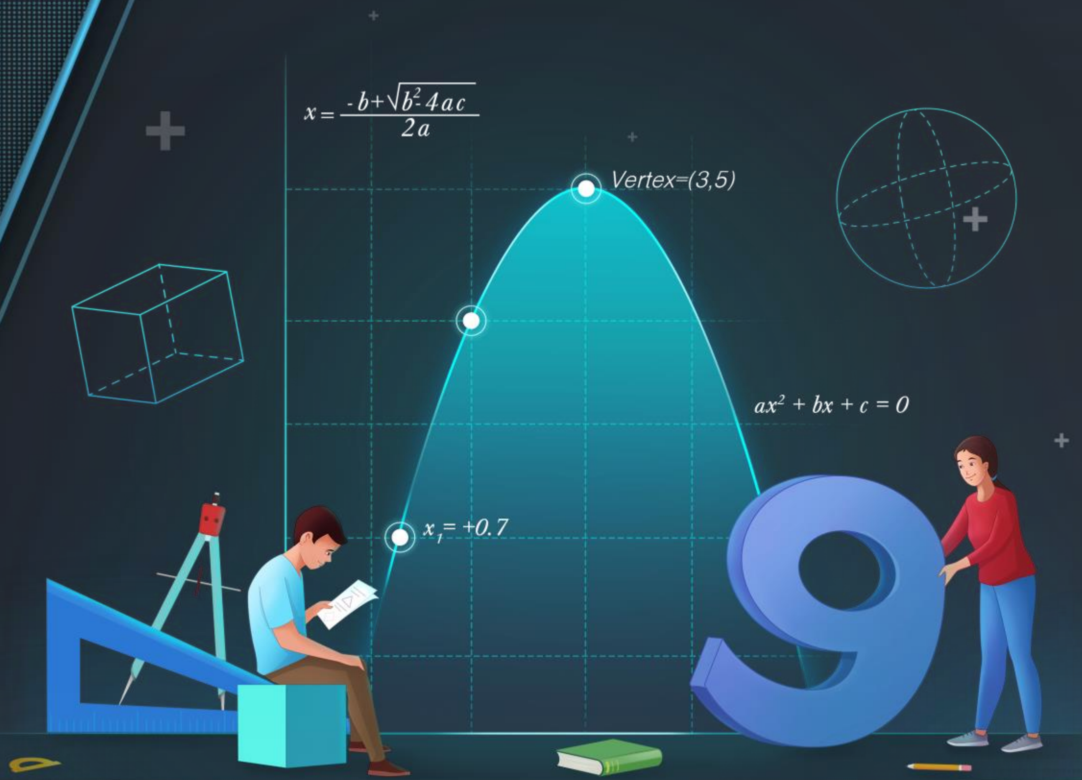
Welcome to

Aakash + BYJU'S LIVE

Quadratic Equations



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Session 01

Introduction to Quadratic Equation

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Key Takeaways

Polynomial

➢ A function f defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$,

Where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ is called a real polynomial of degree n with real coefficients ($a_n \neq 0$, $n \in \mathbb{W}$).

Example

- $f(x) = x^5 - 3x^4 + 2x^2 - 5x + 4$: A polynomial of degree 5
- $f(x) = 2x^3 + x^2 - x + 1$: A polynomial of degree 3
- $f(x) = x^2 + \sqrt{x} + 1$: Not a polynomial
- $f(x) = x^4 + 7x^{\frac{5}{2}} + 6x - 3$: Not a polynomial

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Key Takeaways

Quadratic Equation

- A polynomial of degree two is called a quadratic polynomial.

$f(x) = ax^2 + bx + c = 0$, is a quadratic equation where $a, b, c \in \mathbb{R}$, $a \neq 0$.

Note

- A quadratic equation will have two roots.
- Roots are given by: $x = \frac{-b \pm \sqrt{D}}{2a}$ where D (discriminant) $= b^2 - 4ac$

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Key Takeaways

Roots of Quadratic Equation

➤ Roots of equation $ax^2 + bx + c = 0$ are given by :

$x = \frac{-b \pm \sqrt{D}}{2a}$, Where D(discriminant) $= b^2 - 4ac$

Example

- Roots of equation $x^2 - 3x + 2 = 0$

$\Rightarrow x = \frac{3 \pm \sqrt{9-4(1)(2)}}{2(1)} \Rightarrow x = \frac{3 \pm 1}{2} = 1, 2$ (Real)

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Key Takeaways

Roots of Quadratic Equation

➢ Roots of equation $ax^2 + bx + c = 0$ are given by :

$x = \frac{-b \pm \sqrt{D}}{2a}$, Where D(discriminant) $= b^2 - 4ac$

Example

- Roots of equation $x^2 + x + 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$

Square root of a negative number is an Imaginary number ($\sqrt{-1} = i$ (iota))

$\Rightarrow x = \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$ (Imaginary)

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Roots of quadratic equation $x^2 + 3x + 9 = 0$ are:

Given quadratic equation is $x^2 + 3x + 9 = 0$

Here $a = 1, b = 3, c = 9$

For $ax^2 + bx + c = 0$

roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-3 \pm \sqrt{9-36}}{2}$$

$$= \frac{-3 \pm 3\sqrt{-3}}{2}$$

$$\therefore x = \frac{-3 \pm 3\sqrt{3}i}{2}$$

A $\frac{-3+3\sqrt{3}i}{2}$

B $\frac{-2+2\sqrt{3}i}{3}$

C $\frac{-2-\sqrt{3}i}{2}$

D $\frac{-3-3\sqrt{3}i}{2}$

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Polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ (a_n \neq 0, n \in \mathbb{W}).$$

Where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ is called a real polynomial of degree n.

Note

- Any polynomial equation of degree n with real coefficients has n roots.
- A root of multiplicity k is counted as k roots.

Example

$x^2 - 2x + 1 = 0$ has 2 roots.

$x^2 - 2x + 1 = (x-1)(x-1)$ Roots are : 1 , 1

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Equation v/s Identity

- A polynomial equation of degree n, having more than n roots is called an identity.

  Thus, an identity has infinite roots.

Example

$(x-1)^2 = x^2 - 2x + 1$ is an identity.

It is true for infinite values of x

- If quadratic equation $ax^2 + bx + c = 0$ has more than two roots,

  it becomes an identity.

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Equation v/s Identity

$ax^2 + bx + c = 0$ is

- A quadratic equation : if $a \neq 0$
- A linear equation : if $a = 0, b \neq 0$
- A contradiction : if $a = b = 0, c \neq 0$
- An identity : if $a = b = c = 0$

Find the value of a for which the equation

$(a^2 - a - 2)x^2 + (a^2 - 4)x + (a^2 - 3a + 2) = 0$, has more than 2 roots ?

Solution :

For $(a^2 - a - 2)x^2 + (a^2 - 4)x + (a^2 - 3a + 2) = 0$ to be an identity :

$(a^2 - a - 2) = 0 \Rightarrow (a+1)(a-2) = 0 \Rightarrow a = -1\,,2 \cdots (i)$

$(a^2 - 4) = 0 \Rightarrow a^2 = 4 \Rightarrow a = -2{,}2 \cdots (ii)$

$(a^2 - 3a + 2) = 0 \Rightarrow (a-1)(a-2) = 0 \Rightarrow a = 1{,}2 \cdots (iii)$

By (i) , (ii) & (iii) ,

$a = 2$

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Prove that the equation $\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$ is an identity.

Solution :

It is a quadratic equation

On putting $x=a$

$$\frac{(a-a)(a-b)}{(c-a)(c-b)}+\frac{(a-b)(a-c)}{(a-b)(a-c)}+\frac{(a-c)(a-a)}{(b-c)(b-a)}=1\Rightarrow 1=1$$

So, $x=a$ is a root of equation

On putting $x=b$

$$\frac{(b-a)(b-b)}{(c-a)(c-b)}+\frac{(b-b)(b-c)}{(a-b)(a-c)}+\frac{(b-c)(b-a)}{(b-c)(b-a)}=1\Rightarrow 1=1$$

So, $x=b$ is a root of equation

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Prove that the equation $\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$ is an identity.

Solution :

Similarly, On putting $x=c$

$$\frac{(c-a)(c-b)}{(c-a)(c-b)}+\frac{(c-b)(c-c)}{(a-b)(a-c)}+\frac{(c-c)(c-a)}{(b-c)(b-a)}=1 \Rightarrow 1=1$$

So, $x=c$ is a root of equation

Thus, it is an identity.

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Key Takeaways

Relation Between Roots and Coefficients

- Let a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. has roots α & β.

So, by factor theorem: $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

$ax^2 + bx + c = a(x^2 - (\alpha + \beta)x + \alpha\beta)$

On comparing ,

$\alpha + \beta = -\frac{b}{a}$ = sum of roots (S)

$\alpha\beta = \frac{c}{a}$ = product of roots (P)

Key Takeaways

Relation Between Roots and Coefficients

- Let a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ has roots α & β.

$$\alpha + \beta = -\frac{b}{a} = \text{sum of roots } (S)$$

$$\alpha\beta = \frac{c}{a} = \text{product of roots } (P)$$

Note

- If sum (S) and product (P) of roots is known then, the equation can be written as : $x^2 - Sx + P = 0$

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Relation Between Roots and Coefficients

Note

- If sum (S) and product (P) of roots is known then, the equation can be written as : $x^2 - Sx + P = 0$
- If sum of coefficients of a quadratic equation is zero, then one root is always 1.

Example

$x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$

The roots of the given equation are $x = 1$ & 2

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If α & β are the roots of the equation $ax^2 + bx + c = 0$, then the value of

$(i) \frac{1}{\beta} + \frac{1}{\alpha}$ $(ii)\ \alpha^2 + \beta^2$ is

Solution :

$(i) \frac{1}{\beta} + \frac{1}{\alpha}$

$= \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$

$(ii)\ \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - \frac{2c}{a}$

$= \frac{b^2 - 2ac}{a^2}$

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If α and β be two roots of the equation $x^2 - 64x + 256 = 0$.

Then the value of $\left(\frac{\alpha^3}{\beta^5}\right)^{\frac{1}{8}} + \left(\frac{\beta^3}{\alpha^5}\right)^{\frac{1}{8}}$ is:

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A 1

B 3

C 2

D 4

Solution:

Given : α and β be two roots of the equation $x^2 - 64x + 256 = 0$.

To find : $\left(\frac{\alpha^3}{\beta^5}\right)^{\frac{1}{8}} + \left(\frac{\beta^3}{\alpha^5}\right)^{\frac{1}{8}}$

$x^2 - 64x + 256 = 0$

$\Rightarrow \alpha + \beta = 64 , \alpha\beta = 256$

$$\left(\frac{\alpha^3}{\beta^5}\right)^{\frac{1}{8}} + \left(\frac{\beta^3}{\alpha^5}\right)^{\frac{1}{8}} = \frac{\alpha+\beta}{(\alpha\beta)^{\frac{5}{8}}}$$

$$= \frac{64}{(256)^{\frac{5}{8}}}$$

$$= \frac{64}{32} = 2$$

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Difference of Roots

➤ Let roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ be α and β.

Difference of roots = $|\alpha - \beta|$ $= \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$= \sqrt{\left(-\frac{b}{a}\right)^2 - \frac{4c}{a}} = \sqrt{\frac{b^2-4ac}{a^2}}$$

$\alpha + \beta = -\frac{b}{a}$

$\alpha\beta = \frac{c}{a}$

$\Rightarrow |\alpha - \beta| = \frac{\sqrt{D}}{|a|}$, where $D = b^2 - 4ac$

The quadratic equation whose difference of roots is 3 & and their product is 4 can be:

Solution :

Given : The quadratic equation whose difference of roots is 3 & and their product is 4

Let α and β are the roots of the quadratic equation

$|\alpha - \beta| = 3$

$\alpha\beta = 4 \cdots (i)$

If sum (S) and product (P) of roots is known then, the equation can be written as : $x^2 - Sx + P = 0$

$$(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$$

$$= 9 + 16$$

$$= 25$$

$\alpha + \beta = \pm 5 \cdots (ii)$

From equation (i) & (ii)

$x^2 \pm 5x + 4 = 0$

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The quadratic equation whose difference of roots is 3 & and their product is 4 can be:

A $x^2 + 5x - 4 = 0$

B $x^2 - 5x + 4 = 0$

C $x^2 + 5x + 4 = 0$

D $x^2 - 3x + 4 = 0$

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If α & β are the roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are : $\alpha + 2, \beta + 2$

Solution :

Given : α & β are the roots of the equation $ax^2 + bx + c = 0$,

To find : $\alpha + 2, \beta + 2$

Method 1:

$\alpha + 2, \beta + 2$

Sum of roots = $\alpha + \beta + 4 = -\frac{b}{a} + 4 = \frac{4a-b}{a}$

Product of roots = $(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$

$$= \frac{c}{a} - \frac{2b}{a} + 4 = \frac{4a-2b+c}{a}$$

So, the equation is : $x^2 - \frac{4a-b}{a}x + \frac{4a-2b+c}{a} = 0$

$ax^2 - (4a - b)x + 4a - 2b + c = 0$

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If α & β are the roots of the equation $ax^2 + bx + c = 0$,
then the equation whose roots are : $\alpha + 2, \beta + 2$

Solution :

Given : α & β are the roots of the equation $ax^2 + bx + c = 0$,

To find : $\alpha + 2 , \beta + 2$

Method 2: Using transformation (only possible when roots are expressed symmetrically and explicitly in terms of α & β)

$\alpha + 2 , \beta + 2$

Let the equation whose roots are $\alpha + 2 , \beta + 2$ is $Ay^2 + By + C = 0$

So, $y = \alpha + 2 , \beta + 2$

i.e. $y = x + 2$, express x in terms of y

$x = y - 2$, substitute in the given equation $(ax^2 + bx + c = 0)$

$a(y-2)^2 + b(y-2) + c = 0$

Thus, the required equation is $ay^2 - (4a - b)y + 4a - 2b + c = 0$

$y \to x \quad \therefore ax^2 - (4a - b)x + 4a - 2b + c = 0$

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If α & β are the roots of the equation $ax^2 + bx + c = 0$,

then the equation whose roots are $2\alpha, 2\beta$:

Solution : Given : α & β are the roots of the equation $ax^2 + bx + c = 0$,

To find : 2α , 2β

Using transformation (only possible when roots are expressed symmetrically and explicitly in terms of α & β)

Let the equation whose roots are 2α , 2β is $Ay^2 + By + C = 0$

So, $y = 2\alpha, 2\beta$

i.e. $y = 2x$, express x in terms of y

$x = \frac{y}{2}$, substitute in the given equation $(ax^2 + bx + c = 0)$

$$a\left(\frac{y}{2}\right)^2 + b\left(\frac{y}{2}\right) + c = 0$$

Thus, the required equation is : $ay^2 + 2by + 4c = 0$

$y \to x$ $\quad \therefore ax^2 + 2bx + 4c = 0$

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Session 02

Nature of roots & Common roots

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If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m-4)x + 2 = 0$, least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is :

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Solution :

$$3m^2x^2 + m(m-4)x + 2 = 0$$

$$\lambda + \frac{1}{\lambda} = 1$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 1$$

$$\alpha^2 + \beta^2 = \alpha\beta$$

$$(\alpha+\beta)^2 = 3\alpha\beta \Rightarrow \left(-\frac{m(m-4)}{3m^2}\right)^2 = \frac{3(2)}{3m^2}$$

$$\frac{(m-4)^2}{9m^2} = \frac{2}{m^2}$$

$$\Rightarrow (m-4)^2 = 18$$

$$m = 4 \pm \sqrt{18}$$

$$\Rightarrow m = 4 \pm 3\sqrt{2}$$

A. $-2+\sqrt{2}$

B. $4-3\sqrt{2}$

C. $2-\sqrt{3}$

D. $4-2\sqrt{3}$

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Key Takeaways

Nature of Roots

- Consider the equation : $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0, D = b^2 - 4ac$

  - $D > 0$
    Real and distinct roots
  - $D = 0$
    Real and equal roots
  - $D < 0$
    Imaginary roots

- Imaginary roots occur in conjugate pairs roots: $\alpha + i\beta$, $\alpha - i\beta$ where $i = \sqrt{-1}$

Example

$$x^2 - 3x + 4 = 0$$

$$D = 9 - 4(1)(4) \quad = -7$$

$$x = \frac{3 \pm \sqrt{7}i}{2(1)} \quad = \frac{3+\sqrt{7}i}{2}, \frac{3-\sqrt{7}i}{2}$$

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Key Takeaways

Nature of Roots

- Consider the equation : $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0, D = b^2 - 4ac$

$D \geq 0$
Real roots

- $D > 0$
  Real and distinct roots
- $D = 0$
  Real and equal roots

Note

- For a quadratic expression to be perfect square of a linear factor i.e., $(px + q)^2$ type, it's $D = 0$ (since both roots will be equal)

Example $(x - 1)^2$, it's $D = 0$

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The values of k, for which the expression $x^2 + (k+1)x + 2k$, can be expressed as a perfect square of a linear factor is :

Solution :

Given expression: $x^2 + (k+1)x + 2k$

$D = 0$

$\Rightarrow (k+1)^2 - 8k = 0$

$\Rightarrow k^2 - 6k + 1 = 0$

$\Rightarrow k = 3 \pm 2\sqrt{2}$

A $4 + 2\sqrt{3}, 4 - 2\sqrt{3}$

B $2 + \sqrt{2}, 2 - \sqrt{2}$

C $-1 + \sqrt{5}, 1 - \sqrt{5}$

D $3 + 2\sqrt{2}, 3 - 2\sqrt{2}$

The least possible value of a, for which the equation $2x^2 + (a-10)x + \frac{33}{2} = 2a$ has real roots

A. 2

B. 3

C. 4

D. 8

Solution :

Given: equation $2x^2 + (a-10)x + \frac{33}{2} = 2a$ has real roots.

$$\Rightarrow D \geq 0$$

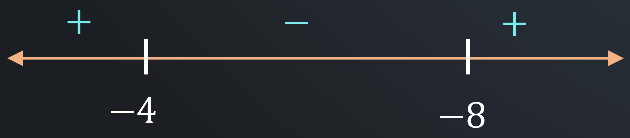
$$\Rightarrow (a-10)^2 - 4 \times 2 \times \left(\frac{33}{2} - 2a\right) \geq 0$$

$$\Rightarrow (a-10)^2 - 4(33 - 4a) \geq 0$$

$$\Rightarrow a^2 - 4a - 32 \geq 0$$

$$\Rightarrow (a+4)(a-8) \geq 0$$

$$\Rightarrow a \in (-\infty, -4] \cup [8, \infty)$$



Thus, the least positive value of a is 8.

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The sum of all integral values of $k (k \neq 0)$ for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is

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Solution:

$$\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$$

$$\Rightarrow \frac{2x-4-x+1}{(x-1)(x-2)} = \frac{2}{k}$$

$$\Rightarrow 2x^2 - 6x + 4 = k(x-3)$$

$$\Rightarrow 2x^2 - x(6+k) + (4+3k) = 0$$

For no solution: $D < 0$

$$(6+k)^2 - 4 \cdot 2(4+3k) < 0$$

$$\Rightarrow k^2 - 12k + 4 < 0$$

$$k = \frac{+12 \pm \sqrt{144-16}}{2} \Rightarrow k = 6 \pm 4\sqrt{2}$$

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The sum of all integral values of $k(k \neq 0)$ for which the equation $\frac{2}{x-1} - \frac{1}{x-2} = \frac{2}{k}$ in x has no real roots, is

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Solution:

$$k = \frac{+12 \pm \sqrt{144-16}}{2} \Rightarrow k = 6 \pm 4\sqrt{2}$$

$$\Rightarrow k \in \left(6 - 4\sqrt{2}, 6 + 4\sqrt{2}\right) \Rightarrow k = 1, 2, 3, \cdots, 11$$

$$S_n = \frac{n(n+1)}{2}$$

Sum of all values of k

$$= 11\left(\frac{11+1}{2}\right)$$

$$= 66$$

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If one root of the equation $ix^2 - 2(1+i)x + 2 - i = 0$ is $2 - i$. then the other root of the equation is :

Solution :

Let the roots of the equation $ix^2 - 2(1+i)x + 2 - i = 0$ are : $2 - i, \alpha$

Sum of roots = $\frac{2(1+i)}{i} = -2i(1+i) = 2 - 2i$

$\Rightarrow 2 - i + \alpha = 2 - 2i$

$\Rightarrow \alpha = -i$

$i = \sqrt{-1} \Rightarrow i^2 = -1$

$\frac{1}{i} = \frac{1.i}{i.i} = -i$

A $2 + i$

B $i - 2$

C $-i$

D $2i$

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Nature of Roots

- For the quadratic equation : $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{Q}$, $a \neq 0$, if $D = b^2 - 4ac$ is a perfect square then roots are rational.

Example

- $2x^2 - 5x + 2 = 0$

$D = 25 - 4(2)(2) = 9$ (Perfect square)

$x = \frac{5 \pm 3}{2(2)} = \frac{1}{2}, 2$

Nature of Roots

- For the quadratic equation : $ax^2 + bx + c = 0,\ a, b, c \in \mathbb{Q}, a \ \neq 0,$ if $D = b^2 - 4ac$ is not a perfect square then roots are irrational which occur in conjugate pairs: $p + \sqrt{q}, p - \sqrt{q}$.

Example

- $x^2 - 2x - 1 = 0$

$D = 4 - 4(1)(-1) \ = 8$: (Not a perfect square)

$x = \frac{2 \pm 2\sqrt{2}}{2} \quad = 1 - \sqrt{2}\,, 1 + \sqrt{2}$

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The quadratic equation with rational coefficients, if one root of the equation is $\sqrt{3}+2$:

Solution :

Given: one root of the equation is $\sqrt{3}+2$

The roots of the equation are : $\sqrt{3}+2$, $-\sqrt{3}+2$

Sum$(S) = 4$

Product$(P) = 1$

$$x^2 - Sx + P = 0$$

So, the equation will be

$$x^2 - 4x + 1 = 0$$

A $x^2 - 4x - 1 = 0$

B $x^2 - 4x + 1 = 0$

C $x^2 + 4x + 1 = 0$

D $x^2 + 4x - 1 = 0$

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The number of all possible positive integral values of α, for which the roots of the quadratic equation $6x^2 - 11x + \alpha = 0$ are rational numbers ?

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A. 2

B. 3

C. 4

D. 5

Solution :

Given equation: $6x^2 - 11x + \alpha = 0$

For rational roots of $6x^2 - 11x + \alpha = 0$

D should be a perfect square

$D = 121 - 24\alpha = k^2$

So, $\alpha = 1 : D = 97$ (not a perfect square)

$\alpha = 2 : D = 73$ (not a perfect square)

$\alpha = 3 : D = 49$ (a perfect square)

$\alpha = 4 : D = 25$ (a perfect square)

$\alpha = 5 : D = 1$ (a perfect square)

So, 3 values are possible

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Key Takeaways

Nature of Roots

- For the quadratic equation : $ax^2 + bx + c = 0$, $a = 1, b, c \in \mathbb{Z}$, if $D = b^2 - 4ac$ is a square of an integer then roots are integers.

Example

- $x^2 - 5x + 6 = 0$

  $D = 25 - 4(6)(1) \quad = 1$: (Perfect square)

  $x = \frac{5 \pm 1}{2} \quad = 2, 3$

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Key Takeaways

Nature of Roots

- $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2}$ if b is integer then,

Integer Integer

$\alpha = \frac{-b+\sqrt{b^2-4c}}{2} = \frac{\text{integer}+\text{integer}}{2} = \frac{\text{integer}}{2}$, which need not be an integer.

If b is odd then,

$b^2 - 4a = (\text{odd})^2$

odd even

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Key Takeaways

Nature of Roots

$b^2 - 4a = (\text{odd})^2$

↓ odd ↓ even

odd odd

$$\alpha = \frac{-b + \sqrt{b^2 - 4a}}{2} = \frac{\text{odd} + \text{odd}}{2} = \frac{\text{even}}{2} = \text{integer}$$

Similarly roots will be even when b is even integer

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Key Takeaways

Common Roots

- Consider two quadratic equations:

$$a_1x^2 + b_1x + c_1 = 0; \qquad a_2x^2 + b_2x + c_2 = 0$$

Case 1: Exactly one common root:

Let the common root be α

So, $a_1\alpha^2 + b_1\alpha + c_1 = 0 \cdots (i)$

$a_2\alpha^2 + b_2\alpha + c_2 = 0 \cdots (ii)$

By cross multiplication,

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{-\alpha}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

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Key Takeaways

Common Roots

- Consider two quadratic equations:

$$a_1x^2 + b_1x + c_1 = 0; \qquad a_2x^2 + b_2x + c_2 = 0$$

Case 1: Exactly one common root:

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{-\alpha}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$-\alpha = \frac{b_1c_2 - b_2c_1}{a_1c_2 - a_2c_1}; \qquad -\alpha = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Eliminating α,

$$(a_1c_2 - a_2c_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$$

Condition for one common root

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The value(s) of k for which the equations $x^2 - 5x + 6 = 0$ and $x^2 + 2kx - 3 = 0$ have exactly one common root ?

Solution :

Method 1 :

$x^2 - 5x + 6 = 0$

$x^2 + 2kx - 3 = 0$

Condition for exactly one common root :

$(-3 - 6)^2 = (2k + 5)(15 - 12k)$

$\Rightarrow 24k^2 + 30k + 6 = 0$

$\Rightarrow 4k^2 + 5k + 1 = 0$

$\Rightarrow (4k + 1)(k + 1) = 0$

$\Rightarrow k = -\frac{1}{4}, -1$

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The value(s) of k for which the equations $x^2 - 5x + 6 = 0$ and $x^2 + 2kx - 3 = 0$ have exactly one common root ?

Solution :

Method 2 :

$x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

Substituting values in 2^{nd} equation

$x^2 + 2kx - 3 = 0$

If $x = 2 \Rightarrow 4 + 4k - 3 = 0 \Rightarrow k = -\frac{1}{4}$

If $x = 3 \Rightarrow 9 + 6k - 3 = 0 \Rightarrow k = -1$

$\therefore k = -\frac{1}{4}, -1$

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Common Roots

➤ Consider two quadratic equations:

$$a_1x^2 + b_1x + c_1 = 0; \qquad a_2x^2 + b_2x + c_2 = 0$$

Case 2: Both roots are common:

Condition:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

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If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $(a, b, c \in \mathbb{R})$, have a common root, then $a : b : c$ is :

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Solution :

$$x^2 + 2x + 3 = 0 \begin{cases} -1 + i\sqrt{2} \\ -1 - i\sqrt{2} \end{cases} \qquad D < 0$$

Since $a, b, c \in \mathbb{R}$

Equations will have both roots common

(Imaginary roots occur in conjugate pairs)

$a : b : c = 1 : 2 : 3$

A $3 : 1 : 2$

B $1 : 2 : 3$

C $3 : 2 : 1$

D $1 : 3 : 2$

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Session 03

Graph & Sign of Quadratic Expression

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Let $\lambda \neq 0$ be in $\mathbb{R}$. If α and β are the roots of the equation $x^2 - x + 2\lambda = 0$, and α and γ are the roots of the equation $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to

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Since α is the common root,

$\alpha^2 - \alpha + 2\lambda = 0 \cdots (i)$

$3\alpha^2 - 10\alpha + 27\lambda = 0 \cdots (ii)$

$3(i) - (ii),\ \alpha = 3\lambda$ Substituting in (i),

$9\lambda^2 - \lambda = 0$

$\lambda(9\lambda - 1) = 0 \quad (\lambda \neq 0)$

$\Rightarrow \lambda = \frac{1}{9}$

$\alpha\beta = 2\lambda\ ;\ \alpha\gamma = 9\lambda$

So, $\frac{\beta\gamma}{\lambda} = \frac{18\lambda^2}{\alpha^2\lambda} = \frac{18\lambda^2}{9\lambda^2\lambda} = \frac{2}{\lambda} = 18$

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Key Takeaways

Graph of Quadratic Expression

For $y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

$$y = ax^2 + bx + c \Rightarrow y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$\Rightarrow y = a\left(x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

$$\Rightarrow y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$\Rightarrow y = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right) \Rightarrow y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a > 0$: $y_{min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

If $a < 0$: $y_{max} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

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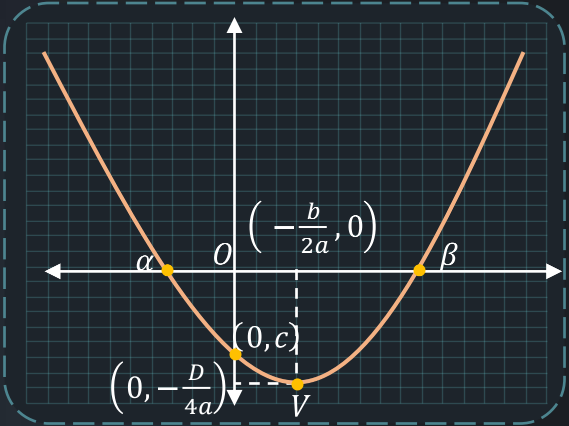
Key Takeaways

Graph of Quadratic Expression

For $y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$



$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a > 0$: Mouth opening upward parabola

$(i)\ D > 0$

Two real and distinct roots: α, β

Vertex V: $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$y_{min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$ Range: $y \in \left[-\frac{D}{4a}, \infty\right)$

$f(0) = c$

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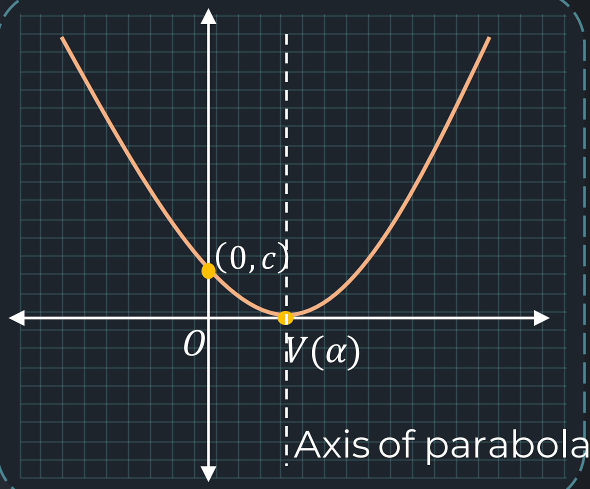
Key Takeaways

Graph of Quadratic Expression

For $y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$



$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a > 0$: Mouth opening upward parabola

$(ii)\ D = 0$

Real and equal roots: α

Vertex V: $\left(-\frac{b}{2a}, 0\right)$

Graph of a parabola is always symmetric w.r.t its axis

$y_{min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

Range: $y \in [0, \infty)$

$f(0) = c$

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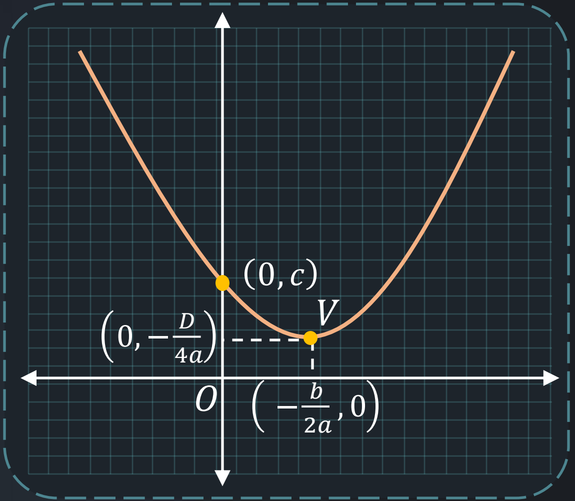
Key Takeaways

Graph of Quadratic Expression

For $y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$



$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a > 0$: Mouth opening upward parabola

$(iii)\ D < 0$

Imaginary Roots

Vertex V: $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$y_{min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$ Range: $y \in \left[-\frac{D}{4a}, \infty\right)$

$f(0) = c$

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Draw the graph and find the range for the following quadratic expression: $y = x^2 - 3x + 1.$

Solution:

$y = x^2 - 3x + 1$

$a = 1 > 0$ (upward parabola)

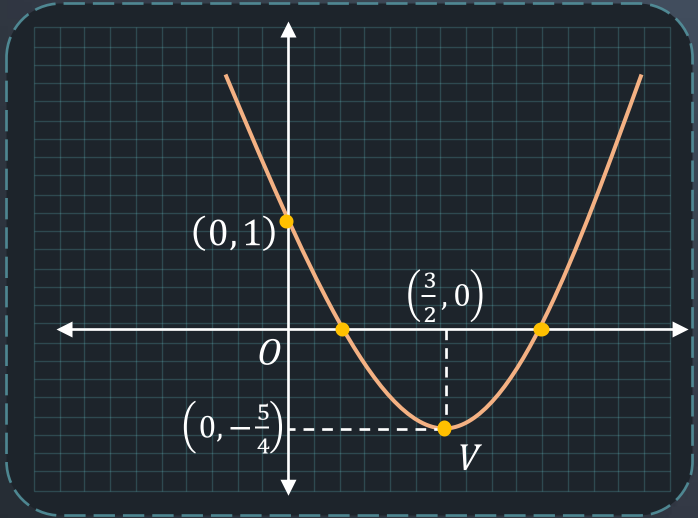
$D = 9 - 4 = 5 > 0$ (cut x-axis at two points)

$V_x = -\frac{b}{2a} = \frac{3}{2}$

$V_y = -\frac{D}{4a} = -\frac{5}{4}$

$c = 1$ (cut y-axis on the positive side)

Range: $y \in \left[-\frac{D}{4a}, \infty\right) \Rightarrow y \in \left[-\frac{5}{4}, \infty\right)$



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Graph of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a < 0$: Mouth opening downward parabola

$(i)\ D > 0$

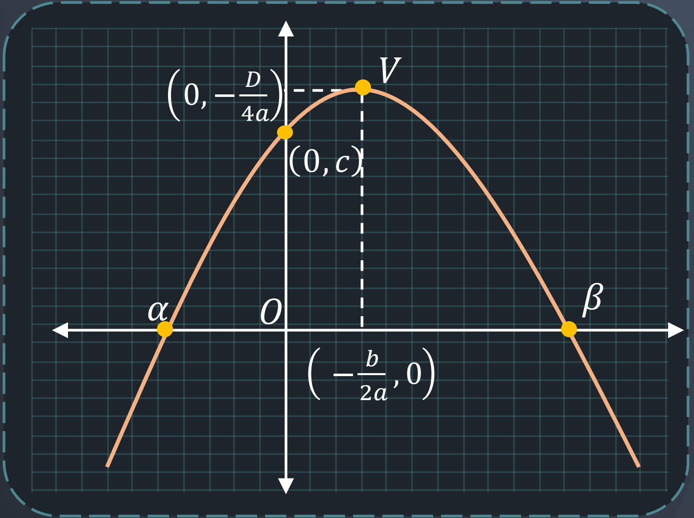
Two real and distinct roots: α, β

Vertex V: $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$f(0) = c$

$y_{max} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

Range: $y \in \left(-\infty, -\frac{D}{4a}\right]$



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Graph of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$

If $a < 0$: Mouth opening downward parabola

$(ii)\ D = 0$

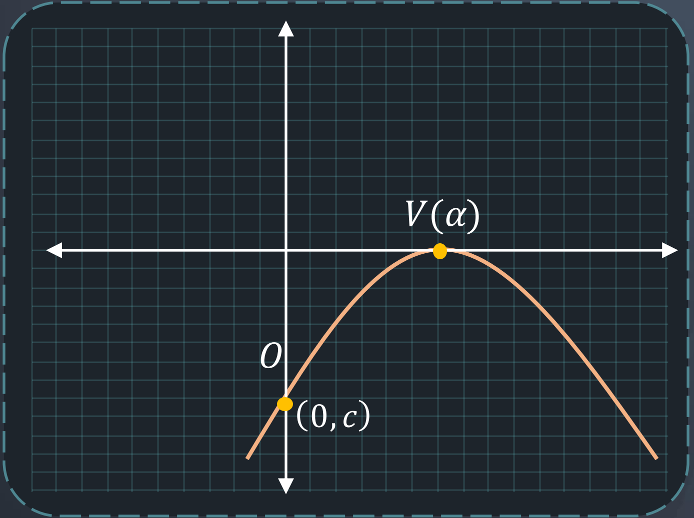
Real and equal roots: α

Vertex V: $\left(-\frac{b}{2a}, 0\right)$

$f(0) = c$

$y_{max} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

Range: $y \in (-\infty, 0]$



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Graph of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

If $a < 0$: Mouth opening downward parabola

$(iii)\ D < 0$

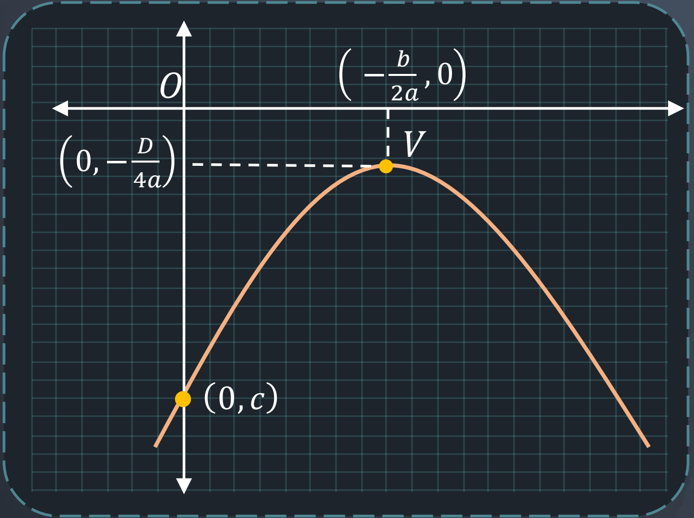
Imaginary Roots

Vertex V: $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$f(0) = c$

$y_{max} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

Range: $y \in \left(-\infty, -\frac{D}{4a}\right]$



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Find the range & draw the graph of the quadratic expression:

$y = -x^2 + 2x - 3$

$y = -x^2 + 2x - 3$

$a = -1 < 0$ (downward parabola)

$D = 4 - 12 = -8 < 0$ (does not cut x-axis)

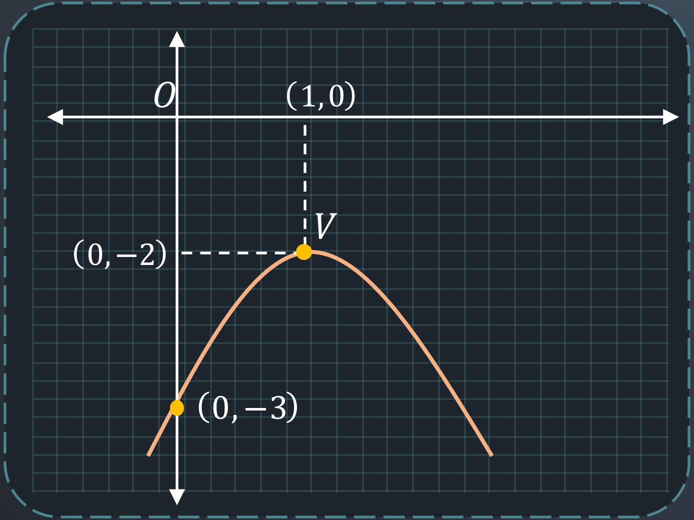
$V_x = -\frac{b}{2a} = -\frac{2}{-2} = 1$

$V_y = -\frac{D}{4a} = -\frac{-8}{4 \times -1} = -2$

Range: $y \in \left(-\infty, -\frac{D}{4a}\right]$

$\Rightarrow y \in (-\infty, -2]$

$c = -3$ (cut y-axis on the negative side)



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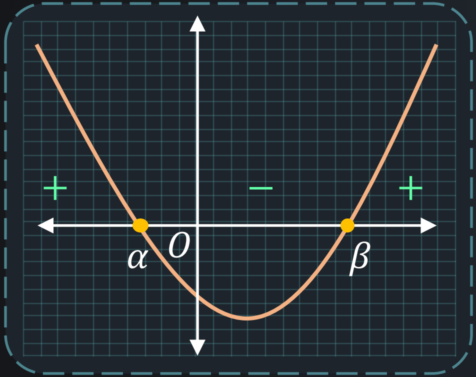
Key Takeaways

Sign of Quadratic Expression

$y = f(x) = ax^2 + bx + c,\ D = b^2 - 4ac;\ a, b, c \in \mathbb{R}, a \neq 0$

For, $a > 0$

$(i)\ D > 0$

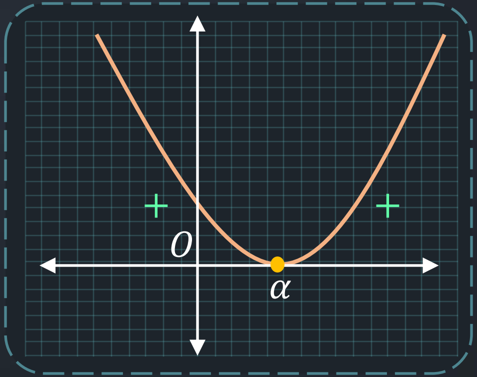


Two real and distinct roots: α, β

$f(x) > 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) < 0\ \forall\ x \in (\alpha, \beta)$

$(ii)\ D = 0$



Real and equal roots: α, α

$f(x) > 0\ \forall\ x \in \mathbb{R} - \{\alpha\}$

$f(x) < 0\ \forall\ x\ \in \phi$

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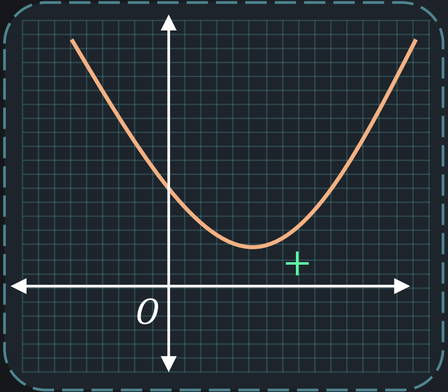
Key Takeaways

Sign of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

For, $a > 0$

$(iii)\ D < 0$



Imaginary roots

$f(x) > 0\ \forall\ x \in \mathbb{R}$

$f(x) < 0\ \forall\ x \in \phi$

Note

If $a > 0, D < 0$, then $f(x) > 0\ \forall\ x \in \mathbb{R}$

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The integer k for which the inequality $x^2 - 2(3k-1)x + 8k^2 - 7 > 0$ is valid for every x in $\mathbb{R}$ is:

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A. 3

B. 2

C. 4

D. 0

Solution:

For $x^2 - 2(3k-1)x + 8k^2 - 7 > 0$

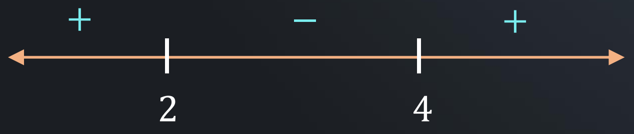
If $a > 0, D < 0$, then $f(x) > 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow \left(2(3k-1)\right)^2 - 4(8k^2 - 7) < 0$

$\Rightarrow (3k-1)^2 - (8k^2 - 7) < 0$

$\Rightarrow k^2 - 6k + 8 < 0$

$\Rightarrow (k-4)(k-2) < 0$



$\Rightarrow 2 < k < 4$

$\therefore k = 3$

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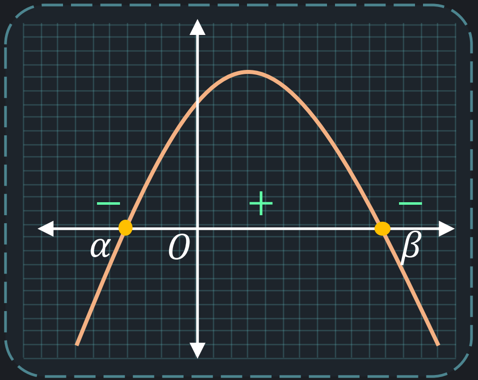
Sign of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

For, $a < 0$

$(i)\ D > 0$

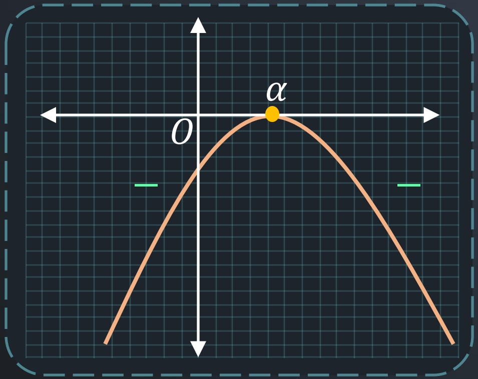


Two real and distinct roots: α, β

$f(x) < 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0\ \forall\ x \in (\alpha, \beta)$

$(ii)\ D = 0$



Real and equal roots: α, α

$f(x) < 0\ \forall\ x \in \mathbb{R} - \{\alpha\}$

$f(x) > 0\ \forall\ x\ \in \phi$

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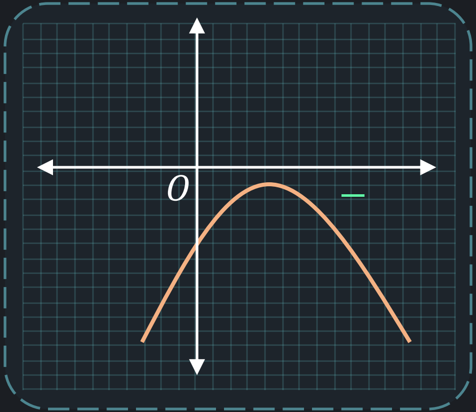
Sign of Quadratic Expression

$y = f(x) = ax^2 + bx + c, D = b^2 - 4ac; a, b, c \in \mathbb{R}, a \neq 0$

For, $a < 0$

$(iii)\ D < 0$



Imaginary roots

$f(x) < 0\ \forall\ x \in \mathbb{R}$

$f(x) > 0\ \forall\ x \in \phi$

Note

If $a < 0, D < 0$, then $f(x) < 0\ \forall\ x \in \mathbb{R}$

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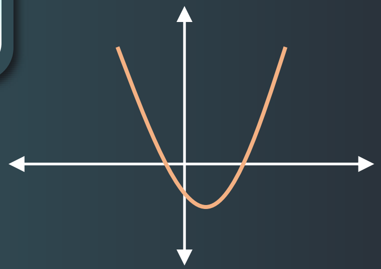
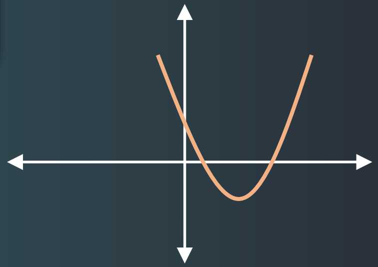
In which of the following graph(s) of the quadratic expression $f(x) = ax^2 + bx + c$, is $abc < 0$?

A



B



$a > 0$

$-\frac{b}{2a} > 0 \Rightarrow b < 0$

$c < 0$

$\therefore abc > 0$

$a > 0$

$-\frac{b}{2a} > 0 \Rightarrow b < 0$

$c > 0$

$\therefore abc < 0$

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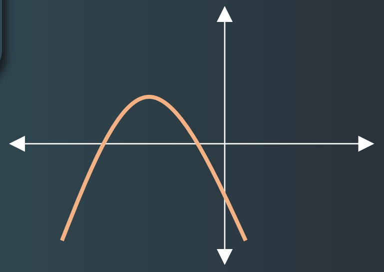
In which of the following graph(s) of the quadratic expression $f(x) = ax^2 + bx + c$, is $abc < 0$?

C



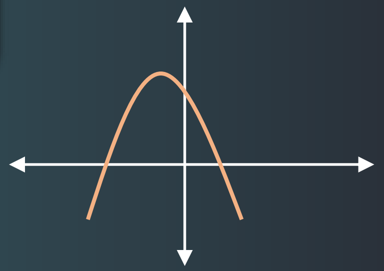
$$a < 0$$

$$-\frac{b}{2a} < 0 \Rightarrow b < 0$$

$$c < 0$$

$$\therefore abc < 0$$

D



$$a < 0$$

$$-\frac{b}{2a} < 0 \Rightarrow b < 0$$

$$c > 0$$

$$\therefore abc > 0$$

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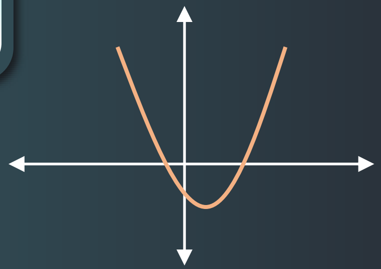
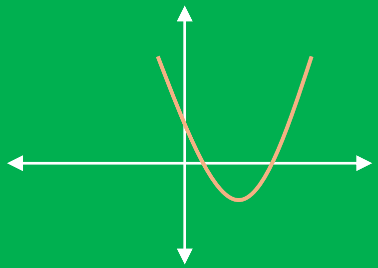
In which of the following graph(s) of the quadratic expression $f(x) = ax^2 + bx + c$, is $abc < 0$?

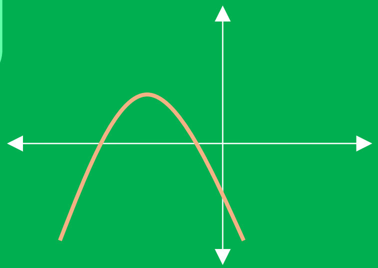
A



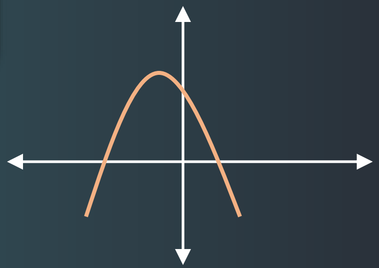
B



C



D



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Session 04

Range of Quadratic Expression & Introduction to Location of Roots

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If the roots of the equation $ax^2 + bx + 1 = 0$ are non-real complex, then which of the following is/are true ?

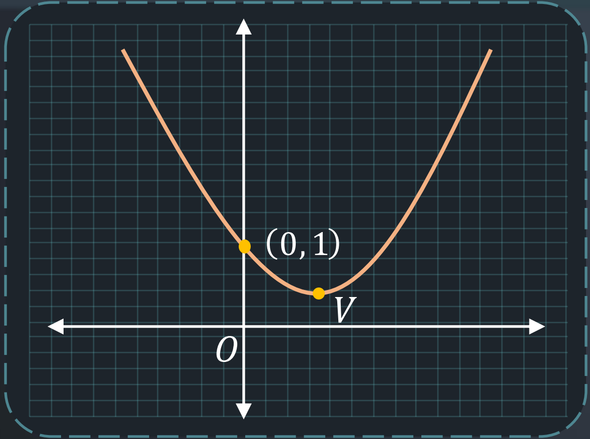
Solution:

Let $f(x) = ax^2 + bx + 1$

Roots are imaginary: $D < 0$

$c = 1 > 0$

Thus, parabola is upward opening.



$f(1) = a + b + 1 > 0$

$f(-2) = 4a - 2b + 1 > 0$

$f\left(\frac{1}{3}\right) = \frac{a}{9} + \frac{b}{3} + 1 > 0 \Rightarrow a + 3b + 9 > 0$

$f(-1) = a - b + 1 > 0$

A $a + b + 1 > 0$

B $4a - 2b + 1 < 0$

C $a + 3b + 9 < 0$

D $a - b + 1 > 0$

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Key Takeaways

Range of Quadratic Expression (in a given interval)

$y = f(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}, a \neq 0 \quad D = b^2 - 4ac$

When $x \in \mathbb{R}$

- If $a > 0$,

  Range: $y \in \left[-\frac{D}{4a}, \infty\right)$

- If $a < 0$,

  Range: $y \in \left(-\infty, -\frac{D}{4a}\right]$

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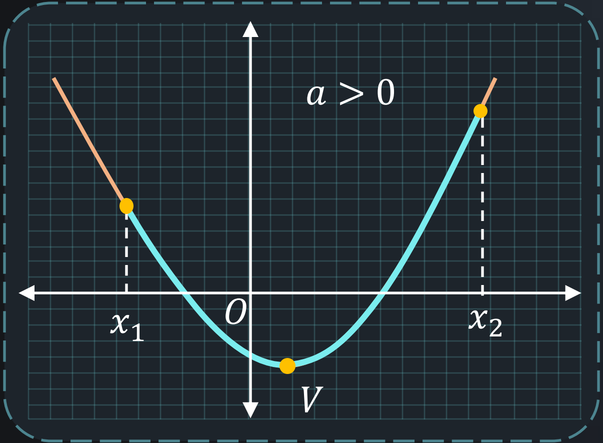
Key Takeaways

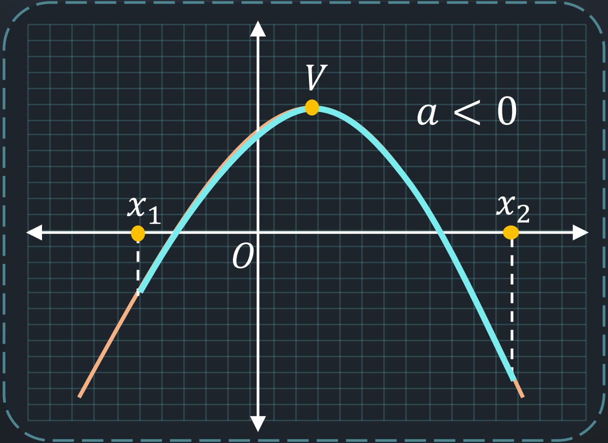
Range of Quadratic Expression (in a given interval)

$y = f(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}, a \neq 0 \quad D = b^2 - 4ac$

When x lies in interval $[x_1, x_2]$

(i) If $-\frac{b}{2a} \in (x_1, x_2)$ Range: $y \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$





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Key Takeaways

Range of Quadratic Expression (in a given interval)

Range: $y \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

Example

- $\min\{-3, 4, 9\} = -3$
- $\max\{-3, 4, 9\} = 9$

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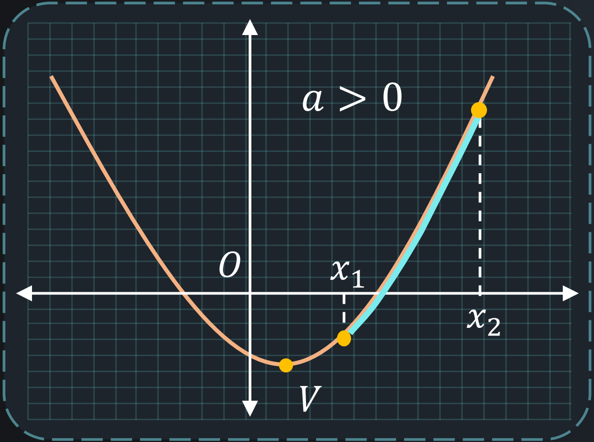
Key Takeaways

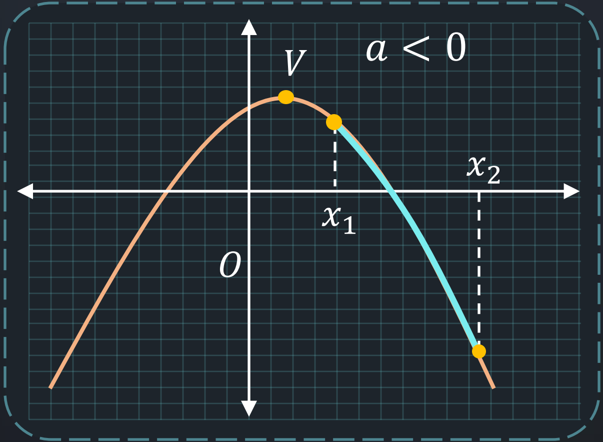
Range of Quadratic Expression (in a given interval)

$y = f(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}, a \neq 0 \quad D = b^2 - 4ac$

When x lies in interval $[x_1, x_2]$

(ii) If $-\frac{b}{2a} \notin (x_1, x_2)$ Range: $y \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$





Find the range of the following quadratic expressions

$(i)\ y = -2x^2 + 5x - 1, x \in [-3, -2]$ $(ii)\ y = x^4 - x^2 - 1, x \in \mathbb{R}$

$(i)\ y = f(x) = -2x^2 + 5x - 1$

$V_x = \frac{5}{4} \notin [-3, -2]$

$f(-3) = -34; f(-2) = -19$

Range: $y \in [-34, -19]$

$y \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$

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Find the range of the following quadratic expressions

$(i)\ y = -2x^2 + 5x - 1, x \in [-3, -2]$ $(ii)\ y = x^4 - x^2 - 1, x \in \mathbb{R}$

$(ii)\ y = x^4 - x^2 - 1, x \in \mathbb{R}$

$$y \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

Let $x^2 = t, t \in [0, \infty)$

So, $y = t^2 - t - 1$

$V_x = \frac{1}{2} \in [0, \infty)$

$f(0) = -1\ ; f\left(\frac{1}{2}\right) = -\frac{5}{4}; f(\infty) \to \infty$

Range: $y \in \left[-\frac{5}{4}, \infty\right)$

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Find the range of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, x \in \mathbb{R}$

Solution:

Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

On cross multiplying,

$$x^2y + xy + y = x^2 - x + 1$$

$$x^2(y-1) + x(y+1) + y - 1 = 0$$

Since roots are real

$$x^2(y-1) + x(y+1) + y - 1 = 0$$

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Find the range of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, x \in \mathbb{R}$

Solution:

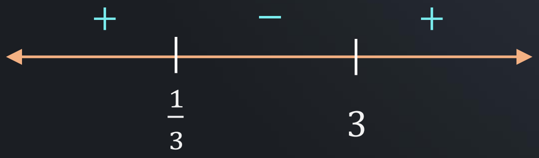
Since roots are real $x^2(y-1) + x(y+1) + y - 1 = 0$

Case 1: $y \neq 1, D \geq 0$

$(y+1)^2 - 4(y-1)^2 \geq 0$

$\Rightarrow (3y-1)(-y+3) \geq 0$

$\Rightarrow (3y-1)(y-3) \leq 0$



$\Rightarrow y \in \left[\frac{1}{3}, 3\right] - \{1\} \cdots (i)$

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Find the range of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, x \in \mathbb{R}$

Solution:

$x^2(y-1) + x(y+1) + y - 1 = 0$

Case 1: $y \neq 1, D \geq 0$

$y \in \left[\frac{1}{3}, 3\right] - \{1\} \cdots (i)$

Case 2: $y = 1$

On putting, $y = 1$, in equation

We get $x = 0$

So, $y = 1 \cdots (ii)$

By (i) & (ii),

$y \in \left[\frac{1}{3}, 3\right]$

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If the range of expression $y^2 + y - 2$ is $[a, b]$ where $y = \frac{2x}{1+x^2}$, & $x \in \mathbb{R}$,
Then find the value of $(b - a)$.

$y = \frac{2x}{1+x^2}$

On cross multiplying,

$x^2y - 2x + y = 0,\ x \in \mathbb{R}$

Case 1: $y \neq 0, D \geq 0$

$4 - 4y^2 \geq 0 \Rightarrow y^2 - 1 \leq 0$

$y \in [-1,1] - \{0\} \cdots (i)$

Case 2: $y = 0 \Rightarrow x = 0$

$y \in \{0\} \cdots (ii)$

By (i) & (ii), $y \in [-1, 1]$

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If the range of expression $y^2 + y - 2$ is $[a, b]$ where $y = \frac{2x}{1+x^2}$, & $x \in \mathbb{R}$,
Then find the value of $(b - a)$.

Range of $f(y) = y^2 + y - 2, y \in [-1, 1]$

$y \in [-1, 1]$

$V_x = -\frac{1}{2} \in [-1,1]$

$f(-1) = -2; f\left(-\frac{1}{2}\right) = -\frac{9}{4}; f(1) = 0$

Range is $\left[-\frac{9}{4}, 0\right]$

$\therefore b - a = \frac{9}{4}$

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Key Takeaways

Location Of Roots

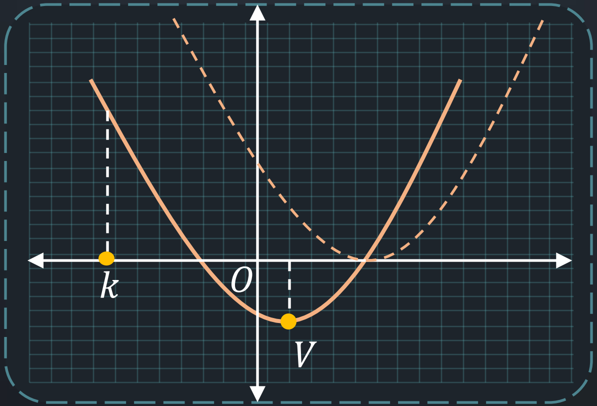
Let the quadratic equation be: $ax^2 + bx + c = 0; a, b, c \in \mathbb{R}$ & $a \neq 0$

$ax^2 + bx + c = 0$ Divide the entire equation with x^2 coefficient

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Case 1: Both roots of are greater than a specified number k

- $D \geq 0$
- $f(k) > 0$
- $-\frac{b}{2a} > k$



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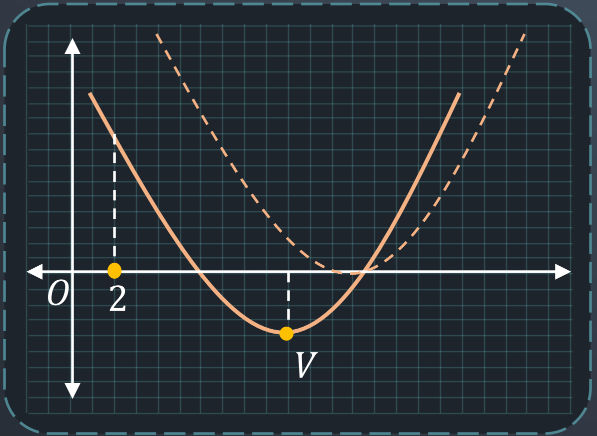
The range of value(s) of m, for which both roots of the equation $x^2-(m-3)x+m=0, m\in\mathbb{R}$, are greater than 2, is:

Solution:

(i) $D \geq 0$

(ii) $f(k) > 0$

(iii) $-\frac{b}{2a} > k$



(i) $D \geq 0$

$\Rightarrow (m-3)^2 - 4m \geq 0$

$\Rightarrow m^2 - 10m + 9 \geq 0$

$\Rightarrow (m-1)(m-9) \geq 0$

$\Rightarrow m \in (-\infty, 1] \cup [9, \infty) \cdots (i)$

(ii) $f(2) > 0$

$\Rightarrow 4 - 2(m-3) + m > 0$

$\Rightarrow m < 10 \cdots (ii)$

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The range of value(s) of m, for which both roots of the equation $x^2 - (m-3)x + m = 0, m \in \mathbb{R}$, are greater than 2, is:

Solution:

$(iii)\ -\frac{b}{2a} > 2$

$\Rightarrow \frac{m-3}{2} > 2$

$\Rightarrow m - 7 > 0$

$\Rightarrow m > 7 \cdots (iii)$

$(i)\ D \geq 0$

$(ii)\ f(k) > 0$

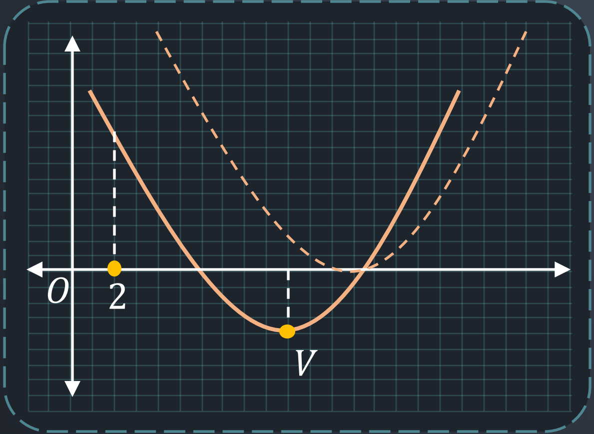
$(iii)\ -\frac{b}{2a} > k$

$m \in (-\infty, 1] \cup [9, \infty) \cdots (i)$

$m < 10 \cdots (ii)$

By $(i), (ii)$ & (iii),

$m \in [9,10)$



A $[9, \infty)$

B $(5,7)$

C $[9,10)$

D $(-\infty, 1] \cup [9, \infty)$

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Location Of Roots

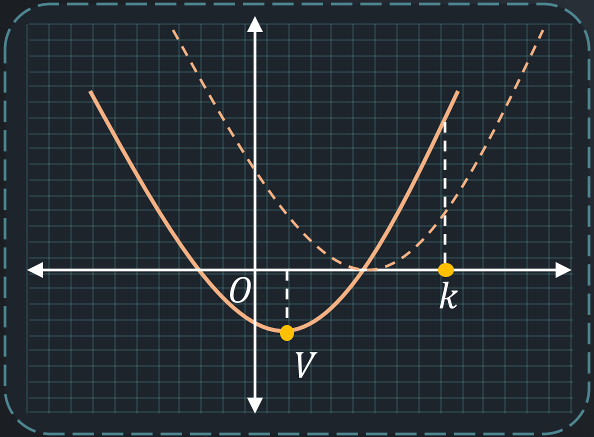
Let the quadratic equation be: $ax^2 + bx + c = 0; a, b, c \in \mathbb{R}$ & $a \neq 0$

$ax^2 + bx + c = 0$ Divide the entire equation with x^2 coefficient

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Case 2: Both roots are less than a specified number k

- $D \geq 0$
- $f(k) > 0$
- $-\frac{b}{2a} < k$



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Let $4x^2 - 4(\alpha - 2)x + \alpha - 2 = 0, (\alpha \in \mathbb{R})$ be a quadratic equation, then the value of α for which both roots are less than $\frac{1}{2}$, is:

$(i)\ D \geq 0$

$4x^2 - 4(\alpha - 2)x + \alpha - 2 = 0$

$D \geq 0 \Rightarrow 16(\alpha - 2)^2 - 16(\alpha - 2) \geq 0$

$\Rightarrow (\alpha - 2)(\alpha - 3) \geq 0$

$\Rightarrow \alpha \in (-\infty, 2] \cup [3, \infty) \cdots (i)$

$(ii)\ f\left(\frac{1}{2}\right) > 0$

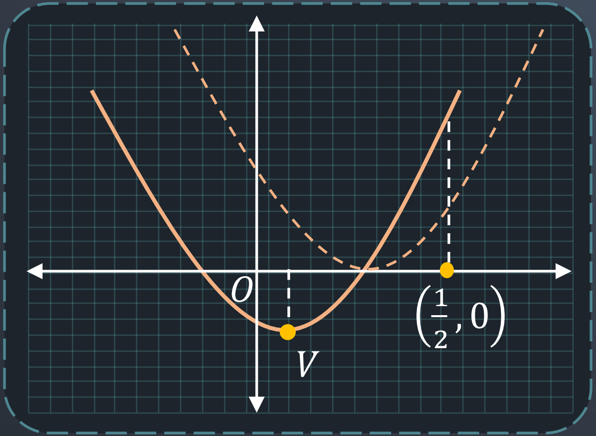
$f\left(\frac{1}{2}\right) > 0 \Rightarrow 1 - 2(\alpha - 2) + \alpha - 2 > 0$

$\Rightarrow 3 - \alpha > 0 \Rightarrow \alpha \in (-\infty, 3) \cdots (ii)$

$(i)\ D \geq 0$

$(ii)\ f(k) > 0$

$(iii)\ -\frac{b}{2a} < k$



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Let $4x^2 - 4(\alpha - 2)x + \alpha - 2 = 0, (\alpha \in \mathbb{R})$ be a quadratic equation, then the value of α for which both roots are less than $\frac{1}{2}$, is:

$\alpha \in (-\infty, 2] \cup [3, \infty) \cdots (i)$

$3 - \alpha > 0 \Rightarrow \alpha \in (-\infty, 3) \cdots (ii)$

$(3)\ -\dfrac{b}{a} < \dfrac{1}{2}$

$-\dfrac{b}{2a} < \dfrac{1}{2}$

$\Rightarrow \dfrac{\alpha - 2}{2} < \dfrac{1}{2}$

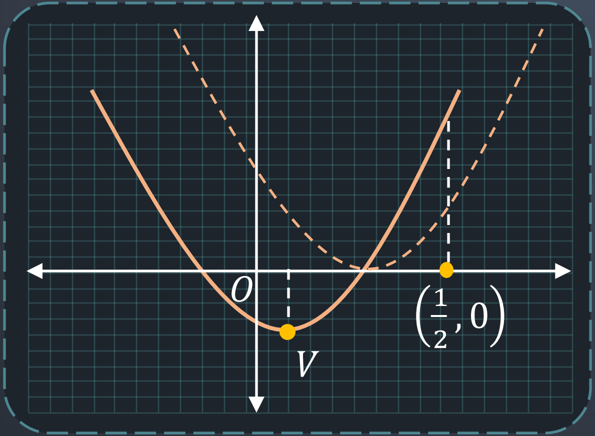
$\Rightarrow \alpha < 3 \cdots (iii)$

By $(i), (ii) \& (iii)$, $\alpha \in (-\infty, 2]$

$(1)\ D \geq 0$

$(2)\ f(k) > 0$

$(3)\ -\frac{b}{2a} < k$



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Session 05

More about Location of Roots

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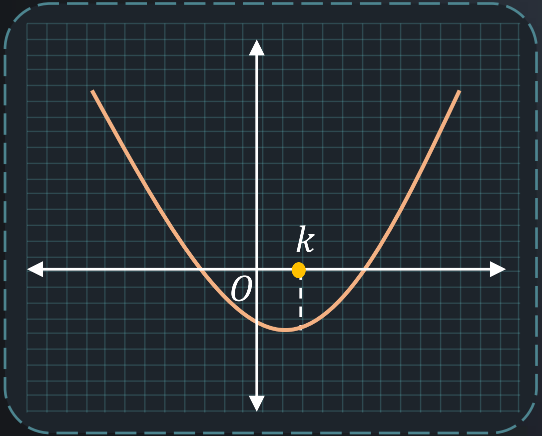
Location of roots

Let the quadratic equation be: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad ; a, b, c \in \mathbb{R} \, \& \, a \neq 0$

Case 3: k lies between the roots OR one root is greater and other is lesser than k

- $f(k) < 0$

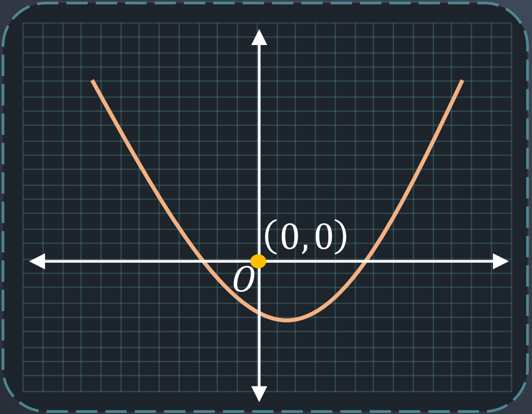


The range of value(s) of a for which roots of the equation $(a^2 - a + 2)x^2 + 2(a-3)x + 9(a^4 - 16) = 0$ are of opposite sign, is:

Solution:

Let $f(x) = \underbrace{(a^2 - a + 2)}_{>0}x^2 + 2(a-3)x + 9(a^4 - 16) = 0$

$$f(x) = x^2 + \frac{2(a-3)}{(a^2 - a + 2)}x + \frac{9(a^4 - 16)}{(a^2 - a + 2)}$$



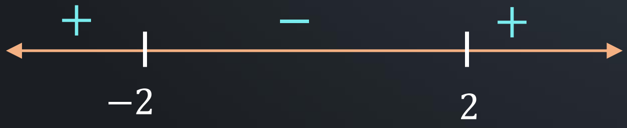
If roots are of opposite sign,

$$f(0) < 0 \Rightarrow \frac{9(a^4 - 16)}{(a^2 - a + 2)} < 0$$

$$\Rightarrow a^4 - 16 < 0$$

$$\Rightarrow (a^2 - 4)(a^2 + 4) < 0$$

$$\Rightarrow (a-2)(a+2) < 0$$



$\therefore a \in (-2, 2)$

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The range of value(s) of a for which roots of the equation $(a^2 - a + 2)x^2 + 2(a - 3)x + 9(a^4 - 16) = 0$ are of opposite sign, is:

A $(-2,2)$

B $(-\infty, 2)$

C $(-\infty, -2) \cup (2, \infty)$

D $(-2, 2) - \{0\}$

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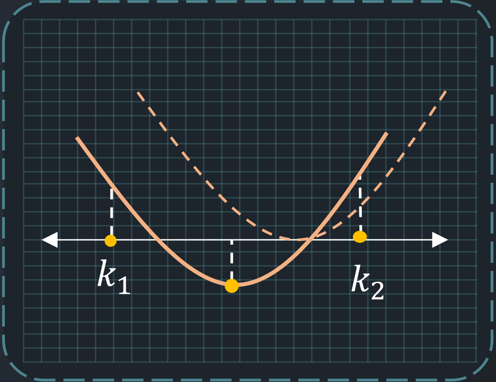
Key Takeaways

Location of roots

Let the quadratic equation be: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad ; a, b, c \in \mathbb{R} \; \& \; a \neq 0$

Case 4 : Both roots lie between k_1 & k_2

- $D \geq 0$
- $f(k_1) > 0$
- $f(k_2) > 0$
- $k_1 < -\frac{b}{2a} < k_2$



The range of value(s) of a, for which both roots of the equation $(a-2)x^2+2ax+a+3=0$, $a\in\mathbb{R}-\{2\}$, lies between $(-2,1)$, is:

Case 4 : Both roots lie between k_1 & k_2

- $D\geq 0$
- $f(k_1)>0$
- $f(k_2)>0$
- $k_1<-\dfrac{b}{2a}<k_2$

Solution:

$$(a-2)x^2+2ax+a+3=0$$

Let $f(x)=x^2+\dfrac{2a}{a-2}x+\dfrac{a+3}{a-2}=0$

- $D\geq 0 \Rightarrow \left(\dfrac{2a}{a-2}\right)^2-4\left(\dfrac{a+3}{a-2}\right)\geq 0$

$$\Rightarrow a^2-(a-2)(a+3)\geq 0\Rightarrow a\leq 6\cdots(i)$$

- $f(-2)>0\Rightarrow 4-\dfrac{4a}{(a-2)}+\dfrac{a+3}{a-2}>0$

$$\Rightarrow \frac{a-5}{a-2}>0$$

$$\Rightarrow a\in(-\infty,2)\cup(5,\infty)\cdots(ii)$$

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The range of value(s) of a, for which both roots of the equation $(a-2)x^2+2ax+a+3=0$, $a \in \mathbb{R}-\{2\}$, lies between $(-2,1)$, is:

Solution:

$$(a-2)x^2+2ax+a+3=0$$

Let $f(x)=x^2+\frac{2a}{a-2}x+\frac{a+3}{a-2}=0$

- $f(1)>0 \Rightarrow 1+\frac{2a}{(a-2)}+\frac{a+3}{a-2}>0$

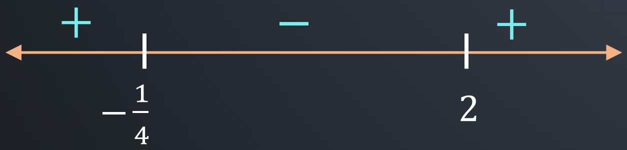
$$\Rightarrow \frac{4a+1}{a-2}>0$$

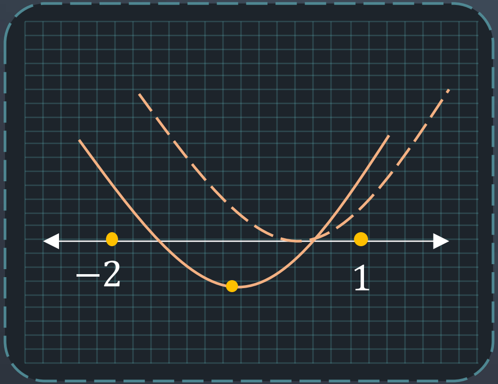
$$\Rightarrow a \in \left(-\infty,-\frac{1}{4}\right) \cup (2,\infty) \cdots (iii)$$

- $-2<-\frac{b}{2a}<1 \Rightarrow -2<-\frac{a}{a-2}<1$

$$\Rightarrow a \leq 6 \cdots (i)$$

$$\Rightarrow a \in (-\infty,2) \cup (5,\infty) \cdots (ii)$$





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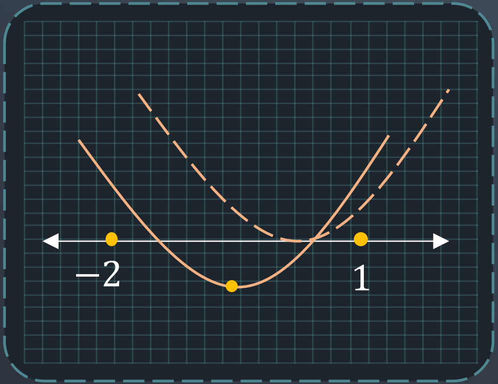
The range of value(s) of a, for which both roots of the equation $(a-2)x^2+2ax+a+3=0\,, a\in\mathbb{R}-\{2\}$, lies between $(-2,1)$, is:

Solution: $(a-2)x^2+2ax+a+3=0\,, a\neq 2$ $\quad \Rightarrow a\leq 6\cdots(i)$

$\Rightarrow a \in (-\infty,2)\cup(5,\infty)\cdots(ii)$

$\Rightarrow a \in \left(-\infty,-\frac{1}{4}\right)\cup(2,\infty)\cdots(iii)$

Let $f(x)=x^2+\frac{2a}{a-2}x+\frac{a+3}{a-2}=0$

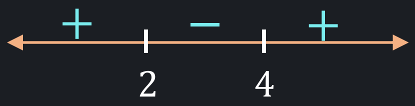


$\Rightarrow -2<-\frac{a}{a-2}<1$

$-2<-\frac{a}{a-2}$

$\frac{-a}{a-2}+2>0 \quad \Rightarrow \frac{-a+2a-4}{a-2}>0$

$\Rightarrow \frac{a-4}{a-2}>0$



$a\in(-\infty,2)\cup(4,\infty)$

$-\frac{a}{a-2}-1<0$

$\Rightarrow \frac{-a-a+2}{a-2}<0 \quad \Rightarrow \frac{2-2a}{a-2}<0$

$\Rightarrow \frac{a-1}{a-2}>0$

$a\in(-\infty,1)\cup(2,\infty)$

$\Rightarrow a \in (-\infty,1)\cup(4,\infty)\cdots(iv)$

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The range of value(s) of a, for which both roots of the equation $(a-2)x^2+2ax+a+3=0$, $a \in \mathbb{R}-\{2\}$, lies between $(-2,1)$, is:

Solution:

$$(a-2)x^2+2ax+a+3=0, a \neq 2$$

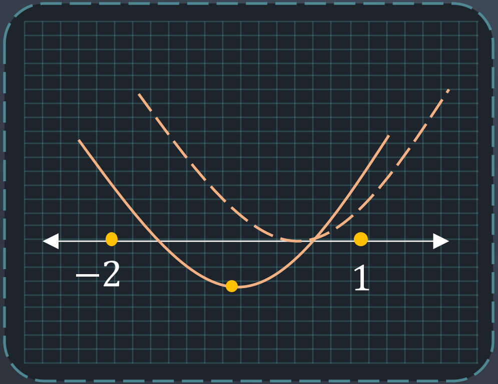
Let $f(x)=x^2+\frac{2a}{a-2}x+\frac{a+3}{a-2}=0$

$$\Rightarrow a \leq 6 \cdots (i)$$

$$\Rightarrow a \in (-\infty,2) \cup (5,\infty) \cdots (ii)$$

$$\Rightarrow a \in \left(-\infty,-\frac{1}{4}\right) \cup (2,\infty) \cdots (iii)$$

$$\Rightarrow a \in (-\infty,1) \cup (4,\infty) \cdots (iv)$$



By $(i),(ii),(iii)$ & (iv),

$$a \in \left(-\infty,-\frac{1}{4}\right) \cup (5,6]$$

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The range of value(s) of a, for which both roots of the equation $(a-2)x^2 + 2ax + a + 3 = 0$, $a \in \mathbb{R} - \{2\}$, lies between $(-2,1)$, is:

A $(-\infty, 6)$

B $\left(-\infty, -\frac{1}{4}\right) \cup (5,6]$

C $\left(-\infty, -\frac{1}{4}\right) \cup (5, \infty)$

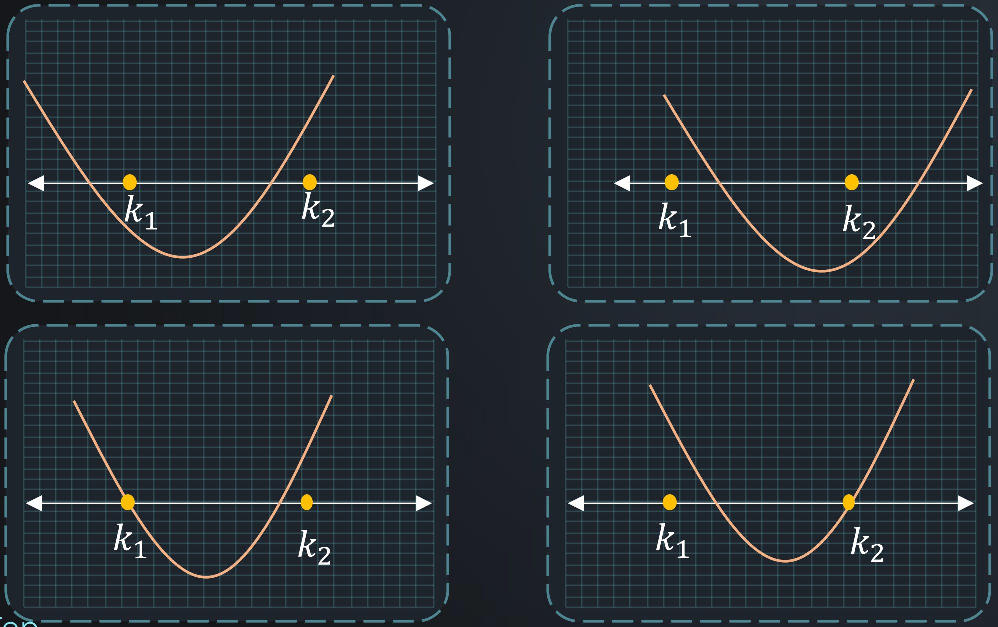
D $(-\infty, 6]$

Key Takeaways

Location of roots

Let the quadratic equation be: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; $a, b, c \in \mathbb{R}$ & $a \neq 0$

Case 5 : Exactly one root lie between k_1 & k_2



- $f(k_1) \cdot f(k_2) < 0$
- $f(k_1) \cdot f(k_2) = 0$

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Key Takeaways

Location of roots

Let the quadratic equation be: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; $a, b, c \in \mathbb{R}$ & $a \neq 0$

Case 5 : Exactly one root lie between k_1 & k_2

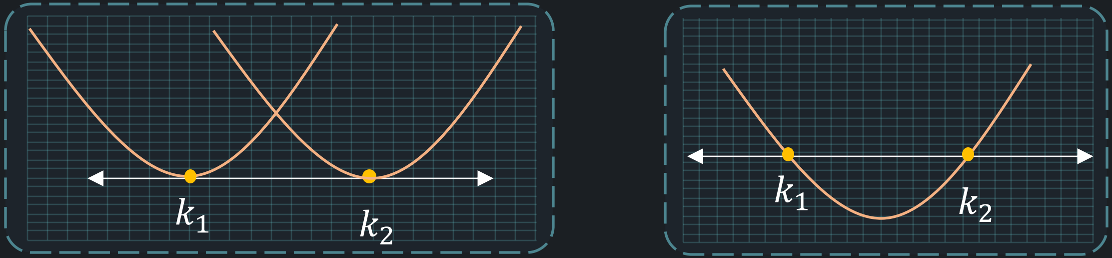
- $f(k_1) \cdot f(k_2) < 0$

OR

- $f(k_1) \cdot f(k_2) = 0$

Note

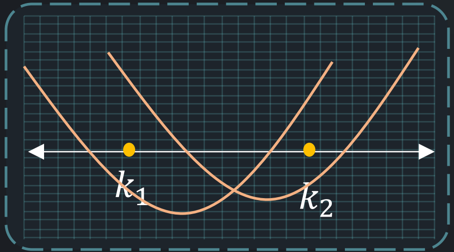
- In the condition $f(k_1) \cdot f(k_2) = 0$ exclude the cases when the roots are k_1 or k_2 or both.



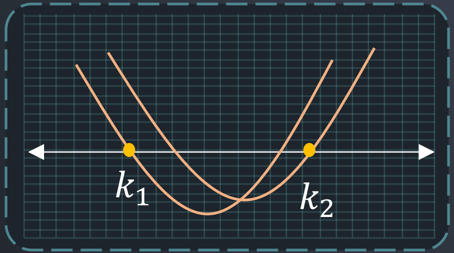
Find all values of b so that the equation $x^2 + (3 - 2b)x + b = 0$ has exactly one root in $(-1, 2)$:

Solution:

$f(x) = x^2 + (3 - 2b)x + b$



OR



- $f(k_1) \cdot f(k_2) < 0$
- $f(k_1) \cdot f(k_2) = 0$

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Find all values of b so that the equation $x^2 + (3 - 2b)x + b = 0$ has exactly one root in $(-1, 2)$:

Solution:

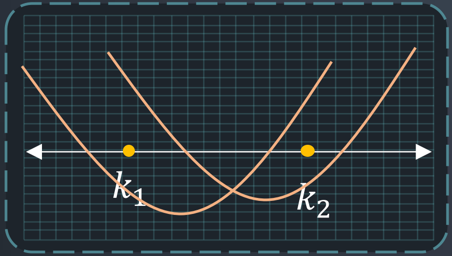
$f(x) = x^2 + (3 - 2b)x + b$

$f(-1) \cdot f(2) < 0$

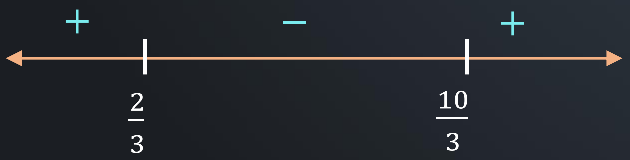
$(1 - (3 - 2b) + b) \cdot (4 + (3 - 2b) \cdot 2 + b) < 0$

$(3b - 2) \cdot (10 - 3b) < 0$

$(3b - 2) \cdot (3b - 10) > 0$



➤ $f(k_1) \cdot f(k_2) < 0$



$b \in \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{10}{3}, \infty\right) \dots (i)$

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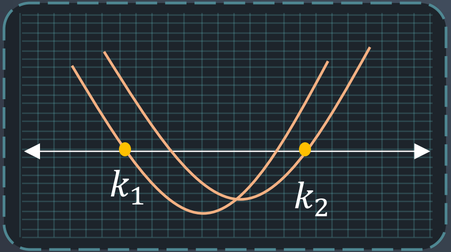
Find all values of b so that the equation $x^2 + (3 - 2b)x + b = 0$ has exactly one root in $(-1, 2)$:

Solution:



$$f(x) = x^2 + (3 - 2b)x + b$$

➢ $f(k_1) \cdot f(k_2) = 0$

$$f(-1) \cdot f(2) = 0$$

$$\Rightarrow (1 - (3 - 2b) + b) \cdot (4 + (3 - 2b) \cdot 2 + b) = 0$$

$$(3b - 2) \cdot (10 - 3b) = 0$$

$$\Rightarrow b = \left\{\frac{2}{3}, \frac{10}{3}\right\}$$

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Find all values of b so that the equation $x^2 + (3 - 2b)x + b = 0$ has exactly one root in $(-1, 2)$:

Solution:

$f(x) = x^2 + (3 - 2b)x + b$

$\Rightarrow b = \left\{\frac{2}{3}, \frac{10}{3}\right\}$

If $b = \frac{2}{3} \cdots (ii)$

$\Rightarrow x^2 + \left(3 - 2 \times \frac{2}{3}\right)x + \frac{2}{3} = 0$

$\Rightarrow 3x^2 + 5x + 2 = 0$

$\Rightarrow 3x^2 + 3x + 2x + 2 = 0$

$\Rightarrow (3x + 2)(x + 1) = 0$

$\Rightarrow x = -1$ and $x = -\frac{2}{3}$ $\quad -\frac{2}{3} \in (-1, 2)$

Note

- In the condition $f(k_1) \cdot f(k_2) = 0$ exclude the cases when both the roots are k_1 or k_2 or k_1, k_2.

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Find all values of b so that the equation $x^2 + (3 - 2b)x + b = 0$ has exactly one root in $(-1, 2)$:

Solution:

$f(x) = x^2 + (3 - 2b)x + b$

$\Rightarrow b = \left\{\frac{2}{3}, \frac{10}{3}\right\}$

If $b = \frac{10}{3} \ldots (iii)$

$\Rightarrow x^2 + \left(3 - 2 \times \frac{10}{3}\right)x + \frac{10}{3} = 0$

$\Rightarrow 3x^2 - 11x + 10 = 0$

$\Rightarrow 3x^2 - 6x - 5x + 10 = 0$

$\Rightarrow (3x - 5)(x - 2) = 0$

$\Rightarrow x = 2$ and $x = \frac{5}{3}$

$\frac{5}{3} \in (-1, 2)$

Note

- In the condition $f(k_1) \cdot f(k_2) = 0$ exclude the cases when both the roots are k_1 or k_2 or k_1, k_2.

$-\frac{2}{3} \in (-1, 2)$

$\therefore b \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{10}{3}, \infty\right)$

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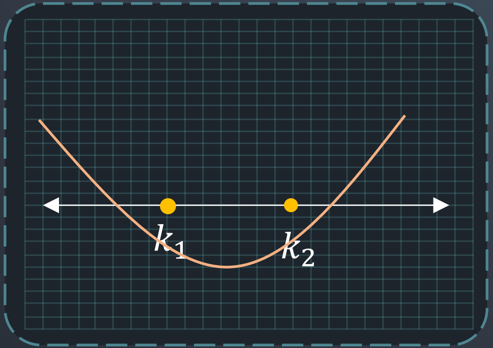
Location of Roots

Let the quadratic equation be: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad ; a, b, c \in \mathbb{R} \;\&\; a \neq 0$

Case 6 : k_1 & k_2 lie between the roots OR

one root is less than k_1 and another is greater than k_2



- $f(k_1) < 0$
- $f(k_2) < 0$

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If α, β are the roots of the quadratic equation $x^2 - 2p(x-4) - 15 = 0$, then the set of value(s) of p, for which one root is less than 1 and other is greater than 2 , is :

Solution : $x^2 - 2p(x-4) - 15 = 0$

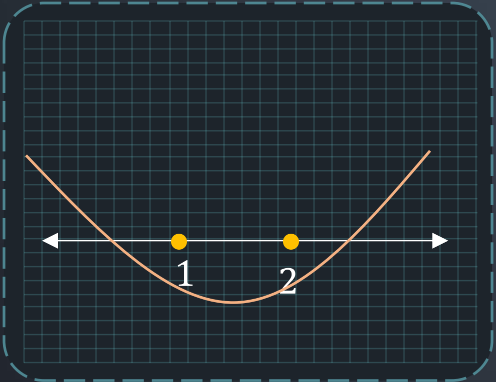
- $f(1) < 0 \Rightarrow 1 - 2p + 8p - 15 < 0$

$\Rightarrow p < \frac{7}{3} \cdots (i)$

- $f(2) < 0 \Rightarrow 4 + 4p - 15 < 0$

$\Rightarrow 4p - 11 < 0 \Rightarrow p < \frac{11}{4} \cdots (ii)$

By (i) & (ii), $p \in \left(-\infty, \frac{7}{3}\right)$



A $\left(-\infty, \frac{7}{3}\right)$

B $\mathbb{R}$

C $\left(\frac{7}{3}, \infty\right)$

D None of these

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Key Takeaways

Pseudo Quadratic Equation

It is an equation that can be transformed into quadratic equation using an appropriate substitution.

Example

- $\sqrt{x} + 2x - 3 = 0$, can be converted into quadratic by substituting $\sqrt{x} = t$

  $\Rightarrow t + 2t^2 - 3 = 0$

- $(x^2 + x)^2 - (x^2 + x) - 5 = 0$, can be converted into quadratic by substituting $x^2 + x = t$

  $\Rightarrow t^2 - t - 5 = 0$

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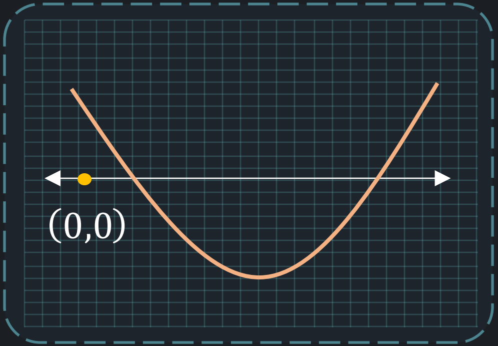
If the equation $x^4 - \lambda x^2 + 9 = 0$, has four distinct real roots, then λ lies in the interval :

Solution: Let $x^2 = t,\ t \in [0, \infty)$

$$f(t) = t^2 - \lambda t + 9 = 0 \cdots (i)$$



$t > 0 \Rightarrow x^2 = +ve$

$\Rightarrow$ two real and distinct values of x

$t = 0 \Rightarrow x^2 = 0$

$\Rightarrow x = 0$ (only one value)

$t < 0 \Rightarrow x^2 = -ve$

$\Rightarrow$ two imaginary values of x

A $(-\infty, -6) \cup (6, \infty)$

B $(0, \infty)$

C $(6, \infty)$

D $(-\infty, -6)$

For 4 real and distinct values of x ,

equation (i) should have two distinct positive roots.

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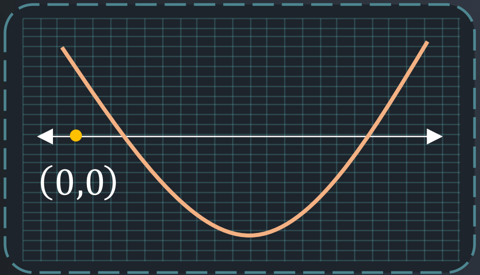
If the equation $x^4 - \lambda x^2 + 9 = 0$, has four distinct real roots, then λ lies in the interval :

Solution : Let $x^2 = t,\ t \in [0, \infty)$ $\quad f(t) = t^2 - \lambda t + 9 = 0 \cdots (i)$

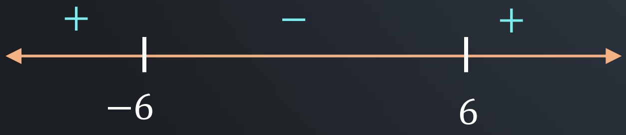


- $D > 0$

$\Rightarrow \lambda^2 - 36 > 0$

$\Rightarrow (\lambda + 6)(\lambda - 6) > 0$

$\Rightarrow \lambda \in (-\infty, -6) \cup (6, \infty) \cdots (ii)$



- $f(0) = 9 > 0 \Rightarrow \lambda \in \mathbb{R} \cdots (iii)$

- $-\dfrac{b}{2a} > 0 \Rightarrow \lambda > 0 \cdots (iv)$

By $(ii), (iii)$ & (iv), $\lambda \in (6, \infty)$

A $(-\infty, -6) \cup (6, \infty)$

B $(0, \infty)$

C $(6, \infty)$

D $(-\infty, -6)$

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Find the values of m for which the equation $x^4 + (1-2m)x^2 + (m^2-1) = 0$ has three real distinct solutions

Solution:

We have,

$$f(x) = x^4 + (1-2m)x^2 + (m^2-1) = 0$$

Let, $x^2 = t$

$$\Rightarrow f(t) = t^2 + (1-2m)t + (m^2-1) = 0$$

$$t_1 = x^2 \qquad\qquad t_2 = x^2$$

$$x_1 = +\sqrt{t_1} \qquad x_2 = -\sqrt{t_1} \qquad x_3 = +\sqrt{t_2} \qquad x_4 = -\sqrt{t_2}$$

Three distinct real roots iff $t_1 > 0$ & $t_2 = 0$

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Find the values of m for which the equation $x^4 + (1-2m)x^2 + (m^2-1) = 0$ has three real distinct solutions

Solution:

We have,

$$f(t) = t^2 + (1-2m)t + (m^2-1) = 0 \quad \begin{matrix} \nearrow t_1 \\ \searrow t_2 \end{matrix}$$

As $t_1 > 0$ & $t_2 = 0$

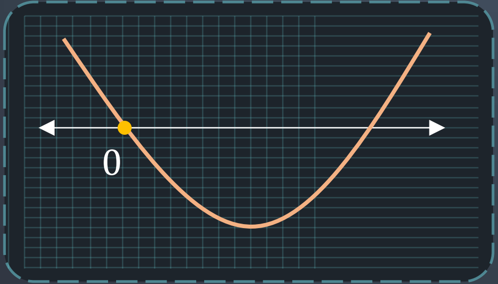
1. $D > 0$

$\Rightarrow b^2 - 4ac > 0$

$\Rightarrow (1-2m)^2 - 4(m^2-1) > 0$

$\Rightarrow -4m + 5 > 0$

$\Rightarrow m \in \left(-\infty, \frac{5}{4}\right) = A$



1. $D > 0$
2. $f(0) = 0$
3. $-\frac{b}{2a} > 0$

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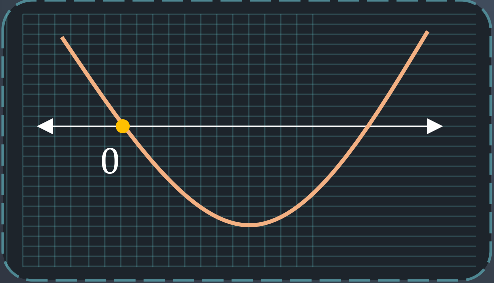
Find the values of m for which the equation $x^4 + (1-2m)x^2 + (m^2-1) = 0$ has three real distinct solutions

Solution: We have,

$$f(t) = t^2 + (1-2m)t + (m^2-1) = 0 \quad \to t_1,\ t_2$$

As $t_1 > 0$ & $t_2 = 0$



1. $D > 0$
2. $f(0) = 0$
3. $-\frac{b}{2a} > 0$

2. $f(0) = 0$

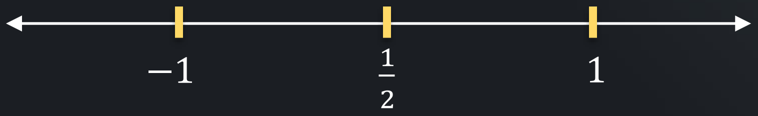
$\Rightarrow m^2 - 1 = 0$

$\Rightarrow m = \pm 1 = B$

3. $-\frac{b}{2a} > 0$

$\Rightarrow -\left(\frac{1-2m}{2}\right) > 0$

$\Rightarrow m > \frac{1}{2} = C$



$\Rightarrow m \in A \cap B \cap C$

$\therefore m = 1$

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Find the values of m for which the equation $x^4 + (1 - 2m)x^2 + (m^2 - 1) = 0$ has three real distinct solutions

A $m \in (1,2)$

B $m \in \left(1, \frac{5}{4}\right)$

C $m \in [0,2)$

D $m \in \{1\}$

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Session 06

Theory of Equations & Transformation of Polynomial Equation

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Key Takeaways

Cubic Equation

Let a cubic equation be : $ax^3 + bx^2 + cx + d = 0$; Where $a, b, c, d \in \mathbb{R}$, $a \neq 0$

- Roots of the equation can be real or imaginary.
- If coefficients are real, then there exists at least one real root for the cubic equation.

  (Imaginary roots occur in conjugate pairs i.e. if $p + iq$ is one root , then $p - iq$ will also be the root)
- If coefficients are rational, then irrational roots occur in conjugate pairs.

  (If $p + \sqrt{q}$ is one root , then $p - \sqrt{q}$ will also be the root)

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Key Takeaways

Cubic Equation

Let a cubic equation be : $ax^3 + bx^2 + cx + d = 0$; Where $a, b, c, d \in \mathbb{R}$, $a \neq 0$

If roots of the equation are : α, β, γ , then

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$

$$= a(x^3 - x^2(\alpha + \beta + \gamma) + x(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma)$$

On comparing ,

- $\alpha + \beta + \gamma = \left(\sum \alpha\right) = -\frac{b}{a}$ (S_1 : Sum of roots taken one at a time)
- $\alpha\beta + \beta\gamma + \gamma\alpha = \left(\sum \alpha\beta\right) = \frac{c}{a}$ (S_2 : Sum of roots taken two at a time)
- $\alpha\beta\gamma = \prod \alpha = -\frac{d}{a}$ (P: Product of roots)

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Key Takeaways

Cubic Equation

Note

- If roots are given, then the cubic equation can also be written as

$$x^3 - (S_1)x^2 + (S_2)x - P = 0$$

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If α, β, γ are the roots of the equation $2x^3 + x^2 - 3x - 1 = 0$, then the value of $(i)\ \alpha + \beta + \gamma$ $(ii)\ \alpha\beta + \beta\gamma + \gamma\alpha$ $(iii)\ \alpha\beta\gamma$, are :

Solution :

$$i)\ \alpha + \beta + \gamma = -\frac{b}{a} = -\frac{1}{2}$$

$$ii)\ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = -\frac{3}{2}$$

$$iii)\ \alpha\beta\gamma = -\frac{d}{a} = \frac{1}{2}$$

If one root of the equation $x^3 - 7x^2 + ax + b = 0, a, b \in \mathbb{R}$, is $3 + 2i$, then the ordered pair (a, b) is :

Solution :

$x^3 - 7x^2 + ax + b = 0$

Let the roots be : $3 + 2i\,, 3 - 2i\,, \alpha$

Sum of roots $= 6 + \alpha = 7 \Rightarrow \alpha = 1$

Sum taken two at a time :

$a = (3 + 2i)(3 - 2i) + (3 + 2i)\alpha + (3 - 2i)\alpha \Rightarrow 13 + 6\alpha = a$

$\Rightarrow a = 19$

Product of roots : $-b = (3 + 2i)(3 - 2i)\alpha$

$\Rightarrow b = -13$

$\therefore (a, b) \equiv (19, -13)$

A. $(-19, 13)$

B. $(19, -13)$

C. $(19, 13)$

D. $(-19, -13)$

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Let α, β, γ be the roots of the equation $(x-a)(x-b)(x-c) = d\ , d \neq 0$, then the roots of the equation $(x-\alpha)(x-\beta)(x-\gamma) + d = 0$ are:

A $a+1, b+1, c+1$

B a, b, c

C $a-1, b-1, c-1$

D $a+1, b-1, c-1$

Solution:

Roots of equation $(x-a)(x-b)(x-c) - d = 0$ are α, β, γ

Thus, $(x-a)(x-b)(x-c) - d = (x-\alpha)(x-\beta)(x-\gamma)$

$$(x-a)(x-b)(x-c) = (x-\alpha)(x-\beta)(x-\gamma) + d$$

$$(x-\alpha)(x-\beta)(x-\gamma) + d = (x-a)(x-b)(x-c)$$

$\therefore$ Roots of the equation $(x-\alpha)(x-\beta)(x-\gamma) + d = 0$ are: a, b, c

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If equations $x^3 + x - 2 = 0$ and $ax^2 + bx + c = 0$, $(a, b, c \in \mathbb{N})$ have two roots common, then the minimum value of $a + b + c$ is:

Solution:

$$x^3 + x - 2 = 0 \Rightarrow (x - 1)(x^2 + x + 2) = 0$$

Roots are 1, $\dfrac{-1 \pm \sqrt{7}i}{2}$

So, the common roots with the equation $ax^2 + bx + c = 0$

will be $\dfrac{-1 \pm \sqrt{7}i}{2}$ (since imaginary roots occur in pairs)

$\therefore$ Equations $\left.\begin{array}{r} x^2 + x + 2 = 0 \\ ax^2 + bx + c = 0 \end{array}\right\}$ Have both roots common

$\therefore a : b : c = 1 : 1 : 2$

Minimum value of $a + b + c$ is 4

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Theory of Equations

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of the equation,

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_0 \neq 0, n \in \mathbb{W}$, then

S_1: Sum of roots taken one at a time $= \sum \alpha_1 = -\frac{a_1}{a_0}$

S_2: Sum of roots taken two at a time $= \sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$

S_3: Sum of roots taken three at a time $= \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$

P: Product of roots $= \prod \alpha_n = (-1)^n\frac{a_n}{a_0}$

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If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $-4x^4 - 3x^2 + 5x - 7 = 0$, then find the value of:

$(i) \sum \alpha$ $(ii) \sum \alpha\beta$ $(iii) \sum \alpha\beta\gamma$ $(iv) \prod \alpha$

Solution:

$$-4x^4 - 3x^2 + 5x - 7 = 0$$

$$-4x^4 + 0 \cdot x^3 - 3x^2 + 5x - 7 = 0$$

$(i) \sum \alpha = 0$

$(ii) \sum \alpha\beta = \frac{3}{4}$

$(iii) \sum \alpha\beta\gamma = \frac{5}{4}$

$(iv) \prod \alpha = \frac{7}{4}$

S_1: Sum of roots taken one at a time $= \sum \alpha_1 = -\frac{a_1}{a_0}$

S_2: Sum of roots taken two at a time $= \sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$

S_3: Sum of roots taken three at a time $= \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$

P: Product of roots $= \prod \alpha_n = (-1)^n \frac{a_n}{a_0}$

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If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^2 + qx + r = 0$, then find the equation whose roots are: $\alpha\beta\gamma, \alpha\beta\delta, \alpha\gamma\delta, \beta\gamma\delta$

Solution:

Roots are $\alpha\beta\gamma, \alpha\beta\delta, \alpha\gamma\delta, \beta\gamma\delta$

Since $\alpha\beta\delta\gamma = r \quad \Rightarrow$ Roots will become: $\frac{r}{\delta}, \frac{r}{\gamma}, \frac{r}{\beta}, \frac{r}{\alpha}$

Let the equation whose roots are $\frac{r}{\alpha}, \frac{r}{\beta}, \frac{r}{\gamma}, \frac{r}{\delta}$, be $Ay^4 + By^3 + Cy^2 + Dy + E = 0$

$y = \frac{r}{x} \quad \Rightarrow x = \frac{r}{y}$

Thus, the transformation will be $x \to \frac{r}{y}$, in the equation $x^4 + px^2 + qx + r = 0$

$$\left(\frac{r}{y}\right)^4 + p\left(\frac{r}{y}\right)^2 + q\left(\frac{r}{y}\right) + r = 0$$

OR replace y by x

So, the equation will be: $\left(\frac{r}{x}\right)^4 + p\left(\frac{r}{x}\right)^2 + q\left(\frac{r}{x}\right) + r = 0$

$\therefore$ The required equation is $x^4 + qx^3 + prx^2 + r^3 = 0$

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Key Takeaways

Transformation of a Polynomial Equation

Let α_i is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, then the equation whose roots are:

- $\alpha_i + k \ \forall\, i$: replace $x \to x - k$
- $\alpha_i \cdot k \ \forall\, i$: replace $x \to \frac{x}{k}$
- $p\alpha_i + q \ \forall\, i$: replace $x \to \frac{x-q}{p}$
- $\frac{k}{\alpha_i} \ \forall\, i$: replace $x \to \frac{k}{x}$
- $(\alpha_i)^n \ \forall\, i, n \in \mathbb{N}$: replace $x \to (x)^{\frac{1}{n}}$
- $(\alpha_i)^{\frac{1}{n}} \ \forall\, i, n \in \mathbb{N}$: replace $x \to (x)^n$

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If roots α, β, γ are of the equation $x^3 - 6x^2 + 10x - 3 = 0$, then the cubic equation with the roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$, is:

Solution:

Roots: α, β, γ

$$x^3 - 6x^2 + 10x - 3 = 0$$

Transformation: $x \to \frac{x-1}{2}$

The required equation having roots $2\alpha + 1,\ 2\beta + 1,\ 2\gamma + 1$ is:

$$\left(\frac{x-1}{2}\right)^3 - 6\left(\frac{x-1}{2}\right)^2 + 10\left(\frac{x-1}{2}\right) - 3 = 0$$

$$\Rightarrow (x-1)^3 - 12(x-1)^2 + 40(x-1) - 3 \cdot 8 = 0$$

$$\Rightarrow x^3 - 1 - 3x^2 + 3x - 12x^2 - 12 + 24x + 40x - 40 - 24 = 0$$

$$\Rightarrow x^3 - 15x^2 + 67x - 77 = 0$$

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If roots α, β, γ of the equation $x^3 + qx + r = 0$, then the equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$, is:

Solution:

Roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$

Since $\alpha + \beta + \gamma = 0 \Rightarrow \alpha + \beta = -\gamma$,

$$\beta + \gamma = -\alpha$$

$$\gamma + \alpha = -\beta$$

So, the roots become $-\alpha, -\beta, -\gamma$

Thus, the equation will be:

$$(-x)^3 + q(-x) + r = 0$$

Transformation: $x \to -x$

A $x^3 - qx - r = 0$

B $qx^3 - x + r = 0$

C $x^3 + qx - r = 0$

D $rx^3 - qx + 1 = 0$

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Let $[x]$ denote the greatest integer less than or equal to x. Then, the values of $x \in \mathbb{R}$ satisfying the equation $[e^x]^2 + [e^x + 1] - 3 = 0$ lies in the interval:

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Solution:

Given: $[e^x]^2 + [e^x + 1] - 3 = 0$

$\Rightarrow [e^x]^2 + [e^x] - 2 = 0$

$\Rightarrow ([e^x] + 2)([e^x] - 1) = 0$

$\therefore [e^x] = 1$, since $[e^x] = -2$ is not possible.

$\therefore e^x \in [1, 2)$

$\therefore x \in [0, \log_e 2)$

A. $\left[0, \frac{1}{e}\right)$

B. $[1, e)$

C. $[0, \log_e 2)$

D. $[\log_e 2, \log_e 3)$

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Session 07

Descartes rule of sign changes & Newtons theorem

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Key Takeaways

Descartes' Rule

- Let us consider an equation: $x^2 - 3x + 2 = 0$,

  Roots of this equation are: $1, 2$ (both positive)

  $x^2 - 3x + 2 = 0$

  $+ \quad - \quad +$

  $1 \quad 2$

  If we look at the sign change of the coefficients, it's occurring two times, and thus 2 positive roots.

- Let us consider an equation: $x^3 - 6x^2 + 11x - 6 = 0$,

  Roots of this equation are: $1, 2, 3$ (all positive)

  $x^3 - 6x^2 + 11x - 6 = 0$

  $+ \quad - \quad + \quad -$

  $1 \quad 2 \quad 3$

  If we look at the sign change of the coefficients, it's occurring three times, and thus 3 positive roots.

Key Takeaways

Descartes' Rule

For an equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$,

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $a_0 \neq 0, n \in \mathbb{W}$

- Maximum number of positive roots is equal to number of sign changes between consecutive (non–zero) coefficients in $f(x) = 0$.
- Maximum number of negative roots is equal to number of sign changes between consecutive (non–zero) coefficients in $f(-x) = 0$.

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Key Takeaways

Descartes' Rule

Example

$$f(x) = x^7 + 5x^6 - x^3 + 7x + 2 = 0$$

$$f(+x) = x^7 + 5x^6 - x^3 + 7x + 2 = 0$$

$+ \quad + \quad - \quad + \quad +$

2 sign changes ⇒ Maximum 2 positive roots are possible for $f(x) = 0$

$$f(x) = x^7 + 5x^6 - x^3 + 7x + 2 = 0$$

$$f(-x) = -x^7 + 5x^6 + x^3 - 7x + 2 = 0$$

$- \quad + \quad + \quad - \quad +$

3 sign changes ⇒ Maximum 3 negative roots are possible for $f(x) = 0$

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Key Takeaways

Descartes' Rule

Example

$$f(x) = x^7 + 5x^6 - x^3 + 7x + 2 = 0$$

- As number of non real roots for $f(x) = 0$ is equal to degree of polynomial – number of real roots

$\therefore$ Maximum 2 positive, 3 negative real roots & at least 2 non-real roots are possible for $f(x) = 0$

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Key Takeaways

Descartes' Rule

Note Number of positive or negative roots for $f(x) = 0$ is equal to number of sign changes between consecutive (non–zero) coefficients or even number times lesser than number of sign changes.

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Key Takeaways

Descartes' Rule

Polynomial Equation	Roots	Sign changes for $f(+x) = 0$	Sign changes for $f(-x) = 0$
$(x-1)(x-2) = 0$ (or) $x^2 - 3x + 2$	1,2 (both roots positive)	2	0
$x^2 - 2x + 3 = 0$	$1 \pm i\sqrt{2}$	2	0
$(x-1)(x-2)(x-3) = 0$ or $x^3 - 6x^2 + 11x - 6 = 0$	1,2, 3 (three positive roots)	3	0
$(x+1)(x^2+1) = 0$ or $x^3 + x^2 + x + 1 = 0$	$-1, \pm i$ (one negative root)	0	3

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Key Takeaways

Descartes' Rule

Example

- $f(x) = 2x^5 + x^4 - x^2 - 6x + 5 = 0$

$$f(+x) = 2x^5 + x^4 - x^2 - 6x + 5 = 0$$

$$+ \quad + \quad - \quad - \quad +$$

2 sign changes.

Maximum 2 positive real roots are possible for $f(x) = 0$

Hence, $f(x) = 0$ will have either 2 positive real roots or none.

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Key Takeaways

Descartes' Rule

Example

- $f(x) = 2x^5 + x^4 - x^2 - 6x + 5 = 0$

$$f(-x) = -2x^5 + x^4 - x^2 + 6x + 5 = 0$$

$-$ $+$ $-$ $+$ $+$

3 sign changes.

Maximum 3 negative real roots are possible for $f(x) = 0$

Hence, $f(x) = 0$ will have either 3 or 1 negative real roots.

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Descartes' Rule

Note

- If the number of sign changes in $f(x) = 0$ is 0 or 1, then number of positive root(s) for $f(x) = 0$ is equal to number of sign change.

- If the number of sign changes in $f(-x) = 0$ is 0 or 1, then number of negative root(s) for $f(x) = 0$ is equal to number of sign change.

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Descartes' Rule

Example

- $f(x) = x^2 + 3x + 2 = 0$

  Roots of this equation are : $-1, -2$ (both negative)

  $x^2 + 3x + 2 = 0$

  $+ \quad + \quad +$

  Zero sign change: 0 positive root

For the equation $3x^4 + 12x^2 + 5x - 4 = 0$, Which of the following is / are true ?

Solution: Given Equation: $3x^4 + 12x^2 + 5x - 4 = 0$

Let $f(x) = 3x^4 + 12x^2 + 5x - 4 = 0$

By Descartes' Rule : number of sign change $= 1$

So, positive root is 1

Similarly,

$f(-x) = 3x^4 + 12x^2 - 5x - 4 = 0$

number of sign change $= 1$

So, negative root is 1

So, other two roots are imaginary.

A. Four real roots

B. Two real and two imaginary roots

C. One positive and one negative roots

D. Four imaginary roots

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Equations Reducible to Quadratic Form

By using some appropriate substitution,
we can convert certain equations to quadratic equation form.

Example

- $(x^2 + x)^2 - (x^2 + x) - 5 = 0$, with $(x^2 + x) \to t$

  Converted into quadratic $\Rightarrow t^2 - t - 5 = 0$

- $\log^2 x + \log x - 5 = 0$, with $\log x \to t$

  Converted into quadratic $\Rightarrow t^2 + t - 5 = 0$

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The number of real roots of the equation, $e^{4x} + 2e^{3x} - e^x - 6 = 0$, is:

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Solution:

Given: $f(x) = e^{4x} + 2e^{3x} - e^x - 6 = 0$

Let $e^x = t \ (t > 0)$

$\Rightarrow f(t) = t^4 + 2t^3 - t - 6$

Number of real roots for $f(x) = 0$ is equal to number of positive roots for $f(t) = 0$

> **Note**
>
> If the number of sign changes in $f(x) = 0$ is 0 or 1, then number of positive root(s) for $f(x) = 0$ is equal to number of sign change.

Using Descartes rule,

Number of sign changes for $f(+t) =$ $+ \quad + \quad - \quad -$ 1 sign change

So 1 positive root exists for $f(t) = 0$

$\Rightarrow 1$ real root exists for $f(x) = 0$

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The number of real roots of the equation, $e^{4x} + 2e^{3x} - e^{x} - 6 = 0$, is:

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A 1

B 2

C 4

D 0

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The number of real roots of the equation, $e^{4x} + 2e^{3x} - e^{x} - 6 = 0$, is:

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Solution:

Given: $e^{4x} + e^{3x} - 4e^{2x} + e^{x} + 1 = 0$

Let $e^{x} = t$,

Equation becomes: $t^4 + t^3 - 4t^2 + t + 1 = 0$

Divide by t^2

$\left(t^2 + \frac{1}{t^2}\right) + \left(t + \frac{1}{t}\right) - 4 = 0$

Put $t + \frac{1}{t} = z$

$z^2 - 2 + z - 4 = 0$

$\Rightarrow z = -3, 2 \Rightarrow t + \frac{1}{t} = -3, 2$ (-3 is not possible as $t > 0$)

$t^2 - 2t + 1 = 0$

$\Rightarrow t = 1 \quad \Rightarrow e^{x} = 1 \Rightarrow x = 0 \quad \therefore$ The number of real roots is 1.

A 3

B 2

C 1

D 4

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Number of distinct real roots of the equation $(x^2 - x - 1)^{(x^2-3x+2)} = 1$, is:

Solution:

Given: $(x^2 - x - 1)^{(x^2-3x+2)} = 1$

$a^n = 1$

Case 1: $a = 1$

Case 2: $n = 0$

Case 3: $a = -1$ (n: even)

Case 1: $a = 1$

$x^2 - x - 1 = 1$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = -1, 2$

Case 2: $n = 0$

$x^2 - 3x + 2 = 0$

$\Rightarrow x = 1, 2$

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Number of distinct real roots of the equation $(x^2 - x - 1)^{(x^2-3x+2)} = 1$, is:

Solution:

Given: $(x^2 - x - 1)^{(x^2-3x+2)} = 1$

Case 3: $a = -1$ (but here n has to be even)

$x^2 - x - 1 = -1$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x = 0, 1$

For $x = 0 \rightarrow x^2 - 3x + 2 = 2$ (even)

For $x = 1 \rightarrow x^2 - 3x + 2 = 0$ (even)

∴ Number of real distinct roots = 4

$a^n = 1$

Case 1: $a = 1$

Case 2: $n = 0$

Case 3: $a = -1$ (n: even)

A 2

B 3

C 4

D 1

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Key Takeaways

Newton's theorem

Let α and β are the roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ and

(i) Sum of n^{th} power of roots i.e., $\alpha^n + \beta^n = S_n$,

(ii) Difference of n^{th} power of roots i.e., $\alpha^n - \beta^n = D_n$

$$a\,S_{n+2} + b\,S_{n+1} + S_n = 0, \qquad a\,D_{n+2} + b\,D_{n+1} + cD_n = 0 \ \ (n \in \mathbb{N})$$

Proof: α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$

$\alpha^n(a\alpha^2 + b\alpha + c) = 0$

$\Rightarrow a\alpha^{n+2} + b\alpha^{n+1} + c\alpha^n = 0 \cdots (i)$

Similarly, $a\beta^{n+2} + b\beta^{n+1} + c\beta^n = 0 \cdots (ii)$

Key Takeaways

Proof: $\alpha^{n+2} + b\alpha^{n+1} + c\alpha^{n} = 0 \cdots (i)$

$$+ \quad a\beta^{n+2} + b\beta^{n+1} + c\beta^{n} = 0 \cdots (ii)$$

$$a(\alpha^{n+2} + \beta^{n+2}) + b(\alpha^{n+1} + \beta^{n+1}) + c(\alpha^{n} + \beta^{n}) = 0$$

$$\Rightarrow a\,S_{n+2} + b\,S_{n+1} + c\,S_{n} = 0$$

Similarly,

$$\alpha^{n+2} + b\alpha^{n+1} + c\alpha^{n} = 0 \cdots (i)$$

$$- \quad a\beta^{n+2} + b\beta^{n+1} + c\beta^{n} = 0 \cdots (ii)$$

$$a(\alpha^{n+2} - \beta^{n+2}) + b(\alpha^{n+1} - \beta^{n+1}) + c(\alpha^{n} - \beta^{n}) = 0$$

$$\Rightarrow a\,D_{n+2} + b\,D_{n+1} + c\,D_{n} = 0$$

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Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then value of $\frac{a_{10} - 2a_8}{3a_9}$ is:

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A 4

B 1

C 2

D 3

Solution:

Method 1:

Given :

$a_n = \alpha^n - \beta^n$ & $x^2 - 6x - 2 = 0$

$\Rightarrow a_{n+2} - 6a_{n+1} - 2a_n = 0$

> $D_n = \alpha^n - \beta^n$
>
> $\Rightarrow aD_{n+2} + bD_{n+1} + cD_n = 0$

Substituting $n = 8$

$\Rightarrow a_{10} - 6a_9 - 2a_8 = 0$

$\Rightarrow a_{10} - 2a_8 = 6a_9$

$\Rightarrow \frac{a_{10} - 2a_8}{3a_9} = \frac{6a_9}{3a_9} = 2$

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Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then value of $\frac{a_{10}-2a_8}{3a_9}$ is:

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A 4

B 1

C 2

D 3

Solution:

Method 2: α and β be the roots of $x^2 - 6x - 2 = 0$.

$$\frac{a_{10}-2a_8}{3a_9} = \frac{(\alpha^{10}-\beta^{10})-2(\alpha^8-\beta^8)}{3(\alpha^9-\beta^9)}$$

$$= \frac{(\alpha^{10}-2\alpha^8)-(\beta^{10}-2\beta^8)}{3(\alpha^9-\beta^9)}$$

$$= \frac{\alpha^8(\alpha^2-2)-\beta^8(\beta^2-2)}{3(\alpha^9-\beta^9)}$$

$$= \frac{\alpha^8(6\alpha)-\beta^8(6\beta)}{3(\alpha^9-\beta^9)}$$

$$= \frac{6(\alpha^9-\beta^9)}{3(\alpha^9-\beta^9)} = \frac{6}{3} = 2$$

$$\alpha^2 - 6\alpha - 2 = 0$$

$$\Rightarrow \alpha^2 - 2 = 6\alpha$$

$$\beta^2 - 6\beta - 2 = 0$$

$$\Rightarrow \beta^2 - 2 = 6\beta$$

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If α, β are roots of equation $x^2 + 5(\sqrt{2})x + 10 = 0, \alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n, then the value of $\frac{P_{17}P_{20}+5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19}+5\sqrt{2}P_{18}^2}$ is equal to:

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Solution:

$$\alpha^{n-2}(\alpha^2 + 5\sqrt{2}\alpha + 10) = 0 \cdots (1)$$

$$\beta^{n-2}(\beta^2 + 5\sqrt{2}\beta + 10) = 0 \cdots (2)$$

From (2) – (1)

$$P_n + 5\sqrt{2}P_{n-1} = -10P_{n-2}$$

$$\frac{P_{17}(P_{20}+5\sqrt{2}P_{19})}{P_{18}(P_{19}+5\sqrt{2}P_{18})} = \frac{P_{17}\cdot(-10P_{18})}{P_{18}\cdot(-10P_{17})} = 1$$

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THANK
YOU

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