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ERRATA.

Page	5,	line	12	from	top	replace	„edges"	by	„limits l_k"
„	14,	„	8	„	„	„	$7a$	„	$7c$
„	„ ,	„	9	„	„	„	$7b$	„	$7a$
„	16,	„	15	„	„	„	S_n	„	S_{n-1}
„	20,	„	1	„	„	„	19β	„	19α
„	21,	„	8	„	„	„	22γ, 22β	„	22π, 22γ

Geometrical deduction of semiregular from regular polytopes and space fillings

BY

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Geometrical deduction of semiregular from regular polytopes and space fillings

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Introduction.

1. The object of this memoir is to give a method by which bodies having a certain kind of semiregularity may be derived from regular bodies in an Euclidean space of any number of dimensions; and space fillings of the former from space fillings of the latter.

These space fillings or nets for threedimensional space have been given in a paper entitled „Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative" by Mr. A. ANDREINI [1]), who deduced them by means of the angles of the different polyhedra. Photographs prepared for the stereoscope, taken from that paper, representing the various semiregular space fillings were sent to me by Prof. SCHOUTE to whom I desire to record here my thanks for the generous help he has given me during the whole course of this investigation. These photographs suggested a method by which at once the semiregular bodies and the manner in which they combine to fill fourdimensional space could be derived from regular polytopes and nets in that space. It will be seen that this method can be applied to spaces of any other number of dimensions.

The semiregularity considered here is that in which there is one kind of vertex and one length of edge [2]), and the symbols used

[1]) *Memorie della Società italiana della Scienze* (detta dei XL), serie 3a, tomo XIV.

[2]) So the greater part of the forms called semiregular here will have a degree of regularity less than ½ in the scale of Mr. E. L. ELTE.

for the polyhedra of this description, almost the same as those given by ANDREINI, are as follows:

T, C, O, D, I indicate the five regular polyhedra, using the initial letters of their ordinary names Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron; and if p_n indicates a regular polygon with n vertices we have

tT = truncated T	limited by	$4p_6$	and	$4p_3$,
tC = " C	" "	$6p_8$	"	$8p_3$,
tO = " O	" "	$8p_6$	"	$6p_4$,
tD = " D	" "	$12p_{10}$	"	$20p_3$,
tI = " I	" "	$20p_6$	"	$12p_5$,
CO = C and O in equilibrium	" "	$6p_4$	"	$8p_3$,
ID = I " D " "	" "	$12p_5$	"	$20p_3$,
RCO = combination of rhombic D, C and O	" "	$18p_4$	"	$8p_3$,
RID = " " " Tr [1], I " D	" "	$12p_5$, $30p_4$	"	$20p_3$,
tCO = truncated [2] CO	" "	$6p_8$, $12p_4$	"	$8p_6$,
tID = " ID	" "	$12p_{10}$, $30p_4$	"	$20p_6$.

Moreover we want:

$P_3, P_4,...$ for threedimensional triangular prisms, square prisms (cubes), etc.

$P_C, P_O,...$ for fourdimensional prisms on a cube, an octahedron,... as base,

P (3 ; 3) or simply (3 ; 3) for a prismotope [3]) with two groups of threedimensional prisms P_3 as limiting bodies,

P (6 ; 8) or simply (6 ; 8) for a prismotope having for limiting bodies six octagonal and eight hexagonal prisms, etc.

2. The transformation of the regular into the semiregular bodies and space fillings can be carried out by means of two inverse operations which may be called *expansion* and *contraction*.

In order to define these operations conveniently, the vertices, edges, faces, limiting bodies,... of a regular polytope are called

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[1]) By Tr we indicate the solid limited by 30 lozenges in planes through the edges of I or D normal to the lines joining the centre to the midpoint of each edge.

[2]) According to custom the word "truncated" is used here, though this body and the next one cannot be derived from the CO and the ID by truncation.

[3]) This body is also a „simplotope" as the describing polygons (placed here in planes perfectly normal to each other) are triangles (compare SCHOUTE's „Mehrdimensionale Geometrie", vol. II, p. 128).

In general a prismotope is generated in the following way:

Let S_p and S_q be two spaces of p and q dimensions having only one point in common; let P be a polytope in S_p, Q a polytope in S_q. Now move S_p with P in it parallel to itself, so as to make any vertex of P describe all the points of Q. Then P generates the prismotope. Here we have to deal only with the case of two planes ($p = q = 2$); by the symbol (6;8) we will indicate the polytope limited by eight hexagonal and six octagonal prisms obtained in the indicated manner if we start from a hexagon and an octagon situated in two planes perfectly normal to each other.

its limits (l) and are denoted respectively by the symbols l_0, l_1, l_2, $l_3 \ldots$

I. The operations of expansion and contraction.

Definition of expansion.

3. Let O be the centre of a regular polytope and M_1, M_2, M_3.. the centres of its limits l_1, l_2, $l_3 \ldots$ The operation of expansion e_k consists in moving the limits l_k to equal distances away from O each in the direction of the line $O M_k$ which joins O to its centre, the limits l_k remaining parallel to their original positions, retaining their original size, and being moved over such a distance that the two new positions of any vertex, which was common to two adjacent edges in the original polytope, shall be separated by a length equal to an edge.

The polytope determined by the new positions of the limits l_k will have the kind of semiregularity described above. The limits l_k are said to be the subject of expansion or briefly the *subject*; and the new polytope is denoted by the symbol of the original regular polytope preceded by the symbol e_k.

A few particular cases, in 2, 3 and 4 dimensions, will now be examined.

Examples of the e_1 expansion.

4. Here the edges (l_1) are the subject.

It is evident that this operation applied to any regular polygon changes it into a regular polygon having the same length of edge and twice as many sides. In Fig. 1*a* a square is changed into an octagon by the application of the e_1 expansion [1]).

Fig. 1*b* shews the e_1 expansion of a cube. The real movement of any edge AB is in the direction of the line OM_1 but that movement may be resolved into two. Thus instead of moving AB directly to the position A_1B_1 it might have been moved to $A'B'$ or to $A''B''$ and then to A_1B_1. If the movements of all the edges be thus resolved the result is the same as if the faces AC, AD... (Fig. 1*c*) of the original cube had been first transformed into octagons by an e_1 expansion of each in its own plane, and then moved

[1]) In these drawings the *thick* lines represent edges of regular polytopes in their original or in new positions, the *thin* lines edges introduced by expansion.

away from the centre O until the edges $A'B'$ and $A''B''$, which correspond to an edge AB of the cube, are coincident and become the common edge of two octagons (transformed squares). It is to be noticed that as each vertex of a cube is common to three edges (three members of the subject) it takes three new positions, which, owing to the regularity of the cube, are the three vertices of an equilateral triangle. Thus the faces of the cube have been expanded into octagons and the vertices into triangles.

Fig. 2*a* shews the e_1 expansion of a tetrahedron. Each face is changed into a hexagon, each vertex into a triangle. Here again a vertex of the tetrahedron is common to three members of the subject; the result is a tT.

Fig. 2*b* shows the same expansion of an octahedron. Each face is changed into a hexagon; but each vertex into a square because in an octahedron each vertex belongs to four edges (four members of the subject); the result is a tO.

From these examples it is easy to find the e_1 expansion of an icosahedron and of a dodecahedron.

5. This investigation leads to the determination of the e_1 expansion applied to the fourdimensional polytopes. For instance in the C_8 each cube is transformed (in its own space) by the e_1 expansion and becomes a tC (Figs. 1*b* and 2*c*). These transformed cubes must be so adjusted that an edge which was in the C_8 common to three cubes [1]) is, in its new position, common to three transformed cubes. Again each vertex in a C_8 is common to four edges and must take four new positions which are the four vertices of a regular tetrahedron. Thus the vertex of the C_8 is expanded into a tetrahedron, which is said to be of vertex *import*. This tetrahedron might have been determined in another way; for four cubes meet in a vertex of a C_8 and in each the vertex is changed into a triangle; therefore a vertex of C_8 is replaced by a body limited by four triangles i. e. a tetrahedron.

The two kinds of limiting body of the new polytope $e_1 C_8$ are shewn in Fig. 2*c*; in Fig. 2*d* are shewn the limiting bodies of $e_1 C_5$.

In C_{16}, where six edges meet in a vertex, the e_1 expansion changes each tetrahedron into a tT (Fig. 2*a*) and each vertex into an octahedron (of vertex import) whose vertices are the six new positions of a vertex of the C_{16}.

Again in C_{24} eight edges meet in a vertex, so that the e_1 expan-

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[1]) In order to facilitate the application of the operation of expansion it is desirable to have at hand a table of incidences; this is provided on Table III.

sion here changes each octahedron into a *tO* (Fig. 2*b*) and each vertex into a cube (of vertex import) whose vertices are the eight new positions taken by a vertex of C_{24}.

In a similar manner the e_1 expansions of C_{120} and C_{600} may be determined.

6. *Rule.* These examples lead to the general rule for the e_1 expansion of a regular fourdimensional polytope P. The limiting bodies of P are transformed by the e_1 expansion and the vertices expanded into regular polyhedra each having as many vertices as there are edges meeting in a vertex of P.

Examples of the e_2 expansion.

7. As the faces are the subject in this expansion there can be no application to a single polygon in twodimensional space.

The e_2 expansion of a cube, an *RCO*, is shewn in figure 3*a*; there are three groups of faces:

1st : squares corresponding to the faces of the original cube
2nd : „ „ „ „ edges „ „ „ „
3rd : triangles „ „ „ vertices „ „ „ „

In this expansion of any regular polyhedron the faces of the first group are like those of the original polyhedron; the faces of the second group are always squares, since they are determined by the two new positions of an edge of the original polyhedron; those of the third group are triangles, squares or pentagons according as a vertex of the original polyhedron belongs to three, four or five faces.

As the cube and the octahedron are reciprocal bodies, the number of vertices lying in a face of one being equal to the number of faces meeting in a vertex of the other, it follows that the e_2 expansion of the octahedron is also an *RCO* (Fig. 3*b*).

Again the tetrahedron is self reciprocal, the number of vertices lying in a face being equal to the number of faces meeting in a vertex; so in the e_2 expansion the faces of vertex import are, like the faces of the tetrahedron, equilateral triangles (Fig. 4).

The e_2 expansion of the icosahedron and dodecahedron, which are reciprocal bodies, is an *RID*.

8. The e_2 expansion of the C_8 transforms each cube into an *RCO* and, as in the C_8 each face is common to two cubes, so those faces in the *RCO* which are faces of the cubes in new positions must now be common to two *RCO*. In the C_8 each edge belongs to three faces, so in the new polytope each edge takes

three new positions which are the three parallel edges of a right prism on an equilateral triangular base.

This manner of determining the prism (expanded edge) bears the most direct relation to the particular expansion under consideration, namely that in which the faces are the subject; but it could have been determined otherwise. Thus in a C_8 three cubes meet in an edge and as each is changed into an *RCO*, its edges are changed into squares, so that instead of three coincident edges there are now three squares, the side faces of a right prism.

Again in the C_8 each vertex belongs to six faces and therefore must assume six positions. From this it is evident that the body taking the place in the new polytope of the vertex in the C_8 has six vertices and it remains to determine its faces.

In figure 5 are represented, in their true relative positions as far as threedimensional space will allow, two of the four *RCO* and two of the four P_3 which have taken the place of the four cubes and the four edges meeting in a vertex of the C_8. It shews that each *RCO* supplies a triangular face and each prism a triangular face — all equilateral — to the body that takes the place of the vertex of the C_8. This body therefore is a regular octahedron, four of whose faces are in contact with *RCO* and four with P_3.

The new polytope then, $e_2 C_8$, is limited by 8 *RCO*, 32 P_3 of edge import, 16 *O* of vertex import.

9. *Rule.* The rule for the e_2 expansion of a regular fourdimensional polytope *P* may be stated thus:

The limiting bodies of *P* are transformed by the e_2 expansion. The edges are expanded into prisms each having as many edges parallel to the axis as there are faces meeting in an edge of *P*. The vertices are expanded into bodies having two groups of faces, one kind of edge, and as many vertices as there are faces meeting in a vertex of *P*. One group of faces is supplied by the bases of the prisms of edge import and of these the number is equal to the number of edges meeting in a vertex of *P*; the other is supplied by the expanded vertices of the transformed limiting bodies, of which the number is equal to the number of limiting bodies meeting in a vertex of *P*.

Examples of the e_3 expansion.

10. Here the limiting bodies are the subject; and it is at once evident that this expansion applied to reciprocal fourdimensional

bodies, e. g. to C_8 and C_{16}, also to C_{120} and C_{600}, must produce the same result; while applied to a self reciprocal form it produces a polytope whose limiting bodies of vertex import are like the original limiting bodies and of the same number, and whose limiting bodies of face import are of the same number and kind as those of edge import.

Thus as in the C_8 each face belongs to two, each edge to three, and each vertex to four cubes, it follows that in the expansion each face takes two, each edge three, and each vertex four positions. The $e_3 C_8$ is therefore limited by 8 cubes of body import (cubes of the original C_8), 24 P_4 of face import, 32 P_3 of edge import, and 16 tetrahedra of vertex import (Fig. 6*a*). In the C_{16} each face belongs to two, each edge to four, each vertex to eight tetrahedra, so in the expansion each face takes two, each edge four, and each vertex eight positions and the $e_3 C_{16}$ is limited by 16 tetrahedra of body import, 32 P_3 of face import, 24 P_4 of edge import, and 8 cubes of vertex import (Fig. 6*b*). These two polytopes are alike except that the *imports* are *reciprocal.*

11. Generally there are four groups of limiting bodies:

1st: polyhedra of body import like the limiting bodies of the original cell,

2nd: prisms of face import defined by their bases (two positions of each face of the original cell),

3rd: prisms of edge import defined by their edges parallel to the axis (as many positions of an edge as there are bodies meeting in an edge of the original cell),

4th: polyhedra of vertex import having as many vertices as there are bodies meeting in a vertex of the original cell.

So in $e_3 C_5$ there are 10 T, 20 P_3; in C_{24} there are 48 O, 192 P_3.

This expansion of a C_{120} and a C_{600} (reciprocal cells) can easily be determined.

12. *Rule.* The rule for the e_3 expansion of a regular polytope P of fourdimensional space is as follows:

The limiting bodies of P are moved apart (untransformed).

The faces are replaced by prisms whose bases are parallel positions of a face of P. The edges are replaced by prisms each having as many edges parallel to the axis as there are limiting bodies meeting in an edge of P. Each vertex is replaced by a regular polyhedron, the number of whose vertices is equal to the number of limiting bodies meeting in a vertex of P.

Generalization.

13. The foregoing result may be generalized thus. If any set of limits e_r be the subject of expansion in a regular polytope P_n in a space of n dimensions the polytope P_n' defined by the new positions of the members of the subject has for its limits l'_{n-1}:

1st: a group consisting of the limits l_{n-1} of P_n transformed by the e_r expansion $(e_r\, l_{n-1})$,

2nd: a group of vertex import, each member of the group being determined by its vertices, the number of which is equal to the number of limits l_r meeting in a vertex of P_n and having one kind of edge. This polytope is regular in the e_1 and the e_{n-1} expansions. These two groups are the principle ones.

3rd: there are besides various kinds of prisms. Those of edge import (1-import) are determined by the new positions of an edge of P_n and the number of these positions is equal to the number of limits l_r meeting in an edge of P_n. The prisms of face import (2-import) are determined each by the new positions of a face of P_n, and the number of these is equal to the number of limits l_r meeting in a face of P_n and so on. The whole series of prisms is as follows: 1-import, 2-import, ...r—1-import.

Combination of operations.

14. The expansions described above have been applied to regular bodies according to the definition given on page 5, transforming them into bodies possessing a particular kind of semiregularity.

The question now arises: can these semiregular bodies be transformed by the application of any further expansion without having lost the kind of semiregularity defined above?

It is evident in the first place that a movement of all the edges or of all the faces would produce bodies with edges of different lenghts. But an inspection of the transformed bodies in three-dimensional space (Figs. 1 *b*, 2 *a* and 2 *b*) shows that in each of the polyhedra *tC*, *tT* and *tO* there are two groups of faces, each of which taken alone defines the polyhedron: one group corresponds to the faces (expanded), the other to the vertices (expanded) of the original polyhedron, and these two groups differ as to a particular characteristic.

The members of the first group are in contact with members of the same group; the members of the second are separated by at least the length of an edge from members of their own group. As the operation of expansion applied to a set of limits has the effect of separating any two adjacent members, it follows that the first group can, the second group cannot, be made the subject of expansion.

For instance in $e_1 C$ (Fig. *7a*) the triangles cannot be moved away from the centre without increasing the length of the edges joining them, but the octagons may be moved away from the centre until the edge AB common to two has assumed two new positions $A'B'$, $A''B''$ which are the opposite sides of a square. The new positions of the octagons define a polyhedron having the required kind of semiregularity. [1])

15. This double operation may be denoted by the symbol $e_2 e_1 C$ where it is understood that the faces forming the subject of the e_2 expansion are only those which have taken the place of faces in the original cube. Similarly the interpretation of the symbol $e_1 e_2 C$ is that the e_2 expansion is applied to a cube and that the subject of further expansion is composed of those faces which have taken the place of edges in the original cube. This is shewn in Fig. *7b* where the group of 12 squares (corresponding to the edges of the original cube) form the subject of expansion. These two figures *7a* and *7b* show that

$$e_1 e_2 C = e_2 e_1 C = tCO$$

and it is evident that the order in which the operations are applied to any regular polyhedron is indifferent, for the two operations could have been carried out simultaneously.

In Fig. *7c* is shewn the result of the double operation $e_2 e_1 O$ applied to an octahedron. This is also a *tCO*.

If the double operation be applied to a I and an D the result in both cases will be a *tID*.

This body and the *tCO* are incapable of further expansion.

16. Thus it appears that three expansions can be applied to the cube, octahedron, dodecahedron, icosahedron, namely e_1, e_2, $e_1 e_2$. But more can be done with the tetrahedron owing to the fact that it is self reciprocal.

Fig. *7d* and *7e* show respectively the result of the $e_2 e_1$ and the $e_1 e_2$ expansion applied to a tetrahedron, and the result in both cases is a *tO* which can be further expanded into a *tCO* (Fig. *7c*). Thus the self reciprocity of the tetrahedron allows an expansion which cannot be carried out in the other four polyhedra. The

[1]) Here the group of octagons may be called the „independent" variable, while the triangles, which are transformed into hexagons, are the „dependent" ones.

combination of operations may be applied in the same way to the cells of fourdimensional space as one or two examples will show.

17. *Case* $e_1 e_2 C_8$. — The e_2 operation applied to a C_8 produces a polytope limited by 8 *RCO*, 32 P_3, 16 *O* (Fig. 5). The symbol $e_1 e_2 C_8$ directs that the new subject of expansion comprising those limiting bodies in $e_2 C_8$ which correspond to edges of C_8, i. e. the 32 P_3, shall, themselves unchanged, be carried away from the centre (of the $e_2 C_8$).

These P_3 in their new positions define the polytope sought. This movement changes the *RCO* and the *O*. Each *RCO* was derived from a cube by the e_2 expansion; the new expansion e_1 carries out the group of 12 squares (corresponding to the edges of the cube), thereby producing a *tCO* (Fig. 7*b*). In order to determine the change in the octahedron of vertex import it is only necessary to observe that four of its faces (those in contact with bases of P_3) are still in contact with them and are only changed in position, while the other four (those which were in contact with *RCO*) are changed into hexagons in contact with *tCO*. Thus the octahedron is changed into a *tT*. The effect on a single octahedron is the same as if its alternate faces had been made the subject of expansion (Fig. 8).

18. *Case* $e_2 e_3 C_8$. — The result of applying the e_3 operation to a C_8 is a polytope limited by cubes (original cubes of the C_8), P_4 of face import, P_3 of edge import, and tetrahedra of vertex import (Fig. 6*a*). The symbol e_2 directs that the square prisms of face import shall be moved away from the centre of $e_3 C_8$, they themselves remaining unchanged except in position. These in their new positions define the new polytope and it only remains to determine in what manner their movement has modified the remaining limiting bodies of the $e_3 C_8$. This can be seen at once in a drawing. In figure 9*a* are shewn seven limiting bodies of the $e_3 C_8$; one is a cube of the original C_8, after having been separated by the e_3 movement from the adjacent cubes; three are cubes of face import interposed by the same movement between the cubes of the C_8; three are P_3 of edge import, their bases being faces of a tetrahedron of vertex import. The symbol e_2 directs that the cubes of face import are to be moved out. The result is shewn in figure 9*b*; the original cube is changed into an *RCO*, the P_3 into a P_6 and the *T* into a *tT*. It is necessary to bear in mind that only one limiting body of any polytope can be in threedimensional space at a time, and in representing several at once in it there must be either distortion of the limiting bodies or separation of faces and edges which ac-

tually coincide. Moreover the direction of the real movement cannot be represented; but valid conclusions may be drawn from diagrams such as these, if the mind always distinguishes between the actual and the apparent relation of parts.

These two examples suffice to show how the result of the combination of operations may be applied to the fourdimensional cells. There are seven expansions of each:

$$e_1, \quad e_2, \quad e_3, \quad e_1 e_2, \quad e_1 e_3, \quad e_2 e_3, \quad e_1 e_2 e_3,$$

but owing to the reciprocity of some of the figures these are not all different.

Thus it appears that in any expansion a set of limits, which define the body and which is such that each member is in contact with other members of the same set, may be made the subject of expansion.

Definition of contraction.

19. In each of the expansions e_1, e_2, $e_3 \ldots$ the resulting semiregular polytope may be reduced to the regular one from which it was derived, by an inverse operation which may be called contraction.

Here the limits which formed the subject of the expansion are moved towards the centre and brought back to their original positions.

The direct operation separates the members of the subject; the inverse operation brings them again into contact, annihilating the edges introduced by expansion. In both positions they define the polytope of which they are the limits.

The conditions necessary to the inverse operation are: 1[st], the limits forming the subject must define the polytope; 2[nd], no two members of the subject can be in contact before contraction.

The polytopes of vertex import always satisfy these conditions and can be made the subject of contraction. The symbol c is used to denote contraction. The import of the limits forming the subject is shown by means of subscripts, as in expansion.

Examples of contraction.

20. The inverse operation will be made clear by one or two examples.

In figure 10 the square $A\ B\ C\ D$ has been expanded by the e_1 operation; the edges of vertex import in the resulting octagon have been made the subject of the inverse operation, that is, they have been moved nearer to the centre so far that the edges of the original square are annihilated, and the final result is the square

$E\ F\ G\ H$, denoted by the symbol $c_0 e_1 S$ where S is the square $A\ B\ C\ D$.

In figure 10*b* is shown a cube transformed by the e_1 operation, i. e. an $e_1 C$; the triangles of vertex import are brought nearer to the centre by the c_0 operation and the result is a CO whose symbol is now $c_0 e_1 C$.

Again, the tCO may be considered in two ways. It may be deduced from either the octahedron or the cube (compare Figs. 7*a* and 7*b*), so it may be denoted by $e_1 e_2 C$ or $e_1 e_2 O$. Though the identity of these results may be expressed in the form of an equation: $e_1 e_2 C = e_1 e_2 O$, it must still be borne in mind that the imports are different. Let each of these symbols be preceded by c_0. What are the results? If the tCO has been derived from the cube, the hexagons are of vertex import; if, on the other hand, it has been derived from the octahedron, the octagons are of vertex import. Thus the symbol $c_0 e_1 e_2 C$ indicates that the hexagons, and the symbol $c_0 e_1 e_2 O$ that the octagons, are the subject of the inverse operation whence $c_0 e_1 e_2 C = tO$ (Fig. 7*c*), $c_0 e_1 e_2 O = tC$ (Fig. 7*a*). But the octagons correspond to the faces of the cube and the hexagons to the faces of the octahedron, so that $c_0 e_1 e_2 C = c_2 e_1 e_2 O$, $c_0 e_1 e_2 O = c_2 e_1 e_2 C$.

21. An example will show the combination of inverse operations. The tCO derived from a cube (Fig. 11*a*) may be reduced to an octahedron by moving the squares and the hexagons nearer to the centre; the tCO derived from an octahedron (Fig. 11*b*) may be reduced to a cube by moving the squares and the octagons nearer to the centre.

These operations are denoted respectively by the equations

$$c_0 c_1 e_1 e_2 C = O \quad , \quad c_0 c_1 e_1 e_2 O = C.$$

22. In figure 5 are shown, the limiting bodies of an $e_2 C_8$. If the octahedra of vertex import be made the subject of the inverse operation, the following changes will take place: each P_3, separating two neighbouring octahedra, is reduced to two coincident triangles. This annihilates the edges of the prism parallel to the axis. But these are the edges of the original C_8 in the new positions due to expansion and if these be annihilated each RCO will be reduced to an octahedron. Thus the new body is a C_{24}, eight of whose limiting bodies are compressed RCO, while sixteen are of vertex import in the expansion $e_2 C_8$.

As in the enumeration of the polytopes and the nets given in the three Tables only the c_o appears, c_o has been replaced by c.

Partial operations.

23. It has been seen that in both expansion and contraction it is a necessary condition that the subject of operation shall define the polytope both before and after the movement.

In expansion, each member of the subject must be in contact with other members. In contraction, each member must be separated from the other members by at least the length of an edge.

It sometimes happens that one of these conditions is satisfied by a *group* consisting of the alternate members of a set of limits. Such a group may then be made the subject of expansion or contraction. If the members be in contact, they may be made the subject of expansion; if they be not in contact, they may be made the subject of contraction.

24. Thus, an octahedron is defined by a group of four alternate triangles, but each of these triangles is in contact with the other three, so that these four may be made the subject of expansion. This partial operation, which changes the octahedron into a truncated tetrahedron, is denoted by the symbol $\frac{1}{2} e_2 O$. So $\frac{1}{2} e_2 O = tT$.

Again, a CO whose symbol is $c_0 e_1 C$ is defined by a group of four alternate triangles. Each of these is separated from the others by the length of an edge. This group may therefore be made the subject of the c operation, which changes the CO into a T. So $\frac{1}{2} c_0 c_0 e_1 C = T$.

It may be remarked that the partial contraction $\frac{1}{2} c_0$ can never take place without a previous complete contraction c_0.

25. The corresponding case in fourdimensional space is expressed by the symbol $\frac{1}{2} c_0 c_0 e_1 C_8$. This indicates that first, the edges of the C_8 are made the subject of expansion; second, the sixteen tetrahedra of vertex import are made the subject of contraction; third, a group of eight alternate tetrahedra are made the subject of still further contraction. This last partial movement changes the cubes of the C_8 into tetrahedra and annihilates eight of the tetrahedra of vertex import, thus changing the C_8 into a C_{16}, eight of whose limiting tetrahedra are derived from the limiting cubes of the C_8, the remaining eight being of vertex import. So $\frac{1}{2} c_0 c_0 e_1 C_8 = C_{16}$.

These examples suffice to show in what manner and under what conditions the partial operations may be applied.

II. Application to space fillings.

Expansion applied to the nets.

26. A space filling or net in any space S_n may be considered as a polytope with an infinite number of limiting spaces of n dimensions in a space S_{n+1} of one dimension higher. [1]) According to this view the operations of expansion and contraction and their combinations may be applied to nets; but the fact that the net is a particular case of a polytope modifies the manner in which the operation is to be applied.

Expansion has been defined as a movement of any set of limits away from the centre of a polytope. This movement in general separates the members of the subject.

In a polytope in S_n with an infinite number of n—1-dimensional limits (a net) the centre is at an infinite distance in a direction normal to the space S_n of the net and no movement away from the centre can separate the limits forming the subject, in other words can expand the net. Now it has been shewn that the real movement taking place in an expansion may be resolved into two, one of which transforms the limits each in its own space and the other adjusts those transformed limits. In this way the operation can be applied to the special case under consideration. Thus if the e_1 expansion be applied to a net of squares (Fig. 12) they are transformed into overlapping octagons and then the octagons must be moved apart until an edge which was common to two squares becomes common to two octagons.

This adjustment leaves a gap $A_1\ A_2\ A_3\ A_4$ (vertex gap) between the octagons corresponding to the vertex A common to four squares. Thus the transformed net of squares is composed of two constituents, octagons corresponding to the squares, and squares corresponding to vertices of the original net.

27. In threedimensional space there is only one regular space filling i. e. the net NC of cubes. The net $N(O, T)$ of octahedra and tetrahedra is semiregular.

If the e_1 expansion be applied to a net of cubes each cube is transformed into a tC. These will overlap and must be moved apart until an edge which was common to four cubes becomes common to four tC (Fig. 13) By this adjustment octahedral gaps (vertex gaps) are left at the vertices. So the net $e_1 NC$ is formed of tC and O.

In order to determine the octahedra it is necessary to observe that as a vertex of the original net belongs to six edges, i. e.

[1]) See the quoted paper of ANDREINI, art. 47.

six members of the subject, each vertex takes six new positions, forming the six vertices of an octahedron (see rule, art. 6) whose eight faces are supplied by the expanded vertices of the eight cubes meeting in a vertex of the original net.

28. The application to fourdimensional space is simple.

For instance if the e_1 expansion be applied to a net of C_8 each C_8 is changed into an $e_1 C_8$ (Fig. 2*c*), two adjacent ones having a tC in common. As a vertex in the net C_8 belongs to eight edges (eight members of the subject) each vertex takes eight new positions which are the eight vertices of a C_{16}.

The limiting bodies of this C_{16} may be identified as follows. In the net C_8 each vertex is surrounded by 16 members. Each vertex of a C_8 is changed by expansion into a tetrahedron, so that the vertex gap in the net is surrounded by 16 tetrahedra, the limiting bodies of a C_{16}. Thus by the e_1 expansion a net of C_8 has been converted into a net $e_1 NC_8$ of two constituents, $e_1 C_8$ and C_{16}, in which two adjacent $e_1 C_8$ have a tC in common, while an $e_1 C_8$ and a C_{16} have a tetrahedron in common.

29. Again the e_2 expansion may be applied to a plane net. In this case the constituents of the net are moved apart until an edge assumes two positions, the opposite sides of a square, and the vertex gap is a polygon with as many vertices as there were constituents meeting in a point in the original net; figure 14 (*a* and *b*) shews this with regard to a net of triangles.

If the e_2 operation be applied to a net of squares, it moves apart the squares and the result is again a net of squares; but they are not all of the same kind, some being the squares of the original net, some of edge import, others of vertex import (Fig. 15). From this simple example it may be seen that the e_n expansion applied to a net of measure polytopes in n-dimensional space produces again a net of measure polytopes; but the latter is composed of constituents with different imports, and the subject of any further expansion must be suitably chosen. For instance if the $e_1 e_2$ expansion be applied to a net of squares the subject of the e_1 expansion comprises only those squares of edge import introduced by the e_2 expansion in a net of squares (Fig. 15*b*). The result is that the squares of the subject remain unchanged except in position. Those of vertex import and those corresponding to the squares of the original net are changed into octagons of different imports. The corresponding double expansion of the net of triangles is shewn in figure 14*c*.

30. If the e_2 expansion be applied to a net of cubes each cube

is changed into an *RCO*. Four of these are shewn in Fig. 16 after having been adjusted so that a face which was common to two cubes becomes common to two *RCO*.

This adjustment leaves edge gaps and vertex gaps.

As an edge belongs to four and a vertex to twelve faces (members of the subject) the edge gap is defined by four new parallel positions of an edge and the vertex gap by twelve new positions of a vertex. Therefore the first is filled by a square prism (a cube) and the second by a *CO*. In the *CO* the triangles are supplied by triangular faces of the eight *RCO* (expanded cubes) and the squares by the bases of the six prisms (expanded edges) surrounding the gap. Thus the net of cubes is changed by the e_2 transformation into a net $e_2 NC$ with the three constituents *RCO*, *C* and *CO* (A. 20) [1]).

The e_2 expansion may be applied to a net $N(O,T)$ of O and T by taking either the group of O or the group of T as independent variable, and the faces of that group as subject. Whichever group is chosen, its faces in their original position define the net $N(O,T)$, in their final position the new net. Thus if the e_2 expansion be applied to the O each O is changed into an *RCO* (Fig. 3*b*) whose triangular faces are in contact with the untransformed tetrahedra. The vertices of each O are now changed into squares (Fig. 3*b*) and as six octahedra meet in a vertex of $N(O,T)$ the vertex gap is a cube. Thus the new net $e_2 N(\underline{O},T)$ has three constituents *RCO*, *C*, *T* (Fig. 17) (A. 19).

In figure 18 is shewn the result $e_1 N(\underline{O},T) = e_1 N(O,\underline{T})$ of the e_1 expansion applied either to the octahedra or to the tetrahedra of the net (O,T).

31. In fourdimensional space an example is given of the e_2 expansion $e_2 NC_{24}$. Each C_{24} is changed into an $e_2 C_{24}$ limited by 24 *RCO*, 96 P_3, 24 *CO* (see rule, art. 9 and Fig. 19π). [2])

The *RCO* are transformed octahedra, the P_3 are expanded edges, and the *CO* expanded vertices. When the transformed C_{24} are adjusted so that an octahedron which in the regular net is common to two C_{24} is changed into an *RCO* common to two $e_2 C_{24}$, there are edge gaps and vertex gaps.

In order to facilitate the determination of these gaps it will be well to state clearly the manner in which the three kinds of limiting bodies are mutually arranged in the $e_2 C_{24}$.

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[1]) This means Fig. 20 in ANDREINI's memoir quoted in art. 1. In order to facilitate comparison a table of threedimensional nets is given on plate III.

[2]) Here and in the following figures π means "principal" constituent, while α, β, etc. stand for the polytopes filling the vertex gap, the edge gap, etc.

A shaded face $A_1B_1C_1$ common to two *RCO* (in Fig. 19) is the new position of a face *ABC* common to two octahedra in C_{24}; A_1B_1, A_2B_2, A_3B_3 are three new positions of an edge *AB* of the C_{24}, and the two positions A_1B_1, A_2B_2 in the *RCO* are identical with the two positions A_1B_1, A_2B_2 in the prism. Again, the vertices of the *CO* are the 12 positions taken by a vertex *A* of the C_{24} of which four $A_1 A_2 A_4 A_5$ are identical with four $A_1 A_2 A_4 A_5$ in the *RCO*.

In the net of C_{24} an edge is common to four and a vertex to 32 faces (members of the subject), so that the edge gap is defined by four positions of an edge and the vertex gap by 32 positions of a vertex. The limiting bodies surrounding these two gaps may be found in the following manner. Four C_{24} meet in an edge and eight in a vertex of the net C_{24}. In each, the edge is changed into a P_3 and the vertex into a *CO*. Thus among the limiting bodies surrounding the edge and vertex gaps there must be four P_3 in parallel positions in the former and 8 *CO* in the latter.

Now in the original net two adjacent C_{24}, let us say *M* & *N*, have a common octahedron, or it may be said that two octahedra, limiting bodies of two adjacent C_{24}, coincide. So in the transformed net two adjacent $e_2 C_{24}$ have an *RCO* (transformed octahedron) in common; or it may be said that two *RCO*, limiting bodies of two adjacent $e_2 C_{24}$, *M* & *N*, coincide.

Thus the *RCO* (Fig. 19π) represents two coincident limiting bodies, one belonging to *M* and the other to *N*. In each the face $(A_1 B_1, A_2 B_2)$ is in contact with a P_3 and these two P_3 can have no other point in common, or else the polytopes *M* and *N*, having already one common limiting body, an *RCO*, would coincide.

Thus two adjacent P_3 surrounding the edge gap have a square face in common. It remains now to seek a polytope which satisfies the following conditions. It must be determined by four parallel positions of an edge and have amongst its limiting bodies four parallel P_3 of which any adjacent two have a square face in common.

A fourdimensional prism on a tetrahedral base is the only body which satisfies these conditions, so that the limiting bodies are 4 P_3, 2 *T* (Fig. 19β).

Each of the tetrahedra is determined by its vertices i. e. four positions assumed by the end point of an edge of the net C_{24} and is therefore of vertex import.

As 16 edges meet in a vertex of the net C_{24}, there are 16 of these tetrahedra surrounding the vertex gap.

The limiting bodies of the polytope which must fill the vertex gap are therefore 16 tetrahedra and 8 CO (Fig. 19β). The regular net of C_{24} has thus been transformed into one of three constituents:

(1) $e_2 C_{24}$ (limited by RCO, P_3, CO), Fig. 19π

(2) P_T, Fig. 19β

(3) $c e_1 C_8$ (limited by 8 CO, 16 T), Fig. 19α.

In this net two polytopes (π) have an RCO, a (π) and a (β) have a P_3, a (π) and an (α) have a CO, and an (α) and a (β) have a T in common.

The e_3 expansion applied to a block of cubes.

32. The figure 20 shews the result $e_3 NC$ clearly. It has already been remarked that this expansion leads to a block of cubes of different kinds, some having face import (a), some edge import (b), and some vertex import (c).

In figure 21 is shewn the result of the operation $e_1 e_3 NC$; the cubes corresponding to those of the original net are changed into tC; the cubes of edge import (subject of the second operation e_1) remain cubes; those of face and vertex import are changed respectively into P_8 and RCO (A. 22).

The e_3 expansion applied to a net of C_{16}.

33. Each C_{16} is expanded according to the rule and produces a polytope limited by T, P_3, P_4, C (Fig. 22π).

When these are adjusted, so that tetrahedra which were common to two C_{16} are common to two $e_3 C_{16}$, there are face, edge, and vertex gaps; these are defined respectively by three parallel positions of a face, 12 parallel positions of an edge, and 96 positions of a vertex; since in the NC_{16} a face is common to three, an edge to 12, and a vertex to 96 tetrahedra (members of the subject). It remains only to determine the limiting bodies surrounding these gaps.

34. In order to find those of the face gap the three new parallel positions of the face ABC are represented by the triangles $A_1 B_1 C_1$, $A_2 B_2 C_2$, $A_3 B_3 C_3$ (Fig. 23).

It follows from the definition of expansion that the lines $A_1 A_2$, $A_2 A_3$, $A_3 A_1$ ..... are normal to the face ABC and equal to an edge. Thus the face gap is surrounded by two groups of three P_3; one group consists of the P_3: $A_1 B_1 C_1 A_2 B_2 C_2$, $A_2 B_2 C_2 A_3 B_3 C_3$, $A_3 B_3 C_3 A_1 B_1 C_1$ of face import and the other of $A_1 A_2 A_3 B_1 B_2 B_3$, $B_1 B_2 B_3 C_1 C_2 C_3$, $C_1 C_2 C_3 A_1 A_2 A_3$ of edge import.

The members of each group are in triangular contact with members of the same and in square contact with members of the other group.

This polytope, called a simplotope, is a special case of a group of polytopes called prismotopes [1]).

Two kinds of limiting bodies surrounding the edge gap have now been found, i. e. square prisms due to the transformed C_{16} (Fig. 22γ) and P_8 due to the expanded face (Fig. 22β); there are six of the former and eight of the latter, since six C_{16} and eight faces meet in an edge of NC_{16}. As the axes of these 14 prisms are parallel, the body must be a fourdimensional prism whose base is a CO of vertex import (since its vertices are the 12 positions taken by the end point of an edge).

The vertex gap is surrounded by cubes (π) and CO (β), and there are 24 of each since 24 C_{16} and 24 edges meet in a vertex of NC_{16}.

Thus there are four constituents in the new net $e_3 NC_{16} : e_3 C_{16}$, prismotope (3 ; 3), P_{CO}; and a polytope $e_2 C_{16}$ limited by 24 C, 24 CO.

The manner in which these different bodies are in contact is indicated by the imports in the drawings and by the vertical lines.

35. Two examples are given in order to show how a second operation may be applied to the result of a single expansion (Figs. 24 & 25).

Let it be desired to apply the e_1 expansion to the net obtained above. Here those constituents taking the place of *edges* in the original NC_{16} are the subject and must be moved unchanged into new positions. Thus the edge gap in the new net is like that in the e_3 expansion (compare Figs. 22β & 24β).

Moreover those limiting bodies of edge import in the transformed C_{16} and in the prismotope (face gap) must also remain unchanged (compare the parts π and γ of Fig. 22 and Fig. 24).

The tetrahedra (Fig. 22π) are transformed by the e_1 expansion into tT (Fig. 24π).

A careful examination of the manner in which the P_3 of face import and the cube of vertex import in the same polytope (π) are in contact with the tetrahedra will show in what manner they must be changed (see Fig. 24π). From these may be traced the changes in the face gap (γ) and vertex gap (α).

36. If it be desired to apply the e_2 expansion to $e_3 NC_{16}$ the

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[1]) Compare the foot note [2]) in art. 1.

face gap remains unchanged (Figs 22γ and 25γ), as well as the limiting body of face import in the $e_3 C_{16}$ (π).

The tetrahedron (Fig. 22π) is changed by the e_2 expansion into a CO (Fig. 25π) and again the manner in which the other limiting bodies of this polytope are affected by the change can be traced by an examination of the manner in which they are connected with the tetrahedra.

The changes in the edge and vertex gaps can also be traced (compare Figs. 22 and 25).

The polytope of vertex import in Fig. 25 is remarkable, as it is limited by 48 semiregular polyhedra *of the same kind.*

The e_4 expansion.

37. The e_4 expansion applied to a net of C_8, C_{16} or C_{24} separates the adjacent constituents by a distance equal to an edge. Thus two neighbouring members of a block are separated by a fourdimensional prism whose two opposite bases are the two limiting bodies that coincided in the regular net. The net of C_8 so treated results in another net of C_8 of different imports.

The net of C_{16} transformed by the e_4 expansion leads to the following result. The C_{16} are separated, so that instead of two having a tetrahedron in common they are separated by a distance equal to an edge.

In other words the tetrahedron common to two adjacent C_{16} has assumed two parallel positions, the bases of a fourdimensional prism (Fig. 26δ).

The side limiting bodies of this fourdimensional prism are four P_3 (of face import). As three C_{16} meet in a face in the net of C_{16} each face must assume three positions which define a prismotope (3 ; 3) (Fig. 26γ).

Again six C_{16} meet in an edge of the net, therefore each edge takes six positions, i. e. the new positions are the side edges of a fourdimensional prism on an octahedral base (β). It may be seen by (π), (δ), (γ) and (β) that only one of these four polytopes possesses a limiting body with vertex import, i. e. the one filling the edge gap (β), so that the vertex gap is surrounded by octahedra, and as in the net of C_{16} there are 24 edges meeting in a vertex it follows that 24 octahedra surround the vertex gap; that is, it is a C_{24}. This new net evidently may also be obtained by applying the e_4 expansion to the net NC_{24}.

38. The foregoing investigation leads to the following conclusion as to the nets of fourdimensional space.

If the edges are the subject there are only vertex gaps.

If the faces are the subject there are edge and vertex gaps.

If the limiting bodies are the subject there are face, edge, and vertex gaps.

If the constituents are the subject there are body, face, edge, and vertex gaps.

The vertex gaps are filled by polytopes determined by their vertices. Their limiting bodies are regular or semiregular polyhedra.

The edge gaps are filled by fourdimensional prisms determined by edges parallel to their axes. Their bases are either regular or semiregular polyhedra and their other limiting bodies are prisms.

The face gaps are filled by prismotopes determined by parallel positions of a face and are limited by two groups of prisms.

The body gaps are filled by fourdimensional prisms determined by two parallel positions of a regular or semiregular polyhedron.

Contraction applied to the nets.

39. One or two examples will suffice to shew the application of this process to the nets.

If in the net $e_1 N(O,T)$ (Fig. 18) (A. 24) the CO corresponding to the vertices of the original octahedra be made the subject of contraction, the tO are reduced to CO, the tT to O, while the CO remain unchanged. Thus $ce_1 N(O,T)$ denotes a net composed of O and CO (A. 18).

40. In the net $e_2 NC_{24}$ (Fig. 19) the polytopes filling the vertex gap (α) may be made the subject of contraction, when the following changes take place. The polytope α remains unchanged except in position; the prism β is reduced to a tetrahedron common to two of the polytopes α; the CO of π remain unchanged while the RCO are reduced to cubes. Thus the net of three constituents is reduced to one of two constituents, one limited by 8 CO and 16 T, the other by 24 C and 24 CO.

Tables.

41. The chief results of this memoir are tabulated in the Tables I and II.

Table I gives the 48 polytopes of expansion (the regular polytopes included) and the 42 polytopes of contraction. The first set has been numbered from 1 to 48; if p stands for any number, p' of the second set is obtained by application of the operation $c\,(=c_0)$ to p of the first set. The first set consists of 39 different polytopes; the second set contains only eight new ones.

Table II gives the 48 nets of expansion (the regular nets included) and of the nets of contraction only the seven new ones, so altogether $39 + 7$ i. e. 46 fourdimensional nets.

Table III gives the nets of threedimensional space and a table of incidences.

TABLE OF POLYTOPES IN S_4. I.

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
	Expansion.					
		5	10	10	5	
1	C_5	T	—	—	—	
2	e_1C_5	tT	—	—	T	
3	e_2C_5	CO	—	P_3	O	
4	e_3C_5	T	P_3	P_3	T	
5	$e_1e_2C_5$	tO	—	P_3	tT	
6	$e_1e_3C_5$	tT	P_6	P_3	CO	= 7
7	$e_2e_3C_5$	CO	P_3	P_6	tT	= 6
8	$e_1e_2e_3C_5$	tO	P_6	P_6	tO	
		24	96	96	24	
25	C_{24}	O	—	—	—	
26	e_1C_{24}	tO	—	—	C	= 21
27	e_2C_{24}	RCO	—	P_3	CO	
28	e_3C_{24}	O	P_3	P_3	O	
29	$e_1e_2C_{24}$	tCO	—	P_3	tC	
30	$e_1e_3C_{24}$	tO	P_6	P_3	RCO	= 31
31	$e_2e_3C_{24}$	RCO	P_3	P_6	tO	= 30
32	$e_1e_2e_3C_{24}$	tCO	P_6	P_6	tCO	
	Contraction.					
		5	10	10	5	
2′	ce_1C_5	O	—	—	T	= 3′
3′	ce_2C_5	T	—	—	O	= 2′
4′	ce_3C_5	—	—	—	T	= 1
5′	$ce_1e_2C_5$	tT	—	—	tT	
6′	$ce_1e_3C_5$	O	P_3	—	CO	= 3
7′	$ce_2e_3C_5$	T	—	—	tT	= 2
8′	$ce_1e_2e_3C_5$	tT	P_3	—	tO	= 5
		24	96	96	24	
26′	ce_1C_{24}	CO	—	—	C	= 19
27′	ce_2C_{24}	C	—	—	CO	= 19
28′	ce_3C_{24}	—	—	—	O	= 25
29′	$ce_1e_2C_{24}$	tC	—	—	tC	
30′	$ce_1e_3C_{24}$	CO	P_3	—	RCO	= 27
31′	$ce_2e_3C_{24}$	C	—	—	tO	= 26
32′	$ce_1e_2e_3C_{24}$	tC	P_3	—	tCO	= 29

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
	Expansion.					
		8	24	32	16	
9	C_8	C	—	—	—	
10	e_1C_8	tC	—	—	T	
11	e_2C_8	RCO	—	P_3	O	
12	e_3C_8	C	P_4	P_3	T	= 20
13	$e_1e_2C_8$	tCO	—	P_3	tT	
14	$e_1e_3C_8$	tC	P_8	P_3	CO	= 23
15	$e_2e_3C_8$	RCO	P_4	P_6	tT	= 22
16	$e_1e_2e_3C_8$	tCO	P_8	P_6	tO	= 24
		120	720	1200	600	
33	C_{120}	D	—	—	—	
34	e_1C_{120}	tD	—	—	T	
35	e_2C_{120}	RID	—	P_3	O	
36	e_3C_{120}	D	P_5	P_3	T	= 44
37	$e_1e_2C_{120}$	tID	—	P_3	tT	
38	$e_1e_3C_{120}$	tD	P_{10}	P_3	CO	
39	$e_2e_3C_{120}$	RID	P_5	P_6	tT	
40	$e_1e_2e_3C_{120}$	tID	P_{10}	P_6	tO	= 48
	Contraction.					
		8	24	32	16	
10′	ce_1C_8	CO	—	—	T	= 19′
11′	ce_2C_8	O	—	—	O	= 25
12′	ce_3C_8	—	—	—	T	= 17
13′	$ce_1e_2C_8$	tO	—	—	tT	= 21′
14′	$ce_1e_3C_8$	CO	P_4	—	CO	= 19
15′	$ce_2e_3C_8$	O	—	—	tT	= 18
16′	$ce_1e_2e_3C_8$	tO	P_4	—	tO	= 21
		120	720	1200	600	
34′	ce_1C_{120}	ID	—	—	T	= 43′
35′	ce_2C_{120}	I	—	—	O	= 42′
36′	ce_3C_{120}	—	—	—	T	= 41
37′	$ce_1e_2C_{120}$	tI	—	—	tT	= 45′
38′	$ce_1e_3C_{120}$	ID	P_5	—	CO	= 43
39′	$ce_2e_3C_{120}$	I	—	—	tT	= 42
40′	$ce_1e_2e_3C_{120}$	tI	P_5	—	tO	= 45

Number	Symbol of expansion	Limiting bodies & import: body	face	edge	vertex	
	Expansion.					
		16	32	24	8	
17	C_{16}	T	—	—	—	
18	e_1C_{16}	tT	—	—	O	
19	e_2C_{16}	CO	—	P_4	CO	
20	e_3C_{16}	T	P_3	P_4	C	= 12
21	$e_1e_2C_{16}$	tO	—	P_4	tO	= 26
22	$e_1e_3C_{16}$	tT	P_6	P_4	RCO	= 15
23	$e_2e_3C_{16}$	CO	P_3	P_8	tC	= 14
24	$e_1e_2e_3C_{16}$	tO	P_6	P_8	tCO	= 16
		600	1200	720	120	
41	C_{600}	T	—	—	—	
42	e_1C_{600}	tT	—	—	I	
43	e_2C_{600}	CO	—	P_5	ID	
44	e_3C_{600}	T	P_3	P_5	D	= 36
45	$e_1e_2C_{600}$	tO	—	P_5	tI	
46	$e_1e_3C_{600}$	tT	P_6	P_5	RID	
47	$e_2e_3C_{600}$	CO	P_3	P_{10}	tD	
48	$e_1e_2e_3C_{600}$	tO	P_6	P_{10}	tID	= 40
	Contraction.					
		16	32	24	8	
18′	ce_1C_{16}	O	—	—	O	= 25
19′	ce_2C_{16}	T	—	—	CO	= 10′
20′	ce_3C_{16}	—	—	—	C	= 9
21′	$ce_1e_2C_{16}$	tT	—	—	tO	= 13′
22′	$ce_1e_3C_{16}$	O	P_3	—	RCO	= 11
23′	$ce_2e_3C_{16}$	T	—	—	tC	= 10
24′	$ce_1e_2e_3C_{16}$	tT	P_3	—	tCO	= 13
		600	1200	720	120	
42′	ce_1C_{600}	O	—	—	I	= 35′
43′	ce_2C_{600}	T	—	—	ID	= 34′
44′	ce_3C_{600}	—	—	—	D	= 33
45′	$ce_1e_2C_{600}$	tT	—	—	tI	= 37′
46′	$ce_1e_3C_{690}$	O	P_3	—	RID	= 35
47′	$ce_2e_3C_{600}$	T	—	—	tD	= 34
48′	$ce_1e_2e_3C_{600}$	tT	P_3	—	tID	= 37

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
			Expansion.				
1	NC_8	C_8	—	—	—	—	
2	e_1NC_8	e_1C_8	—	—	—	C_{16}	
3	e_2NC_8	e_2C_8	—	—	P_O	C_{24}	
4	e_3NC_8	e_3C_8	—	(4 ; 4)	P_{CO}	ce_1C_8	
5	e_4NC_8	C_8	P_C	(4 ; 4)	P_C	C_8	
6	$e_1e_2NC_8$	$e_1e_2C_8$	—	—	P_O	e_1C_{16}	
7	$e_1e_3NC_8$	$e_1e_3C_8$	—	(4 ; 8)	P_{CO}	e_2C_{16}	
8	$e_1e_4NC_8$	e_1C_8	P_{tC}	(4 ; 8)	P_C	e_3C_8	= 11
9	$e_2e_3NC_8$	$e_2e_3C_8$	—	(4 ; 4)	P_{tO}	$ce_1e_2C_8$	
10	$e_2e_4NC_8$	e_2C_8	P_{RCO}	(4 ; 4)	P_{RCO}	e_2C_8	
11	$e_3e_4NC_8$	e_3C_8	P_C	(8 ; 4)	P_{tC}	e_1C_8	= 8
12	$e_1e_2e_3NC_8$	$e_1e_2e_3C_8$	—	(4 ; 8)	P_{tO}	e_1C_{24}	
13	$e_1e_2e_4NC_8$	$e_1e_2C_8$	P_{tCO}	(4 ; 8)	P_{RCO}	$e_2e_3C_8$	= 15
14	$e_1e_3e_4NC_8$	$e_1e_3C_8$	P_{tC}	(8 ; 8)	P_{tC}	$e_1e_3C_8$	
15	$e_2e_3e_4NC_8$	$e_2e_3C_8$	P_{RCO}	(8 ; 4)	P_{tCO}	$e_1e_2C_8$	= 13
16	$e_1e_2e_3e_4NC_8$	$e_1e_2e_3C_8$	P_{tCO}	(8 ; 8)	P_{tCO}	$e_1e_2e_3C_8$	
			Contraction.				
49	ce_1NC_8	ce_1C_8	—	—	—	C_{16}	
50	$ce_1e_2NC_8$	$ce_1e_2C_8$	—	—	—	e_1C_{16}	

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
			Expansion.				
17	NC_{16}	C_{16}	—	—	—	—	
18	e_1NC_{16}	e_1C_{16}	—	—	—	C_{24}	
19	e_2NC_{16}	e_2C_{16}	—	—	P_C	e_2C_{16}	
20	e_3NC_{16}	e_3C_{16}	—	(3 ; 3)	P_{CO}	e_2C_{16}	
21	e_4NC_{16}	C_{16}	P_T	(3 ; 3)	P_O	C_{24}	= 37
22	$e_1e_2NC_{16}$	$e_1e_2C_{16}$	—	—	P_C	$e_1e_2C_{16}$	
23	$e_1e_3NC_{16}$	$e_1e_3C_{16}$	—	(3 ; 6)	P_{CO}	e_2C_{24}	
24	$e_1e_4NC_{16}$	e_1C_{16}	P_{tT}	(3 ; 6)	P_O	e_3C_{24}	= 43
25	$e_2e_3NC_{16}$	$e_2e_3C_{16}$	—	(3 ; 3)	P_{tC}	$ce_1e_2C_{24}$	
26	$e_2e_4NC_{16}$	e_2C_{16}	P_{CO}	(3 ; 3)	P_{RCO}	e_2C_{24}	= 42
27	$e_3e_4NC_{16}$	e_3C_{16}	—	(6 ; 3)	P_{tO}	$e_1e_2C_{16}$	= 40
28	$e_1e_2e_3NC_{16}$	$e_1e_2e_3C_{16}$	P_T	(3 ; 6)	P_{tC}	$e_1e_2C_{24}$	
29	$e_1e_2e_4NC_{16}$	$e_1e_2C_{16}$	P_{tO}	(3 ; 6)	P_{RCO}	$e_1e_3C_{24}$	
30	$e_1e_3e_4NC_{16}$	$e_1e_3C_{16}$	P_{tT}	(6 ; 6)	P_{tO}	$e_1e_3C_{24}$	= 46
31	$e_2e_3e_4NC_{16}$	$e_2e_3C_{16}$	P_{CO}	(6 ; 3)	P_{tCO}	$e_1e_2C_{24}$	= 45
32	$e_1e_2e_3e_4NC_{16}$	$e_1e_2e_3C_{16}$	P_{tO}	(6 ; 6)	P_{tCO}	$e_1e_2e_3C_{24}$	= 48
			Contraction.				
51	ce_2NC_{16}	ce_2C_{16}	—	—	—	e_2C_{16}	
52	$ce_1e_2NC_{16}$	$ce_1e_2C_{16}$	—	—	—	$e_1e_2C_{16}$	
53	$ce_1e_3NC_{16}$	$ce_1e_3C_{16}$	—	(3 ; 3)	—	e_2C_{24}	
54	$ce_2e_3NC_{16}$	$ce_2e_3C_{16}$	—	—	—	$ce_1e_2C_{24}$	
55	$ce_1e_2e_3NC_{16}$	$ce_1e_2e_3C_{16}$	—	(3 ; 3)	—	$c_1e_2C_{24}$	

Number	Symbol of expansion	Transformed original constituent	Gaps: body	Gaps: face	Gaps: edge	Gaps: vertex	
			Expansion.				
33	NC_{24}	C_{24}	—	—	—	—	
34	e_1NC_{24}	e_1C_{24}	—	—	—	C_8	
35	e_2NC_{24}	e_2C_{24}	—	—	P_T	ce_1C_8	
36	e_3NC_{24}	e_3C_{24}	—	(3 ; 3)	P_O	C_{24}	
37	e_4NC_{24}	C_{24}	P_O	(3 ; 3)	P_T	C_{16}	= 21
38	$e_1e_2NC_{24}$	$e_1e_2C_{24}$	—	—	P_T	e_1C_8	
39	$e_1e_3NC_{24}$	$e_1e_3C_{24}$	—	(3 ; 6)	P_O	e_2C_8	
40	$e_1e_4NC_{24}$	e_1C_{24}	P_{tO}	(3 ; 6)	P_T	e_3C_8	= 27
41	$e_2e_3NC_{24}$	$e_2e_3C_{24}$	—	(3 ; 3)	P_{tT}	$ce_1e_2C_8$	
42	$e_2e_4NC_{24}$	e_2C_{24}	P_{RCO}	(3 ; 3)	P_{CO}	e_2C_{16}	= 26
43	$e_3e_4NC_{24}$	e_3C_{24}	P_O	(6 ; 3)	P_{tT}	e_1C_{16}	= 24
44	$e_1e_2e_3NC_{24}$	$e_1e_2e_3C_{24}$	—	(3 ; 6)	P_{tT}	$e_1e_2C_8$	
45	$e_1e_2e_4NC_{24}$	$e_1e_2C_{24}$	P_{tCO}	(3 ; 6)	P_{CO}	$e_1e_3C_8$	= 31
46	$e_1e_3e_4NC_{24}$	$e_1e_3C_{24}$	P_{tO}	(6 ; 6)	P_{tT}	$e_2e_3C_8$	= 30
47	$e_2e_3e_4NC_{24}$	$e_2e_3C_{24}$	P_{RCO}	(6 ; 3)	P_{tO}	$e_1e_2C_{16}$	
48	$e_1e_2e_3e_4NC_{24}$	$e_1e_2e_3C_{24}$	P_{tCO}	(6 ; 6)	P_{tO}	$e_1e_2e_3C_{16}$	= 32

TABLE OF NETS IN S_3. III.

Andreini number	Symbol	Transformed original constituents	Gaps			Andreini number	Symbol	Transformed original constituents	Gaps		
			face	edge	vertex				face	edge	vertex
	NC	C				12	$N(O,T)$	O,T			
17	e_1NC	tC			O	24	$e_1N(O,T)$	tO,tT			CO
20	e_2NC	RCO		P_4	CO	19	$e_2N(\underline{O},T)$	RCO,T			C
	e_3NC	C	P_4	P_4	C	23	$e_1e_3N(\underline{O},T)$	tCO,tT			tC
21	e_1e_2NC	tCO		P_4	tO	14	$e_1e_2N(O,\underline{T})$	tO,tO			tO
22	e_1e_3NC	tC	P_8	P_4	RCO	15	$\frac{1}{2}e_2N(\underline{O},T)$	tT,T			T
22	e_2e_3NC	RCO	P_4	P_8	tC						
24^{bis}	$e_1e_2e_3NC$	tCO	P_8	P_8	tCO						
18	ce_1NC	CO			O						

TABLE OF INCIDENCES.

	Single polytopes						Nets		
	C_5	C_8	C_{16}	C_{24}	C_{120}	C_{600}	C_8	C_{16}	C_{24}
Cells meeting in a body . .							2	2	2
〃 〃 〃 〃 face. . .							4	3	3
〃 〃 〃 an edge . .							8	6	4
〃 〃 〃 a vertex. .							16	24	8
Bodies 〃 〃 〃 face. . .	2	2	2	2	2	2	4	3	3
〃 〃 〃 an edge . .	3	3	4	3	3	5	12	12	6
〃 〃 〃 a vertex .	4	4	8	6	4	20	32	96	24
Faces 〃 〃 an edge. .	3	3	4	3	3	5	6	8	4
〃 〃 〃 a vertex. .	6	6	12	12	6	30	24	96	32
Edges 〃 〃 〃 〃 . .	4	4	6	8	4	12	8	24	16

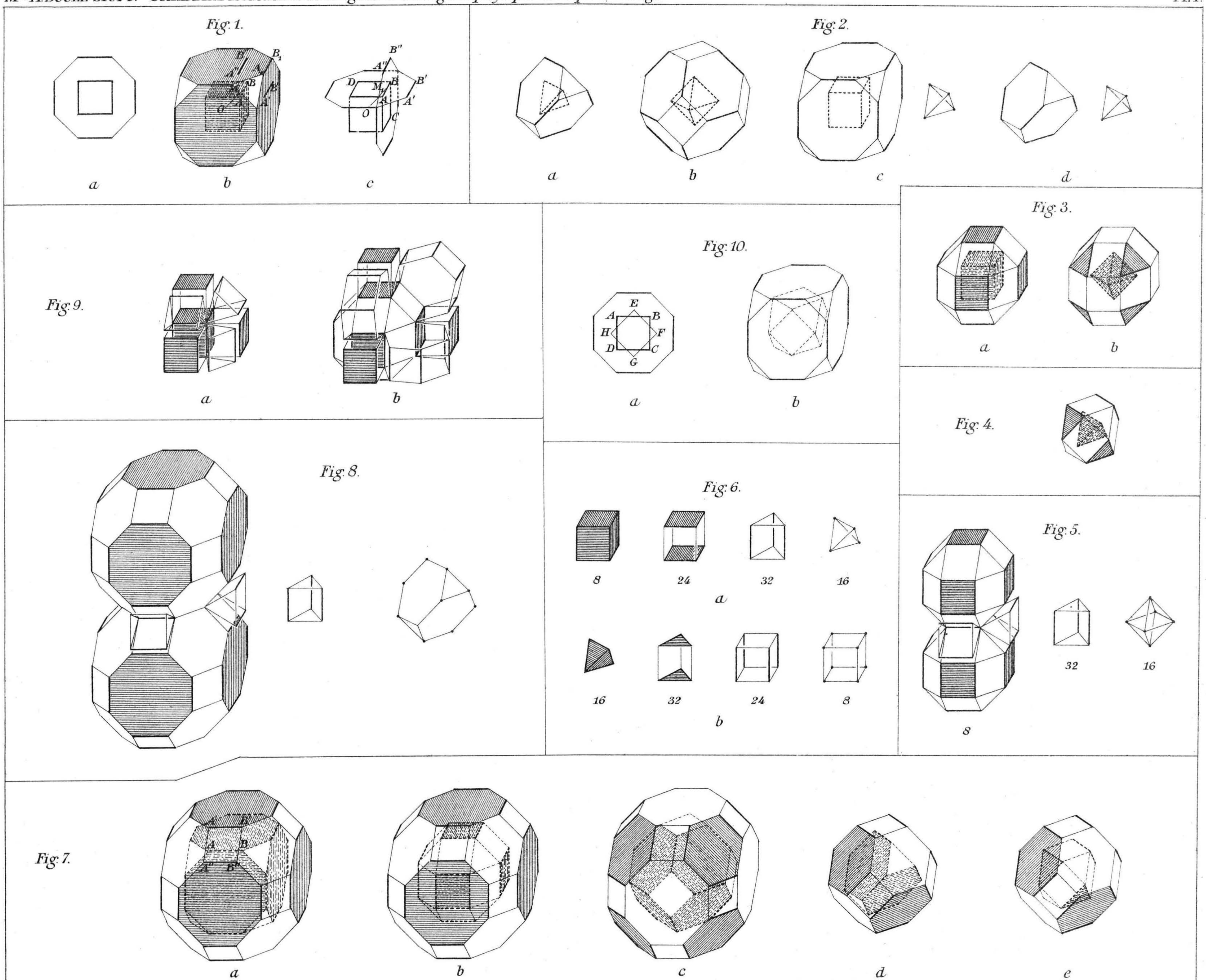

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Fig. 11.

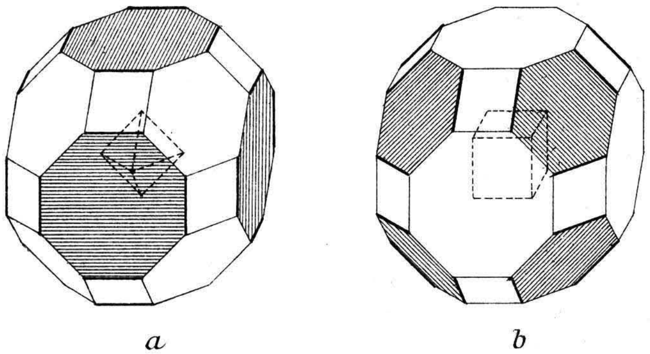


Fig. 12.

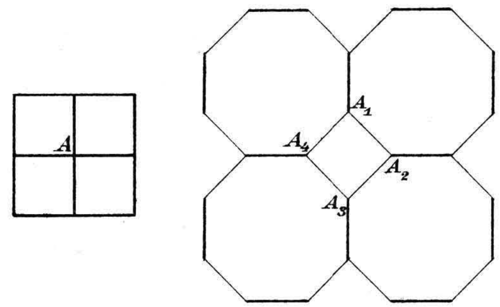


Fig. 13.

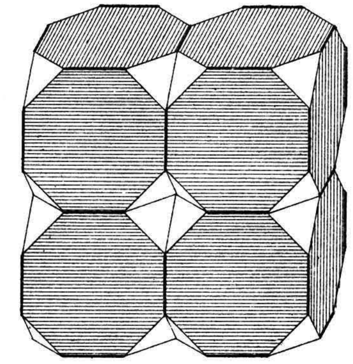


Fig. 14.

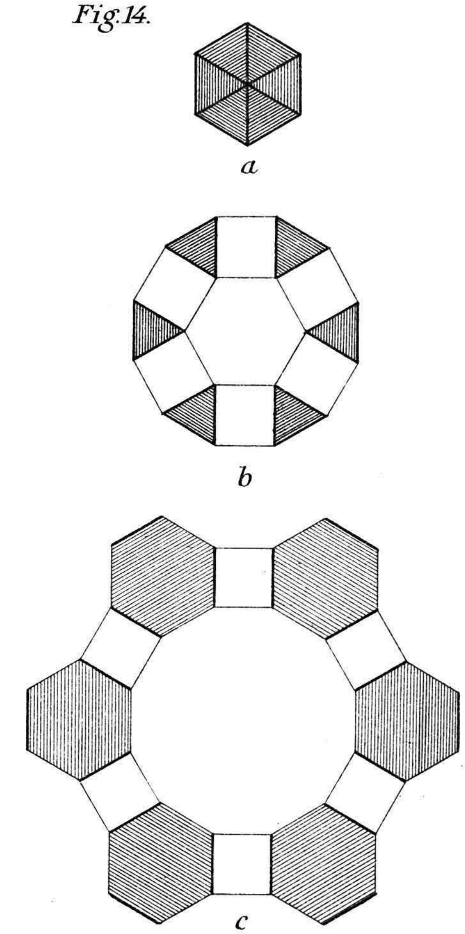


Fig. 19.

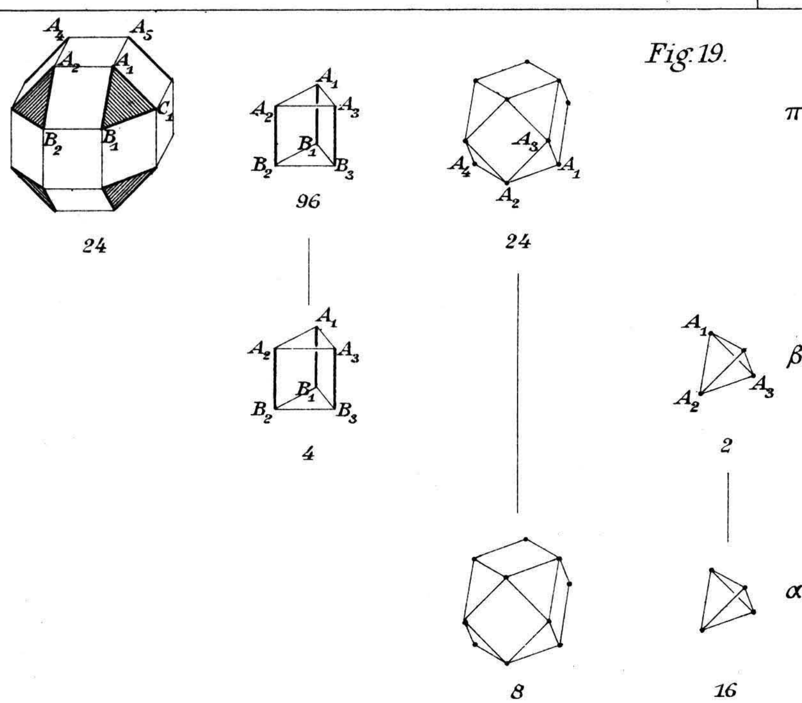


Fig. 18.

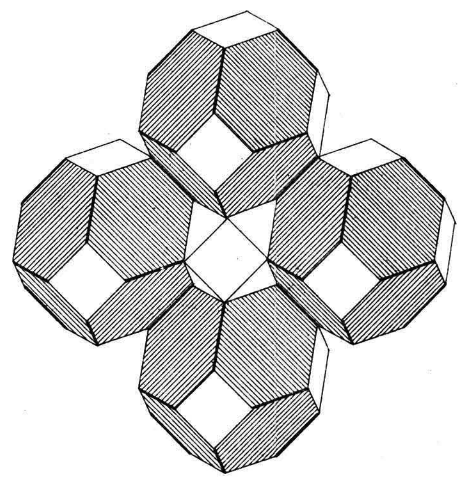


Fig. 17.

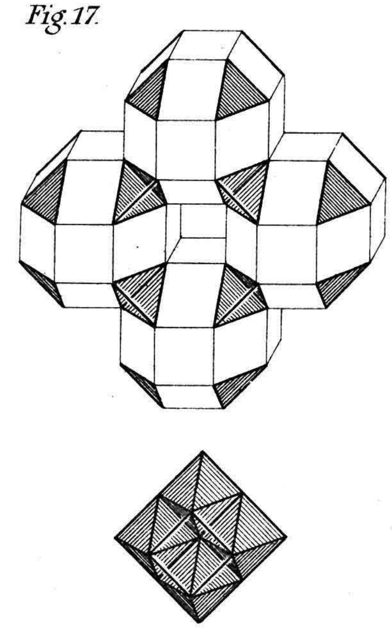


Fig. 15.

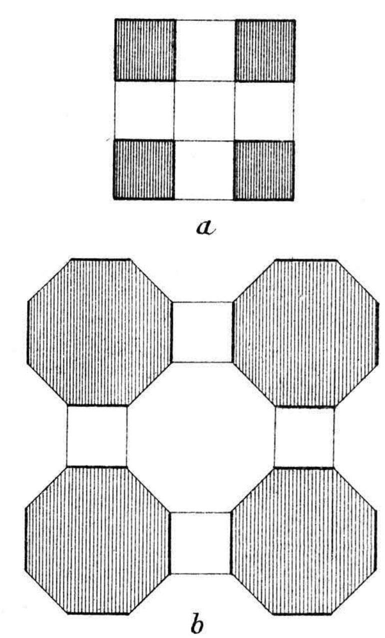


Fig. 20.

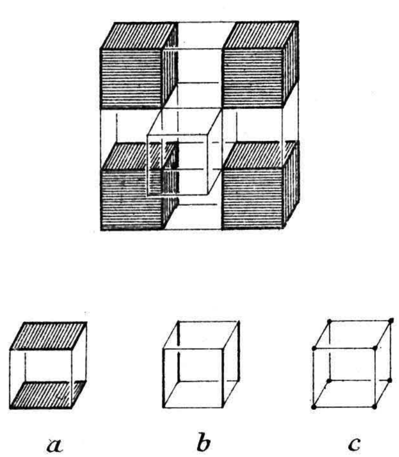


Fig. 21.

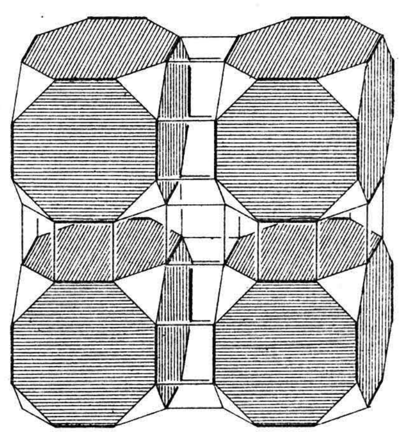
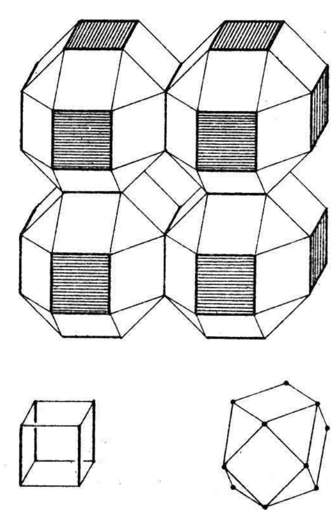


Fig. 16.



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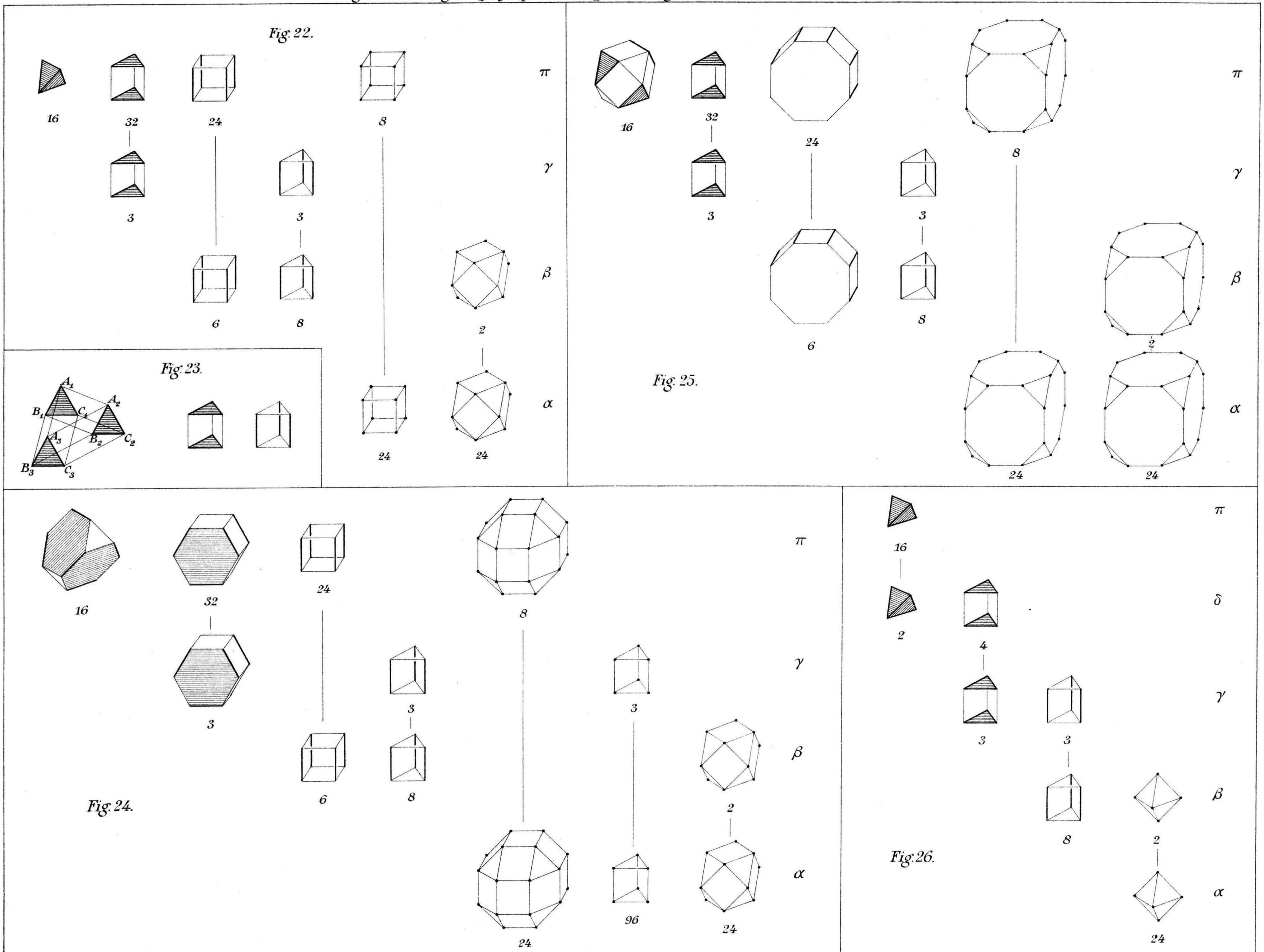

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