Tyche: Probabilistic Selection over Encrypted Data for Generative Language Models

Lars Wolfgang Folkerts and Nektarios Georgios Tsoutsos

University of Delaware
`{folkerts, tsoutsos}@udel.edu`

Abstract. Generative AI, a significant technological disruptor in recent years, has impacted domains like augmented reality, coding assistance, and text generation. However, use of these models requires users to trust the model owners with their sensitive data given as input to the model. Fully Homomorphic Encryption (FHE) offers a promising solution, and many earlier works have investigated the use this technology for machine learning as a service (MLaaS) applications. Still, these efforts do not cater to generative models that operate probabilistically, allowing for diverse and creative outputs. In this work, we introduce three novel probabilistic selection algorithms for autoregressive generative AI: multiplication-scaled cumulative sum, heuristic cumulative sum, and the random-multiplication argmax. Each of these approaches presents distinctive challenges in optimizing the trade-off between precision and timing performance, a balance intricately tied to the specific characteristics of the data under consideration. Our results show that the random multiplication argmax-based method is more scalable than the cumulative sum methods and can accurately mimic the plaintext selection curve.

Keywords: Fully Homomorphic Encryption · Private Language Models · Generative AI.

1 Introduction

Over the past decade, generative AI has become a significant disruptor for modern technology, offering automated solutions for content generation and data processing. Generative AI operates probabilistically, employing statistical models to generate new data that loosely resemble a given training dataset. At its core, an autoregressive generative model functions by iteratively generating output tokens one at a time [8]. Given an input prompt, the model generates a set of probabilities corresponding to the next possible output token. It then randomly selects the next token based on these probabilities and appends it to the input for the subsequent iteration. The generative AI model gradually constructs a coherent and realistic output sequence by repeating this iterative process. This inherent probabilistic nature of generative AI is a key factor contributing to its ability to produce diverse and creative outputs.

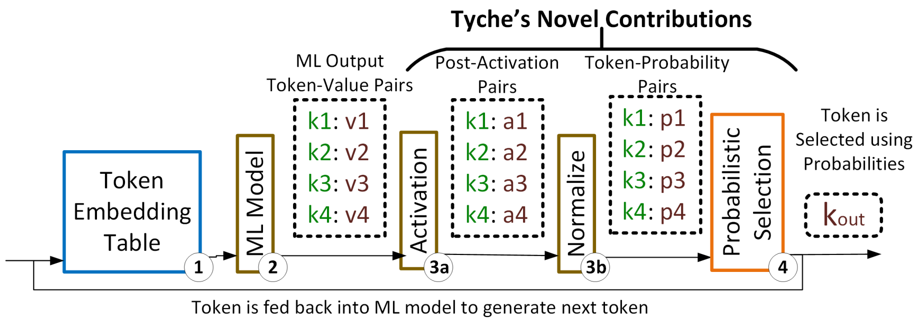


Fig. 1: **Generative Language Models:** In generative models, the process of selecting the most likely next token can be simplified into the four steps shown in this figure. The entire model is run encrypted, but our research focuses the latter two phases: Final Activation/Normalization and Probabilistic Selection. A discussion on how generative models work is presented in Section 2.1, and the associated steps are numbered in this figure.

Nevertheless, the privacy of generative AI systems remains a significant barrier to their widespread adoption. Several prominent companies, including Amazon, Apple, Northrop Grumman, Samsung, Verizon, and various major banks, have banned the use of ChatGPT due to concerns regarding the security and confidentiality of code and customer data [9, 31, 13].

Fully Homomorphic Encryption (FHE) offers a promising solution for preserving privacy in machine learning applications through its ability to perform computations directly on encrypted data. This exceptional capability allows users to securely delegate computations to a cloud service provider by sending encrypted ciphertexts, which prevents the provider from accessing any information about the original plaintext. The cloud performs the computations on the encrypted data and returns the encrypted result to the users, who can ultimately decrypt it to obtain the plaintext outcome.

Interest in building FHE-GPTs have been encouraged by the improvements in FHE performance. Zama AI's Concrete ML library has a working encrypted transformer module. The current demo showcases encrypting a single layer of a GPT-2 model, which is a significant milestone in FHE-ML, and researchers are working to improve on this state-of-the-art contribution to deliver complete, efficient, and cutting-edge private AI models.

This work marks an initial stride towards realizing practical generative machine learning models by assessing multiple techniques for cloud-based encrypted value selection from a set of neural network (NN) outputs. This endeavor tackles two unresolved challenges in the realm of machine learning: (a) the normalization of neural network outputs and (b) the retrieval of probabilistic information. The first challenge entails finding an efficient encrypted counterpart to the softmax function, facilitating the normalization of inputs into probability distributions that sum to 1. The second challenge involves probabilistic information retrieval, where given probabilities p_1 to p_n, $token_1$ is selected p_1 percent of the time, and so forth. Finally, we combine our contributions with several multi-layer perception-based networks to demonstrate the possibility of achieving encrypted generative AI.

Our contributions extend to a diverse set of innovative methodologies aimed at enhancing the probabilistic selection process within encrypted generative machine learning. For each of these techniques, we conduct a comprehensive analysis to evaluate their scalability with respect to algorithmic complexity and the required integer precision, a significant factor influencing the timing performance of FHE. Furthermore, we assess the potential bias introduced from precision limitations and optimizations in FHE. To validate the efficacy of these approaches, we perform empirical evaluations employing text-based generative models that operate on letters. These evaluations encompass measurements of timing and the assessment of loss degradation across various character sets.

2 Background

2.1 Generative AI

At the core of generative ML is the concept of probability. The goal is to model the probability distribution of the training data, allowing the algorithm to generate new samples that are likely to occur in the real world. These models can be used to create realistic images, text, audio, and even entire virtual environments. There are several techniques and architectures used in generative ML, each with its own strengths and limitations. Some of the popular approaches include Variational Autoencoders (VAEs), Generative Adversarial Networks (GANs), and Autoregressive models [8].

In this work, we focus on autoregressive generative inference, which is widely used in various applications, including text generation, image synthesis, handwriting generation, and speech synthesis. It encompasses various approaches such as Google's PixelCNN [24] for images, as well as Bidirectional Encoder Representations from Transformers (BERT) [10] and Generative Pretrained Transformers (GPTs) [26] for large language models. Autoregressive models involve utilizing neural networks to predict the next *token* (such as a pixel, word, or letter) based on previous inputs. The autoregressive approach leverages the sequential nature of the data, allowing the neural network model to capture dependencies and generate coherent and contextually appropriate predictions. The inference server can then select the token based on these probabilities and iterate the process by feeding the selected token back into the model for generating subsequent predictions.

We show an overview of generative language models in Figure 1. There are four major steps:

1. **Token Embedding**: A token embedding table is employed to transform a one-hot vocabulary vector into a lower-dimensional token embedding vector.

2. **ML Model Processing**: In the second step, the data is processed using a machine learning model, which could be a transformer-based architecture or a simpler neural network, depending on the application. Here we assume the output of this step is encoded as a vector v, with each value corresponding to a unique output token k.
3. **Final Activation/Normalization**: The third step involves converting the neural network's output (denoted as v) into an intermediate post-activation value (donated as a) and finally into a normalized probability vector (denoted as p). This is achieved by first applying an activation function, before ensuring that the sum of the outputs equals one, creating a valid probability distribution. In the plaintext domain, softmax, which combines the exponential activation and normalization steps, is typically used for this step, and for FHE-PPML, this step is generally left to the user.
4. **Probabilistic Selection** In the final step, a token, denoted as k_{out} is chosen based on the encrypted probabilities. The selected token can then be fed back into the network for subsequent iterations.

2.2 Fully Homomorphic Encryption (FHE)

Overview. Fully Homomorphic Encryption (FHE) is a class of cryptographic techniques that enables a cloud server to perform computations on encrypted data without the need to decrypt. Lattice-based FHE schemes usually have four key components: key generation, encryption and decryption algorithms, a set of homomorphic operations, and a bootstrapping operation. The key generation process includes the creation of three types of keys: public keys, secret keys, and bootstrapping keys. This key generation step is typically performed by the client and is a one-time computationally intensive setup [18, 5, 15].

During encryption, the public key is used to transform a plaintext message into a many-dimensional ciphertext. In particular, the security of the FHE transformation often relies on hard problems such as LWE [27, 28] and Ring-LWE [20]. Some noise is added to the ciphertext to make it cryptographically hard for an attacker to reverse the plaintext-ciphertext mapping. Only a user with a secret key should be able to decrypt the message.

During homomorphic operations, ciphertexts can be manipulated to preserve the meaningfulness of the underlying plaintext. In particular, homomorphic addition and multiplication directly operate on the encrypted data, generating a new output ciphertext; when decrypted using the secret key, the resulting plaintext corresponds to the desired addition or multiplication computation. This property is what makes the encryption scheme "homomorphic." However, each homomorphic operation contributes to noise accumulation in the resulting ciphertexts. If a large number of operations are performed sequentially, the accumulated noise in the ciphertexts may hinder the successful decryption of the data [5].

The bootstrapping operation offers a noise reduction that mitigates this problem. Here, the bootstrapping key can be viewed by the cloud server as the encryption of the secret key. Therefore, the cloud can perform homomorphic decryption and re-encryption, resulting in a new ciphertext with significantly reduced noise. Bootstrapping can be applied repeatedly to facilitate limitless computational depth. However, bootstrapping is computationally expensive and remains the bottleneck for FHE implementations [5, 6].

TFHE Cryptosystem and Concrete Library. TFHE is a cryptographic scheme that builds upon the foundations of GSW and its ring variants [4]. It has undergone further advancements and enhancements to become the Concrete Library [32]. TFHE operates on either bits or integers and has an efficient bootstrapping process that can be performed on a scale of milliseconds, which allows for deeper neural networks to be implemented. Our work evaluates the proposed probabilistic selection methods using the TFHE scheme.

3 Threat Model

Our work addresses the most prevalent scenario in privacy preserving machine learning, where a cloud service provider possesses a model, and users pay to upload their individual inputs and receive generated results from the cloud. An additional layer of complexity arises from the autoregressive generative model paradigm, which requires the cloud to iteratively return model outputs as inputs to the neural network for subsequent iterations.

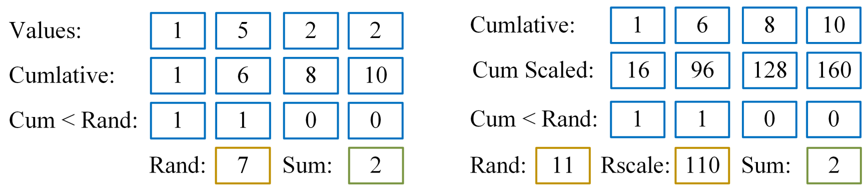
The central focus of our approach revolves around the protection of user data privacy at the outset, as well as the safeguarding of proprietary network characteristics within the cloud, including model weights and biases. To establish a clear threat model, we operate under the assumption of an honest-but-curious cloud provider that faithfully executes operations on encrypted data but has an incentive to eavesdrop on user data; we also defend against external adversaries to attempt to steal the user data through cyberattacks on the server or the network links.

4 Our Proposed Approaches for Cumulative Sum

In this section we introduce two methods for probabilistic selection based on the cumulative sum operation, while our third method based on the argmax operation is discussed on Section 5. For each proposed algorithm, we address the issues of normalization, precision, and bias.

The cumulative sum method enables normalization of output values since there is no efficient division by an encrypted value operation in TFHE. The basic steps of the cumulative sum are as follows, using the example of Fig. 2a:

1. **Activation:** This step assumes all values are non-negative, although not necessarily normalized. Typically, this is done with the softmax function in neural networks, which is a normalized exponential function $y = e^x$. Unfortunately, there is limited support for parallelization in the Concrete Library version 2.5. We therefore use a positive x^2 approximation, $y = max(0, x{+}1)^2$, which maintains the desired non-negative characteristic and can be done with reasonable efficiency.



(a) Cumulative Sum Plaintext Probabilistic Selection

(b) Multiplication Cumulative Sum Method Scaling

Fig. 2: **Cumulative Sum**: To select elements with probabilities proportional to their values, we use cumulative sums. In a plaintext implementation (top), we generate a random number within the range of 0 to the sum of the original vector. In an encrypted domain, where we do not know the sum of the original vector, we employ a multiplication method (bottom) with a fixed value, $Rand_{max}$, to generate a range of random numbers spanning the array.

2. **Cumulative Sum:** This step calculates the cumulative sum of the post-activation neural network output values. While this operation is sequential, addition is very efficient in FHE.
3. **Random Number Generation:** The plaintext version of this algorithm would generate a random number between 0 and the sum of neural network output values, which can be taken from the previous step. However, using FHE, this sum is encrypted, bringing a new challenge on how to approximate the random number generation.
4. **Comparison:** In this step, we run an encrypted comparison of the random number with each index in the cumulative sum array. If the cumulative sum is less than the random number, we return 1; otherwise, 0. Since cumSum is increasing, the 1s are always left justified. This operation is vectorized.
5. **Result:** Finally, we conclude the process by summing the comparison output, yielding the (encrypted) index of the token k_i, which can subsequently be reintroduced into the neural network.
   Alternatively, we obtain a one-hot vector and perform a simple private information retrieval (PIR) for the result. This is achieved by subtracting the less-than-comparison output from itself shifted by one position. To retrieve a single token value, a dot product can be done with the one-hot vector and list of indexes {0, 1,... N-1}, where N is the number of possible tokens, also corresponding to the length of the input x.

In theory, this methodology would provide an unbiased way for encrypted probabilistic selection. However, three main issues prevent this algorithm's success in the encrypted domain. First, some form of **normalization** is needed. In the plaintext algorithm, this manifests in selecting a random number between 0 and the sum of these values, which is already calculated as the last index of the cumulative sum. However, when the value of the sum is encrypted, generating a random number from 0 to this encrypted sum is no longer possible. Therefore, our two proposed cumulative sum methods differ in the way this is achieved. Second, a **bias** may be introduced from approximation, rounding, and precision errors, and for each method, we evaluate this bias. Lastly, FHE computation suffers from **precision** problems, requiring users to choose between accuracy and computational complexity. Therefore, careful tuning is required to obtain the optimal balance of these two constraints.

4.1 Multiplication-Scaled Cumulative Sum

Methodology: The main challenge in the cumulative sum method is generating a random number between 0 and the encrypted cumulative sum array maximum, which we denote as $R_{ideal} = rand(0, cumSum_{max})$. In the multiplication-scaled method, we first select a fixed range for the random number, where $R_{fixed} = rand(0, Rand_{max})$. In the encrypted domain, we can then enforce the scale to be the product of the maximum possible random value and the maximum value of the array, $Cumsum_{max} \cdot Rand_{max}$. This is done by multiplying each value of the cumsum array by $Rand_{max}$, and by multiplying our random number by $R_{scaled} = R_{fixed} \cdot Cumsum_{max}$. We show this method in Algorithm 1.

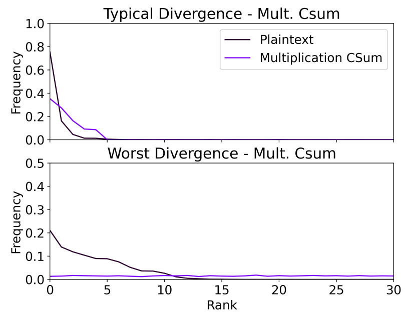
Normalization: Normalization is achieved by adjusting the scale as described above. This normalizes the output and random number to $Cumsum_{max} \cdot Rand_{max}$.

Bias: In a plaintext version of this algorithm with floating point precision or significantly large values of $Rand_{max}$, there is little bias since the cumulative sum operates a normalization alternative. However, in the encrypted domain with a small, low precision value of $Rand_{max}$, the heavy discretization of the random variables introduces a bias.

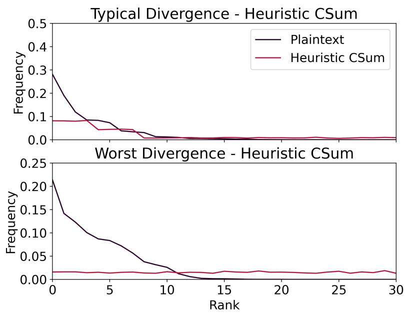
Figure 2b illustrates an example of such bias. The possible values for R_{scaled} are {0, 10, 20, 30...150}. This leads to probabilities of $\{\frac{2}{16}, \frac{8}{16}, \frac{3}{16}, \frac{3}{16}\}$, which are slightly skewed from the original $\{\frac{1}{10}, \frac{5}{10}, \frac{2}{10}, \frac{2}{10}\}$. Such patterns in the data will cause consistent skew toward certain indexes, and will always increase the likelihood of the first index. To prevent these patterns in the data, we propose shuffling the order of the array before applying the cumulative sum. Then, to unshuffle and obtain the result, the PIR approach can be modified by replacing the sequential indexes in the *arrange* assignment in Algorithm 1 with the non-sequential permutation indexes of the cumSum shuffling. This PIR method runs in $\mathcal{O}(n)$ time.

Finally, a natural smoothing effect comes from a combination of rounding and our approximate activation function, $y = max(0, x + 1)^2$, which unlike e^x, sets large negative values to 0. This creates a long tail effect that does not scale for large datasets, as each 0 value has a smoothing value of $1/(N \cdot Rand_{max})$.

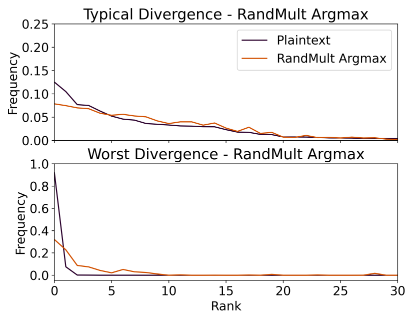
Precision and Complexity: This method has $\mathcal{O}(n)$ multiplications, $\mathcal{O}(n)$ additions and $\mathcal{O}(n)$ comparisons, where n is the vocabulary size (i.e., the number of possible tokens). However, the cumulative sum operation requires high precision to begin with, and multiplying by $Rand_{max}$ only increases this need. Furthermore, $Rand_{max}$ cannot be a static value. To prevent bias, $Rand_{max}$ should increase with cumulative sum. The precision, representing the number of bits



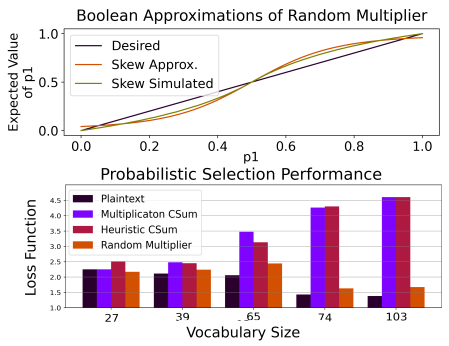
(a) Multiplication Cumulative Sum Divergence



(b) Heuristic Cumulative Sum Divergence



(c) Random Multiplier Argmax Divergence



(d) Randmult Approximation and loss scores

Fig. 3: **Divergence from Plaintext:** This figure highlights the constraints imposed by our approximation techniques and precision errors. In the case of cumulative sums, precision accumulates, and the enforcement of low precision results in only several discretized output values. This can also lead to scenarios where the output vectors a_i are rounded to all zeros, which the algorithm interprets as uniform random selection. On the other hand, the RandMult method exhibits a greater capacity to accommodate higher precision levels and circumvents the biases stemming from random number generation. In the bottom of Figure 3d, we can see the differences affect loss for the three algorithms. Due to the long tail caused by low precision rounding errors, the cumulative sum has an increasing loss as the vocabulary size grows. Conversely, the random multiplier preserves the low model loss as the vocabulary size grows.

required to represent the largest number, is $\mathcal{O}(\log(Rand_{max}) + \log(n))$ bits, so the evaluation times using FHE are less scalable to larger datasets.

Experimental Characterization: In Figure 3a, we present two examples of probability distributions for the encrypted cumulative sum multiplication-scaled method vs. its plaintext counterparts. For the plaintext case, we sorted the output tokens by their selection frequency and compared them with the cumulative sum equivalent distribution for the same ordering. In the average case, we observe that our model could not achieve a very high selection probability; specifically, we observe a shelf (around 0.16) indicating discretization (i.e., ML outputs rounded to the same value). For the worst case, all neural network outputs are discretized to 0, creating a uniform distribution instead of the desired distribution.

Both of these test cases could potentially be enhanced as the Concrete library's proficiency in handling higher-precision inputs improves. Nevertheless, it's crucial to note that this approach exhibits a substantial appetite for precision, making it less favorable in comparison to other algorithms that demand fewer precision-intensive resources.

4.2 Heuristic Cumulative-Sum

Methodology: This method avoids scaling the random multiplier but instead approximates $Rand_{max}$ based on data set heuristics. We assume that the dataset is randomly shuffled, as described in the multiplication-scaled cumulative sum method. The full algorithm is shown in Algorithm 2.

Normalization: Our pseudo-norm is based on heuristics and is not exactly normalized. If the $Rand_{max}$ constant is too small in this method, it will only select the few values at the front of the shuffled array. If the $Rand_{max}$ constant is too large, then the system is biased heavily toward the last single element in the shuffled array. Since selecting a $Rand_{max}$ constant that is too small spreads the bias out on multiple values instead of a single value, it is best to choose a $Rand_{max}$ constant closer to the minimum. In our methodology, the $Rand_{max}$ constant is the cumulative sum value in our dataset's 10^{th} percentile.

Bias: The introduced bias depends on the standard deviation of the sum of post-activation ML outputs in the dataset. With higher standard deviations, the $Rand_{max}$ constant deviates further from the cumulative sum output, causing larger biases. Unfortunately, this was the typical case for our target neural networks, so that, combined with the low precision, it set some test set cases to all 0s. Moreover, this approach still incurs smoothing and long tail biases for small values of $Rand_{max}$, which causes increased bias for larger dataset sizes.

Precision and Complexity: This algorithm uses $\mathcal{O}(n)$ additions and $\mathcal{O}(n)$ comparisons. Here, the maximum precision required is reduced due to the lack of multiplication-scaling but is still dependent on the sum of neural network outputs, which is $\mathcal{O}(\log(n))$ bits.

Experimental Characterization: As shown in Figure 3b, this method still encounters many of the same issues reported for the multiplication-scaled cumulative sum method. Moreover, precision is impacted as the vocabulary size grew; however, there is still room to increase precision at the cost of slowing down execution time. Our analysis shows that the output of the ML model had high variability, which caused outputs where the entire vector was discretized to 0s.

5 Our Proposed Approach for Argmax

While argmax can be useful for ML tasks like classification, its deterministic nature is less useful for generative AI. Nevertheless, if some randomness can be added to the output, achieving the desired stochastic result is possible. This is the core idea behind our proposed argmax method, as we also aim to lower the precision required. Towards that end, our methodology leverages the argmax tournament method, which can be interpreted as a derivative of batcher sort.

In our approach, we attempted to use a batcher-sort network to achieve an argkmax function. Here, pairs of elements are compared and swapped until the max and argmax are found. This has computational complexity $\mathcal{O}(n)$, but can be vectorized to run in $\mathcal{O}(log(n))$. This argmax is the same as the works discussed, but we implemented ours in the Concrete library instead of TFHE-rust, a lower-level version. Unfortunately, implementing a full batcher-sort argkmax using Concrete causes all loops to unroll and be evaluated during compilation time. This significantly impacts the compilation time for any algorithm that uses double loops, which limits the scalability of this approach despite its potential runtime speedups. Given this challenge, we utilized Concrete's built-in maximum function to achieve a speedup.

Methodology: In the random multiplication method, multiplying the post-activation ML outputs with a random vector can be used as the input to the argmax function. This will generate a different answer every time, although it produces some skew for probabilistic selection for the plaintext variant. This method is shown in Algorithm 3.

Normalization: The normalization constraint is mitigated when using argmax methodology. Specifically, we no longer need to normalize the outputs when only seeking the maximum.

Bias: Unlike the cumulative sum generator, this approach introduces a distortion that sharpens the probability distribution. More concretely, suppose we multiply encrypted output distributions $\{p_1, p_2\}$ by uniform random variables $\{\hat{X}_1, \hat{X}_2\}$, respectively. It is important to note that in this context, the variables p_1 and p_2 represent "probabilities" but do not necessarily need to be normalized to one. We wish to find the probability that $p_1 \cdot \hat{X}_1$ is larger than $p_2 \cdot \hat{X}_2$, represented by

$$\mathcal{P}(D_{1,2} > 0) = \mathcal{P}(p_1 \cdot \hat{X}_1 - p_2 \cdot \hat{X}_2 > 0), \tag{1}$$

where $D_{1,2} = (p_1 \cdot \hat{X}_1 - p_2 \cdot \hat{X}_2)$. To calculate this probability, we can first find the expected value

$$\mathbb{E}(D_{1,2}) = \int_0^1 (p_1 \cdot \hat{X}_1) d\hat{X}_1 - \int_0^1 (p_2 \cdot \hat{X}_2) d\hat{X}_2 = \frac{p_1 - p_2}{2} \tag{2}$$

and variance

$$\mathbb{V}(D_{1,2}) = \int_0^1 \int_0^1 (p_1 \cdot \hat{X}_1 - p_2 \cdot \hat{X}_2)^2 d\hat{X}_1 d\hat{X}_2 = \frac{2 \cdot p_1^2 + 2 \cdot p_2^2 - 3 \cdot p_1 \cdot p_2}{6}. \tag{3}$$

With the mean and variance, we can use the normal cumulative distribution function (cdf) $\mathcal{N}\left(\frac{0-\mathbb{E}(D_{1,2})}{\mathbb{V}(D_{1,2})}\right)$ to find the probabilities for different values of p_1.

This gets harder to extrapolate with more variables since the probabilities are not independent, and several constraints between pairs of variables need to be met (i.e., $p_1 > p_2$, $p_2 > p_3$, $p_3 > p_1$ is a contradiction). During algorithmic development, we utilize the bayesian form where contradicted states are removed (the denominator does not sum to 1 when $N > 2$). Thus, the probability token k_1 is selected given probability p_1 is:

$$\frac{\Pi_{i=1}^{N}\mathcal{N}\left(\frac{0-\mathbb{E}(D_{1,i})}{\mathbb{V}(D_{1,i})}\right)}{\sum_{j=1}^{N}\Pi_{i=1,i\neq j}^{N}\mathcal{N}\left(\frac{0-\mathbb{E}(D_{j,i})}{\mathbb{V}(D_{j,i})}\right)}. \tag{4}$$

To illustrate this skew, the top of Figure 3d shows the boolean case, defined as p_1 and $p_2 = 1 - p_1$. The figure shows both the ideal and the distorted probabilities. A slight distortion makes these variables sharper; however, this works in our favor as it reintroduces a similar softmax "S" curve lost through FHE-friendly activation approximation. *This is a surprising result that helps our RandMax algorithm to cancel out the bias introduced through the activation function approximation and achieve a result close to plaintext evaluation.*

Precision and Complexity: The computational complexity of argmax involves $\mathcal{O}(n)$ comparisons. The multiplication step adds $\mathcal{O}(n)$ multiplications. The highest value is the maximum value of the neural network output times $Rand_{max}$ bits. Therefore, the precision is $\mathcal{O}(log(Rand_{max}) + log(max(v))$ bits, as no values are accumulated in the argmax computation. This is much more scalable than the cumulative sum methods, since $max(v) << sum(v)$, and this bitsize does not grow with the vocabulary size.

Experimental Characterization: This method performs significantly better on larger datasets, which is attributed to the lack of precision bitsize growth; this allows our methodology to achieve a higher precision overall. Even with this method's sharpening bias, the model may still not be able to meet the steep probabilities expected from the outputs, as seen in Figure 3c. However, the argmax curve approximates the expected distribution more accurately than cumulative sum methods and achieves a more accurate result with FHE.

6 Experimental Evaluation

6.1 Description of our Datasets

Our generative AI focused on generating letter tokens. This enables us to implement a small encrypted multilayer perceptron (MLP) network, illustrated in Figure 4. This network architecture was kept consistent across all of our datasets and tests to ensure a fair comparison. The network was run encrypted using TFHE, and the results were utilized for our probabilistic selection experiments. Our experiments comprise five different datasets, characterized below:

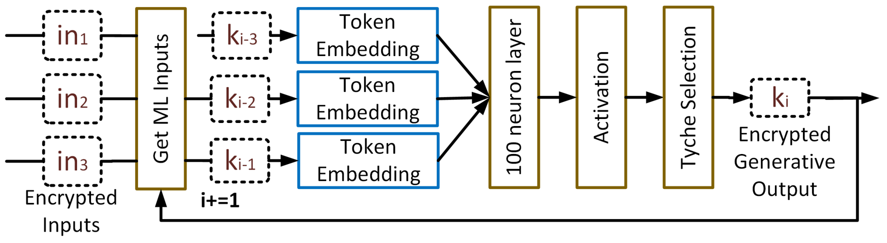


Fig. 4: **Our neural network architechture:** We use a simple mult-layer perception (MLP) model with an embedding table of size 10 and a hidden layer consisting of 100 hidden neurons. We use 3-ngram characters as inputs and the output is the number of possible tokens (i.e., vocabulary size). This output is fed through our probabilistic selection methodology to select an encrypted output.

SSA Names Dataset [17]: This dataset comprises 32K of the most common names taken from ssa.gov for the year 2018. It contains very short phrases (a single name), and has much less training data and higher entropy than our other datasets. This leads to a higher loss score. There are 27 tokens in this dataset, one for each letter of the English alphabet (all lowercase) and one stop character.

Shakespeare Dataset [16] (Lowercase): This dataset consists of all of the works of Shakespeare. We modify this dataset to turn all letters to lowercase. In total, there are 39 tokens in this dataset including the 26 letters of the English alphabet and miscellaneous punctuation.

Shakespeare Dataset [16] (All Case): This dataset is the same as above but regular uppercase and lowercase letters are used for 65 tokens in total. Uppercase letters have a higher level of predictability, leading to lower entropy in this dataset than its lowercase counterpart.

German Parliament (Lowercase): We created this dataset to test the scalability of our algorithms. It consists of proceedings of the German parliament, in the German language. Numbers, umlauts, and characters used in formal government writing such as "§" and parenthesis expand this lowercase dataset to 74 characters. The German language is more predictable than English for length 3 n-grams [30], which leads to more predictable results and less of a long-tail effect than the other datasets.

German Parliament (All Case): This dataset is the same as above, but with 29 extra uppercase characters, bringing the dataset size to 103 tokens.

Synthetic Datasets: In our results, the timing varied based on the magnitude and precision of the neural network outputs, which is related to dataset entropy. We, therefore, created a synthetic dataset to understand how vocabulary size impacts timing performance. We used the names dataset as a baseline for the synthetic dataset but concatenated multiples of outputs to get different size vectors. These synthetic datasets contain 27, 54, 81, and 108 tokens.

6.2 Concrete Library for TFHE

For our experiments, we utilized the Concrete library, which implements the TFHE scheme. While Concrete offers a mature interface and compiler, the current version has two major limitations related to parallelism. First, the library only supports vector-level CPU parallelization using a curated list of numpy primitives. Outside of these numpy library functions, there is little parallelization support, and calculations are limited to one CPU core. This includes for-loops and homomorphic operations not being parallelized, even though there is plenty of opportunity to do so and has been done in prior work [7, 23, 11].

Second, the Concrete compiler does not scale well when using for-loops, which must be unrolled and evaluated sequentially. This means the Concrete implementation of batchersort was unable to support larger vectors and we needed to go with a slower algorithm instead. Consequently, our results are focused on argmax instead of argkmax, as evaluation of argkmax was not feasible.

6.3 Hardware Platform for Evaluations

For our experimental evaluation, we use a 12th Gen Intel(R) Core(TM) i9-12900K 24-core server. To ensure our results are comparable across datasets, we calculate the amortized cost across 10 iterations run in parallel. This allows small vocabulary-size datasets to utilize all 24 cores during vector operations, ensuring a fair comparison, but only 10 cores were used in the sequential parts of our algorithms.

6.4 Latency Performance

Runtime performance, as summarized in Table 1, using FHE depends on algorithm complexity and required precision. In this case, three different factors need to be taken into consideration for our experiments.

Precision and Dataset Characteristics: The output characteristics of the dataset play a key role in runtime performance. In particular, Concrete tunes the TFHE parameters based on the worst-case precision in the plaintext test cases, and uses these parameters for both evaluation and encryption. For the German dataset specifically, the sum of the outputs was significantly lower, dropping the required precision for the cumulative sum operation from 7-bits to 5-bits. This caused a dramatic decrease in runtime for the multiplication cumulative sum method, whose biggest limitation was precision constraints. The cumulative sum heuristic also benefited from this characteristic, causing a decrease in latency.

Computational Complexity: As the vocabulary size grows, the cumulative sum methods are expected to grow faster than the argmax method due to the inherent computational and precision-related complexity. However, the cumulative sum methods are more influenced by the output characteristics of the dataset, particularly the largest sum of the post-activation ML output vector across test cases. Randmax growth is much more predictably since it does

Table 1: **Timing Results (seconds)**: This table summarizes the timing of probablistic selection for the three techniques. The Cumulative Sum (labeled CSum) methods scale more rapidly, and seem very dependent on the worst-case CumSum magnitude (labeled Max CSum). The RandMult method, albeit longer, scales much more linearly and predictably. We show some basic statistics about the datasets including the vocabulary size, the maximum cumlative sum in our test set, the post-softmax entropy, and the post-softmax KL divergence with a uniform random vector.

					Timing Results		
	Vocab	Max	Avg		Multiplication	Heuristic	RandMult
Dataset	Size	CSum	Entropy	KLDiv	CSum	CSum	Argmax
Names	27	41	2.21	1.08	136s	4.0s	191s
Lower Shakespeare	39	84	2.1	1.57	265s	21.8s	224s
Shakespeare	65	78	1.88	2.28	379s	107s	348s
Lower German	74	19	1.69	2.61	228s	119s	386s
German	103	10	1.72	2.91	294s	151s	400s
Synthetic A	27	41	2.21	1.08	136s	4.0s	191s
Synthetic B	54	10	2.78	1.21	256s	64s	365s
Synthetic C	81	10	3.10	1.27	509s	92s	546s
Synthetic D	108	10	3.37	1.31	n/a[1]	124s	721s

[1] Did not fit inside the maximum precision bounds of the Concrete framework.

not rely on this dependency. Still, looking at how size influences runtime performance, the cumulative sum methods start out really efficient and grow to increased runtime overheads. Therefore, the randmax algorithm is a better fit as dataset sizes grow, and for higher entropy datasets that have many high-valued outputs.

Parallelism: The cumulative sum methods have the best parallelism; only FHE-friendly additions are sequential. Conversely, the random multiplication argmax approach requires comparisons to be run sequentially or using a tournament-based method. With the current implementation of the Concrete library, which only performs well on vectorized operations, the random multiplication method has the lowest CPU resource utilization among our results. Improvements to the library, such as a built-in argmax function and homomorphic level parallelism, would further reduce the timing overhead of our methods.

6.5 Model Performance

Finally, we consider the tradeoff in model performance in Figure 3d. The cumulative sum methods showed increasing loss with the dataset size, due to the smoothing effect creating a long tail. Thus, as the dataset grows, more precision is needed to distinguish between probable and improbable values. There was also a higher occurrence of all zero values with the German datasets, resulting from the dataset characteristics and vector size.

The random multiplier argmax method does not suffer from this long tail limitation and can offer higher precision that does not grow with the vector size, resulting in better performance from larger datasets. In addition, the random multiplier dataset is able to support higher input precision, since the max precision is capped to $Rand_{max} \cdot \max(\text{activation}(x))$, unlike the cumulative sum methods that can grow and have a theoretical upper bound equal to the vocabulary size n.

7 Discussion of Related Works

To the best of our knowledge, this is the first work to perform probabilistic selection for language models. To give additional context to our approach, we look at two categories of related works. The first is encrypted language models, which gives some context into the orthogonal work and latencies of the upstream ML algorithms. The second category of related work is enhanced LLMs, which primarily use forms of obfuscation to hide query data.

7.1 Encrypted Language Models

Zama, the designers of the Concrete library, have developed two text-based models. The first one involves sentiment analysis classification [22]. Since this is a classification problem, they do not invoke probabilistic selection, unlike our work, and they use a simple XGBoost to classify the data. They attempt two methodologies for *unencrypted* text preprocessing, one based on term frequency-inverse document frequency (tfidf) and a second using the RoBERTa transformer excluding the final layer.

Zama has also developed a transformer model [21]. Their first implementation is a single transformer block with a single-head GPT2 variant, where layers 2 through 11, over 90% of their model, are run in plaintext. Their second implementation is a multi-head variant with 12 attention heads. This is still implemented as a single layer and with lower precision. The embedding table, layer normalization and probabilistic selection of the words is also performed in plaintext. This is in contrast to our techniques, which allow running every part of the generative AI encrypted.

THE-X is another work that attempts to build a homomorphic transformer, but they remove much of the homomorphic work from the server, including activations, and instead ask the user perform these in the plaintext domain. This defeats the purpose of using homomorphic encryption; here an approach such as multi-party computation (MPC), which allows for multiple computing parties, would be a better choice [3].

7.2 Privacy of Cloud Language Models

There are several techniques that can be used to assist model privacy. The first is differential privacy, which consists of obfuscating the inputs of an input query.

For word tokens, this involves processing the text by replacing synonyms and redacting any sensitive information [2, 29, 19]. However, this technique can harm accuracy, and while the users' direct text may be altered, an attacker can still distill the meaning behind the original query [1].

A second approach entails projecting the inputs into a related subspace. A common technique includes sending a compressed token embedding instead of raw words, which is a lower-dimensional version of the inputs [25]. This work also recommends obfuscation by the rounding of plaintext floating point numbers (which TFHE is very good at in the encrypted domain). These techniques may make the input unreadable to a human attacker, but the meaning of these inputs can still be extracted.

8 Future Work

This work utilizes the new Concrete THFE-based library, which is still rapidly evolving. There are several Concrete features that are still under development that would be of great interest to this work.

The first is an efficient implementation of the maximum, argmax and argkmax functions. In our methodology, we opted to not implement state-of-the-art argkmax techniques due to inefficiencies in the Concrete compiler that led to days-long compile times. These techniques have been proven in THFE-rust, but have not yet migrated to Concrete.

Second, each TFHE operation, such as a single addition, is a complex lattice-based computation that can be parallelized. There are many TFHE research works that accelerate homomorphic operations, such as cuFHE [7], nuFHE [23], REDcuFHE [11], and ArctyrEX [12] that parallelize the lower level operations of TFHE on GPUs, but Concrete does not yet offer this functionality. There are also several works of TFHE hardware accelerators, with MATCHA [14] being the most recent work in the FPGA/ASIC space. We look forward to obtain a greater speedup and utilize more CPU bandwidth through this greater parallelization effort.

9 Concluding Remarks

In this work, we introduce a novel methodology for encrypted probabilistic selection in TFHE. Our approach, we compare two possible methods, the cumulative sum and the argmax, and provide three algorithms for probabilistic selection optimized for generative language models over encrypted data. Our findings show that the argmax-based random multiplication method outperforms the cumulative sum methods in terms of loss stability and precision required, despite offering some bias in the plaintext domain. This result opens new applications into private generative ML, and complements much of the existing work on adapting ML algorithms for TFHE. With many new TFHE developments on the horizon, this work is among the first address one of the main challenges toward encrypted generative AI.

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A Algorithms

Algorithm 1 Multiplication Scaled Cumulative Sum

```
Inputs:
    x: array of pre-softmax output of NN predicting next token
    R_fixed: scalar random number between 0 and $Rand_{max}$
Outputs
    result: integer corresponding to selected next token

  $x = activation(x)$
  $N = len(x)$
  $cumSum_0 = x_0$
  for $i \leftarrow 1, \dots, N-1$ do
      $cumSum_i \leftarrow x_i + cumSum_{i-1}$          ▷ Perform Cumsum
  end for
          ▷ Adjust random number for normalization. Note cumSum is always
            increasing i.e. $cumSum_{max}$ is equivalent to $cumSum_{N-1}$
  $R_{scaled} \leftarrow cumSum_{max} \cdot R_{fixed}$
          ▷ Scale and compare cumSum vector to scalar random.
  $cumSumScaled = cumSum * Rand_{max}$
  $lessThan \leftarrow cumSumScaled < R_{scaled}$
          ▷ Get one hot result index by subtracting lessThan by lessThan shifted one
  $oneHot \leftarrow lessThan_{1::N-1} - lessThan_{0:N-2}$
          ▷ Recover lost index in the case $lessThan$ is all 0s
  $oneHot \leftarrow concatenate(1 - lessThan_0, oneHot)$
          ▷ Can use dot product to recover the index
  $arrange \leftarrow 0, \dots, N-1$
  $result \leftarrow arrange \cdot oneHot$
```

Algorithm 2 Heuristic Cumulative Sum

```
Inputs:
    x: array of pre-softmax output of NN predicting next token
    R_heuristic: scalar random number between 0 and heuristic $cumSum_{heuristicMax}$,
which is determined by the training data
Outputs
    result: integer corresponding to selected next token

  $x = activation(x)$
  $N = len(x)$
  $cumSum_0 = x_0$
  for $i \leftarrow 1, \ldots, N-1$ do
      $cumSum_i \leftarrow x_i + cumSum_{i-1}$                        ▷ Perform Cumsum
  end for
                          ▷ No scaling; $R_{heuristic}$ depends on $cumSum_{heuristicMax}$
  $R_{scaled} = R_{heuristic}$
  $cumSumScaled = cumSum$
  $lessThan \leftarrow cumSumScaled < R_{scaled}$
            ▷ Get one hot result index by subtracting lessThan by lessThan shifted one
  $oneHot \leftarrow lessThan_{1::N-1} - lessThan_{0:N-2}$
  $lessThan \leftarrow cumSumScaled < R_{scaled}$
                          ▷ Recover lost index in the case $lessThan$ is all 0s
  $oneHot \leftarrow concatenate(1 - lessThan_0, oneHot)$
                          ▷ Can use dot product to recover the index
  $arrange \leftarrow 0, \ldots, N-1$
  $result \leftarrow arrange \cdot oneHot$
```

Algorithm 3 Random Multiplication Argmax

```
Inputs:
    x: array of pre-softmax output of NN predicting next token
    Rarray: array of random numbers between 0 and $Rand_{max}$
Outputs
    result: integer corresponding to selected next token

  $x = activation(x)$
  $N = len(x)$
                          ▷ Element wise multiplication with random array
  $x = x * R_{array}$
  $valueMax \leftarrow x_0$
  $argmax \leftarrow 0$
  for $i \leftarrow 1, \ldots, N-1$ do
                          ▷ Find the argument maximum
      $argmax \leftarrow (x_i > valueMax)\ ?\ i : argmax$
                          ▷ Track the value maximum
      $valueMax \leftarrow (x_i > valueMax)\ ?\ x_i : valueMax$
  end for
  $result \leftarrow argmax$
```