Impossibility of Indifferentiable Iterated Blockciphers from 3 or Less Primitive Calls*

Chun Guo[1,2,3](✉), Lei Wang[4](✉), and Dongdai Lin[5](✉)

[1] School of Cyber Science and Technology, Shandong University, Qingdao, China
[2] Key Laboratory of Cryptologic Technology and Information Security of Ministry of Education, Shandong University, Qingdao, Shandong, 266237, China
[3] Shandong Research Institute of Industrial Technology, Jinan, Shandong, 250102, China
[4] Shanghai Jiaotong University
[5] Institute of Information Engineering, Chinese Academy of Sciences, China

Abstract. Virtually all modern blockciphers are iterated. In this paper, we ask: to construct a secure iterated blockcipher "non-trivially", how many calls to random functions and permutations are necessary?
When security means *indistinguishability from a random permutation*, optimality is achieved by the Even-Mansour scheme using 1 call to a public permutation. We seek for the arguably strongest security *indifferentiability from an ideal cipher*, a notion introduced by Maurer et al. (TCC 2004) and popularized by Coron et al. (JoC, 2014).
We provide the first generic negative result/lower bounds: when the key is not too short, no iterated blockcipher making 3 calls is (statistically) indifferentiable. This proves optimality for a 4-call positive result of Guo et al. (Eprint 2016). Furthermore, using 1 or 2 calls, even indifferentiable iterated blockciphers with polynomial keyspace are impossible.
To prove this, we develop an abstraction of idealized iterated blockciphers and establish various basic properties, and apply Extremal Graph Theory results to prove the existence of certain (generalized) non-random properties such as the boomerang and yoyo.

Keywords: Blockcipher, ideal cipher, indifferentiability, lower bounds

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1 Introduction

Iterated blockciphers. Virtually all modern blockciphers, e.g., DES, AES, PRESENT, Skinny, are designed via iteration [2]. These even include theoretical constructions such as the Luby-Rackoff [40], Iterated Even-Mansour (IEM) ciphers [23,11,1,30] and others [21,29]. In fact, the initialization algorithms of some stream ciphers [51] also follow the iteration paradigm.

The idea of iteration dates back to Shannon [48] or even earlier practice of product ciphers. In general, an iterated structure creates a (usually weak) keyed permutation, typically called its *round*, in a "non-trivial" manner, and then composes such rounds till gaining enough security. By "non-trivial", the round has to employ smart ideas to resolve non-invertibility of functions [40] or combine keys with keyless permutations [23,11,1]. Such constructs also constitute natural transformations between (pseudo)random functions and (pseudo)random permutations [40,23,1,16,21], which are fundamental in modern cryptography.

While provably secure blockciphers remain out of reach, there is a definite belief that with sufficient iterations, the iterated paradigm does yield enough security. The primary security notion for a blockcipher is *indistinguishability from a random permutation*, i.e., no adversary with bounded oracle queries and black-box access to a permutation can distinguish whether it is interacting with the blockcipher under a random key or a perfectly random permutation. This has probably been the most widely used security assumption for blockciphers. In fact, with certain idealized assumptions and sufficient iterations, the aforementioned Luby-Rackoff [40], IEM [23,11,30] and Swap-Or-Not [29] have been proven indistinguishable (and bounds usually increase with rounds).

The ideal cipher model. Albeit the de-facto standard, indistinguishability is insufficient for a number of important blockcipher-based cryptosystems. For example, some real-world protocols such as f8 and f9 [33] crucially rely on the stronger related-key security of blockciphers [6]. Even worse, in blockcipher-based hash functions [10,9], the adversary can control both the message and the key of the blockcipher and exploit "known-key" or "chosen-key" attacks [37,8] to break collision- or preimage-resistance of the hash. In fact, a mere PRP cannot yield black-box construction of collision resistant hash [49].

Hence, cryptographers have modeled a reliable (κ, n)-blockcipher (i.e., a blockcipher with κ-bit keys and n-bit blocks) as an *ideal cipher* (IC), i.e., a family of 2^κ independent n-bit random permutations that is public to all entities. This is known as the *ideal cipher model* (ICM), and it turned out crucial for proving security for blockcipher-based schemes when the PRP assumption is not enough [9,10,35]. While remaining a heuristic approach [12,41], a proof in the ideal cipher model is typically considered a good indication of security from the point of view of practice. Meanwhile, "being close to ideal" becomes a new standard for blockcipher design and evaluations—much like "being close to a random oracle" for hash functions [15,7]. In fact, distinguishing blockcipher algorithms from "ideal" has been recognized as an important attack vector [37,8].

Indifferentiability. While ICs are unachievable in the standard model [12,41], it remains an interesting problem to build ICs from other public ideal functions. This class of problem shall be addressed with *indifferentiability* introduced by Maurer et al. [41] and popularized by Coron et al. [15]. Indifferentiability is a simulation-based framework that helps assess whether a construction of a target primitive $A^{\mathbf{B}}$ from a lower-level ideal primitive $\mathbf{B}$ is "structurally close" to $\mathbf{C}$, the ideal version of $A^{\mathbf{B}}$ (e.g., the case where A is the IEM cipher, $\mathbf{B}$ is the random permutation $\mathbf{P}$ and $\mathbf{C}$ is an IC was considered in [1]). $A^{\mathbf{B}}$ is *indifferentiable from* $\mathbf{C}$, if for any *differentiator* D there exists an *efficient simulator* $S^{\mathbf{C}}$ querying $\mathbf{B}$ such that the two systems $(A^{\mathbf{B}}, \mathbf{B})$ and $(\mathbf{C}, S^{\mathbf{C}})$ are indistinguishable in the view of D. Indifferentiability comes equipped with a composition theorem [41] which implies that a large class of protocols (see [44,20] for restrictions) are provably secure in the ideal-$\mathbf{B}$ model if and only if they are provably secure in the ideal-$\mathbf{C}$ model. Since stronger notions are unachievable in general [44,20], indifferentiability is arguably the strongest security notion for cryptosystems. Due to this and due to the importance of composition, indifferentiability has been applied to various cryptosystems, including iterated hash [15,7], blockciphers [16,1,21], authenticated encryption [5] and public-key schemes [52].

Therefore, it has been an important direction to evaluate indifferentiability of popular blockcipher constructions [16,1]. The first feasibility was the key-prepended Feistel cipher of Coron et al. [16], which iterates $\Psi^{\mathbf{F}}(K, x_L \| x_R) := x_R \oplus \mathbf{F}(K \| x_L) \| x_L$ with $x_L, x_R \in \{0,1\}^{n/2}$ and $\mathbf{F}$ a public random function. Coron et al. proved indifferentiability with 14 rounds [16] and established equivalence of ideal models. This was later improved to 10 [17] and 8 rounds [19].

Another line of work established indifferentiability for the mentioned IEM ciphers. Concretely, a t-round IEM cipher employs t n-bit random permutations $\mathbf{P}_1, \ldots, \mathbf{P}_t$ and $t+1$ key derivation functions $\mathsf{kd}_0, ..., \mathsf{kd}_t : \{0,1\}^\kappa \to \{0,1\}^n$, and is defined by iterating $\mathrm{EM}_\ell^{\mathbf{P}}(K, x) := \mathsf{kd}_\ell(K) \oplus \mathbf{P}_\ell(\mathsf{kd}_{\ell-1}(K) \oplus x)$. When $\mathsf{kd}_0 = ... = \mathsf{kd}_t = \mathbf{F}$ for a random function $\mathbf{F} : \{0,1\}^\kappa \to \{0,1\}^n$, positive results were first proven at 5 rounds [1] and later tightened to 3 rounds [28]. When $\mathsf{kd}_0, ..., \mathsf{kd}_t$ are the identity function id, positive results were first proven at 12 rounds [39] and later tightened to 5 rounds [18].

Lower bounds? We seek for understanding the *complexity* and ask: to have a "non-trivial", provably secure iterated (κ, n)-blockcipher, how many calls to the primitives are *necessary*? Such results may shed lights on *limits on efficiency of widely used paradigms* as well as *boundary of blockcipher designs*.

By "non-trivial", we mean the construction must use some ideas. E.g., if an oracle $\mathcal{O}$ already contains an exponential number of independent n-bit random permutations, then $E^{\mathcal{O}}$ can trivially instantiate an indifferentiable blockcipher. With this in mind, we introduce an oracle $\mathcal{P}$ that "provides all but the goal".

In detail, $\mathcal{P} = (\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_{|\mathcal{I}|})$ is a family of independent random permutations indexed by $i \in \mathcal{I}$, where $\mathbf{P}_i : \{0,1\}^{m(i)} \to \{0,1\}^{m(i)}$ for an integer function $m : \mathcal{I} \to \mathrm{poly}(n)$. The set $\mathcal{I}$ is partitioned into $\mathcal{I}_{\le n}$ and $\mathcal{I}_{>n}$, such that $i \in \mathcal{I}_{\le n}$ if and only if $m(i) \le n$. To avoid trivial results, we require $|\mathcal{I}_{\le n}| = O(\mathrm{poly}(n))$, so that $\mathcal{P}$ cannot offer exponentially many n-bit permutations. For $i \in \mathcal{I}_{>n}$, it can be $m(i) \gg n$, and an indifferentiable random function/injection can be built by calling such a wide permutation once [14,5]. Thus, such an oracle $\mathcal{P}$ essentially offers the "maximal" power to the constructions. As will be detailed in Sect. 5.1, existing constructions [16,1,39] can be seen as defined upon $\mathcal{P}$.

The status, of course, depends on the security notion. W.r.t. *indistinguishability*, a single permutation-call is already sufficient using the Even-Mansour scheme [23]. We seek for bounds w.r.t. *indifferentiability*. *Specific lower bounds* have been shown: Feistel ciphers [16,19] consume *at least* 6 random function calls, while IEM ciphers need 4 random function/permutation calls [1,18,28]. Despite this and the fruitful positive results mentioned before, no *general lower bounds* are publicly known (except that a polynomial-length random string is insufficient [41]) due to its challenging nature: the adversarial goal is not as clear as [9,45,3] (which simply finds collisions or pre-images), and one has to pinpoint "non-random" properties that are exploitable within polynomial-queries (unlike [45,3]) in various cases, and further prove that interactions with ideal ciphers and *all possible simulators* are unlikely to admit such properties.

Our results. We prove the first general lower bound: *no iterated blockcipher making 3 or less calls to the oracle $\mathcal{P}$ is statistically indifferentiable from ideal ciphers.* This proves *optimality* for the mentioned 4-call positive result [28].

Model and settings. We consider *iterated blockciphers* that can be written as the composition of *rounds* using keys or derived subkeys. Every *round* is essentially a simpler "1-call" blockcipher *making exactly 1 call to* $\mathcal{P}$, and the total number of $\mathcal{P}$-calls made by the rounds and the key derivation function is a constant.

More concretely, to model rounds/1-call ciphers, we define $E1^{\mathcal{P}}(K,x) := \varphi^{out}\big(K, \mathcal{P}(\varphi^{in}(K,x)), x\big)$ with keyspace $\mathcal{K}$ and domain $\{0,1\}^n$. The *input function* φ^{in} maps $(K,x) \in \mathcal{K} \times \{0,1\}^n$ into a query (i,δ,z) to $\mathcal{P}$, where $\delta \in \{+,-\}$ indicates the direction, $i \in \mathcal{I}$ indexes the queried permutation and $z \in \{0,1\}^{m(i)}$ is the concrete query. The *output function* φ^{out} maps the key K, the $\mathcal{P}$ response $z' = \mathcal{P}(\varphi^{in}(K,x))$ and the plaintext x to the ciphertext y.

$E1^{\mathcal{P}}$ must admit efficient inversion within 1 $\mathcal{P}$-call as well. Thus, it is defined $(E1^{-1})^{\mathcal{P}}(K,y) := \gamma^{out}\big(K, \mathcal{P}(\gamma^{in}(K,y)), y\big)$ for two other input and output functions γ^{in} and γ^{out}. Arguably, this covers all blockciphers using a single oracle call (which resembles [9]). See Fig. 1 for illustration.

Then, for our model of a t-call iterated blockcipher $Et^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$, the keyspace $\mathcal{K}$ is partitioned into disjoint sets $\mathcal{K}^{(0)}, \mathcal{K}^{(1)}, ...\mathcal{K}^{(t-1)}$, such that for all $K \in \mathcal{K}^{(\ell)}$, it has

$$Et^{\mathcal{P}}(K,x) = \Pi^{\mathcal{P}}_{j_{\ell,t-\ell}}\Big(K\|s, ...\Pi^{\mathcal{P}}_{j_{\ell,2}}\big(K\|s, \Pi^{\mathcal{P}}_{j_{\ell,1}}(K\|s,x)\big)...\Big), \tag{1}$$

where:

(i) $s = \mathsf{kd}^{\mathcal{P}}(K)$ is a subkey and $\mathsf{kd}^{\mathcal{P}}$ makes ℓ calls to $\mathcal{P}$, and
(ii) For each $\ell \in \{0,...,t-1\}$ and each $\alpha \in \{1,...,t-\ell\}$, $j_{\ell,\alpha} = \ell(\ell-1)/2+\alpha$ (so that $Et^{\mathcal{P}}$ is defined upon $\ell(\ell+1)/2$ distinct rounds $\Pi_1, ..., \Pi_{\ell(\ell+1)/2}$), and the round $\Pi^{\mathcal{P}}_{j_{\ell,\alpha}}$ is a 1-call cipher.

See Fig. 2 for illustration of $E2^{\mathcal{P}}$ and Figs. 12 and 15 for pseudocode of $E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$. This unifies virtually all existing blockcipher constructions. While the same oracle $\mathcal{P}$ is used everywhere in $Et^{\mathcal{P}}$, our subsequent attacks never utilize this oracle reusing, and are applicable even if multiple $\mathcal{P}_1, \mathcal{P}_2, ...$ are used.

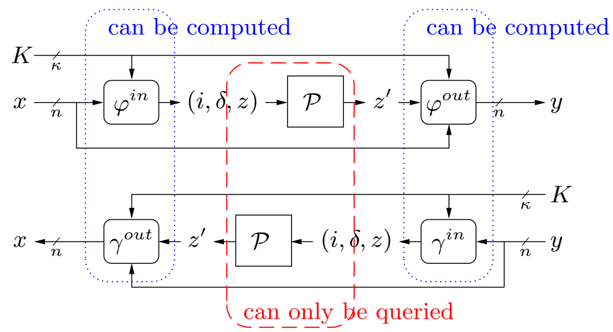

Fig. 1. The general blockcipher $E1^{\mathcal{P}}$ making a single call to its oracle $\mathcal{P}$ for enciphering (up) and deciphering (bottom). $\varphi^{in}, \varphi^{out}, \gamma^{in}$ and γ^{out} are arbitrary (e.g., can be highly non-linear) deterministic and oracle-independent functions, and are computable by the differentiator (as indicated). $\mathcal{P} = (\mathbf{P}_1, \mathbf{P}_2, ...)$ is the mentioned family of random permutations, and it only offers oracle access to the differentiator.

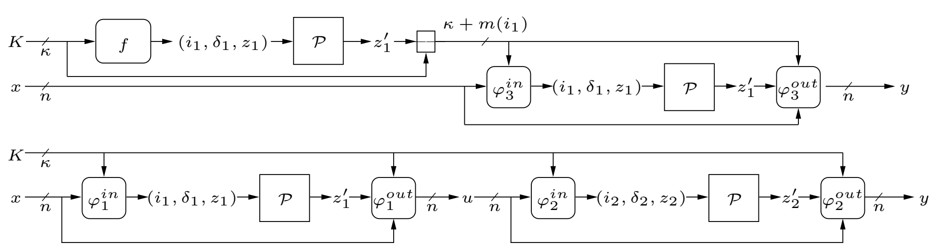

Fig. 2. Encipherment of 2-call iterated blockciphers. (Top) Using one $\mathcal{P}$-call for a key derivation $\mathsf{kd}^{\mathcal{P}}(K) = \mathcal{P}(f(K))$. The function f is deterministic and oracle-independent; (Bottom) Using two $\mathcal{P}$-calls for two rounds, without idealized key derivations.

Our reasoning relies on four *Fundamental Properties* that stem from *the notions of blockciphers* and *of t-call oracle procedures*. Namely, a blockcipher oracle procedure $E^{\mathcal{P}}$ should be **efficiently invertible**, **deterministic**, and enjoy an **oracle-independent** description. Moreover, it should be **non-degenerate** (i.e., $E^{\mathcal{P}}$ cannot be "simplified" in terms of $\mathcal{P}$ calls). We refer to Sect. 3.1 or 4 for details. Our setting may find broader applications in symmetric cryptography. As a side remark, our crucial use of invertibility solves an open problem of [5].

Differentiability of $E1$*,* $E2$ *and* $E3$*.* With the above models, we prove our main result by characterizing $E1^{\mathcal{P}}$ and extending to $E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$.

In detail, for 1-call ciphers $E1^{\mathcal{P}}$, we fully characterize its properties, solely based on the *Fundamental Properties*. In summary, as long as the keyspace has $|\mathcal{K}| \geq 2|\mathcal{I}_{\leq n}| + 1 = O(\text{poly}(n))$ (thus, even polynomial keyspace is unachievable!),[6] we can find either $\Omega(\text{poly}(n))$ "inverse-free" encipherments that collide on the $\mathcal{P}$-call and use an entropy-based differentiating approach [41], or find two "non-inverse-free" encipherments $E1^{\mathcal{P}}(K,x)$ and $E1^{\mathcal{P}}(K',x')$ with $\varphi^{in}(K,x) = \varphi^{in}(K',x')$ and use a special regularity property of φ^{in} and γ^{in} to distinguish. We refer to Sect. 3.2 or Theorem 1 for details.

For 2-call iterated cipher $E2^{\mathcal{P}}$, if $\mathcal{K}^{(1)}$ is large enough, i.e., $E2^{\mathcal{P}}$ invokes key derivation for sufficiently many keys, then our differentiators derives $O(\text{poly}(n))$ keys to "collapse" the cipher to a 1-call instance, which has been attacked. On the other hand, if $\mathcal{K}^{(0)}$ dominates, i.e., $E2^{\mathcal{P}}$ is a general 2-round blockcipher for most keys, then as long as $\mathcal{K}^{(0)}$ is large enough $|\mathcal{K}^{(0)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}} + 5\big)|\mathcal{I}_{\leq n}| + 1 = O(\text{poly}(n))$, we can exhibit a general yoyo distinguisher and breaks its *correlation intractability* (a weaker security notion than indifferentiability). We refer to Sect. 3.3 or Theorem 2 for details.

For 3-call iterated ciphers $E3^{\mathcal{P}}$, if $\mathcal{K}^{(2)}$ or $\mathcal{K}^{(1)}$ is large enough then we again "collapse" it to 1- or 2-call ciphers by deriving poly(n) keys. If $E3^{\mathcal{P}}$ is a general 3-round cipher for most keys $\mathcal{K}^{(0)} \subseteq \{0,1\}^{\kappa}$ with $\kappa \geq 2m_{max}\log_2|\mathcal{I}_{\leq n}| + 2m_{max}n + 6m_{max} + 4 = \Theta(\text{poly}(n))$, $m_{max} := \max_{i\in\mathcal{I}} m(i)$, we exhibit a universal differentiator that (interestingly) has attack advantage either at least $1/\text{poly}(n) - \text{negl}(n)$ or at least $1 - \text{negl}(n)$, where the concrete polynomial and negligible functions depend on the input functions in the three rounds. We refer to Sect. 3.4 or Theorem 3 for details.

A crucial step is to show the existence of certain non-random properties, which is non-obvious in the general 2- and 3-call ciphers. To this end, we apply Extremal Graph Theory [32,38,24], which bound the maximal number of edges in (bipartite) graphs that do not contain certain structures (a.k.a. *Zarankiewicz numbers* [24]). We refer to Sect. 3 for more detailed overview.

Discussion: blockcipher designs. A recent trend is to revisit blockcipher structures and squeeze efficiency for MPC and ZKP settings: see [26] and the references therein. We hope that our work could be a step towards unifying

[6] Though, trivial constructions with $|\mathcal{K}| \leq |\mathcal{I}_{\leq n}|$ exist since $\mathcal{P}$ may offer $|\mathcal{I}_{\leq n}|$ independent n-bit RPs.

relevant theoretical discussions and shed lights on the "boundary" of designs. We summarize some of our conclusions as follows.

(i) *Expense of overcoming non-invertibility*: if a round/1-call cipher $E1^{\mathcal{P}}(K,x)$ want to be inverse-free for some (K,x) (e.g., when using non-invertible primitives), then $E1^{\mathcal{P}}(K,\cdot)$ must admit severe weakness, regardless of its design.
(ii) *Unhelpfulness of wide permutations*: wide permutations with width$>n$ are not "more helpful" in constructing n-bit blockciphers, even if exponentially many are available. This might be another explanation on the difficulty in designing format-preserving encryption schemes (see e.g., [22]).
(iii) *Optimality of popular structures (e.g., the IEM ciphers [18,28])*, in the sense that *no other choice can be better*. This provides the first "excluding-type" theoretical support for practical paradigms.

Besides, since an indifferentiable iterated cipher needs at least 4 calls, our result may be viewed as a theoretical evidence of the advantage (in terms of efficiency) of permutation-based cryptography. Though, we remark that the usual caveats regarding the ideal model apply to this paper: as we consider information-theoretic adversaries, our results do not imply security upper bounds on real-world, computationally bounded adversaries.

Lower bounds: functionality transformations vs. small-to-big. Cryptographic constructs consist of two categories: *functionality transformations* and *small-to-big transformations*. The former achieves "non-trivial" new functionality (e.g., our case), while the latter achieve domain or range extension (e.g., PRGs extend range, while hashes extend domain).

A number of existing efficiency lower bounds concerned with *small-to-big transformations*, including hash functions [36,25,9,45,5], PRGs [25,31], signatures [25,3], encryption [25] and injections [5]. A core idea typically employed by these proofs is to apply the pigeonhole principle to force the scheme making the same sequence of primitive calls for exponentially many inputs. This results in either attacks [9,45,5] or unconditional cryptography [36,25,3].

Despite exciting black-box separations [47,34,4], efficiency lower bounds on *functionality transformations* are relatively rare. Our problem *is functionality transforming*: we allow to use wide permutations on $\geq \kappa + n \gg n$ bits, and the domain of our target $E : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$ is thus *not larger*. This difference is crucial, as pigeonhole principle cannot ensure collisions and we have to rely on other properties such as non-degeneracy (see Lemma 3).

Notably, with our oracle $\mathcal{P}$, relevant impossibility results become possible:

(i) A compression function with enough collision or even indifferentiability security can be built using just 1 call to $\mathcal{P}$ via *truncation* [14];
(ii) An indifferentiable injection (or authenticated encryption) can be built using just 1 call to $\mathcal{P}$ via *Encode-then-Encipher* [5].

Still, indifferentiable iterated blockciphers *cannot* be built within 3 calls. These sharp contrasts emphasize the differences between our setting and [9,5].

In a more restricted setting termed *Linicrypt* [13], i.e., cryptosystems are built from random block functions and *linear* diffusion functions, impossibility results regarding encryption [42,13] and circuit garbling [13] exist.

Future directions. Indeed, blockciphers are *not* necessarily iterated: we serve examples in Appendix A. Intuitively, such designs are weaker than iterated ones with the same number of calls. Though, it is difficult to have a rigorous and clean argument, especially for ciphers with 3 calls. The most intriguing direction is thus to address fully general 2- and 3-call blockciphers, which may shed more lights on iterations. Another intriguing question is whether there are smart ideas to unify the complicated cases in $E3$ analysis. Influences of other aspects such as memory restrictions on adversaries and simulators are also of interest.

On the constructive side, it is intriguing to study the achievability of computational indifferentiability with 3 calls: hardness assumptions on graph problems or key derivation functions might be helpful.

Unlike most practice in symmetric cryptography, our Theorem 3 is asymptotic. The key issue is that our differentiator has to "know" the simulator limitations for its case decision. Classically, simulator (query) complexity is only *polynomially bounded* and seems incompatible with concrete treatments. Fully concrete characterizations may need a new paradigm and are left for future work.

Roadmap. We serve notations and definitions in Sect. 2. Then, as mentioned, we provide a technical overview in Sect. 3.

For the main elaborations, we first formalize the *Fundamental Properties* in Sect. 4. We then give detailed elaborations and characterizations for our 1-call cipher in Sect. 5. With the help of these characterizations, we present differentiators against $E1$, $E2$ and $E3$ in Sect. 6, 7 and 8 respectively. Case-study for general 3-iteration is lengthy and takes a separate section Sect. 9. Some detailed proofs are available in the full version.

2 Preliminaries

Fix n as the security parameter, and write poly(n) and negl(n) for arbitrary polynomial and negligible functions respectively. Denote by $\perp$ the empty string. Given $x \in \{0,1\}^n$ and $a \leq n$, denote by $\mathsf{left}_a(x)$ (resp., $\mathsf{right}_a(x)$) the a leftmost (resp., rightmost) bits of x. When two sets A and B are disjoint, we denote $A \sqcup B$ their (disjoint) union. For any domain, denote by id the identity function.

An m-bit random permutation is a permutation that is uniformly selected from $\mathsf{Perm}(m)$, the set of all $(2^m)!$ possible m-bit permutations. Throughout the remaining, we denote by $\mathbf{IC} : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$ an ideal cipher (which is randomly picked from all (κ, n)-blockciphers) with $\kappa = \text{poly}(n)$.

We stress that, the phrases "such as" and "if and only if" will be abbreviated as **s.t.** and **iff.** in our pseudocode respectively.

Permutation family $\mathcal{P}$. As briefed in the Introduction, we consider constructing an n-bit blockcipher from a permutation family oracle $\mathcal{P}$ that "provides all but the goal". In detail, $\mathcal{P} = (\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_{|\mathcal{I}|})$ provides independent random permutations indexed by $i \in \mathcal{I}$, where $\mathbf{P}_i \in \mathsf{Perm}(m(i))$ for a fixed function $m : \mathcal{I} \to \mathrm{poly}(n)$ (viewed as parameters of $\mathcal{P}$). It can be $m(i) \gg n$ for some $i \in \mathcal{I}$. The index set is thus partitioned as $\mathcal{I} = \mathcal{I}_{\leq n} \sqcup \mathcal{I}_{>n}$, where $i \in \mathcal{I}_{\leq n}$ if and only if $m(i) \leq n$. We require $|\mathcal{I}_{\leq n}| = O(\mathrm{poly}(n))$, while $\mathcal{I}_{>n}$ can be exponentially large. We call permutations with width$> n$ *wide*. Denote by $m_{max} := \max_{i\in\mathcal{I}} m(i)$ and $m_{min} := \min_{i\in\mathcal{I}} m(i)$ the size of largest, resp. smallest permutation in $\mathcal{P}$.

Oracle $\mathcal{P}$ accepts queries of the form (i, δ, z), where $i \in \mathcal{I}$ is the index, $\delta \in \{+, -\}$ indicates if forward $\mathbf{P}_i$ or backward $\mathbf{P}_i^{-1}$ is queried, and $z \in \{0,1\}^{m(i)}$ is the actual $m(i)$-bit input. For $\delta \in \{+, -\}$, we denote $\overline{\delta}$ the opposite of δ.

Indifferentiability. Let $E^{\mathcal{P}}$ be a cryptographic construction that internally queries $\mathcal{P}$, $\mathbf{IC}$ be the ideal crypto object of $E^{\mathcal{P}}$, and $S^{\mathbf{IC}}$ be a simulator that queries $\mathbf{IC}$ and provides the same interfaces as $\mathcal{P}$. Then, for any distinguisher D, the indifferentiability advantage of D against $E^{\mathcal{P}}$ is

$$\mathrm{Adv}^{indif}_{E^{\mathcal{P}},\mathbf{IC},S}(D) = \big|\Pr\big[D^{E^{\mathcal{P}},\mathcal{P}} = 1\big] - \Pr\big[D^{\mathbf{IC},S^{\mathbf{IC}}} = 1\big]\big|.$$

$E^{\mathcal{P}}$ is indifferentiable (in the asymptotic sense), as long as for any polynomial-query D: (a) the advantage $\mathrm{Adv}^{indif}_{E^{\mathcal{P}},\mathbf{IC},S}(D)$ is $\mathrm{negl}(n)$ w.r.t. the security parameter n for any D, and (b) the number of queries made by S to $\mathbf{IC}$ is $\mathrm{poly}(n)$.

Notations for differentiators. Consider blockciphers $E1^{\mathcal{P}}, E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$ built upon the oracle $\mathcal{P}$. By the above, to break indifferentiability, we shall exhibit a differentiator D that "fools" any query-efficient simulator $S^{\mathbf{IC}}$ with non-negligible probability. Notice, D has access to two oracles (E, P) where $\mathrm{E} \in \{E1^{\mathcal{P}}, E2^{\mathcal{P}}, E3^{\mathcal{P}}, \mathbf{IC}\}$ and $\mathrm{P} \in \{\mathcal{P}, S^{\mathbf{IC}}\}$. To describe the interaction between $D^{\mathrm{E},\mathrm{P}}$ and its oracles E, P, we use the expressions $\mathrm{P}(i, \delta, z) \to z'$ to mean that D queries P on (i, δ, z) and P answers with z', and $\mathrm{E}(K, x) \to y$ to mean that E is queried on (K, x) and returns y. Note that in the latter case, the query may be made by S. The notation $\mathrm{E}^{-1}(K, y) \to x$ is similar.

As convention, our differentiators always output 1 when it guesses the "real world", and output 0 when it guesses the "ideal" or "simulated world".

Tools from Extremal Graph Theory. Consider a bipartite graph $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$. Intuitively, if $|\mathcal{E}|$ is sufficiently large, then $\mathcal{G}$ must have short cycles (since long cycles will be truncated). This was proven by Hoory [32], and will help establishing the existence of certain structures. To ease applying, we restate Hoory's result [32, Eqs. (1) and (2)] as follows.

Proposition 1. *Let $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$ be a bipartite graph such that:*

(i) $|\mathcal{V}_L|$ and $|\mathcal{V}_R|$ have a common upper bound, i.e., there exists an integer $M > 0$ such that $|\mathcal{V}_L| \leq M$, $|\mathcal{V}_R| \leq M$; and
(ii) $|\mathcal{E}| \geq \big((M)^{\frac{1}{t-1}} + 1\big) \times M$ for some positive integer t.

Then, $\mathcal{G}$ contains a cycle $C_{2\ell}$ with $\ell \leq t$.

Proof. By Hoory [32, Eqs. (1) and (2)], if the shortest cycle in $\mathcal{G}$ has length $2t$, then $\mathcal{V}_L$ and $\mathcal{V}_R$ have to fulfill

$$\sum_{i=0}^{t-1}\Big(\frac{|\mathcal{E}|}{|\mathcal{V}_L|}-1\Big)^{\lceil i/2\rceil}\Big(\frac{|\mathcal{E}|}{|\mathcal{V}_R|}-1\Big)^{\lfloor i/2\rfloor}\le \max\big\{|\mathcal{V}_L|,|\mathcal{V}_R|\big\}\le M. \qquad (2)$$

The left hand side of Eq. (2) is lower bounded by

$$\ge \sum_{i=0}^{t-1}\Big(\frac{|\mathcal{E}|}{\max\{|\mathcal{V}_L|,|\mathcal{V}_R|\}}-1\Big)^{i}\ge\Big(\frac{|\mathcal{E}|}{\max\{|\mathcal{V}_L|,|\mathcal{V}_R|\}}-1\Big)^{t-1}. \qquad (3)$$

By this, as long as $\big(\frac{|\mathcal{E}|}{\max\{|\mathcal{V}_L|,|\mathcal{V}_R|\}}-1\big)^{t-1}\ge M$, a cycle of length $2t$ is guaranteed to exist. This condition further translates into

$$|\mathcal{E}|\ge\big((M)^{\frac{1}{t-1}}+1\big)\times M \qquad (4)$$

and yields the claim. □

If $|\mathcal{E}|$ is large, then $\mathcal{G}$ contains a small complete bipartite graph (a.k.a. biclique). This was proven by Kővári, Sós and Turán [38], and is restated as follows.

Proposition 2. *Let $\mathcal{G}=(\mathcal{V}_L,\mathcal{V}_R,\mathcal{E})$ be a bipartite graph such that:*

(i) There exist two integers $M,N>0$ such that $|\mathcal{V}_L|\le M$ and $|\mathcal{V}_R|\le N$; and
(ii) $|\mathcal{E}|\ge(b-1)^{\frac{1}{a}}\cdot MN^{1-\frac{1}{a}}+(a-1)N$.

Then, $\mathcal{G}$ contains the complete bipartite graph $K_{a,b}$ as a sub-graph.

Proof. The maximum number of edges in an $M\times N$ bipartite $K_{a,b}$-free graph is known as the Zarankiewicz number $Z(M,N,a,b)$. By [24, Eq. (3.1)] (proven by Kővári, Sós and Turán [38]), it holds $Z(M,N,a,b)\le(b-1)^{\frac{1}{a}}\cdot MN^{1-\frac{1}{a}}+(a-1)N$. $K_{a,b}$ thus exists when $|\mathcal{V}_L|\le M$, $|\mathcal{V}_R|\le N$ while $|\mathcal{E}|\ge Z(M,N,a,b)$.[7] □

3 Technical Overview

As mentioned, we characterize the 1-call model $E1^{\mathcal{P}}$ and then extend the discussion to 2- and 3-call iterated models $E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$. Below in Sect. 3.1, we first elaborate more on *Fundamental Properties* underlying our reasoning. We then provide intuitions for $E1^{\mathcal{P}}$, $E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$ in Sect. 3.2, 3.3 and 3.4 in turn.

3.1 Fundamental Properties

As mentioned, our analyses rely on four properties that we believe fundamental to blockcipher oracle procedures. First, the *definition of the notion of blockciphers* yield two properties for a blockcipher oracle procedure $E^{\mathcal{P}}$:

[7] Nikiforov [43] proved a tradeoff, but does not improve in our applications.

(i) **Efficient invertibility**: blockciphers should be efficiently invertible. Namely, there is a corresponding oracle procedure $(E^{-1})^{\mathcal{P}}$ computing its inverse;

(ii) **Deterministic**: blockciphers should be *deterministic*. For $E^{\mathcal{P}}$, it means for a fixed oracle $\mathcal{P}$, *evaluating* $E^{\mathcal{P}}(K,x) \to y$ *and the corresponding decipherment* $(E^{-1})^{\mathcal{P}}(K,y)$ *always yield the same transcript of* $\mathcal{P}$*-queries and responses.*

Besides, since an oracle procedure $E^{\mathcal{P}}$ shall have a fixed description that is independent from $\mathcal{P}$, sub-procedures in $E^{\mathcal{P}}$ are **oracle-independent**. We further assume that $E^{\mathcal{P}}$ is **non-degenerate** and cannot be "simplified" in terms of $\mathcal{P}$ calls, i.e., no encipherment $E^{\mathcal{P}}(K,x)$ can be approximately computed using less $\mathcal{P}$ calls than $E^{\mathcal{P}}$. Formal definitions will be given in Sect. 4.

3.2 Full characterization of 1-call cipher $E1$

As per mentioned, $E1^{\mathcal{P}}(K,x) := \varphi^{out}\big(K, \mathcal{P}(\varphi^{in}(K,x))\big)$ and $(E1^{-1})^{\mathcal{P}}(K,y) := \gamma^{out}\big(K, \mathcal{P}(\gamma^{in}(K,y))\big)$, where φ^{in}, φ^{out}, γ^{in} and γ^{out} can be arbitrary oracle-independent functions. The *Fundamental Properties* already ensure a number of non-trivial properties (on oracle procedures of blockciphers).

Inv-freeness and its oracle-independence. Our first observation is about inverse-freeness of $E1^{\mathcal{P}}$. An encipherment $E1^{\mathcal{P}}(K,x)$ is *inverse-free* (inv-free for short), if $E1^{\mathcal{P}}(K,x) \to y$ and its corresponding decipherment $(E1^{-1})^{\mathcal{P}}(K,y) \to x$ call $\mathcal{P}(i,\delta,\star)$ on the same direction δ; otherwise, $E1^{\mathcal{P}}(K,x)$ is *non-inverse-free* (non-inv-free). In common designs (Feistel, Misty, IEM, etc.), encipherments under a fixed key are either all inv-free or non-inv-free for all plaintext. However, in general, the inv-freeness of $E1^{\mathcal{P}}(K,x)$ may depend on x, admitting *data-dependent inv-freeness*. We serve an example in Fig. 3.

Our observation is that *in* $E1^{\mathcal{P}}$*, inv-freeness cannot depend on the oracle* $\mathcal{P}$, i.e., one can decide if an encipherment $E1^{\mathcal{P}}(K,x)$ is inv-free without querying $\mathcal{P}$. Intuitively, it is because the query directions of encipherment and decipherment are determined by the input functions φ^{in} and γ^{in}, which are oracle-independent. The formal presentation will be given in Lemma 1. As will be seen, exploitable weakness in an encipherment $E1^{\mathcal{P}}(K,x)$ depends on its inv-freeness, the oracle-independence of which turns out crucial in our attacks.

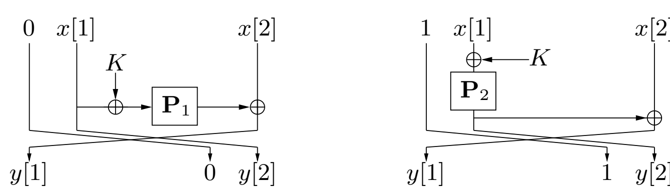

Fig. 3. A 1-call cipher/round that has data-dependent inverse-freeness. (Left) when the leftmost bit of $x = 0\|x[1]\|x[2]$ is 0 (a permutation-based Feistel round); (Right) when the leftmost bit of $x = 1\|x[1]\|x[2]$ is 1 (a Misty-like round).

Properties of inv-free $E1^{\mathcal{P}}(K,x)$. For intuitions, consider the key-prepended Feistel round $y = \Psi^{\mathbf{F}}(K,x) := \big(\mathsf{right}_{n/2}(x) \oplus \mathbf{F}(K\|\mathsf{left}_{n/2}(x))\big)\|\mathsf{left}_{n/2}(x)$. This round is inv-free: an encipherment $\Psi^{\mathbf{F}}(K,x)$ and its corresponding decipherment $(\Psi^{-1})^{\mathbf{F}}(K,y)$ make the same "forward" call to $\mathbf{F}(z)$, $z = K\|\mathsf{left}_{n/2}(x) = K\|\mathsf{right}_{n/2}(y)$. This means some information of x is kept in the ciphertext y without "protection". The same property is shared by various Feistel variants [2, Chapter 1.3.1] (including the Lai-Massey scheme [2, Chapter 1.5]). Casting it into our general model $E1^{\mathcal{P}}$, it means *inv-free* $E1^{\mathcal{P}}(K,x) \to y$ *must have* $\varphi^{in}(K,x) = \gamma^{in}(K,y)$.

As a less obvious fact in $\Psi^{\mathbf{F}}$, there necessarily exist many distinct encipherments that make the same $\mathbf{F}$-call. I.e., $\Psi^{\mathbf{F}}(K,x)$ calls $\mathbf{F}(z)$ as long as $\mathsf{right}_{n/2}(x) = \mathsf{right}_{n/2}(z)$, and there are $2^{n/2}$ possible x for every z. Similarly for other inv-free designs. It turns out that: *with the non-degeneracy assumption on* $E1$*, if there is one inv-free* $E1^{\mathcal{P}}(K,x)$ *then there are* $\Omega(poly(n))$ *distinct inv-free* $E1^{\mathcal{P}}(K,x_1)$*,* $E1^{\mathcal{P}}(K,x_2), ...$ *that collide on the* $\mathcal{P}$*-call, i.e.,* $\varphi^{in}(K,x_1) = \varphi^{in}(K,x_2) = ... = \varphi^{in}(K,x)$. This further implies that under each key K, all inv-free $E1^{\mathcal{P}}(K,x)$ give rise to $o\big(\frac{2^n}{\mathrm{poly}(n)}\big)$ distinct $\mathcal{P}$-calls (even if they can query exponentially many permutations with width$> n$).

We refer to Lemmas 2–4 in Sect. 5.2 for formal elaborations.

Properties of non-inv-free $E1^{\mathcal{P}}(K,x)$. For intuitions, consider the IEM round $y = \mathrm{EM}^{\mathbf{P}}(K,x) := K \oplus \mathbf{P}(K \oplus x)$, which is non-inv-free since $(\mathrm{EM}^{-1})^{\mathbf{P}}$ always calls $\mathbf{P}^{-1}$. $\mathrm{EM}^{\mathbf{P}}$ is more secure than $\Psi^{\mathbf{F}}$. In fact, attacks against $\mathrm{EM}^{\mathbf{P}}$ have to exploit at least 2 keys K, K' [1, Sect. 3.1, full version] and seek for encipherments $\mathrm{EM}^{\mathbf{P}}(K,x)$ and $\mathrm{EM}^{\mathbf{P}}(K',x')$ that collide on the $\mathbf{P}$-call, i.e., with $K \oplus x = K' \oplus x'$. Such collided encipherments *do exist*, because $EM^{\mathbf{P}}$ *cannot use wide* $\mathbf{P}$. Concretely, to invoke a wide $\mathbf{P}$, $\mathrm{EM}^{\mathbf{P}}$ must pad $x \in \{0,1\}^n$ with some "non-trivial" information (e.g., $\mathbf{P}(x\|0)$, or $\mathbf{P}(x\|K)$); but then, by invoking $\mathbf{P}^{-1}$, decipherments are unlikely to "recover" correctly padded $\mathbf{P}$-inputs. In fact, this irrecoverability is the core idea of *Encode-then-Encipher* [5].

It turns out that this irrecoverability stems from oracle-independence. In detail, assuming oracle-independence of φ^{in} and γ^{in}, *non-inv-free encipherments* $E1^{\mathcal{P}}(K,x)$ *can only query permutations with width*$\leq n$. Thus, non-inv-free give rise to at most $|\mathcal{I}_{\leq n}|2^{n+1}$ distinct $\mathcal{P}$-calls. We refer to Lemma 5 in Sect. 5.2 for formal elaborations.

Attack $E1^{\mathcal{P}}$. With the above properties, we are able to bump into our differentiator $D1$ on $E1^{\mathcal{P}}$. In detail, the cipher $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ may fall into two cases.

Case 1: there exists at least 1 inv-free encipherment $E1^{\mathcal{P}}(K,x)$. As discussed, this means we can find $t = \Omega(\mathrm{poly}(n))$ distinct inv-free $E1^{\mathcal{P}}(K,x_1), ..., E1^{\mathcal{P}}(K,x_t)$ that make the same $\mathcal{P}$-call $\mathcal{P}(i,\delta,z)$, $(i,\delta,z) = \varphi^{in}(K,x_1) = ... = \varphi^{in}(K,x_t)$. Thus, the restriction of $E1^{\mathcal{P}}(K,\cdot)$ to $\{x_1,...,x_t\}$ is a bijection defined upon a polynomial-length random string $z' = \mathcal{P}(i,\delta,z)$, and we can apply an entropy-based differentiating approach [41].

Case 2: $E1^{\mathcal{P}}(K,x)$ is non-inv-free for all $(K,x) \in \mathcal{K} \times \{0,1\}^n$. Then, $E1^{\mathcal{P}}(K,x)$ can only invoke the permutations in $\mathcal{P}$ with width$\leq n$ (as discussed). Therefore, the number of possible images of φ^{in} is at most $|\mathcal{I}_{\leq n}|2^{n+1}$. As long as $|\mathcal{K}|2^n \geq |\mathcal{I}_{\leq n}|2^{n+1}$, i.e., the keyspace has $|\mathcal{K}| \geq 2|\mathcal{I}_{\leq n}|+1$ (which is $O(\text{poly}(n))$, though), the pigeonhole principle guarantees the existence of $(K,x),(K',x') \in \mathcal{K} \times \{0,1\}^n$ with collision $\varphi^{in}(K,x) = \varphi^{in}(K',x')$. $D1$ thus finds such a pair of collided $(K,x),(K',x')$ and attacks by checking if $\gamma^{in}\big(K,\mathrm{E}(K,x)\big) = \gamma^{in}\big(K',\mathrm{E}(K',x')\big)$.

The formal proof is more technical and relies on a sort of "regularity" of the input functions φ^{in} and γ^{in} (Lemma 6). Please see Sect. 6 for details.

3.3 Attack 2-call iterated cipher $E2$

Built upon our above results on $E1^{\mathcal{P}}$, we further consider our 2-call model $E2^{\mathcal{P}}$. Recall that the keyspace $\mathcal{K}$ of $E2^{\mathcal{P}}$ can be partitioned $\mathcal{K} = \mathcal{K}^{(0)} \sqcup \mathcal{K}^{(1)}$, such that $E2^{\mathcal{P}}(K,x) = \Pi_3^{\mathcal{P}}(K\|\mathsf{kd}^{\mathcal{P}}(K),x)$ for all $K \in \mathcal{K}^{(1)}$, whereas $E2^{\mathcal{P}}(K,x) = \Pi_2^{\mathcal{P}}\big(K,\Pi_1^{\mathcal{P}}(K,x)\big)$ for all $K \in \mathcal{K}^{(0)}$. The sub-procedures $\mathsf{kd}^{\mathcal{P}}$ has $\mathsf{kd}^{\mathcal{P}}(K) = \mathcal{P}(f(K))$ for another oracle-independent function f. In addition, for $j = 1,2,3$, $\Pi_j^{\mathcal{P}}$ is a 1-call cipher with input and output functions $\varphi_j^{in}, \varphi_j^{out}, \gamma_j^{in}$ and γ_j^{out}. We refer to Fig. 2 for illustration and Fig. 12 for a pseudocode description.

This model $E2^{\mathcal{P}}$ may fall into two cases.

Case 1: $E2^{\mathcal{P}}$ invokes kd for sufficiently many keys. Formally, if the key sets have $|\mathcal{K}^{(1)}| \geq 2|\mathcal{I}_{\leq n}|+1$, we simply pick $\lambda = 2|\mathcal{I}_{\leq n}|+1$ keys $K_1,...,K_\lambda \in \mathcal{K}^{(1)}$ and derive subkeys $s_1 = \mathsf{kd}^{\mathcal{P}}(K_1),...,s_\lambda = \mathsf{kd}^{\mathcal{P}}(K_\lambda)$. This consumes at most $\lambda = O(\text{poly}(n))$ P-queries. We then view the round $\Pi_3^{\mathcal{P}}$ as a 1-call cipher with keyspace $\{K_1\|s_1,...,K_\lambda\|s_\lambda\}$ and apply our differentiator $D1$. (It is thus crucial that $D1$ can break $E1$ with polynomial-keyspace.)

Case 2: $E2^{\mathcal{P}}$ is 2-iteration for sufficiently many keys. The concrete condition is $|\mathcal{K}^{(0)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}}+3\big)|\mathcal{I}_{\leq n}| = O(\text{poly}(n))$. Our idea (non-trivially) generalizes existing specific attacks, which is elaborated as follows.

Outset: boomerang property. Our initial intuition lies in a chosen-key boomerang differentiator against the 2-round IEM cipher $y = K \oplus \mathbf{P}_2(K \oplus \mathbf{P}_1(K \oplus x))$, $K \in \{0,1\}^n$ (which is motivated by Andreeva et al.'s [1, Sect. 3.2, full version]). Briefly, for any x, let $u = \mathbf{P}_1(K \oplus x)$. The attack begins by computing four distinct pairs $(K_1,u_1),(K_2,u_2),(K_3,u_3),(K_4,u_4)$ with $u_1 = u_2$, $u_3 = u_4$; $K_1 \oplus u_1 = K_3 \oplus u_3$ and $K_2 \oplus u_2 = K_4 \oplus u_4$. I.e., they induce two collided inputs to $\mathbf{P}_1^{-1}$ and two collide inputs to $\mathbf{P}_2$. Once such four pairs are derived, the differentiator can computes a 4-tuple of cipher inputs/outputs $\big((K_1,x_1,y_1),...,(K_4,x_4,y_4)\big)$ that has $K_1 \oplus x_1 = K_2 \oplus x_2$, $K_3 \oplus x_3 = K_4 \oplus x_4$; $K_1 \oplus y_1 = K_3 \oplus y_3$, $K_2 \oplus y_2 = K_4 \oplus y_4$; as shown in Fig. 4 (left). Such a 4-tuple satisfies an evasive relation [39] and is hard to found in the ideal world. Actually the involved structure is the basis of the boomerang attack developed in [50].

A similar boomerang can be exhibited in the 2-round Feistel. Motivated by these, our differentiator against the general 2-iteration cipher tries to find

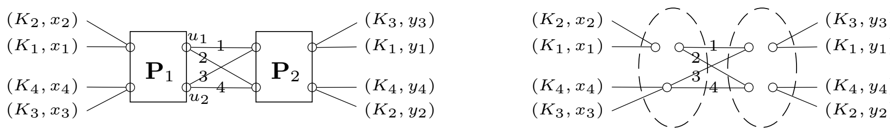

Fig. 4. (Top) Boomerang distinguisher in 2-round IEM. Circles indicate values in domain and range of $\mathbf{P}_1$ and $\mathbf{P}_2$ (in particular, u_1 and u_2 are marked), and lines indicate encipherment flows. To simplify, for lines between $\mathbf{P}_1$ and $\mathbf{P}_2$ the pair (K_j, u_j) is simplified as j. (Bottom) An example of boomerang distinguisher in 2-round general $E2$. Circles indicate values in domain and range of $\mathcal{P}$, and lines indicate encipherment flows. When a line "crosses" a pair of circles, it means the encipherment is non-inv-free in that round, and the two circles (naturally) indicate the $\mathcal{P}$ inputs and outputs; when a line "crosses" a single circle, it means the encipherment is non-inv-free in that round (so that rightward and leftward evaluations reach the same $\mathcal{P}$ input). Thus, the four encipherments are all non-inv-free in the 2nd round; in the 1st round, $E2^{\mathcal{P}}(K_1, x_1)$ and $E2^{\mathcal{P}}(K_2, x_2)$ are non-inv-free, while $E2^{\mathcal{P}}(K_3, x_3)$ and $E2^{\mathcal{P}}(K_4, x_4)$ are inv-free.

pairs $(K_1, u_1), (K_2, u_2), (K_3, u_3), (K_4, u_4) \in \mathcal{K}^{(0)} \times \{0,1\}^n$ that induce similar collided $\mathcal{P}$-calls, i.e., $\gamma_1^{in}(K_1, u_1) = \gamma_1^{in}(K_2, u_2)$, $\gamma_1^{in}(K_3, u_3) = \gamma_1^{in}(K_4, u_4)$; $\varphi_2^{in}(K_1, u_1) = \varphi_2^{in}(K_3, u_3)$ and $\varphi_2^{in}(K_2, u_2) = \varphi_2^{in}(K_4, u_4)$, as shown in Fig. 4 (right). This is a general boomerang property. Unlike Fig. 4 (left), the four encipherments may not be non-inv-free in two rounds: actually, Fig. 4 (right) serves an example where the 1st round of $E2^{\mathcal{P}}(K_3, x_3)$ and $E2^{\mathcal{P}}(K_4, x_4)$ are inv-free.

From boomerang to yoyo. But does such a 4-tuple ever exist? Unlike "concrete" ciphers such as IEM and Feistel, this is unclear in $E2$. To solve this, we apply Hoory's [32] result on girth (i.e., maximal length of cycles in a graph). Briefly, if we view the possible inputs to $\mathcal{P}$ as shores and the pairs $(K, u) \in \mathcal{K}^{(0)} \times \{0,1\}^n$ as edges, then we can build a bipartite graph $\mathcal{G}$, and the above 4-tuple becomes a 4-cycle C_4 (i.e., cycle of length 4) in $\mathcal{G}$. By Hoory [32] (which is restated in Sect. 2, Proposition 1), as long as the number of edges is large enough, such a 4-cycle or 4-tuple is guaranteed to exist.

However, as will be clear in the analysis (see Remark at page 34), the above requires $\mathcal{K}^{(0)}$ to be of exponential size, which would prohibit its application in our later attack against $E3$. To remedy, we consider longer cycles $C_{2\lambda}$, $\lambda \le n+1$. I.e., our differentiator seeks for a 2λ-tuple $\big((K_1, u_1), ..., (K_{2\lambda}, u_{2\lambda})\big)$ that has

$$\begin{aligned}
&\varphi_2^{in}(K_1, u_1) = \varphi_2^{in}(K_2, u_2), && \gamma_1^{in}(K_2, u_2) = \gamma_1^{in}(K_3, u_3),\\
&\varphi_2^{in}(K_3, u_3) = \varphi_2^{in}(K_4, u_4), && \gamma_1^{in}(K_4, u_4) = \gamma_1^{in}(K_5, u_5), \quad ...\\
&\varphi_2^{in}(K_{2\lambda-1}, u_{2\lambda-1}) = \varphi_2^{in}(K_{2\lambda}, u_{2\lambda}), && \gamma_1^{in}(K_{2\lambda}, u_{2\lambda}) = \gamma_1^{in}(K_1, u_1). \qquad (5)
\end{aligned}$$

Once such a 2λ-tuple is found, our differentiator can computes a 2λ-tuple of $E2$ inputs/outputs $\big((K_1, x_1, y_1), ..., (K_{2\lambda}, x_{2\lambda}, y_{2\lambda})\big)$ that has a "cycle of collisions". I.e., $\gamma_2^{in}(K_1, y_1) = \gamma_2^{in}(K_2, y_2), \varphi_1^{in}(K_2, x_2) = \varphi_1^{in}(K_3, x_3), ..., \gamma_2^{in}(K_{2\lambda-1}, y_{2\lambda-1}) = \gamma_2^{in}(K_{2\lambda}, y_{2\lambda}), \varphi_1^{in}(K_{2\lambda}, x_{2\lambda}) = \varphi_1^{in}(K_1, x_1)$. An example with $\lambda = 4$ is shown in Fig. 5. This is actually a general version of the yoyo distinguisher [46]. By

Hoory [32], $|\mathcal{K}^{(0)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}}+3\big)|\mathcal{I}_{\leq n}| = O(\mathrm{poly}(n))$ already suffices for the existence of $\big((K_1,u_1),...,(K_{2\lambda},u_{2\lambda})\big)$. Note that Hoory does not apply when $\mathcal{G}$ is a multigraph, but this implies existence of C_2. These solve our first problem.

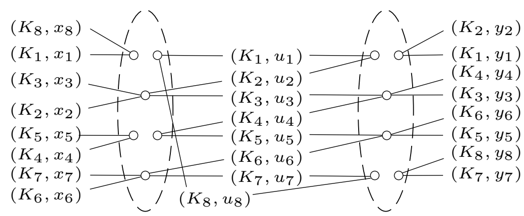

Fig. 5. An example of general yoyo distinguisher with $\lambda = 4$ in $E2$. Meanings of the objects follow Fig. 4.

Non-degenerate input functions. Subtleties remain. To argue that no polynomial-query simulator can work out a similar 2λ-tuple of ideal cipher inputs/outputs $\big(\mathbf{IC}(K_1,x_1) = y_1, ..., \mathbf{IC}(K_{2\lambda},x_{2\lambda}) = y_{2\lambda}\big)$, the input functions φ_1^{in} and γ_2^{in} must be somewhat "non-degenerate". Roughly, $\Pr[x \xleftarrow{\$} \{0,1\}^n : \varphi_1^{in}(K,x) = (i,\delta,z)] = \mathrm{negl}(n)$ and $\Pr[y \xleftarrow{\$} \{0,1\}^n : \gamma_2^{in}(K,y) = (i,\delta,z)] = \mathrm{negl}(n)$ for any K and any (i,δ,z).

Wlog consider φ_1. Indeed, due to the aforementioned "regularity" (Lemma 6), it can be proven $\Pr_x[\varphi_1^{in}(K,x) = (i,\delta,z) \mid \Pi_1^{\mathcal{P}}(K,x)$ non-inv-free$] = \mathrm{negl}(n)$. But $\varphi_1^{in}(K,\cdot)$ may lead $\Omega(2^n/\mathrm{poly}(n))$ distinct inv-free $\Pi_1^{\mathcal{P}}(K,x)$ to the same call $\mathcal{P}(i,\delta,z)$ (i.e., being highly biased), which enables the simulator to cheat.

A complete case-study thus has to consider whether φ_1^{in} and γ_2^{in} are "non-degenerate". However, input functions in virtually all blockciphers are indeed "non-degenerate" (please see Sect. 7): otherwise, the round is ridiculously weak. Meanwhile, complete case-study would take us quite far afield. We thereby decide to simplify and introduce *non-degenerate input functions* as an additional assumption for $E2^{\mathcal{P}}$ and $E3^{\mathcal{P}}$, i.e., $\Pr_x[\varphi_1^{in}(K,x) = (i,\delta,z) \mid \Pi_1^{\mathcal{P}}(K,x)$ inv-free$] = \mathrm{negl}(n)$ and $\Pr_y[\gamma_2^{in}(K,y) = (i,\delta,z) \mid (\Pi_2^{-1})^{\mathcal{P}}(K,y)$ inv-free$] = \mathrm{negl}(n)$. We refer to Sect. 7 for more details. With this additional restriction, we prove that no polynomial-query simulator can work out the aforementioned 2λ-tuple. In fact, Eq. (5) defines a novel evasive relation in 2-round general ciphers, which is stronger than differentiability. We refer to Sect. 7 for details.

It is crucial to restrict our discussion to iterated blockciphers: since the set of valid intermediate values u between the rounds is simply $\{0,1\}^n$, an attacker can pick such a u and compute forward or backward. Indeed, this middle-to-sides approach is common in known- and chosen-key attacks [37].

3.4 Attack 3-call iterated cipher $E3$

We further consider our 3-call model $E3^{\mathcal{P}}$. Recall that $E3^{\mathcal{P}} : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$ has $\kappa = \Theta(\mathrm{poly}(n))$, and its keyspace can be partitioned $\{0,1\}^\kappa = \mathcal{K}^{(0)} \sqcup \mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}$, such that:

(i) $E3^{\mathcal{P}}(K,x) = \Pi_6^{\mathcal{P}}(K\|\mathsf{kd}_1^{\mathcal{P}}(K),x)$ for all $K \in \mathcal{K}^{(2)}$;
(ii) $E3^{\mathcal{P}}(K,x) = \Pi_5^{\mathcal{P}}\big(K\|\mathsf{kd}_2^{\mathcal{P}}(K),\Pi_4^{\mathcal{P}}(K\|\mathsf{kd}_2^{\mathcal{P}}(K),x)\big)$ for all $K \in \mathcal{K}^{(1)}$;
(iii) $E3^{\mathcal{P}}(K,x) = \Pi_3^{\mathcal{P}}\big(K,\Pi_2^{\mathcal{P}}\big(K,\Pi_1^{\mathcal{P}}(K,x)\big)\big)$ for all $K \in \mathcal{K}^{(0)}$.

The sub-procedures $\mathsf{kd}_1^{\mathcal{P}}(K)$ and $\mathsf{kd}_2^{\mathcal{P}}(K)$ derive corresponding subkeys via two and one calls to $\mathcal{P}$ respectively. In addition, for $j = 1,2,...,6$, $\Pi_j^{\mathcal{P}}$ is a 1-call cipher with input and output functions $\varphi_j^{in},\varphi_j^{out},\gamma_j^{in}$ and γ_j^{out}. We refer to Fig. 15 for pseudocode of $E3^{\mathcal{P}}$.

When $E3^{\mathcal{P}}$ invokes kd_1 or kd_2 for sufficiently many keys $K \in \{0,1\}^{\kappa}$, we again derive poly(n) subkeys to reduce $E3^{\mathcal{P}}$ to $E1$ or $E2$ instances with polynomial keyspace, and apply our previous differentiators (thanks to that our differentiators break $E1$ and $E2$ with polynomial keyspace).

The crux is the case where $E3^{\mathcal{P}}(K,x) = \Pi_3^{\mathcal{P}}\big(K,\Pi_2^{\mathcal{P}}\big(K,\Pi_1^{\mathcal{P}}(K,x)\big)\big)$ for virtually all 2^{κ} keys K. Depending on whether the $\Theta(2^{\kappa+n})$ encipherments are "mostly" inv-free or not in the 3 rounds, exploitable non-random properties significantly vary in the $2^3 = 8$ cases and cannot be unified. We thereby have to appeal for a (lengthy) case-study.

Furthermore, note that inv-freeness can be data-dependent, which causes a subtle technical challenge. Namely, without querying P, one cannot fully decide if a certain encipherment is inv-free in the 3 rounds.[8] But querying P would trigger simulator actions in the ideal world, and the simulated P may be defined to change the inv-freeness of the encipherments in question. This turns out a technical challenge, and we call it *decisional inv-free problem*. Our solution is two-fold. First, we identified relevant conditions that are decidable without querying P, so that our differentiator could invoke the right subroutine for case-study without attracting simulator's attention. Meanwhile, to compute (intermediate) values of the encipherments in question, our differentiator (tries the best to) query the enciphering oracle E instead of P to avoid "waking" the simulator. Our case conditions ensure that the ideal cipher responses (in the ideal world) will satisfy our expectations on inv-freeness. We will elaborate more later.

Below we denote by $x \in \{0,1\}^n$ the plaintext, $u = \Pi_1^{\mathcal{P}}(K,x)$ the 1st round output, $w = \Pi_2^{\mathcal{P}}(K,u)$ the 2nd round output and $y = \Pi_3^{\mathcal{P}}(K,w)$ the ciphertext.

Case 1: there are $\boldsymbol{\Theta(2^{\kappa})}$ keys $\boldsymbol{K}$ s.t. only $\boldsymbol{o(2^n/\mathrm{poly}(n))}$ $\boldsymbol{\Pi_1^{\mathcal{P}}(K,x)}$ are non-inv-free, and only $\boldsymbol{o(2^n/\mathrm{poly}(n))}$ $\boldsymbol{(\Pi_3^{-1})^{\mathcal{P}}(K,y)}$ are non-inv-free. Roughly, this means most of the $\Theta(2^{\kappa+n})$ encipherments $E3^{\mathcal{P}}(K,x)$ are inv-free in 1st and 3rd rounds. A famous example is the 3-round Feistel Feistel3$(K,x) := \Psi^{\mathbf{F}_3}\big(K,\Psi^{\mathbf{F}_2}\big(K,\Psi^{\mathbf{F}_1}(K,x)\big)\big)$. Since the 2nd round could be arbitrary, the "hybrid" cipher Hyb$(K,x) := \Psi^{\mathbf{F}_3}\big(K,K\oplus\mathbf{P}\big(K\oplus\Psi^{\mathbf{F}_1}(K,x)\big)\big)$ is another example. A fact shared by the two examples is that *there are many encipherments that collide on* **F**- *or* **P**-*calls in the 2nd round*. Concretely,

[8] E.g., given K and a 1st round output $u \in \{0,1\}^n$, one can decide the inv-freeness of the corresponding encipherment in the 1st and 2nd rounds, since $\Pi_1^{\mathcal{P}}(K,u)$ and $(\Pi_2^{-1})^{\mathcal{P}}(K,u)$ can be decided. But without querying P, one cannot derive $w = \Pi_2^{\mathcal{P}}(K,u)$, and thus cannot decide if the process is inv-free in the 3rd round.

– In Feistel3, let $u = \Psi^{\mathbf{F}_1}(K, x)$. Then, for any $z \in \{0,1\}^{n/2}$, all the $2^{n/2}$ encipherments with key K and 1st round output $u = \star \| z$ call $\mathbf{F}_2(K \| z)$;
– In Hyb1, let $u = \Psi^{\mathbf{F}_1}(K, x)$. Then, for any $z \in \{0,1\}^n$, all the 2^n encipherments with (K, u), $K \oplus u = z$, call $\mathbf{P}(z)$.

It turns out that this can be proven in the general 3-round cipher (in this case): there exist $t = \Omega(\text{poly}(n))$ distinct intermediate values $(K_1, u_1), ..., (K_t, u_t)$ that collide on 2nd round $\mathcal{P}$-call, i.e., $\varphi_2^{in}(K_1, u_1) = ... = \varphi_2^{in}(K_t, u_t) = (i_2, \delta_2, z_2)$, as shown in Fig. 6 (left).

With such a "star" structure, we issue the "central" query $\mathrm{P}(i_2, \delta_2, z_2) \to z_2'$. In the real world, the response z_2' is consistent with $\Omega(\text{poly}(n))$ encipherments. Namely, for all $j \in \{1, ..., t\}$, suppose we evaluate $w_j \leftarrow \varphi_2^{out}(K_j, z_2', u_j)$, $x_j \leftarrow (\Pi_1^{-1})^{\mathrm{P}}(K_j, u_j)$ and $\mathrm{E}(K_j, x_j) \to y_j$. In the real world, if $\Pi_3^{\mathrm{P}}(K_j, w_j)$ is inv-free then it holds $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j)$. In the ideal world, S (roughly) has to find ideal cipher inputs/outputs $\mathbf{IC}(K_j, x_j) = y_j$ that have both inputs and outputs involved in certain collisions, i.e., $\varphi_1^{in}(K_j, x_j) = \gamma_1^{in}(K_j, u_j)$ and $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j)$, the probability of which can be proven negligible. This slightly oversimplifies, and we refer to Sect. 9.1 for details.

The question is: how the *decisional inv-free problem* affects in this case? The point is that: the above strategy only works for encipherments that are inv-free in both 1st and 3rd rounds. When we query $\mathrm{P}(i_2, \delta_2, z_2) \to z_2'$, S may define z_2' such that many of the involved $\Pi_3^{\mathrm{P}}(K_j, w_j)$ become non-inv-free. It seems cumbersome to argue that there remain many (useful) inv-free $\Pi_3^{\mathrm{P}}(K_j, w_j)$.

Such simulator strategies are prohibited by our case condition. In detail, if S want to define z_2' such that $\Pi_3^{\mathrm{P}}(K_j, w_j)$ is non-inv-free for some j, S must find an ideal cipher input/output $\mathbf{IC}(K_j, x_j) = y_j$ such that $(\Pi_3^{-1})^{\mathrm{P}}(K_j, y_j)$ is non-inv-free (otherwise, there appears inconsistency). Though,

– Since it must satisfy $\varphi_1^{in}(K_j, x_j) = \gamma_1^{in}(K_j, u_j)$ ($\Pi_1(K_j, x_j)$ is also inv-free), it cannot be due to a backward query $\mathbf{IC}^{-1}(K_j, y_j) \to x_j$;
– Since only $o(2^n/\text{poly}(n))$ $(\Pi_3^{-1})^{\mathcal{P}}(K_j, y)$ are non-inv-free for the involved key K_j, it cannot be due to a forward query $\mathbf{IC}(K_j, x_j) \to y_j$ either.

Thus, our attack strategy will reach $(\Pi_3^{-1})^{\mathrm{P}}(K_j, y_j)$ non-inv-free and succeed.

In our formal elaborations, this case is actually Subcase 3.1. We refer to Sect. 9.1 for details.

For the remaining, we first focus on the case that there are $\Theta(2^\kappa)$ keys K **s.t.** $\Omega(2^n/\text{poly}(n))$ $\Pi_1^{\mathcal{P}}(K, x)$ are non-inv-free, and that $\Omega(2^n/\text{poly}(n))$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free. Depending on whether the $\Omega(2^n/\text{poly}(n))$ 1st round outputs u have $\Pi_2^{\mathcal{P}}(K, u)$ non-inv-free or not, we further distinguish Case 2 and 3.

Case 2: there are $\Theta(2^\kappa)$ keys K s.t. $\Omega\big(\frac{2^n}{\text{poly}(n)}\big)$ u have $(\Pi_1^{-1})^{\mathcal{P}}(K, u)$ non-inv-free and $\Pi_2^{\mathcal{P}}(K, u)$ non-inv-free, and $\Omega\big(\frac{2^n}{\text{poly}(n)}\big)$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free. A crucial example is the 3-round IEM cipher $\text{IEM3}(K, x) := K \oplus \mathbf{P}_3\big(K \oplus \mathbf{P}_2\big(K \oplus \mathbf{P}_1(K \oplus x)\big)\big)$. Let $u = \mathbf{P}_1(K \oplus x)$ in IEM3. Let's see an attack

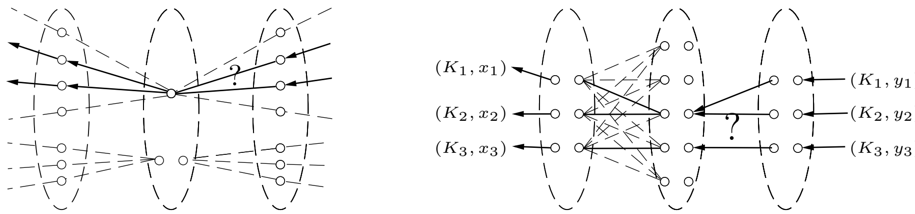

Fig. 6. Query structures used in attacking 3-round general ciphers. (Left) Structures for Case 1. The dashed lines show two examples of "stars": the top "star" centers around an inv-free 2nd round encipherment, while the bottom "star" centers around a non-inv-free 2nd round. (Right) Structures for Case 2. The dashes lines show a simple example of biclique $K_{3,5}$ (we certainly cannot draw "exponential-size"). The bold lines indicate the encipherments sampled by our attack, the arrows indicate the direction of our attack's evaluations, and the ? indicates where our attack checks equalities.

for intuition. We begin with three intermediate values $(K_1, u_1), (K_2, u_2), (K_3, u_3)$ that have $K_1 \oplus u_1 = K_2 \oplus u_2 \neq K_3 \oplus u_3$, and then query P_1^{-1}, compute the plaintexts $x_1 \leftarrow K_1 \oplus \mathrm{P}_1^{-1}(u_1)$, $x_2 \leftarrow K_2 \oplus \mathrm{P}_1^{-1}(u_2)$ and $x_3 \leftarrow K_3 \oplus \mathrm{P}_1^{-1}(u_3)$, and acquire the ciphertexts $\mathrm{E}(K_1, x_1) \rightarrow y_1$, $\mathrm{E}(K_2, x_2) \rightarrow y_2$ and $\mathrm{E}(K_3, x_3) \rightarrow y_3$. With these, if we query $\mathrm{P}_3^{-1}(K_1 \oplus y_1) \rightarrow w$ and $\mathrm{P}_3^{-1}(K_2 \oplus y_2) \rightarrow w'$, then the simulator S shall define them such that $w \oplus K_1 = w' \oplus K_2$; if we query $\mathrm{P}_3^{-1}(K_1 \oplus y_1) \rightarrow w$ and $\mathrm{P}_3^{-1}(K_3 \oplus y_3) \rightarrow w'$, then S shall define $w \oplus K_1 \neq w' \oplus K_3$. S cannot know our choice and thus won't be prepared correctly.

To translate this attack to the general 3-round model, we need to grasp its core idea. It turns out to be a structure of exponential size: if we view the range of P_1 and the domain of P_2 as two shores and the pairs (K, u) as edges, then we can build a biclique $K_{3,2^n}$. Due to this, given the P_1^{-1} and P_3^{-1} queries, there remain exponential possibilities for the three relevant encipherments $(K_1, x_1), (K_2, x_2)$ and (K_3, x_3), and S cannot pinpoint them. Furthermore, S does not know our choice of P_3^{-1}-queries either. These ideas were also used by Andreeva et al.'s attack on IEM3 [1, Sect. 3.3, full version] (though details slightly deviate).

In the general 3-round cipher, we should view possible inputs to $\mathcal{P}$ as shores and intermediate pairs (K, u) as edges to build a bipartite graph $\mathcal{G}$ (which resembles our previous treatments of 2-iteration), as shown in Fig. 6 (right). Again we need to prove that there indeed exists a biclique $K_{3,2^n}$ as a sub-graph in $\mathcal{G}$, and we resort to *Zarankiewicz numbers* [24]. Concretely, by Kővári, Sós and Turán (KST) [38] (restated in Sect. 2, Proposition 2), as long as κ is large enough (though still $\Theta(n)$), the number of edges is large enough and $K_{3,2^n}$ is guaranteed to exist. This enables finding and exploiting the three encipherments.

Regarding the *decisional inv-free problem*, the setting is simpler than Case 1. In detail, it can be proven that we can sample encipherments $(K_1, x_1), (K_2, x_2)$ and (K_3, x_3) that were unlikely queried by the simulator S. By this and by the case condition, we reach $(\Pi_3^{-1})^{\mathcal{P}}(K_1, y_1), (\Pi_3^{-1})^{\mathcal{P}}(K_2, y_2)$ and $(\Pi_3^{-1})^{\mathcal{P}}(K_3, y_3)$ with non-negligible probability $\Omega(1/\mathrm{poly}(n))$ after querying $\mathrm{E}(K_1, x_1) \rightarrow y_1$, $\mathrm{E}(K_2, x_2) \rightarrow y_2$ and $\mathrm{E}(K_3, x_3) \rightarrow y_3$. This fits into our expectations.

There remain subtleties: similarly to the 2-round case (Sect. 3.3), KST's result [38] only applies to simple graphs. When $\mathcal{G}$ is a multigraph with high multiplicity, we have to resort to a dedicated treatment. Interestingly, using the fact that there can be many edges between a single pair of vertexes, we are able to find three encipherments $(K_1, u_1), (K_2, u_2)$ and (K_3, u_3) that are similar to the above "simple" case. The involved structure is given in Fig. 7 (left).

We refer to Sect. 9.2 and 9.3 (Subcase 3.2) for details.

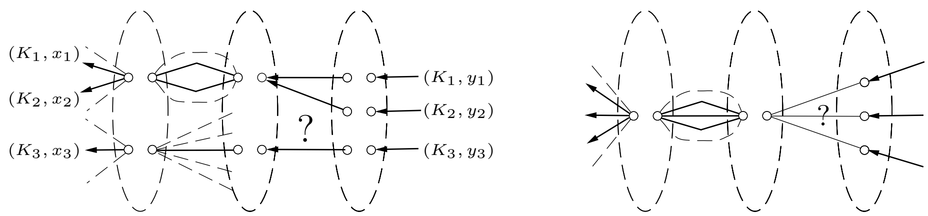

Fig. 7. Query structures used in attacking 3-round general ciphers, when the involved graphs contain heavy multi-edges. (Left) Structures for Case 2. The idea is adapted from Fig. 6 (right). The dashed arcs indicate that there are many (superpolynomial) distinct encipherments "crossing" the same pair of inputs in 1st and 2nd rounds. The dashed lines show that the number of possible encipherments "crossing" the two relevant $\mathcal{P}$-inputs are exponential. (Right) Structures for Case 4. The idea is adapted from Fig. 8 (right): we can find λ useful encipherments within a single pair of $\mathcal{P}$-inputs (in the 1st and 2nd rounds). The use of bold lines, arrows and ? follows Fig. 6.

Case 3: there are $\Theta(2^{\kappa})$ keys K s.t. $\Omega(\frac{2^n}{\mathrm{poly}(n)})$ u have $(\Pi_1^{-1})^{\mathcal{P}}(K, u)$ non-inv-free and $\Pi_2^{\mathcal{P}}(K, u)$ inv-free, and $\Omega(\frac{2^n}{\mathrm{poly}(n)})$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free. Our attack in this case reuses already discussed ideas. In detail, (roughly) we sample a pair (K, u) from the $\Omega(2^{\kappa+n}/\mathrm{poly}(n))$ pairs that have 1st round non-inv-free while 2nd round inv-free. We then evaluate backward to $x \leftarrow (\Pi_1^{-1})^{\mathrm{P}}(K, u)$, "wrap" by querying $\mathrm{E}(K, x) \rightarrow y$ and further $w \leftarrow (\Pi_3^{-1})^{\mathrm{P}}(K, y)$. Since $\Omega(2^n/\mathrm{poly}(n))$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free, we reach $(\Pi_3^{-1})^{\mathrm{P}}(K, y)$ non-inv-free with a non-negligible probability and overcome the *decisional inv-free problem*, as shown in Fig. 8 (left). Since $\Pi_2(K, u)$ is inv-free, if we are interacting with the general 3-round cipher then it holds $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$ (as discussed in Sect. 3.2). On the other hand, if we are interacting with the ideal world $(\mathbf{IC}, S^{\mathbf{IC}})$, the simulator S only gains two P-calls $\mathrm{P}(\gamma_1^{in}(K, u))$ and $\mathrm{P}(\gamma_3^{in}(K, y))$. As discussed in Case 2, they won't enable S to pinpoint the encipherment (K, x). Consequently, S is unable to define simulated P and enforce the equality $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$.

In our attack (Sect. 9), this corresponds to Subcase 3.3, and we refer to Sect. 9.4 for the formal elaboration (which uses pigeonhole principle and non-degeneracy of φ_2^{in}).

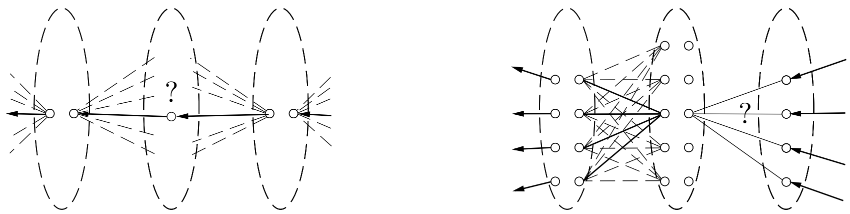

Fig. 8. Query structures used in attacking 3-round general ciphers. (Left) Structures for Case 3. The dashed lines show that the number of possible encipherments "crossing" the two relevant $\mathcal{P}$-inputs are exponential (though, they may not overlap). (Right) Structures for Case 4. The figure shows a simple example with $\lambda = 4$. The use of bold lines, arrows and ? follows Fig. 6.

We then focus on the case that there are $\Theta(2^\kappa)$ keys K **s.t.** $\Omega(2^n/\text{poly}(n))$ $\Pi_1^{\mathcal{P}}(K, x)$ are non-inv-free, and that $o(2^n/\text{poly}(n))$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free. Similarly to Case 2 and 3, we further distinguish Case 4 and 5.

Case 4: there are $\Theta(2^\kappa)$ keys K s.t. $\Omega\big(\frac{2^n}{\text{poly}(n)}\big)$ u have $(\Pi_1^{-1})^{\mathcal{P}}(K, u)$ non-inv-free and $\Pi_2^{\mathcal{P}}(K, u)$ non-inv-free, and $o\big(\frac{2^n}{\text{poly}(n)}\big)$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free. In this case, we reuse the query structures found in Case 2. We also reuse the idea that the inv-free 3rd round allows checking consistency.

In detail, consider the bipartite graph $\mathcal{G}$ built between the 1st and 2nd round (which resembles Case 2). When $\mathcal{G}$ is (roughly) simple, we can find a biclique $K_{\lambda,2^n}$ with $\lambda = m_{max}$, which resembles Case 2. We then sample one vertex (i_2, δ_2, z_2) from the right shore of $K_{\lambda,2^n}$, pinpointing λ encipherments $(K_1, u_1), ..., (K_\lambda, u_\lambda)$ that invoke $\text{P}(i_2, \delta_2, z_2)$ in the 2nd round. See Fig. 8 (left). We then evaluate backward $x_1 \leftarrow (\Pi_1^{-1})^{\text{P}}(K_1, u_1), ..., x_\lambda \leftarrow (\Pi_1^{-1})^{\text{P}}(K_\lambda, u_\lambda)$, "wrap" $\text{E}(K_1, x_1) \to y_1, ..., \text{E}(K_\lambda, x_\lambda) \to y_\lambda$. As only $o\big(\frac{2^n}{\text{poly}(n)}\big)$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free, we likely reach $(\Pi_3^{-1})^{\text{P}}(K_1, y_1), ..., (\Pi_3^{-1})^{\text{P}}(K_\lambda, y_\lambda)$ inv-free and overcome the *decisional inv-free problem*, as shown in Fig. 8 (right).

In the real world, the 2nd round outputs $(K_1, w_1), ..., (K_\lambda, w_\lambda)$ of these encipherments are derivable from $(K_1, w_1), ..., (K_\lambda, w_\lambda)$ using a fixed $z_2' \in \{0,1\}^{m(i_2)}$. Meanwhile, they have $\varphi_3^{in}(K_1, w_1) = \gamma_3^{in}(K_1, y_1), ..., \varphi_3^{in}(K_\lambda, w_\lambda) = \gamma_3^{in}(K_\lambda, y_\lambda)$. To simulate consistently, the simulator S in the ideal world has to find a corresponding $z_2' \in \{0,1\}^{m(i_2)}$ satisfying the λ equalities for the ideal cipher responses $y_1, ..., y_\lambda$. Since $\lambda = m_{max}$, this can be proven infeasible.

When $\mathcal{G}$ is a multigraph with high multiplicity, a single pair of vertexes already suffices to pinpoint λ encipherments $(K_1, u_1), ..., (K_\lambda, u_\lambda)$ that invoke the same $\text{P}(i_2, \delta_2, z_2)$ in the 2nd round, as shown in Fig. 7 (right). Our above idea thus remains applicable.

In our attack (Sect. 9), this corresponds to Subcase 3.4, and we refer to Sect. 9.5 for the formal elaboration.

Case 5: there are $\Theta(2^\kappa)$ keys K s.t. $\Omega\big(\frac{2^n}{\text{poly}(n)}\big)$ u have $(\Pi_1^{-1})^{\mathcal{P}}(K, u)$ non-inv-free and $\Pi_2^{\mathcal{P}}(K, u)$ inv-free, and $o\big(\frac{2^n}{\text{poly}(n)}\big)$ $(\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are

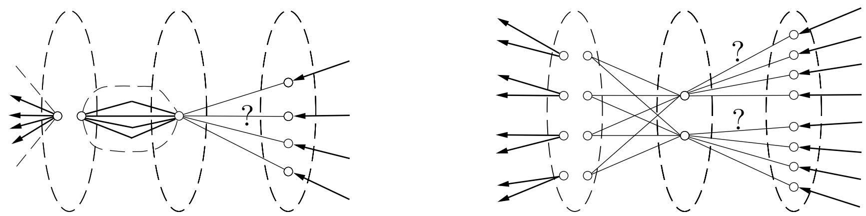

Fig. 9. Query structures used in attacking 3-round general ciphers, Case 5. (Left) When the graph contains heavy multi-edges, we can find λ useful encipherments within a single pair of $\mathcal{P}$-inputs (in the 1st and 2nd rounds). The figure shows a simple example with $\lambda = 4$. (Right) When the graph does not contain too many multi-edges (and the biclique $K_{\lambda,2}$ exists).

non-inv-free. Again, consider the bipartite graph $\mathcal{G}$ built between the 1st and 2nd round (which resembles Cases 2 and 4). When $\mathcal{G}$ is a multigraph with high multiplicity, we reuse the idea of Case 4 and exploit the structure shown in Fig. 9 (left). The case that $\mathcal{G}$ is (roughly) simple turns out to be the most complicated, and many of our earlier attempts failed. Our eventual idea is built upon a polynomial-size boomerang structure in the 1st and 2nd rounds, which is depicted in Fig. 9 (right).

In detail, we seek for a biclique $K_{\lambda,2}$, $\lambda = O(m_{max})$, in $\mathcal{G}$, as shown in Fig. 9 (right). Again by KST [38], such bicliques exist as long as κ is large enough (though still $\Theta(m_{max}n) = \Theta(\text{poly}(n))$).

The biclique $K_{\lambda,2}$ pinpoints two groups of encipherments, with each group colliding on a 2nd round $\mathcal{P}$-call, as shown in Fig. 9. Therefore, in the real world, there are two $\mathcal{P}$-outputs that are consistent with all the 2λ encipherments. Meanwhile, every encipherment in one group is paired with an encipherment in the other group, such that the two encipherments collide on the 1st round $\mathcal{P}$-call. By these, in the ideal world, the simulator S has to seek for 2λ ideal cipher queries that have both inputs and outputs involved in certain collisions. Namely, the 2λ ideal cipher queries can be arranged in a $2 \times \lambda$ matrix, such that:

– For every pair of ideal cipher queries in every column, the corresponding simulated encipherments collide on the 1st round $\mathcal{P}$-call; and
– For each group of λ ideal cipher queries in each row, there exists a response z_2' that satisfy certain relation with their λ ciphertexts.

When $\lambda = O(m_{max})$, this can be proven infeasible.

In our attack (Sect. 9), this corresponds to Subcase 3.5, and we refer to Sect. 9.6 for the formal presentations. Some of our earlier failed attempts are also available there.

Other cases: there are $\boldsymbol{\Theta(2^\kappa)}$ keys $\boldsymbol{K}$ s.t. $\boldsymbol{o\big(\frac{2^n}{\text{poly}(n)}\big)}$ $\boldsymbol{\Pi_1^{\mathcal{P}}(K,x)}$ are non-inv-free and $\boldsymbol{\Omega\big(\frac{2^n}{\text{poly}(n)}\big)}$ $\boldsymbol{(\Pi_3^{-1})^{\mathcal{P}}(K,y)}$ are non-inv-free. Then, if most $w = (\Pi_3^{-1})^{\mathcal{P}}(K,y)$ are inv-free w.r.t. Π_2, it follows the above Case 4 by symmetry; if most $w = (\Pi_3^{-1})^{\mathcal{P}}(K, y)$ are non-inv-free w.r.t. Π_2, it follows the above Case 5 by symmetry. We thereby complete the case-study.

Again, we refer to Sect. 8 for the complicated details.

4 Fundamental Properties

By the *notion* of blockciphers, a blockcipher shall be *deterministic* and *efficiently invertible*. The latter has been reflected in Fig. 10 (and Figs. 12 and 15 as well). Below we formalize the former for blockcipher oracle procedures.

Definition 1 (Deterministicness). *An oracle procedure $E^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ instantiating a blockcipher must be deterministic, meaning that for any $(K, x) \in \mathcal{K} \times \{0,1\}^n$, let $y = E^{\mathcal{P}}(K, x)$. Then, the transcripts of $\mathcal{P}$-queries and responses obtained during encipherment $E^{\mathcal{P}}(K, x)$ and decipherment $(E^{-1})^{\mathcal{P}}(K, y)$ are always identical. (E.g., if $E^{\mathcal{P}}(K, x)$ queries $\mathcal{P}(i, \delta, z) \to z'$ at some stage, then $(E^{-1})^{\mathcal{P}}(K, y)$ queries either $\mathcal{P}(i, \delta, z) \to z'$ or $\mathcal{P}(i, \overline{\delta}, z') \to z$ at some stage.)*

Two more properties/assumptions that we rely on are **oracle-independence of sub-procedures** and **non-degeneracy of $E^{\mathcal{P}}$**.

Oracle-independence means sub-procedures in $E^{\mathcal{P}}$ must be *oracle-independent*. Since the oracle procedure $E^{\mathcal{P}}$ (or black-box cryptographic construction) has a fixed description, this seems obvious (and indeed common in black-box constructions [25] and impossibility proofs [9,45]). Though, we highlight it for clarity. Interestingly, ad hoc blockciphers also strive for such independence (probably to avoid unexpected internal dependency). For example, in AES, the ShiftRows and MixColumns steps are rather independent from SubBytes.

Non-degeneracy means no encipherment $E^{\mathcal{P}}(K, x)$ can be approximately computed using less $\mathcal{P}$ calls than $E^{\mathcal{P}}$, i.e., $E^{\mathcal{P}}$ cannot be "simplified". Formally,

Definition 2 ((Everywhere) Non-degenerate Oracle Procedure). *An oracle procedure $E^{\mathcal{P}}$ is (everywhere) $\varepsilon_{de(E)}$-non-degenerate, if*

$$\max_{E',K,x}\Big\{\Pr_{\mathcal{P}}\big[(E')^{\mathcal{P}}(K, x) = E^{\mathcal{P}}(K, x)\big]\Big\} \le \varepsilon_{de(E)} = negl(n). \tag{6}$$

where the maximum is taken over all $(K, x) \in \{0,1\}^\kappa \times \{0,1\}^n$ and all oracle procedures $(E')^{\mathcal{P}}$ such that the number of $\mathcal{P}$-calls made during computing $(E')^{\mathcal{P}}(K, x)$ is less than $E^{\mathcal{P}}(K, x)$.

Why non-degenerate? If $E1^{\mathcal{P}}$ is $1/\text{poly}(n)$-non-degenerate, then there is an obvious differentiator with advantage $1/\text{poly}(n) - 2^{-n}$. More importantly, a t-call blockcipher "uses" all of its t $\mathcal{P}$-calls "effectively" only if it is non-degenerate.

5 General 1-Call Blockciphers

We first elaborate on our 1-call cipher model $E1^{\mathcal{P}}$ in Sect. 5.1. Then, we characterize the properties of $E1^{\mathcal{P}}$ in Sect. 5.2.

Algorithm $E1^{\mathcal{P}}(K,x)$ // $(K,x) \in \mathcal{K} \times \{0,1\}^n$ $(i,\delta,z) \leftarrow \varphi^{in}(K,x)$ $z' \leftarrow \mathcal{P}(i,\delta,z)$ $y \leftarrow \varphi^{out}(K,z',x)$ **return** y	**Algorithm** $(E1^{-1})^{\mathcal{P}}(K,x)$ $(i,\delta,z) \leftarrow \gamma^{in}(K,y)$ $z' \leftarrow \mathcal{P}(i,\delta,z)$ $x \leftarrow \gamma^{out}(K,z',y)$ **return** x

Fig. 10. Definition of the 1-call blockcipher $E1^{\mathcal{P}}$. $\varphi^{in}, \varphi^{out}, \gamma^{in}$, and γ^{out} are all deterministic and oracle-independent.

5.1 General Model of 1-call Blockciphers/Rounds

We consider any blockcipher oracle procedure $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ that is built from the permutation family $\mathcal{P}$ in the following way. Let φ^{in} and φ^{out} be two arbitrary deterministic functions that are computable by the computational class of the differentiator(s). Then, $E1^{\mathcal{P}}(K,x) := \varphi^{out}\big(K, \mathcal{P}\big(\varphi^{in}(K,x)\big), x\big)$.

Since blockciphers are *efficiently invertible* by definitions, $E1^{\mathcal{P}}$ is accomplished by $(E1^{-1})^{\mathcal{P}}(K,y) := \gamma^{out}\big(K, \mathcal{P}\big(\gamma^{in}(K,y)\big), y\big)$ using two other deterministic functions γ^{in} and γ^{out}. We stress that to ensure $(E1^{-1})^{\mathcal{P}}\big(K, E1^{\mathcal{P}}(K,x)\big) \equiv x$, γ^{in} and γ^{out} are strongly correlated with φ^{in} and φ^{out}, and this will be crucial for our attack. A formal description using pseudocode is given in Fig. 10.

Examples to facilitate understanding. First, the key-prepended Feistel round [16] uses $\mathbf{F} : \{0,1\}^{\kappa+n/2} \to \{0,1\}^{n/2}$ and defines $\Psi^{\mathbf{F}}(K,x) := \mathsf{right}_{n/2}(x) \oplus \mathbf{F}(K \| \mathsf{left}_{n/2}(x)) \| \mathsf{left}_{n/2}(x)$. It is an $E1$ instance with

$$\begin{aligned} \varphi^{in}(K,x) &:= \big(i, +, K \| \mathsf{right}_{n/2}(x) \| [0]_{n/2}\big), \\ \varphi^{out}(K,z',x) &:= \mathsf{right}_{n/2}(x) \oplus \mathsf{right}_{n/2}(z') \| \mathsf{left}_{n/2}(x) \end{aligned} \tag{7}$$

using an index i with $m(i) = \kappa + n$ (and truncated permutation [14]).

Second, the "key-alternating Feistel" round [27] uses $\mathbf{F} : \{0,1\}^{n/2} \to \{0,1\}^{n/2}$ and defines $\mathsf{KAF}^{\mathbf{F}}(K,x) := \mathsf{right}_{n/2}(x) \oplus \mathbf{F}(K \oplus \mathsf{left}_{n/2}(x)) \| \mathsf{left}_{n/2}(x)$. It is an $E1$ instance with

$$\begin{aligned} \varphi^{in}(K,x) &:= \big(i, +, K \oplus \mathsf{right}_{n/2}(x) \| [0]_{n/2}\big), \\ \varphi^{out}(K,z',x) &:= \mathsf{right}_{n/2}(x) \oplus \mathsf{right}_{n/2}(z') \| \mathsf{left}_{n/2}(x) \end{aligned} \tag{8}$$

using an index i with $m(i) = n$ (and truncated permutation).

Third, the IEM round [23] defines $\mathrm{EM}^{\mathbf{P}}(K,x) := K \oplus \mathbf{P}(K \oplus x)$ for $\mathbf{P} \in \mathsf{Perm}(n)$. It is an $E1$ instance with

$$\varphi^{in}(K,x) := (i, +, K \oplus x), \ \ \varphi^{out}(K,z',x) := K \oplus z' \tag{9}$$

using an index i with $m(i) = n$.

Finally, a "key-alternating" Misty-R cipher round [2, Chapter 3.18.8] defines $\text{Misty-R}^{\mathbf{P}}(K,x) := \mathbf{P}\big(K \oplus \mathsf{right}_{n/2}(x)\big) \big\| \big(\mathsf{left}_{n/2}(x) \oplus \mathbf{P}\big(K \oplus \mathsf{right}_{n/2}(x)\big)\big)$ for $\mathbf{P} \in \mathsf{Perm}(n/2)$. It is an $E1$ instance with

$$\varphi^{in}(K,x) := \big(i, +, K \oplus \mathsf{right}_{n/2}(x)\big), \ \ \varphi^{out}(K,z',x) := z' \big\| \big(\mathsf{left}_{n/2}(x) \oplus z'\big) \tag{10}$$

using an index i with $m(i) = n/2$.

It is easy to see unbalanced Feistel [2, Chapter 1.3.1], Lai-Massey [2, Chapter 1.5] and keyed Feistel rounds are instances of $E1$ as well. Though, $E1$ does not cover multi-line generalized Feistel [2, Chapter 1.3.1] (which makes multiple $\mathcal{P}$ calls per round) and Swap-Or-Not [29] (which uses small-range functions).

5.2 Properties of 1-call Blockciphers/Rounds

We first introduce several helper sets. We then discuss properties of *data-dependent encipherments*, *inverse-free* and *non-inverse-free encipherments* in turn.

Notations. For any 1-call cipher $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ and any K in its keyspace $\mathcal{K}$, define

$$\begin{aligned}\mathsf{Dom}_{\mathsf{if}}(E1, K) &:= \Big\{x \in \{0,1\}^n : \delta = \delta',\ \text{where } (i,\delta,z) = \varphi^{in}(K,x),\\ &\qquad (i,\delta',z') = \gamma^{in}(K,y),\ y = E1^{\mathcal{P}}(K,x)\Big\},\\ \mathsf{Rng}_{\mathsf{if}}(E1, K) &:= \Big\{y \in \{0,1\}^n : \delta = \delta',\ \text{where } (i,\delta,z) = \gamma^{in}(K,y),\\ &\qquad (i,\delta',z') = \varphi^{in}(K,x),\ x = (E1^{-1})^{\mathcal{P}}(K,y)\Big\},\\ \mathsf{Dom}_{\mathsf{ni}}(E1, K) &:= \{0,1\}^n \backslash \mathsf{Dom}_{\mathsf{if}}(E1,K),\quad \mathsf{Rng}_{\mathsf{ni}}(E1,K) := \{0,1\}^n \backslash \mathsf{Rng}_{\mathsf{if}}(E1,K).\end{aligned} \tag{11}$$

For $x \in \mathsf{Dom}_{\mathsf{if}}(E1, K)$, the encipherment $E1^{\mathcal{P}}(K,x)$ and the corresponding decipherment $(E1^{-1})^{\mathcal{P}}(K,y)$ call $\mathcal{P}$ on the same direction. Therefore, $E1^{\mathcal{P}}(K,x)$ is *inverse-free* (inv-free for short), as reflected by the subscript if. Otherwise, $E1^{\mathcal{P}}(K,x)$ is *non-inverse-free* (non-inv-free), as reflected by ni. We remark that $E1^{\mathcal{P}}(K,x) = E'(\mathcal{P}(f(K)),x)$ is also inv-free, although it may not match classical understandings.

For $\mathsf{tag} \in \{\mathsf{ni}, \mathsf{if}\}$, define sets for plaintexts/ciphertexts in $\mathsf{Dom}_{\mathsf{tag}}/\mathsf{Rng}_{\mathsf{tag}}$ that are mapped to a certain $\mathcal{P}$ input (i,δ,z):

$$\begin{aligned}\mathsf{Dom}_{\mathsf{tag}}(E1,K,i,\delta,z) &:= \big\{x \in \mathsf{Dom}_{\mathsf{tag}}(E1,K) : \varphi^{in}(K,x) = (i,\delta,z)\big\},\\ \mathsf{Rng}_{\mathsf{tag}}(E1,K,i,\delta,z) &:= \big\{y \in \mathsf{Rng}_{\mathsf{tag}}(E1,K) : \gamma^{in}(K,y) = (i,\delta,z)\big\}.\end{aligned} \tag{12}$$

$$\begin{aligned}\mathsf{Dom}_{\mathsf{tag}}(E1,K,i,\delta) &:= \cup_{z\in\{0,1\}^{m(i)}} \mathsf{Dom}_{\mathsf{tag}}(E1,K,i,\delta,z),\\ \mathsf{Rng}_{\mathsf{tag}}(E1,K,i,\delta) &:= \cup_{z\in\{0,1\}^{m(i)}} \mathsf{Rng}_{\mathsf{tag}}(E1,K,i,\delta,z).\end{aligned} \tag{13}$$

We slightly abuse the notation $\mathsf{Rng}_\star$ to denote the actual ranges of the input functions φ^{in} and γ^{in}. In detail, for $\mathsf{tag} \in \{\mathsf{ni}, \mathsf{if}\}$, define

$$\begin{aligned}\mathsf{Rng}_{\mathsf{tag}}(\varphi^{in},K) &:= \big\{(i,\delta,z) : (i,\delta,z) = \varphi^{in}(K,x) \text{ for some } x \in \mathsf{Dom}_{\mathsf{tag}}(E1,K)\big\},\\ \mathsf{Rng}_{\mathsf{tag}}(\gamma^{in},K) &:= \big\{(i,\delta,z) : (i,\delta,z) = \gamma^{in}(K,y) \text{ for some } y \in \mathsf{Rng}_{\mathsf{tag}}(E1,K)\big\}.\\ \mathsf{Rng}_{\mathsf{tag}}(\varphi^{in}) &:= \cup_{K\in\mathcal{K}} \mathsf{Rng}_{\mathsf{tag}}(\varphi^{in},K),\quad \mathsf{Rng}_{\mathsf{tag}}(\gamma^{in}) := \cup_{K\in\mathcal{K}} \mathsf{Rng}_{\mathsf{tag}}(\gamma^{in},K).\end{aligned} \tag{14}$$

On data-dependence. As indicated by the partition $\{0,1\}^n = \mathsf{Dom}_{\mathsf{if}}(E1,K) \sqcup \mathsf{Dom}_{\mathsf{ni}}(E1,K)$, the inv-freeness of $E1^{\mathcal{P}}(K,x)$ can be data-dependent. Though, as mentioned in Sect. 3.1, (surprisingly) *one can decide whether an encipherment* $E1^{\mathcal{P}}(K,x)$ *is inv-free without querying* $\mathcal{P}$. This turns out crucial in our attacks.

Lemma 1 (Inv-freeness is oracle-independent). *Consider the blockcipher* $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ *in Fig. 10. Then, for any pair* $(K,x) \in \mathcal{K} \times \{0,1\}^n$*, resp.* $(K,y) \in \mathcal{K} \times \{0,1\}^n$*, whether* $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)$*, resp.* $y \in \mathsf{Rng}_{\mathsf{if}}(E1,K)$*, can be determined without querying* $\mathcal{P}$*.*

Proof. Assume otherwise, and let (K,x) be the input such that whether $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)$ depends on $\mathcal{P}$. Let $(i,\delta,z) = \varphi^{in}(K,x)$ and $y = E1^{\mathcal{P}}(K,x)$. Since $E1^{\mathcal{P}}(K,x)$ only makes one query $\mathcal{P}(i,\delta,z)$ to $\mathcal{P}$, $E1^{\mathcal{P}}(K,x)$ is inv-free if and only if $\mathcal{P}(i,\delta,z)$ is in a certain subset of $\{0,1\}^{m(i)}$. Namely, there exists a partition $\{0,1\}^{m(i)} = \mathcal{Z}_\delta \cup \mathcal{Z}_{\overline{\delta}}$ such that $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)$ if and only if $\mathcal{P}(i,\delta,z) \in \mathcal{Z}_\delta$.

However, let $(i,\delta',z') = \gamma^{in}(K,y)$, then $y \in \mathsf{Rng}_{\mathsf{if}}(E1,K)$ if and only if $\delta' = \delta$. This means it always holds $\mathcal{P}(i,\delta,z) \in \mathcal{Z}_{\delta'}$, where δ' is *fixed* by the definition of the function γ^{in}. This violates our assumption that γ^{in} is oracle-independent.

Therefore, one can decide if $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)$ solely by computations. The argument for $y \in \mathsf{Rng}_{\mathsf{if}}(E1,K)$ is similar by symmetry. □

Properties of inv-free encipherments. We now formalize the intuitive weaknesses of inv-free encipherments discussed in Sect. 3.1.

Lemma 2 (Inv-freeness preserves partial inputs). *Consider the 1-call blockcipher* $E1^{\mathcal{P}}$ *in Fig. 10. Then, for any pair* (K,x)*,* $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)$*, it holds* $\gamma^{in}(K,y) = \varphi^{in}(K,x)$ *for* $y = E1^{\mathcal{P}}(K,x)$*. It further implies* $|\mathsf{Dom}_{\mathsf{if}}(E1,i,\delta,z)| = |\mathsf{Rng}_{\mathsf{if}}(E1,i,\delta,z)|$ *for any* $i \in \mathcal{I}$*,* $\delta \in \{+,-\}$ *and* $z \in \{0,1\}^{m(i)}$*. Proof: this is a straightforward implication of Fig. 10 and Definition 1.*

The second observation follows by non-degeneracy: if there exists one inv-free encipherment $E1^{\mathcal{P}}(K,x)$, then there must exist superpolynomially many.

Lemma 3 (Inv-freeness can't be unique). *Consider the 1-call blockcipher* $E1^{\mathcal{P}}$ *in Fig. 10. If* $E1^{\mathcal{P}}$ *is* $\varepsilon_{de(E1)}$*-non-degenerate in the sense of Definition 2, then for any* (K,i,δ,z) *such that* $\mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z) \neq \emptyset$*, it holds*

$$\big|\mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z)\big| > 1/\varepsilon_{de(E1)} = \Omega(poly(n)). \tag{15}$$

Proof. Assume otherwise, i.e., $|\mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z)| \leq \varepsilon_{de(E1)}^{-1}$ for some (K,i,δ,z). By Lemma 2, for any $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z)$, the corresponding ciphertext $y = E1^{\mathcal{P}}(K,x)$ must have $y \in \mathsf{Rng}_{\mathsf{if}}(E1,K,i,\delta,z)$. By Lemma 2, $|\mathsf{Rng}_{\mathsf{if}}(E1,K,i,\delta,z)| = |\mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z)| \leq 1/\varepsilon_{de(E1)}$. By these, for any $x \in \mathsf{Dom}_{\mathsf{if}}(E1,K,i,\delta,z)$, one can uniformly pick $y \xleftarrow{\$} \mathsf{Rng}_{\mathsf{if}}(E1,K,i,\delta,z)$ to *encipher* (K,x) *without querying* $\mathcal{P}$ *at all*, and the success probability is at least $\varepsilon_{de(E1)}$. This contradicts the assumption that $E1^{\mathcal{P}}$ is $\varepsilon_{de(E1)}$-non-degenerate as Eq. (6). □

An implication of Lemma 3 is that the ranges of φ^{in} and γ^{in} cannot be too large.

Lemma 4 (Functions in inv-free encipherments). *Consider the 1-call blockcipher $E1^{\mathcal{P}}$ in Fig. 10. If $E1^{\mathcal{P}}$ is $\varepsilon_{de(E1)}$-non-degenerate (see Definition 2), then it holds $\left|\mathsf{Rng}_{\mathsf{if}}(\varphi^{in}, K)\right| = \left|\mathsf{Rng}_{\mathsf{if}}(\gamma^{in}, K)\right| \leq |\mathsf{Dom}_{\mathsf{if}}(E1, K)| \cdot \varepsilon_{de(E1)} \leq 2^n \cdot \varepsilon_{de(E1)}$.*

Proof. The claim $\left|\mathsf{Rng}_{\mathsf{if}}(\varphi^{in}, K)\right| = \left|\mathsf{Rng}_{\mathsf{if}}(\gamma^{in}, K)\right|$ clearly holds by Lemma 2. To show $\left|\mathsf{Rng}_{\mathsf{if}}(\varphi^{in}, K)\right| \leq |\mathsf{Dom}_{\mathsf{if}}(E1, K)|\varepsilon_{de(E1)} = 2^n \cdot \varepsilon_{de(E1)}$ (since $|\mathsf{Dom}_{\mathsf{if}}(E1, K)| = 2^n$), assume otherwise, i.e., $\varphi^{in}(K, \cdot)$ maps the values in $\mathsf{Dom}_{\mathsf{if}}(E1, K)$ to $> |\mathsf{Dom}_{\mathsf{if}}(E1, K)|\varepsilon_{de(E1)}$ images. But by Lemma 3, it holds $|\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)| > \frac{1}{\varepsilon_{de(E1)}}$ for every $(i, \delta, z) \in \mathsf{Rng}_{\mathsf{if}}(\varphi^{in}, K)$. By this,

$$\begin{aligned}\sum_{(i,\delta,z)\in \mathsf{Rng}_{\mathsf{if}}(\varphi^{in},K)} |\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)| &> |\mathsf{Dom}_{\mathsf{if}}(E1, K)|\varepsilon_{de(E1)} \times \frac{1}{\varepsilon_{de(E1)}} \\ &> |\mathsf{Dom}_{\mathsf{if}}(E1, K)|,\end{aligned}$$

a contradiction. □

Properties of non-inv-free encipherment. For $E1^{\mathcal{P}}(x)$, $x \in \mathsf{Dom}_{\mathsf{ni}}(E1, K)$, our first observation is that $E1^{\mathcal{P}}(x)$ cannot query *wide random permutations* (as discussed in Sect. 3.2). We now elaborate on the "regularity" of φ^{in} and γ^{in}. For example, in the IEM round (see Eq. (9)), for every $K \in \{0,1\}^n$ we have $|\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +)| = 2^n = 2^{m(i)}$, and for every $z \in \{0,1\}^n$ we have $|\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +, z)| = 1 = |\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +)|/2^{m(i)}$. In the "key-alternating" Misty-R round (see Eq. (9)), we have $|\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +)| = 2^n$ for $K \in \{0,1\}^{n/2}$ and $|\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +, z)| = 2^n = |\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, +)|/2^{m(i)}$ with $m(i) = n/2$ for every $z \in \{0,1\}^{n/2}$.

Below we formalize the above first idea and show that the actual ranges of the functions φ^{in} and γ^{in} must be somewhat limited.

Lemma 5 (Non-inv-free encipherments cannot query wide P). *Consider the 1-call blockcipher $E1^{\mathcal{P}}$ in Fig. 10. Then:*

- *For any key $K \in \mathcal{K}$ and any $x \in \mathsf{Dom}_{\mathsf{ni}}(E1, K)$, let $(i, \delta, z) = \varphi^{in}(K, x)$, then it holds $i \in \mathcal{I}_{\leq n}$;*
- *Similarly, for any key $K \in \mathcal{K}$ and any $y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)$, let $(i, \delta, z) = \gamma^{in}(K, y)$, then it holds $i \in \mathcal{I}_{\leq n}$.*

Consequently, $\left|\mathsf{Rng}_{\mathsf{ni}}(\varphi^{in})\right| \leq |\mathcal{I}_{\leq n}| 2^{n+1}$, $\left|\mathsf{Rng}_{\mathsf{ni}}(\gamma^{in})\right| \leq |\mathcal{I}_{\leq n}| 2^{n+1}$.

Proof. Assume otherwise, then there exists (K, x) such that $x \in \mathsf{Dom}_{\mathsf{ni}}(E1, K)$, and $(i, \delta, z) = \varphi^{in}(K, x)$ has $i \in \mathcal{I}_{>n}$. This means $|z| = m(i) > n$.

Furthermore, the oracle response $\mathcal{P}(\varphi^{in}(K, x)) = z'$ must be that there exists $y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)$ such that $\gamma^{in}(K, y) = (i, \overline{\delta}, z')$. Since $x \in \mathsf{Dom}_{\mathsf{ni}}(E1, K) \subseteq \{0,1\}^n$ and $y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K) \subseteq \{0,1\}^n$, the number t of z and z' related by such relation is at most 2^n, meaning that $\mathcal{P}(i,\cdot,\cdot)$ must map a set of $t \le 2^n$ possible z values (that are determined by φ^{in}) to a set of $t \le 2^n$ possible z' values (that are determined by γ^{in}). This violates the oracle-independence assumption on φ^{in} and γ^{in}. Therefore, for any (K,x), $x \in \mathsf{Dom_{ni}}(E1,K)$, let $(i,\delta,z) = \varphi^{in}(K,x)$, then it holds $i \in \mathcal{I}_{\le n}$, i.e., $m(i) \le n$. It thus follows $\left|\mathsf{Rng_{ni}}(\varphi^{in})\right| \le \left|\{+,-\}\right| \times |\mathcal{I}_{\le n}| \times 2^n \le |\mathcal{I}_{\le n}|2^{n+1}$ and $\left|\mathsf{Rng_{ni}}(\gamma^{in})\right| \le |\mathcal{I}_{\le n}|2^{n+1}$. □

We then formalize the above (somewhat surprising) "regularity" idea.

Lemma 6 (Regularity in non-inv-free encipherments). *Consider the 1-call blockcipher $E1^{\mathcal{P}}$ in Fig. 10. Then, for any $K \in \mathcal{K}$ and any $(i,\delta) \in \mathcal{I}_{\le n} \times \{+,-\}$, the restriction of φ^{in} to $\mathsf{Dom_{ni}}(E1,K,i,\delta)$ (resp., the restriction of γ^{in} to $\mathsf{Rng_{ni}}(E1,K,i,\delta)$) is regular. I.e., the following holds for any $z,z' \in \{0,1\}^{m(i)}$*

$$\left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z)\right| = \left|\mathsf{Rng_{ni}}(E1,K,i,\overline{\delta},z')\right| = \frac{\left|\mathsf{Dom_{ni}}(E1,K,i,\delta)\right|}{2^{m(i)}}.$$

This also means $\left|\mathsf{Dom_{ni}}(E1,K,i,\delta)\right|$ must be divisible by $2^{m(i)}$.

Proof. First, by the deterministicness property (Definition 1), it can be seen that $\left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z)\right| = \left|\mathsf{Rng_{ni}}(E1,K,i,\overline{\delta},z')\right|$ holds for any $z' = \mathcal{P}(i,\delta,z)$.

Then, to prove $\left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z)\right| = \left|\mathsf{Dom_{ni}}(E1,K,i,\delta)\right|/2^{m(i)}$, assume that $\left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z)\right| = C_{(i,\delta,z)}$ for any $(i,\delta) \in \mathcal{I}_{\le n} \times \{+,-\}$ and $z \in \{0,1\}^{m(i)}$. We show that $C_{(i,\delta,z)}$ is a constant for all $z \in \{0,1\}^{m(i)}$, which immediately implies $\left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z)\right| = \frac{|\mathsf{Dom_{ni}}(E1,K,i,\delta)|}{2^{m(i)}}$.

Towards a contradiction, assume that there exists $z^\circ \in \{0,1\}^{m(i)}$ such that $C_{(i,\delta,z^\circ)} \ne C_{(i,\delta,z)}$ for all $z \ne z^\circ$. Then, as we have shown, there necessarily exists $z'^\circ \in \{0,1\}^{m(i)}$ such that $\left|\mathsf{Rng_{ni}}(E1,K,i,\overline{\delta},z'^\circ)\right| = \left|\mathsf{Dom_{ni}}(E1,K,i,\delta,z^\circ)\right| = C_{(i,\delta,z^\circ)} \ne C_{(i,\delta,z)}$ for all $z \ne z^\circ$. But this means $\mathcal{P}(i,\cdot,\cdot)$ must map z° to z'°, both of which are fixed by φ^{in} and γ^{in}. This violates our assumption that φ^{in} and γ^{in} are oracle-independent. □

By Lemma 5, we can derive the collision probability among images of φ^{in} and γ^{in} as follows.

Corollary 1 (Probability of collisions). *For any (i,δ,z) and any set $\mathcal{S} \subseteq \{0,1\}^n$ with $|\mathcal{S}| = poly(n)$, when n is sufficiently large it holds*

$$\Pr\left[y \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \gamma^{in}(K,y) = (i,\delta,z) \;\middle|\; y \in \mathsf{Rng_{ni}}(E1,K)\right] \le \frac{2}{2^{m_{min}}}, \quad (16)$$

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \varphi^{in}(K,x) = (i,\delta,z) \;\middle|\; x \in \mathsf{Dom_{ni}}(E1,K)\right] \le \frac{2}{2^{m_{min}}}. \quad (17)$$

Proof. To have Eq. (16):

$$\begin{aligned} &\Pr\left[y \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \gamma^{in}(K,y) = (i,\delta,z) \;\middle|\; y \in \mathsf{Rng_{ni}}(E1,K)\right] \\ = &\frac{|\mathsf{Rng_{ni}}(E1,K,i,\delta,z) \backslash \mathcal{S}|}{|\mathsf{Rng_{ni}}(E1,K) \backslash \mathcal{S}|} \le \frac{|\mathsf{Rng_{ni}}(E1,K,i,\delta)|}{2^{m(i)} \times \left(|\mathsf{Rng_{ni}}(E1,K)| - |\mathcal{S}|\right)} \text{ (Lemma 6)}. \end{aligned}$$

1: **Algorithm** $D1^{\mathrm{E,P}}(\mathcal{K})$
2: **Determines** the sets $\mathsf{Dom}_{\mathsf{if}}(E1, K)$ and $\mathsf{Dom}_{\mathsf{ni}}(E1, K)$ for all $K \in \mathcal{K}$
3: **if** $\exists K \in \mathcal{K}$ **s.t.** $\mathsf{Dom}_{\mathsf{if}}(E1, K) \neq \emptyset$ **then** // Case 1
4: **Picks** $\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)$ **s.t.** $\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z) \neq \emptyset$
5: $\lambda \leftarrow \lceil \frac{m(i)}{n} \rceil + 1$, $\mathrm{P}(i, \delta, z) \to z'$
6: **Picks** distinct $x_1, ..., x_\lambda \in \mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)$
7: **Outputs** 1 **iff.** $\varphi^{out}(K, z', x_j) = \mathrm{E}(K, x_j)$ for all $j \in \{1, ..., \lambda\}$
8: **else** // Case 2: $\mathsf{Dom}_{\mathsf{ni}}(E1, K) = \{0,1\}^n$ for all $K \in \mathcal{K}$
9: **Picks** (K, x) and (K', x') with $\varphi^{in}(K, x) = \varphi^{in}(K', x')$
10: $\mathrm{E}(K, x) \to y$, $\mathrm{E}(K', x') \to y'$
11: **Outputs** 1 **iff.** $\gamma^{in}(K, y) = \gamma^{in}(K', y')$
12: **end if**

Fig. 11. Differentiator $D1^{\mathrm{E,P}}$ used in Theorem 1.

Since $|\mathcal{S}| = \mathrm{poly}(n)$, we have $|\mathsf{Rng}_{\mathsf{ni}}(E1, K)| - |\mathcal{S}| \geq |\mathsf{Rng}_{\mathsf{ni}}(E1, K)|/2$ for large enough n, so that

$$\begin{aligned}&\Pr\big[y \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \gamma^{in}(K, y) = (i, \delta, z) \;\big|\; y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)\big] \\ \leq\; & \frac{2|\mathsf{Rng}_{\mathsf{ni}}(E1, K, i, \delta)|}{2^{m(i)} \times |\mathsf{Rng}_{\mathsf{ni}}(E1, K)|} \leq \frac{2}{2^{m(i)}} \leq \frac{2}{2^{m_{min}}}.\end{aligned}$$

Eq. (17) follows similarly. These complete the proof. □

6 Attack 1-Call Blockciphers

After the preparations in Sect. 5.2, we are able to establish insecurity of 1-call ciphers $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ of Fig. 10.

Theorem 1 (Differentiability of $E1^{\mathcal{P}}$). *Let $E1^{\mathcal{P}} : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ be a blockcipher defined by Fig. 10. Assume that $E1^{\mathcal{P}}$ is deterministic and $\varepsilon_{de(E1)}$-non-degenerate in the sense of Definition 2, and its keyspace has $|\mathcal{K}| \geq 2|\mathcal{I}_{\leq n}| + 1 = O(poly(n))$. Then, when n is sufficiently large, there exists a differentiator $D1^{\mathrm{E,P}}$ making at most $\lceil m_{max}/n \rceil + 2$ queries and has an advantage at least $1 - \frac{{m_{max}}^2}{2^n} - \frac{2}{2^{m_{min}}} = 1 - negl(n)$.*

It is crucial to restrict $|\mathcal{K}| > |\mathcal{I}_{\leq n}|$: otherwise, $\mathcal{P}$ may already offer $|\mathcal{K}|$ independent n-bit random permutations. There are two purposes to consider $|\mathcal{K}| = \mathrm{poly}(n)$. First, it strengthens the negative result (i.e., even indifferentiable cipher of logarithmic key length is impossible). Second, such $D1$ can function as subroutines of $D2$ and $D3$ in Sect. 7 and 8.

The differentiator $D1^{\mathrm{E,P}}$ is formally described in Fig. 11. Below we provide intuitions and analyses. First, by Lemma 1, line 2 is well-defined: the sets $\mathsf{Dom}_{\mathsf{if}}(E1, K)$ and $\mathsf{Dom}_{\mathsf{ni}}(E1, K)$ can be computed without querying $\mathcal{P}$. The actions of $D1$ then depends on these sets, and consist of two cases.

Case 1: $\exists K \in \{0,1\}^\kappa$ such that $\mathsf{Dom}_{\mathsf{if}}(E1, K) \neq \emptyset$. There certainly exists (i, δ, z) such that $\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, \delta, z) \neq \emptyset$ (so that $D1$ succeeds at line 4). By

Lemma 3, it holds $\left|\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)\right| > 1/\varepsilon_{de(E1)} = \Omega(\text{poly}(n))$ for any such (i, δ, z).

Then, $\varphi^{out}(K, z', \cdot) : \mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z) \to \mathsf{Rng}_{\mathsf{if}}(E1, K, i, \delta, z)$ is a bijection defined using a random string $z' = \mathcal{P}(i, \delta, z)$ of polynomial size $m(i) = \text{poly}(n)$. The problem now reduces to differentiating $\varphi^{out}\big(K, v, \cdot\big)$ from a random bijection, and our approach in Fig. 11 follows [41, Sect. 6]. Since φ^{out} is deterministic, $\mathbf{y} = \big(E1^{\mathcal{P}}(K, x_1), ..., E1^{\mathcal{P}}(K, x_\lambda)\big)$ is fully specified by the string z', and there are at most $2^{m(i)}$ possible values for $\mathbf{y}$. Let $\mathcal{Y}(z')$ be the set of these $2^{m(i)}$ values. Clearly, $D1$ always outputs 1 when interacting with $(E1^{\mathcal{P}}, \mathcal{P})$. In contrast, when $D1$ is interacting with the ideal world, it outputs 1 only if $\big(\mathbf{IC}(K, x_1), ..., \mathbf{IC}(K, x_\lambda)\big) \in \mathcal{Y}(z')$, the probability of which is at most $2^{m(i)}/2^{\lambda n} + \lambda^2/2^n \le 1/2^{\lambda n - m(i)} + \lambda^2/2^{n+1}$ (plus $\lambda^2/2^{n+1}$ the distance between $\big(\mathbf{IC}(K, x_1), ..., \mathbf{IC}(K, x_\lambda)\big)$ and a λn-bit uniform string). Since $\lambda = \lceil \frac{m(i)}{n} \rceil + 1 \le m_{max}$ (as long as $n \ge 2$ and $m_{max} \ge 2$), we have $1/2^{\lambda n - m(i)} \le 1/2^n$ and advantage at least $1 - 1/2^n - {m_{max}}^2/2^{n+1} \ge 1 - {m_{max}}^2/2^n$.

Case 2: $\mathsf{Dom}_{\mathsf{ni}}(E1, K) = \{0,1\}^n$ for all $K \in \mathcal{K}$, i.e., all $E1^{\mathcal{P}}(K, x)$ are non-inverse-free. Since $\left|\mathsf{Rng}_{\mathsf{ni}}(\varphi^{in})\right| \le |\mathcal{I}_{\le n}|2^{n+1}$ by Lemma 5, there exist $(|\mathcal{K}| - 2|\mathcal{I}_{\le n}|)2^n \ge 2^n$ pairs $\big((K, x), (K', x')\big)$ with $\varphi^{in}(K, x) = \varphi^{in}(K', x')$. Hence, $D1^{\mathrm{E},\mathrm{P}}$ can find (K, x) and (K', x') that have $\varphi^{in}(K, x) = \varphi^{in}(K', x')$ at line 9 and then queries $\mathrm{E}(K, x) \to y$ and $\mathrm{E}(K', x') \to y'$.

When $D1^{\mathrm{E},\mathrm{P}}$ is interacting with the real world $(E1^{\mathcal{P}}, \mathcal{P})$, it necessarily holds $\gamma^{in}(K, y) = \gamma^{in}(K', y')$. When $D1^{\mathrm{E},\mathrm{P}}$ is interacting with the ideal world $(\mathbf{IC}, S^{\mathbf{IC}})$, the response y' is uniformly distributed in either $\{0,1\}^n$ or $\{0,1\}^n \backslash \{y\}$. By Corollary 1 Eq. (16), the probability to have $\gamma^{in}(K, y) = \gamma^{in}(K', y')$ is at most $2/2^{m_{min}}$ in the ideal world, and the advantage in this case is $1 - 2/2^{m_{min}}$.

Summary. By the above, when n is sufficiently large, it holds

$$\mathrm{Adv}^{indif}_{E1^{\mathcal{P}}, \mathbf{IC}, S}(D1) \ge \min\left\{1 - \frac{{m_{max}}^2}{2^n}, 1 - \frac{2}{2^{m_{min}}}\right\} \ge 1 - \frac{{m_{max}}^2}{2^n} - \frac{2}{2^{m_{min}}}.$$

Discussion. For clearness, we summarize the uses of *Fundamental Properties*:

(i) Case 1 relies on Lemma 3 due to non-degeneracy of $E1$;
(ii) Case 2 relies on Corollary 1 due to invertibility and oracle-independence.

Case 1 relies on non-degeneracy of $E1$. In Appendix B we present another attack without non-degeneracy, which may serve additional insights.

7 Attack 2-Call Iterated Blockciphers

For 2- and 3-call ciphers, we restrict to *iterated blockciphers* that are built from *key derivation functions* and *rounds*. For a 2-call cipher $E2^{\mathcal{P}}$, this means encipherment $E2^{\mathcal{P}}(K, x)$ must proceed with either of the following flows:

– **Type-I**: $E2^{\mathcal{P}}(K, x) = \Pi_3^{\mathcal{P}}\big(K \| \mathsf{kd}^{\mathcal{P}}(K), x\big)$ for a 1-call function $\mathsf{kd}^{\mathcal{P}} : \{0,1\}^\kappa \to \{0,1\}^{m_{max}}$ and a 1-call cipher $\Pi_3^{\mathcal{P}} : \{0,1\}^{\kappa + m_{max}} \times \{0,1\}^n \to \{0,1\}^n$, or

– **Type-II**: $E2^{\mathcal{P}}(K,x) = \Pi_2^{\mathcal{P}}\big(K, \Pi_1^{\mathcal{P}}(K,x)\big)$ for two 1-call ciphers/rounds $\Pi_1^{\mathcal{P}}, \Pi_2^{\mathcal{P}} : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$.

The keyspace is partitioned $\mathcal{K} = \mathcal{K}^{(0)} \sqcup \mathcal{K}^{(1)}$, such that $E2^{\mathcal{P}}(K,\cdot)$ follows **Type-I** encipherment if and only if $K \in \mathcal{K}^{(1)}$. Formally, we consider the cipher $E2^{\mathcal{P}}$ defined in Fig. 12. As mentioned in the Introduction, there is no need to use multiple $\mathcal{P}_1, \mathcal{P}_2, ...$, since $\mathcal{P}$ already provides multiple independent permutations.

Below, before describing our attack and analysis in Sect. 7.2, we first introduce our non-degenerate assumption on φ^{in}.

7.1 Non-degenerate input functions

As discussed the overview (Sect. 3.3), we make an additional non-degenerate assumption on the input functions φ^{in}. Formally,

Definition 3 (Non-degenerate Keyed Function). *A keyed function $\varphi^{in}(\cdot,\cdot)$ is $\varepsilon_{de(\varphi^{in})}$-non-degenerate, if the following two upper bounds hold (recall from Lemma 1 that the set $\mathsf{Dom}_{\mathsf{if}}(E1,K)$ is fully determined by φ^{in}):*

$$\max_{K,(i,\delta,z)} \Big\{ \Pr\big[x \xleftarrow{\$} \{0,1\}^n : \varphi^{in}(K,x) = (i,\delta,z) \;\big|\; x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)\big]\Big\} \le \varepsilon_{de(\varphi^{in})}.$$

$$\max_{x,(i,\delta,z)} \Big\{ \Pr\big[K \xleftarrow{\$} \{0,1\}^\kappa : \varphi^{in}(K,x) = (i,\delta,z) \;\big|\; x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)\big]\Big\} \le \varepsilon_{de(\varphi^{in})}.$$

By default, we assume $\varepsilon_{de(\varphi^{in})} = negl(n)$ is negligible.

Input functions in common inv-free blockciphers are indeed non-degenerate: e.g., key-prepended Feistel round has $\varphi^{in}(K,x) = \big(i, +, K \| \mathsf{right}_{n/2}(x) \| [0]_{n/2}\big)$ (see Eq. (7)) and $\varepsilon_{de(\varphi^{in})} = \max\{1/2^n, 1/2^\kappa\}$; key-alternating Feistel round has $\varphi^{in}(K,x) = \big(i, +, K \oplus \mathsf{right}_{n/2}(x) \| [0]_{n/2}\big)$ (see Eq. (8)) and $\varepsilon_{de(\varphi^{in})} = 1/2^n$.

As shown in Corollary 1 and as interesting insights, the analogue of Definition 3 for non-inverse-free enciphering *can be proven*. However, for inverse-free enciphering we have to make this assumption to simplify. A round using degenerate input functions is ridiculously weak, and one can figure out trivial attacks. However, extending such "trivial attacks" to 2- and 3-call ciphers turns out to be rather cumbersome, and this would take us quite far afield. We hope that future works could develop new analytic approaches to overcome this difficulty.

If x or K is sampled from $\{0,1\}^n \backslash \mathcal{S}$ or $\{0,1\}^\kappa \backslash \mathcal{S}$ for $|\mathcal{S}| = \mathrm{poly}(n)$, then for sufficiently large n, it can proven

$$\max_{K,(i,\delta,z)} \Big\{ \Pr_{x \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S}} \big[\varphi^{in}(K,x) = (i,\delta,z) \;\big|\; x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)\big]\Big\} \le 2\varepsilon_{de(\varphi^{in})}.$$

$$\max_{K,(i,\delta,z)} \Big\{ \Pr_{y \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S}} \big[\gamma^{in}(K,y) = (i,\delta,z) \;\big|\; y \in \mathsf{Rng}_{\mathsf{if}}(E1,K)\big]\Big\} \le 2\varepsilon_{de(\varphi^{in})}. \quad (18)$$

$$\max_{x,(i,\delta,z)} \Big\{ \Pr_{K \xleftarrow{\$} \{0,1\}^\kappa \backslash \mathcal{S}} \big[\varphi^{in}(K,x) = (i,\delta,z) \;\big|\; x \in \mathsf{Dom}_{\mathsf{if}}(E1,K)\big]\Big\} \le 2\varepsilon_{de(\varphi^{in})}.$$

$$\max_{y,(i,\delta,z)} \Big\{ \Pr_{K \xleftarrow{\$} \{0,1\}^\kappa \backslash \mathcal{S}} \big[\gamma^{in}(K,y) = (i,\delta,z) \;\big|\; y \in \mathsf{Rng}_{\mathsf{if}}(E1,K)\big]\Big\} \le 2\varepsilon_{de(\varphi^{in})}. \quad (19)$$

```
Algorithm E2^P(K, x)                      Algorithm (E2^{-1})^P(K, y)
if K ∈ K^(1) then                         if K ∈ K^(1) then
  return Π_3^P(K‖kd^P(K), x)                return (Π_3^{-1})^P(K‖kd^P(K), y)
else        // K ∈ K^(0)                  else        // K ∈ K^(0)
  u ← Π_1^P(K, x)                           u ← (Π_2^{-1})^P(K, y)
  return Π_2^P(K, u)                        return (Π_1^{-1})^P(K, u)
end if                                    end if

Algorithm Π_j^P(K, x) // j ∈ {0, 1, ..., 5}   Algorithm (Π_j^{-1})^P(K, y) //
(i_j, δ_j, z_j) ← φ_j^in(K, x)            j ∈ {0, 1, ..., 5}
z'_j ← P(i_j, δ_j, z_j)                   (i_j, δ_j, z_j) ← γ_j^in(K, y)
y ← φ_j^out(K, z'_j, x)                   z'_j ← P(i_j, δ_j, z_j)
return y                                  x ← φ_j^out(K, z'_j, y)
                                          return x
Algorithm kd^P(K)
(i, δ, z) ← f(K)
z' ← P(i, δ, z)
return z'
```

Fig. 12. Definition of the 2-call iterated blockcipher $E2^{\mathcal{P}}$.

To see these, note that by Definition 3, for any $K \in \mathcal{K}$ and (i, δ, z), we have

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n : \varphi^{in}(K, x) = (i, \delta, z) \;\middle|\; x \in \mathsf{Dom_{if}}(E1, K)\right] \tag{20}$$
$$= \frac{|\mathsf{Dom_{if}}(E1, K, i, \delta, z)|}{|\mathsf{Dom_{if}}(E1, K)|} \leq \varepsilon_{de(\varphi^{in})}.$$

This means

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \varphi^{in}(K, x) = (i, \delta, z) \;\middle|\; x \in \mathsf{Dom_{if}}(E1, K)\right]$$
$$= \frac{|\mathsf{Dom_{if}}(E1, K, i, \delta, z) \backslash \mathcal{S}|}{|\mathsf{Dom_{if}}(E1, K) \backslash \mathcal{S}|} \leq \frac{|\mathsf{Dom_{if}}(E1, K, i, \delta, z)|}{|\mathsf{Dom_{if}}(E1, K)| - |\mathcal{S}|}.$$

By Lemma 3, it holds $|\mathsf{Dom_{if}}(E1, K, i, \delta, z)| = \Omega(\mathrm{poly}(n))$. By Lemma 3, it holds $|\mathsf{Dom_{if}}(E1, K)| = \Omega(\mathrm{poly}(n))$. On the other hand, $|\mathcal{S}| = \mathrm{poly}(n)$. Therefore, when n is sufficiently large, it holds $|\mathcal{S}| \leq |\mathsf{Dom_{if}}(E1, K)|/2$, and thus

$$\Pr\left[x \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S} : \varphi^{in}(K, x) = (i, \delta, z) \;\middle|\; x \in \mathsf{Dom_{if}}(E1, K)\right]$$
$$\leq \frac{|\mathsf{Dom_{if}}(E1, K, i, \delta, z)|}{|\mathsf{Dom_{if}}(E1, K)| - |\mathcal{S}|} \leq \frac{|\mathsf{Dom_{if}}(E1, K, i, \delta, z)|}{|\mathsf{Dom_{if}}(E1, K)|/2} \leq 2\varepsilon_{de(\varphi^{in})}$$

holds for any $K \in \mathcal{K}$ and (i, δ, z). By this,

$$\max_{K,(i,\delta,z)} \left\{ \Pr_{y \xleftarrow{\$} \{0,1\}^n \backslash \mathcal{S}} \left[\gamma^{in}(K, y) = (i, \delta, z) \;\middle|\; y \in \mathsf{Rng_{if}}(E1, K)\right] \right\} \leq 2\varepsilon_{de(\varphi^{in})}$$

follows from Lemma 2. For the two probabilities regarding $K \xleftarrow{\$} \{0,1\}^\kappa \backslash \mathcal{S}$, since we only apply them for Theorem 3 which has $\kappa = \mathrm{poly}(n)$, the proofs are similar.

```
Algorithm D2^{E,P}(K^(0), K^(1))
if |K^(1)| ≥ 2|I_{≤n}| + 1 then          // Case 1: key derivation plus idealized round
  Let K_1 ⊆ K^(1) be such that |K_1| ≥ 2|I_{≤n}| + 1, K_1^{ex} ← ∅
  for K ∈ K_1 do          // Derives subkeys for keys in K_1
    z' ← P(f(K)), K_1^{ex} ← K_1^{ex} ∪ {K||z'}
  end for
  Invokes D1^{E,P}(K_1^{ex}) (see Fig. 11) on Π_3^P : K_1 × {0,1}^n → {0,1}^n
else          // The other case: |K^(0)| ≥ (6(3|I_{≤n}|)^{1/n} + 3)|I_{≤n}|
  Invokes Subroutine Yoyo(Π_1, Π_2, K^(0))          // See Fig. 14
end if
```

Fig. 13. Differentiator $D2^{\mathrm{E,P}}$ used in Theorem 2.

Subroutine Yoyo$(\Pi_1, \Pi_2, \mathcal{K}^{(0)})$

// Recall that $E2^{\mathcal{P}}(K,x) = \Pi_2^{\mathcal{P}}\big(K, \Pi_1^{\mathcal{P}}(K,x)\big)$ for all $K \in \mathcal{K}^{(0)}$

Picks 2λ distinct pairs $\big((K_j,u_j)\big)_{j=1,\ldots,2\lambda} \in \big(\mathcal{K}^{(0)} \times \{0,1\}^n\big)^{2\lambda}$, that satisfy $\lambda \le n+1$, and

$$\begin{aligned}
\varphi_2^{in}(K_1,u_1) &= \varphi_2^{in}(K_2,u_2), & \gamma_1^{in}(K_2,u_2) &= \gamma_1^{in}(K_3,u_3),\\
\varphi_2^{in}(K_3,u_3) &= \varphi_2^{in}(K_4,u_4), & \gamma_1^{in}(K_4,u_4) &= \gamma_1^{in}(K_5,u_5),\ldots\\
\varphi_2^{in}(K_{2\lambda-1},u_{2\lambda-1}) &= \varphi_2^{in}(K_{2\lambda},u_{2\lambda}), & \gamma_1^{in}(K_{2\lambda},u_{2\lambda}) &= \gamma_1^{in}(K_1,u_1).
\end{aligned} \tag{21}$$

$\mathrm{P}\big(\gamma_1^{in}(K_2,u_2)\big) \to z_1'^{(1)}$, $\mathrm{P}\big(\gamma_1^{in}(K_4,u_4)\big) \to z_1'^{(2)}$, ..., $\mathrm{P}\big(\gamma_1^{in}(K_{2\lambda},u_{2\lambda})\big) \to z_1'^{(\lambda)}$

$\mathrm{P}\big(\varphi_2^{in}(K_1,u_1)\big) \to z_2'^{(1)}$, $\mathrm{P}\big(\varphi_2^{in}(K_3,u_3)\big) \to z_2'^{(2)}$, ..., $\mathrm{P}\big(\varphi_2^{in}(K_{2\lambda-1},u_{2\lambda-1})\big) \to z_2'^{(\lambda)}$

Derives the 2λ corresponding plaintexts and ciphertexts via querying E:

$$\begin{aligned}
x_1 &\leftarrow \gamma_1^{out}(K_1, z_1'^{(\lambda)}, u_1), \quad \mathrm{E}(K_1,x_1) \to y_1,\\
x_2 &\leftarrow \gamma_1^{out}(K_2, z_1'^{(1)}, u_2), \quad \mathrm{E}(K_2,x_2) \to y_2, \ldots,\\
x_{2\lambda} &\leftarrow \gamma_1^{out}(K_{2\lambda}, z_1'^{(\lambda)}, u_{2\lambda}), \quad \mathrm{E}(K_{2\lambda},x_{2\lambda}) \to y_{2\lambda}.
\end{aligned}$$

Outputs 1 **iff.** the following 4λ equations hold:

$$\begin{aligned}
&\gamma_2^{in}(K_1,y_1) = \gamma_2^{in}(K_2,y_2), \varphi_1^{in}(K_2,x_2) = \varphi_1^{in}(K_3,x_3),\\
&\gamma_2^{in}(K_3,y_3) = \gamma_2^{in}(K_4,y_4), \varphi_1^{in}(K_4,x_4) = \varphi_1^{in}(K_5,x_5), \ldots\\
&\gamma_2^{in}(K_{2\lambda-1},y_{2\lambda-1}) = \gamma_2^{in}(K_{2\lambda},y_{2\lambda}), \varphi_1^{in}(K_{2\lambda},x_{2\lambda}) = \varphi_1^{in}(K_1,x_1).
\end{aligned} \tag{22}$$

Fig. 14. Subroutine Yoyo used by the differentiator $D2^{\mathrm{E,P}}$.

7.2 Attack and advantage

We are now able to establish insecurity of 2-call iterated ciphers $E2^{\mathcal{P}}$ of Fig. 12.

Theorem 2 (Differentiability of $E2^{\mathcal{P}}$). *Let $E2^{\mathcal{P}}$ be a blockcipher defined by Fig. 10 with keyspace $|\mathcal{K}| \ge \big(6(3|\mathcal{I}_{\le n}|)^{\frac{1}{n}} + 5\big)|\mathcal{I}_{\le n}| + 1 = O(poly(n))$. Assume that: (i) for all $j \in \{0,1,2\}$, the round $\Pi_j^{\mathcal{P}}$ is deterministic and $\varepsilon_{de(\Pi_j)}$-non-degenerate, and (ii) φ_1^{in} and φ_2^{in} are $\varepsilon_{de(\varphi_1^{in})}$- and $\varepsilon_{de(\varphi_2^{in})}$-non-degenerate respectively (see Definition 3). Then, when n is sufficiently large, there exists a differentiator $D2^{\mathrm{E,P}}$ making poly(n) queries and has advantage at least*

$1 - {m_{max}}^2/2^n - 2q^2\varepsilon_{de(\varphi_1^{in})} - 2q^2\varepsilon_{de(\varphi_2^{in})} - 6q^2/2^{m_{min}} = 1 - negl(n)$, where q is the number of **IC***-queries made by $D2$ and S in total.*

The differentiator $D2^{\mathrm{E,P}}$ is formally described in Fig. 13. Below we provide intuitions and analyses. $E2^{\mathcal{P}}$ may consist of two cases.

Case 1: $|\mathcal{K}^{(1)}| \geq 2|\mathcal{I}_{\leq n}| + 1$. Recall that $E2^{\mathcal{P}}(K,x) = \Pi_3^{\mathcal{P}}\big(K\|\mathsf{kd}^{\mathcal{P}}(K),x\big)$ for all $(K,x) \in \mathcal{K}_1 \times \{0,1\}^n$. Recall from Fig. 13 that $D2$ simply applies the differentiator $D1$ (Fig. 11) to the 1-call cipher $\Pi_3^{\mathcal{P}} : \mathcal{K}_1^{ex} \times \{0,1\}^n \to \{0,1\}^n$— since $|\mathcal{K}_1| = 2|\mathcal{I}_{\leq n}| + 1 = \mathrm{poly}(n)$, it is feasible to build $\mathcal{K}_1^{ex}$ and start $D1$. The advantage in this case is thus at least $1 - {m_{max}}^2/2^n - 2/2^{m_{min}}$ by Theorem 1.

Case 2: the other case. By our assumption $|\mathcal{K}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}} + 5\big)|\mathcal{I}_{\leq n}| + 1$, in this case we have $|\mathcal{K}^{(0)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}} + 3\big)|\mathcal{I}_{\leq n}|$. Below in the first paragraph, we prove that the subroutine Yoyo always finds the 2λ pairs at line 3. As a result, $D2$ always outputs 1 when interacting with $(E2^{\mathcal{P}}, \mathcal{P})$. Then, in the second paragraph we prove $D2$ outputs 1 with a negligible probability when interacting with $(\mathbf{IC}, S^{\mathbf{IC}})$ to complete the analysis.
Existence of the 2λ pairs. Consider the set of intermediate values $\mathcal{K}^{(0)} \times \{0,1\}^n$. We construct a bipartite graph $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$ with left shore $\mathcal{V}_L = \big(\cup_{K\in\mathcal{K}^{(0)}} \mathsf{Rng}_{\mathsf{if}}(\gamma_1^{in}, K)\big) \cup \mathsf{Rng}_{\mathsf{ni}}(\gamma_1^{in})$, right shore $\mathcal{V}_R = \big(\cup_{K\in\mathcal{K}^{(0)}} \mathsf{Rng}_{\mathsf{if}}(\varphi_2^{in}, K)\big) \cup \mathsf{Rng}_{\mathsf{ni}}(\varphi_2^{in})$ (see Definition 4) and $|\mathcal{E}| = |\mathcal{K}^{(0)}|2^n$. It then holds $\sum_{K\in\mathcal{K}^{(0)}} |\mathsf{Rng}_{\mathsf{if}}(\gamma_1^{in}, K)| \leq \sum_{K\in\mathcal{K}^{(0)}} |\mathsf{Dom}_{\mathsf{if}}(\Pi_1, K)| \times \varepsilon_{de(\Pi_1)} \leq 2^n|\mathcal{K}^{(0)}|\varepsilon_{de(\Pi_1)}$ and $|\mathsf{Rng}_{\mathsf{ni}}(\gamma_1^{in})| \leq |\mathcal{I}_{\leq n}|2^{n+1}$ by Lemmas 4 and 5. Since $|\mathcal{K}^{(0)}| = \mathrm{poly}(n)$ and $\varepsilon_{de(\Pi_1)} = \mathrm{negl}(n)$, it holds $|\mathcal{V}_L| \leq 2^n + |\mathcal{I}_{\leq n}|2^{n+1} \leq 3|\mathcal{I}_{\leq n}|2^n$. Similarly, $|\mathcal{V}_R| \leq 3|\mathcal{I}_{\leq n}|2^n$. $\mathcal{G}$ contains an edge $\big((i_1, \delta_1, z_1), (i_2, \delta_2, z_2)\big) \in \mathcal{E}$ if and only if there exists $(K, u) \in \mathcal{K}^{(0)} \times \{0,1\}^n$ such that $\gamma_1^{in}(K, u) = (i_1, \delta_1, z_1)$ and $\varphi_2^{in}(K, u) = (i_2, \delta_2, z_2)$. Then, 2λ pairs $\big((K_1, u_1), (K_2, u_2), ..., (K_{2\lambda}, u_{2\lambda})\big)$ satisfying Eq. (21) indicates a cycle $C_{2\lambda}$ of length 2λ in the graph $\mathcal{G}$.

Now, if $\mathcal{G}$ is a multigraph, then it already contains C_2. Otherwise, we can apply Proposition 1: let $M = 3|\mathcal{I}_{\leq n}|2^n$, then $\mathcal{G}$ satisfies the two conditions of Proposition 1. Therefore, for any t, $\mathcal{G}$ contains $C_{2\lambda}$, $\lambda \leq t$ as long as

$$|\mathcal{E}| \geq \Big(\big(M\big)^{\frac{1}{t-1}} + 1\Big) \times M = \Big(\big(3|\mathcal{I}_{\leq n}|2^n\big)^{\frac{1}{t-1}} + 1\Big) \times 3|\mathcal{I}_{\leq n}|2^n. \tag{23}$$

Setting $t = n+1$, we conclude that the cycle $C_{2\lambda}$ can be found (with the unlimited computations) as long as $|\mathcal{E}| = |\mathcal{K}^{(0)}|2^n \geq \big(\big(3|\mathcal{I}_{\leq n}|2^n\big)^{\frac{1}{n}} + 1\big) \times 3|\mathcal{I}_{\leq n}|2^n$ (or: $|\mathcal{K}^{(0)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}} + 3\big)|\mathcal{I}_{\leq n}|$). As discussed, this condition is fulfilled, and there thus always exists $C_{2\lambda}$ with $\lambda \leq n+1$.
Remark. If $|\mathcal{K}^{(0)}|$ is exponential, then 4-cycles exist (this can be seen by injecting $t = 2$ into Eq. (23)), enabling a more classical boomerang distinguisher.
Attack advantage. Define a 2λ-ary relation $\mathcal{R}_{yoyo}$: $\big((K_j, x_j, y_j)\big)_{j=1,...,2\lambda} \in \mathcal{R}_{yoyo}$ if and only if the 2λ triples satisfy Eq. (22).

Then, $D2$ outputs 1 if and only if it reaches $\big((K_j, x_j, y_j)\big)_{j=1,...,2\lambda} \in \mathcal{R}_{yoyo}$ at line 7 (see Fig. 14). When $D2$ is interacting with $(E2^{\mathcal{P}}, \mathcal{P})$, this always holds due to Eq. (21). When $D2$ is interacting with $(\mathbf{IC}, S^{\mathbf{IC}})$, the probability that $D2$ outputs 1 equals the probability that $S^{\mathbf{IC}}$ finds such a 2λ-tuple $\big((K_j, x_j, y_j)\big)_{j=1,...,2\lambda} \in \mathcal{R}_{yoyo}$ with $\mathbf{IC}(K_j, x_j) = y_j$, $j = 1, ..., 2\lambda$. Below we prove that $\mathcal{R}_{yoyo}$ is actually an *evasive relation* [12], which implies our goal.

To this end, consider the interaction between $(D2, S)$ and $\mathbf{IC}$. We define two bad event Bad^+ and Bad^-. Bad^+ happens, if right after $D2$ or S issuing a new forward query $\mathbf{IC}(K, x) \to y$, there exists an earlier adversarial $\mathbf{IC}$-query record (K', x', y') such that $\gamma_2^{in}(K, y) = \gamma_2^{in}(K', y')$. Similarly by symmetry, Bad^- happens, if right after a new backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K, y) \to x$, there exists an earlier $\mathbf{IC}$-query record (K', x', y') such that $\varphi_1^{in}(K, x) = \varphi_1^{in}(K', x')$.

To bound $\Pr[\mathsf{Bad}^+]$, consider a new forward query $\mathbf{IC}(K, x) \to y$. For each earlier $\mathbf{IC}$-query record (K', x', y'), let $(i', \delta', z') = \gamma_2^{in}(K', y')$, then we have:

$$\Pr\big[\gamma_2^{in}(K, y) = (i', \delta', z')\big] \le \underbrace{\Pr\big[\gamma_2^{in}(K, y) = (i', \delta', z') \mid y \in \mathsf{Rng_{if}}(\Pi_2, K)\big]}_{\le 2\varepsilon_{de(\varphi_2^{in})} \text{ (Eq. (18))}} + \underbrace{\Pr\big[\gamma_2^{in}(K, y) = (i', \delta', z') \mid y \in \mathsf{Rng_{ni}}(\Pi_2, K)\big]}_{\le 2/2^{m_{min}} \text{ (Corollary 1)}}.$$

The number of possible combinations of forward query $\mathrm{E}(K, x) \to y$ and earlier query (K', x', y') is at most q^2. Therefore, $\Pr[\mathsf{Bad}^+] \le q^2(2\varepsilon_{de(\varphi_2^{in})} + 2/2^{m_{min}})$. Similarly by symmetry, $\Pr[\mathsf{Bad}^-] \le q^2(2\varepsilon_{de(\varphi_1^{in})} + 2/2^{m_{min}})$.

Finally, it can be seen as long as neither Bad^+ nor Bad^- occurs, S cannot obtain $\big((K_j, x_j, y_j)\big)_{j=1,...,2\lambda} \in \mathcal{R}_{yoyo}$. This establishes the evasiveness of $\mathcal{R}_{yoyo}$, and attack advantage is at least $1 - q^2(2\varepsilon_{de(\varphi_1^{in})} + 2\varepsilon_{de(\varphi_2^{in})} + 4/2^{m_{min}})$.

Discussion. For the attack in this case (and in subsequent Sect. 8), it is crucial to restrict discussion to iterated blockciphers: since the set of valid intermediate values u between the rounds is simply $\{0,1\}^n$, an attacker can pick such a u and compute forward or backward. Indeed, this middle-to-sides approach is common in known- and chosen-key attacks [37].

Lampe and Seurin [39] considered a similar evasive relation, i.e., the boomerang relation in 3-round IEM, which resembles Fig. 4 (left). Their bound is inferior in some sense, because they simplified the description of the relation.

Summary. When n is sufficiently large, $\mathrm{Adv}^{indif}_{E2^{\mathcal{P}},\mathbf{IC},S}(D2)$ is lower bounded by

$$\min\Big\{1 - \frac{{m_{max}}^2}{2^n} - \frac{2}{2^{m_{min}}}, 1 - q^2(2\varepsilon_{de(\varphi_1^{in})} + 2\varepsilon_{de(\varphi_2^{in})} + 4/2^{m_{min}})\Big\}$$
$$\ge 1 - {m_{max}}^2/2^n - 2q^2\varepsilon_{de(\varphi_1^{in})} - 2q^2\varepsilon_{de(\varphi_2^{in})} - 6q^2/2^{m_{min}}.$$

8 Attack 3-Call Iterated Blockciphers

For 3-call iterated ciphers, $E3^{\mathcal{P}}(K, x)$ can take one of the following three flows:

– **Type-I**: $E3^{\mathcal{P}}(K, x) = \Pi_6^{\mathcal{P}}\big(K \| \mathsf{kd}_1^{\mathcal{P}}(K), x\big)$ for a 2-call KDF $\mathsf{kd}_1^{\mathcal{P}} : \{0,1\}^\kappa \to \{0,1\}^{2m_{max}}$ and a 1-call cipher $\Pi_6^{\mathcal{P}} : \{0,1\}^{\kappa+2m_{max}} \times \{0,1\}^n \to \{0,1\}^n$, or

Algorithm $E3^{\mathcal{P}}(K, x)$
if $K \in \mathcal{K}^{(2)}$ **then**
  return $\Pi_6^{\mathcal{P}}\big(K \| \mathsf{kd}_1^{\mathcal{P}}(K), x\big)$
else if $K \in \mathcal{K}^{(1)}$ **then**
  $s \leftarrow \mathsf{kd}_2^{\mathcal{P}}(K)$
  return $\Pi_5^{\mathcal{P}}\big(K \| s, \Pi_4^{\mathcal{P}}(K \| s, x)\big)$
else // $K \in \mathcal{K}^{(0)}$
  $u \leftarrow \Pi_1^{\mathcal{P}}(K, x)$
  return $\Pi_3^{\mathcal{P}}\big(K, \Pi_2^{\mathcal{P}}(K, u)\big)$
end if

Algorithm $\mathsf{kd}_1^{\mathcal{P}}(K)$
$(i_1, \delta_1, z_1) \leftarrow f_{1,1}(K)$
$z_1' \leftarrow \mathcal{P}(i_1, \delta_1, z_1)$
$(i_2, \delta_2, z_2) \leftarrow f_{1,2}(K, z_1')$
$z_2' \leftarrow \mathcal{P}(i_2, \delta_2, z_2)$
return $z_1' \| z_2'$

Algorithm $(E3^{-1})^{\mathcal{P}}(K, y)$
if $K \in \mathcal{K}^{(2)}$ **then**
  return $(\Pi_6^{-1})^{\mathcal{P}}\big(K \| \mathsf{kd}_1^{\mathcal{P}}(K), y\big)$
else if $K \in \mathcal{K}^{(1)}$ **then**
  $s \leftarrow \mathsf{kd}_2^{\mathcal{P}}(K)$
  return $(\Pi_4^{-1})^{\mathcal{P}}\big(K \| s, (\Pi_5^{-1})^{\mathcal{P}}(K \| s, y)\big)$
else // $K \in \mathcal{K}^{(0)}$
  $w \leftarrow (\Pi_3^{-1})^{\mathcal{P}}(K, y)$
  return $(\Pi_1^{-1})^{\mathcal{P}}\big(K, (\Pi_2^{-1})^{\mathcal{P}}(K, w)\big)$
end if

Algorithm $\mathsf{kd}_2^{\mathcal{P}}(K)$
$(i, \delta, z) \leftarrow f_{2,1}(K)$
$z' \leftarrow \mathcal{P}(i, \delta, z)$
return z'

// Definitions of $\Pi_j^{\mathcal{P}}(K, x)$ and $(\Pi_j^{-1})^{\mathcal{P}}(K, y)$ are the same as Fig. 12

Fig. 15. Definition of the 3-call iterated blockcipher $E3^{\mathcal{P}}$.

– **Type-II**: $E3^{\mathcal{P}}(K, x) = \Pi_5^{\mathcal{P}}\big(K \| \mathsf{kd}_2^{\mathcal{P}}(K), \Pi_4^{\mathcal{P}}(K \| \mathsf{kd}_2^{\mathcal{P}}(K), x)\big)$ for a 1-call KDF $\mathsf{kd}_2^{\mathcal{P}} : \{0,1\}^{\kappa} \to \{0,1\}^{m_{max}}$ and two 1-call ciphers $\Pi_4^{\mathcal{P}}, \Pi_5^{\mathcal{P}} : \{0,1\}^{\kappa + m_{max}} \times \{0,1\}^n \to \{0,1\}^n$, or
– **Type-III**: $E3^{\mathcal{P}}(K, x) = \Pi_3^{\mathcal{P}}\big(K, \Pi_2^{\mathcal{P}}\big(K, \Pi_1^{\mathcal{P}}(K, x)\big)\big)$ for three 1-call ciphers $\Pi_1^{\mathcal{P}}, \Pi_2^{\mathcal{P}}, \Pi_3^{\mathcal{P}} : \{0,1\}^{\kappa} \times \{0,1\}^n \to \{0,1\}^n$.

The keyspace is partitioned $\mathcal{K} = \mathcal{K}^{(0)} \sqcup \mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}$, such that $E3^{\mathcal{P}}(K, \cdot)$ follows **Type-I**, resp. **Type-II** encipherment if and only if $K \in \mathcal{K}^{(2)}$, resp. $K \in \mathcal{K}^{(1)}$. Formally, $E3^{\mathcal{P}}$ is defined in Fig. 15.

Theorem 3 (Differentiability of $E3^{\mathcal{P}}$). *Let $E3^{\mathcal{P}}$ be a blockcipher defined by Fig. 15 with keyspace $\{0,1\}^{\kappa}$, $\kappa \geq 2m_{max} \log_2 |\mathcal{I}_{\leq n}| + 2m_{max}n + 6m_{max} + 4 = \Theta(poly(n))$. Assume that for $j = 1,2,3,4,5,6$, (i) the round $\Pi_j^{\mathcal{P}}$ is deterministic and $\varepsilon_{de(\Pi_j)}$-non-degenerate, and (ii) φ_j^{in} is $\varepsilon_{de(\varphi_j^{in})}$-non-degenerate (see Definition 3). Then, when n is sufficiently large, there exists a differentiator $D3^{\mathrm{E,P}}$ making $poly(n)$ queries to E and P and having advantage either $1/poly(n) - negl(n)$ or $1 - negl(n)$ for some $poly(n)$ and $negl(n)$ determined by φ_j^{in}, $j = 1,2,3,4,5,6$.*

The differentiator $D3^{\mathrm{E,P}}$ is formally described in Fig. 16. Due to the complicated form, the advantage is only given in asymptotic form. We refer to Sect. 3.4 for the overview.

Below we analyze the attack advantage. Let q be the number of **IC**-queries made by $D3$ and S in total. We also distinguish three cases.

Case 1: $|\mathcal{K}^{(2)}| \geq 2|\mathcal{I}_{\leq n}| + 1$. Recall that $E3^{\mathcal{P}}(K, x) = \Pi_6^{\mathcal{P}}\big(K \| \mathsf{kd}_1^{\mathcal{P}}(K), x\big)$ for all $(K, x) \in \mathcal{K}^{(2)} \times \{0,1\}^n$. Similarly to Case 1 in Appendix **??**, $D3$ derives $2|\mathcal{I}_{\leq n}| + 1$ subkeys and invokes $D1$ to attack $\Pi_6^{\mathcal{P}}$. The advantage is thus $1 - {m_{max}}^2/2^n - 2/2^{m_{min}}$ by Theorem 1.

```
Algorithm D3^{E,P}(κ, K^(1), K^(2))
if |K^(2)| ≥ 2|I_{≤n}| + 1 then          // Case 1: 2-call key derivation plus 1-call round
  Let K_2 ⊆ K^(2) be such that |K_2| ≥ 2|I_{≤n}| + 1, K_2^{ex} ← ∅
  for K ∈ K_2 do          // Derives subkeys for keys in K_2
    z'_1 ← P(f_{1,1}(K)), z'_2 ← P(f_{1,2}(K, z'_1)), K_2^{ex} ← K_2^{ex} ∪ {K‖z'_1‖z'_2}
  end for
  Invokes D1^{E,P}(K_2^{ex}) (see Fig. 11) on Π_6^P : K_2^{ex} × {0,1}^n → {0,1}^n
else if |K^(1)| ≥ (6(3|I_{≤n}|)^{1/n} + 3)|I_{≤n}| then
  // Case 2: 1-call key derivation plus two 1-call rounds
  Let K_1 ⊆ K^(1) be such that |K_1| = (6(3|I_{≤n}|)^{1/n} + 3)|I_{≤n}|, K_1^{ex} ← ∅
  for K ∈ K_1 do          // Derives subkeys for keys in K_1
    z' ← P(f_{2,1}(K)), K_1^{ex} ← K_1^{ex} ∪ {K‖z'}
  end for
  Invokes Subroutine Yoyo(Π_4, Π_5, K_1^{ex}) (see Fig. 13)
else          // Case 3: it necessarily holds |K^(0)| ≥ 2^κ/2
  Picks K_0 ⊆ {0,1}^κ \ (K^(1) ∪ K^(2)) be such that |K_0| = 2^κ/2
  Invokes Subroutine Handle3Iter(K_0)          // See Fig. 17
end if
```

Fig. 16. Differentiator $D3^{\mathrm{E,P}}$ used in Theorem 3.

Case 2: $|\mathcal{K}^{(1)}| \geq \big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}}+3\big)|\mathcal{I}_{\leq n}|$. Recall from Figs. 15 and 16 that $E3^{\mathcal{P}}(K,x) = \varPi_5^{\mathcal{P}}\big(K\|\mathsf{kd}_2^{\mathcal{P}}(K), \varPi_4^{\mathcal{P}}(K\|\mathsf{kd}_2^{\mathcal{P}}(K),x)\big)$ for all $(K,x) \in \mathcal{K}^{(1)} \times \{0,1\}^n$, and $D3$ derives $\big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}}+3\big)|\mathcal{I}_{\leq n}|$ subkeys and calls $\big(6\big(3|\mathcal{I}_{\leq n}|\big)^{\frac{1}{n}}+3\big)|\mathcal{I}_{\leq n}|$ to attack the 2-iteration $\varPi_5^{\mathcal{P}} \circ \varPi_4^{\mathcal{P}}$. The advantage is thus at least $1 - {m_{max}}^2/2^n - 2q^2\varepsilon_{de(\varphi_1^{in})} - 2q^2\varepsilon_{de(\varphi_2^{in})} - 6q^2/2^{m_{min}} = 1 - \mathrm{negl}(n)$ by Theorem 2.

Case 3: the others. In the remaining cases, since both $|\mathcal{K}^{(1)}|$ and $|\mathcal{K}^{(2)}|$ are $O(\mathrm{poly}(n))$, it necessarily holds $|\mathcal{K}^{(0)}| \geq 2^\kappa/2$ for large enough n. Recall from Figs. 15 that $E3^{\mathcal{P}}(K,x) = \varPi_3^{\mathcal{P}}\big(K, \varPi_2^{\mathcal{P}}\big(K, \varPi_1^{\mathcal{P}}(K,x)\big)\big)$ for all $K \in \mathcal{K}^{(0)}$.

For any $K \in \mathcal{K}$ and $\mathsf{tag}_1, \mathsf{tag}_2 \in \{\mathsf{if}, \mathsf{ni}\}$, define

$$\begin{aligned}
&\mathcal{X}_{\mathsf{tag}_1}(K) := \mathsf{Dom}_{\mathsf{tag}_1}(\varPi_1, K), \qquad \mathcal{Y}_{\mathsf{tag}_1}(K) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_3, K),\\
&\mathcal{U}_{\mathsf{tag}_1,\mathsf{tag}_2}(K) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_1, K) \bigcap \mathsf{Dom}_{\mathsf{tag}_2}(\varPi_2, K),\\
&\mathcal{W}_{\mathsf{tag}_1,\mathsf{tag}_2}(K) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_2, K) \bigcap \mathsf{Dom}_{\mathsf{tag}_2}(\varPi_3, K).
\end{aligned} \tag{24}$$

E.g., $\mathcal{U}_{\mathsf{if},\mathsf{if}}(K)$ contains all the intermediate values u that has both 1st round $(\varPi_1^{-1})^{\mathcal{P}}(K,u)$ and 2nd round $\varPi_2^{\mathcal{P}}(K,u)$ inv-free under the key K. Furthermore, for any $K \in \mathcal{K}$, any $\mathsf{tag}_1, \mathsf{tag}_2 \in \{\mathsf{if}, \mathsf{ni}\}$, any $(i,\delta) \in \mathcal{I} \times \{+,-\}$ and any $z \in \{0,1\}^{m(i)}$, define

$$\begin{aligned}
&\mathcal{Y}_{\mathsf{tag}_1}(K,i,\delta) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_3, K, i, \delta),\\
&\mathcal{U}_{\mathsf{tag}_1,\mathsf{tag}_2}(K,i,\delta) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_1, K, i, \delta) \bigcap \mathsf{Dom}_{\mathsf{tag}_2}(\varPi_2, K, i, \delta),\\
&\mathcal{U}_{\mathsf{tag}_1,\mathsf{tag}_2}(K,i,\delta,z) := \mathsf{Rng}_{\mathsf{tag}_1}(\varPi_1, K, i, \delta, z) \bigcap \mathsf{Dom}_{\mathsf{tag}_2}(\varPi_2, K, i, \delta, z).
\end{aligned} \tag{25}$$

for the sets of values giving rise to queries $\mathcal{P}(i,\delta,\star)$ or $\mathcal{P}(i,\delta,z)$.

As shown in Fig. 16, we use a subroutine Handle3Iter to handle the case of 3-iterations. We refer to Sect. 3.4 for its ideas. The detailed case-study is lengthy and deferred to Sect. 9. In all, the case-study yields the claim in Theorem 3.

Subroutine HANDLE3ITER($\mathcal{K}_0$)
Picks a set $\mathcal{K}_1 \subseteq \mathcal{K}_0$ **s.t.** $|\mathcal{K}_1| \geq |\mathcal{K}_0|/4$, and CATEGORYXY($K$) is a constant for all $K \in \mathcal{K}_1$
if CATEGORYXY(K) $= 0$ for all $K \in \mathcal{K}_1$ **then**
  Invokes Subroutine SUBCASE31($\mathcal{K}_1$)
else if CATEGORYXY(K) $= 1$ for all $K \in \mathcal{K}_1$ **then**
  $R \leftarrow \min_{K \in \mathcal{K}_1} \left\{ \frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n}, \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2^n} \right\}$
  Picks $\mathcal{K}_2 \subseteq \mathcal{K}_1$ **s.t.** $|\mathcal{K}_2| \geq |\mathcal{K}_1|/2$, and CATEGORYU($K, R$) is a constant for all $K \in \mathcal{K}_2$
  if CATEGORYU(K, R) $= 0$ for all $K \in \mathcal{K}_2$ **then**
    Invokes Subroutine SUBCASE32($\mathcal{K}_2, R$)
  else // CATEGORYU(K, R) $= 1$ and CATEGORYXY(K) $= 1$ for all $K \in \mathcal{K}_2$
    Invokes Subroutine SUBCASE33($\mathcal{K}_2, R$)
  end if
else if CATEGORYXY(K) $= 2$ for all $K \in \mathcal{K}_1$ **then**
  $R \leftarrow \min_{K \in \mathcal{K}_1} \left\{ \frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n} \right\}$
  Picks $\mathcal{K}_2 \subseteq \mathcal{K}_1$ **s.t.** $|\mathcal{K}_2| \geq |\mathcal{K}_1|/2$, and CATEGORYU($K, R$) is a constant for all $K \in \mathcal{K}_2$
  if CATEGORYU(K, R) $= 0$ for all $K \in \mathcal{K}_2$ **then**
    Invokes Subroutine SUBCASE34($\mathcal{K}_2, R$)
  else // CATEGORYU(K, R) $= 1$ and CATEGORYXY(K) $= 2$ for all $K \in \mathcal{K}_2$
    Invokes Subroutine SUBCASE35($\mathcal{K}_2$)
  end if
else // CATEGORYXY(K) $= 3$ for all $K \in \mathcal{K}_1$
  $R \leftarrow \min_{K \in \mathcal{K}_1} \left\{ \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2^n} \right\}$
  Picks $\mathcal{K}_2 \subseteq \mathcal{K}_1$ **s.t.** $|\mathcal{K}_2| \geq |\mathcal{K}_1|/2$, and CATEGORYW($K, R$) is a constant for all $K \in \mathcal{K}_2$
  if CATEGORYW(K, R) $= 0$ for all $K \in \mathcal{K}_2$ **then**
    // This case is similar to SUBCASE34 by symmetry
  else // CATEGORYW(K, R) $= 1$ and CATEGORYXY(K) $= 3$ for all $K \in \mathcal{K}_2$
    // This case is similar to SUBCASE35 by symmetry
  end if
end if

Function CATEGORYXY(K)
if $|\mathcal{X}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ **then return** 0
else if $|\mathcal{X}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ **then return** 1
else if $|\mathcal{X}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ **then return** 2
else // $|\mathcal{X}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ **return** 3
end if

Function CATEGORYU(K, R)
if $|\mathcal{U}_{\mathsf{ni,ni}}(K)| \geq 2^n R/2$ **then return** 0
else if $|\mathcal{U}_{\mathsf{ni,if}}(K)| \geq 2^n R/2$ **then return** 1
end if

Function CATEGORYW(K, R)
if $|\mathcal{W}_{\mathsf{ni,ni}}(K)| \geq 2^n R/2$ **then return** 0
else if $|\mathcal{W}_{\mathsf{if,ni}}(K)| \geq 2^n R/2$ **then return** 1
end if

Fig. 17. Subroutine HANDLE3ITER used by the differentiator $D3^{\mathrm{E,P}}$.

9 Case-Study of Case 3 of Sect. 8

As sketched in Sect. 3, the (sub)cases are all non-trivial. We thereby adopt another style of presentation. In detail, instead of first describing the differentiator and then analyzing advantage (as we did in Sect. 6 and 7), we address the (sub)cases in turn in Sect. 9.1–9.7. For each (sub)case, the pseudocode of the corresponding differentiator subroutine is immediately followed by the advantage analysis. We remark that our differentiator $D3$ remains universal, i.e., it is effective regardless of the (sub)case the $E3$ instance falls in.

9.1 Subcase 3.1: $|\mathcal{X}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_1$

As mentioned, this corresponds to Case 1 in Sect. 3.4.

1: **Subroutine** $\textsc{Subcase31}(\mathcal{K}_1)$
2: **Picks** (i_2, δ_2, z_2) such that $|\mathcal{KU}| \geq \frac{1}{4\varepsilon_{de(\Pi_2)}}$, where

$$\mathcal{KU} := \big\{(K,u) : u \in (\mathcal{U}_{\mathsf{if,if}}(K) \cup \mathcal{U}_{\mathsf{if,ni}}(K)) \text{ and } \varphi_2^{in}(K,u) = (i_2,\delta_2,z_2)\big\}$$

3: Samples $\lambda = m_{max}$ distinct $(K_1,u_1),...,(K_\lambda,u_\lambda) \xleftarrow{\$} \mathcal{KU}$
4: $\mathrm{P}(i_2,\delta_2,z_2) \to z_2'$
5: **for** $j \in \{1,...,\lambda\}$ **do**
6: $\quad (i_1^{(j)}, \delta_1^{(j)}, z_1^{(j)}) \leftarrow \gamma_1^{in}(K_j,u_j)$, $\mathrm{P}(i_1^{(j)}, \delta_1^{(j)}, z_1^{(j)}) \to z_1^{'(j)}$
7: $\quad x_j \leftarrow \gamma_1^{out}(K_j, z_1^{'(j)}, u_j)$, $\mathrm{E}(K_j,x_j) \to y_j$, $w_j \leftarrow \varphi_2^{out}(K_j, z_2^{'(j)}, u_j)$
8: **end for**
9: **if** $\exists j : y_j \in \mathcal{Y}_{\mathsf{ni}}(K_j)$ **then**
10: $\quad$ **Outputs** 1
11: **else**
12: $\quad$ **Outputs** 1 **iff.** $\varphi_3^{in}(K_j,w_j) = \gamma_3^{in}(K_j,y_j)$ for all $j \in \{1,...,\lambda\}$
13: **end if**

Analysis of Subcase 3.1. Let $\varepsilon_{\mathsf{ni}} := \max_{K\in\mathcal{K}_1}\big\{|\mathcal{X}_{\mathsf{ni}}(K)|/2^n, |\mathcal{Y}_{\mathsf{ni}}(K)|/2^n\big\}$. By the condition, it holds $\varepsilon_{\mathsf{ni}} = \mathrm{negl}(n)$. We refer to Fig. 6 (left) for the involved query structure. With this, below we first establish the existence of the set $\mathcal{KU}$. Then, we lower bound advantage.

Existence of $\mathcal{KU}$. By Lemma 4, for any $K \in \mathcal{K}_1$ it holds $|\mathsf{Rng}_{\mathsf{if}}(\varphi_2^{in}, K)| \leq 2^n \cdot \varepsilon_{de(\Pi_2)}$; by Lemma 5, it holds $|\mathsf{Rng}_{\mathsf{ni}}(\varphi_2^{in})| \leq |\mathcal{I}_{\leq n}|2^{n+1}$. Therefore, for large enough n it holds $|\mathsf{Rng}_{\mathsf{ni}}(\varphi_2^{in})| + \sum_{K\in\mathcal{K}_1}|\mathsf{Rng}_{\mathsf{if}}(\varphi_2^{in},K)| \leq |\mathcal{I}_{\leq n}|2^{n+1} + |\mathcal{K}_1| \cdot 2^n \cdot \varepsilon_{de(\Pi_2)} \leq |\mathcal{K}_1| \cdot 2^{n+1} \cdot \varepsilon_{de(\Pi_2)}$. On the other hand, $|\mathcal{X}_{\mathsf{ni}}(K)| < 2^n\delta$ implies $|\mathcal{X}_{\mathsf{if}}(K)| = |\mathcal{U}_{\mathsf{if,if}}(K)| + |\mathcal{U}_{\mathsf{if,ni}}(K)| \geq 2^n/2$, meaning that $\sum_{K\in\mathcal{K}_1}\big(|\mathcal{U}_{\mathsf{if,if}}(K)| + |\mathcal{U}_{\mathsf{if,ni}}(K)|\big) \geq |\mathcal{K}_1| \cdot 2^{n-1}$. By these and by the pigeonhole principle, there exists an input $(i_2,\delta_2,z_2) \in \mathsf{Rng}_{\mathsf{ni}}(\varphi_2^{in})$ and a corresponding set $\mathcal{KU}$ such that:

- $\varphi_2^{in}(K,w) = (i_2,\delta_2,z_2)$ for all $(K,u) \in \mathcal{KU}$, and
- $|\mathcal{KU}| \geq \frac{|\mathcal{K}_1|\cdot 2^{n-1}}{|\mathcal{K}_1|\cdot 2^{n+1}\cdot\varepsilon_{de(\Pi_2)}} = \frac{1}{4\varepsilon_{de(\Pi_2)}} = \Omega(\mathrm{poly}(n))$.

Hence, subroutine $\textsc{Subcase31}(\mathcal{K}_1)$ will succeed in finding the set at line 2.

Attack advantage. In the real world, $\textsc{Subcase31}(\mathcal{K}_1)$ either outputs 1 due to line 9, or outputs 1 at line 12 when $\varphi_3^{in}(K_j,w_j) = \gamma_3^{in}(K_j,y_j)$ for all j. Since the equalities always hold in the real world, we have $\Pr[\textsc{Subcase33}(\mathcal{K}_2) = 1] = 1$.

In the ideal world, assume that S knows the λ pairs $(K_1,u_1),...,(K_\lambda,u_\lambda)$ sampled by $\textsc{Subcase33}(\mathcal{K}_2)$ at line 3, and define three bad events during the ideal world execution:

- Badni occurs, if there appears a forward $\mathbf{IC}$-query $\mathbf{IC}(K,x) \to y$ such that $y \in \mathcal{Y}_{\mathsf{ni}}(K)$, or a backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K,y) \to x$ such that $x \in \mathcal{X}_{\mathsf{ni}}(K)$;
- Bad^- occurs, if there appears a backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K,y) \to x$ and $j \in \{1,...,\lambda\}$ such that $x \in \mathcal{Y}_{\mathsf{if}}(K)$ and $\varphi_1^{in}(K,x) = \gamma_1^{in}(K_j,u_j)$;

– $\mathsf{BadGroup}$ occurs, if S succeeds in finding a tuple of λ $\mathbf{IC}$ inputs/outputs $\mathbf{IC}(K_1, x_1^*) = y_1^*, ..., \mathbf{IC}(K_\lambda, x_\lambda^*) = y_\lambda^*$ such that:
  (i) $x_j^* \in \mathcal{X}_{\mathsf{if}}(K_j)$ and $y_j^* \in \mathcal{Y}_{\mathsf{if}}(K_j)$ for all $j \in \{1, ..., \lambda\}$; and
  (ii) There exists $z_2' \in \{0,1\}^{m(i_2)}$ such that $\varphi_1^{in}(K_j, x_j^*) = \gamma_1^{in}\big(K_j, u_j\big)$ and $\varphi_3^{in}\big(K_j, w_j\big) = \gamma_3^{in}(K_j, y_j^*)$ for all $j \in \{1, ..., \lambda\}$, where $w_j = \varphi_2^{out}(K_j, z_2', u_j)$.

Consider Badni first. For every forward $\mathbf{IC}$-query $\mathbf{IC}(K, x) \to y$, the response y is uniform in at least $2^n - q$ possibility. Since $|\mathcal{Y}_{\mathsf{ni}}(K)| < 2^n \varepsilon_{\mathsf{ni}}$, the probability to have $y \in \mathcal{Y}_{\mathsf{ni}}(K)$ is at most $2^n \varepsilon_{\mathsf{ni}}/(2^n - q) \leq 2\varepsilon_{\mathsf{ni}}$. Similarly, the probability to have $x \in \mathcal{X}_{\mathsf{ni}}(K)$ for every backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K, y) \to x$ is at most $2\varepsilon_{\mathsf{ni}}$. Summing over the at most q $\mathbf{IC}$-queries yield $\Pr[\mathsf{Badni}] \leq 2q\varepsilon_{\mathsf{ni}} = \mathrm{negl}(n)$.

Then, consider Bad^-. For every $j \in \{1, ..., \lambda\}$ and every backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K, y^*) \to x^*$, it holds $\Pr\big[\varphi_1^{in}(K, x^*) = \gamma_1^{in}(K_j, u_j) \mid \neg\mathsf{Badni}\big] = \Pr\big[\varphi_1^{in}(K, x^*) = \gamma_1^{in}(K_j, u_j) \mid x \in \mathcal{Y}_{\mathsf{if}}(K)\big]$, which is at most $2\varepsilon_{de(\varphi_1^{in})}$ by Eq. (18) and by the assumption that φ_1^{in} is $\varepsilon_{de(\varphi_1^{in})}$-non-degenerate. Therefore, $\Pr\big[\mathsf{Bad}^- \mid \neg\mathsf{Badni}\big] \leq 2\lambda q \varepsilon_{de(\varphi_1^{in})}$.

Finally, for $\mathsf{BadGroup}$, consider any λ-tuple of $\mathbf{IC}$ inputs/outputs $\mathbf{IC}(K_1, x_1^*) = y_1^*, ..., \mathbf{IC}(K_\lambda, x_\lambda^*) = y_\lambda^*$ obtained by S. Conditioned on $\neg\mathsf{Bad}^-$, they must be due to λ forward $\mathbf{IC}$-queries: otherwise, they cannot have $\varphi_1^{in}(K_j, x_j^*) = \gamma_1^{in}\big(K_j, u_j\big)$ for any $j \in \{1, ..., \lambda\}$.

Then, for every fixed $z_2' \in \{0,1\}^{m(i_2)}$, the probability to have $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j^*)$ for all $j \in \{1, ..., \lambda\}$ and $w_j = \varphi_2^{out}(K_j, z_2', u_j)$ is at most $(2\varepsilon_{de(\varphi_3^{in})})^\lambda$ by Eq. (18). Since $|z_2'| = m(i_2) \leq m_{max}$, the probability that $\mathsf{BadGroup}$ occurs w.r.t. $\mathbf{IC}(K_1, x_1^*) = y_1^*, ..., \mathbf{IC}(K_\lambda, x_\lambda^*) = y_\lambda^*$ is at most $2^{m_{max}} \times (2\varepsilon_{de(\varphi_3^{in})})^\lambda$.

Finally, since S makes at most q queries, the number of choices for such λ-tuples $\big(\mathbf{IC}(K_1, x_1^*) = y_1^*, ..., \mathbf{IC}(K_\lambda, x_\lambda^*) = y_\lambda^*\big)$ is at most q^λ. Thus,

$$\Pr\big[\mathsf{BadGroup}\big] \leq q^\lambda \times 2^{m_{max}} \times (2\varepsilon_{de(\varphi_3^{in})})^\lambda \leq 2^{m_{max}} \times (2q\varepsilon_{de(\varphi_3^{in})})^\lambda. \quad (26)$$

If $\mathsf{BadGroup}$ does not occur then there is no $z_2' \in \{0,1\}^{m(i_2)}$ for S to answer line 4, and thus subsequently at line 12 $\textsc{Subcase31}(\mathcal{K}_1)$ will not output 1. Therefore, in the ideal world we have (noting $\lambda = m_{max}$)

$$\begin{aligned}\Pr\big[\textsc{Subcase31}(\mathcal{K}_1) = 1\big] &\leq \Pr\big[\mathsf{Bad}^+\big] + \Pr\big[\mathsf{Bad}^- \mid \neg\mathsf{Badni}\big] + \Pr\big[\mathsf{BadGroup}\big] \\ &\leq 2q\varepsilon_{\mathsf{ni}} + 2\lambda q \varepsilon_{de(\varphi_1^{in})} + 2^{m_{max}} \times (2q\varepsilon_{de(\varphi_3^{in})})^\lambda \\ &\leq 2q\varepsilon_{\mathsf{ni}} + 2m_{max} q \varepsilon_{de(\varphi_1^{in})} + (4q\varepsilon_{de(\varphi_3^{in})})^{m_{max}} = \mathrm{negl}(n),\end{aligned}$$

and the advantage is at least $1 - (2q\varepsilon_{\mathsf{ni}} + 2m_{max} q \varepsilon_{de(\varphi_1^{in})} + (4q\varepsilon_{de(\varphi_3^{in})})^{m_{max}}) = 1 - \mathrm{negl}(n)$ in Subcase 3.1.

9.2 Subcase 3.2: $|\mathcal{U}_{\mathsf{ni},\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_2$

As mentioned, this corresponds to Case 2 in Sect. 3.4.

1: **Subroutine** $\textsc{Subcase32}(\mathcal{K}_2, R)$

Picks $(i_1^*, \delta_1^*), (i_3^*, \delta_3^*) \in \mathcal{I}_{\le n} \times \{+,-\}$ and $\mathcal{K}_3 \subseteq \mathcal{K}_2$ such that:

– $|\mathcal{K}_3| \ge \frac{|\mathcal{K}_2|}{4|\mathcal{I}_{\le n}|^2}$, and

– $|\mathcal{U}_{\mathsf{ni,ni}}(K, i_1^*, \delta_1^*)| \ge \frac{2^n R}{4|\mathcal{I}_{\le n}|}$ and $|\mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)| \ge \frac{2^n R}{2|\mathcal{I}_{\le n}|}$ for all $K \in \mathcal{K}_3$.

Determines the set $\mathcal{KU}_{\mathsf{ni,ni}} := \big\{(K,u) : K \in \mathcal{K}_3 \wedge u \in \mathcal{U}_{\mathsf{ni,ni}}(K, i_1^*, \delta_1^*)\big\}$

Determines the sets $\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1, \delta_1, z_1), (i_2, \delta_2, z_2)\big) := \big\{(K,u) : (K,u) \in \mathcal{KU}_{\mathsf{ni,ni}} \wedge \gamma_1^{in}(K,u) = (i_1, \delta_1, z_1) \wedge \varphi_2^{in}(K,u) = (i_2, \delta_2, z_2)\big\}$ for all (i_1, δ_1, z_1) and (i_2, δ_2, z_2)

if $\exists z_1, (i_2, \delta_2, z_2)$ **s.t.** $|\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1), (i_2, \delta_2, z_2)\big)| = \Omega(\mathrm{poly}(n))$ **then**

  Picks $z_1^* \in \{0,1\}^{m(i_1^*)}$ and $(i_2^\circ, \delta_2^\circ, z_2^\circ)$ **s.t.** $|\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1^*), (i_2^\circ, \delta_2^\circ, z_2^\circ)\big)| = \Omega(\mathrm{poly}(n))$

  Picks $z_1^{**} \in \{0,1\}^{m(i_1^*)}$ **s.t.** $\sum_{(i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ}) \ne (i_2^\circ, \delta_2^\circ, z_2^\circ)} |\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1^{**}), (i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ})\big)| = \Omega(\mathrm{poly}(n))$

  $\mathrm{P}(i_1^*, \delta_1^*, z_1^*) \to z_1^{'*}$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^{**}) \to z_1^{'**}$

  Samples distinct pairs $(K_1, u_1), (K_2, u_2) \xleftarrow{\$} \mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1^*), (i_2^\circ, \delta_2^\circ, z_2^\circ)\big)$

  Samples $(K_3, u_3) \xleftarrow{\$} \cup_{(i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ}) \ne (i_2^\circ, \delta_2^\circ, z_2^\circ)} \mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1^{**}), (i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ})\big)$

  $y_1 \leftarrow \gamma_1^{out}(K_1, z_1^{'*}, u_1)$, $y_2 \leftarrow \gamma_1^{out}(K_2, z_1^{'*}, u_2)$, $y_3 \leftarrow \gamma_1^{out}(K_3, z_1^{'**}, u_3)$

  Invokes Subroutine $\textsc{Scase32Check}(K_1, x_1, K_2, x_2, K_3, x_3, i_3^*, \delta_3^*)$

else // Multiplicity $= O(\mathrm{poly}(n))$

  $t \leftarrow 2^n$

  Picks $3+t$ inputs/outputs $\big((i_1^*, \delta_1^*, z_1^{(j)})\big)_{j=1,2,3}$, $\big((i_2^{(j)}, \delta_2^{(j)}, z_2^{(j)})\big)_{j=1,2,\ldots,t}$ and $3t$ pairs $\big((K_{1,j}, u_{1,j}), (K_{2,j}, u_{2,j}), (K_{3,j}, u_{3,j})\big)_{j=1,2,\ldots,t}$ from $\mathcal{KU}_{\mathsf{ni,ni}}$ that satisfy

$$\gamma_1^{in}\big(K_{j,1}, u_{j,1}\big) = \ldots = \gamma_1^{in}\big(K_{j,t}, u_{j,t}\big) = (i_1^*, \delta_1^*, z_1^{(j)}) \text{ for all } j \in \{1,2,3\};$$
$$\varphi_2^{in}\big(K_{1,j}, u_{1,j}\big) = \varphi_2^{in}\big(K_{2,j}, u_{2,j}\big) = \varphi_2^{in}\big(K_{3,j}, u_{3,j}\big) = (i_2^{(j)}, \delta_2^{(j)}, z_2^{(j)}) \text{ for } j \in \{1,\ldots,t\}. \tag{27}$$

  $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(1)}) \to z_1^{'(1)}$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(2)}) \to z_1^{'(2)}$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(3)}) \to z_1^{'(3)}$

  Samples distinct indices $\ell_1, \ell_2 \xleftarrow{\$} \{1,2,\ldots,t\}$

  $x_{1,\ell_1} \leftarrow \gamma_1^{out}(K_{1,\ell_1}, z_1^{'(1)}, u_{1,\ell_1})$, $x_{2,\ell_1} \leftarrow \gamma_1^{out}(K_{2,\ell_1}, z_1^{'(2)}, u_{2,\ell_1})$, $x_{3,\ell_2} \leftarrow \gamma_1^{out}(K_{3,\ell_2}, z_1^{'(3)}, u_{3,\ell_2})$

  Invokes Subroutine $\textsc{Scase32Check}(K_{1,\ell_1}, x_{1,\ell_1}, K_{2,\ell_1}, x_{2,\ell_1}, K_{3,\ell_2}, x_{3,\ell_2}, i_3^*, \delta_3^*)$

end if

Subroutine $\textsc{Scase32Check}(K_1, x_1, K_2, x_2, K_3, x_3, i_3^*, \delta_3^*)$

$\mathrm{E}(K_1, x_1) \to y_1$, $\mathrm{E}(K_2, x_2) \to y_2$, $\mathrm{E}(K_3, x_3) \to y_3$

$(i_3^{(1)}, \delta_3^{(1)}, z_3^{(1)}) \leftarrow \gamma_3^{in}(K_1, y_1)$, $(i_3^{(2)}, \delta_3^{(2)}, z_3^{(2)}) \leftarrow \gamma_3^{in}(K_2, y_2)$, $(i_3^{(3)}, \delta_3^{(3)}, z_3^{(3)}) \leftarrow \gamma_3^{in}(K_3, y_3)$

if $y_1 \notin \mathcal{Y}_{\mathsf{ni}}(K_1, i_3^*, \delta_3^*)$ or $y_2 \notin \mathcal{Y}_{\mathsf{ni}}(K_2, i_3^*, \delta_3^*)$ or $y_3 \notin \mathcal{Y}_{\mathsf{ni}}(K_3, i_3^*, \delta_3^*)$ **then**

  Outputs 1

else

  $b \xleftarrow{\$} \{0,1\}$

  if $b = 0$ **then**

    $(K_1^\circ, y_1^\circ) \leftarrow (K_1, y_1)$, $(K_2^\circ, y_2^\circ) \leftarrow (K_2, y_2)$

  else // $b = 1$

    $(K_1^\circ, y_1^\circ) \leftarrow (K_1, y_1)$, $(K_2^\circ, y_2^\circ) \leftarrow (K_3, y_3)$

  end if

33: $(i_3^*, \delta_3^*, z_3^{(1)}) \leftarrow \gamma_3^{in}(K_1^\circ, y_1^\circ)$, $(i_3^*, \delta_3^*, z_3^{(2)}) \leftarrow \gamma_3^{in}(K_2^\circ, y_2^\circ)$
34: **if** $z_3^{(1)} = z_3^{(2)}$ **then**
35: **Outputs** 1
36: **else** // $z_3^{(1)} \neq z_3^{(2)}$
37: $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(1)}) \to z_3^{'(1)}$, $w_1^\circ \leftarrow \gamma_3^{out}(K_1^\circ, z_3^{'(1)}, y_1^\circ)$, $(i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)}) \leftarrow \gamma_2^{in}(K_1^\circ, w_1^\circ)$
38: $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(2)}) \to z_3^{'(2)}$, $w_2^\circ \leftarrow \gamma_3^{out}(K_2^\circ, z_3^{'(2)}, y_2^\circ)$, $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) \leftarrow \gamma_2^{in}(K_2^\circ, w_2^\circ)$
39: **if** $b = 0$ **then**
40: **Outputs** 1 **iff.** $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) = (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$
41: **else** // $b = 1$
42: **Outputs** 1 **iff.** $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) \neq (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$
43: **end if**
44: **end if**
45: **end if**

9.3 Analysis of Subcase 3.2

Recall from Fig. 17 that the concrete condition is $|\mathcal{U}_{\mathsf{ni,ni}}(K)| \geq 2^n R/2$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| \geq 2^n R$, where $R = \min_{K \in \mathcal{K}_1} \left\{ \frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n}, \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2^n} \right\}$. We first show that $\textsc{Subcase32}(\mathcal{K}_2, R)$ can find the values at lines 2 and 15.

Existence of $(i_1^*, \delta_1^*), (i_3^*, \delta_3^*)$ and $\mathcal{K}_3$. For every $K \in \mathcal{K}_2$, consider the sets $\mathcal{BI}_1 := \left\{(i_1, \delta_1) : \mathcal{U}_{\mathsf{ni,ni}}(K, i_1, \delta_1) \neq \emptyset\right\}$ and $\mathcal{BI}_3 := \left\{(i_3, \delta_3) : \mathcal{Y}_{\mathsf{ni}}(K, i_3, \delta_3) \neq \emptyset\right\}$. Then, it holds $i_1, i_3 \in \mathcal{I}_{\leq n}$ by Lemma 5, and thus $|\mathcal{BI}_1 \times \mathcal{BI}_3| \leq 4|\mathcal{I}_{\leq n}|^2$. With these, we present a helper lemma as follows.

Lemma 7. *Fix $K \in \{0,1\}^\kappa$ such that $E3^{\mathcal{P}}(K, x) = \Pi_3^{\mathcal{P}}\left(K, \Pi_2^{\mathcal{P}}\left(K, \Pi_1^{\mathcal{P}}(K, x)\right)\right)$ for all $x \in \{0,1\}^n$. Then, for any $\mathsf{tag} \in \{\mathsf{if}, \mathsf{ni}\}$, there exists $\left((i_1, \delta_1), (i_3, \delta_3)\right) \in \mathcal{BI}_1 \times \mathcal{BI}_3$ such that $|\mathcal{U}_{\mathsf{ni,tag}}(K, i_1, \delta_1)| \geq \frac{|\mathcal{U}_{\mathsf{ni,tag}}(K)|}{2|\mathcal{I}_{\leq n}|}$ and $|\mathcal{Y}_{\mathsf{ni}}(K, i_3, \delta_3)| \geq \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2|\mathcal{I}_{\leq n}|}$.*

Proof. We first establish the existence of $(i_1, \delta_1) \in \mathcal{BI}_1$ with $|\mathcal{U}_{\mathsf{ni,tag}}(K, i_1, \delta_1)| \geq \frac{|\mathcal{U}_{\mathsf{ni,tag}}(K)|}{2|\mathcal{I}_{\leq n}|}$. Assume otherwise, i.e., $|\mathcal{U}_{\mathsf{ni,tag}}(K, i_1, \delta_1)| < \frac{|\mathcal{U}_{\mathsf{ni,tag}}(K)|}{2|\mathcal{I}_{\leq n}|}$ for all possible (i_1, δ_1). Then, it holds $i_1 \in \mathcal{I}_{\leq n}$ by Lemma 5, and the total number of choices of (i_1, δ_1) is thus at most $2|\mathcal{I}_{\leq n}|$. This means $\sum_{(i_1,\delta_1) \in \mathcal{I}_{\leq n} \times \{+,-\}} |\mathcal{U}_{\mathsf{ni,tag}}(K, i_1, \delta_1)| < |\mathcal{U}_{\mathsf{ni,tag}}(K)|$, an obvious contradiction. Similarly, there exists $(i_3, \delta_3) \in \mathcal{BI}_3$ with $|\mathcal{Y}_{\mathsf{ni}}(K, i_3, \delta_3)| \geq \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2|\mathcal{I}_{\leq n}|}$. Thus, the pair $\left((i_1, \delta_1), (i_3, \delta_3)\right)$ exists as claimed. $\square$

By Lemma 7, every $K \in \mathcal{K}_2$ has a corresponding $\left((i_1^{(K)}, \delta_1^{(K)}), (i_3^{(K)}, \delta_3^{(K)})\right) \in \mathcal{BI}_1 \times \mathcal{BI}_3$ with $|\mathcal{U}_{\mathsf{ni,ni}}(K, i_1^{(K)}, \delta_1^{(K)})| \geq \frac{|\mathcal{U}_{\mathsf{ni,ni}}(K)|}{2|\mathcal{I}_{\leq n}|} \geq \frac{2^n R}{4|\mathcal{I}_{\leq n}|}$ and $|\mathcal{Y}_{\mathsf{ni}}(K, i_3^{(K)}, \delta_3^{(K)})| \geq \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2|\mathcal{I}_{\leq n}|} \geq \frac{2^n R}{2|\mathcal{I}_{\leq n}|}$. Since the number of pairs $\left((i_1, \delta_1), (i_3, \delta_3)\right) \in \mathcal{BI}_1 \times \mathcal{BI}_3$ is at most $4|\mathcal{I}_{\leq n}|^2$, the pigeonhole principle yields that there exists $\left((i_1^*, \delta_1^*), (i_3^*, \delta_3^*)\right) \in \mathcal{BI}_1 \times \mathcal{BI}_3$ and $\mathcal{K}_3 \subseteq \mathcal{K}_2$ such that:

- $|\mathcal{K}_3| \geq \frac{|\mathcal{K}_2|}{4|\mathcal{I}_{\leq n}|^2}$, and
- $(i_1^{(K)}, \delta_1^{(K)}) = (i_1^*, \delta_1^*)$ and $(i_3^{(K)}, \delta_3^{(K)}) = (i_3^*, \delta_3^*)$ for all $K \in \mathcal{K}_3$.

I.e., $\textsc{Subcase32}(\mathcal{K}_2, R)$ can find the values at line 2.

Subcase 3.2.1: Subcase32 enters the branch at line 6. We refer to Fig. 7 for the involved query structures in this subcase. For the analysis, let $\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2,\delta_2,z_2)) := \cup_{z_1\in\{0,1\}^{m(i_1^*)}}\mathcal{KU}_{\mathsf{ni,ni}}((i_1^*,\delta_1^*,z_1),(i_2,\delta_2,z_2))$. We first show the existence of $z_1^{**}\in\{0,1\}^{m(i_1^*)}$ satisfying conditions of line 7. Assume otherwise, i.e., $\sum_{(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ})\neq(i_2^\circ,\delta_2^\circ,z_2^\circ)}|\mathcal{KU}_{\mathsf{ni,ni}}((i_1^*,\delta_1^*,z_1),(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ}))| = O(\mathrm{poly}(n))$ for all $z_1\in\{0,1\}^{m(i_1^*)}$. This means

$$\begin{aligned}&\sum_{(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ})\neq(i_2^\circ,\delta_2^\circ,z_2^\circ)}\big|\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ}))\big|\\ &=\sum_{z_1\in\{0,1\}^{m(i_1^*)}}\ \sum_{(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ})\neq(i_2^\circ,\delta_2^\circ,z_2^\circ)}\big|\mathcal{KU}_{\mathsf{ni,ni}}((i_1^*,\delta_1^*,z_1),(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ}))\big|\\ &\le 2^{m(i_1^*)}\times O(\mathrm{poly}(n)) = O(\mathrm{poly}(n)2^n).\end{aligned}$$

Further,

$$\begin{aligned}&|\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2^\circ,\delta_2^\circ,z_2^\circ))| = |\mathcal{KU}_{\mathsf{ni,ni}}| - \sum_{(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ})\neq(i_2^\circ,\delta_2^\circ,z_2^\circ)}|\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ}))|\\ &\ge \frac{2^{\kappa+n-8}R}{|\mathcal{I}_{\le n}|^3} - O(\mathrm{poly}(n)2^n) = \Theta(2^{\kappa+n}/\mathrm{poly}(n)).\end{aligned}$$

Therefore, $\big|\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2^\circ,\delta_2^\circ,z_2^\circ))\big| \gg \big|\mathcal{KU}_{\mathsf{ni,ni}}(\star,(i_2^\circ,\delta_2^\circ,z_2^{\circ\circ}))\big|$ for $z_2^{\circ\circ}\neq z_2^\circ$. This contradicts Lemma 6. Therefore, SUBCASE32 can pick the desired triple $(i_1^*,\delta_1^*,z_1^{**})$ at line 6.

By design, SUBCASE32 then invokes SCASE32CHECK. There are three chances for SCASE32CHECK to output 1:

(i) At line 25 (when $y_1\notin\mathcal{Y}_{\mathsf{ni}}(K_1,i_3^*,\delta_3^*)$ or $y_2\notin\mathcal{Y}_{\mathsf{ni}}(K_2,i_3^*,\delta_3^*)$ or $y_3\notin\mathcal{Y}_{\mathsf{ni}}(K_3,i_3^*,\delta_3^*)$). For simplicity, denote this event by Badni.

(ii) At line 35 (when $z_3^{(1)}=z_3^{(2)}$). For simplicity, denote this event by Coll.

(iii) At line 40 (when $b=0$ and $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})=(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$) or line 42 (when $b=1$ and $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})\neq(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$).

In the real world, it always holds $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})=(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$ when $b=0$ and $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})\neq(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$ when $b=1$: see Fig. 6 (right) for illustration. Therefore, $\Pr\big[\text{SCASE32CHECK} = 1 \text{ in real world}\big]=1$.

Now, consider the ideal world interaction. For this, let

$$\begin{aligned}M_1 &:= \big|\mathcal{KU}_{\mathsf{ni,ni}}((i_1^*,\delta_1^*,z_1^*),(i_2^\circ,\delta_2^\circ,z_2^\circ))\big|,\\ M_2 &:= \sum_{(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ})\neq(i_2^\circ,\delta_2^\circ,z_2^\circ)}\big|\mathcal{KU}_{\mathsf{ni,ni}}((i_1^*,\delta_1^*,z_1^{**}),(i_2^{\circ\circ},\delta_2^{\circ\circ},z_2^{\circ\circ}))\big|\end{aligned}$$

for the sets picked at lines 6 and 7. By the conditions, it holds $M_1, M_2 = \Omega(\mathrm{poly}(n))$.

Since (i_1^*,δ_1^*,z_1^*) and $(i_1^*,\delta_1^*,z_1^{**})$ are queried at line 8, S knows these two sets since then. Let $\mathsf{Queried}$ be the event that at least one of the **IC**-queries

$\mathbf{IC}(K_1, x_1) \to y_1$, $\mathbf{IC}(K_2, x_2) \to y_2$ and $\mathbf{IC}(K_3, x_3) \to y_3$ has been made by S before SCASE32CHECK is invoked.

Since: (i) $(K_1, u_1), (K_2, u_2) \xleftarrow{\$} \mathcal{KU}_{\mathsf{ni},\mathsf{ni}}\big((i_1^*, \delta_1^*, z_1^*), (i_2^\circ, \delta_2^\circ, z_2^\circ)\big)$ and $(K_3, u_3) \xleftarrow{\$} \cup_{(i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ}) \neq (i_2^\circ, \delta_2^\circ, z_2^\circ)} \mathcal{KU}_{\mathsf{ni},\mathsf{ni}}\big((i_1^*, \delta_1^*, z_1^{**}), (i_2^{\circ\circ}, \delta_2^{\circ\circ}, z_2^{\circ\circ})\big)$ are uniformly sampled, and (ii) S makes at most q $\mathbf{IC}$-queries, meaning that the number of "good" choices for $(K_1, u_1), (K_2, u_2)$ that do not lead to $\mathsf{Queried}$ (i.e., do not contain simulator-queried inputs) is at least $\binom{M_1-q}{2}$ and the number of "good" choices for (K_3, u_3) that do not lead to $\mathsf{Queried}$ is at least $M_2 - q$, and (iii) subsequent line 11 only has "simulator-unaware" computations and does not provide any information about the three sampled pairs, it holds

$$\Pr\big[\mathsf{Queried}\big] = 1 - \Pr\big[\neg\mathsf{Queried}\big] \le 1 - \frac{\binom{M_1-q}{2}}{\binom{M_1}{2}} \times \frac{M_2 - q}{M_2} \le \frac{2q}{M_1 - 1} + \frac{q}{M_2}.$$

Conditioned on that $\mathsf{Queried}$ didn't occur, we analyze the 3 chances in turn.

Line 25. With the conditions of this lemma, the three $\mathbf{IC}$-queries $\mathrm{E}(K_1, x_1) \to y_1$, $\mathrm{E}(K_2, x_2) \to y_2$, $\mathrm{E}(K_3, x_3) \to y_3$ at line 22 are all new and fresh. By this and by $|\mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)| \ge \frac{2^n R}{2|\mathcal{I}_{\le n}|}$ for all $K \in \mathcal{K}_3$, we have

$$\begin{aligned}\Pr\big[\mathsf{Badni}\big] &= 1 - \Pr\big[y_1 \in \mathcal{Y}_{\mathsf{ni}}(K_1, i_3^*, \delta_3^*) \wedge y_2 \in \mathcal{Y}_{\mathsf{ni}}(K_2, i_3^*, \delta_3^*) \\ &\qquad \wedge y_3 \in \mathcal{Y}_{\mathsf{ni}}(K_3, i_3^*, \delta_3^*)\big] \\ &\le 1 - \left(\frac{\frac{2^n R}{2|\mathcal{I}_{\le n}|} - q}{2^n}\right)^3 \le 1 - \left(\frac{R}{2|\mathcal{I}_{\le n}|} - \frac{q}{2^n}\right)^3. \end{aligned} \tag{28}$$

Line 35. As argued, regardless of the value of b, both y_1° and y_2° are fresh and uniform. Thus, by Corollary 1 we have

$$\Pr\big[\mathsf{Coll} \mid \neg\mathsf{Badni}\big] = \Pr\big[z_3^{(1)} = z_3^{(2)}\big] \le \frac{2}{2^{m_{min}}}.$$

Lines 40 and 42. In the ideal world, S always receives two P-queries $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(1)})$ and $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(2)})$. Subsequently derived $(i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$ and $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)})$ depend on the responses $z_3^{'(1)}$ and $z_3^{'(2)}$ chosen by S. To let SCASE32CHECK output 1 at lines 40 or 42, S should choose $z_3^{'(1)}$ and $z_3^{'(2)}$ such that:

- $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) = (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$ when $b = 0$, and
- $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) \neq (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$ when $b = 1$.

(Let's be generous to S and assume that such $z_3^{'(1)}$ and $z_3^{'(2)}$ do *exist*.)

Throughout the execution, S gains four P-queries from SUBCASE32, i.e., $\mathrm{P}(i_1^*, \delta_1^*, z_1^*)$ and $\mathrm{P}(i_1^*, \delta_1^*, z_1^{**})$ at line 8 and $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(1)})$ and $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(2)})$ at lines 37 and 38. It might hold $z_1^* = z_1^{**}$, but this has no influence. Conditioned on these, S cannot determine the crucial pairs $(K_1, u_1), (K_2, u_2), (K_3, u_3)$ and b: every combination $\big((K_1', u_1'), (K_2', u_2'), (K_3', u_3'), b'\big)$ has the same probability of occurring. This can be seen from the fact that our arguments above can be carried for *all such triples*.[9]

Therefore, S's best strategy to choose $z_3^{'(1)}$ and $z_3^{'(2)}$ is to guess the challenge bit b, and the success probability is 1/2. By these, we have

$$\Pr\big[\textsc{Scase32Check} \text{ outputs 1 at line 40 or 42} \mid \neg\mathsf{Queried} \wedge \neg\mathsf{Badni} \wedge \neg\mathsf{Coll}\big] \leq \frac{1}{2}.$$

<u>*Summary.*</u> Summing over the above probabilities, we reach

$$\begin{aligned}
&\Pr\big[\textsc{Scase32Check} = 1 \text{ in ideal world}\big] \\
\leq\ &\Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] + \Pr\big[\mathsf{Coll} \wedge \neg\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] \\
&+ \Pr\big[\textsc{Scase32Check} \text{ outputs 1 at line 40 or 42} \wedge \neg\mathsf{Queried} \wedge \neg\mathsf{Badni} \wedge \neg\mathsf{Coll}\big] \\
\leq\ &\Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Badni} \mid \neg\mathsf{Queried}\big] + \Pr\big[\mathsf{Coll} \mid \neg\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] \\
&+ \frac{1}{2} \times \Pr\big[\neg\mathsf{Badni} \mid \neg\mathsf{Queried}\big] \\
=\ &\Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Coll} \mid \neg\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] + \frac{1}{2} + \frac{1}{2} \times \Pr\big[\mathsf{Badni} \mid \neg\mathsf{Queried}\big] \\
\leq\ &\frac{2q}{M_1 - 1} + \frac{q}{M_2} + \frac{2}{2^{m_{min}}} + 1 - \frac{1}{2} \times \left(\frac{R}{2|\mathcal{I}_{\leq n}|} - \frac{q}{2^n}\right)^3.
\end{aligned} \tag{29}$$

These indicate the advantage lower bound $\frac{1}{2} \times \big(\frac{R}{2|\mathcal{I}_{\leq n}|} - \frac{q}{2^n}\big)^3 - \frac{2q}{M_1-1} - \frac{q}{M_2} - \frac{2}{2^{m_{min}}}$. Since $M_1, M_2 = \Omega(\mathrm{poly}(n))$ by the conditions, we have R, $\frac{R}{2|\mathcal{I}_{\leq n}|} = \Omega(1/\mathrm{poly}(n))$ and $\frac{q}{2^n}, \frac{2q}{M_1-1}, \frac{q}{M_2} = \mathrm{negl}(n)$, and the lower bound is $1/\mathrm{poly}(n) - \mathrm{negl}(n)$ in Subcase 3.2.1.

Subcase 3.2.2: Subcase32 enters the branch at line 13. I.e., there exists $\lambda = O(\mathrm{poly}(n))$ such that $|\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1^*, \delta_1^*, z_1^*), (i_2^\circ, \delta_2^\circ, z_2^\circ)\big)| \leq \lambda$ for all z_1^* and $(i_2^\circ, \delta_2^\circ, z_2^\circ)$. We refer to Fig. 6 (right) for the involved query structure in this subcase.

<u>*Existence of the 3t pairs.*</u> Since $|\mathcal{U}_{\mathsf{ni,ni}}(K, i_1, \delta_1)| \geq \frac{2^n R}{4|\mathcal{I}_{\leq n}|}$ for all $K \in \mathcal{K}_3$ and since $|\mathcal{K}_3| \geq \frac{|\mathcal{K}_2|}{4|\mathcal{I}_{\leq n}|^2}$, the set $\mathcal{KU}_{\mathsf{ni,ni}} := \big\{(K, u) : K \in \mathcal{K}_3 \wedge u \in \mathcal{U}_{\mathsf{ni,ni}}(K, i_1, \delta_1)\big\}$ has $|\mathcal{KU}_{\mathsf{ni,ni}}| \geq \frac{2^n R}{4|\mathcal{I}_{\leq n}|} \times \frac{|\mathcal{K}_2|}{4|\mathcal{I}_{\leq n}|^2} \geq \frac{2^{\kappa+n-8} R}{|\mathcal{I}_{\leq n}|^3}$.

We build a bipartite graph $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$ with left shore $\mathcal{V}_L = \{(i_1^*, \delta_1^*, z_1) : z_1 \in \{0,1\}^{m(i_1)}\}$, right shore $\mathcal{V}_R = \mathsf{Rng}_{\mathsf{ni}}(\varphi_2^{in})$ (see Eq. (14)) and $|\mathcal{E}| = |\mathcal{KU}_{\mathsf{ni,ni}}|$. $\mathcal{G}$ contains an edge $\big((i_1^*, \delta_1^*, z_1), (i_2, \delta_2, z_2)\big) \in \mathcal{E}$ if and only if there exists $(K, u) \in \mathcal{KU}_{\mathsf{ni,ni}}$ such that $\gamma_1^{in}(K, u) = (i_1^*, \delta_1^*, z_1)$ and $\varphi_2^{in}(K, u) = (i_2, \delta_2, z_2)$. $\mathcal{G}$ $\mathcal{G}$ is a simple bipartite graph (without duplicated edges): when there are multiple pairs $(K_1, u_1), (K_2, u_2), \ldots$ "linking" the same pair of vertexes, $\mathcal{E}$ only includes one edge. Therefore, $|\mathcal{V}_L| \leq 2^n$ and $|\mathcal{V}_R| \leq |\mathcal{I}_{\leq n}| 2^{n+1}$ by Lemma 5, while $|\mathcal{E}| \geq |\mathcal{KU}_{\mathsf{ni,ni}}|/\lambda \geq \frac{2^{\kappa+n-8} R}{\lambda |\mathcal{I}_{\leq n}|^3}$ by our assumptions.

[9] A formal argument can be given by analyzing the conditional probability $\Pr\big[\big((K_1', u_1'), (K_2', u_2'), (K_3', u_3'), b'\big) \mid (i_1^*, \delta_1^*, z_1^*), \ldots, (i_3^*, \delta_3^*, z_3^{(2)})\big]$, but this would be somewhat tedious.

Then, the $3+t$ vertexes and $3t$ pairs satisfying Eq. (27) indicate a biclique $K_{3,t}$ in $\mathcal{G}$, which takes $\big((i_1^*, \delta_1^*, z_1^{(j)})\big)_{j=1,2,3}$ and $\big((i_2^{(j)}, \delta_2^{(j)}, z_2^{(j)})\big)_{j=1,2,\dots,t}$ as vertexes and $\big((K_{1,j}, u_{1,j}), (K_{2,j}, u_{2,j}), (K_{3,j}, u_{3,j})\big)_{j=1,2,\dots,t}$ as edges. We apply Proposition 2: let $M = 2^n$, $N = |\mathcal{I}_{\le n}|2^{n+1}$, $a = m_{max}$ and $b = 2^n$ (since $m_{max} > 3$ for sufficiently large n and $t \le n$, if $\mathcal{G}$ contains $K_{m_{max},2^n}$ then it contains $K_{3,t}$, though using $a = m_{max}$ prepares for Subcase 3.4 while $b = 2^n$ simplifies calculations), then the above graph $\mathcal{G}$ satisfies the two conditions of Proposition 2. Moreover, when n is large enough it holds

$$\begin{aligned}
&(b-1)^{\frac{1}{a}} \cdot MN^{1-\frac{1}{a}} + (a-1)N \\
\le\ & (2^n)^{\frac{1}{m_{max}}} \cdot 2^n \cdot |\mathcal{I}_{\le n}|^{1-\frac{1}{m_{max}}} (2^{n+1})^{1-\frac{1}{m_{max}}} + (m_{max}-1)|\mathcal{I}_{\le n}|2^{n+1} \\
\le\ & 2^{2n} \cdot (2|\mathcal{I}_{\le n}|)^{\frac{m_{max}-1}{m_{max}}} + (m_{max}-1)|\mathcal{I}_{\le n}|2^{n+1} \le |\mathcal{I}_{\le n}|2^{2n+1}.
\end{aligned}$$

Therefore, $\mathcal{G}$ contains $K_{m_{max},2^n} \supseteq K_{3,t}$ as long as $|\mathcal{E}| \ge \frac{2^{\kappa+n-8}R}{\lambda|\mathcal{I}_{\le n}|^3} \ge |\mathcal{I}_{\le n}|2^{2n+1}$, meaning that $\kappa \ge n+9-\log_2 R+\log_2 \lambda+4\log_2 |\mathcal{I}_{\le n}|$. Note that $R = \Omega(1/\mathrm{poly}(n))$ and $\lambda = O(\mathrm{poly}(n))$, thus $-\log_2 R + \log_2 \lambda = \Theta(1)$. This is fulfilled by our assumption $\kappa \ge 2m_{max}\log_2|\mathcal{I}_{\le n}| + 2m_{max}n + 6m_{max} + 4$ as long as $m_{max} \ge 2$ and n is large enough, and these establish the existence of $3t$ pairs satisfying Eq. (27).

Attack advantage. By design, SUBCASE32 then makes a call to the subroutine SCASE32CHECK$(K_{j_1,\ell_1}, x_{j_1,\ell_1}, K_{j_2,\ell_1}, x_{j_2,\ell_1}, K_{j_3,\ell_2}, x_{j_3,\ell_2}, i_3^*, \delta_3^*)$ at line 19, which produces the final output. The analysis follows Subcase 3.2.1.

First, $\Pr\big[$SCASE32CHECK $= 1$ in real world$\big] = 1$, and we mainly need to address the ideal world interaction. To this end, assume that the simulator S completely knows the biclique found by SUBCASE32$(\mathcal{K}_2, R)$ at line 15, and let Queried be the event that at least one of the **IC**-queries at line 22 (which are $\mathbf{IC}(K_{j_1,\ell_1}, x_{j_1,\ell_1}) \to y_{j_1,\ell_1}$, $\mathbf{IC}(K_{j_2,\ell_1}, x_{j_2,\ell_1}) \to y_{j_2,\ell_1}$ and $\mathbf{IC}(K_{j_3,\ell_2}, x_{j_3,\ell_2}) \to y_{j_3,\ell_2}$ using the notations in SUBCASE32) has been made by S before SUBCASE32 executing this line. Since: (i) S makes at most q **IC**-queries, meaning that the number of "good" indices $\ell^* \in \{1, ..., t\}$ that do not lead to Queried (i.e., do not contain simulator-queried inputs) is at least $t - q$, and (ii) ℓ_1, ℓ_2 are uniformly sampled from $\{1, ..., t\}$ at line 17, and (iii) subsequent line 18 only has "simulator-unaware" computations and does not provide any information about ℓ_1, ℓ_2, it holds ($t - 1 \ge t/2$ when $m_{min} = \mathrm{poly}(n)$ is large enough)

$$\Pr\big[\mathsf{Queried}\big] = 1 - \Pr\big[\neg\mathsf{Queried}\big] \le 1 - \frac{\binom{t-q}{2}}{\binom{t}{2}} \le \frac{3q}{t} \le \frac{3q}{2^n}. \tag{30}$$

Conditioned on that Queried didn't occur, the analyses of lines 25 and 35 are exactly the same as Subcase 3.2.1. Minor differences appears regarding lines 40 and 42. In detail, throughout the execution, S gains five P-queries from SUBCASE32, i.e., $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(j_1)})$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(j_2)})$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^{(j_3)})$, $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(1)})$ and $\mathrm{P}(i_3^*, \delta_3^*, z_3^{(2)})$. Conditioned on these, S cannot determine the crucial values ℓ_1, ℓ_2

and b: every triple $(\ell_1, \ell_2, b) \in \{1, ..., t\} \times \{1, ..., t\} \times \{0, 1\}$ has the same probability of occurring. Therefore, S's best strategy to choose $z_3^{'(1)}$ and $z_3^{'(2)}$ is to guess the challenge bit b, and the success probability is 1/2. By these and following Eq. (29), we have

$$\begin{aligned}
&\Pr\big[\textsc{Scase32Check} = 1 \text{ in ideal world}\big] \\
\le\ &\Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Badni} \mid \neg\mathsf{Queried}\big] + \Pr\big[\mathsf{Coll} \mid \neg\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] \\
&+ \frac{1}{2} \times \Pr\big[\neg\mathsf{Badni} \mid \neg\mathsf{Queried}\big] \\
=\ &\Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Coll} \mid \neg\mathsf{Badni} \wedge \neg\mathsf{Queried}\big] + \frac{1}{2} + \frac{1}{2} \times \Pr\big[\mathsf{Badni} \mid \neg\mathsf{Queried}\big] \\
\le\ &\frac{3q}{2^n} + \frac{2}{2^{m_{min}}} + 1 - \frac{1}{2} \times \left(\frac{R}{2|\mathcal{I}_{\le n}|} - \frac{q}{2^n}\right)^3.
\end{aligned}$$

Since R, $\frac{R}{2|\mathcal{I}_{\le n}|} = \Omega(1/\text{poly}(n))$ and $\frac{q}{2^n}, \frac{3q}{2^n} = \text{negl}(n)$, the bound is $1/\text{poly}(n) - \text{negl}(n)$ in Subcase 3.2.2. In all, attack advantage is always at least $1/\text{poly}(n) - \text{negl}(n)$ in Subcase 3.2.

Discussion. Our use of $b \xleftarrow{\$} \{0, 1\}$ gains inspirations from Andreeva et al.'s differentiator against a type of 3-round IEM cipher [1, Sect. 3.3, full version]. In particular, an earlier version of our attack closely follows Andreeva et al., but its effectiveness relies on the distribution of the functions γ_3^{out} and φ_2^{in}, which is difficult. We thus introduce two different pairs $\big((K_1, y_1), (K_2, y_2)\big)$ and $\big((K_1, y_1), (K_3, y_3)\big)$: the former collide in the 2nd round, while the latter do not collide. By this, S has to guess whether it should ensure $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}) = (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$ and cannot win easily.

9.4 Subcase 3.3: $|\mathcal{U}_{\mathsf{ni,if}}(K)| = \Omega(2^n/\text{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\text{poly}(n))$ for all $K \in \mathcal{K}_2$

As mentioned, this corresponds to Case 2 in Sect. 3.4.

Subroutine $\textsc{Subcase33}(\mathcal{K}_2, R)$
Picks $(i_1^*, \delta_1^*), (i_3^*, \delta_3^*) \in \mathcal{I}_{\le n} \times \{+, -\}$ and $\mathcal{K}_3 \subseteq \mathcal{K}_2$ such that:
– $|\mathcal{K}_3| \ge \frac{|\mathcal{K}_2|}{4|\mathcal{I}_{\le n}|^2}$, and
– $|\mathcal{U}_{\mathsf{ni,if}}(K, i_1^*, \delta_1^*)| \ge \frac{2^n R}{4|\mathcal{I}_{\le n}|}$ and $|\mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)| \ge \frac{2^n R}{2|\mathcal{I}_{\le n}|}$ for all $K \in \mathcal{K}_3$.
$z_1^* \xleftarrow{\$} \{0, 1\}^{m(i_1^*)}$, $\mathrm{P}(i_1^*, \delta_1^*, z_1^*) \to z_1^{'*}$
$K \xleftarrow{\$} \mathcal{K}_3$
Picks $u \in \mathcal{U}_{\mathsf{ni,if}}(K)$ **s.t.** $\gamma_1^{in}(K, u) = (i_1^*, \delta_1^*, z_1^*)$
$x \leftarrow \gamma_1^{out}\big(K, z_1^{'*}, u\big)$, $\mathrm{E}(K, x) \to y$
if $y \notin \mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)$ **then**
  Outputs 1
else   // $y \in \mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)$
  $(i_3^*, \delta_3^*, z_3) \leftarrow \gamma_3^{in}(K, y)$, $\mathrm{P}(i_3^*, \delta_3^*, z_3) \to z_3'$, $w \leftarrow \gamma_3^{out}(K, z_3', y)$
  Outputs 1 **iff.** $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$
end if

Analysis of Subcase 3.3. Recall from Fig. 17 that the concrete condition is $|\mathcal{U}_{\mathsf{ni,if}}(K)| \geq 2^n R/2$, where $R = \min_{K\in\mathcal{K}_1}\left\{\frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n}, \frac{|\mathcal{Y}_{\mathsf{ni}}(K)|}{2^n}\right\}$. We refer to Fig. 8 (left) for the involved query structure.

Using Lemma 7 and following Subcase 3.2 (Sect. 9.3), it can be shown that $\textsc{Subcase33}(\mathcal{K}_2, R)$ will succeed in finding $(i_1^*, \delta_1^*), (i_3^*, \delta_3^*)$ at line 2. Below we focus on the attack advantage. To this end, $\textsc{Subcase33}(\mathcal{K}_2, R)$ either outputs 1 at line 8, or outputs 1 at line 11 when $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$ holds. In the real world, the latter equality always holds, and thus $\Pr[\textsc{Subcase33}(\mathcal{K}_2, R) = 1] = 1$.

In the ideal world, let $\mathsf{Queried}$ be the event that either $\mathbf{IC}\big(K, x\big) \to y$ or $\mathbf{IC}^{-1}\big(K, y\big) \to x$ has been made by S before executing line 6. Since S makes at most q $\mathbf{IC}$-queries, and since K is uniformly picked from $\mathcal{K}_3$ at line 5, it holds

$$\Pr\big[\mathsf{Queried}\big] \leq \frac{q}{|\mathcal{K}_3|} \leq \frac{4q|\mathcal{I}_{\leq n}|^2}{2^{\kappa-4}} \leq \frac{2^6 q|\mathcal{I}_{\leq n}|^2}{2^\kappa}. \tag{31}$$

If $y \notin \mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*)$ then $\textsc{Subcase33}(\mathcal{K}_2)$ outputs 1 at line 8. Conditioned on that $\mathsf{Queried}$ did not occur, the key K never appeared in earlier $\mathbf{IC}$-queries and $\mathbf{IC}\big(K, x\big) \to y$ yields a uniform y. Following Eq. (28), we have

$$\Pr\big[y \notin \mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*) \text{ at line } 8\big] \leq 1 - \left(\frac{R}{2|\mathcal{I}_{\leq n}|} - \frac{q}{2^n}\right). \tag{32}$$

Once $\textsc{Subcase33}(\mathcal{K}_2, R)$ enters the branch after line 9, to let it output 1, S should "guess" K and ensure the equality $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$ before executing line 11. Every key in $\mathcal{K}_3$ has encipherments "anchored" at the queries $\mathcal{P}(i_1^*, \delta_1^*, z_1^*) \to z_1^{'*}$ and $\mathcal{P}(i_3^*, \delta_3^*, z_3^*) \to z_3^{'*}$. Therefore, even if line 10 has been executed and S has been aware of the two adversarial queries to P, since S does not know the adversarial query to $\mathbf{IC}(K, x) \to y$, all keys in $\mathcal{K}_3$ still have the same chance of being the selected K. We can view the experiment as $K \xleftarrow{\$} \mathcal{K}_3$ happens *after* line 10, so that the probability to have $\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w)$ for the w chosen by S is at most $2\varepsilon_{de(\varphi_2^{in})}$ by Eq. (19). Gathering this with Eqs. (31) and (32) yields

$$\begin{aligned}\Pr\big[\textsc{Subcase32}(\mathcal{K}_2, R) = 1\big] \leq\ & \Pr\big[\mathsf{Queried}\big] + \Pr\big[y \notin \mathcal{Y}_{\mathsf{ni}}(K, i_3^*, \delta_3^*) \text{ at line } 8\big] \\ & + \Pr\big[\varphi_2^{in}(K, u) = \gamma_2^{in}(K, w) \text{ at line } 11\big] \\ \leq\ & \frac{2^6 q|\mathcal{I}_{\leq n}|^2}{2^\kappa} + 1 - \left(\frac{R}{2|\mathcal{I}_{\leq n}|} - \frac{q}{2^n}\right) + 2\varepsilon_{de(\varphi_2^{in})}.\end{aligned}$$

The advantage in Subcase 3.2 thus has lower bound $\frac{R}{2|\mathcal{I}_{\leq n}|} - \frac{q}{2^n} - \frac{2^6 q|\mathcal{I}_{\leq n}|^2}{2^\kappa} - 2\varepsilon_{de(\varphi_2^{in})}$. Since $R, \frac{R}{2|\mathcal{I}_{\leq n}|} = \Omega(1/\text{poly}(n))$ and $\frac{q}{2^n}, \frac{2^6 q|\mathcal{I}_{\leq n}|^2}{2^\kappa}, 2\varepsilon_{de(\varphi_2^{in})} = \text{negl}(n)$, the lower bound is $1/\text{poly}(n) - \text{negl}(n)$.

9.5 Subcase 3.4: $|\mathcal{U}_{\mathsf{ni,ni}}(K)| = \Omega(2^n/\text{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\text{poly}(n))$ for all $K \in \mathcal{K}_2$

As mentioned, this corresponds to Case 4 in Sect. 3.4.

Subroutine SUBCASE34($\mathcal{K}_2, R$)
Determines the set $\mathcal{KU}_{\mathsf{ni,ni}} := \big\{(K,u) : K \in \mathcal{K}_2 \wedge u \in \mathcal{U}_{\mathsf{ni,ni}}(K)\big\}$
Determines the sets $\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1,\delta_1,z_1),(i_2,\delta_2,z_2)\big) := \big\{(K,u) : (K,u) \in \mathcal{KU}_{\mathsf{ni,ni}} \wedge \gamma_1^{in}(K,u) = (i_1,\delta_1,z_1) \wedge \varphi_2^{in}(K,u) = (i_2,\delta_2,z_2)\big\}$ for all (i_1,δ_1,z_1) and (i_2,δ_2,z_2)
if $\exists(i_1,\delta_1,z_1),(i_2,\delta_2,z_2) : |\mathcal{KU}_{\mathsf{ni,ni}}\big((i_1,\delta_1,z_1),(i_2,\delta_2,z_2)\big)| \geq m_{max}$ **then**
  Invokes SCASE34CHECK($\mathcal{KU}_{\mathsf{ni,ni}}, i_1, \delta_1, z_1, i_2, \delta_2, z_2$) // See below
else // Multiplicity $< m_{max}$
  $\lambda \leftarrow m_{max}$, $t \leftarrow 2^n$
  Picks $\lambda + t$ inputs/outputs $\big((i_1^{(j)},\delta_1^{(j)},z_1^{(j)})\big)_{j=1,2,...,\lambda}$, $\big((i_2^{(j)},\delta_2^{(j)},z_2^{(j)})\big)_{j=1,2,...,t}$ and λt pairs $\big((K_{1,j},u_{1,j}),...,(K_{\lambda,j},u_{\lambda,j})\big)_{j=1,2,...,t}$ from $\mathcal{KU}_{\mathsf{ni,ni}}$ that satisfy

$$\begin{aligned}&\gamma_1^{in}\big(K_{j,1},u_{j,1}\big) = ... = \gamma_1^{in}\big(K_{j,t},u_{j,t}\big) = (i_1^{(j)},\delta_1^{(j)},z_1^{(j)}) \text{ for all } j \in \{1,...,\lambda\};\\ &\varphi_2^{in}\big(K_{1,j},u_{1,j}\big) = ... = \varphi_2^{in}\big(K_{\lambda,j},u_{\lambda,j}\big) = (i_2^{(j)},\delta_2^{(j)},z_2^{(j)}) \text{ for } j \in \{1,...,t\}.\end{aligned} \tag{33}$$

  $\mathrm{P}(i_1^{(1)},\delta_1^{(1)},z_1^{(1)}) \to z_1^{'(1)}$, $\mathrm{P}(i_1^{(2)},\delta_1^{(2)},z_1^{(2)}) \to z_1^{'(2)}$, ..., $\mathrm{P}(i_1^{(\lambda)},\delta_1^{(\lambda)},z_1^{(\lambda)}) \to z_1^{'(\lambda)}$
  $\ell \xleftarrow{\$} \{1,2,...,t\}$
  Computes λ plaintexts $x_{1,\ell} \leftarrow \gamma_1^{out}(K_{1,\ell}, z_1^{'(1)}, u_{1,\ell})$, $x_{2,\ell} \leftarrow \gamma_1^{out}(K_{2,\ell}, z_1^{'(2)}, u_{2,\ell})$, ..., $x_{\lambda,\ell} \leftarrow \gamma_1^{out}(K_{\lambda,\ell}, z_1^{'(\lambda)}, u_{\lambda,\ell})$
  $\mathrm{E}(K_{1,\ell},x_{1,\ell}) \to y_{1,\ell}$, $\mathrm{E}(K_{2,\ell},x_{2,\ell}) \to y_{2,\ell}$, ..., $\mathrm{E}(K_{\lambda,\ell},x_{\lambda,\ell}) \to y_{\lambda,\ell}$ // λ queries
  if $\exists j \in \{1,...,\lambda\}$ such that $y_{j,\ell} \in \mathcal{Y}_{\mathsf{ni}}(K_{j,\ell})$ **then**
    Outputs 1
  else // I.e., every ciphertext $y_{j,\ell}$ has $y_{j,\ell} \in \mathcal{Y}_{\mathsf{if}}(K_{j,\ell})$
    $\mathrm{P}(i_2^{(\ell)},\delta_2^{(\ell)},z_2^{(\ell)}) \to z_2^{'(\ell)}$
    $w_{1,\ell} \leftarrow \varphi_2^{out}\big(K_{1,\ell}, z_2^{'(\ell)}, u_{1,\ell}\big)$, ..., $w_{\lambda,\ell} \leftarrow \varphi_2^{out}\big(K_{\lambda,\ell}, z_2^{'(\ell)}, u_{\lambda,\ell}\big)$
    Outputs 1 **iff.** $\varphi_3^{in}(K_{j,\ell}, w_{j,\ell}) = \gamma_3^{in}(K_{j,\ell}, y_{j,\ell})$ for all $j = 1,2,...,\lambda$
  end if
end if

Subroutine SCASE34CHECK($\mathcal{KU}, i_1, \delta_1, z_1, i_2, \delta_2, z_2$)
Picks $\lambda = m_{max}$ distinct pairs $(K_1,u_1),...,(K_\lambda,u_\lambda)$ from $\mathcal{KU}$ **s.t.** $\gamma_1^{in}(K_j,u_j) = (i_1,\delta_1,z_1)$ and $\varphi_2^{in}(K_j,u_j) = (i_2,\delta_2,z_2)$ for all $j \in \{1,...,\lambda\}$
$\mathrm{P}(i_1,\delta_1,z_1) \to z_1'$
Computes λ plaintexts $x_1 \leftarrow \gamma_1^{out}(K_1,z_1',u_1)$, $x_2 \leftarrow \gamma_1^{out}(K_2,z_1',u_2)$, ..., $x_\lambda \leftarrow \gamma_1^{out}(K_\lambda,z_1',u_\lambda)$
$\mathrm{E}(K_1,x_1) \to y_1$, $\mathrm{E}(K_2,x_2) \to y_2$, ..., $\mathrm{E}(K_\lambda,x_\lambda) \to y_\lambda$ // λ queries
Let $\mathcal{J} \subseteq \{1,...,\lambda\}$ be the set of indices such that $y_j \in \mathcal{Y}_{\mathsf{if}}(K_j)$ for all $j \in \mathcal{J}$
if $|\mathcal{J}| < \lambda - 1$ **then**
  Outputs 1
else
  $\mathrm{P}(i_2,\delta_2,z_2) \to z_2'$
  for $j \in \mathcal{J}$ **do**
    $w_j \leftarrow \varphi_2^{out}(K_j,z_2',u_j)$
  end for
  Outputs 1 **iff.** $\varphi_3^{in}(K_j,w_j) = \gamma_3^{in}(K_j,y_j)$ for all $j \in \mathcal{J}$
end if

Analysis of Subcase 3.4. Recall from Fig. 17 that the concrete condition is $|\mathcal{U}_{\mathsf{ni},\mathsf{ni}}(K)| \geq 2^n R/2$, where $R = \min_{K\in\mathcal{K}_1}\left\{\frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n}\right\}$. On the other hand, let $\varepsilon_{\mathsf{ni}} := \max_{K\in\mathcal{K}_1}\left\{|\mathcal{Y}_{\mathsf{ni}}(K)|/2^n\right\}$. By the condition, it holds $\varepsilon_{\mathsf{ni}} = \mathrm{negl}(n)$.

Depending on whether the condition at line 4 holds, we distinguish two subcases.

Subcase 3.4.1: SUBCASE34 *enters the branch at line 5.* In detail, SUBCASE34 invokes SCASE34CHECK. Since SCASE34CHECK will also be used in Sect. 9.6, we provide a lemma for this subroutine.

Lemma 8.

$$\Big|\Pr\big[\textsc{Scase34Check} = 1 \textit{ in real world}\big] - \Pr\big[\textsc{Scase34Check} = 1 \textit{ in ideal world}\big]\Big| \geq 1 - 2q^2/2^{m_{min}} - 2q\varepsilon_{\mathsf{ni}} - 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1} = 1 - \textit{negl}(n).$$

For cleanness, its proof is deferred to the end of Sect. 9.5, and we refer to Fig. 7 (right) for the involved query structure. This lemma immediately yields the advantage lower bound $1-2q^2/2^{m_{min}} - 2q\varepsilon_{\mathsf{ni}} - 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1} = 1-\mathrm{negl}(n)$ in this subcase.

Subcase 3.4.2: SUBCASE34 *enters the branch at line 6.* We refer to Fig. 8 (right) for the involved query structure in this subcase.

The proof that SUBCASE34 can find the vertexes and pairs at line 8 just follows Subcase 3.2 (Sect. 9.3). In the real world, the equalities $\varphi_3^{in}\big(K_{j,\ell}, w_{j,\ell}\big) = \gamma_3^{in}\big(K_{j,\ell}, y_{j,\ell}\big)$ for $j = 1,2,...,\lambda$ at line 18 always hold and SUBCASE34 always outputs 1.

We now consider the ideal world interaction. Partly following Subcase 3.2 (Sect. 9.3), we also introduce an event $\mathsf{Queried}$, which happens if at least one of the λ adversarial **IC**-queries $\mathbf{IC}(K_{1,\ell}, x_{1,\ell}) \to y_{1,\ell}$, ..., $\mathbf{IC}(K_{\lambda,\ell}, x_{\lambda,\ell}) \to y_{\lambda,\ell}$ at line 12 has been made by S before SUBCASE34 executing this line. The analysis follows Subcase 3.2 (Sect. 9.3) and yields $\Pr\big[\mathsf{Queried}\big] \leq q/t \leq q/2^n = \mathrm{negl}(n)$.

Furthermore, let Bad^+ be the event that any forward **IC**-query $\mathbf{IC}(K, x) \to y$ yields $y \in \mathcal{Y}_{\mathsf{if}}(K)$. Using $|\mathcal{Y}_{\mathsf{ni}}(K)| < 2^n\varepsilon_{\mathsf{ni}}$ and summing over the q **IC**-queries, we reach $\Pr[\mathsf{Bad}^+] \leq 2q\varepsilon_{\mathsf{ni}} = \mathrm{negl}(n)$.

Now, in the ideal world, we have

$$\Pr\big[\textsc{Subcase34}(\mathcal{K}_2, R) = 1\big] \leq \Pr\big[\mathsf{Queried}\big] + \Pr\big[\mathsf{Bad}^+\big] + \Pr\big[\textsc{Subcase34}(\mathcal{K}_2, R) = 1 \mid \neg\mathsf{Queried} \wedge \neg\mathsf{Bad}^+\big].$$

When $\textsc{Subcase34}(\mathcal{K}_2, R)$ executes line 16, S gets a "chance" of choosing a "good" $z_2^{\prime(\ell)}$ to ensure the equalities, and $\Pr\big[\textsc{Subcase34}(\mathcal{K}_2) = 1 \mid \neg\mathsf{Queried} \wedge \neg\mathsf{Bad}^+\big]$ is thus bounded by the probability that there exists $z_2' \in \{0,1\}^{m(i_2^{(\ell)})}$ such that $\varphi_3^{in}\big(K_{j,\ell}, \varphi_2^{out}(K_{j,\ell}, z_2', u_{j,\ell})\big) = \gamma_3^{in}(K_{j,\ell}, y_{j,\ell})$ for $j = 1, 2, ..., \lambda$.

Conditioned on $\neg\mathsf{Queried}$, all the λ adversarial **IC**-queries made at line 12 are fresh and are thus forward. By this, by $\neg\mathsf{Bad}^+$ and further by an argument similarly to Subcase 3.1, for each particular $z'_2 \in \{0,1\}^{m(i_2^{(\ell)})}$, the probability to have $\varphi_3^{in}\big(K_{j,\ell}, \varphi_2^{out}(K_{j,\ell}, z'_2, u_{j,\ell})\big) = \gamma_3^{in}(K_{j,\ell}, y_{j,\ell})$, $j = 1, 2, ..., \lambda$, is at most $(2\varepsilon_{de(\varphi_3^{in})})^{\lambda}$ by Eq. (18). Therefore, with $\lambda = m_{max}$, we have

$$\Pr\big[\textsc{Subcase34}(\mathcal{K}_2, R) = 1 \mid \neg\mathsf{Eve} \wedge \neg\mathsf{Bad}^+\big] \leq 2^{m(i_2^{(\ell)})}(2\varepsilon_{de(\varphi_3^{in})})^{\lambda} \leq (4\varepsilon_{de(\varphi_3^{in})})^{m_{max}}.$$

These yield the ideal world probability $\Pr\big[\textsc{Subcase34}(\mathcal{K}_2, R) = 1\big] \leq q/2^n + 2q\varepsilon_{\mathsf{ni}} + (4\varepsilon_{de(\varphi_3^{in})})^{m_{max}}$ and attack advantage lower bound of $1 - q/2^n - 2q\varepsilon_{\mathsf{ni}} - (4\varepsilon_{de(\varphi_3^{in})})^{m_{max}} = 1 - \mathrm{negl}(n)$.

Proof of Lemma 8. By design, $\textsc{Scase34Check}$ outputs 1 either at line 28 due to $|\mathcal{J}| < \lambda - 1$ or at line 34 due to the equalities $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j)$ for all $j \in \mathcal{J}$. In the real world, the latter equalities always hold and thus $\Pr\big[\textsc{Scase34Check} = 1 \text{ in real world}\big] = 1$. Regarding the ideal world probability, we introduce two events:

(i) First, Bad^- occurs during the interaction, if there appears a backward **IC**-query $\mathbf{IC}^{-1}(K, y) \to x$ that has $\varphi_1^{in}(K, x) = \varphi_1^{in}(K', x')$ for a previously appeared **IC** input/output $\mathbf{IC}(K', x') = y'$. Since there are at most q **IC**-queries, it has $\Pr\big[\mathsf{Bad}^-\big] \leq 2q^2/2^{m_{min}}$ by Corollary 1.

(ii) Second, Badni occurs during the interaction, if there appears a forward **IC**-query $\mathbf{IC}(K, x) \to y$ that has $y \in \mathcal{Y}_{\mathsf{ni}}(K)$. Clearly, $\Pr\big[\mathsf{Badni}\big] \leq 2q\varepsilon_{\mathsf{ni}}$.

Now, in the ideal world, we have

$$\begin{aligned}&\Pr\big[\textsc{Scase34Check} = 1 \text{ in ideal world}\big]\\ \leq\ &\Pr\big[\mathsf{Bad}^-\big] + \Pr\big[\mathsf{Badni}\big] + \Pr\big[\textsc{Scase34Check} = 1 \mid \neg\mathsf{Bad}^- \wedge \neg\mathsf{Badni}\big].\end{aligned}$$

Conditioned on $\neg\mathsf{Bad}^-$, at least $\lambda - 1$ among the **IC**-queries $\mathrm{E}(K_1, x_1) \to y_1$, $\mathrm{E}(K_2, x_2) \to y_2$, ..., $\mathrm{E}(K_\lambda, x_\lambda) \to y_\lambda$ at line 25 are forward. Let their indices be $1, ..., \lambda - 1$. Further conditioned on $\neg\mathsf{Badni}$, it holds $y_1 \in \mathcal{Y}_{\mathsf{if}}(K_1), ..., y_{\lambda-1} \in \mathcal{Y}_{\mathsf{if}}(K_{\lambda-1})$, meaning that $\textsc{Scase34Check}$ never outputs 1 at line 28 in the ideal world.

When $\textsc{Scase34Check}$ executes line 30, S gets a "chance" of choosing a "good" z'_2 to ensure the equalities at line 34, and $\Pr\big[\textsc{Scase34Check} = 1 \mid \neg\mathsf{Bad}^- \wedge \neg\mathsf{Badni}\big]$ is thus bounded by the probability that there exists $z'_2 \in \{0,1\}^{m(i_2)}$ such that $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j)$ for $j = 1, 2, ..., \lambda - 1$. Conditioned on $\neg\mathsf{Bad}^-$, all the **IC**-queries indexed by $\mathcal{J}$ are forward. By this, by an argument similarly to Subcase 3.1, for each particular $z'_2 \in \{0,1\}^{m(i_2)}$, the probability to have $\varphi_3^{in}(K_j, w_j) = \gamma_3^{in}(K_j, y_j)$, $j = 1, 2, ..., \lambda - 1$, is at most $(2\varepsilon_{de(\varphi_3^{in})})^{\lambda-1}$ by Eq. (18). Therefore, with $\lambda = m_{max}$, we have

$$\begin{aligned}\Pr\big[\textsc{Scase34Check} = 1 \mid \neg\mathsf{Bad}^- \wedge \neg\mathsf{Badni}\big] &\leq 2^{m(i_2)}(2\varepsilon_{de(\varphi_3^{in})})^{\lambda-1}\\ &\leq 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1}.\end{aligned}$$

These yield $\Pr\big[\textsc{Scase34Check} = 1 \text{ in ideal world}\big] \leq 2q^2/2^{m_{min}} + 2q\varepsilon_{\mathsf{ni}} + 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1}$ and difference $\geq 1 - 2q^2/2^{m_{min}} - 2q\varepsilon_{\mathsf{ni}} - 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1} = 1 - \mathrm{negl}(n)$ in Subcase 3.4.2. In all, advantage is always at least $1 - \mathrm{negl}(n)$ in Subcase 3.4.

9.6 Subcase 3.5: $|\mathcal{U}_{\mathsf{ni,if}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_2$

This corresponds to Case 5 in Sect. 3.4.

1: **Subroutine** SUBCASE35($\mathcal{K}_2$)
2: **Determines** the set $\mathcal{KU}_{\mathsf{ni,if}} := \{(K, u) : K \in \mathcal{K}_2 \wedge u \in \mathcal{U}_{\mathsf{ni,if}}(K)\}$
3: **Determines** the sets $\mathcal{KU}_{\mathsf{ni,if}}\big((i_1, \delta_1, z_1), (i_2, \delta_2, z_2)\big) := \{(K, u) : (K, u) \in \mathcal{KU}_{\mathsf{ni,if}} \wedge \gamma_1^{in}(K, u) = (i_1, \delta_1, z_1) \wedge \varphi_2^{in}(K, u) = (i_2, \delta_2, z_2)\}$ for all (i_1, δ_1, z_1) and (i_2, δ_2, z_2)
4: **if** $\exists (i_1, \delta_1, z_1), (i_2, \delta_2, z_2) : |\mathcal{KU}_{\mathsf{ni,if}}\big((i_1, \delta_1, z_1), (i_2, \delta_2, z_2)\big)| \geq m_{max}$ **then**
5: **Invokes** SCASE34CHECK($\mathcal{KU}_{\mathsf{ni,if}}, i_1, \delta_1, z_1, i_2, \delta_2, z_2$) // See Sect. 9.5
6: **else** // Multiplicity $< m_{max}$
7: $\lambda \leftarrow 2m_{max}$
8: **Picks** 2λ pairs $\big((K_{1,j}, u_{1,j}), (K_{2,j}, u_{2,j})\big)_{j=1,2,...,\lambda}$ and $\lambda + 2$ oracle inputs $\big((i_1^{(j)}, \delta_1^{(j)}, z_1^{(j)})\big)_{j=1,2,...,\lambda}, (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)})$ and $(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)})$, **s.t.**

$$\begin{aligned} &\gamma_1^{in}\big(K_{1,j}, u_{1,j}\big) = \gamma_1^{in}\big(K_{2,j}, u_{2,j}\big) = (i_1^{(j)}, \delta_1^{(j)}, z_1^{(j)}),\ j = 1, ..., \lambda, \\ &\varphi_2^{in}\big(K_{1,1}, u_{1,1}\big) = \varphi_2^{in}\big(K_{1,2}, u_{1,2}\big) = ... = \varphi_2^{in}\big(K_{1,\lambda}, u_{1,\lambda}\big) = (i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)}), \\ &\varphi_2^{in}\big(K_{2,1}, u_{2,1}\big) = \varphi_2^{in}\big(K_{2,2}, u_{2,2}\big) = ... = \varphi_2^{in}\big(K_{2,\lambda}, u_{2,\lambda}\big) = (i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}). \end{aligned} \quad (34)$$

9: Makes $\lambda + 2$ queries $\mathrm{P}\big(i_1^{(1)}, \delta_1^{(1)}, z_1^{(1)}\big) \to z_1^{'(1)}$, ..., $\mathrm{P}\big(i_1^{(\lambda)}, \delta_1^{(\lambda)}, z_1^{(\lambda)}\big) \to z_1^{'(\lambda)}$, and $\mathrm{P}\big(i_2^{(1)}, \delta_2^{(1)}, z_2^{(1)}\big) \to z_2^{'(1)}$, $\mathrm{P}\big(i_2^{(2)}, \delta_2^{(2)}, z_2^{(2)}\big) \to z_2^{'(2)}$
10: Derives 2λ plaintexts $x_{1,j} \leftarrow \varphi_1^{out}\big(K_{1,j}, z_1^{'(j)}, u_{1,j}\big)$ and $x_{2,j} \leftarrow \varphi_1^{out}\big(K_{2,j}, z_1^{'(j)}, u_{2,j}\big)$ for $j = 1, ..., \lambda$
11: Queries $\mathrm{E}(K_{1,j}, x_{1,j}) \to y_{1,j}$ and $\mathrm{E}(K_{2,j}, x_{2,j}) \to y_{2,j}$ for $j = 1, ..., \lambda$
12: Let $\mathcal{J}_1 = \{j'_1, j'_2, ..., j'_{s_1}\} \subseteq \{1, ..., \lambda\}$ be the set of indices such that $y_{1,j'} \in \mathcal{Y}_{\mathsf{if}}(K_{1,j'})$ for all $j' \in \mathcal{J}_1$
13: Let $\mathcal{J}_2 = \{j''_1, j''_2, ..., j''_{s_2}\} \subseteq \{1, ..., \lambda\}$ be the set of indices such that $y_{2,j''} \in \mathcal{Y}_{\mathsf{if}}(K_{2,j''})$ for all $j'' \in \mathcal{J}_2$
14: **if** $|\mathcal{J}_1| < \lambda/2 \ \wedge \ |\mathcal{J}_2| < \lambda/2$ **then**
15: **Outputs** 1
16: **else**
17: **if** $|\mathcal{J}_1| \geq \lambda/2$ **then**
18: $w_{1,j'} \leftarrow \varphi_2^{out}\big(K_{1,j'}, z_2^{'(1)}, u_{1,j'}\big)$ for all $j' \in \mathcal{J}_1$
19: **Outputs** 1 **iff.** $\varphi_3^{in}\big(K_{1,j'}, w_{1,j'}\big) = \gamma_3^{in}\big(K_{1,j'}, y_{1,j'}\big)$ for all $j' \in \mathcal{J}_1$
20: **else** // $|\mathcal{J}_2| \geq \lambda/2$
21: $w_{2,j''} \leftarrow \varphi_1^{out}\big(K_{2,j''}, z_2^{'(2)}, u_{2,j''}\big)$ for all $j'' \in \mathcal{J}_2$
22: **Outputs** 1 **iff.** $\varphi_3^{in}\big(K_{2,j''}, w_{2,j''}\big) = \gamma_3^{in}\big(K_{2,j''}, y_{2,j''}\big)$ for all $j'' \in \mathcal{J}_2$
23: **end if**
24: **end if**
25: **end if**

Analysis of Subcase 3.5. Recall from Fig. 17 that the concrete condition is $|\mathcal{U}_{\mathsf{ni,if}}(K)| \geq 2^n R/2$, where $R = \min_{K \in \mathcal{K}_1} \big\{\frac{|\mathcal{X}_{\mathsf{ni}}(K)|}{2^n}\big\}$. On the other hand, let $\varepsilon_{\mathsf{ni}} := \max_{K \in \mathcal{K}_1} \big\{|\mathcal{Y}_{\mathsf{ni}}(K)|/2^n\big\}$.

Subcase 3.5.1: SUBCASE35 *enters the branch at line 5.* In this subcase, SUBCASE35 invokes SCASE34CHECK, and Lemma 8 immediately yields the advantage lower

bound $1 - 2q^2/2^{m_{min}} - 2q\varepsilon_{\mathsf{ni}} - 2(4\varepsilon_{de(\varphi_3^{in})})^{m_{max}-1} = 1 - \text{negl}(n)$. We refer to Fig. 9 (right) for the involved query structures.
Subcase 3.5.2: SUBCASE35 enters the branch at line 6. We refer to Fig. 9 (left) for the involved query structure in this subcase.

EXISTENCE OF THE 2λ PAIRS. We construct a bipartite graph $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$ with $\mathcal{V}_L = \mathsf{Rng}_{\mathsf{ni}}(\gamma_1^{in})$ (see Eq. (14)), $\mathcal{V}_R = \cup_{K\in\mathcal{K}_2}\mathsf{Rng}_{\mathsf{if}}(\varphi_2^{in}, K)$ (see Eq. (14)) and $|\mathcal{E}| = |\mathcal{KU}_{\mathsf{ni,if}}|$. $\mathcal{G}$ contains an edge $\big((i_1,\delta_1,z_1),(i_2,\delta_2,z_2)\big) \in \mathcal{E}$ (duplication excluded), if and only if there exists $(K,u) \in \mathcal{KU}_{\mathsf{ni,if}}$ such that $\gamma_1^{in}(K,u) = (i_1,\delta_1,z_1) \wedge \varphi_2^{in}(K,u) = (i_2,\delta_2,z_2)$. Since $|\mathcal{KU}_{\mathsf{ni,if}}\big((i_1,\delta_1,z_1),(i_2,\delta_2,z_2)\big)| < m_{max}$ for all (i_1,δ_1,z_1) and (i_2,δ_2,z_2), we get a simple bipartite graph $\mathcal{G} = (\mathcal{V}_L, \mathcal{V}_R, \mathcal{E})$ with $|\mathcal{V}_L| \le |\mathcal{I}_{\le n}|2^{n+1}$ (Lemma 5), $|\mathcal{V}_R| \le \sum_{K\in\mathcal{K}_2}|\mathsf{Dom}_{\mathsf{if}}(\Pi_2,K)| \cdot \varepsilon_{de(\Pi_2)} \le |\mathcal{K}_2| \cdot 2^n \cdot \varepsilon_{de(\Pi_2)} \le \varepsilon_{de(\Pi_2)} 2^{\kappa+n-4}$ (Lemma 4) and $\mathcal{E} \ge |\mathcal{KU}_{\mathsf{ni,if}}|/m_{max} \ge \frac{2^{\kappa+n-5}R}{m_{max}}$.

Then, 2λ pairs $\big((K_{1,j},u_{1,j}),(K_{2,j},u_{2,j})\big)_{j=1,2,\dots,\lambda}$ satisfying Eq. (34) indicate a biclique $K_{\lambda,2}$ in $\mathcal{G}$. We apply Proposition 2: let $M = |\mathcal{I}_{\le n}|2^{n+1}$, $N = \varepsilon_{de(\Pi_2)}2^{\kappa+n-4}$, $a = \lambda$ and $b = 2$. Then, the bound from Proposition 2 has

$$\begin{aligned}(b-1)^{\frac{1}{a}} \cdot MN^{1-\frac{1}{a}} + (a-1)N \le |\mathcal{I}_{\le n}|2^{n+1} \times \big(\varepsilon_{de(\Pi_2)}2^{\kappa+n-4}\big)^{1-\frac{1}{\lambda}} \\ + (\lambda-1)\varepsilon_{de(\Pi_2)}2^{\kappa+n-4}.\end{aligned}$$

The above graph $\mathcal{G}$ satisfies the two conditions of Proposition 2. Since we have $|\mathcal{E}| \ge 2^{\kappa+n-5}R/m_{max}$, $\mathcal{G}$ contains $K_{\lambda,2}$ as long as[10]

$$|\mathcal{E}| \ge \frac{2^{\kappa+n-5}R}{m_{max}} \ge |\mathcal{I}_{\le n}|2^{n+1} \times \big(\varepsilon_{de(\Pi_2)}2^{\kappa+n-4}\big)^{1-\frac{1}{\lambda}} + (\lambda-1)\varepsilon_{de(\Pi_2)}2^{\kappa+n-4}.$$

Since $R = \Omega(1/\text{poly}(n))$, $\lambda = 2m_{max} = O(\text{poly}(n))$ and $\varepsilon_{de(\Pi_2)} = \text{negl}(n)$, when n is large enough it holds $\lambda\varepsilon_{de(\Pi_2)} \ll R/4m_{max}$, and

$$2^{\kappa+n-5}R/m_{max} - (\lambda-1)\varepsilon_{de(\Pi_2)}2^{\kappa+n-4} \ge 2^{\kappa+n-6}R/m_{max}.$$

Moreover, when n is large enough it holds $R/m_{max} \ge (\varepsilon_{de(\Pi_2)})^{1-\frac{1}{\lambda}}$. Thus, $\mathcal{G}$ contains $K_{\lambda,2}$ as long as

$$\begin{aligned}
2^{\kappa+n-6}R/m_{max} &\ge |\mathcal{I}_{\le n}|2^{n+1} \times \big(\varepsilon_{de(\Pi_2)}2^{\kappa+n-4}\big)^{1-\frac{1}{\lambda}}, \\
2^{\kappa+n-6} &\ge |\mathcal{I}_{\le n}|2^{n+1} \times \big(2^{\kappa+n-4}\big)^{1-\frac{1}{\lambda}}, \\
\kappa+n-6 &\ge \log_2|\mathcal{I}_{\le n}| + n + 1 + \big(\kappa+n-4\big) - \frac{1}{\lambda}\big(\kappa+n-4\big), \\
\frac{1}{2m_{max}}\big(\kappa+n-4\big) &\ge \log_2|\mathcal{I}_{\le n}| + n + 3, \\
\kappa &\ge 2m_{max}\log_2|\mathcal{I}_{\le n}| + (2m_{max}-1)n + 6m_{max} + 4.
\end{aligned}$$

[10] We can also consider a symmetrical setting, i.e., for $\mathcal{G}^* = (\mathcal{V}_L^*, \mathcal{V}_R^*, \mathcal{E}^*)$, $|\mathcal{V}_L^*| \le M = \varepsilon_{de(\Pi_2)}2^{\kappa+n-4}$, $|\mathcal{V}_R^*| \le N = |\mathcal{I}_{\le n}|2^{n+1}$, $a = 2$ and $b = \lambda$. The upper bound on $|\mathcal{E}^*|$ such that $\mathcal{G}^*$ does not contain $K_{2,\lambda}$. Then Proposition 2 indicates $|\mathcal{E}^*| \le (\lambda-1)^{1/2} \times \varepsilon_{de(\Pi_2)}2^{\kappa+n-4} \times |\mathcal{I}_{\le n}|^{1/2}2^{(n+1)/2} + |\mathcal{I}_{\le n}|2^{n+1}$. This can be fulfilled only if $\varepsilon_{de(\Pi_2)}$ is exponentially small rather than merely negligible.

This is fulfilled under our assumption $\kappa \geq 2m_{max}\log_2|\mathcal{I}_{\leq n}|+2m_{max}n+6m_{max}+4$. Therefore, with our parameters, Subcase35($\mathcal{K}_2$) can find the $\lambda+2$ vertexes $\big((i_1^{(j)},\delta_1^{(j)},z_1^{(j)})\big)_{j=1,2,...,m_{max}}$, $(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$ and $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})$ and the 2λ pairs (edges) $\big((K_{1,j},u_{1,j}),(K_{2,j},u_{2,j})\big)_{j=1,2,...,\lambda}$ at line 8 that satisfy Eq. (34).

Attack advantage. Subcase35($\mathcal{K}_2$) outputs 1 at either line 15 or lines 19 or 22 according to the equalities.

In the real world, the equalities checked at lines 19 or 22 always hold, and thus Subcase35($\mathcal{K}_2$) always outputs 1.

Analysis of the ideal world probability partly follows Subcase 3.1. Concretely, we define two events Bad^- and Bad^+ in the interaction between $D3$, S and $\mathbf{IC}$:

(i) Bad^- happens, if right after a backward $\mathbf{IC}$-query $\mathbf{IC}^{-1}(K,y)\to x$, there exists an earlier-obtained $\mathbf{IC}$ input/output $\mathbf{IC}(K',x')=y'$ such that $x\in\mathcal{X}_{\mathsf{ni}}(K)\wedge x'\in\mathcal{X}_{\mathsf{ni}}(K')\wedge\varphi_1^{in}(K,x)=\varphi_1^{in}(K',x')$. Using Corollary 1, it can be seen $\Pr[\mathsf{Bad}^-]\leq 2q^2/2^{m_{min}}$.

(ii) Bad^+ occurs, if there appears a forward $\mathbf{IC}$-query $\mathbf{IC}(K,x)\to y$ that has $y\in\mathcal{Y}_{\mathsf{ni}}(K)$. Since $|\mathcal{Y}_{\mathsf{ni}}(K)|<2^n\varepsilon_{\mathsf{ni}}$ for all $K\in\mathcal{K}_2$, it holds $\Pr[\mathsf{Bad}^+]\leq 2q\varepsilon_{\mathsf{ni}}$.

By construction, after executing line 13, there have been two groups of λ $\mathbf{IC}$ inputs/outputs $\mathsf{G}_1=\big(\mathbf{IC}(K_{1,1},x_{1,1})=y_{1,1},...,\mathbf{IC}(K_{1,\lambda},x_{1,\lambda})=y_{1,\lambda}\big)$ and $\mathsf{G}_2=\big(\mathbf{IC}(K_{2,1},x_{2,1})=y_{2,1},...,\mathbf{IC}(K_{2,\lambda},x_{2,\lambda})=y_{2,\lambda}\big)$ in the ideal world interaction, such that:

(i) $x_{b,j}\in\mathcal{X}_{\mathsf{ni}}(K_{b,j})$ for all $(b,j)\in\{1,2\}\times\{1,...,\lambda\}$; and

(ii) $\varphi_1^{in}\big(K_{1,j},x_{1,j}\big)=\varphi_1^{in}\big(K_{2,j},x_{2,j}\big)$ for all $j=1,...,\lambda$.

Conditioned on $\neg\mathsf{Bad}^-$, in each pair of $\mathbf{IC}$ inputs/outputs $\big(\mathbf{IC}(K_{1,j},x_{1,j})=y_{1,j},\mathbf{IC}(K_{2,j},x_{2,j})=y_{2,j}\big)$ such that $x_{1,j}\in\mathcal{X}_{\mathsf{ni}}(K_{1,j})\wedge x_{2,j}\in\mathcal{X}_{\mathsf{ni}}(K_{2,j})\wedge\varphi_1^{in}\big(K_{1,j},x_{1,j}\big)=\varphi_1^{in}\big(K_{2,j},x_{2,j}\big)$, either $\mathbf{IC}(K_{1,j},x_{1,j})=y_{1,j}$ or $\mathbf{IC}(K_{2,j},x_{2,j})=y_{2,j}$ must be obtained via a forward $\mathbf{IC}$-query. By this, either G_1 contains at least $\lambda/2$ forward $\mathbf{IC}$-queries, or G_2 contains at least $\lambda/2$ forward $\mathbf{IC}$-queries. Further conditioned on $\neg\mathsf{Bad}^+$, every forward $\mathbf{IC}$-query $\mathbf{IC}(K,x)\to y$ has $y\in\mathcal{Y}_{\mathsf{if}}(K)$. By these, Subcase35($\mathcal{K}_2$) always enter the branch of line 16 (thus, it won't output 1 at line 15).

Assume that S completely knows the involved values $\big((i_1^{(j)},\delta_1^{(j)},z_1^{(j)})\big)_{j=1,2,...,\lambda}$, $(i_2^{(1)},\delta_2^{(1)},z_2^{(1)})$ and $(i_2^{(2)},\delta_2^{(2)},z_2^{(2)})$ and 2λ edges $\big((K_{1,j},u_{1,j}),(K_{2,j},u_{2,j})\big)_{j=1,2,...,\lambda}$ found by Subcase35 at line 8. Conditioned on $\neg\mathsf{Bad}^-$ and on that Subcase35 has entered the branch of line 16, Subcase35 outputs 1 only if S succeeds in pinpointing $b\in\{1,2\}$ and $\mathcal{J}^*=\{j_1^*,j_2^*,...,j_s^*\}\subseteq\{1,...,\lambda\}$, $s\geq\lambda/2$, and making s forward $\mathbf{IC}$-queries $\mathbf{IC}(K_{b,j_1^*},x_1^*)\to y_1^*,...,\mathbf{IC}(K^*_{b,j_s^*},x_s^*)\to y_s^*$ that have:

- There exists $z_2'\in\{0,1\}^{m(i_2^{(b)})}$ such that $\varphi_3^{in}\big(K_{b,j_\ell^*},\varphi_2^{out}\big(K_{b,j_\ell^*},z_2',u_{b,j_\ell^*}\big)\big)=\gamma_3^{in}\big(K_{b,j_\ell^*},y_\ell^*\big)$ for $\ell=1,...,s$.

(Once this is done, S may use them to fool SUBCASE35($\mathcal{K}_2$) outputting 1 at line 19 or 22).

Let BadGroup be the event that S succeeds in making a tuple of $s = \lambda/2$ forward **IC**-queries as above. The remaining argument follows Subcase 3.1. Concretely, for each s-tuple of **IC**-queries $\mathbf{IC}(K_{b,j_1^*}, x_1^*) \to y_1^*, ..., \mathbf{IC}(K^*_{b,j_s^*}, x_s^*) \to y_s^*$, the probability that $\exists z_2' \in \{0,1\}^{m(i_2^{(b)})}$ with $\varphi_3^{in}\big(K_{b,j_\ell^*}, \varphi_2^{out}\big(K_{b,j_\ell^*}, z_2', u_{b,j_\ell^*}\big)\big) = \gamma_3^{in}\big(K_{b,j_\ell^*}, y_\ell^*\big)$ for $\ell = 1, ..., s$ is at most $2^{m(i_2^{(b)})} \times (2\varepsilon_{de(\varphi_3^{in})})^s \le 2^{m_{max}} \times (2\varepsilon_{de(\varphi_3^{in})})^s$. Summing over the at most q^s choices of possible s-tuples $\mathbf{IC}(K_{b,j_1^*}, x_1^*) \to y_1^*, ..., \mathbf{IC}(K^*_{b,j_s^*}, x_s^*) \to y_s^*$ and using $s = \lambda/2 = m_{max}$, we reach

$$\begin{aligned}&\Pr\big[\mathsf{BadGroup} \mid \neg\mathsf{Bad}^- \wedge \neg\mathsf{Bad}^+\big] \\ \le\ & q^s \times 2^{m_{max}} \times (2\varepsilon_{de(\varphi_3^{in})})^s \le (2q)^{m_{max}} \times (2\varepsilon_{de(\varphi_3^{in})})^{m_{max}} \le (4q\varepsilon_{de(\varphi_3^{in})})^{m_{max}}.\end{aligned}$$

Therefore, the total probability that SUBCASE35($\mathcal{K}_2$) outputs 1 in the ideal world cannot exceed $\Pr[\mathsf{Bad}^-] + \Pr[\mathsf{Bad}^+] + \Pr[\mathsf{BadGroup} \mid \neg\mathsf{Bad}^- \wedge \neg\mathsf{Bad}^+] \le 2q^2/2^{m_{min}} + 2q\varepsilon_{\mathsf{ni}} + (4q\varepsilon_{de(\varphi_3^{in})})^{m_{max}}$, and the advantage is at least $1 - 2q^2/2^{m_{min}} - 2q\varepsilon_{\mathsf{ni}} - (4q\varepsilon_{de(\varphi_3^{in})})^{m_{max}} = 1 - \mathrm{negl}(n)$ in Subcase 3.5.2. In all, advantage is always at least $1 - \mathrm{negl}(n)$ in Subcase 3.5.

Further intuitions for Subcase 3.5. For $E3$, differentiator actions and analysis in Subcase 3.5 constituted the most complicated part. Meanwhile, it requires a large key size $\kappa \ge \lambda \log_2 |\mathcal{I}_{\le n}| + (\lambda - 1)n - \lambda \log_2 m_{max} - \log_2 \varepsilon_{de(\Pi_2)} + 4$. To justify the complexity, we provide some earlier failed attempts as follows.

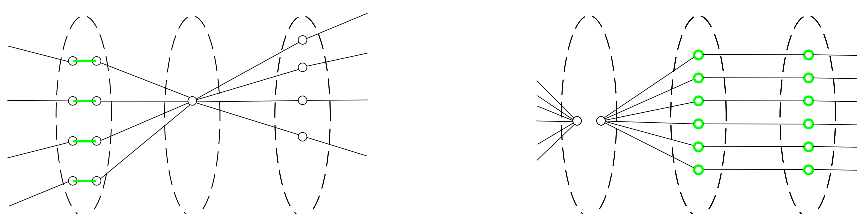

Fig. 18. Query structures used in two failed attempts to attacking 3-round general ciphers, Case 5.

First, a natural attempt is to consider a "star" structure "centered" at the 2nd round, as depicted in Fig. 18 (left). More clearly, the t encipherments collide on a 2nd round $\mathcal{P}$-call $\mathcal{P}(i_2, \delta_2, z_2)$. This structure resembles that used in Subcase 3.1 (see Fig. 6 (left)), and actually its size t can be exponential. But unlike Fig. 6 (left), the exponential encipherments may be non-inv-free in the 1st round. This means they may all call $\mathcal{P}(i_1, \cdot, \cdot)$ with $m(i_1) = n$ (i.e., an n-bit **P**) in the 1st round. It is easy to see the simulator S can program $\mathcal{P}(i_1, \cdot, \cdot)$ (a standard method in indifferentiability simulator for IEM ciphers [1]) to make the t simulated encipherments consistent with the ideal cipher inputs/outputs, as indicated by the green bold lines in Fig. 18 (left).

A second attempt is to consider a "star" structure "centered" at the 1st round, as depicted in Fig. 18 (right). More clearly, the t encipherments collide on a 1st round $\mathcal{P}$-call $\mathcal{P}(i_1, \delta_1, z_1)$. This structure resembles that used in Subcase 3.3 (see Fig. 8 (left)), and t can also be exponential. S cannot program $\mathcal{P}(i_1, \delta_1, z_1)$ to ensure consistency for all the encipherments any more. However, the t encipherments likely invoke distinct $\mathcal{P}$-calls in 2nd and 3rd rounds, as shown in Fig. 18 (right)—at least, with polynomial queries, differentiators cannot find more collisions in 2nd and 3rd rounds. It can be see S can program their 2nd and 3rd round $\mathcal{P}$-call responses (a standard method in indifferentiability simulator for Feistel [16]) to make the t simulated encipherments consistent with the ideal cipher inputs/outputs, as indicated by the green bold circles in Fig. 18 (right).

In comparison, in the structure used in our attack (see Fig. 9), the involved encipherments have collisions in both the 1st and the 2nd rounds. The simulation strategy against our failed attempts thus do not apply. As a final remark, since the right shore of the constructed graph $\mathcal{G}$ is rather large ($|\mathcal{V}_R| \approx \varepsilon_{de(\Pi_2)} 2^{\kappa+n-4}$), it is easy to see attacks cannot use exponential-size bicliques. If $\varepsilon_{de(\Pi_2)}$ is exponentially small (which indeed holds in common blockciphers) then the attack may admit improvements.

9.7 Other subcases: $|\mathcal{X}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_1$

In this subcase, it holds $|\mathcal{W}_{\mathsf{ni,ni}}(K)| + |\mathcal{W}_{\mathsf{if,ni}}(K)| = |\mathcal{Y}_{\mathsf{ni}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_1$. The pigeonhole principle indicates that there exists a subset $\mathcal{K}_2 \subseteq \mathcal{K}_1$ such that $|\mathcal{K}_2| \geq |\mathcal{K}_1|/2$, and either of the following holds for all $K \in \mathcal{K}_2$:

- $|\mathcal{W}_{\mathsf{ni,ni}}(K)| \geq 2^n R/2$; or
- $|\mathcal{W}_{\mathsf{if,ni}}(K)| \geq 2^n R/2$,

where $R = \Omega(1/\mathrm{poly}(n))$ is the ratio computed at line 22 (Fig. 17).

Therefore, our design of the subroutine HANDLE3ITER in this branch (from line 21 in Fig. 17) is *sound*. Then,

- If $|\mathcal{W}_{\mathsf{ni,ni}}(K)| \geq 2^n R/2$ for all $K \in \mathcal{K}_2$, then since we also have $|\mathcal{X}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ for all $K \in \mathcal{K}_2$, the case is similar to Subcase 3.4 ($|\mathcal{U}_{\mathsf{ni,if}}(K)| = \Omega(2^n/\mathrm{poly}(n))$ and $|\mathcal{Y}_{\mathsf{ni}}(K)| = o(2^n/\mathrm{poly}(n))$ for all K in that $\mathcal{K}_2$). The analysis is also similar to Sect. 9.5 by symmetry, and we omit.
- If $|\mathcal{W}_{\mathsf{if,ni}}(K)| \geq 2^n R/2$ for all $K \in \mathcal{K}_2$, then the case is similar to Subcase 3.5. The analysis is similar to Sect. 9.6, and we omit.

9.8 Summary

In summary:

- In Case 1 and 2 (Sect. 8), attack advantage is at least $1 - \mathrm{negl}(n)$;
- In Subcases 3.1, 3.4 and 3.5, there exists a negligible function $\mathrm{negl}(n)$ such that the attack advantage is at least $1 - \mathrm{negl}(n)$;

– In Subcases 3.2 and 3.3, there exists a polynomial $\mathrm{poly}(n)$ and a negligible function $\mathrm{negl}(n)$ such that attack advantage is at least $1/\mathrm{poly}(n) - \mathrm{negl}(n)$.

These yield the claim in Theorem 3.

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A Non-Iterated Blockciphers

As mentioned in the Introduction, blockciphers are indeed *not* necessarily iterated, and we serve two examples in Fig. 19.

Intuitively, with the same number of primitive calls, iterated blockciphers are *more secure* than non-iterated. E.g., note that in both of the instances in Fig. 19, the two calls crowd into a single round encipherment, and their weakness are somewhat obvious. Furthermore, advantage of iterated ciphers are also supported by various security amplification results initiated in [40].

Though, it is challenging to formally inject the above intuitions into our argument to address fully general 2- and 3-call ciphers. In general ciphers it is non-trivial to determine "valid" intermediate values. In addition, the proof has to address the interference between various types of data-dependent encipherment.

With the above considerations, we leave further explorations to future work.

B Attack 1-Call Ciphers without Non-degeneracy

We first introduce a helper combinatorial lemma.

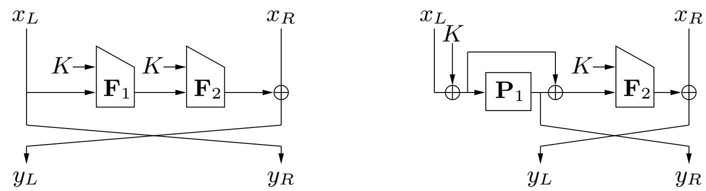

Fig. 19. Examples of non-iterated 2-call blockciphers. (left) an instance that makes the two calls in a single Feistel round, and its decipherment *has to* begin by querying $\mathbf{F}_1$; (right) decipherment *has to* begin by querying $\mathbf{P}_1^{-1}$.

Lemma 9. *Consider a function $f : \mathcal{X} \to \mathcal{Y}$. If f is not a constant function, then there exists $x^\circ \in \mathcal{X}$ such that*

$$\Pr\big[x \xleftarrow{\$} \mathcal{X} : f(x) = f(x^\circ)\big] \leq \frac{1}{2}. \tag{35}$$

Proof. Let $y^* \in \mathcal{Y}$, and let $\mathcal{X}(y^*) \subseteq \mathcal{X}$ be such that $f(x) = y^*$ if and only if $x \in \mathcal{X}(y^*)$. Then,

– If $|\mathcal{X}(y^*)| \leq \frac{|\mathcal{X}|}{2}$, then for any $x^\circ \in \mathcal{X}(y^*)$ it holds $\Pr\big[x \xleftarrow{\$} \mathcal{X} : f(x) = f(x^\circ)\big] = \frac{|\mathcal{X}(y^*)|}{|\mathcal{X}|} \leq \frac{1}{2}$;

– If $|\mathcal{X}(y^*)| > \frac{|\mathcal{X}|}{2}$, then for any $x^\circ \in \mathcal{X} \backslash \mathcal{X}(y^*)$ it holds $\Pr\big[x \xleftarrow{\$} \mathcal{X} : f(x) = f(x^\circ)\big] \leq \frac{|\mathcal{X}| - |\mathcal{X}(y^*)|}{|\mathcal{X}|} \leq \frac{1}{2}$.

Thus the claim. □

In Sect. 6, the Case 2 argument crucially relies on the $\varepsilon_{de(E1)}$-non-degeneracy of $E1$ (so that $\big|\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)\big| > 1/\varepsilon_{de(E1)} = \Omega(\mathrm{poly}(n))$ for any such (i, δ, z)). When $E1$ is degenerate, we have to modify the case-study. The modified differentiator $\overline{D1}^{\mathrm{E,P}}$ is given in Fig. 20.

Below we provide intuitions and analyses. The actions of $\overline{D1}$ consist of three cases.

Case 1: $\exists K \in \{0,1\}^\kappa$ such that $\mathsf{Dom}_{\mathsf{if}}(E1, K) \neq \emptyset$, $\mathsf{Dom}_{\mathsf{ni}}(E1, K) \neq \emptyset$. Then, $E1^{\mathcal{P}}$ necessarily maps $x \in \mathsf{Dom}_{\mathsf{if}}(E1, K)$ ($x \in \mathsf{Dom}_{\mathsf{ni}}(E1, K)$, resp.) to $y \in \mathsf{Rng}_{\mathsf{if}}(E1, K)$ ($y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)$, resp.), and this constitutes our attack idea.

Wlog assume $|\mathsf{Dom}_{\mathsf{if}}(E1, K)| \geq |\mathsf{Dom}_{\mathsf{ni}}(E1, K)|$. Recall from Fig. 20 that $\overline{D1}^{\mathrm{E,P}}$ samples $x \xleftarrow{\$} \mathsf{Dom}_{\mathsf{ni}}(E1, K)$ and queries $\mathrm{E}(K, x) \to y$. When $\overline{D1}^{\mathrm{E,P}}$ is interacting with the real world $(E1^{\mathcal{P}}, \mathcal{P})$, it always holds $y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)$. In the ideal world we have $\Pr\big[y \xleftarrow{\$} \{0,1\}^n : y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)\big] = \frac{|\mathsf{Rng}_{\mathsf{ni}}(E1,K)|}{2^n} \leq 1/2$. Therefore, the attack advantage is at least $1/2$ in Case 1.

Case 2: $\exists K \in \{0,1\}^\kappa$ such that $\mathsf{Dom}_{\mathsf{if}}(E1, K) = \{0,1\}^n$. This means the whole permutation $E1^{\mathcal{P}}(K, \cdot)$ is inverse-free. It further distinguishes two subcases as follows.

Algorithm $\overline{D1}^{\mathrm{E,P}}(\mathcal{K})$
Determines the sets $\mathsf{Dom}_{\mathsf{if}}(E1, K)$ and $\mathsf{Dom}_{\mathsf{ni}}(E1, K)$ for all $K \in \mathcal{K}$
if $\exists K \in \mathcal{K}$ **s.t.** $\mathsf{Dom}_{\mathsf{if}}(E1, K) \neq \emptyset$ and $\mathsf{Dom}_{\mathsf{ni}}(E1, K) \neq \emptyset$ **then** // Case 1
  if $|\mathsf{Dom}_{\mathsf{if}}(E1, K)| \geq |\mathsf{Dom}_{\mathsf{ni}}(E1, K)|$ **then**
    $x \xleftarrow{\$} \mathsf{Dom}_{\mathsf{ni}}(E1, K)$, $\mathrm{E}(K, x) \to y$
    Outputs 1 **iff.** $y \in \mathsf{Rng}_{\mathsf{ni}}(E1, K)$
  else // $|\mathsf{Dom}_{\mathsf{ni}}(E1, K)| \geq |\mathsf{Dom}_{\mathsf{if}}(E1, K)|$
    $x \xleftarrow{\$} \mathsf{Dom}_{\mathsf{if}}(E1, K)$, $\mathrm{E}(K, x) \to y$
    Outputs 1 **iff.** $y \in \mathsf{Rng}_{\mathsf{if}}(E1, K)$
  end if
else if $\exists K \in \{0,1\}^{\kappa}$ **s.t.** $\mathsf{Dom}_{\mathsf{if}}(E1, K) = \{0,1\}^n$ **then** // Case 2
  if $\varphi^{in}(K, \cdot)$ is not a constant function **then** // Subcase 2.1
    Picks $x^{\circ} \in \{0,1\}^n$ **s.t.** $\Pr\big[x \xleftarrow{\$} \{0,1\}^n : \varphi^{in}(K, x) = \varphi^{in}(K, x^{\circ})\big] \leq 1/2$
    $\mathrm{E}(K, x^{\circ}) \to y^{\circ}$
    Outputs 1 **iff.** $\gamma^{in}(K, y^{\circ}) = \varphi^{in}(K, x^{\circ})$
  else // Subcase 2.2: $\varphi^{in}(K, x) = \gamma^{in}(K, y)$ is a constant for all $x, y \in \{0,1\}^n$
    // The same as Fig. 11
  end if
else // Case 3: $\mathsf{Dom}_{\mathsf{ni}}(E1, K) = \{0,1\}^n$ for all $K \in \mathcal{K}$
  // The same as Fig. 11
end if

Fig. 20. Differentiator $\overline{D1}^{\mathrm{E,P}}$ against degenerate $E1$.

Subcase 2.1: $\varphi^{in}(K, \cdot)$ is not a constant function. Then, line 13 is well-defined: by Lemma 9, there exists $x^{\circ} \in \{0,1\}^n$ such that $\Pr_x\big[\varphi^{in}(K, x) = \varphi^{in}(K, x^{\circ})\big] = |\mathsf{Rng}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})|/2^n \leq 1/2$. Let $(i^{\circ}, \delta^{\circ}, z^{\circ}) = \varphi^{in}(K, x^{\circ})$. By Lemma 2, we have $|\mathsf{Rng}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})| = |\mathsf{Dom}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})|$ (recall from Eq. (13) for the notations $\mathsf{Dom}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})$ and $\mathsf{Rng}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})$), which implies

$$\Pr\big[y \xleftarrow{\$} \{0,1\}^n : \gamma^{in}(K, y) = \gamma^{in}(K, y^{\circ})\big] = \frac{|\mathsf{Rng}_{\mathsf{if}}(E1, K, i^{\circ}, \delta^{\circ}, z^{\circ})|}{2^n} \leq \frac{1}{2}. \quad (36)$$

With these observations, we analyze the attack advantage. Recall from Fig. 20 that $\overline{D1}^{\mathrm{E,P}}$ pinpoints the special x° and queries $\mathrm{E}(K, x^{\circ}) \to y^{\circ}$. When $\overline{D1}$ is interacting with the real world, it always holds $\gamma^{in}(K, y^{\circ}) = \varphi^{in}(K, x^{\circ})$ by Lemma 2. On the other hand, $y^{\circ} = \mathbf{IC}(K, x^{\circ})$ is uniform in $\{0,1\}^n$ in the ideal world, and $\Pr[\gamma^{in}(K, y^{\circ}) = \varphi^{in}(K, x^{\circ})] \leq 1/2$ by Eq. (36). These yield attack advantage at least 1/2.

Subcase 2.2: $\varphi^{in}(K, x) = \gamma^{in}(K, y) = (i, \delta, z)$ is a constant. In this subcase, for this key K the encipherment becomes $E1^{\mathcal{P}}(K, x) = \varphi^{out}\big(K, \mathcal{P}(i, \delta, z), x\big)$ using $\mathcal{P}(i, \delta, z)$ as a subkey.

Then, $\varphi^{out}\big(K, \mathcal{P}(i, \delta, z), \cdot\big)$ defines a bijection between $\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z)$ and $\mathsf{Rng}_{\mathsf{if}}(E1, K, i, \delta, z)$. Since $\mathsf{Dom}_{\mathsf{if}}(E1, K, i, \delta, z) = \mathsf{Rng}_{\mathsf{if}}(E1, K, i, \delta, z) = \{0,1\}^n$ and since $|\mathcal{P}(i, \delta, z)| = \mathrm{poly}(n)$, $\varphi^{out}\big(K, \mathcal{P}(i, \delta, z), \cdot\big)$ is a permutation on $\{0,1\}^n$ defined using a random string $\mathcal{P}(i, \delta, z)$ of polynomial size. The attack and analysis then follow [41, Sect. 6] or Sect. 6, Case 1.

Case 3: $\mathsf{Dom}_{\mathsf{ni}}(E1, K, i, \delta, z) = \{0,1\}^n$ for all $K \in \mathcal{K}$. This part is the same as Case 2 in Sect. 6, and advantage is at least $1 - 2/2^{m_{min}}$.

Summary. By the above, when n is sufficiently large, it holds

$$\mathrm{Adv}^{indif}_{E1^{\mathcal{P}},\mathbf{IC},S}(\overline{D1}) \geq \min\left\{\frac{1}{2}, 1-\frac{1}{2^n}-\frac{{m_{max}}^2}{2^n}, 1-\frac{2}{2^{m_{min}}}\right\} \geq \frac{1}{2}.$$