DOCUMENT RESUME

ED 069 539 SE 015 361

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TITLE	Binary Arithmetic From Hariot (CA, 1600 A.D.) to the Computer Age.
PUB DATE	72
NOTE	11p.; Paper presented at the International Congress of Mathematical Education Meeting (2nd, August 29-September 2, 1972, Exeter, England)
EDRS PRICE	MF-$0.65 HC-$3.29
DESCRIPTORS	Biographies; *History; Mathematical Enrichment; *Mathematicians; *Mathematics; *Number Concepts; Symbols (Mathematics)
IDENTIFIERS	*Binary Arithmetic

ABSTRACT

This history of binary arithmetic begins with details of Thomas Hariot's contribution and includes specific references to Hariot's manuscripts kept at the British Museum. A binary code developed by Sir Francis Bacon is discussed. Briefly mentioned are contributions to binary arithmetic made by Leibniz, Fontenelle, Gauss, Euler, Benzout, Barlow, DeMorgan, Cantor, Muller, Peano, and Bouton. (DT)

BINARY ARITHMETIC FROM HARIOT (CA. 1600 A.D.) TO THE COMPUTER AGE

by

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presented at

THE SECOND INTERNATIONAL CONGRESS
OF MATHEMATICAL EDUCATION
29th August to 2nd Sept 1972
at The University of Exeter
in Exeter, England

THOMAS HARIOT (1560-1621)

Hariot's work on binary and other *r*-ary arithmetic was done about 1600 A.D.; it was not published and, since it was not to come to light until the computer age was well under way, failed to have any direct influence on the development of this topic. Yet it is worth mentioning--especially in view of the great pride Leibniz was to take 100 years later (in 1703) in publishing "Explication de l'arithmétique binaire." Fontenelle, the editor of the journal involved, not only hailed this arithmetic as *new*, but felt obliged to mention one Thomas Fantet de Lagny (1660-1734), who had discovered binary arithmetic independently, but had sent in *his* paper *after* Leibniz's. Fontenelle's attempt to thus forestall a priority fight between Leibniz and Lagny seems rather futile considering that all three (and seemingly everyone else) had overlooked "Meditatio," a chapter in Bishop Caramuel y Lobkowitz's book *Mathesis biceps* published in Italy in 1670. Here the Bishop had asked: "Is there one arithmetic or are there many?" He concluded there were many--arithmetics of base 2, 3, 4, 5, 6, 7, 8, 9, 12 and 60, for example. He showed number representations in each of these.

Octonaria	Denaria
1	1
2	2
4	4
10	8
20	16
40	32
100	64
...	...
...	...
4 000 000 000	536 870 912

Fig. 1. Portion of table transcribed from BM (British Museum) Hariot MS 6782, Folio No. 1.

While Bishop Caramuel had beaten Leibniz by *33* years in publishing on this topic, the Englishman Hariot, to whom we want to return now, had worked on it in his manuscripts about *100* years before that 1703 article. Figure 1 shows some work with octal number representations from one of the more than 5000 folio-sized pages of Hariot manuscripts. Figure 2 shows some 'base 3' tables, which Hariot had placed amidst some work on binary arithmetic. Other examples of binary arithmetic can be found on portions of MS 6782 f.247, MS 6788 f. 245, and MS 6786 f. 347. The binary arithmetic shown on the last named folio was already brought to light through J.W. Shirley's 1951 article in the Journal of Physics. This included the details of the multiplication of

$$1101101 \times 1101101 = 10111001101001$$

i.e., $$109 \times 109 = 11881.$$

1	1
2	2
10	3
11	4
12	5
20	6
21	7
22	8
100	9

1	1
10	3
100	9
1000	27
10000	81

Fig. 2. Tables transcribed from BM Hariot MS 6786 Folio No. 517. It shows number representations in base 3 and their equivalents in base 10.

As we have seen, the work on this topic is widely scattered among unrelated calculations within Hariot's manuscripts. One finds doodling and repetition. It is doubtful that the work on this or other topics appears in chronological order. It seems that blank portions of some pages were filled in later with unrelated work.

Some of the work scattered elsewhere in his manuscripts may, however, be related directly to Hariot's work on binary arithmetic. The strongest example that I have found appears in Figure 3, where it seems that he was convincing himself that natural numbers may be expressed as the sum of powers of two.

		16	16
1	1	17	16+1
2	2	18	16+2
3	2+1	19	16+2+1
4	4	20	16+4
5	4+1	21	16+4+1
6	4+2	22	16+4+2
7	4+2+1	23	16+4+2+1
8	8	24	16+8
9	8+1	25	16+8+1
10	8+2	26	16+8+2
11	8+2+1	27	16+8+2+1
12	8+4	28	16+8+4
13	8+4+1	29	16+8+4+1
14	8+4+2	30	16+8+4+2
15	8+4+2+1	31	16+8+4+2+1

Fig. 3. Table transcribed from Hariot MS 6783 f.29. (ca.1600 A.D.)

What the British Museum has designated as MS 6783 consists of 426 folios. After folio 29, Hariot seems preoccupied with systematically listing all possible combinations (non-empty subsets) of n things. He would do so for $n = 1, 2, 3, 4$, and 5 and then append a vigorous "etc." as if to say:

> The truth when it is seen
> is known without other evidence.

Indeed, this sentence appears in Hariot's handwriting on MS 6788 f. 132. One of Hariot's techniques of listing combinations is illustrated here for $n=3$:

```
a    ⎫
b    ⎪
c    ⎪
ab   ⎬  7
ac   ⎪
bc   ⎪
abc  ⎭
```

He soon shifted to techniques that used the symbols + and -. Again he would do this for $n=1$ to $n=5$ and append his "etc." In Figure 4 one such technique is illustrated for $n=3$. He seems to imagine that the three columns are headed *a*, *b*, and *c*. A "plus" would mean "yes, this letter is included" and a "minus" that "no, the letter is not included." While this seems to include the empty set, his focus is on the 7 (or in general the 2^n-1) non-empty subsets. His bracket and "7" would so indicate.

```
- - -
- - + ⎫
+ + - ⎪
+ + + ⎪
- + - ⎬  7
- + + ⎪
+ - - ⎪
+ - + ⎭
```

Fig. 4. Transcribed from MS 6783 f.407.

Figure 5 shows a variation of this technique which brought Hariot to a "binary order," albeit in reverse.. If we replace "minus" by "0" and "plus" by "1" we get the eight 3-bit strings from 111 to 000.

```
+ + + ⎫
+ + - ⎪
+ - + ⎪
+ - - ⎬  7
- + + ⎪
- + - ⎪
- - + ⎭
- - -
```

Fig. 5. Transcribed from MS 6783 f. 98

Whether Hariot himself connected his combination listings with his binary arithmetic or even with his table of Fig. 3 is not clear.

Several pages of Hariot's work deal with infinite series and may or may not have been connected with his interest in *r*-ary arithmetic. One example of this

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1 + \frac{1}{1} = \frac{2}{1}$$

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\ldots$$

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + - \ldots = 1 + \frac{1}{9} = \frac{10}{9}$$

Fig. 6. Transcribed from MS 6784 f.428.

is shown in Figure 6. Apparently he was convincing himself that

$$1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \ldots = 1 + \frac{1}{r-1} = \frac{r}{r-1}$$

for any base or radix r. In more compact notation:

$$(1.1111\ldots)_r = \frac{r}{r-1} .$$

We turn next to Sir Francis Bacon, who invented a code with which one could hide secret messages inside of some cover text. The modern reader sees this as a binary code, similar to the alpha-numeric codes used in computers. Bacon's code enjoyed some noteriety for its *misuse* by Baconians who 'discovered' secret messages 'proving' their contention that Bacon had written Shakespeare's plays.

FRANCIS BACON--HIS BILITERAL CODE OF 1623

I	II	III
0	AAAAA	*a*
1	AAAAB	*b*
2	AAABA	*c*
3	AAABB	*d*
4	AABAA	*e*
5	AABAB	*f*
6	AABBA	*g*
7	AABBB	*h*
8	ABAAA	*i*
9	ABAAB	*j*
10	ABABA	*k*
11	ABABB	*l*
12	ABBAA	*m*
13	ABBAB	*n*
14	ABBBA	*o*
15	ABBBB	*p*
16	BAAAA	*q*
17	BAAAB	*r*
18	BAABA	*s*
19	BAABB	*t*
20	BABAA	*u*
21	BABAB	*v*
22	BABBA	*w*
23	BABBB	*x*
24	BBAAA	*y*
25	BBAAB	*z*
26	BBABA	
27	BBABB	
28	BBBAA	
29	BBBAB	
30	BBBBA	
31	BBBBB	

Column II lists the 32 possible 5-letter, biliteral 'words' from AAAAA to BBBBB. Bacon called these 'words' *biliteral* because they contained only the two letters, A and B. The first 26 of these have been assigned to the 26 letters of our modern alphabet. The original Bacon code differed slightly, since *j* and *u* were not part of the 1623 alphabet.

Please note that BAABA BAABB AAAAA BBAAA hides the message *stay*. Also note that the coded version is five times as long as the hidden message, since each character of the hidden message is coded into a 5-letter biliteral word.

The biliteral words in turn can be hidden in any other plain message of equal or greater length. We shall hide BAABA BAABB AAAAA BBAAA in the plain message GO AT ONCE TO WARMINSTER, which has the requisite 20 characters. We shall replace each A by a capital letter and each B by a lower case letter, with the following result:

gO At OnCE to WARMInsTER

To summarize,

the overt message . . . . . gO At OnCE to WARMInsTER

hides the code . . . . BA AB ABAA BB AAAAABBAAA

which in turn hides
the secret message . . . . . S T A Y

Problem No. 1 Find the secret message hidden in the overt message shown at the right.

Go At ONcE TO WaRmiNsTer

Problem No. 2 Find the secret message in:

oGOnTz CAmpUS IS BeaUTIful

If you have been able to do problems 1 and 2, then you understand the Bacon code sufficiently for our purposes. Problems 3 and 4 make greater demands on recognizing differences between closely related typstyles. For problem 3:

typestyle A: *italic*

typestyle B: roman

Problem No. 3 Find the secret message in:

I hate you Jim Hendershot

For problem 4:

typestyle A: abcdefghijklmnopqrstuvwxyz

typestyle B: abcdefghijklmnopqrstuvwxyz

Problem No. 4 Find the secret message in:

Pennsylvania State University

That we are justified in calling Bacon's code a *binary* one becomes clear upon observing that "A" and "B" could have been replaced by "0" and "1". We could have spoken of typestyles 0 and 1. Moreover, when the replacement is made, then column I (see page 6) is simply the decimal equivalent of column II. For the misuse to which this code was put by the Baconians, we refer the reader to Kahn's *The Codebreakers*.

A rough idea of this misuse can be gotten by observing the following: Paper and printing methods were poor in Shakespeare's days. Many letters in his first folio were defective for one of several reasons:

(i) unequal shrinkage of the paper that would make one "o" look smaller than another "o"

(ii) defects in typeface

(iii) poor flow of ink.

The Baconian, hoping to find a short secret message in such a book, would assume that perfect letters were typestyle A and defective ones typestyle B (or vice versa). The whole book could then be viewed as one very long biliteral word or binary string. If he then took sufficient liberties as to where the supposed secret message was hidden and liberties as to whether or not to declare a particular letter defective, then he could find any short message he wanted.

AFTER BACON TO THE COMPUTER AGE

Binary arithmetic entered the mainstream of mathematics with Leibniz's 1703 article and Fontenelle's attendant fanfare. It stimulated only a handful of follow-up articles during the next century--quite contrary to Leibniz's hopes. He had believed that binary arithmetic would prove to be a useful research tool and would help settle such then outstanding questions as the irrationality of the number π.

Nevertheless this handful sufficed to cause Gauss to dismiss *r*-ary (for $r \neq 10$) expressions in the following footnote:

> For brevity we will restrict the following discussion to the system which is commonly called decimal, but it can easily be extended to any other.

This footnote appeared on page 377 of *Disquisitiones Arithmeticae* and referred to paragraph 312, in which the periods of the decimal equivalents of rational fractions were being treated. This was in 1801.

Some of this handful of writers between Leibniz and Gauss deserve special mention. Euler divided $(1+2^7+2^9)$ into $(2^{32}+1)$. He found the result to be the whole number

$$(1+2^7+2^8+2^{10}+2^{11}+2^{12}+2^{13}+2^{17}+2^{18}+2^{21}2^{22})$$

and thus spoiled the Fermat Conjecture which assummed that $(2^{32}+1)$ would be prime. Euler was doing binary arithmetic in expanded notation. In 1764 Etienne Bezout's *Course de mathématique* showed refined techniques for converting from one base to any other base that left no room for further improvement except for special cases such as converting from base 2 to a base that is a power of 2. That this is particularly easy was demonstrated through an example (in a mere footnote) by Legendre in his *Essai sur la théorie des nombre* in 1798. Actually, the insight involved is no more profound than that involved in our readiness to read "1984" as <u>19</u> hundred and <u>84</u>, thus interpreting a base 10 expression as if it were a base 100 expression.

After Gauss, the following deserve special mention. Peter Barlow's 1811 *An Elementary Investigation of the Theory of Numbers* contained an entire chapter "On the different Scales of Notation and their Application to the Solution of Arithmetical Problems." In 1855

Augustus DeMorgan's *The Elements of Arithmetic* appeared. It is probably the earliest school text in the English language to include nondecimal numeration. As DeMorgan was to write later:

> The student should accustom himself to work questions in different systems of numeration, which will give him clearer insight into the nature of arithmetical processes than he could obtain by any other method.

Starting with 1875 the *Zeitschrift für Mathematik und Physik* carried an exchange between Moritz Cantor (the mathematical historian) and a secondary teacher from Berlin, Felix Müller. The former had reported that it was the binary system that was behind certain "Tell your age" cards he had seen demonstrated at a carnival. The latter produced sophisticated variations on these he had for some time been using with his 11th graders in Berlin. These cards had for him been an outgrowth of some difficult algebraic identities he had covered with his students. In 1899 Giuseppe Peano (known to most undergraduate mathematics students for his "Peano Axioms") proposed a new system of stenography that had the binary system as its foundation. Charles L. Bouton (1901) wrote an article on the game NIM, whose complete mathematical theory involved binary arithmetic. Improved and generalized versions of this appeared in a steady stream from other mathematicians during the next 70 years.

Thus, when the computer age began in 1946, binary and other r-ary arithmetic was well known in the mathematical community.

This paper is based on research done for my *History of Binary and Other Nondecimal Numeration* (published in 1971) and further research since that time--especially examination of a full set of Hariot manuscripts during my recent sabbatical leave.
