W. de Sitter, On the curvature of space, in:
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Astronomy. — *"On the curvature of space"*. By Prof. W. DE SITTER.

(Communicated in the meeting of 1917, June 30).

1. In order to make possible an entirely relative conception of inertia, EINSTEIN [1]) has replaced the original field equations of his theory by the equations

$$G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \lambda = - \varkappa T_{\mu\nu} + \tfrac{1}{2} \varkappa g_{\mu\nu} T \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In my last paper [2]) I have pointed out two different systems of $g_{\mu\nu}$ which satisfy these equations. The system A is EINSTEIN's, in which the whole of space is filled with matter of the average density ϱ_0. In a stationary state, and if all matter is at rest without any stresses or pressure, then we have $T_{\mu\nu} = 0$ with the exception of $T_{44} = g_{44} \varrho_0$. In the system B this "world-matter" does not exist: we have $\varrho_0 = 0$ and consequently all $T_{\mu\nu} = 0$. The line element in the two systems was there found to be

$$ds^2 = - R^2 \left\{ d\chi^2 + \sin^2 \chi \left[d\psi^2 + \sin^2 \psi \, d\vartheta^2 \right] \right\} + c^2 dt^2, \quad . \quad . \quad (2A)$$

$$ds^2 = - R^2 \left\{ d\omega^2 + \sin^2 \omega \left[d\chi^2 + \sin^2 \chi \left(d\psi^2 + \sin^2 \psi \, d\vartheta^2 \right) \right] \right\} . \quad (2B)$$

In the system A we have

$$\lambda = \frac{1}{R^2} \quad , \quad \varkappa \varrho_0 = 2\lambda , \quad . \quad . \quad . \quad . \quad (3A)$$

and in B:

$$\lambda = \frac{3}{R^2} \quad , \quad \varrho_0 = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3B)$$

In the system A χ, ψ, ϑ are real angles; in B ψ and ϑ are also real, but ω and χ are imaginary. If, however, we put

$$\sin \omega \sin \chi = \sin \zeta \quad , \qquad r = R\zeta,$$

$$\tan \omega \cos \chi = \tan i\eta \, , \qquad t = R\eta,$$

[1]) A. EINSTEIN, *Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie,* Sitzungsber., Berlin 1917 Febr. 8, p. 142.

[2]) W. DE SITTER, *On the relativity of inertia,* these Proceedings, 1917 March 31, vol. XIX, p. 1217.

In the footnote to page 1220 of that paper it is stated that the four-dimensional world of the system B can be represented as a hyperboloid of two sheets in a space of five dimensions, which is projected on a euclidean space of four dimensions by a "stereographic projection". This is erroneous. The hyperboloid has only *one* sheet. Its projection fills only part of the euclidean space of four dimensions; the part outside the limiting hyperboloid $1 + ah^2 = 0$ (which is called (a) in the quoted footnote) is the projection of the conjugated hyperboloid (which is of two sheets).

where $i = \sqrt{-1}$, then ζ and η are real and $(2B)$ becomes:

$$ds^2 = -dr^2 - R^2 \sin^2 \frac{r}{R} [d\psi^2 + \sin^2 \psi \, d\vartheta^2] + \cos^2 \frac{r}{R} c^2 dt^2. \quad . \quad (4B)$$

If in A we also take $r = R\chi$, then $(2A)$ becomes

$$ds = -dr^2 - R^2 \sin^2 \frac{r}{R} [d\psi^2 + \sin^2 \psi \, d\vartheta^2] + c^2 dt^2 \, . \quad . \quad . \quad . \quad (4A)$$

The two systems A and B now differ only in g_{44}. For the sake of comparison we add the system C, with

$$\lambda = 0 \quad , \quad \rho_0 = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3C)$$

in which the line-element is

$$ds^2 = -dr^2 - r^2 [d\psi^2 + \sin^2 \psi \, d\vartheta^2] + c^2 dt^2 \, . \quad . \quad . \quad (4C)$$

Both A and B become identical with C for $R = \infty$.

If in A the origin of coordinates is displaced to a point $\chi_1, \psi_1, \vartheta_1$, and in B to a time-space point $\omega_1, \chi_1, \psi_1, \vartheta_1$, then the line-element conserves the forms $(2A)$ and $(2B)$ respectively. These can then again by the same transformations be altered to $(4A)$ and $(4B)$. In A the variable t, which takes no part in the transformation, remains of course the same. In B on the other hand the new variable t after the transformation is generally not the same as before

I will put, for both systems A and B

$$\chi = \frac{r}{R}$$

In the system B this χ is not the same as in $(2B)$, but it is the angle which was called ζ above. I will continue to use r as an independent variable, and not χ.

2. In the theory of general relativity there is no essential difference between inertia and gravitation. It will, however, be convenient to continue to make this difference. A field in which the line-element can be brought in one of the forms $(4A)$, $(4B)$ or $(4C)$ with the corresponding condition $(3A)$, $(3B)$, or $(3C)$, will be called a field of pure inertia, without gravitation. If the $g_{\mu\nu}$ deviate from these values we will say that there is gravitation. This is produced by matter, which I call "ordinary" or "gravitating" matter. Its density is ρ_1. In the systems B and C there is no other matter than this ordinary matter. In the system A the whole of space is filled with matter, which, in the simple case that the line-element is represented by $(2A)$ or $(4A)$ produces no "gravitation", but only "inertia". This matter I have called "world-matter". Its density is ρ_0. When taken over sufficiently large units of volume this ϱ_0 is a constant. Locally however it may be variable the world-matter can be condensed to bodies of greater density, or it can have a smaller density than the average, or be absent altogether. According to EINSTEIN's view we must assume that *all* ordinary matter (sun, stars, nebulae etc.) consist of condensed world-matter, and perhaps also that all world-matter is thus condensed.

3. To begin with we will neglect gravitation and consider only the inertial field. The three-dimensional line-element is in the two systems A and B.

$$d\sigma^2 = dr^2 + R^2 \sin^2 \frac{r}{R} \left[d\psi^2 + \sin^2 \psi \, d\vartheta^2\right].$$

If R^2 is positive and finite, this is the line-element of a three-dimensional space with a constant positive curvature. There are two forms of this, viz: the space of RIEMANN [1]), or *spherical space*, and the *elliptical space*, which has been investigated by NEWCOMB [2]). In the spherical space all "straight" (i.e. geodetic) lines which start from one point, intersect again in another point: the "antipodal point", whose distance from the first point, measured along any of these lines, is πR. In the elliptical space any two straight lines have only one point in common. In both spaces the straight line is closed; in the spherical space its total length is $2\pi R$, in the elliptical space it is πR. In the spherical space the largest possible distance between two points is πR, in the elliptical space $\frac{1}{2}\pi R$. Both spaces are finite, though unlimited The volume of the whole of spherical space is $2\pi^2 R^3$, of elliptical space $\pi^2 R^3$. For values of r which are small compared with R, the two spaces differ only inappreciably form the euclidean space.

The existence of the antipodal point, where all rays of light starting from a point again intersect, and where also, as will be shown below, the gravitational action of a material point (however small its mass may be) becomes infinite, certainly is a drawback of the spherical space, and it will be preferable to assume the true physical space to be elliptical.

The elliptical space can be projected on euclidean space by the transformation

$$\mathrm{r} = R \tan \chi \quad . \; . \; . \; . \; . \; . \; . \quad (5).$$

The line-element in the systems A and B then becomes

1) *Ueber die Hypothesen welche der Geometrie zu Grunde liegen* (1854).

2) *Elementary theorems relating to geometry of three dimensions and of uniform positive curvature*, CRELLE's Journal Bd. 83, p. 293 (1877).

$$ds^2 = -\frac{dr^2}{\left(1+\frac{r^2}{R^2}\right)^2} - \frac{r^2\,[d\psi^2 + \sin^2\psi\, d\vartheta^2]}{1+\frac{r^2}{R^2}} + c^2\, dt^2 \quad . \quad . \quad (6A)$$

$$ds^2 = -\frac{dr^2}{\left(1+\frac{r^2}{R^2}\right)^2} - \frac{r^2\,[d\psi^2 + \sin^2\psi\, d\vartheta^2]}{1+\frac{r^2}{R^2}} + \frac{c^2\, dt^2}{1+\frac{r^2}{R^2}} \quad . \quad (6B)$$

For $r = \infty$ in the system A all $g_{\mu\nu}$ become zero, with the exception of g_{44}, which remains 1. In the system B g_{44} also becomes zero.

4. The world-lines of light-vibrations are geodetic lines ($ds = 0$) in the four-dimensional time-space. Their projections on the three-dimensional space are the rays of light. In the system A, with the coordinates r, ψ, ϑ, these light-rays are also geodetic lines of the three-dimensional space, and the velocity of light is constant. In the system B this is not so. The velocity of light in that system is, in the radial direction, $v = c \cos \chi$. It is possible, however, in B to introduce space-coordinates, measured in which the velocity of light shall be constant in the radial direction. If the radius-vector in this new measure is called h, we have

$$\cos \chi\, dh = dr$$

The integral of this equation is

$$\sinh \frac{h}{R} = \tan \frac{r}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In the system A we can, of course, also perform the same transformation. The line-element becomes

$$ds^2 = \frac{-dh^2 - \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2\psi\, d\vartheta^2]}{\cosh^2 \frac{h}{R}} + c^2\, dt^2 \quad . \quad . \quad (8A)$$

$$ds^2 = \frac{-dh^2 - \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2\psi\, d\vartheta^2] + c^2\, dt^2}{\cosh^2 \frac{h}{R}} \quad . \quad . \quad . \quad (8B)$$

The three-dimensional line-element

$$d\sigma'^2 = dh^2 + \sinh^2 \frac{h}{R}\,[d\psi^2 + \sin^2\psi\, d\vartheta^2],$$

is that of a space of constant negative curvature: the *hyperbolical space* or space of LOBATSCHEWSKY. When described in the coordinates of this space, the rays of light in the system B are straight (i.e. geodetic) lines, and the velocity of light is constant in all directions, although the system of reference was determined by the condition that it should be constant in the radial direction.

In this system of reference also all $g_{\mu\nu}$ are zero at infinity in the system B, and in A all $g_{\mu\nu}$ excepting g_{44}, which remains 1.

To $h = \infty$ corresponds $r = \frac{1}{2} \pi R$. The whole of elliptical space is therefore by the transformation (7) projected on the whole of hyperbolical space. For values of r exceeding $\frac{1}{2} \pi R$, h becomes negative. Now a point $(-h, \psi, \vartheta)$ is the same as $(h, \pi - \psi, \pi + \vartheta)$. The projection of the spherical space therefore fills the hyperbolical space twice. The same thing is true of the projection, by (5), of the elliptical and spherical spaces on the euclidian space.

5. Let the sun be placed in the origin of coordinates, and let the distance from the sun to the earth be a. We still neglect all gravitation.

In the system A the rays of light are straight lines, when described in the coordinates r, ψ, ϑ, i. e. in the elliptical or spherical space.

In the system B the same is true for the coordinates h, ψ, ϑ (hyperbolical space).

In the system A, consequently to triangles formed by rays of light, the ordinary formulas of spherical trigonometry are applicable. The parallax p of a star whose distance from the sun is r, is thus given by the formula

$$tan\, p = sin \frac{a}{R} cot \frac{r}{R},$$

The square of a/R being negligible, we can write this

$$p = \frac{a}{R} cot \frac{r}{R} = \frac{a}{\mathfrak{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9A)$$

In the system B we have similarly, in the coordinates h, ψ, ϑ:

$$tan\, p = sinh \frac{a}{R} \, coth \frac{h}{R},$$

or

$$p = \frac{a}{R} coth \frac{h}{R} = \frac{a}{R \sin \chi} = \frac{a}{\mathfrak{r}} \sqrt{1 + \frac{\mathfrak{r}^2}{R^2}} \quad . \quad . \quad . \quad (9B)$$

In the system A we have consequently $p = 0$ for $r = \frac{1}{2} \pi R$, i.e. for the largest distance which is possible in the elliptical space. If we admitted still larger distances, which are only possible in the spherical space, then p would become negative, and for $r = \pi R$ we should find $p = -90°$.

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In the system B p has a minimum value

$$p_0 = \frac{a}{R},$$

which it reaches for $h = \infty$, i.e. $r = \frac{1}{2}\pi R$. For values of r exceeding this distance p increases again, and for $r = \pi R$ we should find $p = +90°$.

Already in 1900 SCHWARZSCHILD[1]) gave a discussion of the possible curvature of space, starting from the formulae (9A) and (9B) For the system B we can from the observed parallaxes[2]) derive a lower limit for R. SCHWARZSCHILD finds $R > 4.10^6$ astronomical units. In the system A the measured parallaxes cannot give a limit for R.

In both systems we can, of course, derive such a limit from *distances* which have been determined, or estimated, otherwise than from the measured parallaxes. These distances must, in the elliptical space, be smaller than $\frac{1}{2}\pi R$. This undoubtedly leads to a much higher limit, of the order of 10^{10} or more.

6 The straight line being closed, we should, at the point of the heavens 180° from the sun, see an image of the back side of the sun. This not being the case, practically all the light must be absorbed on the long "voyage round the universe". SCHWARZSCHILD estimates that an absorption of 40 magnitudes would be sufficient[3]). If we adopt the result found by SHAPLEY[4]), viz. that the absorption in intergalactic space is smaller than $0^m.01$ in a distance of 1000 parsecs, then for an absorption of 40 mags we need a distance of 7.10^{11} astronomical units. In the elliptical space we have thus $R > \frac{1}{4} . 10^{12}$.

In the system A we can suppose that this absorption is produced

1) *Ueber das zulassige Krümmungsmaass des Raumes*, Vierteljahrsschrift der Astron Gesellschaft, Bd. 35 p. 337.

2) The meaning is of course actually measured parallaxes, not parallaxes derived by the formula $p = a/r$ from a distance which is determined from other sources (comparison of radial and transversal velocity, absolute magnitude, etc.). SCHWARZSCHILD assumes that there are certainly stars having a parallax of 0″05. All parallaxes measured since then are *relative* parallaxes, and consequently we must at the present time still use the same limit.

3) It might be argued that we should not see the back of the actual sun but of the sun as it was when the light left it. We could thus do without absorption, if the time taken by light to traverse the distance πR exceeded the age of the sun. With any reasonable estimate of this age, we should thus be led to still larger values of R.

4) Contributions from the Mount Wilson Solar Observatory Nrs. 115–117.

by the world-matter. It is about $^1/_{50}$ of the absorption which KING [1]) used in his calculation of the density of matter in interstellar space. The density of the world-matter would thus be about $^1/_{50}$ of the density found by KING, or $\varrho_0 = \frac{2}{2} \cdot 10^{-14}$ in astronomical units. The corresponding value of R (see art. 8) is $R = 2 \cdot 10^{10}$. The total absorption in the distance πR would then be only 3.6 magnitudes. To get the required absorption of 40 magnitudes we must increase ϱ_0, and consequently diminish R. We then find $\varrho_0 = 2 \cdot 10^{-12}$, $R = 2 \cdot 10^9$. This value of course has practically no weight, as it is very doubtful whether the considerations by which KING derived the density from the coefficient of absorption are applicable to the world-matter.

The whole argument is inapplicable to the system B, since in this system the light requires an infinite time for the "voyage round the world" One half of this time is

$$T = \int\limits_0^{1/2\pi R} \frac{1}{v}\, dr\,,$$

and, since $v = c \cos \chi$, we find $T = \infty$.

7. In the system A g_{44} is constant, in B g_{44} diminishes with increasing r. Consequently in B the lines is the spectra of very distant objects must appear displaced towards the red. This displacement by the inertial field is superposed on the displacement produced by the gravitational field of the stars themselves. It is well known that the Helium-stars show a systematic displacement corresponding to a radial velocity of $+ 4.3$ Km/sec. If we assume that about $^1/_3$ of this is due to the gravitational field of the stars themselves [2]), then there remains for the displacement by the inertial field about 3 Km/sec. We should thus have, at the average distance of the Helium stars

$$f = 1 - 2 \,.\, 10^{-5} = \cos^2 \frac{r}{R}\,.$$

If for this average distance we take $r = 3 \cdot 10^7$ (corresponding to a parallax of 0″ 007 by the formula $p = a/r$), this gives $R = \frac{2}{3} \,.\, 10^{10}$. Also for the M-stars, whose average distance is probably the largest after that of the Helium-stars, CAMPBELL [3]) finds a systematic displacement of the same order. The other stars, whose average dis-

[1]) Nature, Vol. 95, p. 701 (Aug. 26, 1915).

[2]) Cf. DE SITTER, *On* EINSTEIN's *theory of gravitation and its astronomical consequences*, Monthly notices, Vol. 76, p. 719.

[3]) Lick Bulletin, Vol. 6, p. 127.

tances are smaller, also have a much smaller systematic displacement towards the red, which can very well be explained by the gravitational field of the stars themselves.

Lately some radial velocities of nebulae [1]) have been observed, which are very large; of the order of 1000 Km/sec. If we take 600 Km/sec., and explain this as a displacement towards the red produced by the inertial field, we should, with the above value of R, find for the distance of these nebulae $r = 4 \cdot 10^8 = 2000$ parsecs. It is probable that the real distance is much larger. [2])

About a *systematic* displacement towards the red of the spectral lines of nebulae we can, however, as yet say nothing with certainty. *If in the future it should be proved that very distant objects have* systematically positive apparent radial velocities, this would be an indication that the system B, and not A, would correspond to the truth. If such a systematic displacement of spectral lines should be shown not to exist, this might be interpreted either as pointing to the system A in preference to B, or as indicating a still larger value of R in the system B.

8. In the paper which has already repeatedly been quoted, SCHWARZSCHILD determined the value of R for elliptical space by the condition that space should be large enough to contain the whole of our galactic system, the star-density being taken constant and equal to the value near the sun. This reasoning cannot be applied to the system A, since the field-equations give a relation between M and ρ, which contradicts SCHWARZSCHILD's condition.

We have

$$\varkappa \rho_0 = \frac{2}{R^2}.$$

The volume of the elliptical space is $\pi^2 R^3$. The total mass is therefore $\pi^2 R^3 \rho_0$, or

1) N.G.C. 4594 { PEASE + 1180 km/sec.; SLIPHER + 1190 „ }
N.G.C. 1068 { SLIPHER + 1100 „; PEASE + 765 „; MOORE + 910 „ }

The nebula in Andromeda however appears to have a considerable negative velocity, viz.:

{ WRIGHT — 304 km/sec.; PEASE — 329 „; SLIPHER — 300 „ }

2) EDDINGTON (Monthly Notices, Vol. 77, p. 375) estimates $r > 100000$ parsecs. This, combined with an apparent velocity of $+600$ km/sec., would give $R > 3.10^{11}$.

$$M = \frac{2\pi^2}{\varkappa} . R .$$

If we take for M the mass of our galactic system, which can be estimated [1]) at $\frac{1}{5} \cdot 10^{10}$ (sun $= 1$), then the last formula gives $R = 41$, or only about $1\frac{1}{2}$ times the distance of Neptune from the sun. This, of course, is absurd. If we use the other formula we can take for ρ_0 the star-density in the immediate neighbourhood of the sun, which we estimate at 80 stars per unit of volume of KAPTEYN (cube of 10 parsecs side), or $\rho_0 = 10^{-17}$ in astronomical units. We then find $R = 9 \cdot 10^{11}$. The total mass then becomes $M = 7 \cdot 10^{19}$, and consequently the galactic system would only represent an entirely negligible portion of the total world-matter.

It appears probable for many different reasons that outside our galactic system there are many more similar systems, whose mutual distances are large compared with their dimensions. If we take for the average mutual distance 10^{10} astronomical units, then an elliptical space with $R = 9 \cdot 10^{11}$ could contain $7 \cdot 10^6$ galactic systems, of which of course only a small number are known to us by direct observation. If, however, they all actually existed, and their average mass were the same as of our own galaxy, then their combined mass would be about $2 \cdot 10^{16}$, and consequently only one three-thousandth part of the world-matter would be condensed to "ordinary" matter. It is very well possible to construct a world in which the whole of the world-matter would, or at least could, be thus condensed. We must then for ρ_0 take the density not within the galactic system, but the average density over a unit of volume which is large compared with the mutual distances of the galactic systems. With the numerical data adopted above, this leads to $R = 5 \cdot 10^{13}$, and there would then be more than a billion galactic systems.

All this of course is very vague and hypothetical. Observation only gives us certainty about the existence of our own galactic system, and probability about some hundreds more. All beyond this is extrapolation.

9. We now come to the case that there is gravitation, which is produced by "ordinary" matter, with the density ρ_1. I will consider the field produced by a small sphere at the origin of the system of coordinates, which I will call the "sun". Its radius is R.

In the system A the world-matter has thus everywhere the constant density ρ_0, except for values of r which are smaller than

[1]) Communicated by Prof. KAPTEYN.

R, i.e. within the sun. There the density[1]) is $\varrho = \varrho_0 + \varrho_1$. In the system B, we have $\varrho = \varrho_1$, and this is zero except for $r < R$.

The line-element then has the form

$$ds^2 = - a dr^2 - b \left[d\psi^2 + \sin^2 \psi \, d\vartheta^2 \right] + f c^2 dt^2 ,$$

and in a stationary state a, b, f are functions of r only. The equations become somewhat simpler if we introduce

$$l = lg\, a \ , \qquad m = lg\, b \ , \qquad n = lg\, f \, .$$

If differential coefficients with respect to r are indicated by accents we find

$$G_{11} = m'' + \tfrac{1}{2} n'' + \tfrac{1}{2} m' (m' - l') + \tfrac{1}{4} n' (n' - l'),$$

$$\frac{a}{b} G_{22} = - \frac{a}{b} + \tfrac{1}{2} m'' + \tfrac{1}{4} m' (2 m' + n' - l'),$$

$$- \frac{a}{f} G_{44} = \tfrac{1}{2} n'' + \tfrac{1}{4} n' (2 m' + n' - l'),$$

$$G_{33} = \sin^2 \psi \, G_{22}.$$

In order to write down the equations (1) we must know the values of $T_{\mu\nu}$. If all matter is at rest, and if there is no pressure or stress in it, these are: $T_{44} = g_{44}\, \varrho$, all other $T_{\mu\nu} = 0$. These values I call $T_{\mu\nu}^0$. If we adopt these, then the equations (1) become, after a simple reduction

$$n'' + n' (m' + \tfrac{1}{2} n' - \tfrac{1}{2} l') = a (\varkappa \varrho - 2 \lambda), \quad . \quad . \quad . \quad (10)$$

$$m'' + \tfrac{1}{2} m' (m' - n' - l') = - a \varkappa \varrho, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$- \frac{a}{b} + \tfrac{1}{2} m' (n' + \tfrac{1}{2} m') = - a \lambda . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It is easily verified that these are satisfied if we take $\varrho = \varrho_0$, and for $g_{\mu\nu}$ we take the values corresponding to one of the forms (4A), (4B), or (4C) of the line-element, with the conditions (3A), (3B), or (3C) respectively. Similarly for (6A), (6B) and (8A), (8B), if the accents in (10), (11), (12) denote differential coefficients with respect to r, or h respectively. Consequently in the field of pure inertia we have $T_{\mu\nu} = T_{\mu\nu}^0$, i.e. by the action of inertia alone there are produced no pressures or stresses in the world-matter.

[1]) This, of course, is not strictly in accordance with EINSTEIN's hypothesis, by which the condensation of the world-matter in the sun should be compensated by a rarefying, or entire absence, of it elsewhere. The mass of the sun however is extremely small compared with the total mass in a unit of volume of such extent as must be taken in order to treat the density of the world-matter as constant. Therefore, if we neglect the compensation, the mass present in the unit of volume containing the sun is only *very* little in excess of that present in the other units. In the real physical world such small deviations from perfect homogeneity must always be considered as possible, and they must produce only small differences in the gravitational field.

If however the mass of the sun is not neglected, then a stationary state of equilibrium, with all matter at rest, cannot exist without internal forces within this matter. The $T_{\mu\nu}$ are then different from $T_{\mu\nu}^0$. If the world-matter is considered as a continuous "fluid", then this fluid can only be at rest if there is in it a pressure or stress. If it is considered as consisting of separated material points then these cannot be at rest. The difference $T_{\mu\nu} - T_{\mu\nu}^0$ vanishes with ϱ, for if $\varrho = 0$, both $T_{\mu\nu}$ and $T_{\mu\nu}^0$ are zero. This difference, therefore, is of the form $\varepsilon . \varrho$, ε being of the order of the gravitation produced by the sun. The right-hand-members of the equations (1), and therefore also of (10), (11), (12) require corrections of the order $\varkappa . \varepsilon . \varrho$. If these are neglected, the equations are no longer exact.

10. The mass of the sun being small, the values of a, b, f will not differ much from those of the inertial field. We can then, in the system A, and for the coordinates r, ψ, ϑ, put

$$a = 1 + \alpha \quad , \quad b = R^2 \sin^2 \chi (1 + \beta) \quad , \quad f = 1 + \gamma,$$

and in a first approximation we can neglect the squares and products of α, β, γ. The equations then became:

$$\gamma'' + \frac{2}{R} \gamma' \cot \chi = a \varkappa \varrho_1, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (13)$$

$$\beta'' + \frac{\cot \chi}{R} (2\beta' - \alpha' - \gamma') + \frac{2\alpha}{R^2} = - a \varkappa \varrho_1 \quad . \ . \quad (14)$$

$$\beta \operatorname{cosec}^2 \chi - \alpha \cot^2 \chi + (\beta' + \gamma') \frac{\cot \chi}{R} = 0. \quad . \ . \quad (15)$$

From (13) we find, remembering that the accents denote differentiations with respect to $r = R . \chi$.

$$\gamma' \sin^2 \chi = \int_0^r a \varkappa \varrho_1 \sin^2 \chi \, dr$$

Outside the sun we have $\varrho_1 = 0$. Thus if we put

$$\mathrm{a} = R^2 \int_0^{\mathrm{R}} a \varkappa \varrho_1 \sin^2 \chi \, dr$$

then outside the sun

$$\gamma' = \frac{\mathrm{a}}{R^2 \sin^2 \chi},$$

from which

$$\gamma = - \frac{\mathrm{a}}{R} \cot \chi = - \frac{\mathrm{a}}{\mathrm{r}}. \quad . \ . \ . \ . \ . \quad (16)$$

For $r = \frac{1}{2}\pi R$, i.e. for the largest distance which is possible in the elliptical space, we have thus $\gamma = 0$. For still larger distances, which are only possible in the spherical space, γ becomes positive, and finally for $r = \pi R$ we should have $g_{44} = \infty$, however small the mass of the sun may be, as has already been remarked above (art. 3).

If now from (14) and (15) we endeavour to determine α and β, we are met by difficulties. It appears that the equations (13), (14), (15) are contradictory to each other. If we make the combination

$$(13) + (14) - 2\,.(15) - R \tan\chi \frac{d\,(15)}{dr}$$

we find

$$\gamma' \tan\chi = 0, \quad \ldots \ldots \quad (17)$$

which is absurd. If the equations were exact, they should, in consequence of the invariance, be dependent on each other. They are however not exact, since on the right-hand-sides terms of the order of $\varepsilon\,.\,\varkappa\varrho$ have been neglected, ε being of the order of α, β, γ. In the world-matter we have[1] $\varkappa\varrho = \varkappa\varrho_0 = 2\lambda$, and these corrections can only be neglected if λ is also of the order ε. This has not been assumed in the equations (13), (14), (15). If we wish to assume it, then we must also develop in powers of λ. We can then use the coordinates r, ψ, ϑ. We put thus

$$a = 1 + \alpha \quad , \quad b = \mathrm{r}^2(1 + \beta) \quad , \quad f = 1 + \gamma.$$

The equations, in which now the accents denote differentiations with respect to r, then become, to the first order

$$\gamma'' + \frac{2}{\mathrm{r}}\gamma' = \varkappa\varrho_1,$$

$$\beta'' + \frac{2}{\mathrm{r}}\beta' - \frac{1}{\mathrm{r}}(\alpha' + \gamma') = -\varkappa\varrho_1 - 2\lambda,$$

$$\beta - \alpha + \mathrm{r}(\beta' + \gamma') = -\lambda \mathrm{r}^2,$$

which are easily verified to be dependent on each other.

We can thus add an arbitrary condition. If we take e.g.

$$\alpha = 2\beta,$$

then we find, to the first order, outside the sun

$$\alpha = -2\lambda\mathrm{r}^2 + \frac{\mathrm{a}}{\mathrm{r}} \quad , \quad \beta = -\lambda\mathrm{r}^2 + \frac{1}{2}\frac{\mathrm{a}}{\mathrm{r}} \quad , \quad \gamma = -\frac{\mathrm{a}}{\mathrm{r}},$$

[1]) Of course, if beside the world matter there is also "ordinary matter", i. e. if the density of the world matter is not constant, this relation is also only approximatively true, and requires a correction of the order $\lambda\,.\,\varepsilon$. (See also art. 11).

where $a = \int\limits_0^R \varkappa \varrho_1 r^2 dr$. If a is neglected these are the terms of the first order in the development of (6 A) in powers of $\lambda = {}^1/R^2$.

11. Consider again the equations (10), (11), (12). If these were exact, they would be dependent on each other. They are, however, not exact, and consequently they are contradictory. If we make the combination:

$$2 \, . \, \frac{d\,(12)}{dr} + 2\,[m' - l'] \;\; (12) - [m' + n'] \, . \, (11) - m' \, . \, (10),$$

we find [1])

$$0 = n' \, a \, \varkappa \varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Consequently the equations are dependent on each other, i.e. a stationary equilibrium, all matter being at rest without internal forces, is only possible, when either $\varrho = 0$ or $n' = 0$, i.e. $g_{44} =$ constant. In the system A ϱ is never zero, since outside the sun $\varrho = \varrho_0$. A stationary equilibrium is then only possible if g_{44} is constant, i.e. if no "ordinary" matter exists, for all ordinary matter will, by the mechanism of the equation (10) or (13) produce a term γ in g_{44} which is not constant. If ordinary or gravitating matter does exist then not only in those portions of space which are occupied by it, but throughout the whole of the world-matter $T_{\mu\nu}$ will differ from $T_{\mu\nu}{}^0$. We can e.g. consider the world-matter as an adiabatic incompressible fluid. If this is supposed to be at rest, we have

$$T_{ii} = - g_{ii} \, p \quad , \quad T_{44} = g_{44} \, \varrho_0,$$

where p is the pressure in the world-matter. I then find

$$p = \varrho_0 \left(\frac{1}{\sqrt{f}} - 1 \right)$$

and, to the first order, and for the coordinates r, ψ, ϑ:

$$\alpha = \beta = - \gamma = a \, . \left(\frac{\cos 2\chi}{R \sin \chi} + \frac{1}{R} \right),$$

$$\varkappa \varrho_0 = 2\lambda - 3 \frac{a}{R^3} = 2\lambda \left(1 - \tfrac{3}{2} \frac{a}{R} \right).$$

For our sun a/R is of the order of 10^{-20}.

For $\chi = \frac{1}{2} \pi$ we have $\gamma = 0$, and for $\chi = \pi$ we should find

[1]) It is easily verified that (18) becomes identical with (17) if all terms of higher orders than the first are neglected.

$\gamma = \infty$, as in the approximate solution (16), in which p was neglected.

For the planetary motion we must go to the second order. I find a motion of the perihelion amounting to

$$d\tilde{\omega} = -\tfrac{3}{2}\lambda\, a^2\, nt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which is of course entirely negligible on account of the smallness of λa^2. In my last paper [1]) it was stated that there is no motion of the perihelion. In that paper the values $T_{\mu\nu}{}^0$ were used, i.e. the pressure p was neglected. The motion (19) can thus be said to be produced by the pressure of the world-matter on the planet. It will disappear if we suppose that in the immediate neighbourhood of the sun the world-matter is absent.

12. In the system B outside the sun we have $\varrho = 0$, and the equations are dependent on each other and can be integrated.

Within the sun $n' a \varkappa \varrho_1$ must be of the second order, and consequently n' must be of the first order. If we put

$$f = \cos^2 \chi\, (1 + \gamma),$$

then $n' = -\dfrac{2}{R} \tan\chi + \dfrac{\gamma'}{1+\gamma}$, thus $\dfrac{\tan\chi}{R}$ must be of the first order. Since $\chi = r/R$ we find that $1/R^2$ must be of the first order, as in system A.

Developing f in powers of $1/R$ we find, to the first order

$$f = 1 - \frac{r^2}{R^2} + \gamma,$$

In the first approximation we find for γ the same value as in the systems A and C, viz: $\gamma = -\,\mathrm{a}/r$. Here however we have also the term $-{}^{r^2}/_{R^2}$. Thus classical mechanics according to NEWTON'S law can only be used as a first approximation if this term, and consequently also $\lambda = {}^3/_{R^2}$ is of the *second* order. Investigating the effect of this term on planetary motion, we find a motion of the perihelion [2]) amounting to

$$d\tilde{\omega} = \frac{3a^3}{2\mathrm{a}\, R^2}\, nt.$$

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[1]) These Proceedings, Vol. XIX, page 1224.

[2]) In my last paper (these Proceedings Vol. XIX, p. 1224) I found

$$d\tilde{\omega} = \frac{3a^3}{4\mathrm{a}\, R^2}\, nt - \frac{cnt^2}{2R^2}.$$

The difference is due to the use of a different system of reference, with a different time and different radius-vector, in the two cases, the formulas for the transformation of the space-variables (especially the radius-vector) from one system to the other depending on the time.

From the condition that this shall for the earth not exceeds ay 2″ per century we find

$$R > 10^8.$$

Then $^1/_{R^2} < 10^{-16}$ is actually of the second order compared with $\varkappa = 25 \cdot 10^{-8}$. This limit of R is still considerably lower than the value which was found above from the displacement of the spectral lines. For the planetary motion — and generally for all mechanical problems which do not involve very large values of r — we can therefore in both systems A and B neglect the effect of λ entirely.