MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let R_1 and R_2 be two relations defined on $\mathbb{R}$ by a $R_1\, b \Leftrightarrow ab \geq 0$ and $a\, R_2\, b \Leftrightarrow a \geq b$. Then,

   (A) R_1 is an equivalence relation but not R_2

   (B) R_2 is an equivalence relation but not R_1

   (C) Both R_1 and R_2 are equivalence relations

   (D) Neither R_1 nor R_2 is an equivalence relation

Answer (D)

Sol. $a\, R_1\, b \Leftrightarrow ab \geq 0$

So, definitely $(a, a) \in R_1$ as $a^2 \geq 0$

If $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$

But if $(a, b) \in R_1$, $(b, c) \in R_1$

$\Rightarrow$ Then (a, c) may or may not belong to R_1

{Consider $a = -5$, $b = 0$, $c = 5$ so (a, b) and $(b, c) \in R_1$ but $ac < 0$}

So, R_1 is not equivalence relation

$a\, R_2\, b \Leftrightarrow a \geq b$

$(a, a) \in R_2 \Rightarrow$ so reflexive relation

If $(a, b) \in R_2$ then (b, a) may or may not belong to R_2

$\Rightarrow$ So not symmetric

Hence it is not equivalence relation

2. Let $f, g : \mathbb{N} - \{1\} \to \mathbb{N}$ be functions defined by $f(a) = \alpha$, where α is the maximum of the powers of those primes p such that p^α divides a, and $g(a) = a + 1$, for all $a \in \mathbb{N} - \{1\}$. Then, the function $f + g$ is

   (A) one-one but not onto

   (B) onto but not one-one

   (C) both one-one and onto

   (D) neither one-one nor onto

Answer (D)

Sol. $f, g : N - \{1\} \to N$ defined as

$f(a) = \alpha$, where α is the maximum power of those primes p such that p^α divides a.

$g(a) = a + 1$,

Now, $f(2) = 1,\ g(2) = 3 \Rightarrow (f + g)(2) = 4$

$f(3) = 1,\ g(3) = 4 \Rightarrow (f + g)(3) = 5$

$f(4) = 2,\ g(4) = 5 \Rightarrow (f + g)(4) = 7$

$f(5) = 1,\ g(5) = 6 \Rightarrow (f + g)(5) = 7$

$\because\ (f + g)(5) = (f + g)(4)$

$\therefore\ f + g$ is not one-one

Now, $\because\ f_{min} = 1,\ g_{min} = 3$

So, there does not exist any $x \in N - \{1\}$ such that $(f + g)(x) = 1, 2, 3$

$\therefore\ f + g$ is not onto

3. Let the minimum value v_0 of $v = |z|^2 + |z - 3|^2 + |z - 6i|^2$, $z \in \mathbb{C}$ is attained at $z = z_0$. Then $\left|2z_0^2 - \bar{z}_0^3 + 3\right|^2 + v_0^2$ is equal to

   (A) 1000

   (B) 1024

   (C) 1105

   (D) 1196

Answer (A)

Sol. Let $z = x + iy$

$v = x^2 + y^2 + (x - 3)^2 + y^2 + x^2 + (y - 6)^2$

$= (3x^2 - 6x + 9) + (3y^2 - 12y + 36)$

$= 3(x^2 + y^2 - 2x - 4y + 15)$

$= 3[(x - 1)^2 + (y - 2)^2 + 10]$

v_{min} at $z = 1 + 2i = z_0$ and $v_0 = 30$

so $|2(1 + 2i)^2 - (1 - 2i)^3 + 3|^2 + 900$

$= |2(-3 + 4i) - (1 - 8i^3 - 6i\,(1 - 2i) + 3|^2 + 900$

$= |-6 + 8i - (1 + 8i - 6i - 12) + 3|^2 + 900$

$= |8 + 6i|^2 + 900$

$= 1000$

4. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & -5 \end{pmatrix}$. Let $\alpha, \beta, \in \mathbb{R}$ be such that $\alpha A^2 + \beta A = 2I$. Then $\alpha + \beta$ is equal to

(A) –10 (B) –6

(C) 6 (D) 10

Answer (D)

Sol. $A^2 = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -3 & -8 \\ 8 & 21 \end{bmatrix}$

$$\alpha A^2 + \beta A = \begin{bmatrix} -3\alpha & -8\alpha \\ 8\alpha & 21\alpha \end{bmatrix} + \begin{bmatrix} \beta & 2\beta \\ -2\beta & -5\beta \end{bmatrix}$$

$$= \begin{bmatrix} -3\alpha + \beta & -8\alpha + 2\beta \\ 8\alpha - 2\beta & 21\alpha - 5\beta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

On Comparing

$8\alpha = 2\beta$, $-3\alpha + \beta = 2$, $21\alpha - 5\beta = 2$

$\Rightarrow \alpha = 2, \beta = 8$

So, $\alpha + \beta = 10$

5. The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is

(A) 0 (B) 1

(C) 2 (D) 6

Answer (A)

Sol. $(2021)^{2022} + (2022)^{2021}$

$= (7k - 2)^{2022} + (7k_1 - 1)^{2021}$

$= \left[(7k-2)^3\right]^{674} + (7k_1)^{2021} - 2021(7k_1)^{2020} + ... -1$

$= (7k_2 - 1)^{674} + (7m - 1)$

$= (7n + 1) + (7m - 1) = 7(m + n)$ (multiple of 7)

$\therefore$ Remainder = 0

6. Suppose $a_1, a_2, \ldots a_n, \ldots$ be an arithmetic progression of natural numbers. If the ration of the sum of first five terms to the sum of first nine terms of the progression is 5 : 17 and $110 < a_{15} < 120$, then the sum of the first ten terms of the progression is equal to

(A) 290

(B) 380

(C) 460

(D) 510

Answer (B)

Sol. $\because$ $a_1, a_2, \ldots a_n \ldots$ be an A.P of natural numbers and

$$\frac{S_5}{S_9} = \frac{5}{17} \Rightarrow \frac{\frac{5}{2}[2a_1 + 4d]}{\frac{9}{2}[2a_1 + 8d]} = \frac{5}{17}$$

$\Rightarrow 34a_1 + 68d = 18a_1 + 72d$

$\Rightarrow 16a_1 = 4d$

$\therefore$ $\boxed{d = 4a_1}$

And $110 < a_{15} < 120$

$\therefore$ $110 < a_1 + 14d < 120 \Rightarrow 110 < 57a_1 < 120$

$\therefore$ $a_1 = 2$ ($\because a_i \in N$)

$d = 8$

$\therefore$ $S_{10} = 5\,[4 + 9 \times 8] = 380$

7. Let $\mathbb{R} \to \mathbb{R}$ be function defined as $f(x) = a\sin\left(\frac{\pi[x]}{2}\right) + [2 - x], a \in \mathbb{R}$, where $[t]$ is the greatest integer less than or equal to t. If $\lim_{x \to -1} f(x)$ exists, then the value of $\int_0^4 f(x)dx$ is equal to

(A) –1 (B) –2

(C) 1 (D) 2

Answer (B)

Sol. $f(x) = a\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]\ a \in R$

Now,

$\because \lim_{x \to -1} f(x)$ exist

$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$\Rightarrow a\sin\left(\frac{-2\pi}{2}\right) + 3 = a\sin\left(\frac{-\pi}{2}\right) + 2$

$\Rightarrow -a = 1 \Rightarrow \boxed{a = -1}$

Now, $\int_0^4 f(x)dx = \int_0^4 \left(-\sin\left(\frac{\pi[x]}{2}\right) + [2 - x]\right)dx$

$= \int_0^1 1dx + \int_1^2 -1dx + \int_2^3 -1dx + \int_3^4 (1-2)dx$

$= 1 - 1 - 1 - 1 = -2$

8. Let $I = \int_{\pi/4}^{\pi/3}\left(\frac{8\sin x - \sin 2x}{x}\right)dx$. Then

(A) $\frac{\pi}{2} < I < \frac{3\pi}{4}$ (B) $\frac{\pi}{5} < I < \frac{5\pi}{12}$

(C) $\frac{5\pi}{12} < I < \frac{\sqrt{2}}{3}\pi$ (D) $\frac{3\pi}{4} < I < \pi$

Answer (*)

Sol. I comes out around 1.536 which is not satisfied by any given options.

$$\int_{\pi/4}^{\pi/3}\frac{8x-2x}{x}dx > I > \int_{\pi/4}^{\pi/3}\frac{8\sin x - 2x}{x}dx$$

$$\frac{\pi}{2} > I > \int_{\pi/4}^{\pi/3}\left(\frac{8\sin x}{x} - 2\right)dx$$

$\frac{\sin x}{x}$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$ so it attains maximum at $x = \frac{x}{4}$

$$I > \int_{\pi/4}^{\pi/3}\left(\frac{8\sin \pi/3}{\pi/3} - 2\right)dx$$

$$I > \sqrt{3} - \frac{\pi}{6}$$

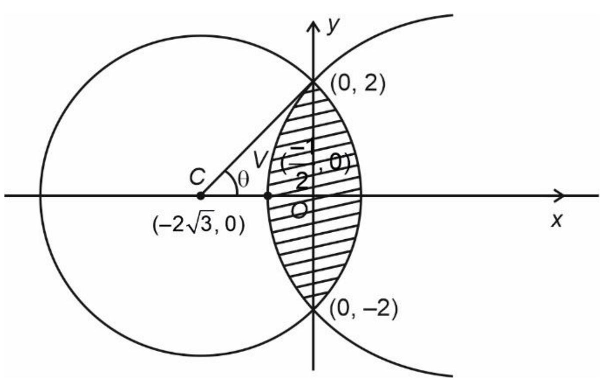
9. The area of the smaller region enclosed by the curves $y^2 = 8x + 4$ and $x^2 + y^2 + 4\sqrt{3}x - 4 = 0$ is equal to

(A) $\frac{1}{3}\left(2 - 12\sqrt{3} + 8\pi\right)$ (B) $\frac{1}{3}\left(2 - 12\sqrt{3} + 6\pi\right)$

(C) $\frac{1}{3}\left(4 - 12\sqrt{3} + 8\pi\right)$ (D) $\frac{1}{3}\left(4 - 12\sqrt{3} + 6\pi\right)$

Answer (C)

Sol.



$$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$$

Area of the required region

$$= \frac{2}{3}\left(4 \times \frac{1}{2}\right) + 4^2 \times \frac{\pi}{6} - \frac{1}{2} \times 4 \times 2\sqrt{3}$$

$$= \frac{4}{3} + \frac{8\pi}{3} - 4\sqrt{3} = \frac{1}{3}\left\{4 - 12\sqrt{3} + 8\pi\right\}$$

10. Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solution of the differential equation $\frac{dy}{dx} = x + y$, with $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y = y_2(x)$ is

(A) 0 (B) 1

(C) 2 (D) 3

Answer (A)

Sol. $\frac{dy}{dx} = x + y$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - 1 = t \Rightarrow \int\frac{dt}{t+1} = \int dx$$

$$\ln|t+1| = x + C'$$

$$|t+1| = Ce^x$$

$$|x+y+1| = Ce^x$$

For $y_1(x)$, $y_1(0) = 0 \Rightarrow C = 1$

For $y_2(x)$, $y_2(0) = 1 \Rightarrow C = 2$

$y_1(x)$ is given by $|x + y + 1| = e^x$

$y_2(x)$ is given by $|x + y + 1| = 2e^x$

At point of intersection

$e^x = 2e^x$

No solution

So, there is no point of intersection of $y_1(x)$ and $y_2(x)$.

11. Let $P(a, b)$ be a point on the parabola $y^2 = 8x$ such that the tangent at P passes through the centre of the circle $x^2 + y^2 - 10x - 14y + 65 = 0$. Let A be the product of all possible values of a and B be the product of all possible values of b. Then the value of $A + B$ is equal to

(A) 0 (B) 25

(C) 40 (D) 65

Answer (D)

Sol. Centre of circle $x^2 + y^2 - 10x - 14y + 65 = 0$ is at (5, 7).

Let the equation of tangent to $y^2 = 8x$ is

$yt = x + 2t^2$

which passes through (5, 7)

$7t = 5 + 2t^2$

$\Rightarrow\ 2t^2 - 7t + 5 = 0$

$t = 1, \frac{5}{2}$

$$A = 2\times1^2\times2\times\left(\frac{5}{2}\right)^2 = 25$$

$$B = 2\times2\times1\times2\times2\times\frac{5}{2} = 40$$

$A + B = 65$

12. Let $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ be two vectors, such that $\vec{a}\times\vec{b} = -\hat{i} + 9\hat{i} + 12\hat{k}$. Then the projection of $\vec{b} - 2\vec{a}$ on $\vec{b} + \vec{a}$ is equal to

(A) 2 (B) $\frac{39}{5}$

(C) 9 (D) $\frac{46}{5}$

Answer (D)

Sol. $\vec{a} = \alpha\hat{i} + \hat{j} + \beta\hat{k}, \vec{b} = 3\hat{i} - 5\hat{j} + 4\hat{k}$

$\vec{a}\times\vec{b} = -\hat{i} + 9\hat{j} + 12\hat{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & \beta \\ 3 & -5 & 4 \end{vmatrix} = -\hat{i} + 9\hat{j} + 12\hat{k}$$

$4 + 5\beta = -1 \Rightarrow \beta = -1$

$-5\alpha - 3 = 12 \Rightarrow \alpha = -3$

$$\vec{b} - 2\vec{a} = 3\hat{i} - 5\hat{j} + 4\hat{k} - 2\left(-3\hat{i} + \hat{j} - \hat{k}\right)$$

$$\vec{b} - 2\vec{a} = 9\hat{i} - 7\hat{j} + 6\hat{k}$$

$$\vec{b} + \vec{a} = \left(3\hat{i} - 5\hat{j} + 4\hat{k}\right) + \left(-3\hat{i} + \hat{j} - \hat{k}\right)$$

$$\vec{b} + \vec{a} = -4\hat{j} + 3\hat{k}$$

Projection of $\vec{b} - 2\vec{a}$ on $\vec{b} + \vec{a}$ is $= \dfrac{\left(\vec{b} - 2\vec{a}\right)\cdot\left(\vec{b} + \vec{a}\right)}{\left|\vec{b} + \vec{a}\right|}$

$$= \frac{28 + 18}{5} = \frac{46}{5}$$

13. Let $\vec{a} = 2\hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = \alpha\hat{i} + \beta\hat{j} + 2\hat{k}$. If $\left(\left(\vec{a}\times\vec{b}\right)\times\hat{i}\right)\cdot\hat{k} = \frac{23}{2}$, then $\left|\vec{b}\times2\hat{j}\right|$ is equal to

(A) 4 (B) 5

(C) $\sqrt{21}$ (D) $\sqrt{17}$

Answer (B)

Sol. Given, $\vec{a} = 2\hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = \alpha\hat{i} + \beta\hat{j} + 2\hat{k}$

Also, $\left(\left(\vec{a}\times\vec{b}\right)\times i\right)\cdot\hat{k} = \frac{23}{2}$

$$\Rightarrow \left(\left(\vec{a}\cdot\hat{i}\right)\vec{b} - \left(\vec{b}\cdot\hat{i}\right)\cdot\vec{a}\right)\cdot\hat{k} = \frac{23}{2}$$

$$\Rightarrow \left(2\cdot\vec{b} - \alpha\cdot\vec{a}\right)\cdot\hat{k} = \frac{23}{2}$$

$$\Rightarrow 2\cdot2 - 5\alpha = \frac{23}{2} \Rightarrow \alpha = \frac{-3}{2}$$

Now, $\left|\vec{b}\times2\hat{j}\right| = \left|\left(\alpha\hat{i} + \beta\hat{j} + 2\hat{k}\right)\times2\hat{j}\right|$

$$= \left|2\alpha\hat{k} + 0 - 4\hat{i}\right|$$

$$= \sqrt{4\alpha^2 + 16}$$

$$= \sqrt{4\left(\frac{-3}{2}\right)^2 + 16}$$

$= 5$

14. Let S be the sample space of all five digit numbers. It p is the probability that a randomly selected number from S, is multiple of 7 but not divisible by 5, then $9p$ is equal to

(A) 1.0146 (B) 1.2085

(C) 1.0285 (D) 1.1521

Answer (C)

Sol. Among the 5 digit numbers,

First number divisible by 7 is 10003 and last is 99995.

$\Rightarrow$ Number of numbers divisible by 7.

$$= \frac{99995 - 10003}{7} + 1$$

$= 12857$

First number divisible by 35 is 10010 and last is 99995.

$\Rightarrow$ Number of numbers divisible by

$$35 = \frac{99995 - 10010}{35} + 1$$

$= 2572$

Hence number of number divisible by 7 but not by 5

$= 12857 – 2572$

$= 10285$

$$9P. = \frac{10285}{90000} \times 9$$

$= 1.0285$

15. Let a vertical tower AB of height $2h$ stands on a horizontal ground. Let from a point P on the ground a man can see upto height h of the tower with an angle of elevation 2α. When from P, he moves a distance d in the direction of $\overrightarrow{AP}$, he can see the top B of the tower with an angle of elevation α. if $d = \sqrt{7}h$, then tan α is equal to

(A) $\sqrt{5} - 2$

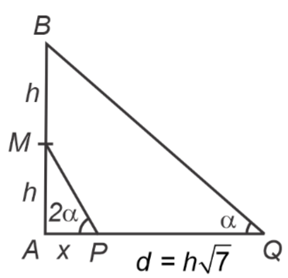
(B) $\sqrt{3} - 1$

(C) $\sqrt{7} - 2$

(D) $\sqrt{7} - \sqrt{3}$

Answer (C)

Sol.



ΔAPM gives

$$\tan 2\alpha = \frac{h}{x} \quad \ldots(i)$$

ΔAQB gives

$$\tan\alpha = \frac{2h}{x+d} = \frac{2h}{x + h\sqrt{7}} \quad \ldots(ii)$$

From (i) and (ii)

$$\tan\alpha = \frac{2 \cdot \tan 2\alpha}{1 + \sqrt{7} \cdot \tan 2\alpha}$$

Let $t = \tan\alpha$

$$\Rightarrow \quad t = \frac{2\dfrac{2t}{1-t^2}}{1 + \sqrt{7} \cdot \dfrac{2t}{1-t^2}}$$

$$\Rightarrow \quad t^2 - 2\sqrt{7}t + 3 = 0$$

$$t = \sqrt{7} - 2$$

16. $(p \wedge r) \Leftrightarrow (p \wedge (\sim q))$ is equivalent to $(\sim p)$ when r is

(A) p (B) $\sim p$

(C) q (D) $\sim q$

Answer (C)

Sol. The truth table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$p \wedge q \Leftrightarrow p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	F	F	T

Clearly $p \wedge q \Leftrightarrow p \wedge \sim q \equiv \sim p$

$\therefore \quad r = q$

17. If the plane P passes through the intersection of two mutually perpendicular planes $2x + ky – 5z = 1$ and $3kx – ky + z = 5$, $k < 3$ and intercepts a unit length on positive x-axis, then the intercept made by the plane P on the y-axis is

(A) $\frac{1}{11}$ (B) $\frac{5}{11}$

(C) 6 (D) 7

Answer (D)

Sol. $P_1 : 2x + ky - 5z = 1$

$P_2 : 3kx - ky + z = 5$

$\because \quad P_1 \perp P_2 \Rightarrow 6k - k^2 + 5 = 0$

$\Rightarrow \quad k = 1, 5$

$\because \quad k < 3$

$\therefore \quad k = 1$

$P_1 : 2x + y - 5z = 1$

$P_2 : 3x - y + z = 5$

$P:(2x+y-5z-1)+\lambda(3x-y+z-5)=0$

Positive x-axis intercept = 1

$\Rightarrow \quad \frac{1+5\lambda}{2+3\lambda}=1$

$\Rightarrow \quad \lambda=\frac{1}{2}$

$\therefore \quad P: 7x+y-4z=7$

y intercept = 7.

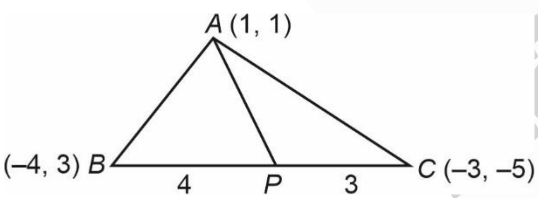
18. Let $A(1, 1)$, $B(-4, 3)$, $C(-2, -5)$ be vertices of a triangle ABC, P be a point on side BC, and Δ_1 and Δ_2 be the areas of triangles APB and ABC, respectively. If $\Delta_1 : \Delta_2 = 4 : 7$, then the area enclosed by the lines AP, AC and the x-axis is

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$

(C) $\frac{1}{2}$ (D) 1

Answer (C)

Sol. $\frac{\Delta_1}{\Delta_2}=\frac{\frac{1}{2}\times BP\times AH}{\frac{1}{2}\times BC\times AH}=\frac{4}{7}$



$P\left(\frac{-20}{7},\frac{-11}{7}\right)$

Line AC : $y-1=2(x-1)$

Intersection with x-axis $=\left(\frac{1}{2},0\right)$

Line $AP: y-1=\frac{2}{3}(x-1)$

Intersection with x-axis $\left(\frac{-1}{2},0\right)$

Vertices are $(1,1),\left(\frac{1}{2},0\right)$ and $\left(\frac{-1}{2},0\right)$

Area $=\frac{1}{2}$ sq. unit

19. If the circle $x^2+y^2-2gx+6y-19c=0$, $g, c\in\mathbb{R}$ passes through the point (6, 1) and its centre lies on the line $x-2cy=8$, then the length of intercept made by the circle on x-axis is

(A) $\sqrt{11}$ (B) 4

(C) 3 (D) $2\sqrt{23}$

Answer (D)

Sol. Circle : $x^2+y^2-2gx+6y-19c=0$

It passes through $h(6, 1)$

$\Rightarrow \quad 36+1-12g+6-19c=0$

$= 12g+19c=43 \quad \ldots(1)$

Line $x-2cy=8$ passes though centre

$\Rightarrow \quad g+6c=8 \quad \ldots(2)$

From (1) & (2)

$g=2, c=1$

$C: x^2+y^2-4x+6y-19=0$

$x\,\text{int}=2\sqrt{g^2-C}$

$=2\sqrt{4+19}$

$=2\sqrt{23}$

20. Let a function $f:\mathbb{R}\to\mathbb{R}$ be defined as :

$$f(x)=\begin{cases}\int_0^x (5-|t-3|)dt, & x>4\\ x^2+bx, & x\le 4\end{cases}$$

where $b\in\mathbb{R}$. If f is continuous at $x = 4$ then which of the following statements is **NOT** true?

(A) f is not differentiable at $x = 4$

(B) $f'(3)+f'(5)=\frac{35}{4}$

(C) f is increasing in $\left(-\infty,\frac{1}{8}\right)\cup(8,\infty)$

(D) f has a local minima at $x=\frac{1}{8}$

Answer (C)

Sol. $\because \quad f(x)$ is continuous at $x = 4$

$\Rightarrow \quad f(4^-)=f(4^+)$

$\Rightarrow \quad 16+4b=\int_0^4(5-|t-3|)dt$

$$= \int_0^3 (2+t)dt + \int_3^4 (8-t)dt$$

$$= 2t + \frac{t^2}{2}\Big)_0^3 + 8t - \frac{t^2}{3}\Big]_3^4$$

$$= 6 + \frac{9}{2} - 0 + (32-8) - \left(24 - \frac{9}{2}\right)$$

$16 + 4b = 15$

$$\Rightarrow \; b = \frac{-1}{4}$$

$$\Rightarrow \; f(x) = \begin{cases} \int_0^x 5-|t-3|\,dt & x > 4 \\ x^2 - \frac{x}{4} & x \le 4 \end{cases}$$

$$\Rightarrow \; f'(x) = \begin{cases} 5-|x-3| & x > 4 \\ 2x - \frac{1}{4} & x \le 4 \end{cases}$$

$$\Rightarrow \; f'(x) = \begin{cases} 8-x & x > 4 \\ 2x - \frac{1}{4} & x \le 4 \end{cases}$$

$$f'(x) < 0 \;\Rightarrow\; x \in \left(-\infty, \frac{1}{8}\right) \cup (8, \infty)$$

$$f'(3) + f'(5) = 6 - \frac{1}{4} = \frac{35}{4}$$

$f'(x) = 0 \;\Rightarrow\; x = \frac{1}{8}$ have local minima

$\therefore$ (C) is only incorrect option.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. For $k \in R$, let the solution of the equation

$$\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}\right)\right)\right)\right)\right)$$

$$= k, 0 < |x| < \frac{1}{\sqrt{2}}$$

Inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$ are $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to _______.

Answer (12)

Sol. $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}\right)\right)\right)\right)\right) = k$

$$\Rightarrow \; \cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right) = k$$

$$\Rightarrow \; \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = k$$

$$\Rightarrow \; \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$$

$$\Rightarrow \; \frac{1-2x^2}{1-x^2} = k^2$$

$\Rightarrow$ $1 - 2x^2 = k^2 - k^2x^2$

$$\therefore \; x^2 - \left(\frac{k^2-1}{k^2-2}\right) = 0 \; \begin{matrix} \alpha \\ \beta \end{matrix}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \qquad \ldots(1)$$

$$\text{and } \frac{\alpha}{\beta} = -1 \qquad \ldots(2)$$

$$\therefore \; 2\left(\frac{k^2-2}{k^2-1}\right)(-1) = -5$$

$$\Rightarrow \; k^2 = \frac{1}{3}$$

and b = S.R = $2\left(\frac{k^2-2}{k^2-1}\right) - 1 = 4$

$$\therefore \; \frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$$

2. The mean and variance of 10 observation were calculated as 15 and 15 respectively by a student who took by mistake 25 instead of 15 for one observation. Then, the correct standard deviation is ________.

Answer (2)

Sol. Given $\frac{\sum_{i=1}^{10} x_i}{10} = 15 \ \ldots(1) \quad \Rightarrow \sum_{i=1}^{10} x_i = 150$

and $\frac{\sum_{i=1}^{10} x_i^2}{10} - 15^2 = 15 \quad \Rightarrow \sum_{i=1}^{10} x_i^2 = 2400$

Replacing 25 by 15 we get

$\sum_{i=1}^{9} x_i + 25 = 150 \quad \Rightarrow \sum_{i=1}^{9} x_i = 125$

$\therefore$ Correct mean $= \frac{\sum_{i=1}^{9} x_i + 15}{10} = \frac{125+15}{10} = 14$

Similarly, $\sum_{i=1}^{2} x_i^2 = 2400 - 25^2 = 1775$

$\therefore$ correct variance $= \frac{\sum_{i=1}^{9} x_i^2 + 15^2}{10} - 14^2$

$= \frac{1775+225}{10} - 14^2 = 4$

$\therefore$ correct S.D = $\sqrt{4} = 2$.

3. Let the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4}$ intersect the plane containing the lines $\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$ and $4ax - y + 5z - 7a = 0 = 2x - 5y - z - 3$, $a \in \mathbb{R}$ at the point $P(\alpha, \beta, \gamma)$. Then the value of $\alpha + \beta + \gamma$ equals _____.

Answer (12)

Sol. Equation of plane containing the line

$4ax - y + 5z - 7a = 0 = 2x - 5y - z - 3$ can be written as

$4ax - y + 5z - 7a + \lambda(2x - 5y - z - 3) = 0$

$(4a + 2\lambda) x - (1 + 5\lambda) y + (5 - \lambda) z - (7a + 3\lambda) = 0$

Which is coplanar with the line

$\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$

$4(4a + 2\lambda) + (1 + 5\lambda) - (7a + 3\lambda) = 0$

$9a + 10\lambda + 1 = 0 \quad \ldots(1)$

$(4a + 2\lambda)1 + (1 + 5\lambda) 2 + 5 - \lambda = 0$

$4a + 11\lambda + 7 = 0 \quad \ldots(2)$

$a = 1, \lambda = -1$

Equation of plane is $x + 2y + 3z - 2 = 0$

Intersection with the line

$\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4}$

$(7t + 3) + 2(-t + 2) + 3(-4t + 3) - 2 = 0$

$-7t + 14 = 0$

$t = 2$

So, the required point is (17, 0, –5)

$\alpha + \beta + \gamma = 12$

4. An ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the vertices of the hyperbola $H: \frac{x^2}{49} - \frac{y^2}{64} = -1$. Let the major and minor axes of the ellipse E coincide with the transverse and conjugate axes of the hyperbola H, respectively. Let the product of the eccentricities of E and H be $\frac{1}{2}$. If the length of the latus rectum of the ellipse E, then the value of $113l$ is equal to _____.

Answer (1552)

Sol. Vertices of hyperbola = (0, ± 8)

As ellipse pass through it i.e.,

$0 + \frac{64}{b^2} = 1 \Rightarrow b^2 = 64 \quad \ldots(1)$

As major axis of ellipse coincide with transverse axis of hyperbola we have $b > a$ i.e.

$e_E = \sqrt{1 - \frac{a^2}{64}} = \frac{\sqrt{64 - a^2}}{8}$

and $e_H = \sqrt{1 + \frac{49}{64}} = \frac{\sqrt{113}}{8}$

$\therefore \quad e_E \cdot e_H = \frac{1}{2} = \frac{\sqrt{64-a^2}\sqrt{113}}{64}$

$\Rightarrow \quad (64 - a^2)(113) = 32^2$

$\Rightarrow \quad a^2 = 64 - \frac{1024}{113}$

L.R of ellipse $= \frac{2a^2}{b} = \frac{2}{8}\left(\frac{113\times 64-1024}{113}\right)$

$= l = \frac{1552}{113}$

$\therefore\quad 113l = 1552$

5. Let $y = y(x)$ be the solution curve of the differential equation

$$\sin\left(2x^2\right)\log_e\left(\tan x^2\right)dy + \left(4xy - 4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx = 0,$$

$0 < x < \sqrt{\frac{\pi}{2}}$, which passes through the point $\left(\sqrt{\frac{\pi}{6}},1\right)$. Then $\left|y\left(\sqrt{\frac{\pi}{3}}\right)\right|$ is equal to ________.

Answer (1)

Sol. $\frac{dy}{dx} + y\left(\frac{4x}{\sin\left(2x^2\right)\ln\left(\tan x^2\right)}\right) = \frac{4\sqrt{2}\,x\sin\left(x^2-\frac{\pi}{4}\right)}{\sin\left(2x^2\right)\ln\left(\tan x^2\right)}$

I.F $= e^{\int\frac{4x}{\sin\left(2x^2\right)}\frac{dx}{\ln\left(\tan x^2\right)}}$

$= e^{\ln\left|\ln\left(\tan x^2\right)\right|} = \ln\left(\tan x^2\right)$

$\therefore\quad \int d\left(y.\ln\left(\tan x^2\right)\right) = \int\frac{4\sqrt{2}\,x\sin\left(x^2-\frac{\pi}{4}\right)}{\sin\left(2x^2\right)}dx$

$\Rightarrow\quad y\ln\left(\tan x^2\right) = \ln\left|\frac{\sec x^2+\tan x^2}{\operatorname{cosec} x^2-\cot x^2}\right| + C$

$\ln\left(\frac{1}{\sqrt{3}}\right) = \ln\left(\frac{\frac{3}{\sqrt{3}}}{2-\sqrt{3}}\right) + C$

$e = \ln\left(\frac{1}{\sqrt{3}}\right) - \ln\left(\frac{\sqrt{3}}{2-\sqrt{3}}\right)$

For $y\left(\sqrt{\frac{\pi}{3}}\right)$

$y\ln\left(\sqrt{3}\right) = \ln\left|\frac{2+\sqrt{3}}{\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}}\right| + \ln\left(\frac{1}{\sqrt{3}}\right) - \ln\left(\frac{\sqrt{3}}{2\sqrt{3}}\right)$

$= \ln\left(2+\sqrt{3}\right) + \ln\left(\frac{1}{\sqrt{3}}\right) + \ln\left(\frac{1}{\sqrt{3}}\right) - \ln\left(\frac{\sqrt{3}}{2-\sqrt{3}}\right)$

$\Rightarrow\quad y\ln\sqrt{3} = \ln\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow\quad \frac{y}{2}\ln 3 = -\frac{1}{2}\ln 3$

$\Rightarrow\quad y = 1$

$\therefore\quad \left|y\left(\sqrt{\frac{\pi}{3}}\right)\right| = 1.$

6. Let M and N be the number of points on the curve $y^5 - 9xy + 2x = 0$, where the tangents to the curve are parallel to x-axis and y-axis, respectively. Then the value of $M + N$ equals ________.

Answer (2)

Sol. Here equation of curve is

$y^5 - 9xy + 2x = 0$ …(i)

On differentiating: $5y^4\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} + 2 = 0$

$\therefore\quad \frac{dy}{dx} = \frac{9y-2}{5y^4-9x}$

When tangents are parallel to x axis then $9y - 2 = 0$

$\therefore\quad M = 1.$

For tangent perpendicular to x-axis

$5y^4 - 9x = 0$ …(ii)

From equation (1) and equation (2) we get only one point.

$\therefore\quad N = 1.$

$\therefore\quad M + N = 2.$

7. Let $f(x) = 2x^2 - x - 1$ and $S = \{n \in \mathbb{Z} : |f(n)| \le 800\}$. Then, the value of $\sum_{n\in S} f(n)$ is equal to ________.

Answer (10620)

Sol. $\because\quad |f(n)| \le 800$

$\Rightarrow\quad -800 \le 2n^2 - n - 1 \le 800$

$\Rightarrow\quad 2n^2 - n - 801 \le 0$

$\therefore \quad n \in \left[\frac{-\sqrt{6409}+1}{4}, \frac{\sqrt{6409}+1}{4}\right]$ and $n \in z$.

$\therefore \quad n = -19, -18, -17, \ldots\ldots\ldots, 19, 20.$

$\therefore \quad \sum(2x^2 - x - 1) = 2\sum x^2 - \sum x - \sum 1.$

$= 2\cdot 2\cdot(1^2 + 2^2 + \ldots + 19^2) + 2.20^2 - 20 - 40$

$= 10620$

8. Let S be the set containing all 3 × 3 matrices with entries from {–1, 0, 1}. The total number of matrices $A \in S$ such that the sum of all the diagonal elements of $A^T A$ is 6 is ________.

Answer (5376)

Sol. Sum of all diagonal elements is equal to sum of square of each element of the matrix.

i.e., $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then $t_r\ (A\cdot A^T)$

$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2$

$\because \quad a_i, b_i, c_i \in \{-1, 0, 1\}$ for i = 1, 2, 3

$\therefore$ Exactly three of them are zero and rest are 1 or – 1.

Total number of possible matrices ${}^9C_3 \times 2^6$

$= \frac{9\times 8\times 7}{6} \times 64$

= 5376

9. If the length of the latus rectum of the ellipse $x^2 + 4y^2 + 2x + 8y - \lambda = 0$ is 4, and l is the length of its major axis, then $\lambda + l$ is equal to ________.

Answer (75)

Sol. Equation of ellipse is: $x^2 + 4y^2 + 2x + 8y - \lambda = 0$

$(x + 1)^2 + 4\,(y + 1)^2 = \lambda + 5$

$\frac{(x+1)^2}{\lambda+5} + \frac{(y+1)^2}{\left(\frac{\lambda+5}{4}\right)} = 1$

Length of latus rectum = $\frac{2\cdot\left(\frac{\lambda+5}{4}\right)}{\sqrt{\lambda+5}} = 4.$

$\therefore \quad \lambda = 59.$

Length of major axis = $2\cdot\sqrt{\lambda+5} = 16 = l$

$\therefore \quad \lambda + l = 75.$

10. Let $S = \{z \in \mathbb{C} : z^2 + \bar{z} = 0\}$. Then $\sum_{z\in S} (\text{Re}(z) + \text{Im}(z))$ is equal to ________.

Answer (0)

Sol. $\because \quad z^2 + \bar{z} = 0$ Let $z = x + iy$

$\therefore \quad x^2 - y^2 + 2ixy + x - iy = 0$

$(x^2 - y^2 + x) + i(2xy - y) = 0$

$\therefore \quad x^2 + y^2 = 0$ and $(2x - 1)y = 0$

if $x = +\frac{1}{2}$ then $y = \pm\frac{\sqrt{3}}{2}$

And if $y = 0$ then $x = 0, -1$

$\therefore \quad z = 0 + 0i,\ -1 + 0i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore \quad \sum(R_e(z) + m(z)) = 0$

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