the Tulane Conference on Calculus Instruction (Douglas, 1986) made similar recommendations.

Several of these groups have observed that using calculators during mathematics instruction will also necessitate changes in the ways that mathematics achievement is assessed. In particular, NACOME stated that "present standards of mathematical achievement will most certainly be invalidated by 'calculator classes'" (p. 42). And the NCTM recommended that test writers integrate the use of calculators into their tests (NCTM, 1986). To study the changes needed in existing standardized tests when calculator use is permitted the College Board and the Mathematical Association of America (MAA) convened a joint symposium; papers presented at that symposium indicated that (a) mathematics tests that permit the use of calculators can be developed that will reliably measure achievement (Leitzel & Waits, 1989), (b) some items on present tests are "calculator sensitive" and need to be revised or replaced (Harvey, 1989a; Leitzel & Waits, 1989), and (c) a variety of issues need to be considered when developing calculator-based mathematics achievement tests (Wilson & Kilpatrick, 1989). Not long after the symposium the College Board's Mathematics Achievement Test Committee began development of a calculator-based version of the Mathematics Level II Achievement Test and the MAA's Committee on Placement Examinations began development of six calculator-based placement tests and two calculator-based high school prognostic (i.e., early placement) tests (Harvey, 1989b; Harvey & Kenelly, 1988); each of these tests will *require* the use of a scientific calculator. Harvey (in press) has discussed some of the issues that must be considered in developing calculator-based tests when the use of scientific calculators is permitted. The active use of graphing utilities both during instruction and testing further obviates the NACOME prediction since when these utilities are used instruction on and student use of problem solving strategies likely will change; the precalculus course developed by Frank Demana and Bert Waits is a case in point.

Calculator and Computer Precalculus Curriculum Project

When the College Board and MAA joint symposium convened in September 1986, the first graphing calculator, the Casio *fx7000G*, had just been introduced; soon after, the Sharp *EL-5200* graphing calculator, and the first graphics and symbol manipulating calculator, the Hewlett-Packard *HP28C* were also marketed. With the appearance of these calculators, school and college students for the first time could have inexpensive mathematics tools of their own and would no longer have to rely upon an inadequate supply of expensive computers that, too often, were not readily available. The recently published NCTM *Curriculum and Evaluation Standards for School Mathematics* (CSSM, 1989) assumes that in Grades 9-12 graphing calculators "will be available to all students at all times" (p. 124).

The Ohio State University Computer and Calculus Precalculus Curriculum (C^2PC) Project antedates the graphing calculators and the NCTM *Standards* though it might seem otherwise. The materials developed by the C^2PC project (Demana & Waits, 1988) are based on the assumption that students normally will have a graphing utility available to them, focuses on a study of functions and their behavior instead of on equations and their manipulation, and expects that students will generate and use both algebraic and graphical representations of problem situations (Demana & Waits, 1988, p. 28) as proposed by the *Standards* (CSSM, 1988, p. 146). Since graphing calculators were not available initially for use with the C^2PC precalculus materials, a computer graphing utility, *Master Grapher* (Waits & Demana, 1988), was developed. *Master Grapher* can display the graphs of functions, conic sections, parametric equations, and polar equations; among its capabilities are the abilities to zoom-in and zoom out, to display several graphs simultaneously, to translate and rotate graphs, and to graph the inverse of a relation. The capabilities of presently available graphing calculators are similar to those described for *Master Grapher*.

The C^2PC precalculus materials include all of the content usually included in high school and college precalculus courses. The procedures used to solve problems in the C^2PC materials are atypical. Consider the problem shown in Figure 1.

> Squares of side length x are removed from a 15 inch by 60 inch piece of cardboard and a box with no top is formed by folding up the resulting tabs. Determine x so that the volume of the resulting box is at least 175 inches.

Figure 1.
Example Algebra Problem

To solve this problem a student studying from *Precalculus Mathematics: A Graphing Approach* (Demana & Waits, 1988) would first generate the function f (i.e., $f(x) = x(10 - 2x)(25 - 2x)$) that is an algebraic representation of the problem situation. Then the student would draw a complete graph of the function f (i.e., a graph of the function in an optimal viewing rectangle that accurately depicts the "end behavior" of the function). Then, in some order the following would ensue. From the original problem situation he or she would determine that the only relevant values of x are those between 0 and 5 and would redraw the graph so as to show only that portion of the graph of f. On this graph the horizontal line $y = 175$ would be superimposed to determine the points on the graph at which to zoom-in to obtain the approximate values of x for which $f(x) = 175$ and thus to solve the problem. In contrast students enrolled in conventionally taught courses would solve the inequality $x(10 - 2x)(25 - 2x) > 175$ by forming the equation $x(10 - 2x)(25 - 2x) - 175 = 0$, find the real roots of that equation, and use those roots to develop a sign chart. The C^2PC materials also teach many of these conventional techniques, including sign charts, usually along with or after instruction using graphical problem solving procedures.

Field Testing the C^2PC Materials

During the academic year 1988-89, *Precalculus Mathematics: A Graphing Approach* (Demana & Waits, 1988) has been used in a field test involving precalculus and college algebra classes at approximately 80 high schools and 40 colleges and universities nationwide. During the field test some students have used graphing calculators while others have used *Master Grapher* (Waits & Demana, 1988). Each of the 10 chapters of *Precalculus Mathematics* is accompanied by a chapter test; many of the schools, colleges, and universities using these materials have administered these tests to their students. In addition, a College Algebra Test consisting of 25 items has been administered at some sites as a pre- and posttest. These data will be analyzed to describe the effectiveness of the C^2PC materials; only preliminary results from one site will be reported here.

College Algebra Test. At the University of Wisconsin-Madison (UW-Madison) a single class of college algebra students were taught the first six chapters of *Precalculus Mathematics* during the first semester, 1988-89; two classes were taught using these materials during the second semester. Primarily, students in these classes used *Master Grapher* though a few students in each class purchased graphing calculators. Each class was given the College Algebra Test as a pre- and a posttest; when these tests were administered, the use of graphing calculators was not permitted. Since the two classes taught during spring semester were taught interchangeably by two different instructors and covered the same material, their data were aggregated. The pretest and posttest means and standard deviations for both semesters as shown in Table 1. There was a statistically significant difference between the pretest and posttest scores for both semesters.

Table 1
College Algebra Test
Means and Standard Deviations

Semester(*N*)	Pretest Mean(S.D.)	Posttest Mean(S.D.)
1(25)	8.76(2.85)	13.32(3.61)*
2(18)	9.67(3.96)	14.06(3.55)*

*$p < .001$

On the College Algebra Test nine of the items ask students to use graphical, algebraic, or both graphical and algebraic information in the item stem to respond. Here are respective examples of items that require interpretation of only graphical, only algebraic, or both graphical and algebraic stem information.

1. A graph of a cubic function is pictured along with the coordinates of the relative maximum and minimum of the function. Students are asked to identify the function that the graph represents.
2. Students are asked to identify the interval on the number line for which $x^2 < 4$.
3. A factored cubic function and its graph are shown. Students are asked to determine the coordinates of the points at which the function intersects the x-axis.

On six of these nine items second semester student performance ($N = 18$) on the posttest was at least 79%; student performance on the remaining three items was 58%, 68%, and 32%. The most difficult of these three items is the one that asked students to identify the graph of the cubic function $f(x) = (1 - x)^2(x - 2)$ given an accurate graph of it. The second most difficult of the three items was the one that asked students to identify the interval for which $x^2 < 4$; the item on which students scored 68% was the one that asked students, given the graph of a function, to identify the intervals on which it is increasing. These data show that even when denied use of a graphing utility student performance on items containing graphical or both graphical and algebraic information in the item stem exceeds overall student performance on the College Algebra Test posttest; it may be possible to attribute this effect to student use of a graphics utility.

Final Examination. During the second semester a two-part final examination was given to the 24 students who completed the course taught using C^2PC materials. The five problems on the take-home part of the final exam required the use of a graphics utility; this part of the exam was untimed and was completed during the last week of classes. The in-class part of the final exam consisted of 11 questions. The mean percent correct and standard deviation for the final examination, the in-class part, and the take home part were 63.71(17.97), 57.71(20.92), and 89.50(16.59), respectively. A t-test revealed that the scores on the take home part were significantly different ($p < .001$) from the scores on the in-class part of the exam. These data would seem to indicate that students can effectively use a graphing utility to assist them in solving those algebra problems in which a graphical representation is appropriate.

Concluding Remarks

Thus far, the test items given to students studying from *Precalculus Mathematics* have been typical in that they have usually asked for symbolic answers to algebra problems. These items have revealed little about the strategies that students use while solving the problems given them or about the knowledge students have about graphical representations or their uses. In this sense the graphical approach to teaching precalculus is unproven and its effects are unknown. In addition, there is not yet any empirical evidence that shows students better understand functions or the relationship between the symbolic and graphical representations of a function. Among our own students we have observed that they more frequently talk about functions and their zeros instead of equations and their roots, that they are quite comfortable with the ideas of relative and local maxima and minima, that they seem to understand that functions belonging to the same class (i.e., cubic polynomial functions) have similar graphs and vice versa, and that they seem to understand the advantages and shortcomings of using graphical representations while solving problems.

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INSIGHTS INTO ASSESSMENT USING FUNCTION-GRAPHING TOOLS GAINED THROUGH COMPUTER-INTENSIVE ALGEBRA

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With the incorporation of function-graphing tools in mathematics curricula, changes in assessment are needed. Data-based observations and hypotheses were generated through pilot tests of Computer-Intensive Algebra, a curriculum that assumed the use of computer tools, including a function grapher and a curve fitter, both in classroom explorations and on tests. The observations address the logistics of testing, the construction of test items, and the development of student preferences concerning the use of graphers.

Computer-Intensive Algebra is a beginning algebra curriculum that has been built on these premises:

1. Realistic situations and the concept of function would be the organizing themes for the rules, principles, and techniques of algebra; and

2. The regular availability of computing tools (including a function grapher, a table-of-values generator, a curve fitter, and a symbolic manipulator) would be assumed.

Work on the curriculum was begun at the University of Maryland (under the direction of James Fey) in 1985 and has continued in linked projects there and at Penn State since 1987. Our students have used function-graphing tools as part of their computer repertoires in each of our four full-year trials. We have conducted trials at two distinctly different sites (varying in ability, grade level, and racial composition). Our data includes classroom observation data, field notes, and results on tests that assumed scientific calculators or a complete array of computing tools. We have traced student performance with the tools on course examinations, in task-based interviews, and on computer-lab explorations.

Our ongoing analyses of data yields some insights into potential impact on student assessment occasioned by the use of function-graphing tools. Three areas of importance emerge:

1. What are logistical issues involving graphing tools in testing situations?

2. What types of test items can be used when students have access to tools including a function grapher and a curve fitter?

3. How do students' initial abilities to use function graphing tools develop?

LOGISTICAL ISSUES

Among the logistical issues surrounding use of function-graphing tools is that of how to test students in a computer lab. Several observations have developed.

Simulate the testing situation

If students are to be tested at the computer, they should have prior experience very similar to the testing situation. Our students did not have home access to the computer software and hardware they would be using on the tests. Most of their pre-test computer experience was in paired problem-solving explorations. Two special provisions were made to prepare students for tests:

a. Extra-class lab time was established, ranging from a special weekly or daily class period to open lab time at lunch and after school.

b. Periodic in-class time was provided for students to work alone at computer stations.

Allow adequate testing time

Computer-based test problems commonly require more decision-making on the part of students than if test problems required only the application of routine algorithms. In a tool-rich testing environment students need to decide which tool to use, how to get the desired accuracy with the chosen tool, how to deal with unexpected results, and how to verify answers through alternate tools and representations. With only half as many computer stations as students, we split our tests into two parts - one that required only calculators and one that required use of computers. Since teachers observed that some students felt overly pressured to complete the computer portion of the test in half of a class period, we made each test portion slightly longer and used two days for each test.

Provide for test security

Two security issues have arisen in our experience with testing that allows the use of function-graphing tools. First, students who are seated next to each other should not be completing identical tests since computer screens are often clearly visible to neighbors. One solution to this problem is to use two different parallel tests for students at adjacent computer stations. Second, when tests extend over two days, there is a natural concern that students will share answers in the intervening time. The fact that the computer-based problems do not often yield easily remembered answers, along with the use of two different test versions, has seemed to alleviate this problem in our trials.

TEST ITEMS

Curricula that assume regular access to function-graphing tools suggest several areas of change in assessment.

Function-graphing items not requiring computing tools

Access to function-graphing tools in class results in increased student familiarity with the interpretation and use of graphs. Many of the items that we have used in testing student ability to interpret and use graphs do not require the on-test use of computing tools:

1. Students can be given the description of a problem setting along with a coordinatized graph (discrete or continuous) of two related variables. They can then be asked to answer questions requiring the identification of a dependent-variable value (or a range of dependent-variable values) for a given independent-variable value (or a given range of independent-variable values), or vice-versa.

2. Students can be given a graph of two related variables without coordinates (for example, water temperature in a pool as a function of hour of the day). They can then be asked to determine whether the graph is reasonable and to explain their rationale.

3. Students can be given the description of an applied situation and then be asked to sketch graphs of related functions and to explain the trends that are depicted in their graphs.

Providing a graph when students are required to generate one

Some of our test items have provided a general sketch of a graph with the expectation that students would produce a computer-generated graph from which to determine answers to questions. In this setting it is not unusual for students to spend a considerable amount of time trying to produce an exact replica of the given graph. For example, if the required graph was a parabola, students would adjust and readjust the units on their graphs to produce a parabola of the same height and width as the given graph. Curriculum writers and testmakers should be aware of this phenomenon.

Items for which a particular strategy is anticipated

If students are provided with an array of computer tools, test writers should expect them to use a variety of solution paths. At times the intention of the question will be subverted by students who make sensible use of the available tools. For example, on one of our first versions of a linear functions test we constructed three different test items to determine student ability to use three different strategies:

1. Students were given a graph of points that fit a linear rule exactly, and asked to determine that rule;

2. Students were given a set of function values that fit a linear rule exactly, and asked to determine that rule; and

3. Students were given a set of input-output values that approximated a linear function, and asked to determine a curve of best fit.

Unpredicted by the test writers, the majority of students found the linear rules quite quickly, all by using the computer least-squares curve fitter!

DEVELOPMENT OF PREFERENCES IN FUNCTION GRAPHERS

Throughout our pilot testing, we have gathered data on students' choice and use of computing tools. One preliminary result is that when students are confronted with an applied problem, and have available an array of computing tools (including function graphers, table generators, curve fitters, and symbolic manipulators), function graphers are usually not their tool of first choice. Even when a graph was provided on test items, if other representations were available, a majority of students (at least 60 percent) chose and used those representations.

One reason for this lack of preference for the function- graphing tool could have been the difficulty of its use. When using our particular graphing tool, students had to choose both the horizontal and the vertical units. The table program required only the choice of a range for input-variable values. A second possible explanation for students avoiding the use of the function grapher in a testing situation may have been the perceived need for more accuracy than the graph seemed to provide.

Here are some excerpts of typical comments drawn from interviews with students immediately following introductory work with the computer tools. Questions were asked about what they liked and didn't like about using a graph (as compared to a table or a function rule) to solve problems.

What students did not like about using a graph:

> In a graph you have to look; you have to estimate what it might be.
>
> [In order to use the graph] you just have to set the right scales.
>
> Sometimes they're hard to read. Cause like if you

have it all in one area or all spread out, you don't really know what your answer is.

When your graph was like in a certain area, when you tried to spread it out, it sometimes went the wrong way. ... Cause like, to make it longer you have to make the numbers smaller or whatever. Instead you'd think to make it longer, you'd make the numbers bigger but you don't.

You have to keep changing the scale to get it to fit.

Sometimes it's confusing about which number's going on which axis.

What students liked about using a graph:

You can see where the peak is and where it starts to go down into the negatives.

I like the way it's all on one screen and you can try once and then you can try again.

Graphs can show you everything in a small area.

It gives you all the answers at once.

I like using the graph the most cause it's neat to see how if you change the [input values] how the graph changes, and it can change the whole outlook.

You can get the overall view of the function rule.

CONCLUSION

The availability of function-graphing tools provides a variety of challenges for test-writers. As we enter the final pilot-testing year of the Computer-Intensive Algebra project, we are just beginning to formulate data-based hypotheses about the impact of function graphers on assessment.

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ASSESSING STUDENTS' KNOWLEDGE OF FUNCTIONS

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Functions, Statistics and Trigonometry with Computers **(FST) is the fifth of six courses developed by the University of Chicago School Mathematics Project for average students. Features of FST include: realistic applications, continuous review, and use of both computer and calculator as well as traditional paper and pencil methods to solve problems. A formative evaluation conducted in eight schools in 1988-89 assessed students' knowledge of functions by multiple choice items on which only a scientific calculator could be used, and by free response items on which a computer or graphing calculator was permitted. This paper describes some free response items, procedures for scoring such items, and preliminary results from the evaluation study.**

In recent years there have been numerous recommendations to update secondary mathematics curriculum and instruction by including such features as statistics, probability, discrete mathematics, realistic applications, and work with calculators and computers (College Board, 1985; NCTM, 1980; NCTM, 1989). Since 1983 the Secondary Component of the University of Chicago School Mathematics Project (UCSMP) has developed a six year sequence of courses for average students which translate these and other recommendations into reality. These courses are called *Transition Mathematics, Algebra, Geometry, Advanced Algebra, Functions, Statistics and Trigonometry with Computers* (FST), and *Precalculus and Discrete Mathematics.* In particular, all UCSMP courses assume scientific calculators are available at all times including on tests; FST assumes that computers are also available; and for the use of *Advanced Algebra* and *Precalculus and Discrete Mathematics* a computer or graphing calculator is highly recommended.

Recently the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) have recommended that both student assessment and program evaluation reflect more accurately new standards for classroom practice. In particular, the *Standards* suggest that methods and tasks for assessing students' learning be aligned with the curriculum's goals and objectives, and with its instructional approaches, including the use of calculators and computers.

Consistent with this recommendation achievement of students in UCSMP courses is always measured on at least some tests that make explicit use of calculator or computer technology. In particular, the field study of FST conducted recently measured students' knowledge of functions with free response items which could be done using any of the software that had been introduced in the course. We hoped that such testing would not only give developers feedback on the effectiveness of their materials, but it would also provide insight into theoretical and practical issues about assessment using computer technology.

This paper describes selected items from the FST evaluation study, procedures used to administer and score the items, and issues raised by use of non-routine items and testing procedures. As the analysis of these data is not yet complete (Sarther, et al., in preparation), results and conclusions presented here should be taken as preliminary.

OVERVIEW OF THE CURRICULUM

Functions, Statistics and Trigonometry with Computers is intended for students in grades 11 or 12 who have either completed UCSMP *Advanced Algebra* (Senk, et al., 1990) or another advanced algebra that includes logarithms and right triangle trigonometry. The content of FST includes work with many typical precalculus functions such as exponential, logarithmic, trigonometric, and some rational functions, as well as substantial work with counting, probability, and both descriptive and inferential statistics. An unusual feature is the extent to which statistical and algebraic ideas are integrated throughout. For instance, students study the effects of translations and scale changes on each of equations, graphs, and sets of data. Other characteristics of FST are its emphasis on reading, realistic applications, continuous review, and open-ended projects.

Computers are used as a tool throughout the course. The amount of hardware recommended for FST is consistent with the *Standards* (NCTM, 1989). A computer should be available in each classroom for demonstration; students should have access to computers during class time whenever needed for individual or small group use; and computers should be available outside of class for doing assignments. Schools using FST provide three types of software: a function grapher, a statistics package, and BASIC language. Lessons referencing this software help students visualize and analyze functions, explore relations between tables, equations, and graphs, simulate experiments, generate and analyze data, and develop limit concepts.

Technology is referenced in exposition and worked examples whenever necessary. In particular, students learn to use both a function grapher and a table-generating program to find zeros, relative extrema, and symmetries of functions, and to solve systems. In the questions or projects sometimes students are told to use a particular tool, but more often than not, a question is merely posed, and they are expected to decide whether to use a computer, calculator, paper and pencil, or solely mental processes to answer it. For instance, when asked to find the zeros of a continuous function some students prefer to use a function grapher to plot the function, and then to zoom or rescale until the x-intercepts are approximated to the desired level of accuracy. Others prefer to use a table-generating program to print values of the function, to look for sign changes in the y-values, and to continue to iterate to the desired level of accuracy.

OVERVIEW OF THE EVALUATION STUDY

During 1988-89 UCSMP conducted a formative evaluation to assess the extent to which FST was implementable and effective in normal classroom settings.

Sample

The study involved nine teachers and about 300 students in eight schools in four states. The seven public schools represent large and medium sized cities and suburban communities; the one private school is coeducational with both boarding and day students.

Six of the nine teachers had taught FST during a pilot study the previous year; and three were new users. Four teachers are leaders in their schools with respect to using computer technology for instruction; three are knowledgeable about computers but had only limited experience using computers in mathematics classes prior to FST; and two had never used microcomputers before - either personally or for teaching. Prior to the start of the school year teachers received the students' texts, and a teacher's manual with answers to questions in the text and some suggestions on how to present material or structure the class; but there was no direct in-service instruction during the study.

About 79% of the students were seniors, and 21% juniors, with about 52% male and 48% female. The sample is about 60% White, 30% Black, and 10% from other groups. Virtually all students reported having used both a scientific calculator and a computer before starting FST, with about 52% having used a computer in the past for doing mathematics. However, virtually none had used statistics packages or function graphers before FST. One class had studied from the UCSMP *Advanced Algebra*, but for all others FST was their first UCSMP course.

Design

At the beginning of the year students took a pretest of advanced algebra content, and both students and teachers completed surveys of background data. Throughout the year teachers completed forms rating the lessons in each chapter, and commenting on the strengths and weaknesses of the materials. During the year each teacher was observed at last twice, and all teachers and some students were interviewed. Additionally, teachers met in Chicago three times for a full day to discuss issues of common concern and to provide further feedback to UCSMP staff on the materials. During the last month of school each student took two tests of achievement on the content in the course, and teachers and students completed final questionnaires.

Posttests

End-of-year achievement was measured by a Multiple Choice Test and a Free Response Test. The Multiple Choice Test consisted of 36 items taken from the pretest, the Second International Mathematics Study (Crosswhite, et al., 1986), or created specifically for the field study. For this test students could use a scientific calculator; but neither computers nor graphing calculators were allowed. Content was chosen by sampling from all chapters in the text, with somewhat more emphasis given to major ideas in the first two thirds of the text than to topics in the last third.

There were two non-overlapping forms of four items each on the Free Response Test, with about half of each class taking each form. Five items deal with standard precalculus (non-statistical) functions. Students could use a graphing calculator or any of the software that had been introduced during the course on the Free Resronse Test.

Three experienced high school teachers who had not been part of the FST study were trained by an FST author, who was also an experienced Advanced Placement Reader, to score the free response items. Each item was graded using a holistic partial credit scheme modeled after the work of Charles, Lester and O'Daffer (1987) and Malone, et al. (1980), according to the general criteria below.

Score	Solution Stage
0	The student began to work, but there was nothing meaningful.
1	Some meaningful work was done, but an early impasse was reached.
2	Sufficient detail indicates that the student proceeded toward a rational solution, but a major error was committed.
3	The problem was very nearly solved, but minor errors were committed.
4	An appropriate method yielded a valid solution.

Students who scored 3 or 4 points were considered successful on that item. To distinguish between students who scored 0 because they had tried but done nothing correct, and those who hadn't started the problem at all, a score of 8 was assigned to solutions that contained no mathematical writing.

Each item was grade blindly by two people. When scores disagreed, a third person graded the problem blindly. If two scores agreed the mode was used as the final score. If no agreement was reached, then the entire group discussed the solution and came to a consensus grade.

RESULTS

In this paper only items about classical precalculus functions are considered. Scores reported are taken from the subsample of 267 students who took all project tests.

Overall, students in FST did as well as students in the precalculus sample from the Second International Mathematics Study. The FST sample averaged 42% on the eight SIMS items dealing with functions, and the U. S. precalculus sample averaged 45% correct.

Graders scoring of the open ended items was generally quite consistent. On the three representative free response items given below the rate of agreement ranged from 72% to 94%.

Figure 1 shows a relatively easy item for FST students. About 62% of the students were successful on the item, and most of the rest earned some points. On this item, as with all the free response items, often the difference between a solution earning three points and one earning four points was the ability of the student to articulate his or her solution. Students typically used a function grapher to find the zeros, but a sizeable minority used a table-generating BASIC program. Common errors included not knowing whether the x- or y-intercept was a zero, and failure to provide sufficient detail on the method used to solve the problem. Most students seemed to have acquired reasonable technical skill in operating the software, and using it to iterate to find zeros of a function.

(a) Determine to the nearest tenth the zeros of the function defined by

$$f(x) = x^4 - 2x^3 - 10x + 5.$$

(b) Explain your method.

score	8	0	1	2	3	4
percent (of n = 131)	1	4	14	18	52	11

Figure 1. An easy free response item, and the distribution of students' scores.

Figure 2 shows an item of greater difficulty. In an application context the student is asked to determine the x-coordinate of the relative maximum of a function. About 23% of the sample was successful, but 38% scored 0 or 8. Common errors included finding the y-coordinate or an intercept of the function. Again, the preferred tool for solving the problem was a function grapher, but some students solved it by generating tables of values.

The polynomial function A defined by

$$A(x) = -.0015x^3 + .1058x$$

gives the approximate alcohol concentration (in percent) in an average person's bloodstream x hours after drinking about 250 ml of 100-proof whiskey. The function is approximately valid for values of x between 0 and 8. How many hours after the consumption of this much alcohol would the percentage of alcohol in a person's blood be the greatest? Express the answer correct to the nearest tenth, and explain how you got your answer.

score	8	0	1	2	3	4
percent (of n = 135)	15	23	18	21	21	2

Figure 2. A difficult free response item, and the distribution of students' scores.

Figure 3 shows the most difficult of the items testing knowledge of functions. More than half the students did not attempt a solution, and of those who did, the vast majority scored 0.

Only 11% were successful. Among those who solved the problem correctly, two different strategies were employed. Some students graphed y = sin 2x and y = 1/2 x on the same set of axes, noted that there were three points of intersection of the line and the sine curve, and found the x-coordinate of the point with the smallest positive x-coordinate. Others created a new function y = sin 2x - 1/2 x, and used the function grapher to locate the smallest positive zero of the function.

Consider the equation $\sin 2x = \frac{1}{2}x$, for x in radians.

(a) How many solutions does this equation have? Justify your answer.
(b) Find the *smallest* positive solution, correct to the nearest hundredth.

score	8	0	1	2	3	4
percent (of n = 131)	56	29	2	4	8	3

Figure 3. A very difficult free response item, and the distribution of students' scores.

DISCUSSION

Overall, the developers are satisfied with the achievement results from the field study. Students in a course that makes substantial use of computer and calculator technology and places less emphasis on paper and pencil drill perform on standard items equally as well as precalculus students who have had considerable work with paper and pencil manipulation but little experience with technology. At the same time FST students learn to use technology to solve problems which had previously been inaccessible to high school students, e.g. finding the zeros of a polynomial function.

However, all is not rosy! The techniques needed to find the relative maximum in the application problem shown in Figure 2 are mathematically no more complex than the techniques needed to find the zero of the function given in Figure 1. Yet performance on the former is much lower than performance on the latter. Although many studies including NAEP report that students score lower on applications than on skills, field studies of other UCSMP courses have not found this to be true. Typically, UCSMP students having experienced a curriculum rich in applications do quite well on such problems. We plan to examine factors, particularly the pattern of review questions, which may explain this anomaly.

The exceptionally low performance on the item shown in Figure 3 suggests that skills for solving equations using a computer may not transfer any easier than skills for solving equations with paper and pencil. Early in the FST texts students had been shown various techniques using both a table generating program and a function grapher to solve an equation of the form f(x) = g(x), where f and g are continuous functions. Most of the examples were polynomial, exponential, or logarithmic. Midway through the course they had learned to graph functions of the form y = a sin bx. But in the text no explicit instruction on how to solve $f(x) = g(x)$, where at least one of f and g is trigonometric, is given until the last chapter. At the time of the testing only three of the classes had gotten to that chapter. In those classes performance was indeed better than in the others, indicating that direct instruction and practice seemed to have great benefit.

Although we have been able to score papers reliably using various partial credit grading schemes, and the information we get is quite useful for program evaluation, we would like to explore ways to get even better information on what students know and can do. Work needs to be done in texts and by teachers on helping students express themselves more fluently both in writing and orally. (Even in UCSMP classes where there are many questions asking students to conjecture, explain, or justify, not all students express their thoughts clearly!) As a research and evaluation community we also need to find better ways of capturing either electronically or on paper what students have done electronically.

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Alternative Conceptions of Probability: Implications for Research, Teaching and Curriculum

REASONING ABOUT CHANCE EVENTS: ASSESSING AND CHANGING STUDENTS' CONCEPTION OF PROBABILITY

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Research on misconceptions of probability indicates that students' conceptions are difficult to change. A recent review of concept learning in science points to the role of contradiction in achieving conceptual change. A software program and evaluation activity were developed to challenge students' misconceptions of probability. Support was found for the effectiveness of the intervention, but results also indicate that some misconceptions are highly resistant to change.

The newly released NCTM Curriculum and Evaluation Standards for School Mathematics (1989) state that "Probability provides concepts and methods for dealing with uncertainty and interpreting predictions based on uncertainty (p. 121). The standards present the case for inclusion of probability in the mathematics curriculum at both the elementary and secondary level.

In grades 5 through 8, goals for students include developing an appreciation for the use of probability in the real world and the power of using a probability model as well as being able to make predictions based on probabilities. Goals for grades 9 through 12 include using probability to represent and solve problems involving uncertainty, using simulations to estimate probabilities, and understanding and using random variables and their probability distributions.

Although the aim of these standards is to better prepare students to use probabilistic reasoning before they enter college, the experience of many college instructors indicates that even college students have difficulty learning basic probability concepts (Garfield & Ahlgren, 1988).

Difficulties Learning Probability: A Conceptual-Learning Perspective

There is abundant research documenting the difficulties people have learning and correctly using probability to reason about chance events (Garfield & Ahlgren, 1988). This paper describes a study based on the concept-learning perspective of learning science as summarized by Eylon and Linn (1988), as it applies to learning probability. This perspective is based on studies of qualitative differences among the concepts students use to explain chance events. The literature identifies prevalent misconceptions of probability, such as the representativeness and availability heuristics described by Kahneman and Tversky (1973), and alternative frameworks which students use to represent knowledge about chance events, such as Konold's outcome orientation (Konold, 1989a).

One aspect of this general concept-learning perspective is the emphasis on the importance of prior knowledge as it affects subsequent learning and the role of contradiction in achieving conceptual change (Eylon & Linn, 1988). Just as in learning science, students begin the formal study of probability with strong conceptions about chance events, some conflicting ideas about chance events, and little knowledge of the many laws and properties of probability. As in learning science, students' ideas are often inconsistent with the formal rules of probability taught in an introductory statistics course. The result is students often fit information presented in class into their own views of the world, rather than alter their conceptual frameworks (Eylon & Linn, 1988). Researchers have found that students' conceptions are often resistant to change and are usually unaffected by traditional instruction.

Research on Overcoming Misconceptions of Probability

Building on the research documenting misconceptions of probability, a few researchers have attempted to confront and change these intuitive beliefs. Konold (1989b) found that some students may actually be using an "outcome" misconception which appears to be very stubborn despite a computer-based modeling intervention.

Shaughnessy (1977) developed an experimental activity-based course in elementary probability and statistics which was more effective than a lecture-based course in overcoming misconceptions attributed to the representative heuristic. When given the same question about the likelihood of different birth orders for six girls and boys, students in the experimental group were more likely to respond

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correctly on a posttest--with fewer choosing answers typical of the representativeness heuristic.

DelMas (1988) developed and used a coin-tossing activity with college students. He explored the role of evaluating the results with students as opposed to having them only experience the activity. Not only did delMas find increased understanding and correct posttest responses with the evaluation activity; there was a marked increase in use of the representativeness heuristic for subjects when the evaluation activity was not included. Shaughnessy also used extensive questioning and interaction with students which may have served a similar function.

A Developmental Theory

Based on the research on learning science in general, and the specific research on learning probability, we have formulated the following theory:

1. Many students bring misconceptions about chance events to a statistics course, or develop misconceptions as they learn about probability for the first time.
2. It is important to engage students in an instructional activity designed to confront these misconceptions.
3. It is important to have students make predictions and to evaluate the results of their predictions in order to draw their attention to inconsistencies between their expectations and observed outcomes.
4. When student attention is drawn to inconsistencies, there is a greater likelihood for conceptual change to occur.

THE RESEARCH STUDY

This paper describes part of a larger study designed to investigate the stability of students' conceptions of probability. An intervention was designed and used to engage students in thinking about probability. This intervention was the "Coin Toss" instructional unit developed by the second author for use on the Apple Macintosh personal computer. The "Reasoning about Chance Events" test was developed to capture changes in student thinking from pre- to

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posttest. Not only were students' misconceptions examined before and after the intervention; patterns of response between pairs of questions were also examined to better understand the stability of students' conceptions of probability.

The major questions explored in the larger study are:

1. What conceptions do students have prior to instruction in probability and how stable are these concepts?
2. How do students' conceptions change as a result of using the "Coin Toss" program?
3. Are certain prior patterns of thinking more or less likely to change as a result of the instructional unit?

The Coin Toss Program

A tutorial software program called "Coin Toss" simulates tosses of a fair coin to demonstrate basic probability concepts. These concepts include:

1. Variability of Samples. Samples will vary and may look quite different from the population. For example, fifty samples of ten tosses of a fair coin will look quite different from each other, and few if any will have a 50:50 split of heads and tails.
2. Effect of Increased Sample Size. As sample size increases, variability decreases. For example, Coin Toss presents sampling distributions of heads for 10 toss, 50 toss, and 100 toss sample sizes; the distributions of heads become quite peaked and narrow as sample size is increased.
3. Independence and Randomness. After a run of heads or tails, you are just as likely to get a head as a tail. Different runs of heads and tails are equally likely to occur. If a fair coin is tossed a large number of times (Coin Toss simulates 10,000 tosses), the different run lengths occur with nearly equal frequencies for heads and tails.

A workbook was designed which serves as both a manual and a place for students to make predictions, record outcomes, and note observations. The current coin Toss software is the fourth revision of the program, based on two years of piloting with different groups of

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students. The program is divided into four sections and takes students about 1 1/2 hours to complete.

The Reasoning about Chance Events Instrument

This test was designed to assess the impact of Coin Toss on students' reasoning. Items used in previous research by Shaughnessy (1981), Konold (1989b), and delMas (1989) were combined to construct this instrument. Some of the items are multiple choice, some are open-ended, and some ask students not only to choose an appropriate answer but also to select the best rationale for the answer chosen.

Method

Subjects in this study were first and second year college students in two introductory statistics courses at a large midwestern university. At the end of the fourth week of the class, following a unit on descriptive statistics, students were given a chapter to read which provided a basic introduction to probability. On the first day of the fifth week of class, students were given Reasoning About Chance Events as a pretest. Following the pretest, students were given instructions on how to use the Coin Toss software along with the workbook in which to record data generated by the computer simulations. When the class met at the end of the week, all students had used Coin Toss. One of the authors met with the students during this class period and engaged the students in discussion about their experience, the results they recorded, and observations and conclusions they noted in their workbooks. Discussion was ended after 30 to 45 minutes, and the students completed Reasoning About Chance Events as a posttest.

RESULTS AND DISCUSSION

A total of 57 students completed both the pretest and posttest. Students' responses on the pretest indicate that they had some incorrect conceptions about probability, confirming other research on misconceptions. Very few students in the study initially displayed stable and correct conceptions about sample variability, and a larger

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number of students had conceptions that were stable, but incorrect and resistant to change. A majority of student conceptions, however, did change, suggesting that the Coin Toss intervention affected their thinking.

A large number of students were not sensitive to the effect of sample size on sample variance. Again, a small number of students initially displayed correct conceptions, and maintained them. A larger percent (nearly three times as many), maintained stable, incorrect conceptions about sample size and variation. However, about two-thirds of the students did show some positive changes in their thinking after using the Coin Toss program.

There was one question which showed a marked change from pretest to posttest, where a majority of students gave the correct response on the posttest. However, a little less than half of these students selected a correct rationale for their correct response. This suggests that their conceptions were still being formed and not totally correct or stable.

This brief account of the results indicate that students tend to enter a statistics class with incorrect conceptions of probability, and that some misconceptions are persistent despite interaction with a software program designed to confront the misconceptions. It is interesting to note, however, that the conceptions of a majority of students are not stable and appear to be affected by use of the software and evaluation activity in class.

More work needs to be done in investigating how student conceptions of probability are affected by different instructional units and approaches. Our work highlights the need to look at both that which changes and that which does not change in students' thinking to properly evaluate the impact of an instructional unit. We must look for methods which confront and change resistant misconceptions and increase and develop the stability of correct conceptions in order to better prepare our students to deal with uncertainty in an uncertain world.

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PROBABILITY AND CURRICULUM

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New goals for learning probability are very different from traditional computational objectives and include heavy emphasis on integration with other topics in mathematics and with other subject areas. Although there is no solid knowledge of how to teach probability ideas well, new materials in a variety of forms are attempting to draw on recent research on students' intuition.

GOALS

The last decade of research in mathematics education and science education has made amply clear that students leave high school with very little understanding of mathematical ideas, or ability to apply them to real-world problems, even when they are able to perform algorithms that we had taken as demonstrating understanding (Garfield & Ahlgren, 1987). Just what can be done about the inefficacy of mathematics education may take years to discover, but surely it is evident that we must begin by specifying the requirements for conceptual understanding, not by merely listing topics or listing calculations that can be made.

Moreover, grasping the concepts of probability requires that other mathematical ideas also be developed, including proportionality and even the idea that numbers can be used to represent abstract variables -- such as strength of conviction. The 0 to 1 scale used for probability is particularly ill suited for naive students, compressing as it does the entire universe of opinion into a tiny range between the first two counting numbers. Indeed, the 0 to 1 convention reduces most of the strong probability statements (very high or low odds) to vanishingly small decimal differences near 0 or 1.

Lists of probability topics for pre-college curriculum have appeared in several contexts, either in the form of recommendations for curriculum or in the form of actual curriculum materials. The best current U.S. example is Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) prepared by the

National Council of Teachers of Mathematics. The recommendations concerning probability consist of the following:

> Ages 5-9: Experiment with concepts of chance.
>
> Ages 10-14: Model situations, and devise and carry out simulations involving probability; construct a sample space and determine the probability of an event; appreciate the power of using a probability model through comparison of experimental results with mathematical expectations; make predictions based on experimental results of mathematical probabilities; develop an appreciation for the pervasive use of probability in the real world.
>
> Ages 15-18: Use experimental probability, theoretical probability, or simulation methods as appropriate, to represent and solve problem situations involving uncertainty; understand and apply properties of probability distributions, including normal, binomial, uniform, and chi-square; and, for college-intending students only, understand the concept of a random variable and apply the concept to generate and interpret probability distributions.
>
> Students at this level should understand the difference between experimental and theoretical probability. Concepts of probability, such as independent and dependent events, and their relationship to compound events and conditional probability should be taught intuitively. Formal definitions and properties should be developed only after a firm conceptual base is established so that students do not indiscriminately apply formulas.

Needless to say, the list is ambitious. In the instructional materials developed by the associated Quantitative Literacy project of the NCTM and the American Statistical Association, the ambitions have reached only as far as the very first part of the ages 15-18 objectives just quoted.

Ideas of probability that are important for everyone to have are represented somewhat differently in the recommendations of Project 2061, Science for All Americans (American Association for the Advancement of Science, 1989). The section titled "Uncertainty" (pp. 104-105) attempts a coherent conceptual story about what probability is about, rather than a particular list of skills. The Project 2061 report distinguishes among three kinds of outcomes: a coherent set of ideas concerning probability, skills in using these ideas, and the inclination to make actual applications of these ideas and skills. The three components are paraphrased liberally below:

> By the end of secondary school (i.e., age 18), students should understand this progression of ideas:
>
> 1. Most events of interest to us are the result of so many complicated causes that we haven't the knowledge or techniques to predict accurately whether they will occur or not. We can, however, still talk intelligently about how likely we believe some outcome of a situation to be.

2. It is often useful to express strength of belief in some outcome numerically, say on a scale of 0 to 1 -- where 0 means we believe there is no possibility of a particular result occurring 1 means that we are completely certain that it will. If our ideas are consistent, then the sum of these probabilities for all possible distinct outcomes of a situation will therefore equal 1.

4. In estimating probabilities for the outcomes of a situation, one useful thing to consider is what has happened in similar situations in the past; if the current situation is like some class of situations in the past, we may expect about the same kind of thing will happen again.

5. Informal estimates of probability based on experience are often strongly influenced by non-scientific aspects of thought: people most readily take into account what is easiest to remember, what fits their prejudices, what seems special about the current circumstances, and so on. We should beware this tendency in others and in ourselves.

6. Another way to inform our judgment is to think about alternatives for what could happen; if we know what all the important alternatives are -- and if we can't see why any one of these alternatives is more likely that any other, we might estimate the probability of any one of them as just what fraction it is of all the possibilities.

7. When a process is believed to be so unpredictable as to be essentially random, it may be useful to simulate the process using random numbers (with random devices or computers), to get an idea of the relative frequencies of different kinds of outcomes.

8. Whenever we are estimating probabilities, we are better off if we don't care about any particular single event, but only about what will happen in groups of similar events. Then we can consider only what will happen on the average for the group.

9. When there are two or more distinct, alternative outcomes of a situation, the probability of one or another of them occurring is the sum of their separate probabilities. Sometimes when an outcome seems surprising it is useful to imagine how many other outcomes would also have seemed surprising -- and consider what the sum of all their probabilities would be. (E.g., the probability is fairly high every year that some small town out of thousands will show a surprisingly high rate of some disease among the dozens on which records are kept.)

10. When there is evidence that two or more events have no causal links between them, the probability of them all occurring can be estimated by the product of their separate probabilities. (E.g., the probability that all the systems will work properly in launching a rocket into space.)

By the end of secondary school, students should have the skills to:

1. Assign relative and numerical likelihoods to outcomes of events and give some rationale, frequentist or classical, for their judgments.

2. Compute relative frequencies from data sets.

3. In familiar contexts, make subjective judgments about the extent of "similarity" in purportedly relevant prior frequencies.

4. Count possible outcomes in simple combinatorial situations.

5. Make realistic estimates of conditional probability in familiar situations.

6. Consider population size as well as probability in judging risks and benefits. (E.g., a probability of only .001 that a medical test will yield false results implies 100,000 mistakes in a population of 100,000,000 people.)

7. Revise strength of belief "appropriately" when given more evidence. (That is, revise in approximately the right direction, not make Bayesian calculations.)

By the end of secondary school, students should be inclined actually to exercise the skills listed above when occasions arise. For example, when the newspaper claims that three home runs in a row are just short of miraculous, or that a new drug is 95% safe.

FORMS OF PROBABILITY CURRICULUM

The school mathematics curriculum is overcrowded. One reason is that topics rarely are dropped from the mathematics curriculum. (A high school mathematics teacher recently implored a mathematics curriculum study group of the American Association for the Advancement of Science, "Please, if you do nothing else, make a strong statement about what we can stop teaching in mathematics.") Another reason for crowding is the tendency to press topics into ever lower grades, so that more serious mathematics can be addressed in the upper grades. And the rapid recent developments in mathematics itself motivate the introduction of additional, modern topics (e.g., computer programing and "discrete" or "finite" mathematics).

Probability is one of the proposed additions to the crowded curriculum. A case can be made that probabilistic problems provide opportunity for learning other concepts (e.g., fractions, proportionality) and practicing computational skills in ways that could be particularly meaningful -- and could therefore be introduced without taking any more time in the long run. But until we know better what it takes to teach probability well, the case is very weak.

On the basis of scope, coherence, and context, there are several approaches to probability curriculum: (1) **ad hoc** suggestions, mostly in the mathematics education literature, for incorporating various games and other activities relating to probability into the regular mathematics classroom (typically not formally connected to one another or to statistics); (2) supplemental units, most notably the Quantitative Literacy materials (Newman, Obremski, and Scheaffer, 1987); (3) integration with traditional math curriculum, such as is done in Real Mathematics (Willoughby, et al., 1987), and the University of Chicago School Mathematics Project (Rubenstein, Schultz, and Flanders, 1987) and the Middle Grades Mathematics Project (Phillips, et al., 1986); (4) integration with other disciplines, seen chiefly in the materials of the Schools Council Project in the U.K. (which has had very limited success in getting non- mathematics teachers to use the materials). Separate courses also appear sporadically in some schools, but the crowding of the math curriculum seems to have minimized that.

It is important to acknowledge that we do not yet know the best (or even a very good) way to teach students probability, and therefore need to continue to look to research when designing or choosing curricula or advocating a particular instructional approach. Far from having a guaranteed place in the curriculum, probability is in competition with many other new mathematical topics to be included at school levels earlier than previously thought possible. Unless probability concepts are taught in contexts of disease transmission, medical testing, accident rates, criminal prosecution, state lottery payoffs, extra-sensory perception, weather prediction, visitors from space, and other "real-world" matters of real interest, putting probability in the curriculum may not be worth the effort. Urns and dice are not enough.

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AN OUTBREAK OF BELIEF IN INDEPENDENCE?

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Results of the most resent administration of NAEP suggest that the majority of secondary students believe in the independence of random events. In the study reported here, a high percentage of high-school and college students answered similar problems correctly. However, about half of the students who appeared to be reasoning normatively on a question concerning the most likely outcome of five flips of a fair coin gave an answer on a follow-up question that was logically inconsistent. It is hypothesized that these students are reasoning according to an "outcome approach" to probability in which they believe they are being asked to predict what <u>will</u> happen. This finding has implications for both test development and curriculum design.

The belief that successive outcomes of a random process are not independent (the so called "gambler's fallacy") is supposedly one of the most common misconceptions about probability. An example of this misconception is the belief that a long run of heads in coin flipping increases the likelihood that the next trial will produce a tails.

One of the possible explanations for the gambler's fallacy is that people reason about such situations according to a "representativeness heuristic" (Kahneman & Tversky, 1972). According to this heuristic, the likelihood that a given sample comes from a particular population is judged on the basis of the degree of similarity between salient features of the sample and the corresponding features of the parent population. After a run of four Hs, and given a choice between the two possible samples HHHHH and HHHHT, the latter is judged as the more likely via the representativeness heuristic since it is closer to the ideal population distribution of 50% Hs.

Use of the representativeness heuristic is often elicited by asking people to choose among possible sequences the most likely to occur. In the case of five flips of a fair coin, all possible ordered sequences are in fact equally likely, the probability of each being $.5^5$. Given a choice among several options, people reasoning according to the representativeness heuristic will chose THHTH as being more likely than THTTT or HTHTH. Kahneman and Tversky (1972) argue that this choice is consistent with the representativeness heuristic in that it reflects both the fact that heads and tails are equally likely and the belief that random series should be "mixed up."

The belief that non-normative expectations such as the gambler's fallacy are widely held has inspired the development of probability and statistics instruction to counter such beliefs. Curriculum designed by Shaughnessy (1977) and Beyth-Maron and Dekel (1983) include units intended to confront and correct judgments based on informal judgment heuristics. However, results on problems involving probability on the most resent administration of the National Assessment of Educational Progress (NAEP) suggest that the majority of secondary students in the United Stated believe in the independence of random events. Asked for the most likely outcome of a fair coin given four successive trials on which the coin landed with tails up, 47% of the 7th graders and 56% of the 11th graders selected the correct alternative. The percentage of responses that were incorrect but consistent with the representativeness heuristic was 38% for the 7th graders and 33% for the 11th graders (Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988). Given that probability is infrequently taught at the secondary level, these data suggest that a concept of independence is more prevalent than non-normative reasoning even prior to formal probability instruction.

In the study reported here, high-school and college students performed even better on similar problems. However, given their inconsistent responses to a follow-up question, it appears that nearly half of the students who answered the problem correctly were reasoning according to a non-normative construct of probability, the "outcome approach" (Konold, 1989; in press).

People who reason according to the outcome approach do not see their goal in uncertainty as specifying probabilities that reflect the distribution of occurrences in a sample, but as predicting results of a single trial in a yes/no fashion. Given the probability of some event, such as "70% chance of rain tomorrow," outcome-oriented individuals adjust the probability value to one of three decision points: 100%, which means "yes," 0%, which means "no," and 50%, which means "I don't know." Thus, the number in the forecast "70% chance of rain" is adjusted up to 100%, after which the forecast is interpreted as "It will rain tomorrow." If it fails to rain, the forecast was "wrong." Given this orientation, a forecast of 50% chance of rain suggests total ignorance on the part of the forecaster about the outcome.

On problems involving coin flipping, the outcome-oriented individual infers from the known probability of "50/50" that there is "no way to know" the outcome of a trial, or series of trials. Although this conclusion appears correct, for the outcome-oriented individual in will be shown to involve a contradiction. When those who reason according to the outcome approach are eliminated from the pool of correct responders in this study, there no longer appears to be an outbreak of belief in independence.

METHOD

Problems and Procedure

This study includes student performance on the following two items:

> Four-heads problem. A fair coin is flipped 4 times, each time landing with heads up. What is the most likely outcome if the coin is flipped a fifth time?
>
> a. Another heads is more likely than a tails.
> b. A tails is more likely than another heads.
> c. The outcomes (heads and tails) are equally likely.
>
> H/T Sequence problem. 1) Which of the following is the most likely result of 5 flips of a fair coin?
>
> a. H H H T T
> b. T H H T H
> c. T H T T T
> d. H T H T H
> e. All four sequences are equally likely.
>
> 2) Which of the above sequences would be least likely to occur?

These two items were included on questionnaires along with other items on probability and statistics. Each item appeared on a separate page, and students were instructed not to return to a page once it had been turned.

Students

Summermath. Both items were administered as part of a nine-item pretest to 16 high-school girls on the first day of a workshop on probability. This workshop was offered in 1987 as part of "Summermath," a six-week residential program sponsored by Mount Holyoke College. Summermath recruits nationwide, and participants represent a range of mathematical ability.

Remedial math. Twenty-five undergraduate students enrolled in the Spring 1987 semester of a remedial-level mathematics course at the University of Massachusetts, Amherst, volunteered to participate in a study on probabilistic reasoning. Probability was not a topic covered in this course. The Four-heads and H/T Sequence problems were among 14 items they completed.

Graduate statistics. Both items were administered as part of a pre-course survey for a graduate-level statistical methods course in the College of Education, University of Minnesota, in the Fall of 1987. This course is the first of a three-semester methods sequence required of all advanced-degree candidates in psychology and education. Dr. Joan Garfield was the instructor, and administered the survey.

RESULTS AND DISCUSSION

Overall, 86% of the students answered the Four-heads problem correctly. Not surprisingly, the performance of the Remedial students was the poorest (70% correct) and the Graduate students the best (96% correct). The most popular alternative answer was the one consistent with the gambler's fallacy, that a tails is more likely after a run of heads. This option was selected by 22% of the Remedial students, 19% of the Summermath students, and 4% of the Graduate students. These results parallel the NAEP results cited earlier and suggest that even without instruction, the majority of students do not commit the gambler's fallacy on this particular problem.

Performance on the H/T Sequence problem is summarized in Table 1. The percentage of students who chose each option as the most likely is listed under the heading "Most." The majority of students (62% overall) correctly chose option e. The alternative f in the table was written in as the correct option by 1 Remedial and 3 Graduate students. They indicated that options a, b, and d were equally likely and that option c was least likely to occur.

Table 1. Performance on H/T Sequence Problem

	Group					
	Remedial		Summermath		Stat Methods	
N=	23	23	16	15	47	41
Sequence	Most	Least	Most	Least	Most	Least
a. HHHTT	17%	9%	0	7%	0	7%
b. THHTH	13%	4%	25%	0	2%	2%
c. THTTT	4%	9%	0	33%	2%	27%
d. HTHTH	0	43%	6%	40%	11%	17%
e. Equal	61%	35%	69%	20%	79%	46%
f. a,b,d	4%				6%	

The most interesting result is the percentage of correct responses to the question of which sequence is least likely. These percentages are listed in Table 1 under the heading "Least." Overall, only 30% of the students responded that all four sequences were equally unlikely. Thus, roughly half of the students who selected the correct option e (all equally likely) for the question regarding the most likely sequence went on to select one of the sequences as least likely rather than respond in a consistent manner that all four sequences were also equally unlikely. This contradiction suggests that even though students may respond correctly to the question of which sequence is most likely, their answers may not be based on normative reasoning.

One hypothesis about why some students respond in contradictory fashion is that these students are reasoning according to the outcome approach. As mentioned in the introduction, outcome-oriented individuals, when asked the probability of some event, interpret the request as one to specify what will happen. In the case of the Four-heads and H/T Sequence problems, they think they are being asked what will happen on the fifth trial, or which sequence will occur, respectively. The 50% probability associated with coin flipping, however, suggests to them that no prediction can be made. Thus they choose the answer "equally likely," and by this they mean they have no basis for making a prediction of what will happen. However, the question in the H/T Sequence problem about which sequence is least likely cannot sensibly be interpreted as, "Which sequence will not occur?" (since none of them may occur). In this case, the outcome oriented individual may switch from a yes/no- to a more probabilistic interpretation of the question and choose the option that they think is least likely.

In addition to choosing an option, Remedial and Summermath students were asked for each problem to "give a brief justification" for their answer. These justifications provide further evidence that a few of the students were reasoning as described above. The responses of four students whose answers to the H/T Sequence problem were inconsistent are given below. Each excerpt is preceded by a code that specifies whether the student was from the Summermath (S) or Remedial (R) group. The answers the students gave for the most likely and the least likely sequences are given in parentheses.

> S15: (e,a). [For e] Anything can happen with probability. The chances of some of the examples are least likely to occur (a,c), but it can happen. [For a] This chance is least likely to occur because they happen the same side in a row.
>
> S16: (e,c). [For e] They all could occur. [For c] Because it is least likely to occur when you have almost a perfect score.
>
> R2: (e,a). It's a chance game. Receiving 3 heads in a row seems unlikely, but could very well occur. No skill is involved, therefore all could likely occur by chance.
>
> R14: (e,d). One never knows which way the coin will drop.

CONCLUSION

These results have some fairly direct implications for curriculum development and testing in probability. The belief that the majority of novices faced with these type of problems will commit the gambler's fallacy has helped to shift the focus in probability instruction away from computational skills towards conceptual development (cf. Garfield & Ahlgren, 1988). This shift has been accompanied by curriculum aimed at the development of concepts such as independence and randomness and the design of items to test for conceptual understanding. Given this focus, problems like those used in this study are likely to become standard fare on course and national exams of mathematical achievement. The results of this study suggest that a sizeable percentage of correct responses to such problems are spurious and reflect an approach to uncertainty that is perhaps more pernicious than the gambler's fallacy. Problems need to be developed that can discriminate individuals who reason according to what has been described here as the outcome approach from those with a normative concept of independence.

The development of probability and statistics curricula for the secondary and even elementary levels has become part of the agenda of current efforts to reform mathematics education in the United States. As mentioned above, one of the directions of new curricula being developed is to help students overcome various of the well-documented misconceptions about probability and statistics. Having students analyze data from actual trials or computer simulations holds some promise of helping them overcome misconceptions based on judgment heuristics -- data that contradict their expectations can lead them to question their beliefs. The outcome orientation, however, may not as easily be challenged through simulations or experiments with objects composed of equally-likely outcomes, because virtually every result would appear to support the expectation that "anything can happen." In fact, the variation in results of replications might serve to strengthen rather than undermine the outcome approach. What may prove effective in the case of the outcome approach is the simulation of phenomena composed of non-equally likely alternatives, especially where one of those alternatives is associated with a very high or low probability. Outcome-oriented individuals predict that events with low probabilities will not occur, and to discover that they do occur, and with about the same relative frequency as their probability, may lead them to alter their belief. As one of the Summermath students observed with some surprise after conducting such simulations, "Even if there is a 1% chance, it could happen!"

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Reaction to Papers on Alternative Conceptions of Probability

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I would like to organize my remarks by pointing out the central issue raised by each paper and then raising some questions for each author.

Konold warns us to look behind the reponses we get from students on multiple choice items. Unless we know the reasons why students choose their answers we cannot infer with any reliability that they actually understand the concept that is being tested, especially when they choose the correct answer. This has major implications for those who construct standardized tests, such as NAEP items, ETS tests, and school district competency tests. Students may choose the correct answers to stochastics items and still harbor deep seated misconceptions.

Questions that I had about Konold's research included:

1) What is the role of language? David Green has done research with adolescents in Great Britain and has found that they have little understanding of terms such as lilely, unlikely, certain, impossible and so forth. What are students' understandings of terms like "least likely" and "most likely?" The different wordings in Konold's research led to inconsistent responses, which he based on the "outcome approach." Does the students' understanding of the language play a further confounding role, adding to the misconceptions based on the outcome approach and on representativeness?

2) Are there task variable effects when sequences of outcomes are involved? In the delMas & Garfield paper, six coins (or babies) were used instead of five. When students are asked to rate the relative likelihoods of outcomes like HTHHHH and HTTHHT, they may be more apt to rely on representativeness, because of the 50-50 ratio. On the other hand in Konold's tasks, students were asked to rate the relative likelihood of sequences like HTHHH and HHTHT. Here, neither outcome fits the 50-50 representativeness ratio, and so "either outcome could happen" as the students might say, thus the outcome approach.

3) Would in-depth structured interviews help to clarify students' thinking on these tasks? Since it is difficult to separate out the students who are relying on representativeness from those who are thinking in terms of single outcomes (outcome approach) from a brief response, an interview methodology is needed to help verify the claims we make about how students are responding to stochastic tasks.

In the paper by delMas and Garfield students were asked to predict outcomes before they used the coin toss software. This is a fundamental issue in both research on and teaching of stochastic concepts. delMas has found that students who are asked to do this type of predicting, evaluation as he would call it, were more successful at overcoming misconceptions of probability and statistics. When students are asked to make a guess before carrying out an experiment or gathering some data, they buy into the task, it becomes their problem. delMas & Garfield's results suggest that we should always ask students to predict, whether we are teaching or researching. The del Mas and Garfield paper prompted me to ask:

1) If the intervention were longer, would probability conceptions have been more stable? The time period here appeared to be just too short. Also, I wonder if it wouldn't be better to try and administer the misconception tasks at the beginning of such a course rather than four weeks into the term. More people appeared to answer the tasks correctly than in prevous research using these tasks, in which the tasks were administered before any instruction. There were potentially confounding interventions in addition to the software intervention.

2) What do we do about the "hard core" students who do not waver from their misconceptions despite our best attempts in instructional intervention? It may be that for some students, even upper secondary and college students, that we must have them carry out physical experiments and simulations with dice, spinners, and random number charts before we put them in a computer software environment. I wonder if the students really make the connection between what the computer is doing quickly, and what they could do themselves, albeit much more slowly.

Ahlgren has appropriately raised the issue of "stochastic literacy" as opposed to a litany of curriculum topics that appear in the NCTM standards document. Clearly, we should all read what AAAS has to say in their 2060 report concerning probability and statistics.

1) Is there room in the curriculum for probability and statistics? I disagree with Ahlgren, I believe there is room in the secondary curriculum for probabililty and statistics. In fact, I believe we would be irresponsible if we did not implement the standards recommendations. A course in probabilty and statistics, or data modelling as some would rather call it, would be a fine alternative to the traditional "general math" hodgepodge that many students are forced to take. There is room in second or third year for all the students who are not in "college track mathematics" to take some stochastics. For those students who are caught in our algebra-geometry-algebra II-precalc/calc upperwardly mobil sequence, stochastics rather than calculus at the fourth year may be better for many of our students. It is also possible to weave substantial probabilty and statistics into our traditional algebra II course, and still cover the main concepts of algebra II. Some teachers have taught algebra II Mon-Wed-Fri, and stochastics on Tues-Thurs, and have managed to finish the Algebra II topics.

You ask what can we cut out, start with throwing out all the emphasis on algebraic manipulations of rational expressions and radical expressions, particularly the latter. How many times have you ever needed to simplify a messy radical expression since you left algebra II?

2) Do we know nothing about how best to teach probabilty and statistics? Again, I have to disagree. We have learned a lot about teaching probability and statistics in the past decade. Start with problems. Ask students to guess. Carry out experiments or simulations to gather data on the problem. Build some discriptive models of the data, emphasing graphing wherever possible. Next, build or use a computer simulation of the problem when possible. And finally, at the end, introduce some theory. No, I think the problem isn't that we dont' know good ways to teach stochastics, it is rather that this knowledge is not widespread. The big problem with implementing the NCTM standards, or the 2060 recommendations, is teacher in-service. Our teachers are not at present prepared to teach these topics, and so we need to mobilize massive stochastic in-service efforts.

Ahlgren has once again pointed out the crucial need for science and mathematics educators to work together, and to help develop "real world" settings for probability and statistics curricula. This is true of all mathematics, not just probability and statistics. Our past has not been filled with very many cooperative efforts. One that comes to mind is USMES, (Unified Science and Mathematics in the Elementary School) which was developed in the early 1970's. This curricula never received its proper day in the sun due to a lack of inservice for teachers and the "back to basics" backlash that hit the schools about that time. But Ahlgren is right in encouraging more cooperative efforts, despite our less than stellar record of past cooperation.

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