Welcome to

Aakash + BYJU'S NOTES

System of Particles and Rotational Motion

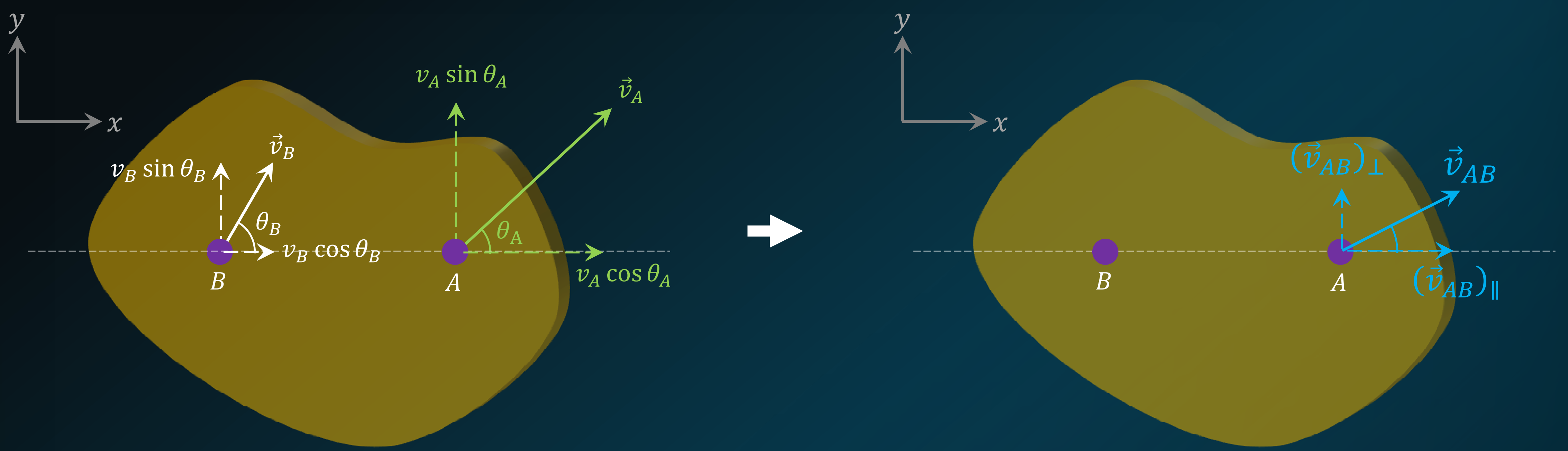
Rigid Body

Initial Final

- Shape and size of the system remains same.
- No change in the distance between any pair of particles.
- No velocity of separation or approach between any two particles.

No change in the distance between any pair of particles.

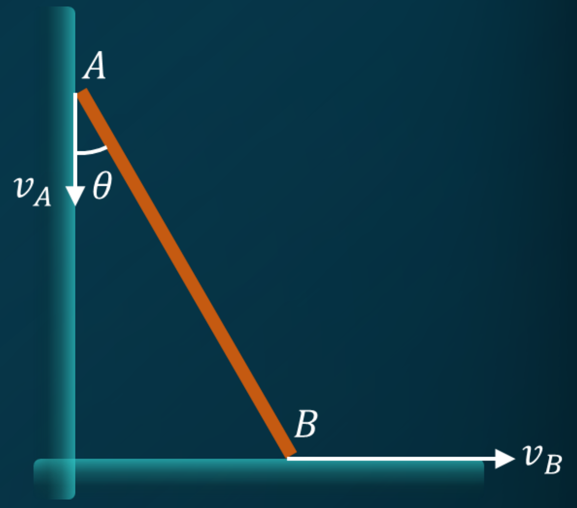
Rigid Body



No velocity of separation or approach between the particles.

- $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$
- $\vec{v}_{AB} = (v_A \cos\theta_A - v_B \cos\theta_B)\hat{\imath} + (v_A \sin\theta_A - v_B \sin\theta_B)\hat{\jmath}$
- $v_{sep} = (v_{AB})_\parallel = 0 \Rightarrow$ $v_A \cos\theta_A = v_B \cos\theta_B$

The velocity of end A of a rigid rod placed between two smooth perpendicular surfaces moves with velocity $10\ m/s$ along the vertical when the angle $\theta = 30°$. Velocity of end B at that exact moment is



Solution : Velocity of separation between the particles at the ends of the rod must be zero since it is rigid.

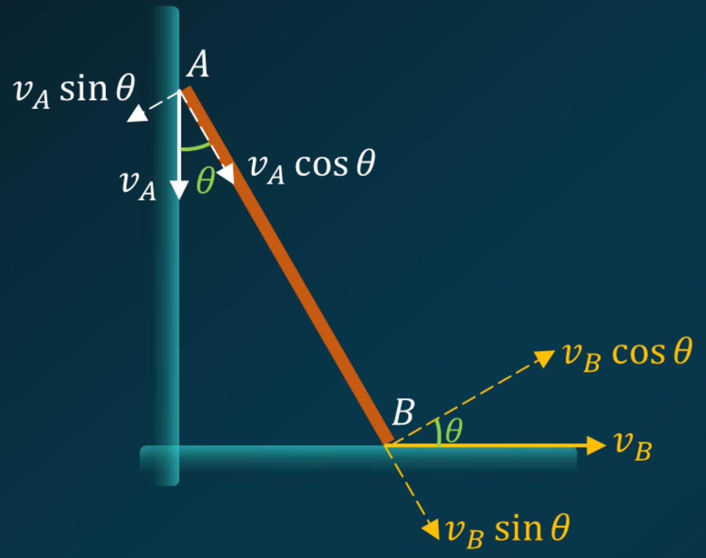
$$v_{sep} = 0$$

$$\Rightarrow\ v_A \cos\theta = v_B \sin\theta$$

$$\Rightarrow 10 \cos 30° = v_B \sin 30°$$

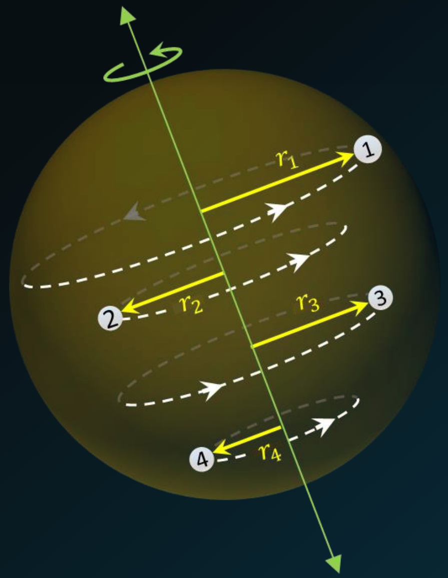
$$\Rightarrow 10 \times \frac{\sqrt{3}}{2} = v_B \times \frac{1}{2}$$

$$\Rightarrow v_B = 10\sqrt{3}\ m/s$$



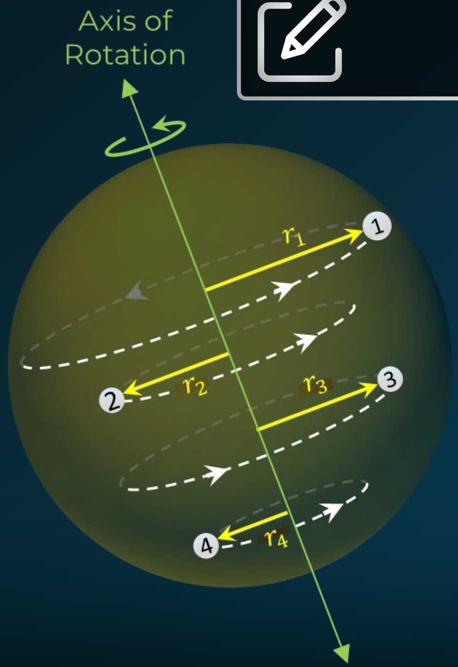
- a) $10\sqrt{3}\ m/s$
- b) $\frac{10}{\sqrt{3}}\ m/s$
- c) $5\sqrt{3}\ m/s$
- d) $\frac{5}{\sqrt{3}}\ m/s$

Circular v/s Rotational Motion



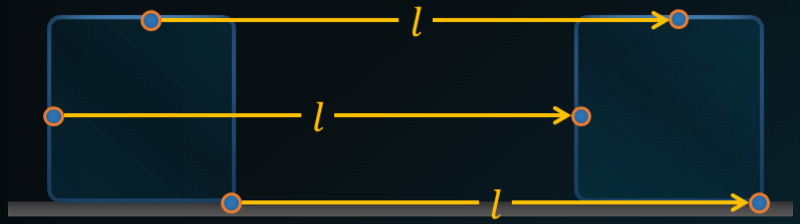
- A circular motion is generally defined for a particle.
- The term rotational motion is used in the case of an extended body.

Axis of Rotation



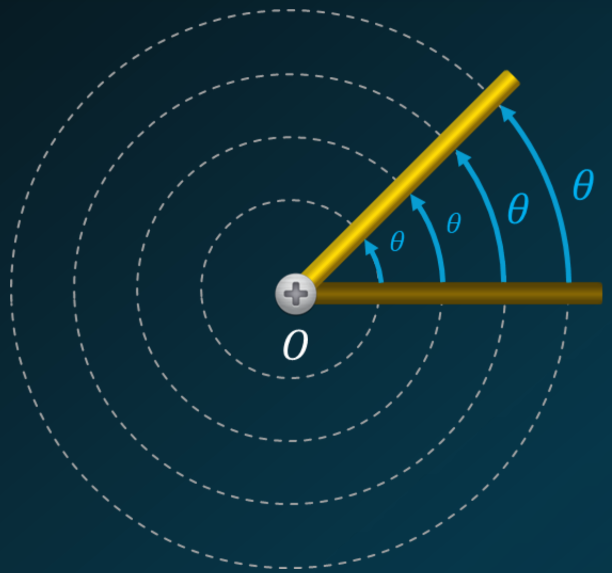
- AOR is the straight line passing through all the fixed points of a rotating rigid body around which all other points of the body move in circles.
- It does not have to pass through the body.
- It does not have to be fixed.
- It does not have to be perpendicular to the surface plane of a two-dimensional object.

Types of Rigid Body Motion



Pure Translational Motion

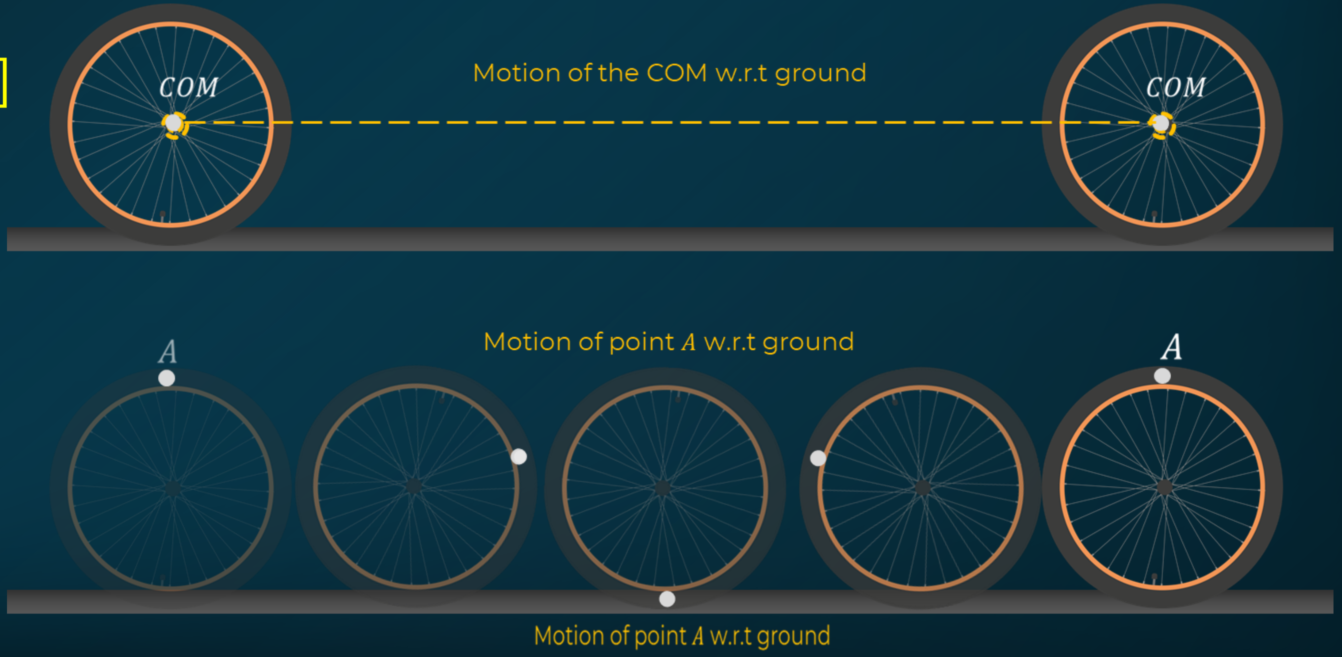
Displacement of each particle within a particular time interval is same.



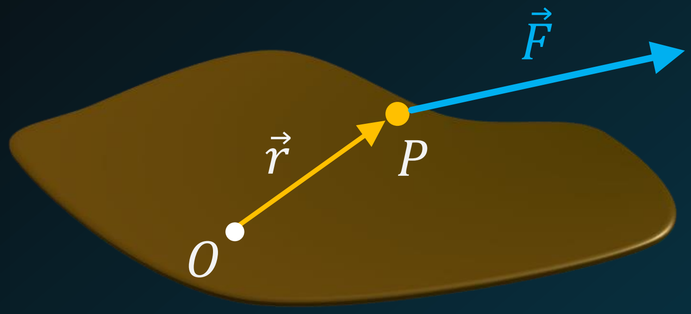
Pure Rotational Motion

Angular displacement of each particle within a particular time interval is same.

Combined Motion [Translation + Rotation]



Torque



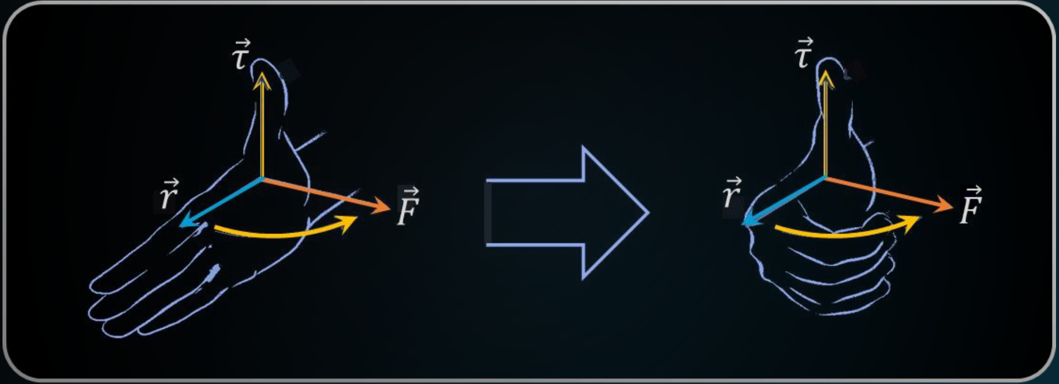
- It is the rotational analogue of Force.
- Represented by Greek letter τ (Tau)
- Mathematically called as Moment of Force
- Torque of the force $\vec{F}$ on the system about point O is given by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where,

$\vec{r}$ = Position vector of the point of application of force w.r.t. point O

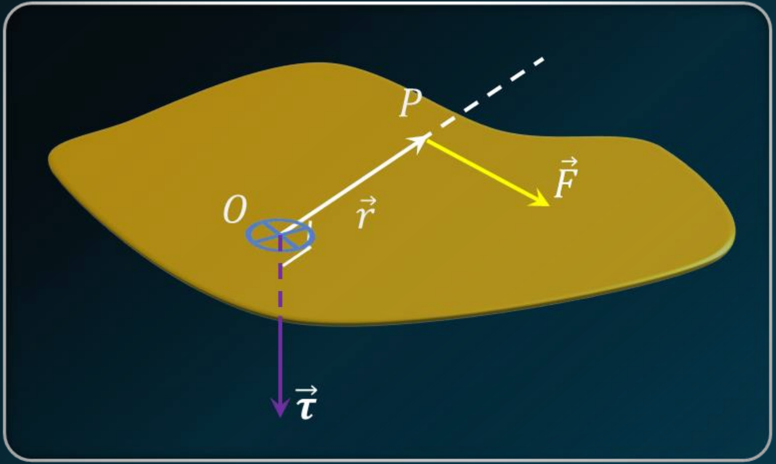
Direction of Torque



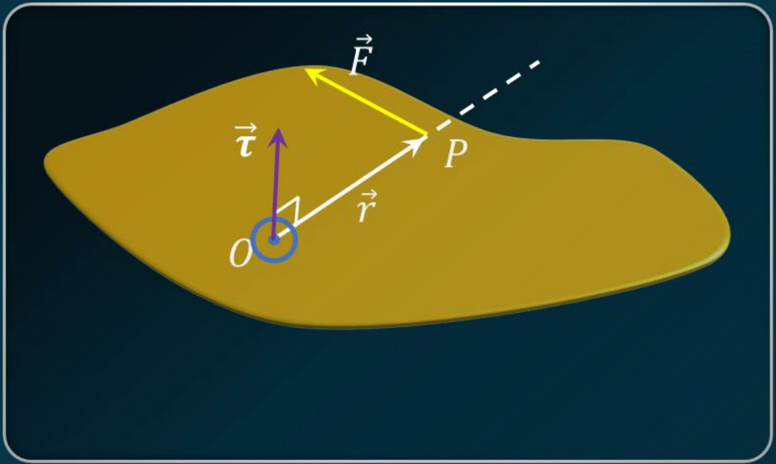
- Torque is an axial vector.
- Direction is determined using the right-hand thumb rule.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

If $\vec{r}$ and $\vec{F}$ are in a plane, then the direction of the $\vec{\tau}$ will be perpendicular to the plane.

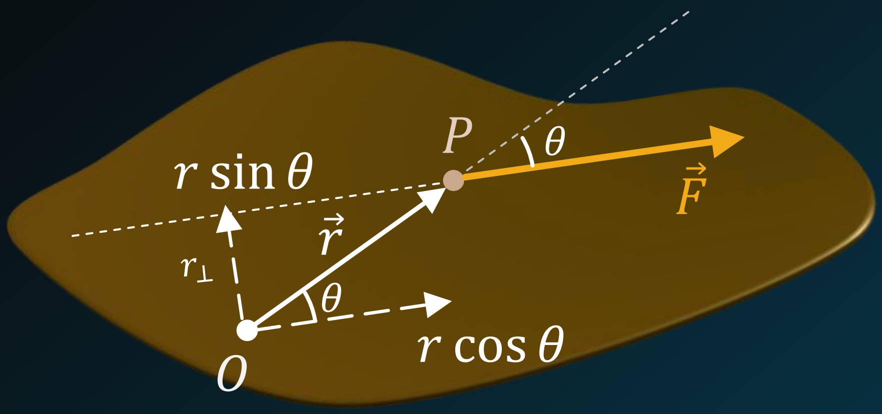


Going into the plane : ⊗



Coming out of the plane : ⊙

Magnitude of Torque

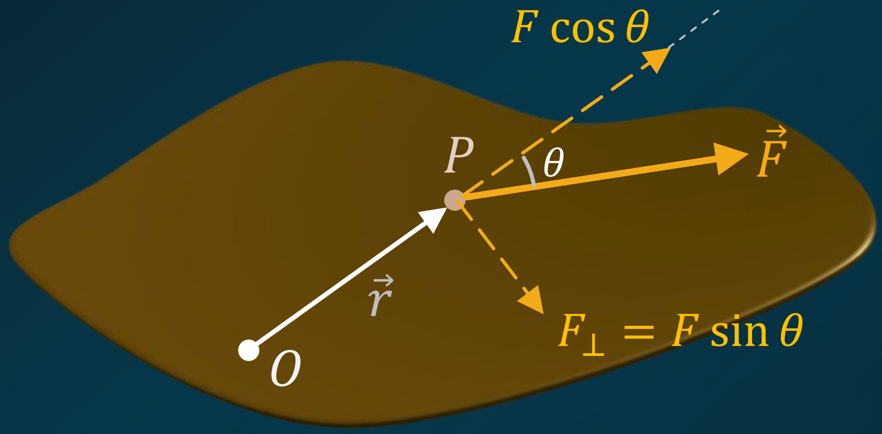


$$\tau = |\vec{r} \times \vec{F}| = rF\sin\theta = (r\sin\theta)F$$

$$\Rightarrow \boxed{\tau = r_\perp F}$$

F –Applied Force

$r_\perp$ – Force arm



$$\tau = |\vec{r} \times \vec{F}| = rF\sin\theta = r(F\sin\theta)$$

$$\Rightarrow \boxed{\tau = rF_\perp}$$

$F_\perp$ – Perpendicular component of applied Force

A particle of mass $2\ kg$ is projected with speed $u = 10\ m/s$ at angle $\theta = 30°$ with horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.

Solution : $m = 2\ kg$

Torque about the point of projection,

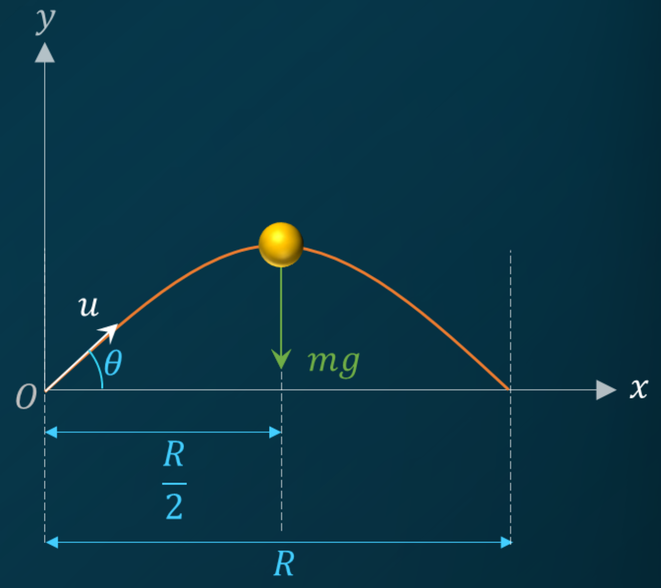
$$\tau = r_{\perp}F$$

$$\Rightarrow \tau = \left(\frac{R}{2}\right)mg$$

$$\Rightarrow \tau = \left(\frac{\left(\frac{u^2 \sin 2\theta}{g}\right)}{2}\right)mg \qquad \left[\because R = \frac{u^2 \sin 2\theta}{g}\right]$$

$$\Rightarrow \tau = \left(\frac{10^2 \sin 60°}{2g}\right) \times 2g$$

$$\Rightarrow \tau = 50\sqrt{3}\ N\ m$$



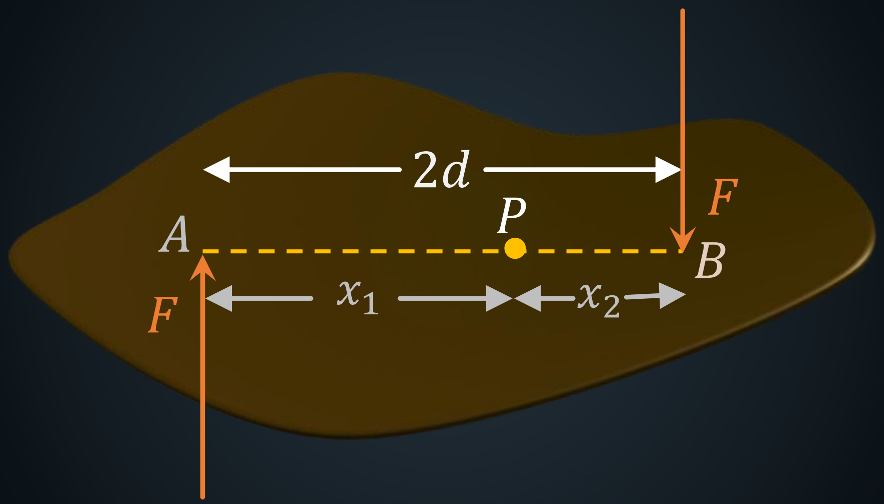
a) $100\ N\ m$

b) $50\ N\ m$

c) $50\sqrt{3}\ N\ m$

d) $100\sqrt{3}\ N\ m$

Force Couple



Consider clockwise direction as $+ve$.

Torque about point P,

$$\tau = Fx_1 + Fx_2$$

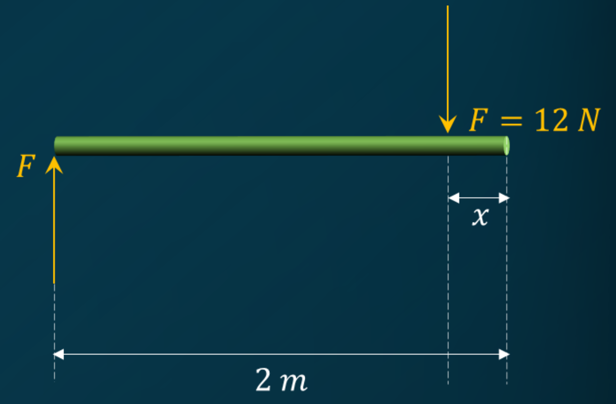
$$\Rightarrow \tau = F(x_1 + x_2)$$

$$\Rightarrow \tau = F(2d)$$

$$\tau = 2Fd$$

Note: Torque is independent of x_1, x_2

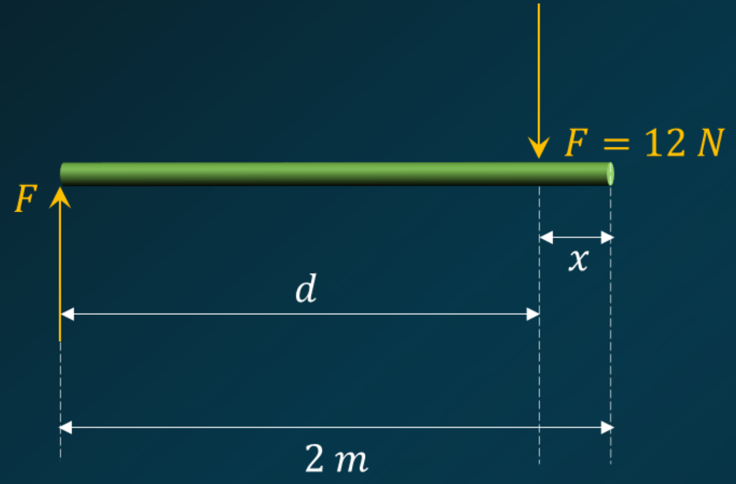
If the torque due to the couple in the given figure is $21\ N\ m$, then the value of x is



Solution : Torque due to the couple, $\tau = Fd$

$\Rightarrow 21 = 12 \times d$

$\Rightarrow d = 1.75\ m$



Now, $d + x = 2\ m$

$\Rightarrow x = 12 - d$

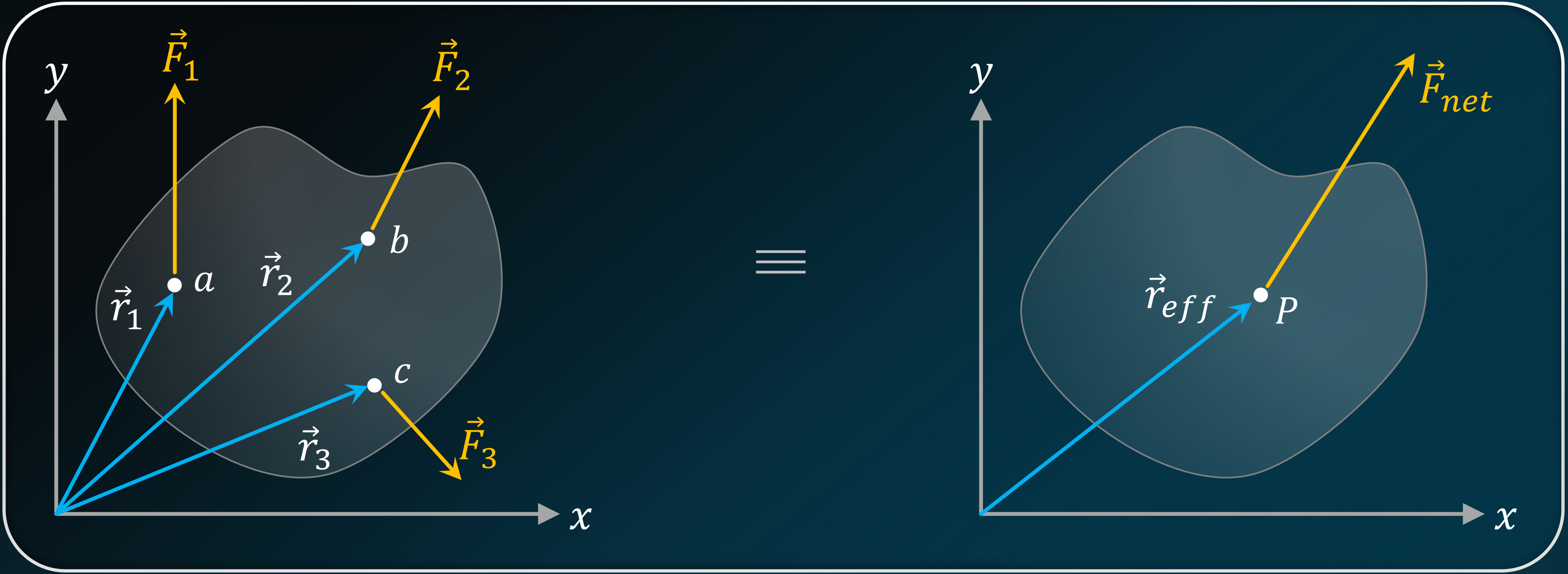
$\Rightarrow x = 12 - 1.75$

$\Rightarrow x = 0.25\ m$

$x = 25\ cm$

a	$25\ cm$
b	$12\ cm$
c	$9\ cm$
d	$10\ cm$

Point of Application of Force



- P is the point at which the resultant of external forces ($\vec{F}_{net}$) can be assumed to be applied.
- $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$
- $\vec{\tau}_{net} = (\vec{r}_1 \times \vec{F}_1) + (\vec{r}_2 \times \vec{F}_2) + (\vec{r}_3 \times \vec{F}_3) = \vec{r}_{eff} \times \vec{F}_{net}$

Mechanical Equilibrium

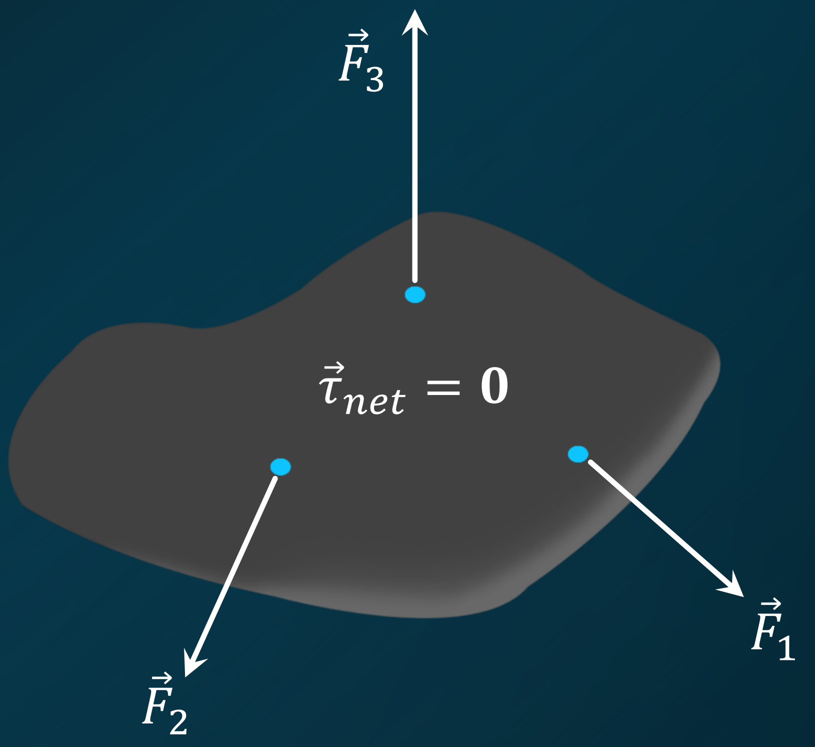
- Translational equilibrium

$$\sum \vec{F}_i = \mathbf{0}$$

- Rotational equilibrium

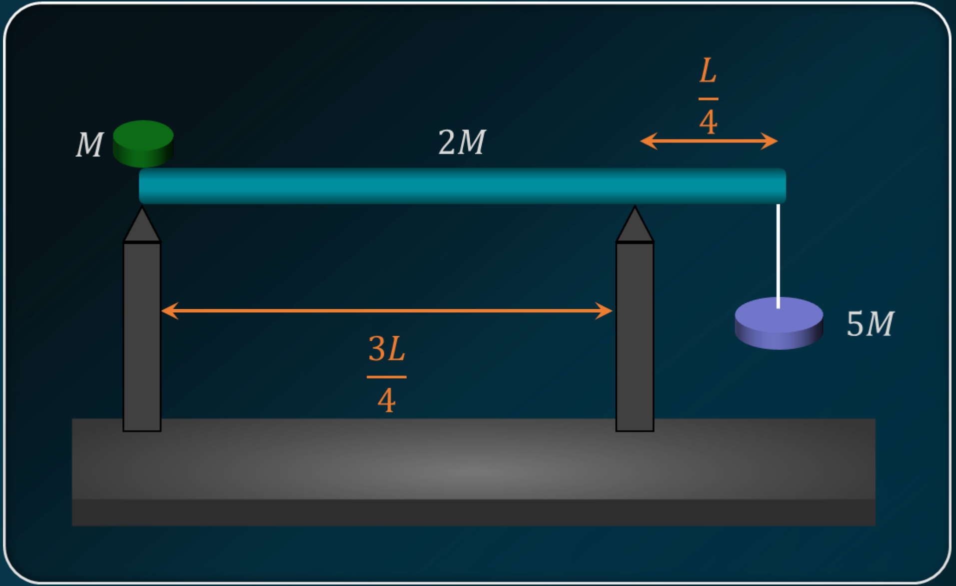
$$\sum \vec{\tau}_i = \mathbf{0}$$ (Torque can be calculated about any axis)

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \mathbf{0}$$

A uniform rod of mass $2M$ and length L is placed on two supports as shown in the figure. A block of mass $5M$ is suspended from one end of the rod. Another mass M is placed on top at the opposite end. The rod is just in equilibrium. Find out the normal reactions provided by the two supports.



Solution :

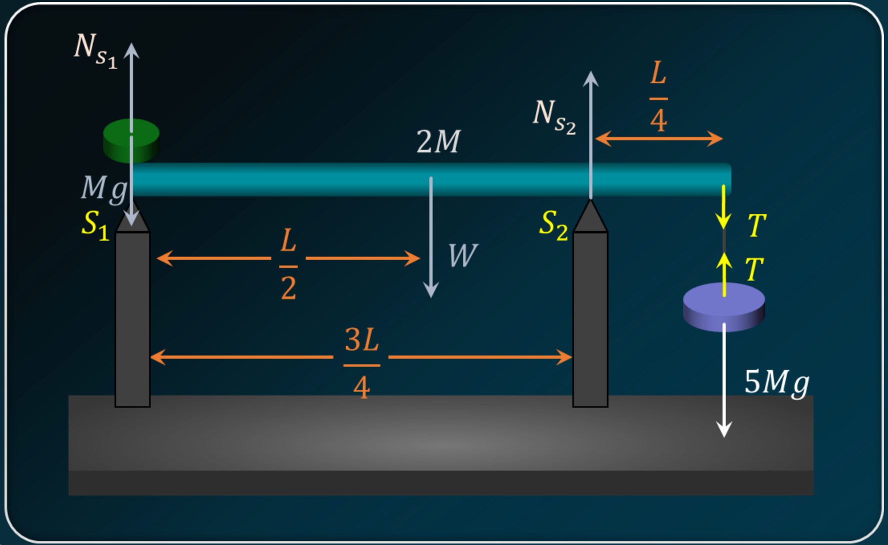
$W = 2Mg$ & $T = 5Mg$

For the rod to be in equilibrium,

$\sum \vec{F}_{net} = 0$

$N_{s_1} + N_{s_2} = Mg + W + T$

$N_{s_1} + N_{s_2} = 8Mg$



Torque about S_1

$\sum (\vec{\tau}_{net})_{S_1} = 0$

$N_{s_2}\left(\frac{3L}{4}\right) - W\left(\frac{L}{2}\right) - T(L) = 0$

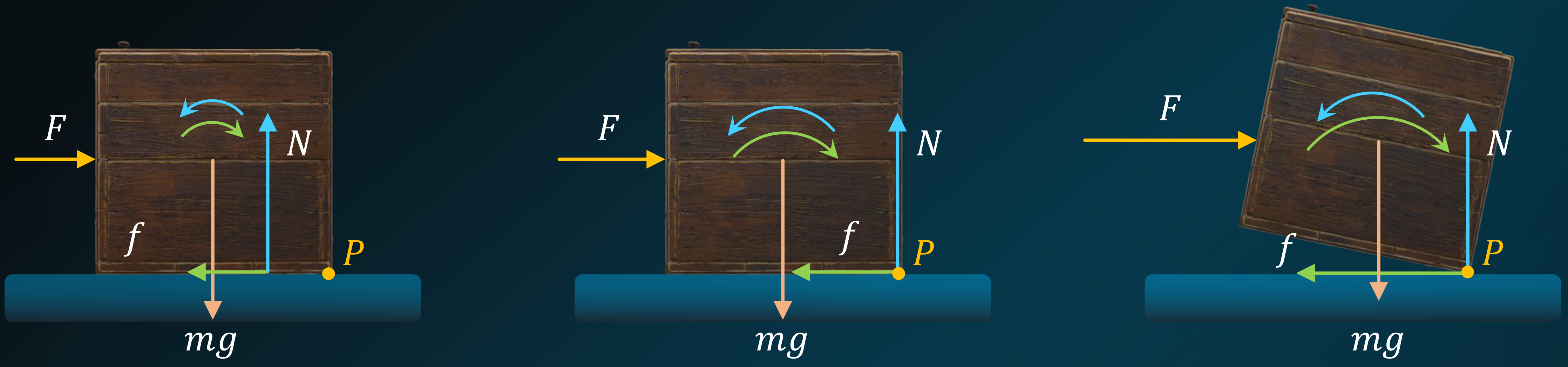
$N_{s_2} = 8Mg$

$\Rightarrow N_{s_1} = 0$

$N_{s_1} = 0\ N$

$N_{s_2} = 8Mg$

Toppling



- As the external force F increases, normal force N adjusts its point of application in order to keep the block from toppling.
- When F and therefore the friction f is high enough, normal force can no longer provide the counter-balancing torque and the block topples about point P.

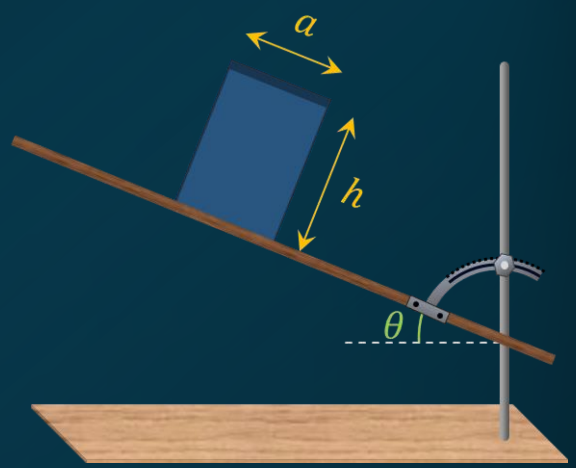
A block with a square base measuring $a \times a$, and height h, is placed on an inclined plane. The coefficient of friction is μ. The angle of inclination θ of the plane is gradually increased. The block will

a) topple before sliding if $\mu > \frac{a}{h}$

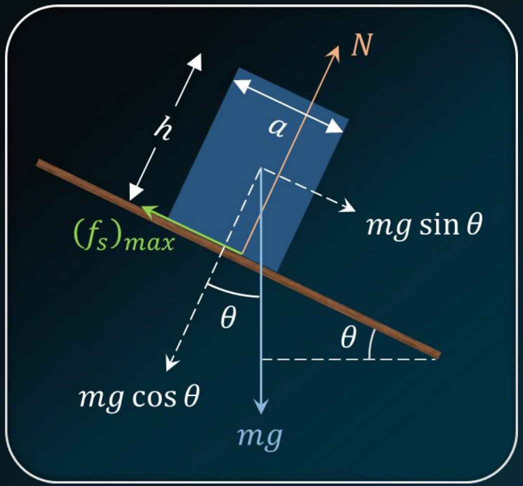
b) topple before sliding if $\mu < \frac{a}{h}$

c) slide before toppling if $\mu > \frac{a}{h}$

d) slide before toppling if $\mu < \frac{a}{h}$



Condition for Sliding



$N = mg\cos\theta$

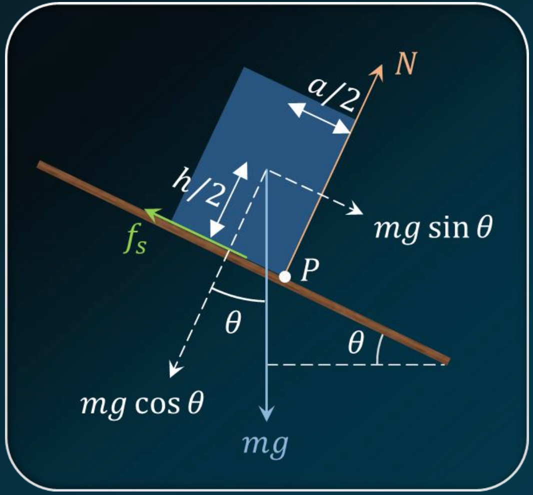
$mg\sin\theta = (f_s)_{max}$ (Body is just about to slide)

$mg\sin\theta = \mu N = \mu mg\cos\theta$

$\tan\theta = \mu$

$\tan\theta > \mu$ (To initiate sliding)

Condition for Toppling



About point P (To initiate toppling)

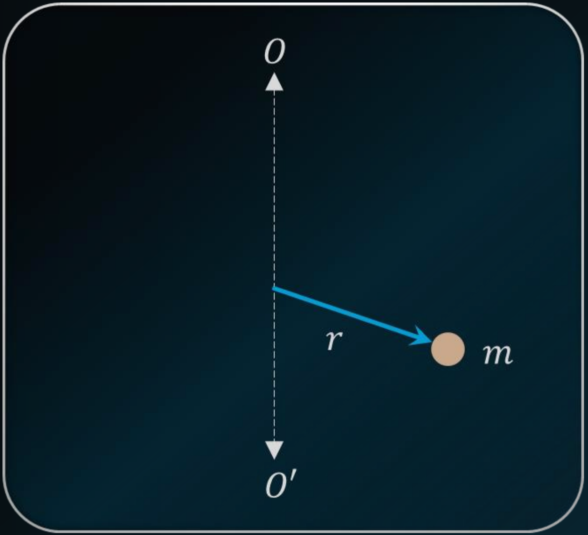
$\tau_{mg\sin\theta} > \tau_{mg\cos\theta}$

$mg\sin\theta \times \frac{h}{2} > mg\cos\theta \times \frac{a}{2}$

$\tan\theta > \frac{a}{h}$ (To initiate toppling)

The block will topple before sliding if $\mu > \frac{a}{h}$ and slide before toppling if $\mu < \frac{a}{h}$.

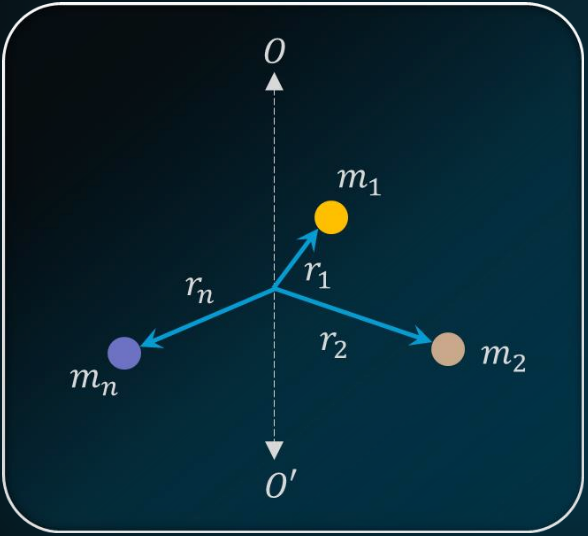
Moment of Inertia



- Rotational analogue of mass.
- Moment of inertia of a particle of mass m located at a perpendicular distance r from an axis in consideration is given by,

$$I = mr^2$$

- It is a scalar quantity.
- Unit of MOI is $kg\ m^2$.



- Moment of inertia of n particles having mass m_1, m_2,...., m_n at distance r_1, r_2,...., r_n from an axis is given by,

$$I = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2 = \sum_{i=1}^{n} (m_i r_i^2)$$

Note: Moment of inertia is added only if they are defined with respect to the same axis of rotation.

A massless equilateral triangle EFG of side a has three particles of mass m situated at its vertices. If the moment of inertia of the system about the line EX perpendicular to EG in the plane of EFG is $\frac{N}{20}ma^2$, then N is

Solution :

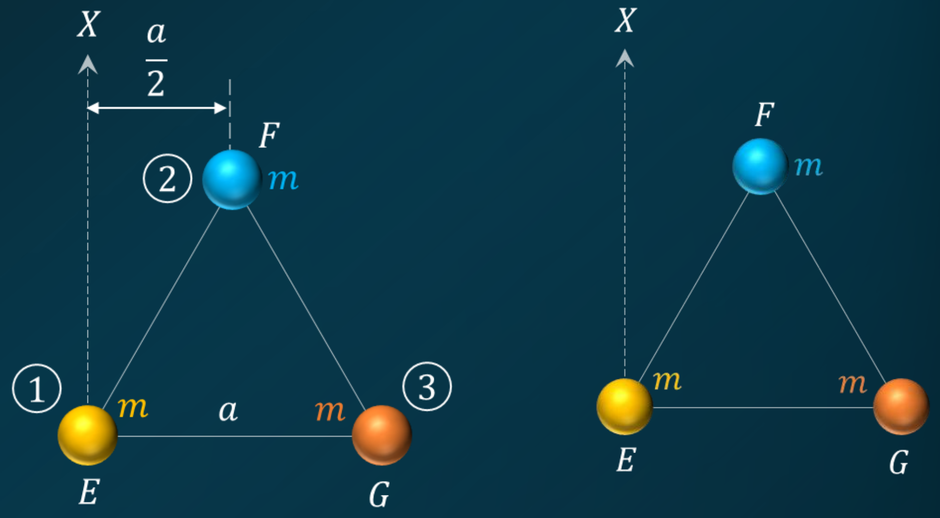
Moment of inertia of the system about EX,

$$I = I_1 + I_2 + I_3$$

$$\Rightarrow I = m(0)^2 + m\left(\frac{a}{2}\right)^2 + ma^2$$

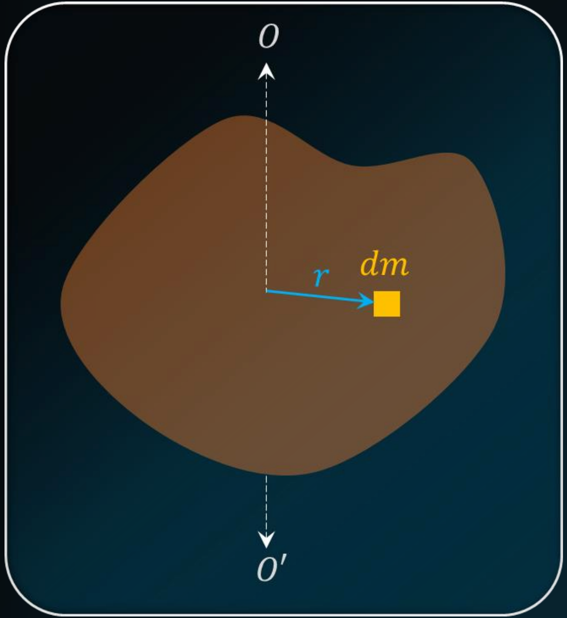
$$\Rightarrow I = \frac{5}{4}ma^2 = \frac{25}{20}ma^2$$

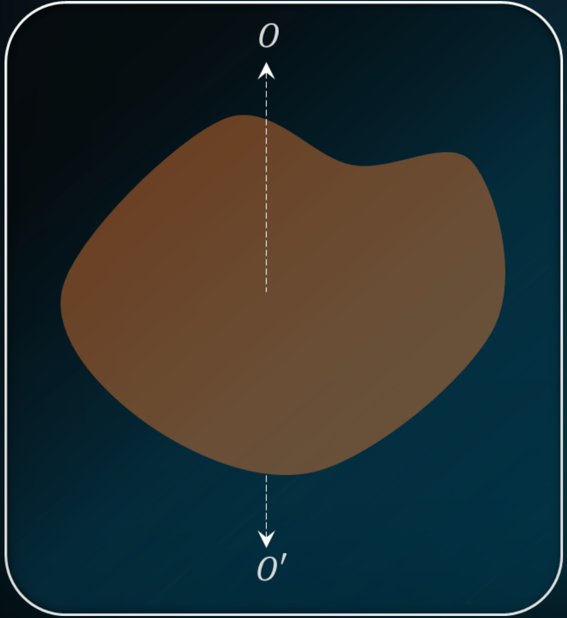
$$\Rightarrow N = 25$$



a	20
b	5
c	25
d	4

Moment of Inertia of Continuous Bodies





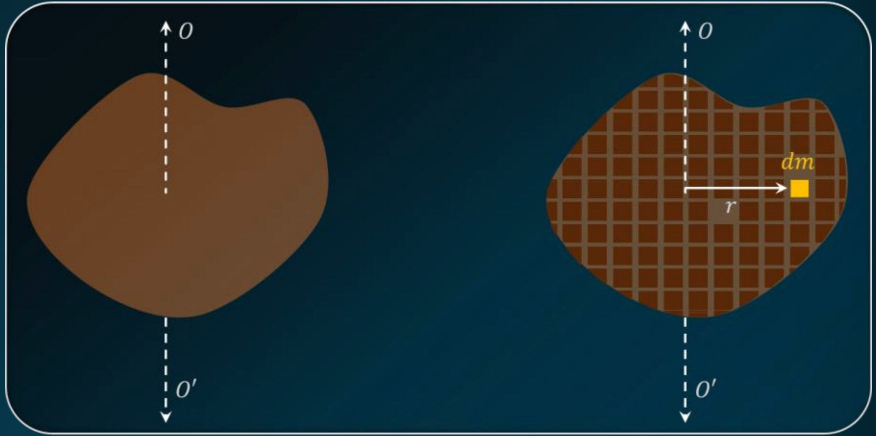
MI of the element about OO',

$$dI = r^2 dm$$

MI of the continuous body about OO',

$$I = \int dI = \int r^2 dm$$

$$I = \int r^2 dm$$



Moment of Inertia depends on

- Axis of rotation,
- Shape and size of the body, and
- Distribution of mass relative to axis of rotation

Calculate the moment of inertia of a uniform rod of length L and mass M about an axis passing through its centre and perpendicular to it.

Solution :

Let λ be the density of the rod.

From the definition of MOI,

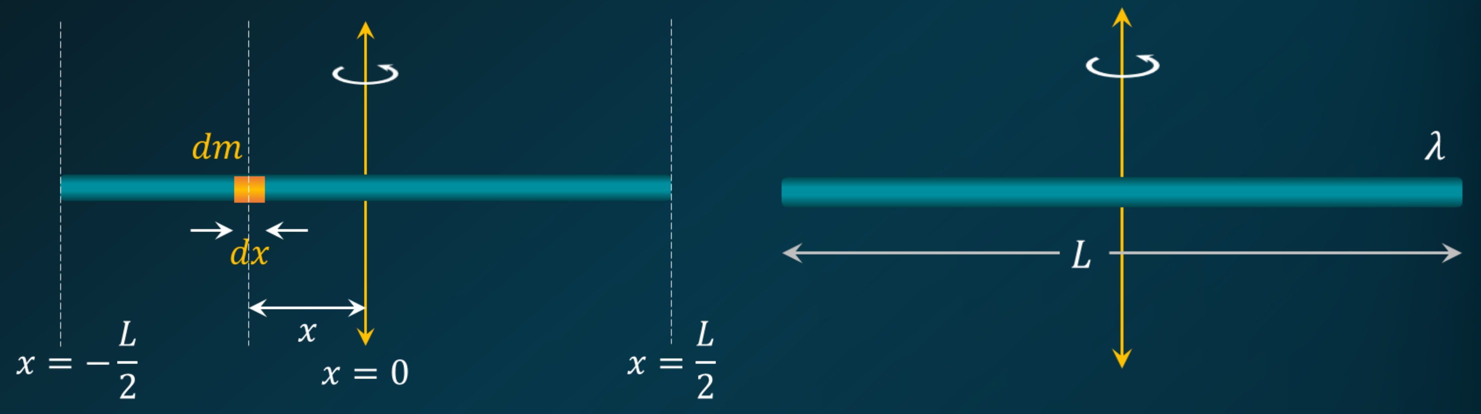
$$dI = dm \times x^2$$

$$\Rightarrow dI = \lambda\, dx \times x^2$$

$$\Rightarrow dI = \frac{M}{L} dx \times x^2$$

$$\therefore I = \int dI = \frac{M}{L} \int_{-L/2}^{L/2} x^2\, dx$$

$$\Rightarrow I = \frac{M}{L} \times \left(\frac{x^3}{3}\right)_{-L/2}^{L/2} \quad \Rightarrow I = \frac{ML^2}{12}$$



a) $\frac{ML^2}{2}$

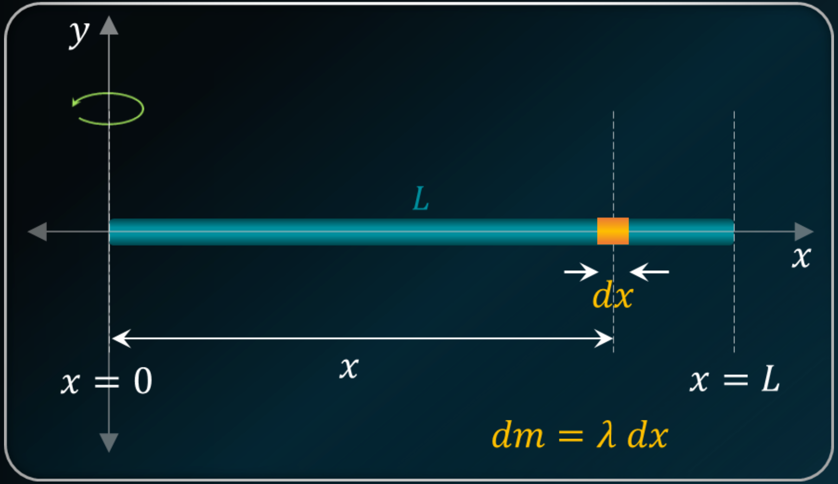
b) $\frac{ML^2}{12}$

c) $\frac{ML^2}{3}$

d) $\frac{ML^2}{4}$

Moment of Inertia of a Thin Uniform Rod

About an axis passing through the end of the rod perpendicular to it



From the definition of MOI,

$$dI = dm \times x^2$$

$$\Rightarrow dI = \lambda\, dx \times x^2$$

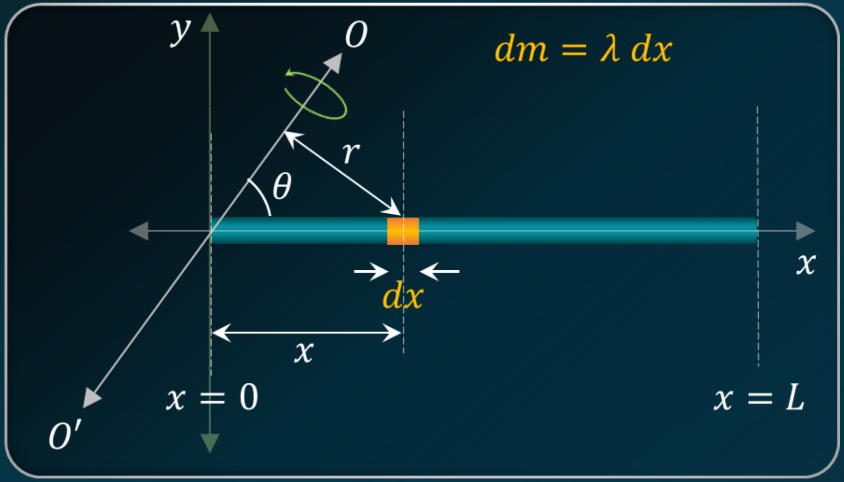
$$\Rightarrow dI = \frac{M}{L} dx \times x^2$$

$$\therefore I = \int dI = \frac{M}{L} \int_0^L x^2\, dx$$

$$\Rightarrow I = \frac{M}{L} \times \left(\frac{x^3}{3}\right)_0^L$$

$$\Rightarrow I = \frac{ML^2}{3}$$

About an axis passing through the end of the rod making an angle θ with it



$$dI = r^2 \times dm$$

$$\Rightarrow dI = (r)^2 \times \lambda\, dx$$

$$\Rightarrow dI = (x \sin\theta)^2 \times \frac{M}{L} dx$$

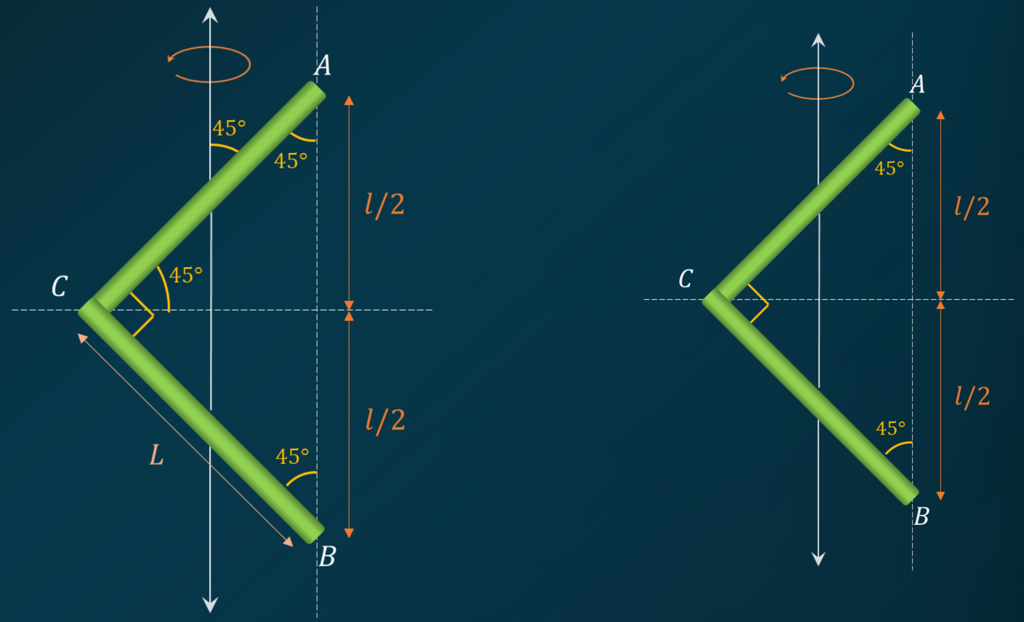
$$(\because r = x \sin\theta)$$

$$I = \int dI = \frac{M}{L} \sin^2\theta \int_0^L x^2\, dx$$

$$I = \frac{M}{L} \sin^2\theta \times \left(\frac{x^3}{3}\right)_0^L$$

$$I = \frac{ML^2}{3} \sin^2\theta$$

Linear mass density of the two rods system, AC and CB is x. Moment of inertia of two rods about an axis passing through their centres as shown is



a	$\frac{xl^3}{24\sqrt{2}}$
b	$\frac{xl^3}{12\sqrt{2}}$
c	$\frac{2xl^3}{\sqrt{3}}$
d	$\frac{xl^3}{6\sqrt{2}}$

Solution : $L = \frac{l/2}{\cos 45^\circ} = \frac{l}{\sqrt{2}}$

Mass of each rod, $m = xL = \frac{xl}{\sqrt{2}}$

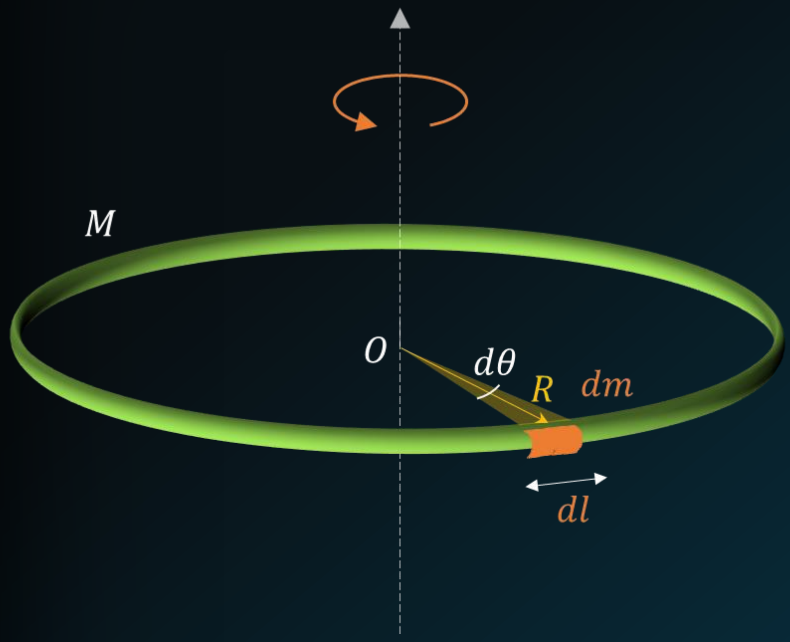
Moment of inertia of the two rods,

$$\Rightarrow I = 2\left[\frac{mL^2}{12}\sin^2 45^\circ\right]$$

$$\Rightarrow I = 2\left[\frac{\left(\frac{xl}{\sqrt{2}}\right)\left(\frac{l}{\sqrt{2}}\right)^2}{12} \times \frac{1}{2}\right]$$

$$\Rightarrow I = \frac{xl^3}{24\sqrt{2}}$$

Moment of Inertia of a Thin Uniform Ring



R represents the distance of dm from the axis in this case

$$dm = \lambda dl = \frac{M}{2\pi R} R d\theta = \frac{M}{2\pi} d\theta$$

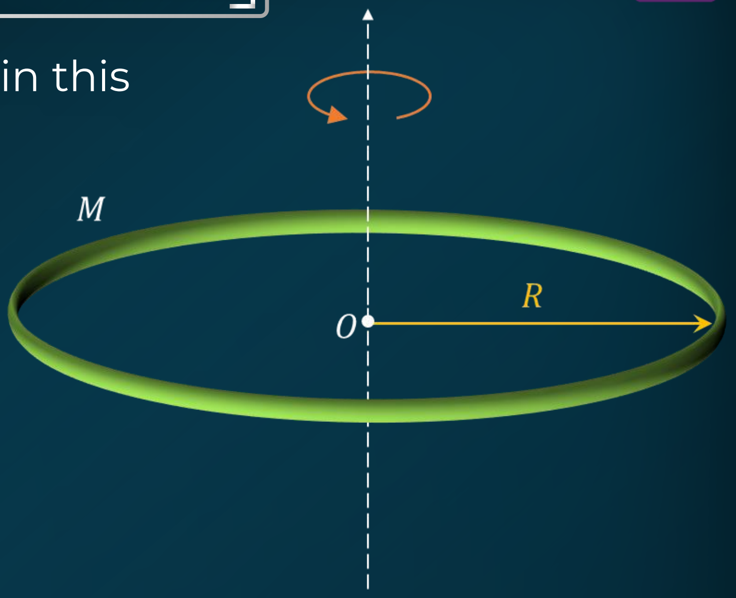
$$dI = R^2(dm) = R^2 \times \frac{M}{2\pi} d\theta$$

$$I = \int dI = \int R^2(dm)$$

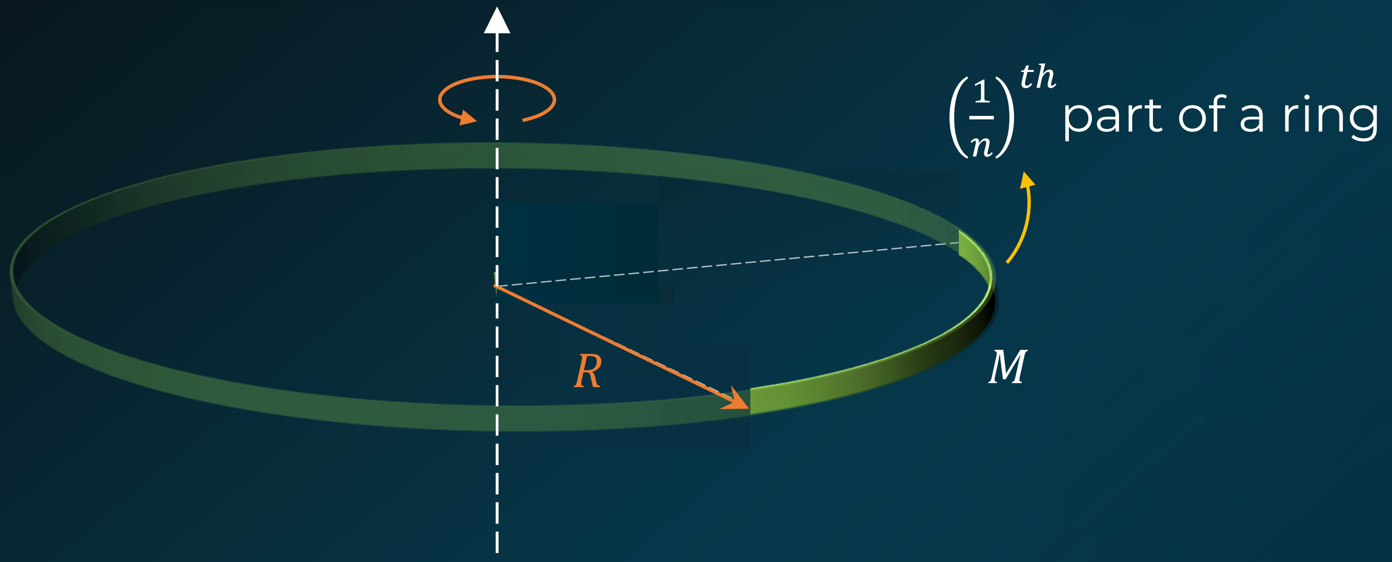
$$I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta$$

$$I = \frac{MR^2}{2\pi}(2\pi)$$

$$I = MR^2$$



Moment of Inertia

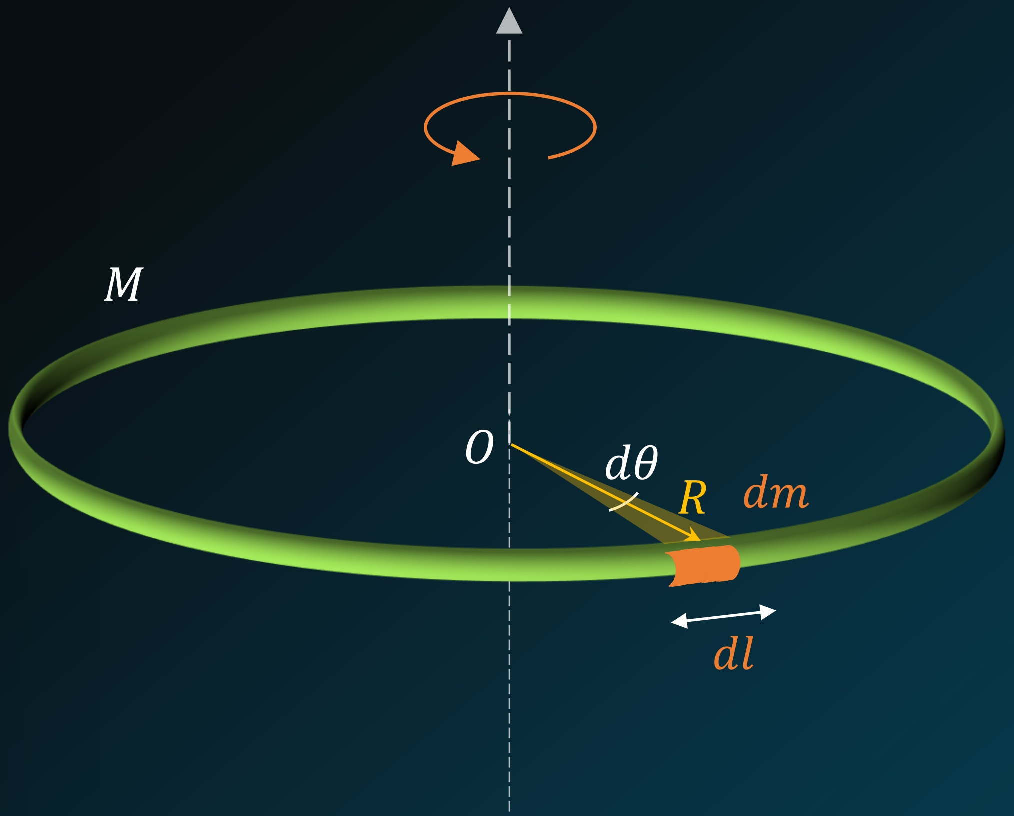


$$I = \frac{1}{n}[I_{ring}]$$

$$\Rightarrow I = \frac{1}{n}[(nM)R^2] \quad (\because m_{ring} = nM)$$

$$\Rightarrow I = MR^2$$

Moment of Inertia of a Thin Non- Uniform Ring



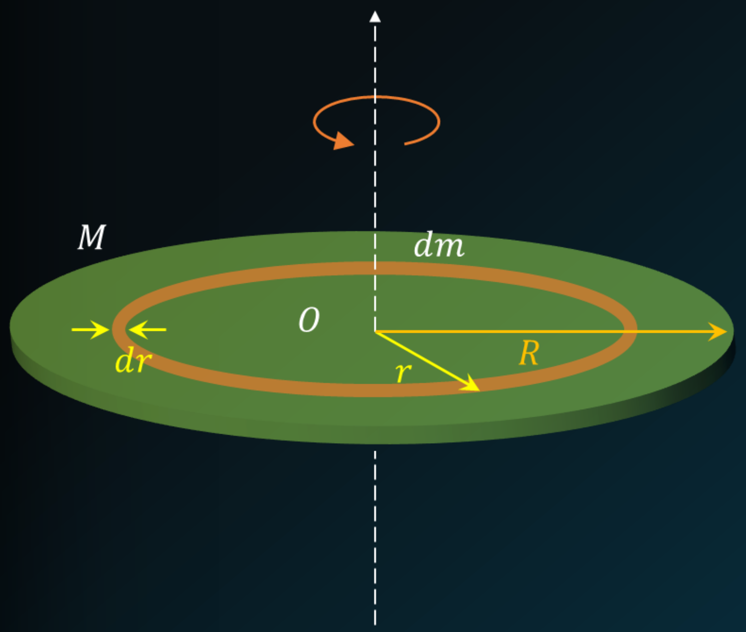
For a non-uniform ring,

$$I = \int (dm)R^2 = R^2 \int (dm)$$

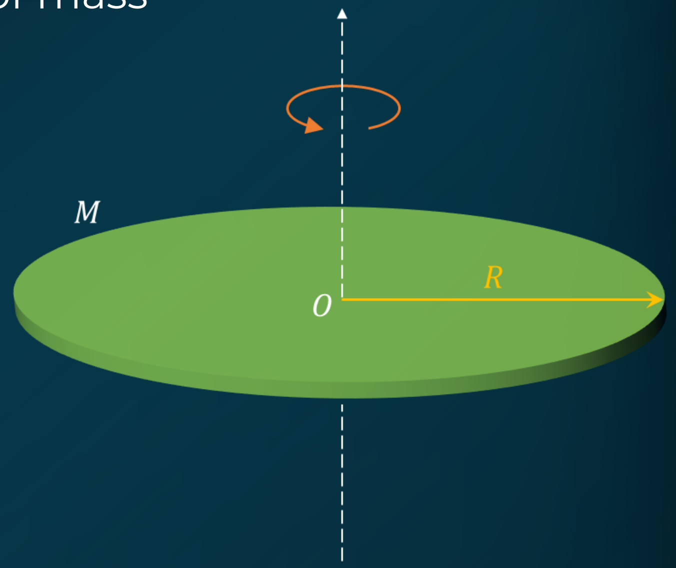
But, $\int (dm) = M$

$$\Rightarrow \boxed{I = MR^2}$$

Moment of Inertia of a Thin Uniform Disc



r represents the distance of differential element of mass dm from the axis in consideration.



$$dm = \sigma(dA) = \left(\frac{M}{\pi R^2}\right)(2\pi r)(dr) = \frac{2M}{R^2} r dr$$

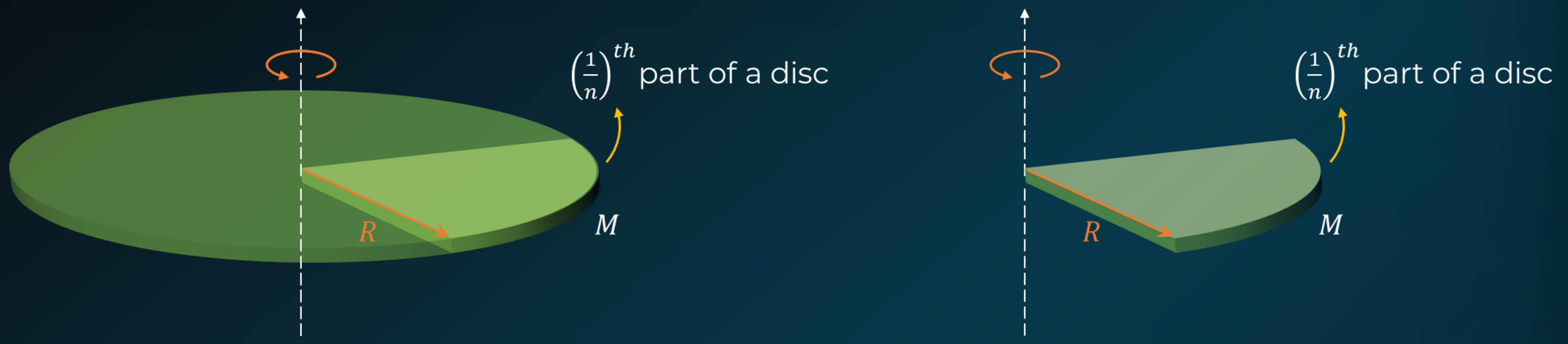
$$\therefore dI = r^2(dm) = (r^2)\frac{2M}{R^2} r dr = \frac{2M}{R^2} r^3 dr$$

$$\Rightarrow \int_0^I dI = \int_0^R \frac{2M}{R^2} r^3 dr = \frac{2M}{R^2}\int_0^R r^3 dr$$

$$\Rightarrow I = \frac{2M}{R^2}\left[\frac{R^4}{4}\right]$$

$$\Rightarrow I = \frac{MR^2}{2}$$

Moment of Inertia

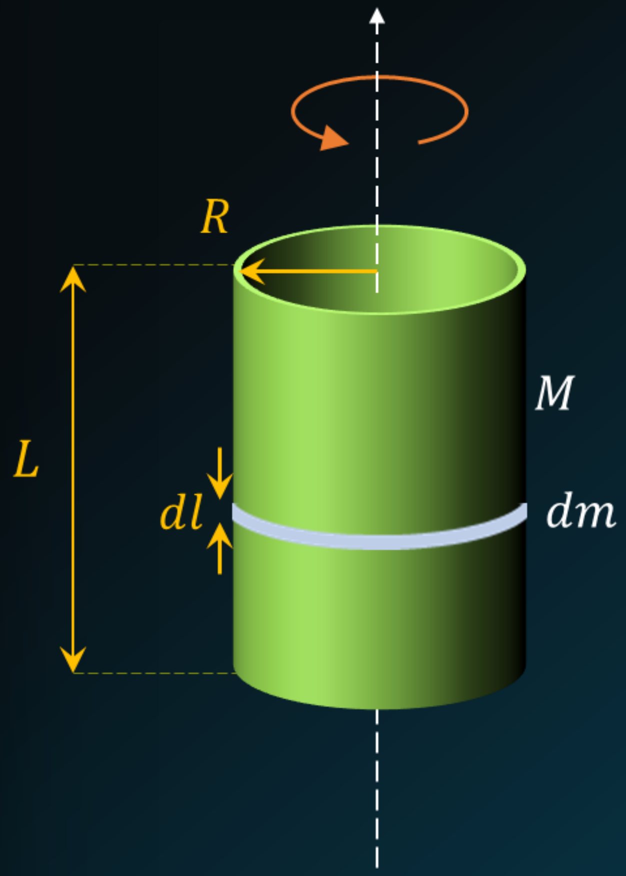


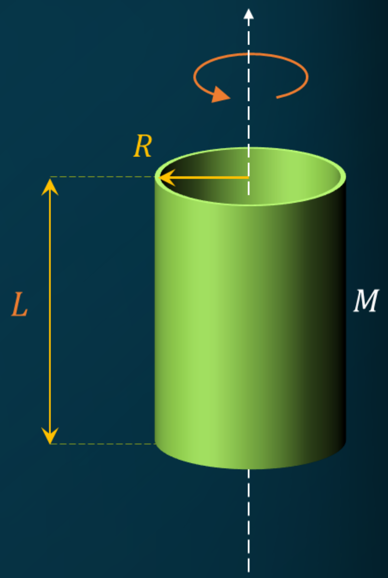
$$I_{section} = \frac{1}{n}[I_{disc}]$$

$$\Rightarrow I_{section} = \frac{1}{n}\left[\frac{(nM)R^2}{2}\right] \quad (\because m_{disc} = nM)$$

$$\Rightarrow \boxed{I_{section} = \frac{MR^2}{2}}$$

MOI of a Thin Uniform Hollow Cylinder





R is the distance of a thin ring of mass dm from the axis

Moment of inertia of the elemental ring,

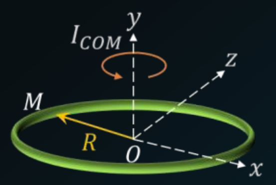
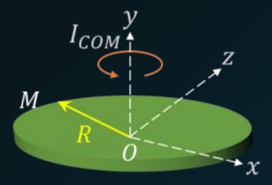
$$dI = R^2 dm$$

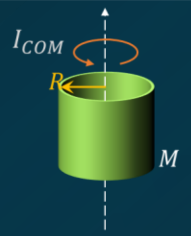
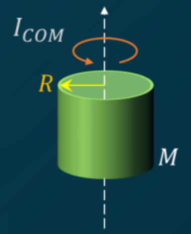
∴Moment of inertia of the cylinder,

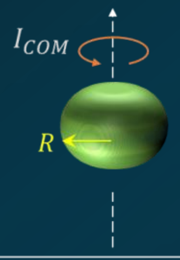
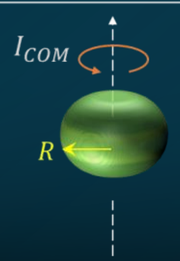
$$\int_0^I dI = \int_0^M R^2 dm$$

$$\Rightarrow \boxed{I = MR^2}$$

MOI of Some Standard Symmetric Bodies

Shape	Diagram	MOI
Thin Uniform Ring		MR^2
Thin Uniform Disc		$\frac{MR^2}{2}$

Shape	Diagram	MOI
Thin Hollow Cylinder		MR^2
Solid Cylinder		$\frac{MR^2}{2}$

Shape	Diagram	MOI
Thin Hollow Sphere		$\frac{2MR^2}{3}$
Solid Sphere		$\frac{2MR^2}{5}$

If I_1 is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and I_2 is the moment of inertia of a ring about an axis perpendicular to its plane and passing through its centre formed by bending the same rod, then

Solution : Length of the thin rod = Perimeter of the ring $\Rightarrow L = 2\pi R$

Moment of inertia of the thin rod,

$$I_1 = \frac{ML^2}{12} = \frac{M(4\pi^2R^2)}{12} = \frac{\pi^2}{3}(MR^2)$$

Moment of inertia of the ring,

$$I_2 = MR^2$$

$$\therefore I_1 = \frac{\pi^2}{3}(I_2)$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{\pi^2}{3}$$

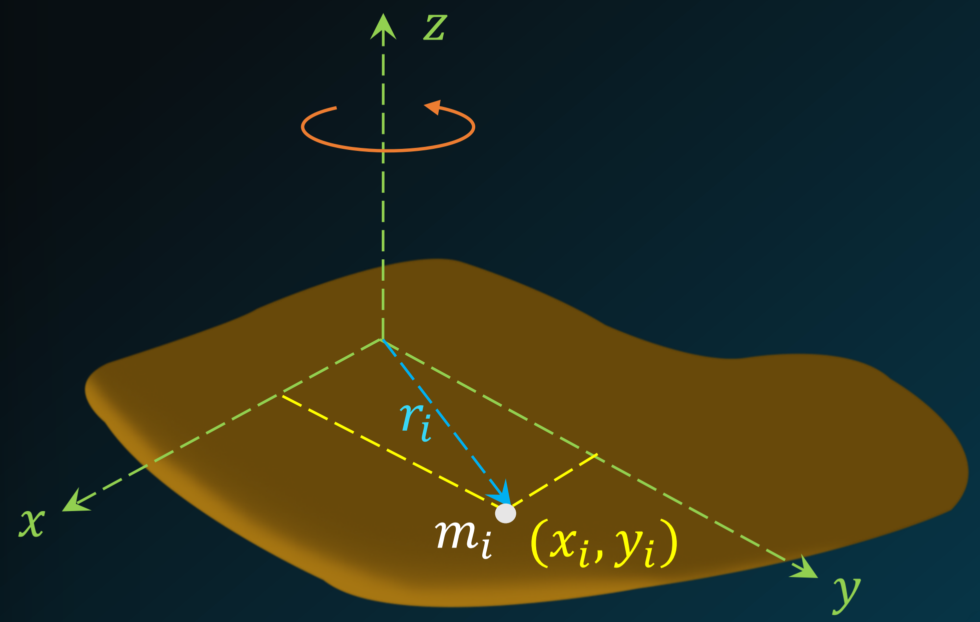
a) $\frac{I_1}{I_2} = \frac{3}{\pi^2}$

b) $\frac{I_1}{I_2} = \frac{2}{\pi^2}$

c) $\frac{I_1}{I_2} = \frac{\pi^2}{2}$

d) $\frac{I_1}{I_2} = \frac{\pi^2}{3}$

Perpendicular Axes Theorem



"The moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body"

$$I_z = I_x + I_y$$

Note: It's only applicable for laminar / planar / 2D objects

Calculate the moment of inertia of a thin uniform ring of mass M and radius R about the axis passing through its diameter.

Solution : By symmetry, $I_x = I_y = I$

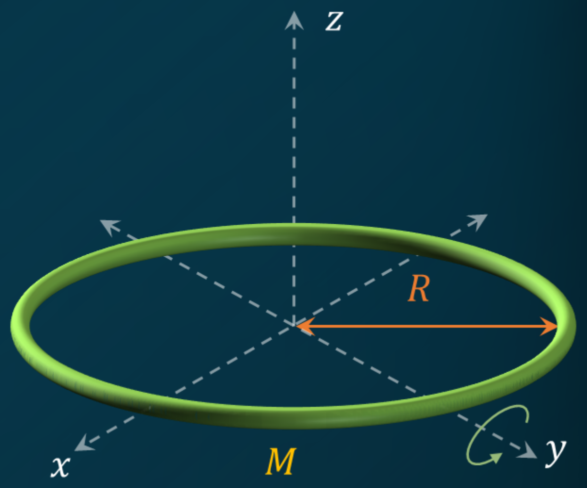
Using Perpendicular Axis Theorem,

$$I_z = I_x + I_y$$

$$\Rightarrow I_z = I + I = 2I$$

$$\Rightarrow MR^2 = 2I$$

$$I = \frac{MR^2}{2}$$



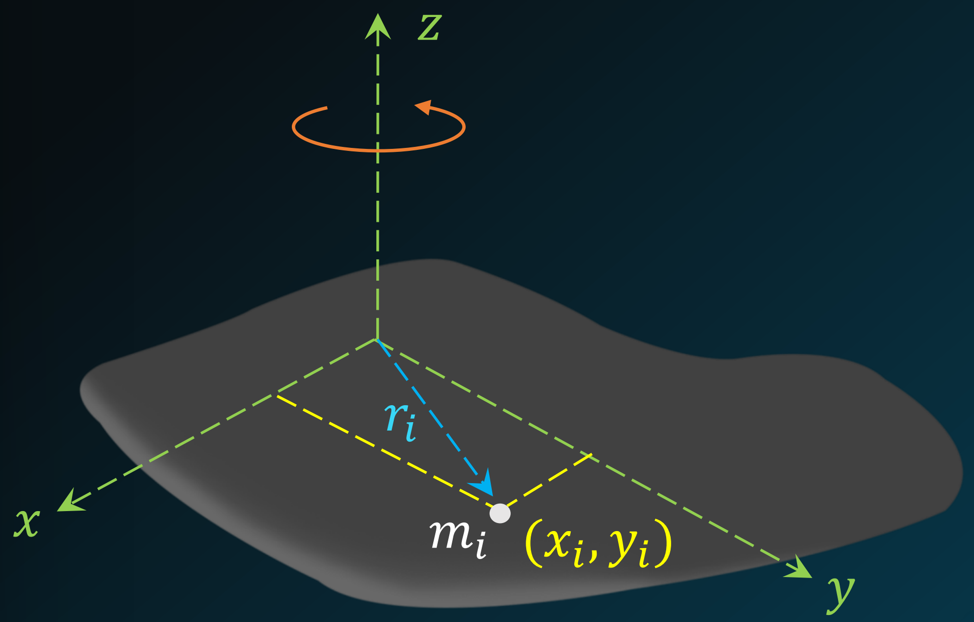
a) $\frac{MR^2}{3}$

b) $\frac{MR^2}{2}$

c) $\frac{5MR^2}{6}$

d) MR^2

Perpendicular Axes Theorem

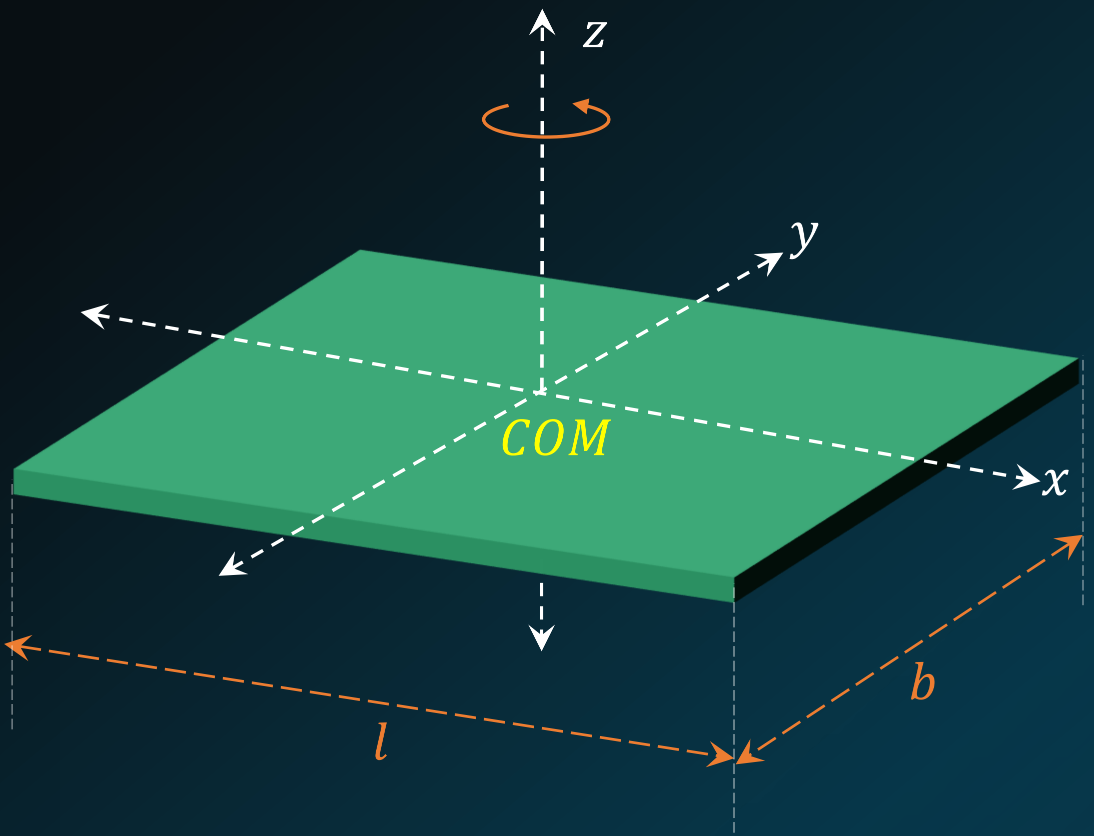


The moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

$$I_z = I_x + I_y$$

Note: It's only applicable for laminar / planar / 2D objects

Thin Uniform Rectangular Lamina



Moment of Inertia about the x axis,

$$I_x = \frac{Mb^2}{12}$$

Moment of Inertia about the y axis,

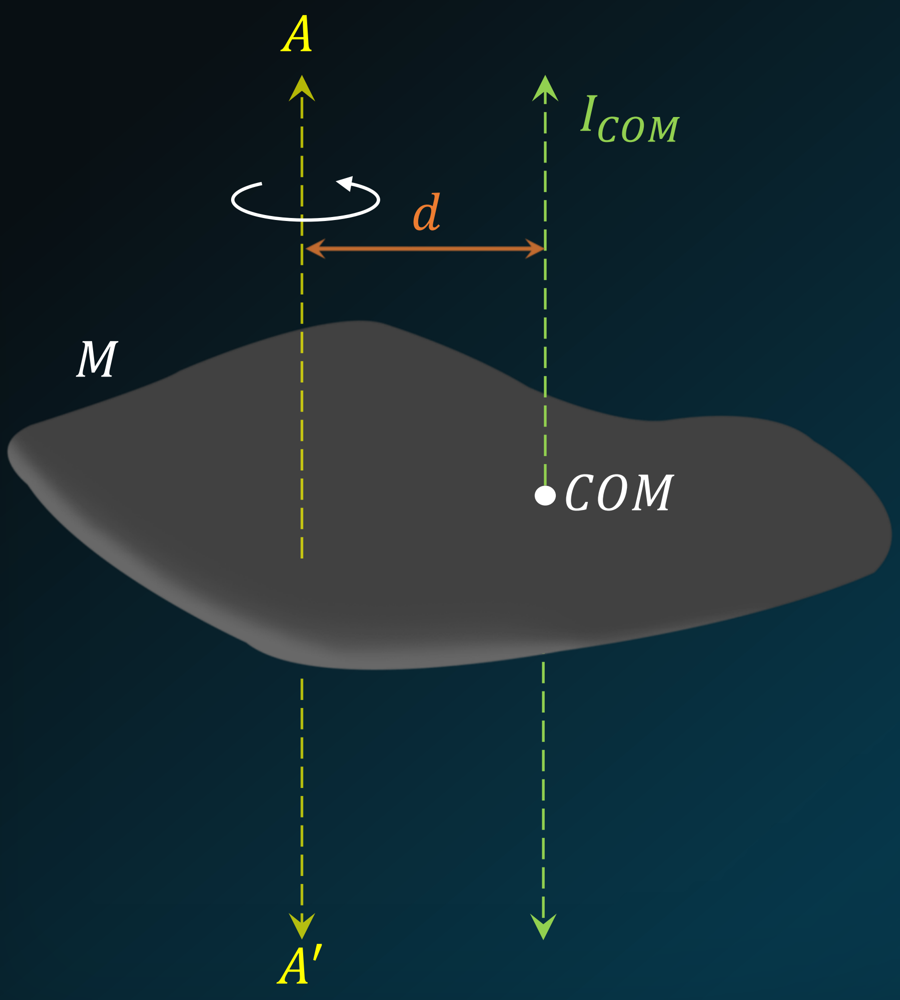
$$I_y = \frac{Ml^2}{12}$$

By Perpendicular Axes Theorem,

$$I_z = I_x + I_y$$

$$I_z = \frac{M}{12}(b^2 + l^2)$$

Parallel Axes Theorem



- Moment of Inertia of a body about an axis parallel to an axis through COM and separated by a perpendicular distance d is given by,

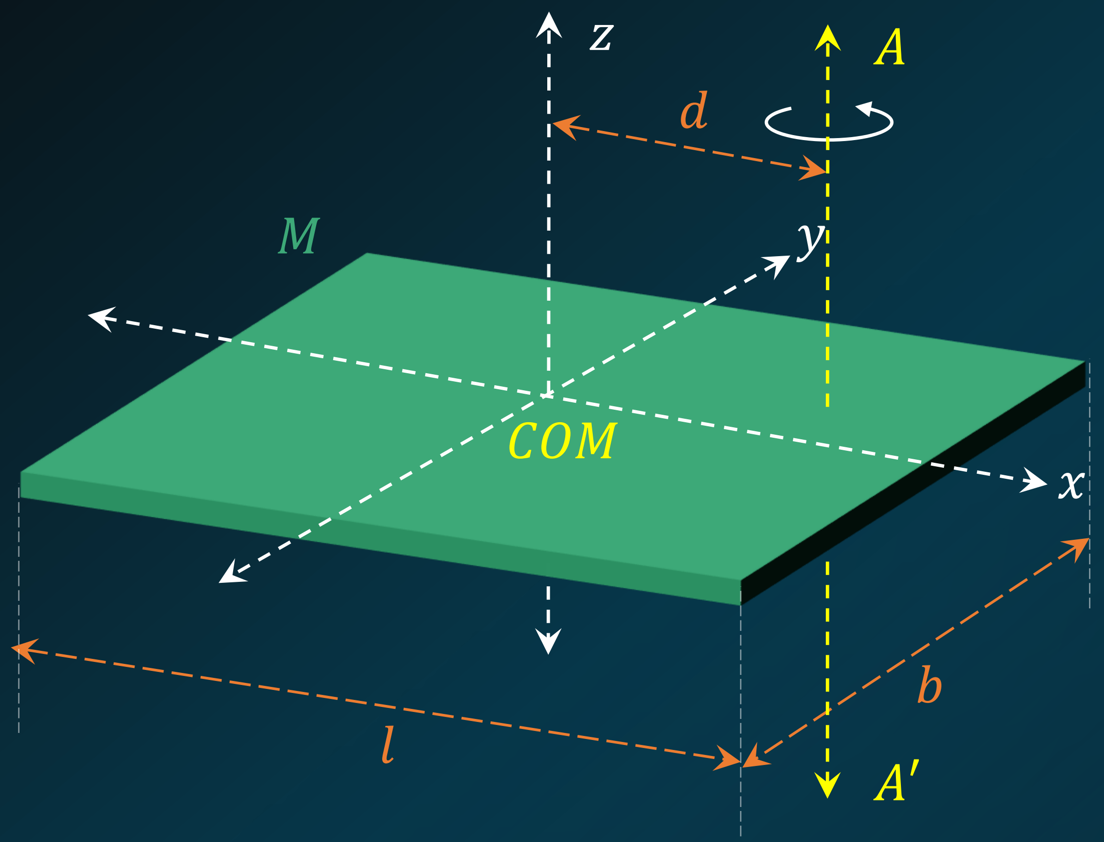
$$I_{AA'} = I_{COM} + Md^2$$

Where,

$$I_{AA'} = (I_{sys})_{AA'}$$

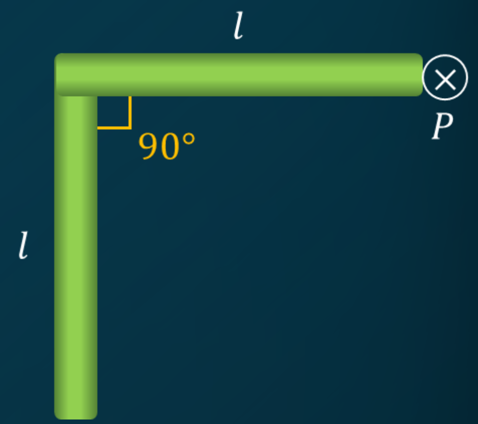
$$I_{COM} = (I_{sys})_{COM}$$

Parallel Axes Theorem



$$I_{AA'} = \frac{M}{12}[b^2 + l^2] + Md^2$$

Find the moment of inertia of the two uniform joint rods having mass m each about point P as shown in the figure, using parallel axes theorem.

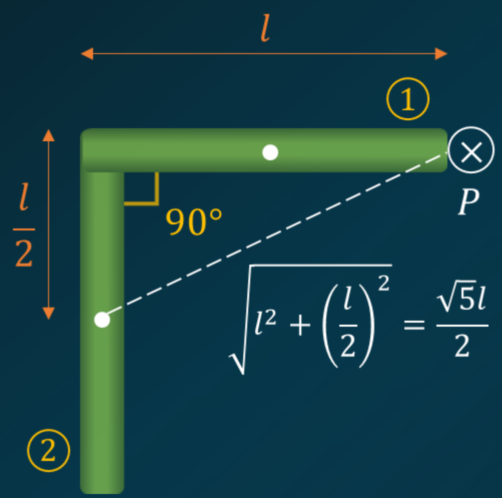


Solution : Moment of inertia of the system about the given axis,

$$I_P = (I_1)_P + (I_2)_P$$

$$= (I_{com} + md_1^2) + (I_{com} + md_2^2)$$

$$= \left[\frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2\right] + \left[\frac{ml^2}{12} + m\left(\frac{\sqrt{5}l}{2}\right)^2\right]$$



$$= \left[\frac{ml^2}{12} + \frac{ml^2}{4}\right] + \left[\frac{ml^2}{12} + \frac{5ml^2}{4}\right]$$

$$= \frac{ml^2}{3} + \frac{4ml^2}{3}$$

$$I_P = \frac{5ml^2}{3}$$

Four solid spheres each of diameter $\sqrt{5}\ cm$ and mass $0.5\ kg$ are placed with their centres at the corners of a square of side $4\ cm$. If the moment of inertia of the system about the diagonal of the square is $N \times 10^{-4}\ kg\ m^2$, then N is

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Solution :

$$a = 4\ cm = 4 \times 10^{-2}\ m$$

$$r = \frac{d}{2} = \frac{\sqrt{5}}{2}\ cm = \frac{\sqrt{5}}{2} \times 10^{-2}\ m$$

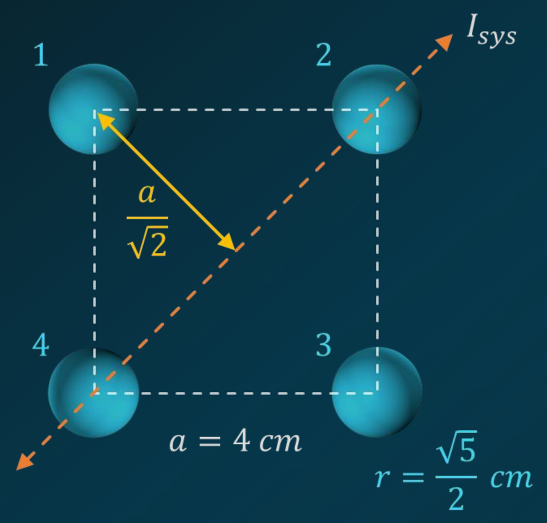
$$I_{sys} = I_1 + I_2 + I_3 + I_4$$

$$= (I_1 + I_3) + (I_2 + I_4)$$

$$= 2\left[\frac{2}{5}mr^2 + m\left(\frac{a}{\sqrt{2}}\right)^2\right] + 2\left(\frac{2}{5}mr^2\right)$$

$$= 9 \times 10^{-4}\ kg\ m^2$$

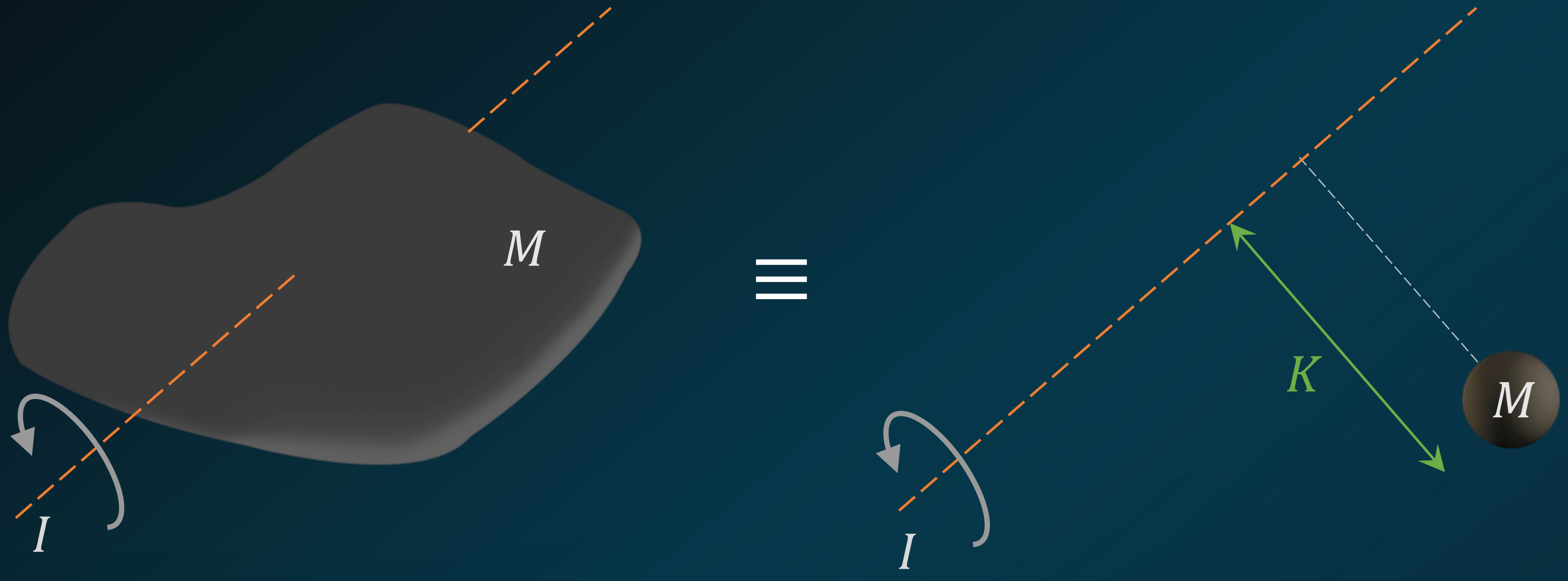
$N = 9$



a	2
b	3
c	4
d	9

Radius of Gyration

Distance (K) from the Axis of Rotation, where the whole mass of the rigid body can be assumed to be concentrated as a point mass such that the MOI of the point mass is the same as that of the rigid body (I).



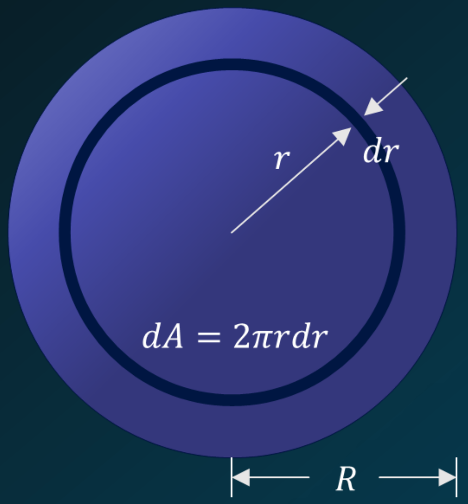
$$I = MK^2$$

$$K = \sqrt{\frac{I}{M}}$$

A thin disc of mass M and radius R has mass per unit area $\sigma(r) = kr^2$, where r is the distance from its centre. Its radius of gyration about an axis through its centre of mass and perpendicular to its plane is

Solution : Mass of the disc is given by,

$$M = \int_0^R \sigma(r)\, dA$$



$$M = \int_0^R kr^2 \times 2\pi r dr$$

$$M = 2\pi k \int_0^R r^3 dr$$

$$M = 2\pi k \left[\frac{r^4}{4}\right]_0^R$$

$$M = \frac{\pi k R^4}{2}$$

MOI of the disc about an axis passing through COM and perpendicular to its plane is given by,

$$I_C = \int_0^R r^2 dm$$

$$I_C = \int_0^R r^2 \times kr^2 \times 2\pi r dr$$

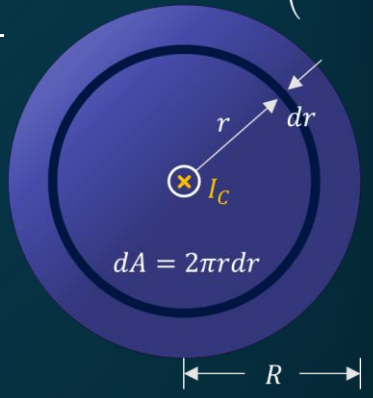
$$I_C = 2\pi k \int_0^R r^5 dr$$

$$I_C = \frac{\pi k R^6}{3}$$

$$\left(M = \frac{\pi k R^4}{2}\right)$$

$$\Rightarrow I_C = \frac{2R^2}{3} \times \frac{\pi k R^4}{2}$$

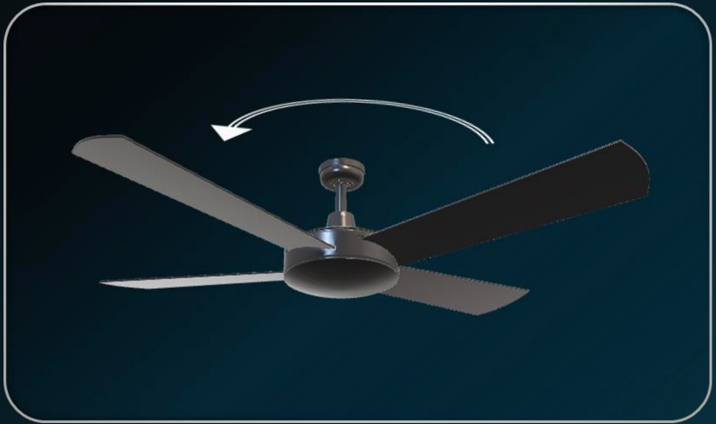
$$\Rightarrow I_C = \frac{2MR^2}{3}$$

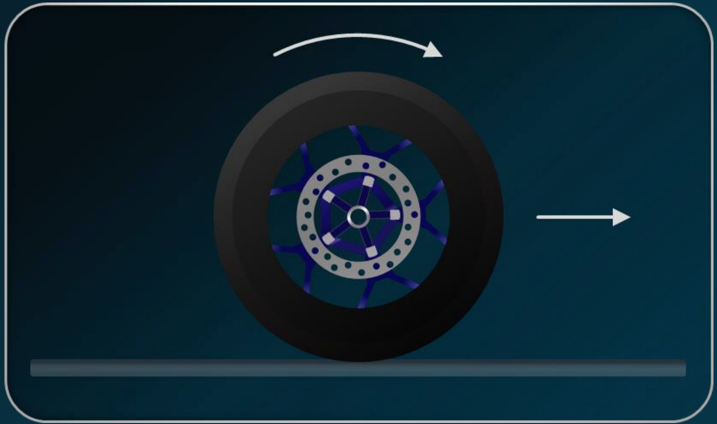


Now, $I_C = MK^2$

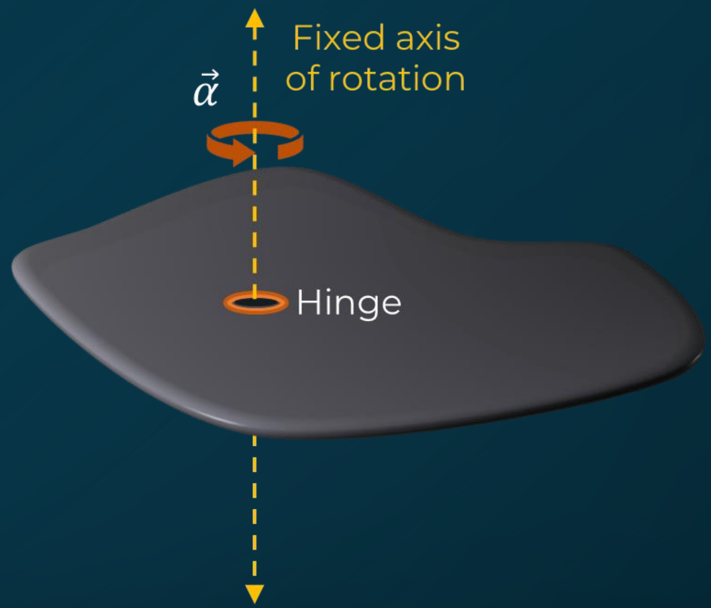
$$\Rightarrow K = \sqrt{\frac{2}{3}}R$$

Pure Rotational Motion





A rigid body in motion, such that its axis of rotation remains fixed with respect to the frame of reference performs pure rotational motion, e. g., a hinged rod.



$\vec{\tau}_{hinge} = I_{hinge}\vec{\alpha}$ (Newton's 2nd law for Rotation)

Where,

I_{hinge} = moment of inertia about hinge

α = angular acceleration of the body

A solid sphere of mass $2\ kg$ and radius $1\ m$ is free to rotate about an axis passing through its centre. Find a constant tangential force F to be applied at the surface of the sphere to make it achieve an angular speed of $10\ rad/s$ in $2\ s$. Also find the number of rotations made by the sphere in that time interval.

Solution :

Given $M = 2\ kg, R = 1\ m, \omega = 10\ rad/s\ and\ t = 2\ s$

Value of F

$\omega = \omega_0 + \alpha t = 0 + 2\alpha = 10\ rad\ s^{-1}$

$\alpha = 5\ rad/s^2$

$\alpha = \frac{\tau}{I} = \frac{F \times R}{\frac{2}{5} MR^2} = \frac{5F}{2MR}$

$\Rightarrow \frac{5F}{2MR} = 5$

$F = \frac{(2)(2)(1)(5)}{5}$

$F = 4\ N$

Number of revolutions

The angle rotated is,

$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}(5)2^2 = 10\ rad$

Number of rotations,

$n = \frac{\theta}{2\pi} = \frac{10}{2\pi}$

$n = \frac{5}{\pi}$

a. $4\ N, \frac{5}{\pi}$

b $2\ N, 5$

c $1\ N, \frac{\pi}{4}$

d $3\ N, \frac{\pi}{5}$

Rotational Kinetic Energy

Rotational kinetic energy for a body rotating about a fixed axis is calculated as-

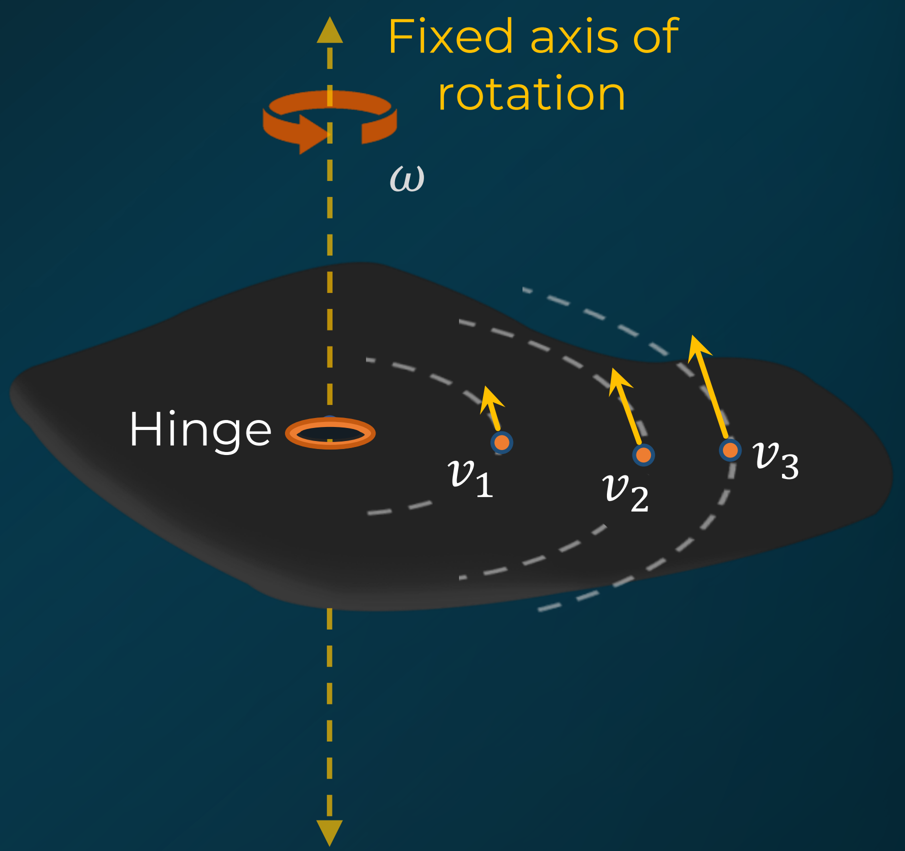
$$(KE)_{rot} = \sum \frac{1}{2} m_i {v_i}^2$$

$$= \frac{1}{2} \sum m_i (\omega r_i)^2$$

$$= \frac{1}{2} \omega^2 \sum m_i {r_i}^2$$

$$(KE)_{rot} = \frac{1}{2} \omega^2 I_{Hinge}$$

Rotational kinetic energy $= \frac{1}{2} I_{Hinge} \omega^2$



Pure Rotational Motion

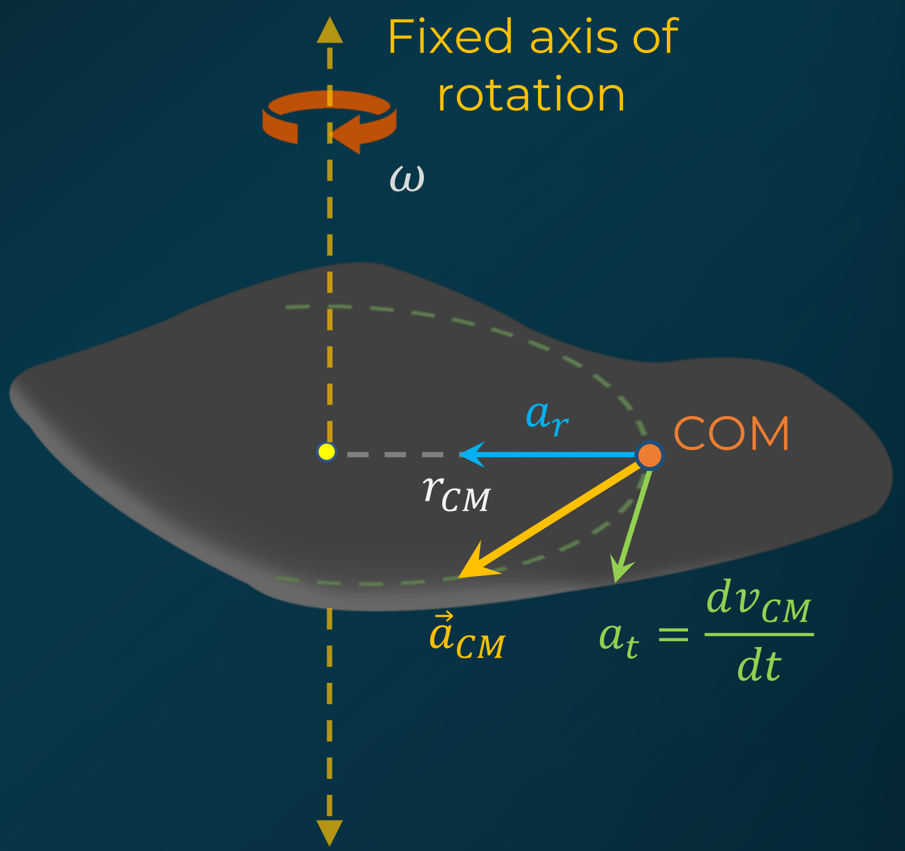
$$\sum(\vec{F}_{ext})_{sys} = m\vec{a}_{CM}$$

Radial

$$\sum\left|(\vec{F}_{ext})_r\right| = m\omega^2 r_{CM} = \frac{mv_{CM}^2}{r_{CM}}$$

Tangential

$$\sum\left|(\vec{F}_{ext})_t\right| = m\frac{dv_{CM}}{dt} = m\alpha r_{CM}$$



For a body performing pure rotational motion-

$$\vec{\tau}_{hinge} = I_{hinge}\vec{\alpha}$$

Total KE = Rotational KE = $\frac{1}{2}I_{Hinge}\omega^2$

Work done by a Torque

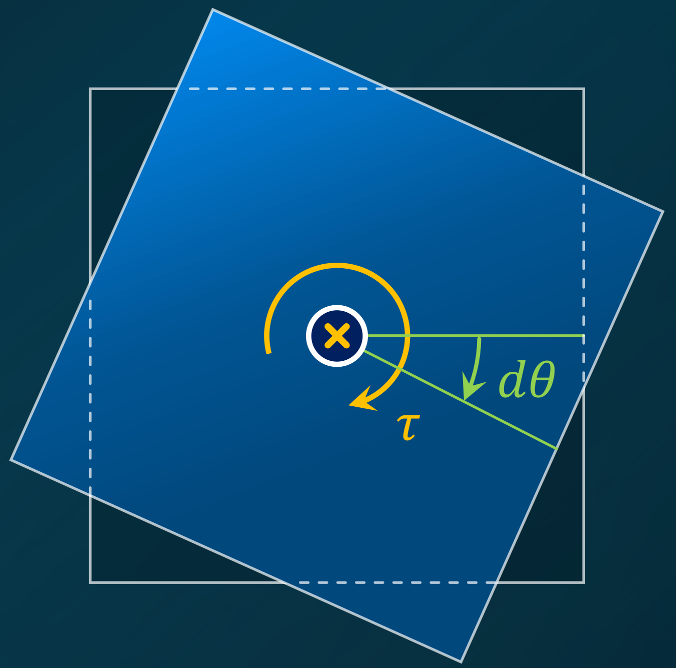
If a torque $\vec{\tau}$ rotates a body through infinitesimal displacement $d\vec{\theta}$, then the infinitesimal work done is

$$dW = \vec{\tau} \cdot d\vec{\theta}$$

If $\vec{\tau}$ and $d\vec{\theta}$ are in the same direction, then

$$dW = \tau d\theta$$

$$\Rightarrow W = \int dW = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$



If a constant torque τ acts on the body, then

$$W = \tau(\theta_2 - \theta_1)$$

$$\Rightarrow W = \tau \Delta\theta$$

A circular disc and a hollow sphere of same mass are rotated about their COM axes as shown. The radius of disc is three times the radius of hollow sphere and disc rotates with half the angular velocity of the hollow sphere. What will be the ratio of their kinetic energies?

Solution :

Mass of the disc $m_d = M$ (Assume)

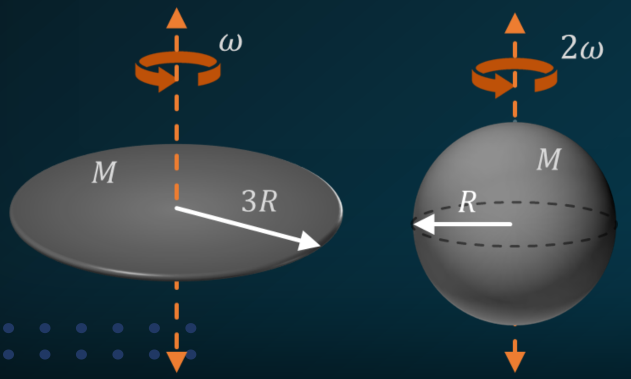
Mass of the hollow sphere $m_s = M$

Radius of the hollow sphere $r_s = R$ (Assume)

Radius of the disc $r_d = 3R$

Angular velocity of the disc $\omega_d = \omega$ (Assume)

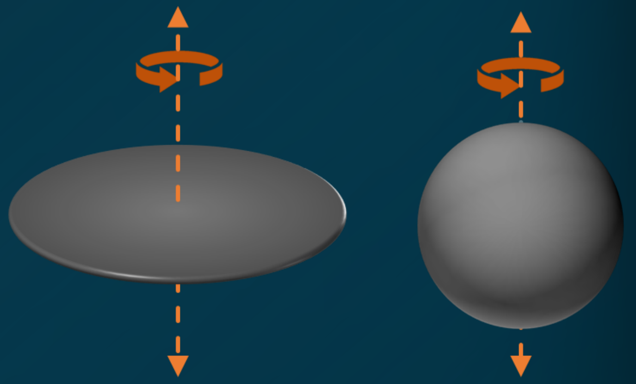
Angular velocity of the hollow sphere $\omega_s = 2\omega$



Rotational kinetic energy of the disc,

$$(KE)_d = \frac{1}{2} I_d \omega_d^2$$

$$= \frac{1}{2} \times \left[\frac{M \times (3R)^2}{2}\right] \times \omega^2 = \frac{9}{4} MR^2\omega^2$$



Rotational kinetic energy of the hollow sphere,

$$(KE)_s = \frac{1}{2} I_s \omega_s^2 \quad = \frac{1}{2} \times \left[\frac{2}{3} MR^2\right] \times (2\omega)^2 = \frac{4}{3} MR^2\omega^2$$

Ratio of $(KE)_d$ to $(KE)_s$,

$$\frac{(KE)_d}{(KE)_s} = \frac{\frac{9}{4} MR^2\omega^2}{\frac{4}{3} MR^2\omega^2} = \frac{27}{16}$$

$$\frac{(KE)_d}{(KE)_s} = \frac{27}{16}$$

a) 9

b) $\frac{27}{4}$

c) $\frac{27}{16}$

d) $\frac{9}{16}$

A mass m is supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release?

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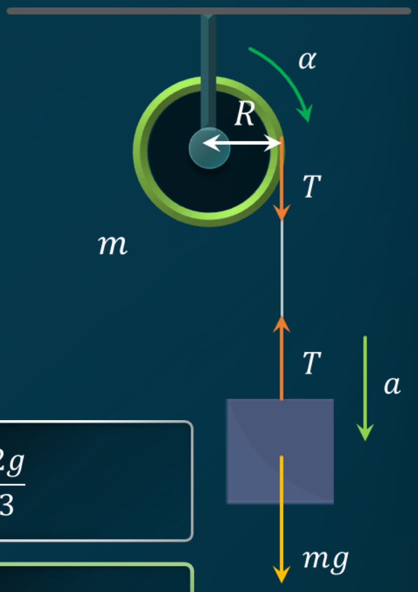


a	$\frac{2g}{3}$
b	$\frac{g}{2}$
c	$\frac{5g}{6}$
d	g

Solution :

For the mass m,

$$mg - T = ma$$

Since the string does not slip on the hollow cylinder,

$$a = R\alpha \quad \dots(1)$$

$$\Rightarrow mg - T = mR\alpha \quad \dots(2)$$

Torque about the centre of the hollow cylinder,

$$RT = I\alpha = mR^2\alpha$$

$$\Rightarrow T = mR\alpha \quad \dots(3)$$

$$a = R\alpha \quad \dots(1)$$

$$mg - T = mR\alpha \quad \dots(2)$$

$$T = mR\alpha \quad \dots(3)$$

Upon solving the equations, we get,

$$mg = 2mR\alpha$$

$$g = 2a$$

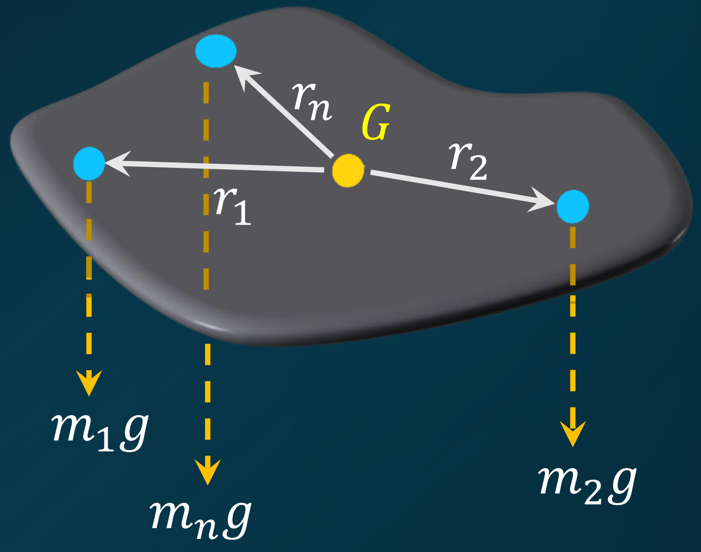
$$a = \frac{g}{2}$$

Centre of Gravity

- The centre of gravity (G) of a body is the point at which the total gravitational torque on the body is zero.

- $\vec{\tau}_g = \sum \vec{\tau}_i = \sum \vec{r}_i \times m_i \vec{g} = \mathbf{0}$

- The COG and COM of a rigid body coincide when the gravitational field is uniform across the body.



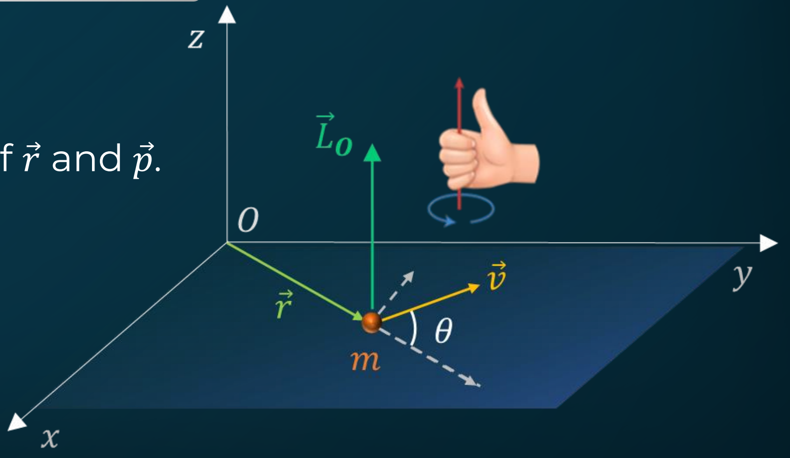
Angular Momentum

Angular momentum is the rotational analogue of linear momentum. It is also called moment of linear momentum.

$\vec{L}_O = \vec{r} \times \vec{p}$ $\because \vec{p} = m\vec{v}$

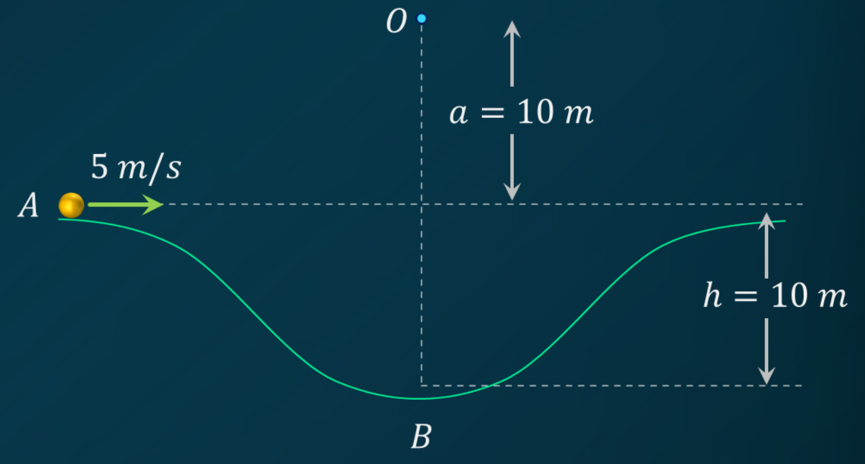
$= m(\vec{r} \times \vec{v})$

- Axial vector
- Always perpendicular to the plane of $\vec{r}$ and $\vec{p}$.
- SI unit: $kg\ m^2/s$



A particle of mass $20\ g$ is released with an initial velocity $5\ m/s$ along the curve from the point A as shown. The point A is at height h from point B. The particle slides along the frictionless surface. When the particle reaches point B, its angular momentum about O will be
(Take, $g = 10\ m/s^2$)

Solution :



Since friction is absent, the mechanical energy of the particle remains constant.

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2 - mgh$$

$$v_B^2 = v_A^2 + 2gh$$

$$= 5^2 + 2(10)(10)$$

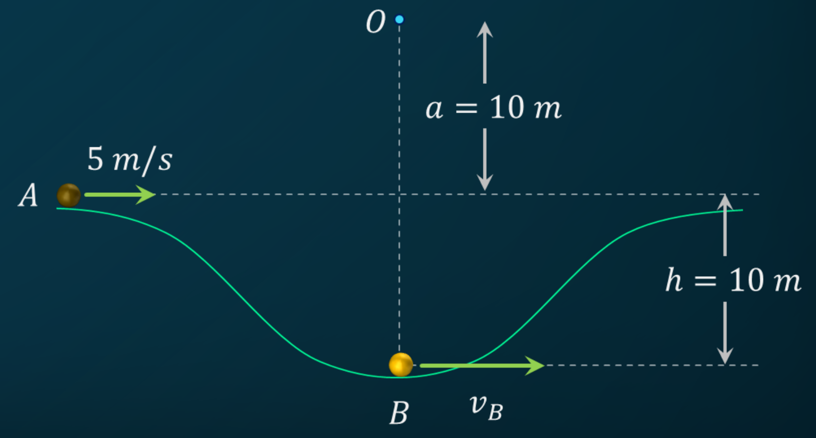
$$v_B = 15\ m/s$$

Angular momentum of the particle about point O,

$$L_O = mv_B(a+h)$$

$$= 20 \times 10^{-3} \times 15 \times (10+10)$$

$$L_O = 6\ kg\ m^2/s$$



Angular Momentum of a System of Particles

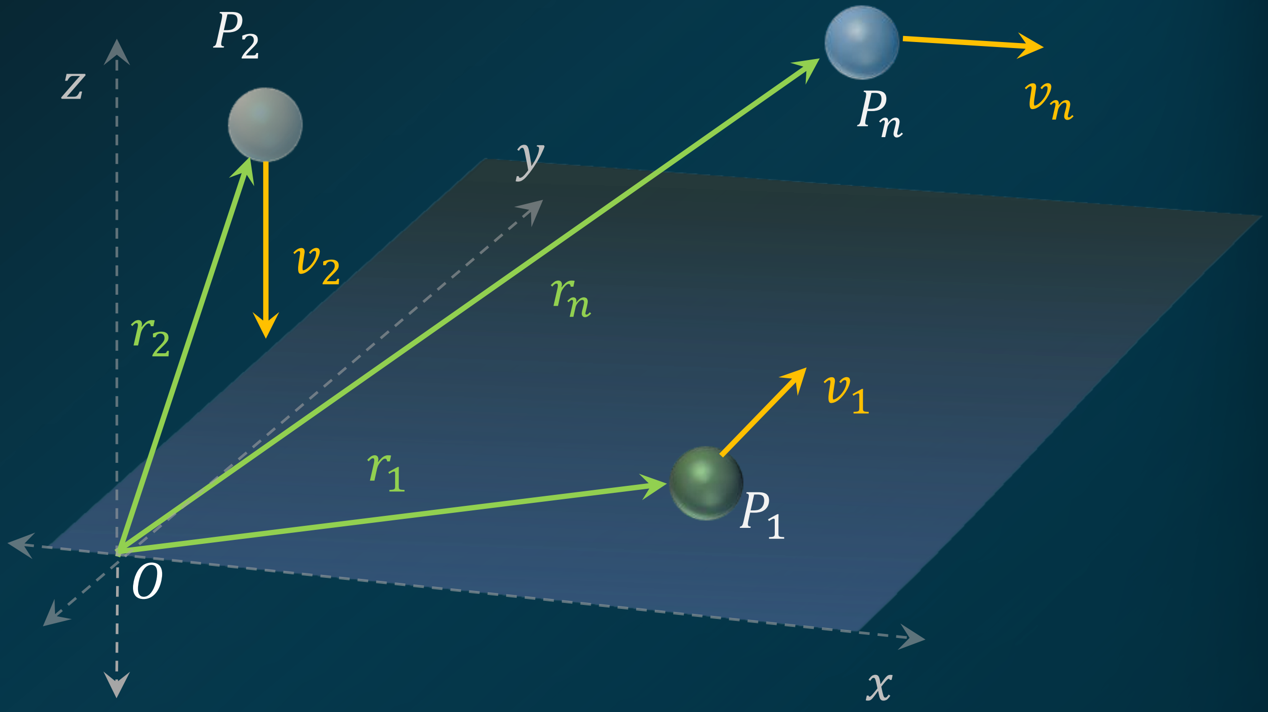
The total angular momentum of a system of particles follows the principle of superposition.

$$\vec{L}_{system,O} = \vec{L}_{1,O} + \vec{L}_{2,O} + \vec{L}_{3,O} \ldots \ldots \vec{L}_{n,O}$$

$$= \sum_{i=1}^{n} \vec{L}_{i,O}$$

$$= \sum_{i=1}^{n} (\vec{r}_i \times \vec{p}_i)_O$$

$$\vec{L}_{system,O} = \sum_{i=1}^{n} m_i (\vec{r}_i \times \vec{v}_i)_O$$



Angular Momentum of a Rigid Body

$$\vec{L}_{sys,O} = \sum_{i=1}^{n} m_i(\vec{r}_i \times \vec{v}_i)$$

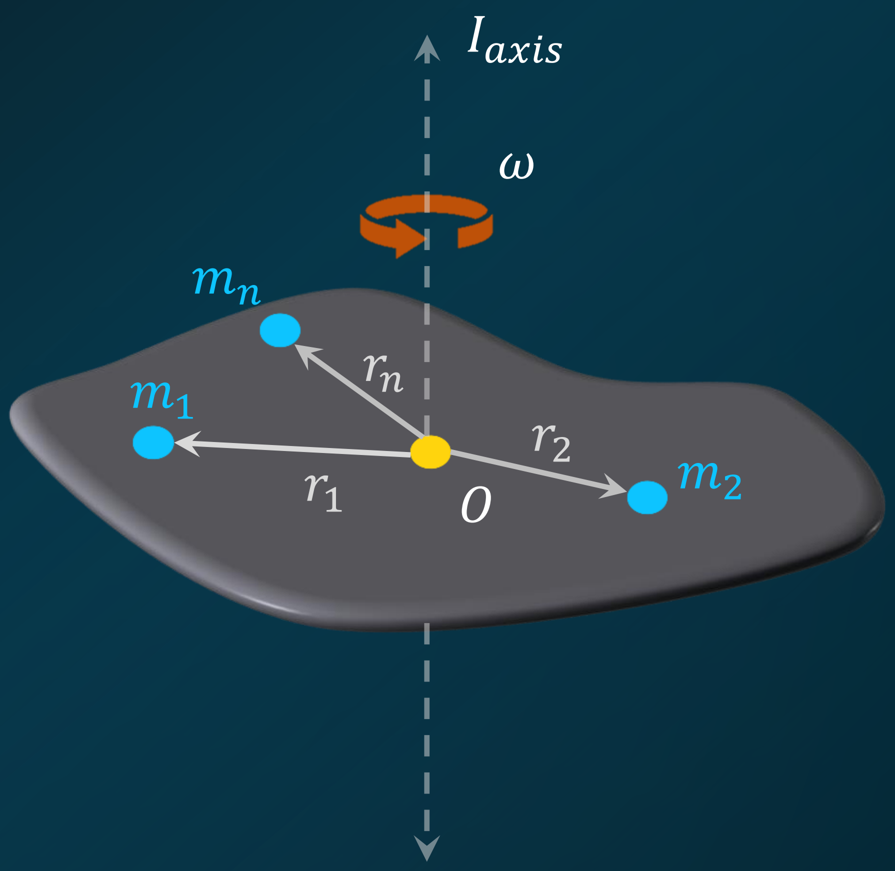
$$= \sum [(r_\perp)(p)]_i$$

$$= \sum (r_\perp m v)_i \qquad (\because v = r_\perp \omega)$$

$$= \omega \sum (m {r_\perp}^2)_i$$

$$L_{sys} = I_O \omega$$

$$(\vec{L}_{sys})_{axis} = I_{axis}\, \vec{\omega}$$



Translational vs Rotational Dynamics

Translational

Applied force causes change in linear momentum of the centre of mass.

$$\sum(\vec{F}_{sys})_{ext} = \frac{d\vec{p}_{sys}}{dt}$$

$$= \frac{d(m\vec{v})_{sys}}{dt}$$

$$\sum(F_{sys})_{ext} = m\vec{a}_{sys}$$

Force is the rate of change of linear momentum.

Rotational

The application of torque causes change in angular momentum of a rigid body at that instant of time.

$$\sum(\vec{\tau}_{ext})_{axis} = \frac{d\vec{L}_{axis}}{dt}$$

$$= \frac{d[I_{axis}\vec{\omega}]}{dt}$$

$$\sum(\vec{\tau}_{ext})_{axis} = I_{axis}\vec{\alpha}$$

Torque is the rate of change of angular momentum.

Conservation of Angular Momentum

When the net torque acting on a system is zero about a given axis, then the total angular momentum of the system about that axis remains constant.

If $\sum(\vec{\tau}_{ext})_{axis} = \mathbf{0}$

$$\sum \frac{d}{dt}(\vec{L}_{axis}) = \mathbf{0}$$

$$\vec{L}_{axis} = \text{constant}$$

Law of conservation of angular momentum is conditional and depends on axis.

A boy of mass M stands at the edge of a platform of radius R that can be freely rotated about its axis. The moment of inertia of the platform is I. The system is at rest when a friend throws a ball of mass m and the boy catches it. If the speed of the ball was v and was moving horizontally along the tangent to the edge of the platform when it was caught by the boy, find the angular speed of the platform after the event.

Solution :

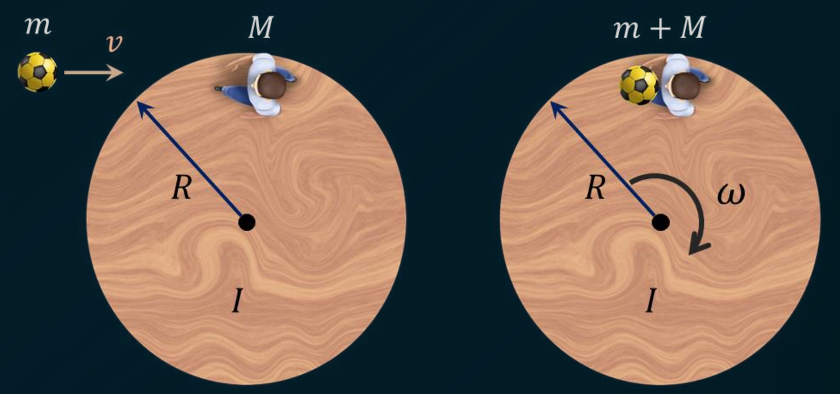
$$\sum (\vec{\tau})_{ext} = \vec{0}$$

On "Platform + Boy + Ball" about axle,

$$\vec{L}_i = \vec{L}_f \quad \text{(about the axle)}$$

$$mvR \otimes + [0] = (I + (M+m)R^2)\omega \otimes$$

$$\omega = \frac{mvR}{(I + (M+m)R^2)}$$

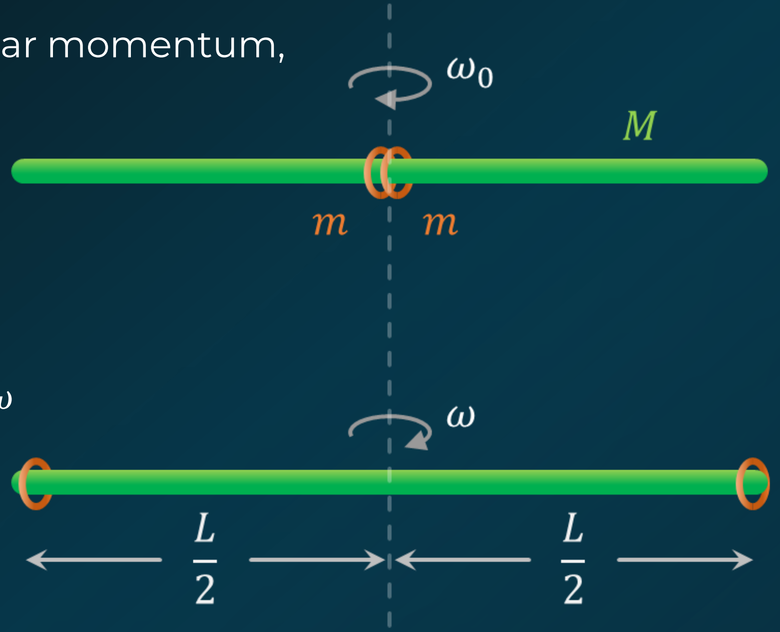


Before catching the ball

After catching the ball

A thin smooth rod of length L and mass M is rotating freely with angular speed ω_0 about an axis perpendicular to the rod and passing through its centre. Two beads of mass m and negligible size are at the centre of the rod initially. The beads are free to slide along the rod. The angular speed of the system, when the beads reach the opposite ends of the rod, will be

Solution : Applying the conservation of angular momentum,



$$I_i\omega_i = I_f\omega_f$$

$$\frac{ML^2}{12}\omega_0 = \left(\frac{ML^2}{12} + m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2\right)\omega$$

$$ML^2\omega_0 = (ML^2 + 6mL^2)\omega$$

$$\omega = \frac{M\omega_0}{M+6m}$$

a) $\frac{M\omega_0}{M+3m}$

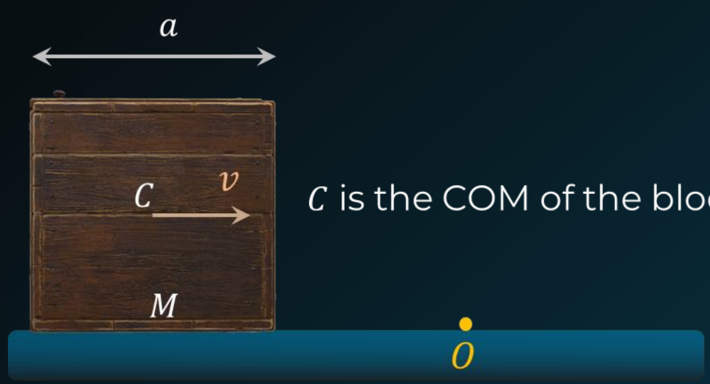
b) $\frac{M\omega_0}{M+m}$

c) $\frac{M\omega_0}{M+2m}$

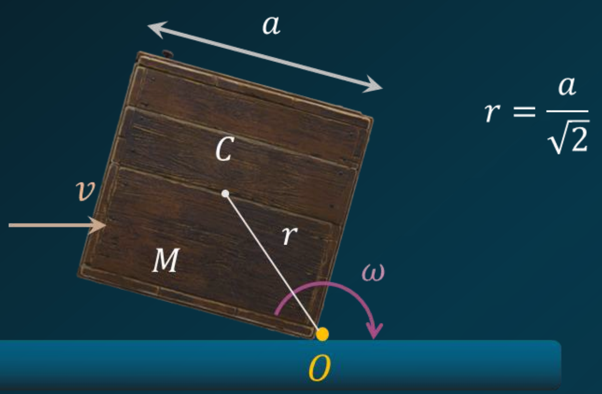
d) $\frac{M\omega_0}{M+6m}$

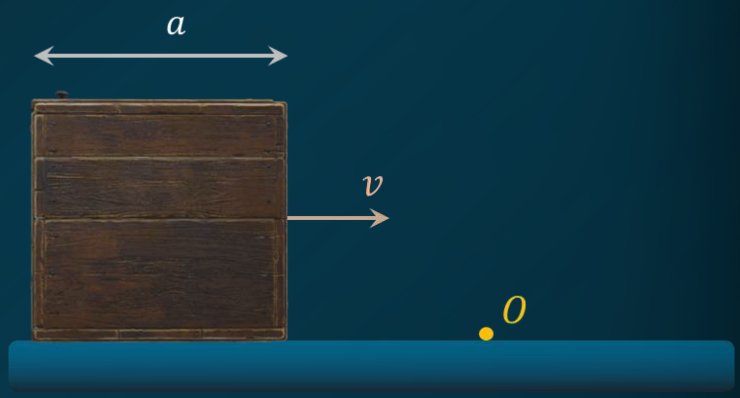
A cubical block of side a is moving with velocity v on a horizontal smooth plane as shown in the figure. It hits a ridge at point O. The angular speed of the block after it hits O is

Solution :



C is the COM of the block





Say, M is the mass of the block

Net torque about O is zero. Thus, angular momentum about O is conserved.

$$L_i = L_f$$

$$\Rightarrow Mv\frac{a}{2} = I_O\omega = (I_{CM} + Mr^2)\omega$$

$$Mv\frac{a}{2} = (I_{CM} + Mr^2)\omega$$

$$\frac{Mva}{2} = \left(\frac{Ma^2}{6} + M\frac{a^2}{2}\right)\omega \Rightarrow \frac{Mva}{2} = \frac{2Ma^2}{3}\omega$$

$$\omega = \frac{3v}{4a}$$

a. $\frac{3v}{4a}$

b. $\frac{3v}{2a}$

c. $\sqrt{\frac{3v}{2a}}$

d. $Zero$

Angular Impulse

When a rigid body is acted upon by an external torque for a short interval of time, it experiences a sudden change in the angular momentum known as angular impulse.

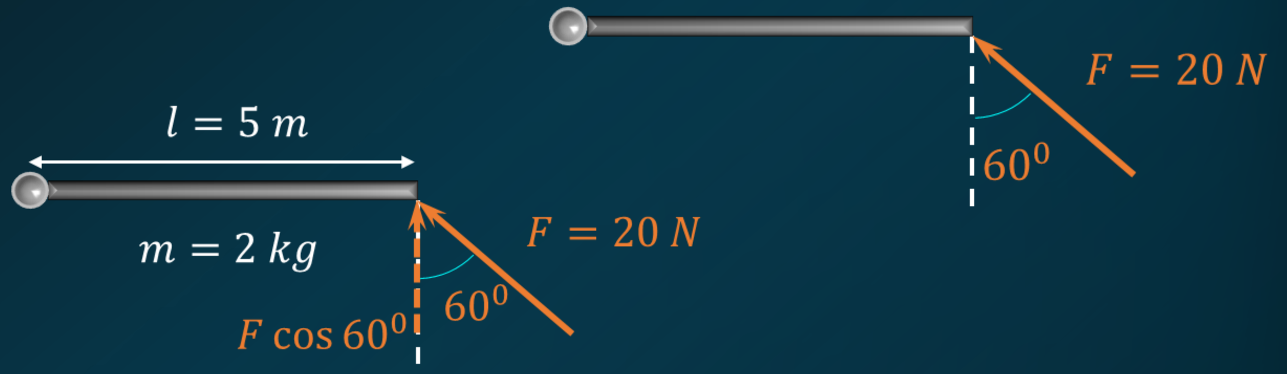
$$\vec{J} = \int_{t_1}^{t_2} \vec{\tau}\, dt = \int_{L_1}^{L_2} d\vec{L}$$

$$\left(\because \vec{\tau} = \frac{d\vec{L}}{dt} \right)$$

Like every other rotational parameter, angular impulse $\vec{J}$ is also defined about an axis.

A rod of mass $2\ kg$ and length $5\ m$ is placed on a frictionless horizontal plane hinged about one of its ends. At the other end, a force $F = 20\ N$ is applied for $0.1\ s$ as shown. Find the angular speed just after the force is applied.

Solution : Only force perpendicular to the length of rod will contribute to change in angular momentum.



$$J = \Delta L = \tau_{ext}\Delta t$$

$$\tau_{ext} = F\cos 60^0 \times l = (20)\left(\frac{1}{2}\right)(5) = 50\ Nm$$

$$\Delta L = I\Delta\omega = \tau_{ext}\Delta t$$

$$\Rightarrow \Delta\omega = \omega = \frac{\tau_{ext}\Delta t}{I} = \frac{3\tau_{ext}\Delta t}{ml^2} \qquad \left(\because I = \frac{ml^2}{3}\right)$$

$$\omega = \frac{3(50)(0.1)}{(2)(5)^2} \Rightarrow \quad \omega = 0.3\ rad/s$$

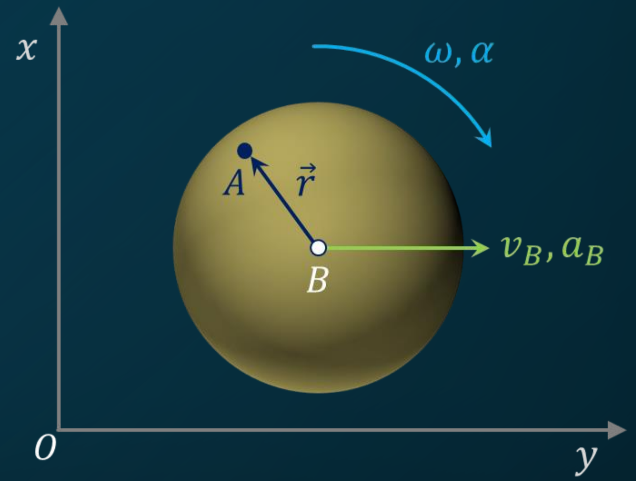
a	$5\ rad/s$
b	$0.3\ rad/s$
c	$0.1\ rad/s$
d	$0.6\ rad/s$

Analysis of Combined Motion

Combined motion is divided into its pure rotational and pure translational counterparts for ease.

Combined rotation and translation | Pure rotation about the COM | Pure translation of the COM

Velocity of point A on the rigid body w.r.t. origin O in the figure is given by,



$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}$$

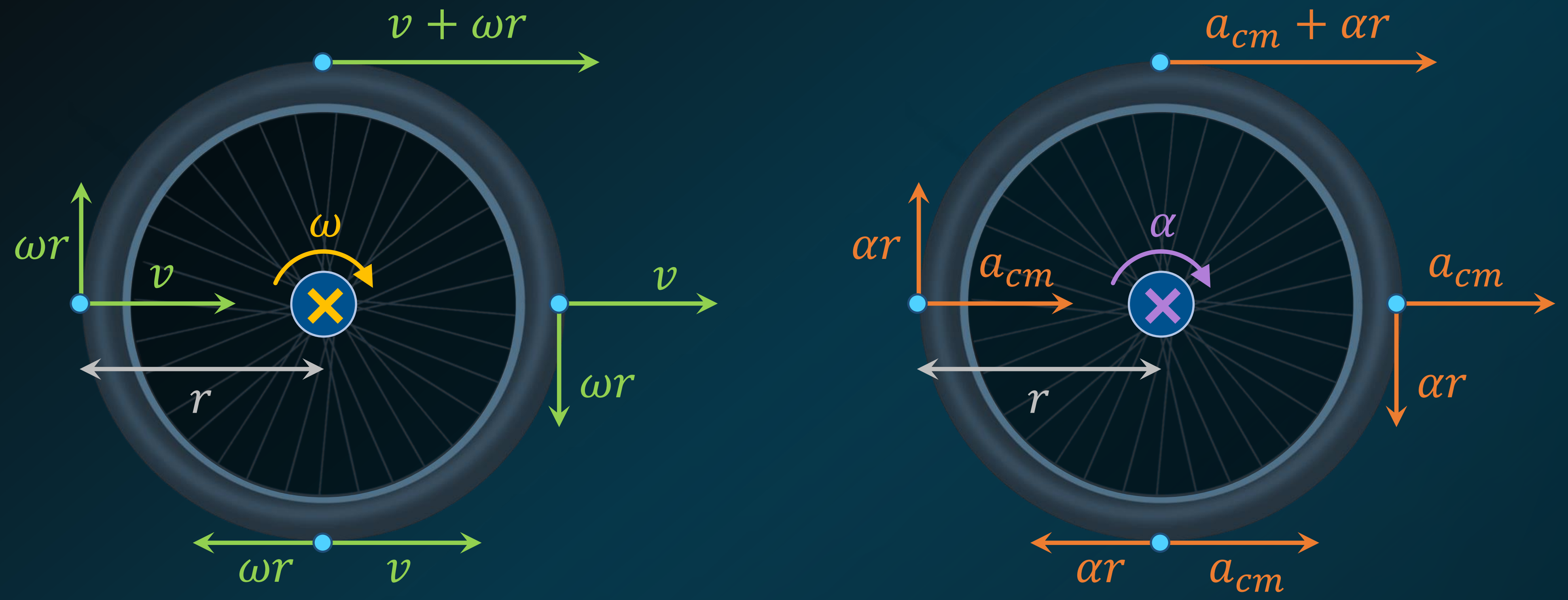
Pure translation

Pure rotation

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}$$

Combined Motion

The velocities and accelerations of various points of a circular rigid body in combined motion are as shown.

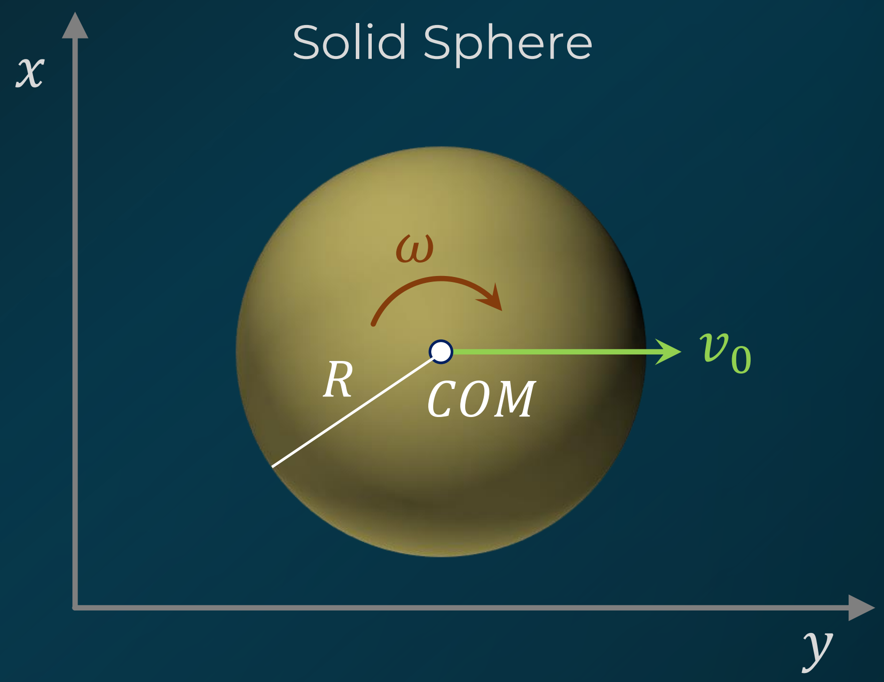


Total Kinetic Energy

The KE of a rigid body in combined motion is obtained by summing the KEs of its rotational and translational counterparts.

$$KE_{total} = KE_{rot} + KE_{trans}$$

$$KE_{total} = \frac{1}{2} I_{COM}\omega^2 + \frac{1}{2} m v_{COM}^2$$

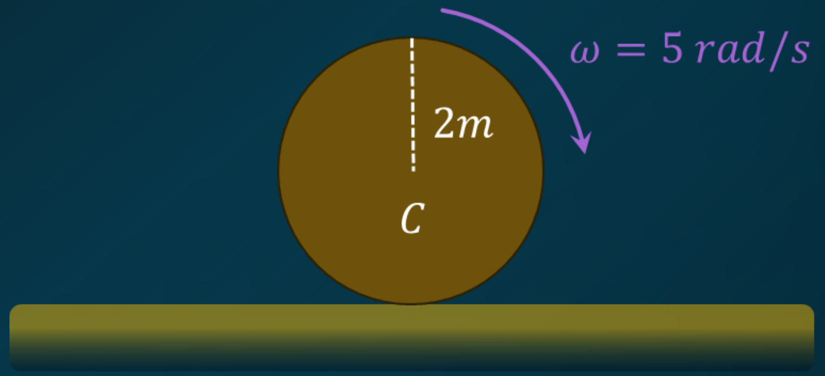


- For a solid sphere of radius R rolling with angular velocity ω and linear speed of v_0 as shown

$$KE_{sphere} = \frac{1}{2} \times \frac{2}{5} m R^2 \omega^2 + \frac{1}{2} m v_0^2$$

$$KE_{sphere} = \frac{m}{10}\left(2R^2\omega^2 + 5v_0^2\right)$$

A cylinder of mass $2\ kg$ and radius $2\ m$ is given a kinetic energy of $150\ J$ and it rolls on a plane as shown. Angular speed of the cylinder is $5\ rad/s$. Find the linear speed of the cylinder.

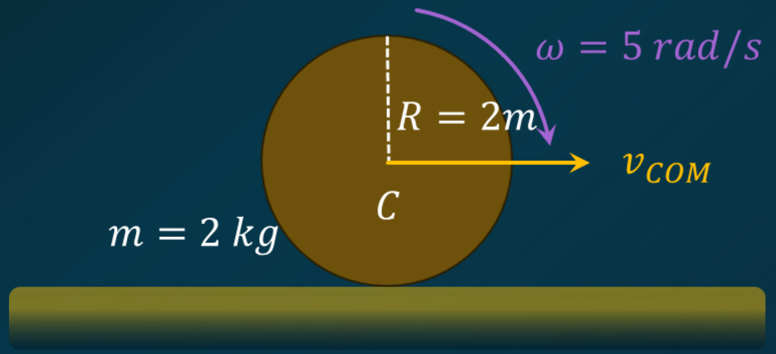


Solution : Total Kinetic energy of the solid cylinder is given by,

$$KE_{total} = \frac{1}{2} I_{COM}\omega^2 + \frac{1}{2} m v_{COM}^2$$

$$KE_{total} = \frac{1}{2} . \frac{mR^2}{2} . \omega^2 + \frac{1}{2} m v_{COM}^2$$

$$\frac{1}{2} m v_{COM}^2 = KE_{total} - \frac{mR^2\omega^2}{4}$$



$KE_{total} = 150\ J$

$$v_{COM}^2 = \frac{2KE_{total}}{m} - \frac{R^2\omega^2}{2} = \frac{2 \times 150}{2} - \frac{(2)^2(5)^2}{2}$$

$$v_{COM}^2 = 150 - 50 = 100 \Rightarrow$$

$$\boldsymbol{v_{COM} = 10\ ms^{-1}}$$

a. $10\ ms^{-1}$

b. $100\ ms^{-1}$

c. $5\ ms^{-1}$

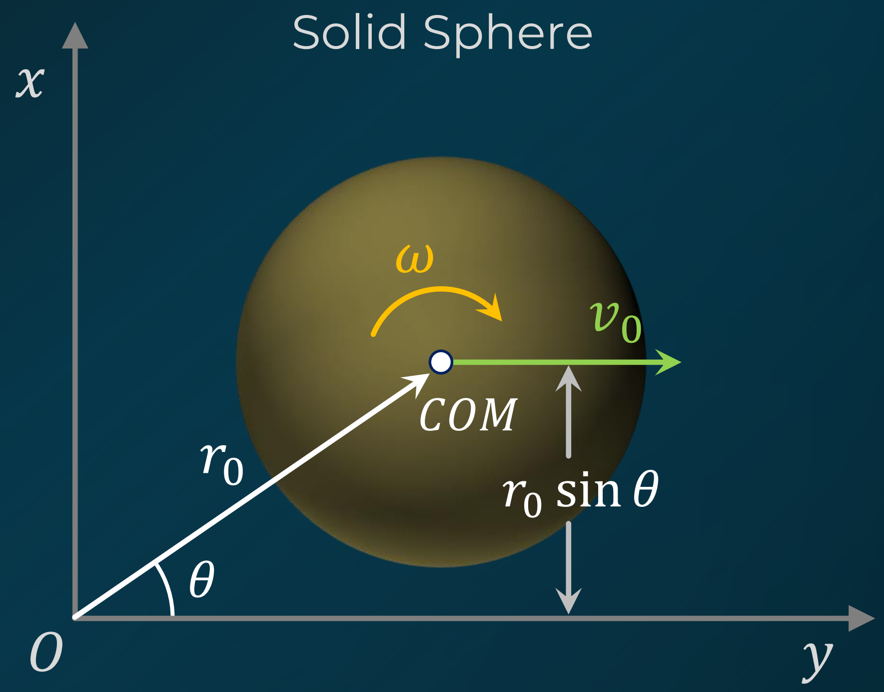
d. $40\ ms^{-1}$

Total Angular Momentum

The total angular momentum of a rigid body about an axis is obtained by adding the angular momenta of (i) the body w.r.t. COM and (ii) COM w.r.t. the desired axis.

$$\vec{L}_O = \vec{L}_{sys,CM} + \vec{L}_{CM,O}$$

$$\vec{L}_O = I_{CM}\vec{\omega} + (\vec{r}_{CM} \times m\vec{v}_{CM})_O$$



- For a solid sphere of radius R rolling with angular velocity ω and linear speed of v_0 as shown

$$\vec{L}_{sphere} = \frac{2}{5}mR^2\omega \;\times\; + mv_0r_0\sin\theta \;\times$$

A solid cylinder of mass 5 kg and radius 2 m is rolling on the ground with translational speed of 30 m/s and angular speed of 10 rad/s at the instant considered. What will be the magnitude of its angular momentum w.r.t. an observer sitting in the apartment at the height of 6 m.

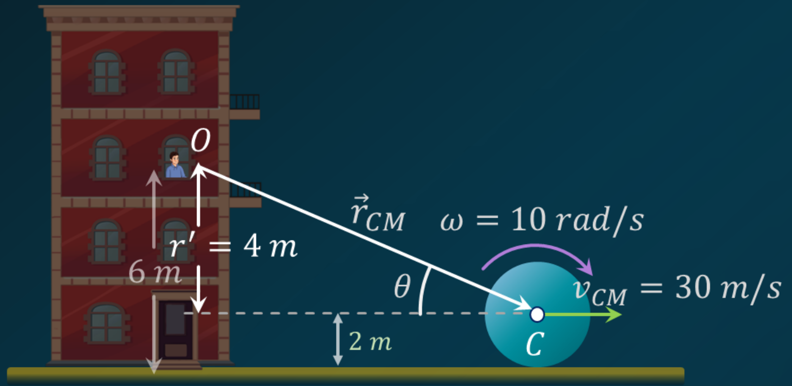
Solution :

Mass of the solid cylinder, $m = 5\ kg$

Radius of the solid cylinder, $r = 2\ m$

Angular velocity, $\omega = 10\ rad/s$ ✕

Perpendicular distance of observer from the line of motion of COM, $r' = 4\ m$



From the definition of total angular momentum,

$$\vec{L}_O = I_{CM}\vec{\omega} + (\vec{r}_{CM} \times m\vec{v}_{CM})_O$$

$$\vec{L}_O = I_{CM}\omega \otimes + m v_{CM} r_{CM} \sin\theta \odot$$

$$L_0 = -\frac{1}{2} m r^2 \omega + m v_{CM} r'$$

$$L_0 = -\left(\frac{1}{2} \times 5 \times 2^2 \times 10\right) + (5 \times 30 \times 4)$$

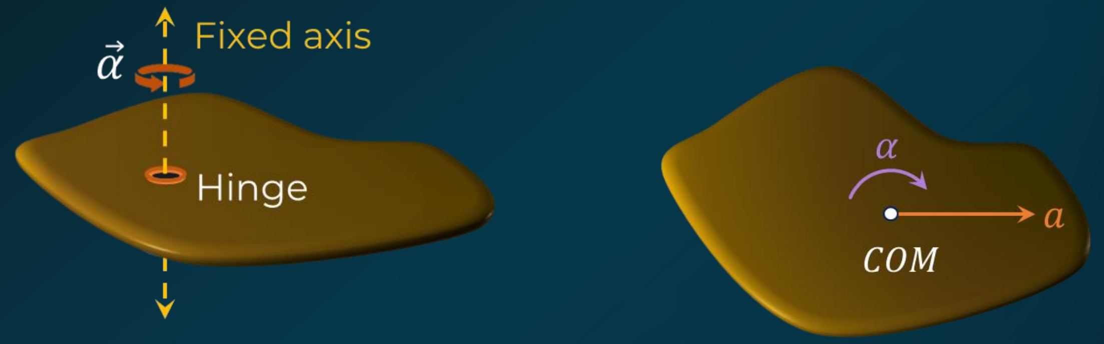
$$L_0 = \mathbf{500}\ kg\ m^2/s$$

a	$600\ kg\ m^2/s$
b	$500\ kg\ m^2/s$
c	$240\ kg\ m^2/s$
d	$400\ kg\ m^2/s$

Dynamics of Rigid Body Motion

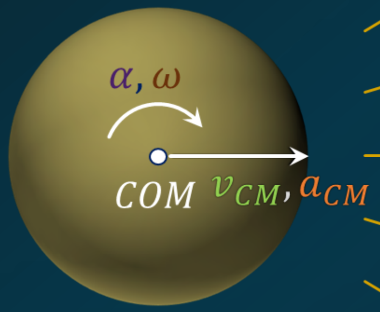
Pure Rotation

The best choice of axis is the one passing through the fixed axis (hinge).



Free Rotation

The best choice of axis is the one passing through the COM (parallel to the angular acceleration)



$$\vec{F}_{ext} = M\vec{a}_{CM}$$

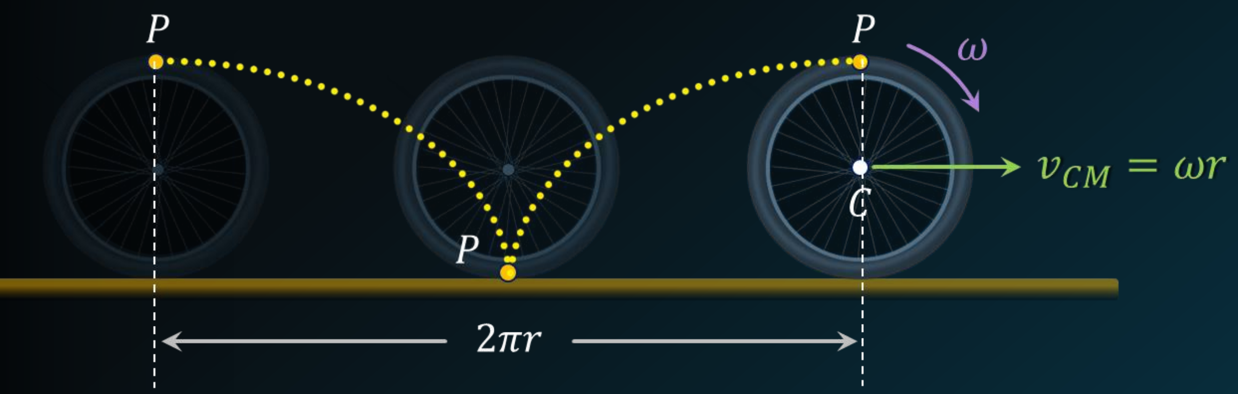
$$\vec{\tau}_{CM} = I_{CM}\vec{\alpha}$$

$$\vec{p}_{sys} = M\vec{v}_{CM}$$

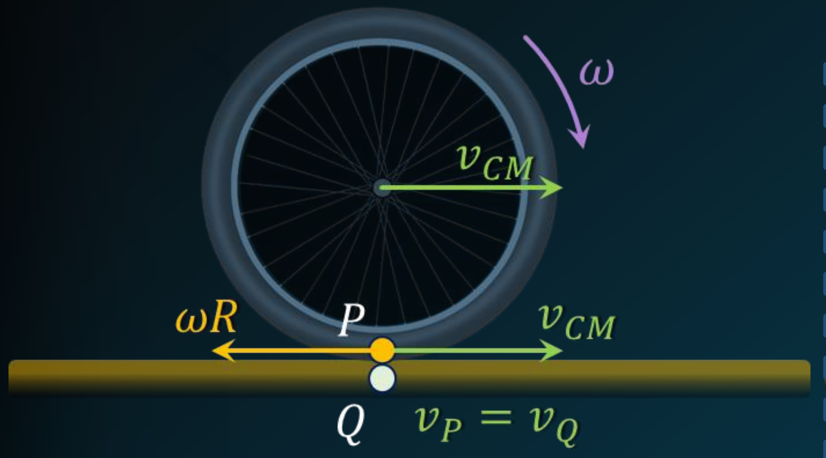
$$KE_{total} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

$$\vec{L}_{total} = I_{CM}\vec{\omega} + (\vec{r}_{CM} \times M\vec{v}_{CM})$$

Pure Rolling



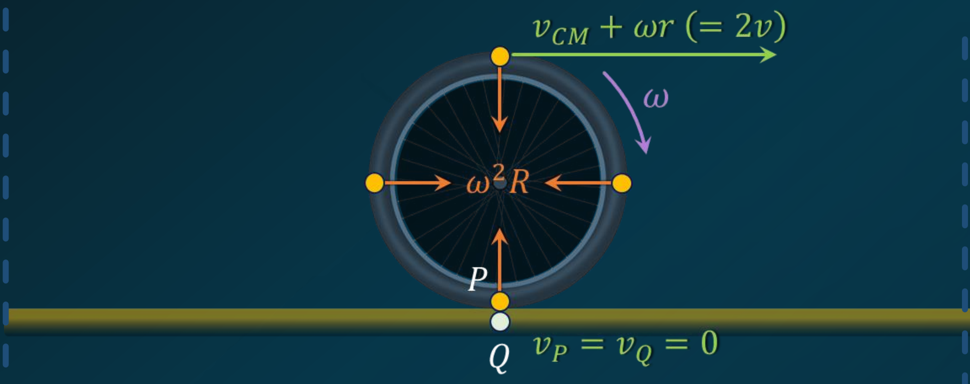
- Particles of the wheel follow a path/loci called cycloid.
- The displacement of the COM in one full rotation is $2\pi r$, where r is the radius of the wheel.
- The instantaneous velocity of the point in contact with the road is zero. (No relative motion/pure rolling)



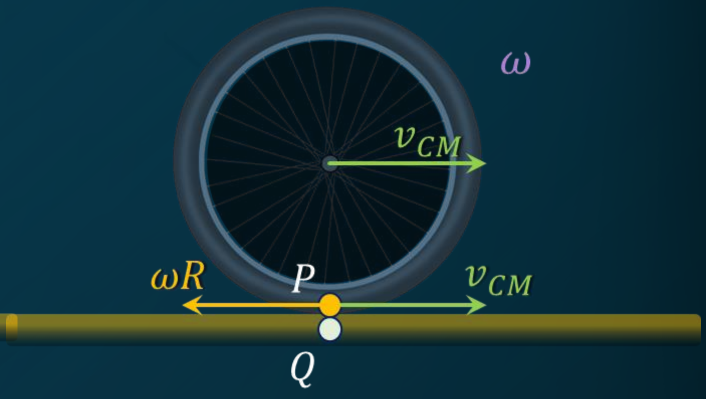
In the case of pure rolling, $v_P = v_Q$

If the ground is at rest, $v_Q = 0$

$\Rightarrow v_P = v_{CM} - \omega R = 0 \qquad v_{CM} = \omega R$



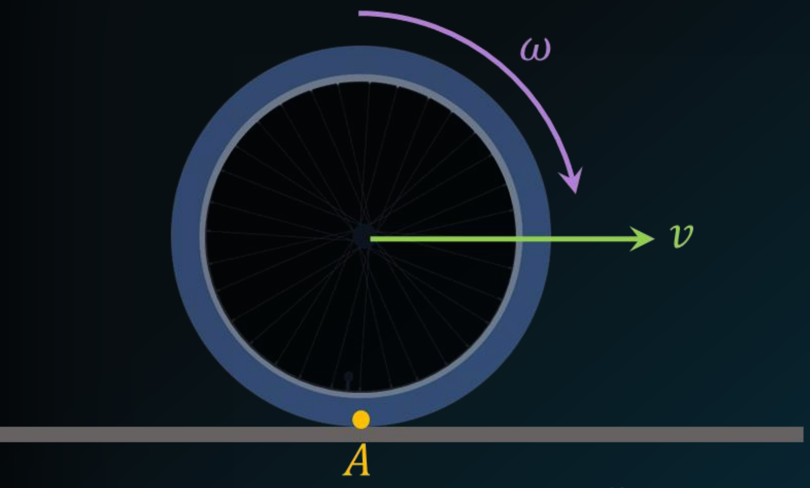
- Despite the instantaneous contact point being stationary, the wheel continues rolling without slipping due to its ongoing rotational motion.
- The centripetal acceleration is same at all points on the periphery equal to $\omega^2 R$.



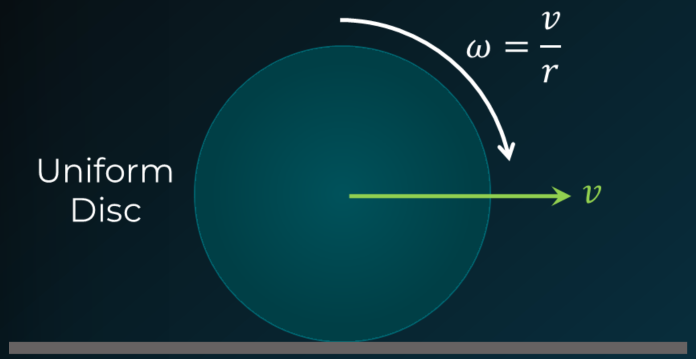
When the ground is at rest,

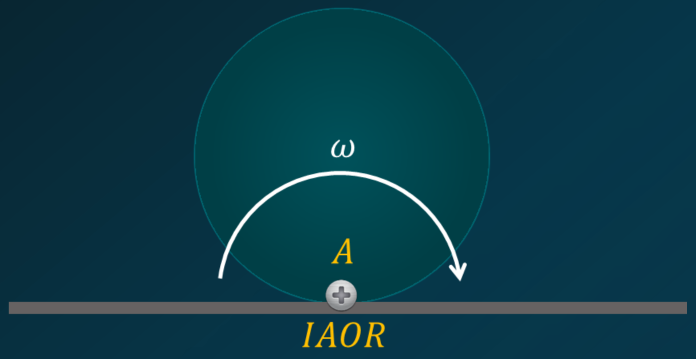
$v_{CM} < \omega r$	Backward Slipping
$v_{CM} = \omega r$	Pure Rolling
$v_{CM} > \omega r$	Forward Slipping

Instantaneous Axis of Rotation



Instantaneous Axis of Rotation concept helps us to treat the case of combined motion as a case of pure rotational motion.





$$KE_{total} = \frac{1}{2}mv^2 + \frac{1}{2}I_{CM}\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\omega^2\right)$$

$$KE_{total} = \frac{3}{4}mv^2$$

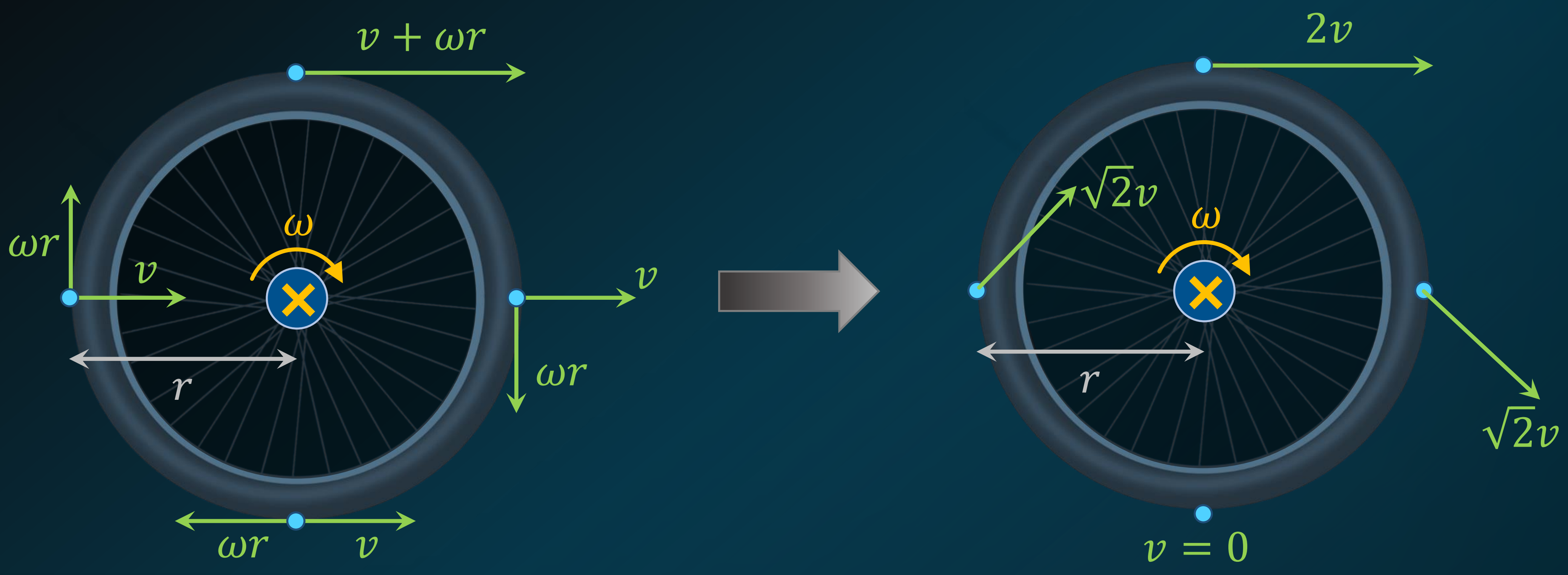
$$KE_{total} = \frac{1}{2}I_{IAOR}\omega^2 = \frac{1}{2}(I_{CM} + mr^2)\omega^2$$

$$= \frac{1}{2}\left(\frac{1}{2}mr^2 + mr^2\right)\omega^2$$

$$KE_{total} = \frac{3}{4}mv^2$$

Velocities of various points

The velocities of various points of a circular rigid body in combined motion are as shown.

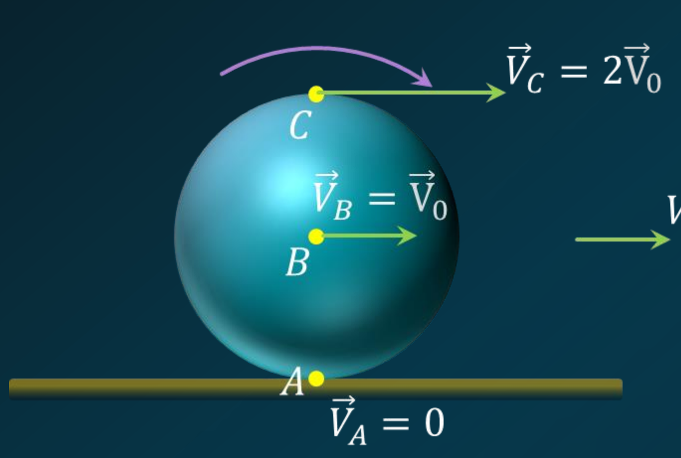


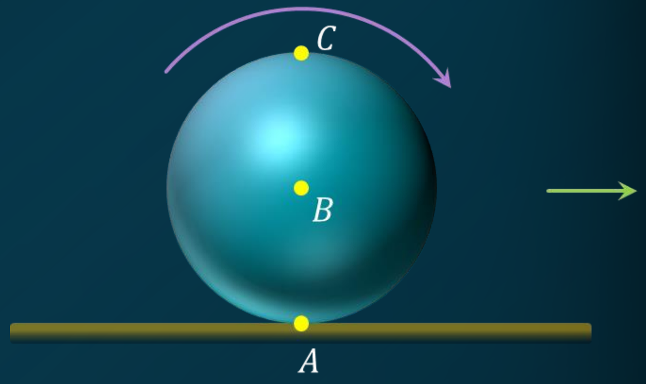
Pure rolling $(v = \omega r)$

A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, *A* is point of contact, *B* is the center of sphere and *C* is the topmost point. Then,

Solution : Say, $\vec{V}_0$ is the velocity of the sphere. Then,

$\vec{V}_A = 0$ $\quad$ $\vec{V}_B = \vec{V}_0$ $\quad$ $\vec{V}_C = 2\vec{V}_0$





$\vec{V}_C - \vec{V}_A = 2\vec{V}_0$

$\vec{V}_B - \vec{V}_C = \vec{V}_0 - 2\vec{V}_0 = -\vec{V}_0$

$\vec{V}_B - \vec{V}_A = \vec{V}_0$

$\vec{V}_C - \vec{V}_A = 2\vec{V}_0 \neq 2(\vec{V}_B - \vec{V}_C)$ — *Option a. is incorrect*

$\vec{V}_C - \vec{V}_B = \vec{V}_0 = \vec{V}_B - \vec{V}_A$ — *Option b. is correct*

$|\vec{V}_C - \vec{V}_A| = 2V_0 = 2|\vec{V}_B - \vec{V}_C|$ — *Option c. is correct*

$|\vec{V}_C - \vec{V}_A| = 2V_0 \neq 4|\vec{V}_B|$ — *Option d. is incorrect*

a) $\vec{V}_C - \vec{V}_A = 2(\vec{V}_B - \vec{V}_C)$

b) $\vec{V}_C - \vec{V}_B = \vec{V}_B - \vec{V}_A$

c) $|\vec{V}_C - \vec{V}_A| = 2|\vec{V}_B - \vec{V}_C|$

d) $|\vec{V}_C - \vec{V}_A| = 4|\vec{V}_B|$

Kinetic Energy (Pure Rolling)

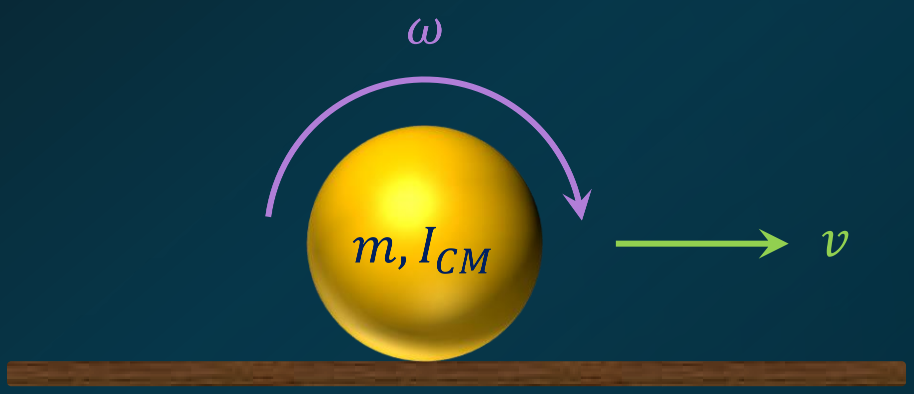
$$KE_{total} = KE_{rot} + KE_{trans}$$

$$= \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} m(\omega r)^2$$

$$= \frac{1}{2} (I_{CM} + mr^2)\omega^2$$

$$\frac{KE_{rot}}{KE_{trans}} = \frac{I_{CM}}{mr^2}$$



A circular disc of mass $2\ kg$ and radius $10\ cm$ rolls without slipping with a speed $2\ m/s$. The total kinetic energy of disc is

Solution :

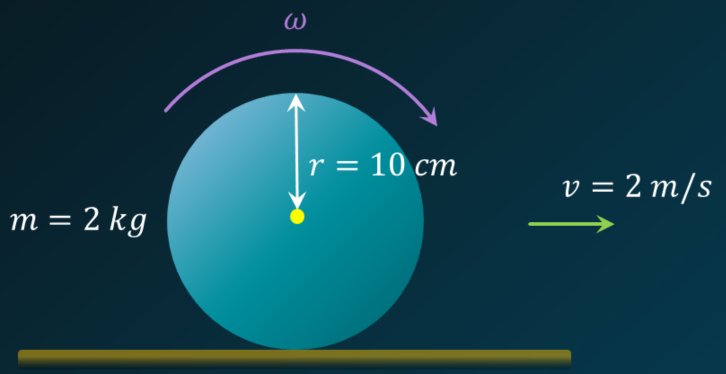
$$KE_{total} = \frac{1}{2}(I_{CM} + mr^2)\omega^2$$

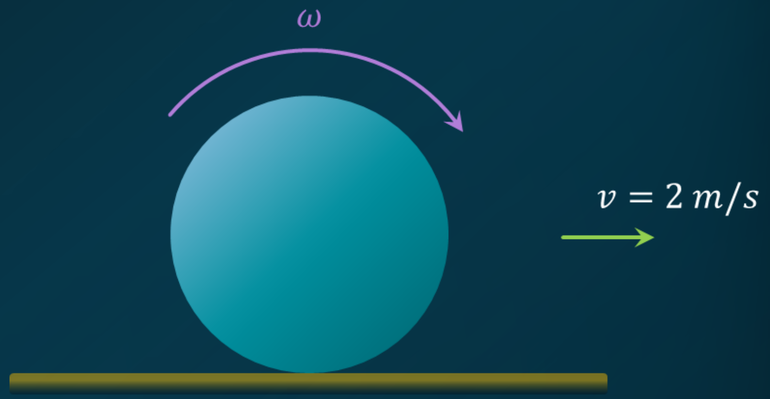
$$KE_{total} = \frac{1}{2}\left(\frac{1}{2}mr^2 + mr^2\right)\left(\frac{v}{r}\right)^2$$

$$KE_{total} = \frac{1}{2}\cdot\frac{3}{2}mr^2\left(\frac{v^2}{r^2}\right) = \frac{3}{4}mv^2$$

$$KE_{total} = \frac{3}{4}(2)(2)^2 = 6\ J$$

$$KE_{total} = 6\ J$$





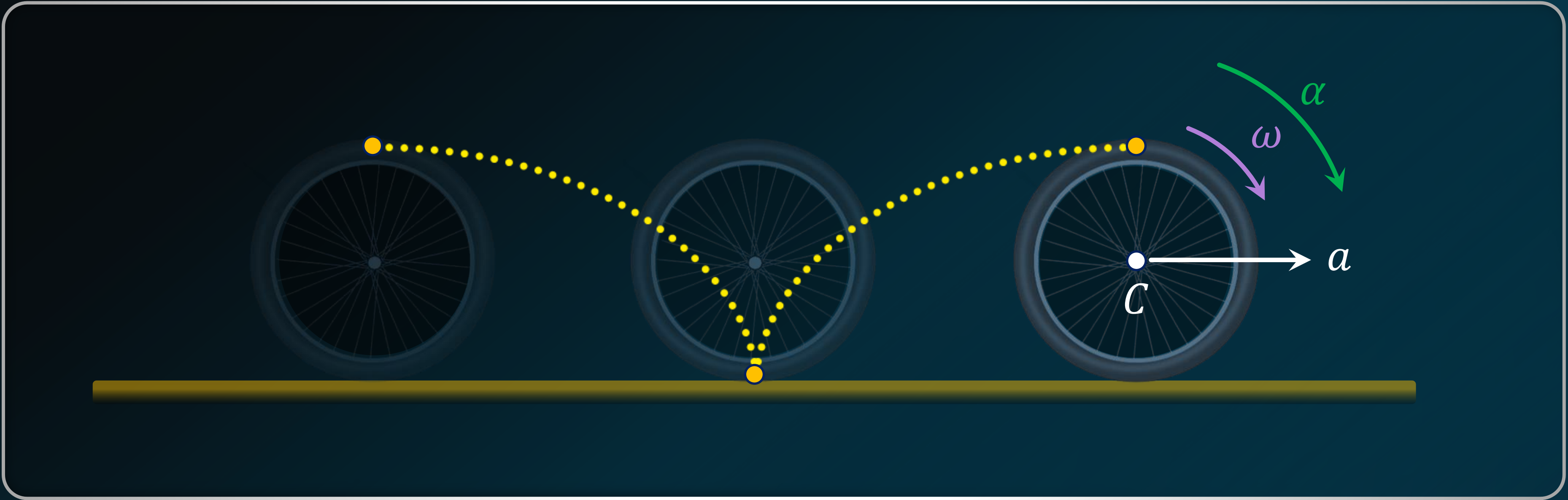
a) $10\ J$

b) $6\ J$

c) $2\ J$

d) $4\ J$

Accelerated Pure Rolling



For the centre of mass of rigid body in pure rolling motion,

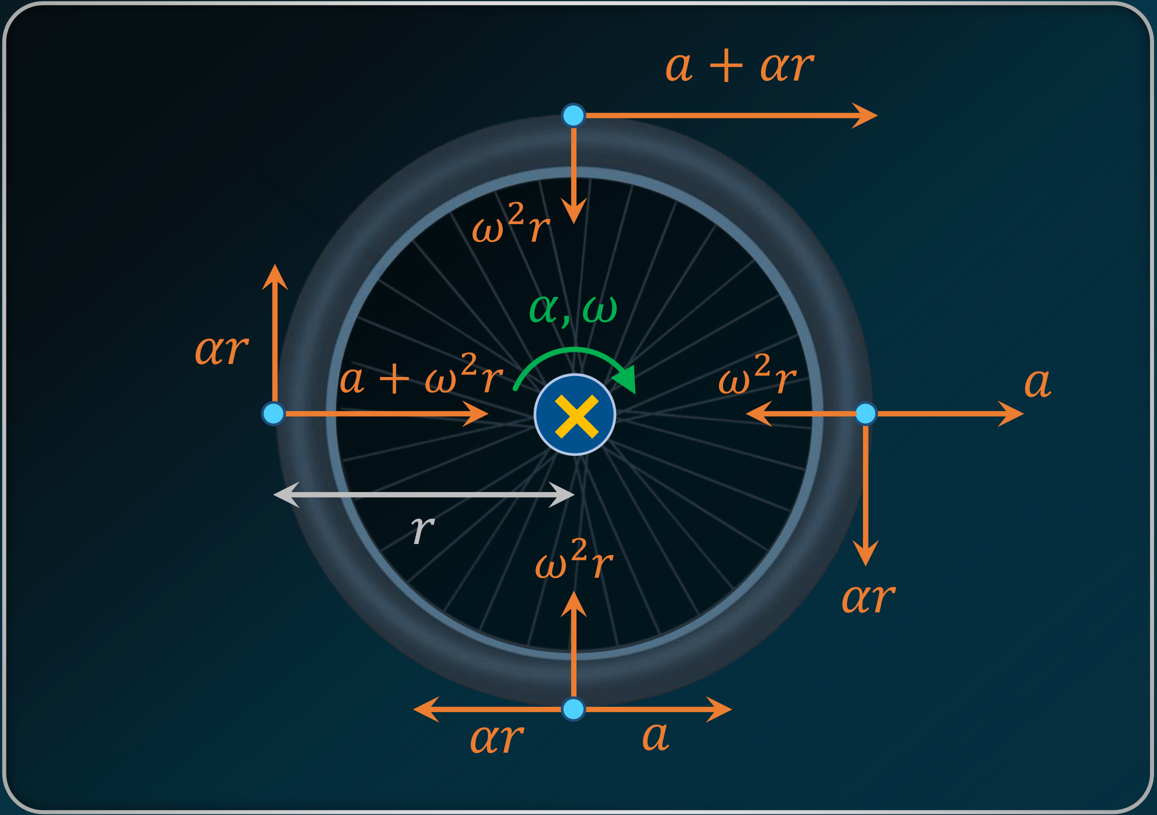
$$a_{CM} = \frac{dv_{CM}}{dt} = \frac{d(r\omega)}{dt}$$

$$= r\frac{d\omega}{dt}$$

$$a_{CM} = r\alpha$$

Acceleration of various points

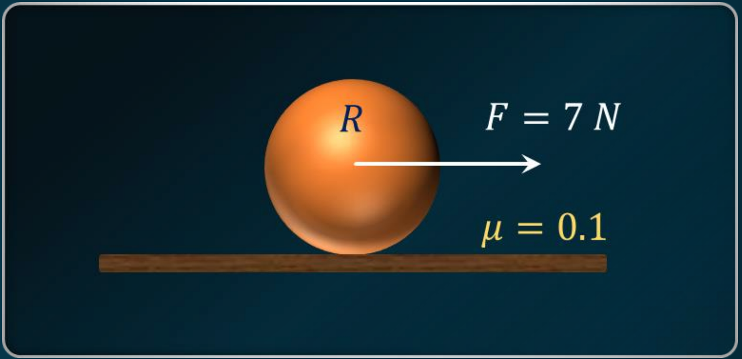
The acceleration of various points of a circular rigid body in combined motion are as shown. (Ground frame)



$a = \alpha r$ (Pure rolling)

A solid sphere of mass $10\ kg$ is placed on a rough surface having coefficient of friction $\mu = 0.1$. A constant force $F = 7\ N$ is applied along a line passing through the centre of the sphere as shown such that it rolls without slipping. The value of frictional force on the sphere is

Solution :



Maximum value of kinetic friction,

$$f_{max} = \mu mg = 10\ N$$

Equation for the translational motion,

$$F - f = ma$$

$$F - f = m\alpha R \quad ...(1) \quad (\because \text{Assuming pure rolling})$$

Equation for the rotational motion,

$$fR = I\alpha$$

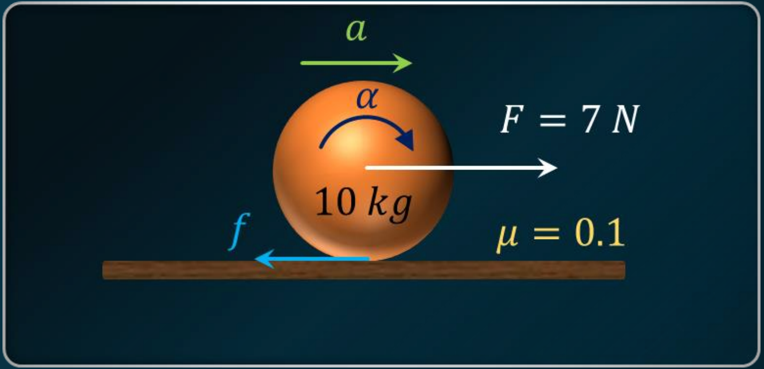
$$\alpha = \frac{fR}{I} \quad ...(2)$$

$$F - f = \frac{fmR^2}{I} \qquad f = \frac{F}{1 + \frac{mR^2}{I}}$$

$$f = \frac{7}{1 + \frac{5}{2}} \qquad \left(\because I = \frac{2}{5}MR^2\right)$$

$$= 2\ N < f_{max}$$

$$f = 2\ N$$



Pure Rolling on an Inclined Plane

For a body rolling w/o slipping on a rough wedge,

$u = 0\ ;\ \omega_0 = 0$

Linear and angular acceleration of the body are constant.

$v = u + at = at$

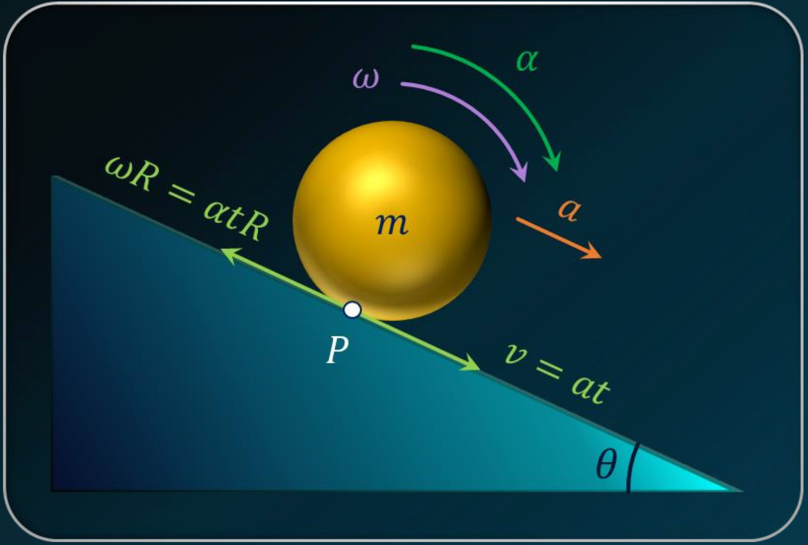
$\omega = \omega_0 + \alpha t = \alpha t$

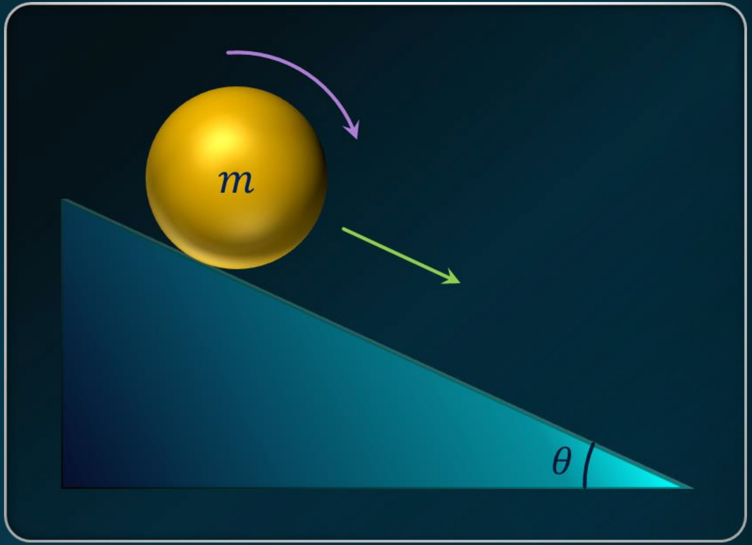
At all instances of pure rolling,

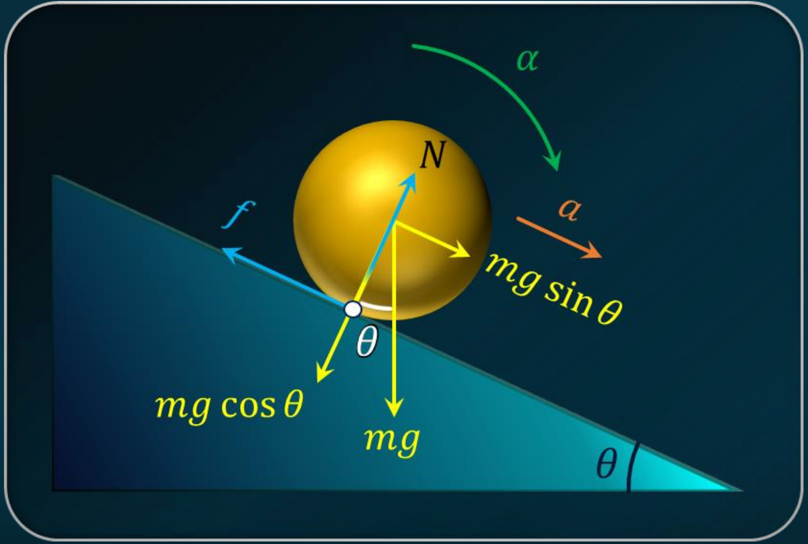
$v = \omega R$

$at = \alpha t R$

$a = \alpha R$







- No force other than friction induces torque in the body about the COM.
- In order to begin (and maintain) pure rolling, frictional force will act in the upward direction of the incline.

A rigid body of mass m, radius R, and moment of inertia I starts pure rolling on a wedge of height h as shown. Find out the time taken by the body to reach the bottom of the inclined plane. K is the radius of gyration of the body about the axis passing through its COM. ($I = MK^2$)

Solution :

Force equation for the rolling body,

$$mg\sin\theta - f = ma$$

Torque equation for the rolling body,

$$fR = I\alpha$$

$$f = \frac{Ia}{R^2} \qquad (\because a = \alpha R)$$

$$mg\sin\theta - \frac{Ia}{R^2} = ma$$

$$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

Time taken in reaching the bottom

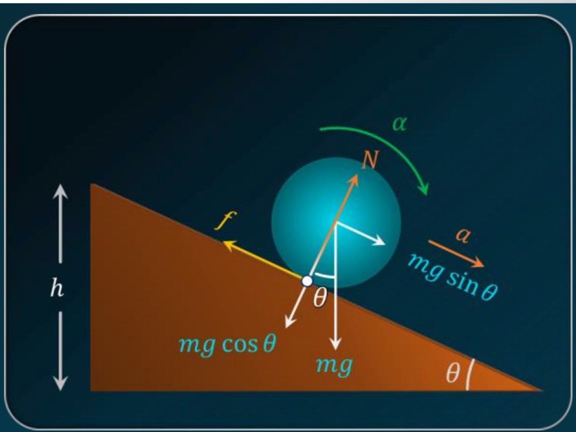
$$d = ut + \frac{1}{2}at^2$$

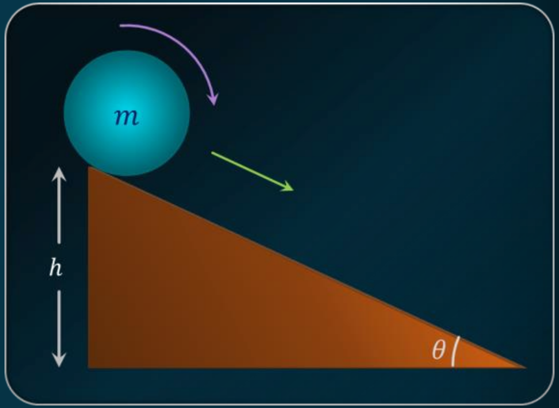
$$\frac{h}{\sin\theta} = 0 + \frac{1}{2} \times \frac{g\sin\theta}{1 + \frac{I}{mR^2}} \times t^2$$

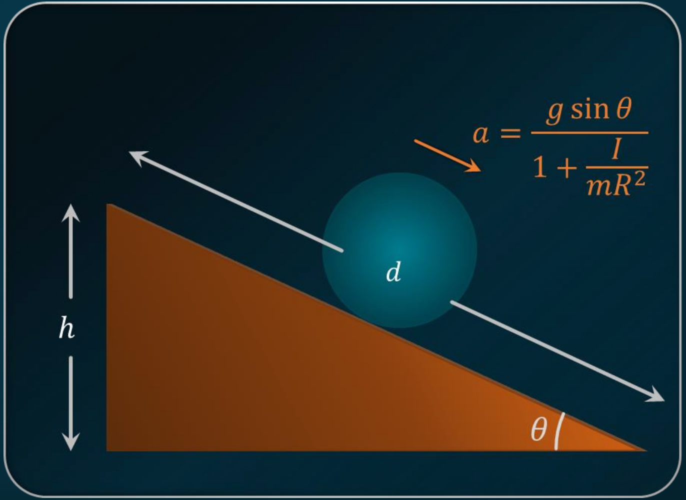
$$t^2 = \frac{2h\left(1 + \frac{I}{mR^2}\right)}{g\sin^2\theta}$$

$$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1 + \frac{K^2}{R^2}\right)}$$

$(\because I = mK^2)$



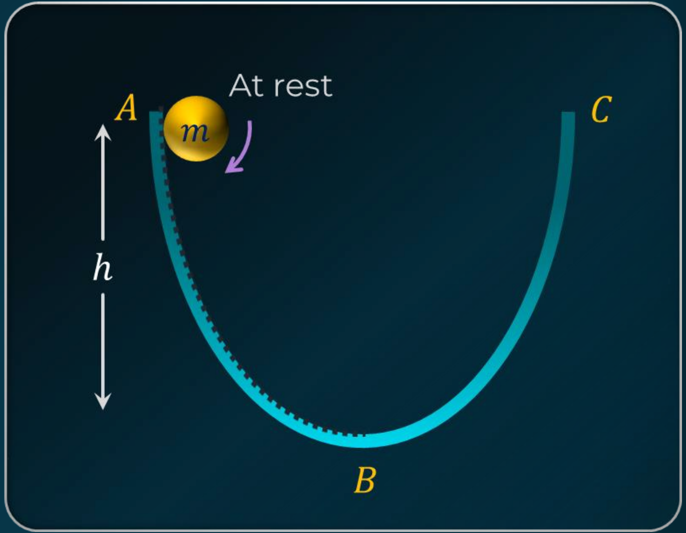




A solid ball of radius r rolls down a parabolic path ABC from a height $h(>> r)$ without slipping as shown in figure. Portion AB of the path is rough while BC is smooth. How high will the ball climb in BC?

Solution :

In portion BC, friction is absent. Therefore only KE_{trans} will be converted into potential energy. KE_{rot} will remain constant.



To find KE_{trans}:

$$\frac{KE_{rot}}{KE_{trans}} = \frac{I_{CM}}{mr^2} = \frac{\frac{2}{5}mr^2}{mr^2}$$

$$\frac{KE_{rot}}{KE_{trans}} = \frac{2}{5}$$

$$\frac{KE_{rot}}{KE_{total}} = \frac{2}{7}$$

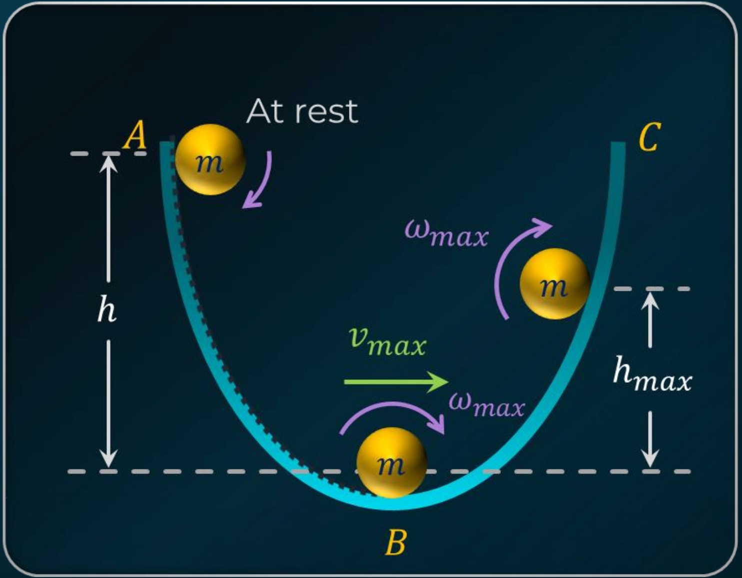
$$KE_{rot} = \frac{2mgh}{7}$$

$$KE_{trans} = \frac{5mgh}{7}$$

$$mgh_{max} = \frac{5mgh}{7}$$

(Energy spent in climbing the other side)

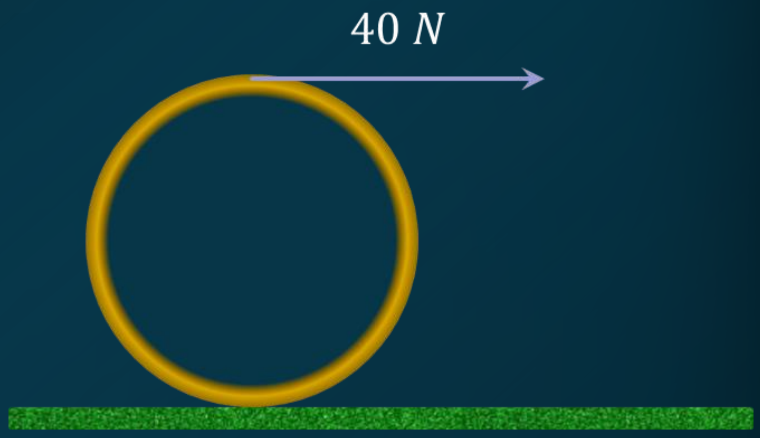
$$h_{max} = \frac{5h}{7}$$



A string is wound around a hollow cylinder of mass $5\ kg$ and radius $0.5\ m$. If the string is now pulled with a horizontal force of $40\ N$ and the cylinder is rolling without slipping on a frictionless horizontal surface (see figure), then the angular acceleration of the cylinder will be (Neglect the mass and thickness of the string)

Solution :

Given, $m = 5\ kg,\ r = 0.5\ m$



As the cylinder is rolling without slipping, horizontal force F produces torque about the centre as shown.

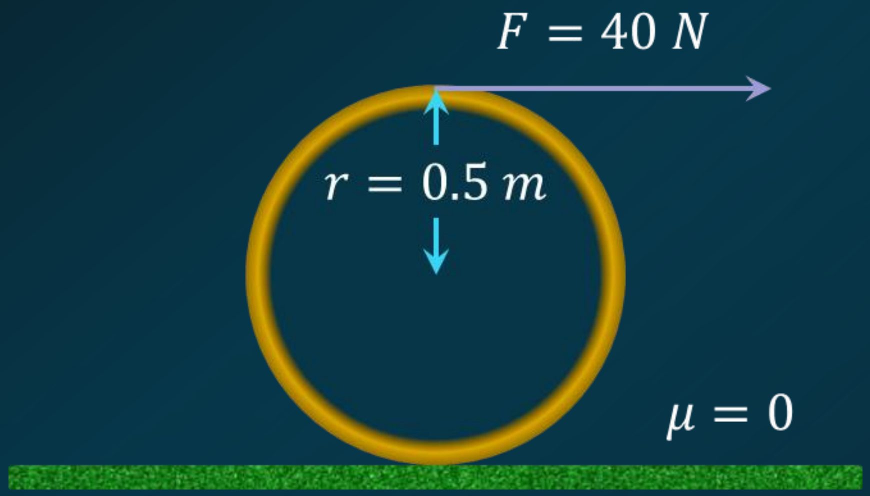
$$\tau = rF \quad (\because r \perp F)$$

$$I\alpha = 0.5 \times 40$$

$$mr^2\alpha = 20$$

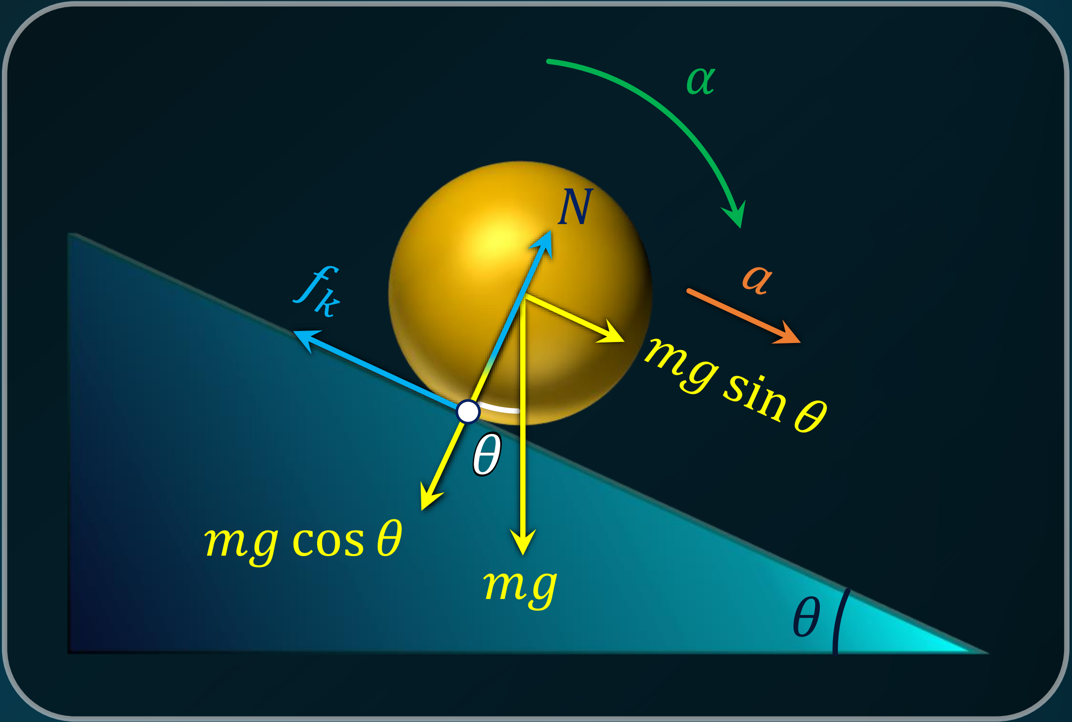
$$\alpha = \frac{20}{5 \times 0.5^2}$$

$$\alpha = 16\ rad\ s^{-2}$$



Pure Rolling on an Inclined Plane

- No force other than friction induces torque in the body about the COM.
- In order to begin (and maintain) pure rolling, frictional force will act in the upward direction of the incline.



A hollow spherical shell starting from O rolls down a hill. At point A, the ball becomes air borne leaving at an angle of $30°$ with the horizontal. The ball strikes the ground at B. What is the value of the distance AB?

Solution :

Since $W_{friction} = 0$, by applying conservation of mechanical energy,

$$(KE_T + KE_R)_i + U_i = (KE_T + KE_R)_f + U_f$$

$$0 + 0 + mgh_1 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh_2$$

$$g(h_1 - h_2) = \frac{1}{2}v^2 + \frac{1}{2}\times\frac{2}{3}R^2\left(\frac{v}{R}\right)^2$$

$$g(2 - 0.2) = \frac{v^2}{2} + \frac{v^2}{3}$$

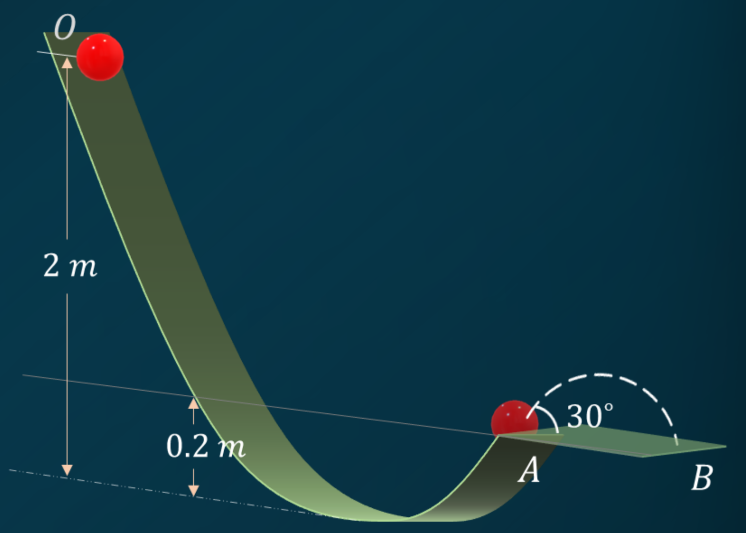
$$\Rightarrow v^2 = \frac{6 \times 1.8 \times 10}{5} = 21.6$$

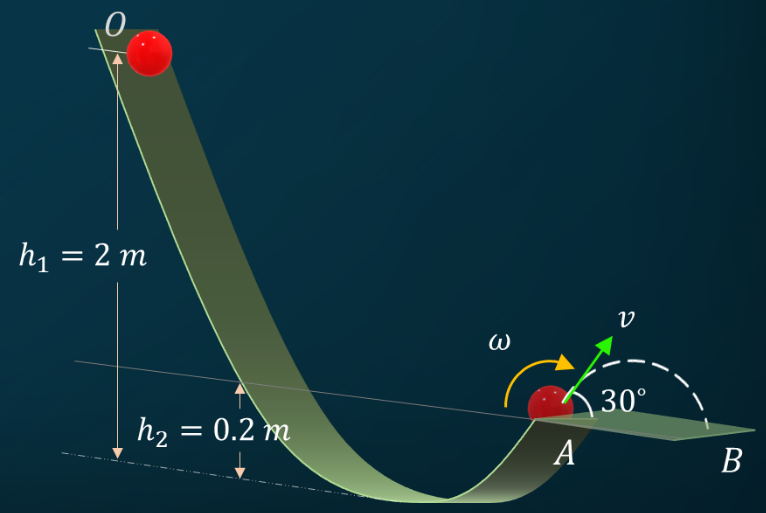
Horizontal range AB:

$$AB = \frac{v^2 \sin 2\theta}{g}$$

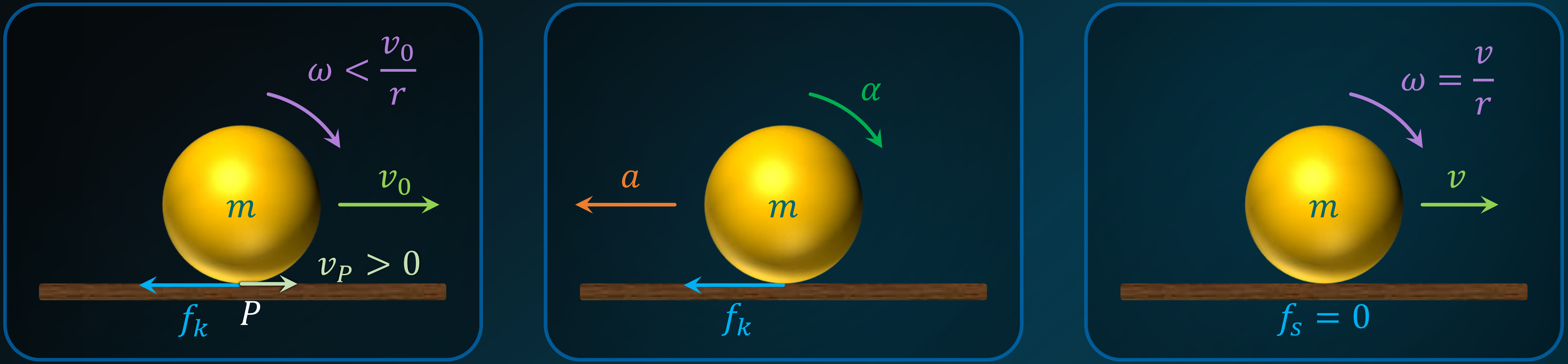
$$= \frac{21.6 \times \sin(2 \times 30°)}{g}$$

$$AB = 1.87\ m$$





Forward Slipping



The sphere is set into combined motion (rolling and slipping)

Frictional force provides linear deceleration and angular acceleration (to initiate pure rolling)

It starts pure rolling and friction diminishes.

A solid cylinder having radius $0.4\ m$, initially rotating with $\omega_0 = 54\ rad/s$ is placed on a rough inclined plane with $\theta = 37°$ having friction coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is ($g = 10\ m/s^2$)

Solution :

Linear acceleration of the cylinder,

$$a = \mu g \cos\theta + g \sin\theta$$

$$= 0.5 \times 10 \cos 37° + 10 \sin 37°$$

$$a = 10\ m/s^2$$

Angular acceleration of the cylinder,

$$\alpha = \frac{f_k R}{I} = \frac{\mu m g R \cos\theta}{\frac{1}{2} m R^2} = \frac{2 \times 0.5 \times 10 \times \frac{4}{5}}{0.4}$$

$$\alpha = 20\ rad/s^2$$

Pure rolling will start when,

$$v = R\omega$$

$$at = R(\omega_0 - \alpha t)$$

$$10t = 0.4(54 - 20t)$$

$$25t = 54 - 20t$$

$$t = 1.2\ s$$