11: Multirate Systems

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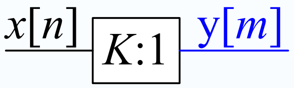
- Can relax analog or digital filter requirements
  e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff
- Reduce computational complexity
  FIR filter length $\propto \frac{f_s}{\Delta f}$ where Δf is width of transition band
  Lower $f_s \Rightarrow$ shorter filter + fewer samples $\Rightarrow$computation $\propto f_s^2$

Building blocks

Downsample



$$y[m] = x[Km]$$

Building blocks

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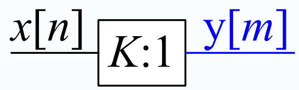

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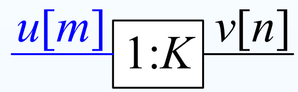
Upsample



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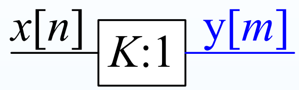

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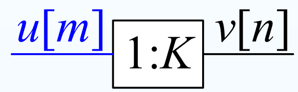

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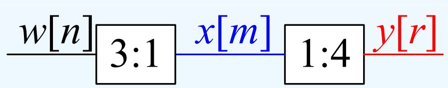
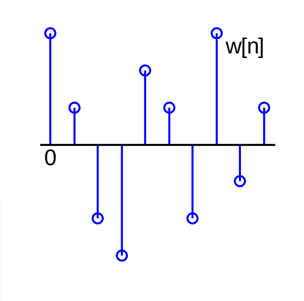
Downsample by 3 then upsample by 4





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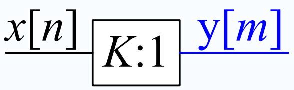

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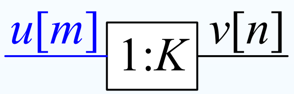

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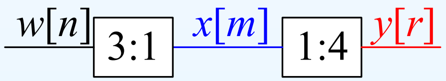

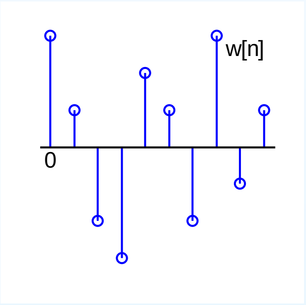

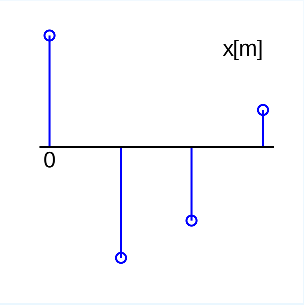

Building blocks

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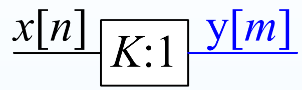

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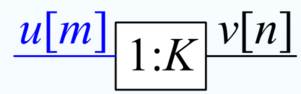

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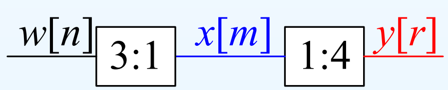

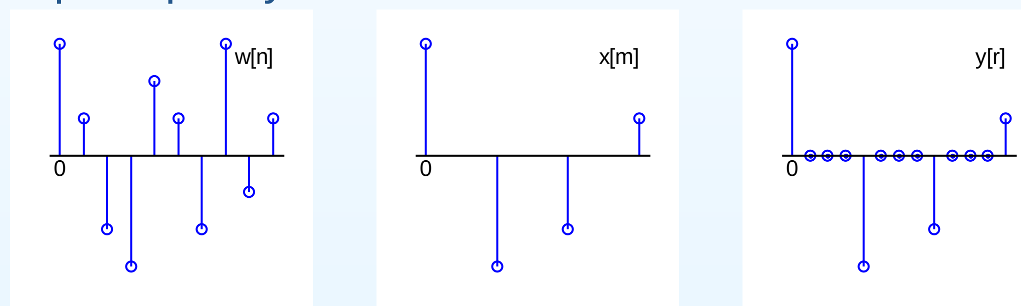

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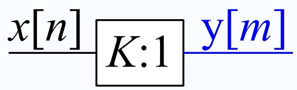

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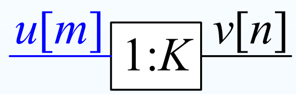

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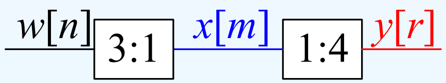

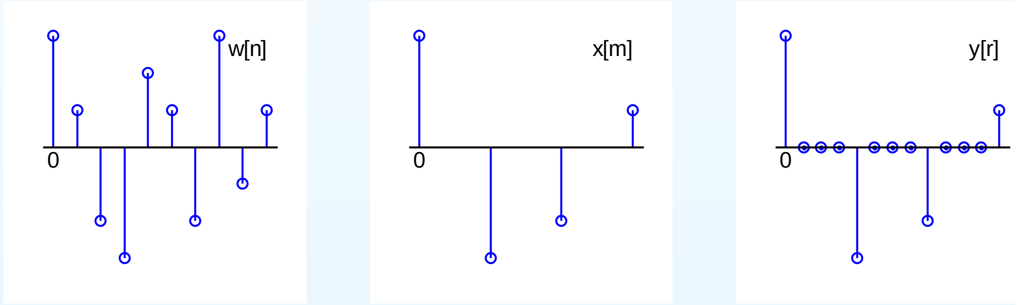

- We use different index variables (n, m, r) for different sample rates

Building blocks

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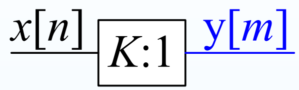

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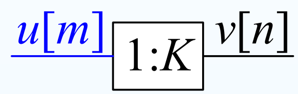

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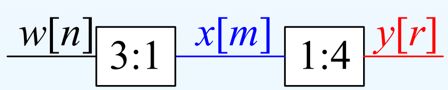

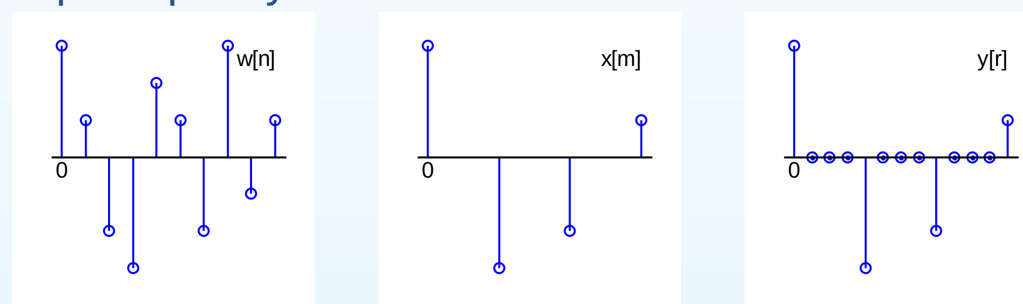

- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)

Building blocks

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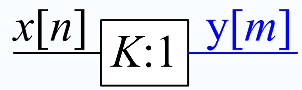

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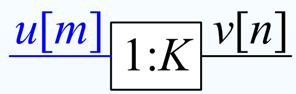

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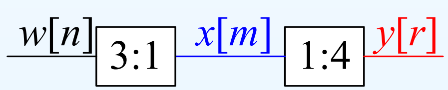
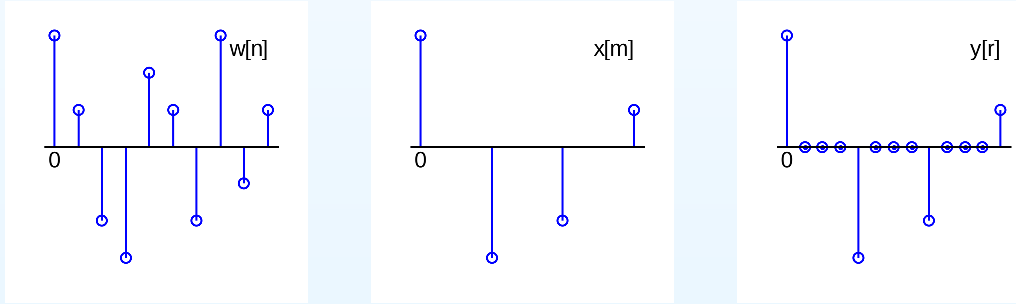
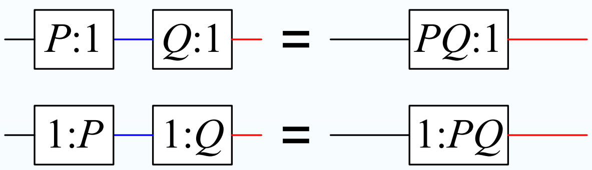




- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)
- Synchronization: all signals have a sample at $n = 0$.

Resampling Cascades

Successive downsamplers or upsamplers can be combined



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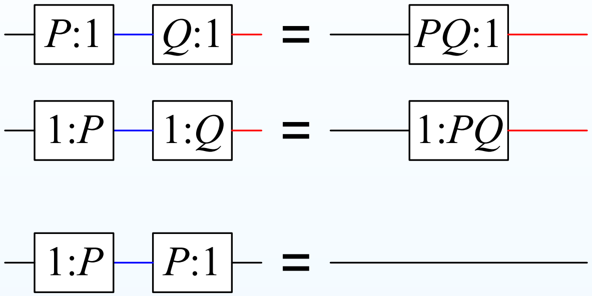
Upsampling can be exactly inverted



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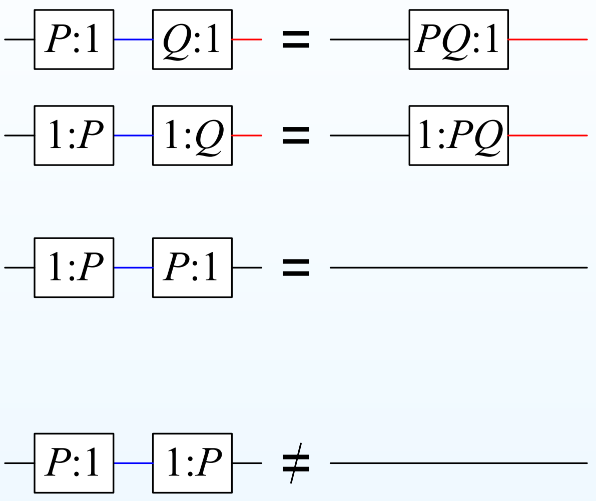
Downsampling destroys information permanently $\Rightarrow$ uninvertible



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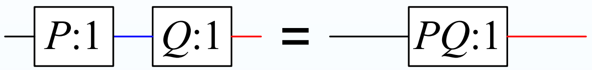

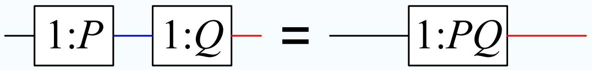

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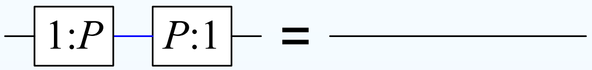

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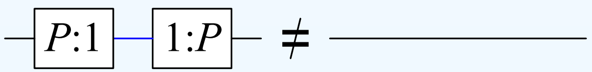
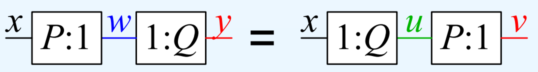


Resampling can be interchanged iff P and Q are coprime (surprising!)



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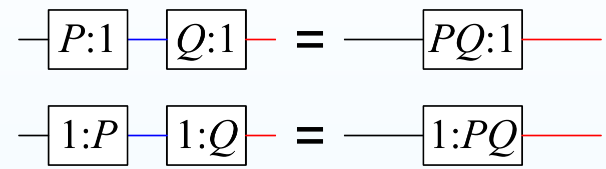

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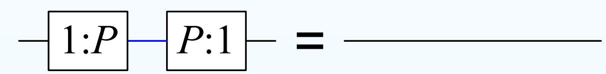

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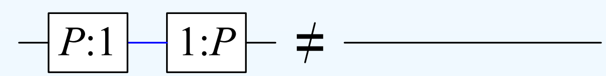

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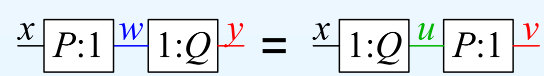

Proof: Left side: $y[n] = w\left[\frac{1}{Q}n\right] = x\left[\frac{P}{Q}n\right]$ if $Q \mid n$ else $y[n] = 0$.

[Note: $a \mid b$ means "a divides into b exactly"]

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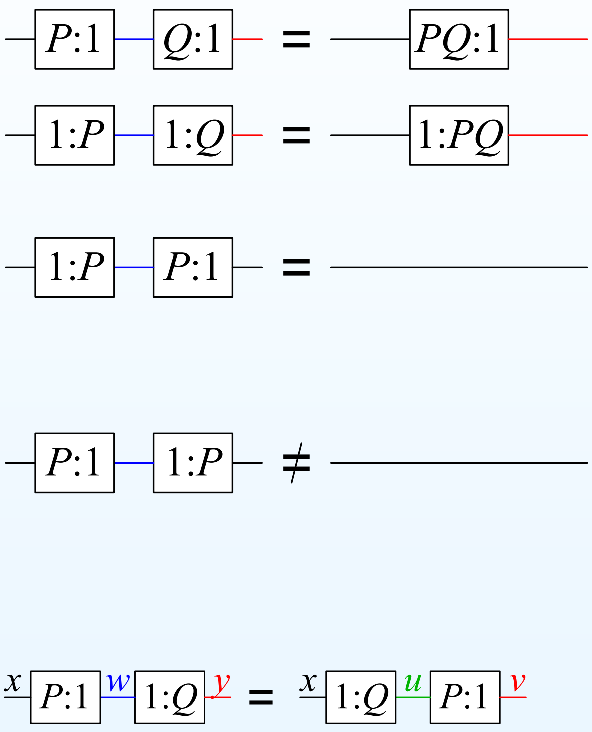

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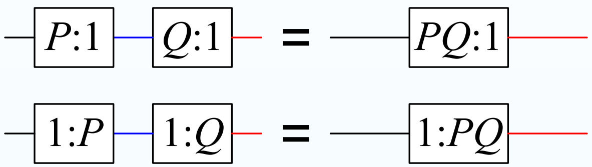

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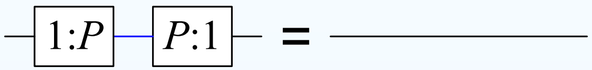

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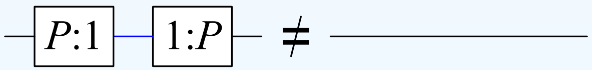

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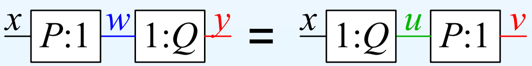

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But $\{Q \mid Pn \;\Rightarrow\; Q \mid n\}$ iff P and Q are coprime.

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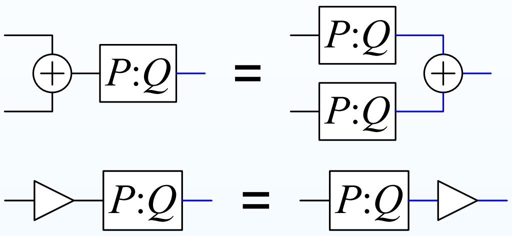
Noble Identities

Resamplers commute with addition and multiplication



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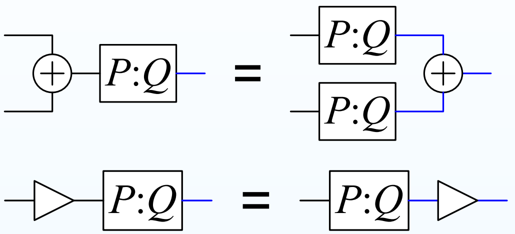
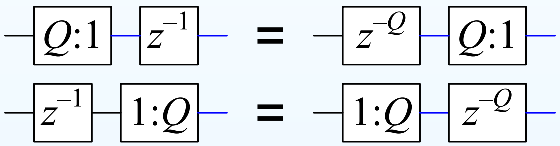


Delays must be multiplied by the resampling ratio



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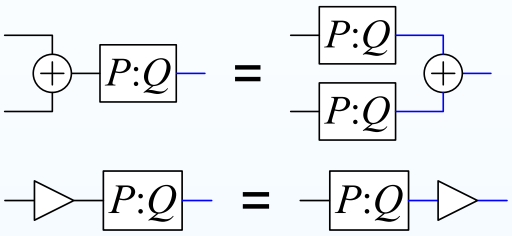

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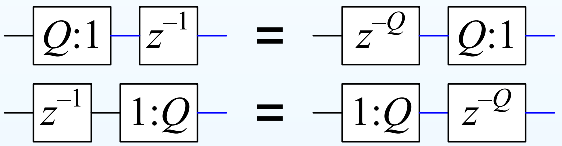
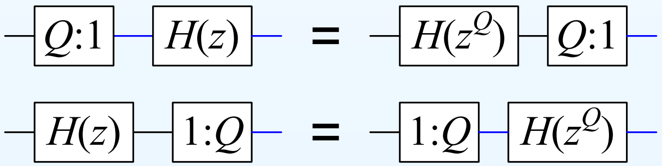


Noble identities:

Exchange resamplers and filters



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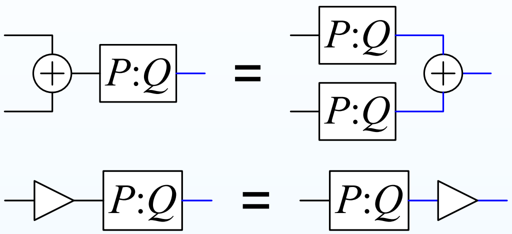

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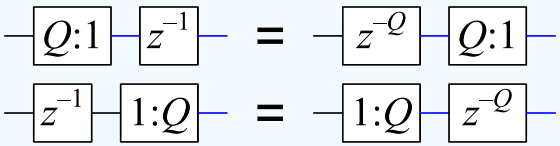

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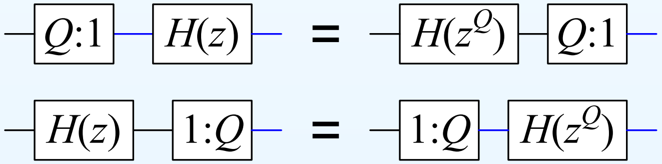

Example: $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots$
$H(z^3) = h[0] + h[1]z^{-3} + h[2]z^{-6} + \cdots$

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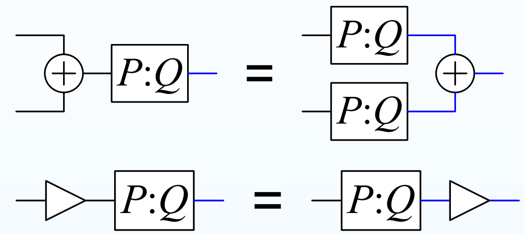

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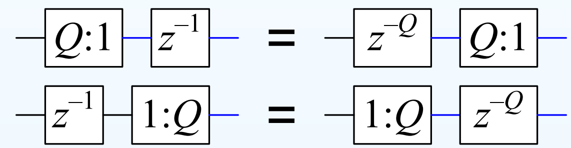

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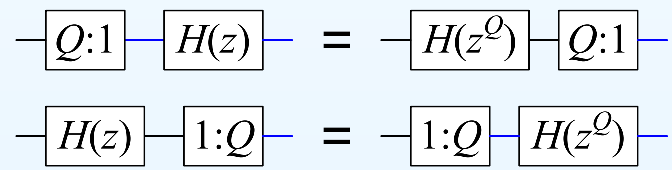
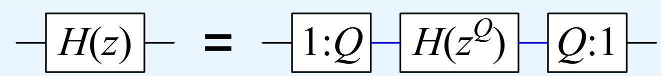


Corrollary



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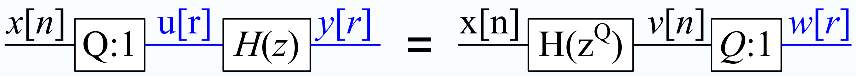
Noble Identities Proof

Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.



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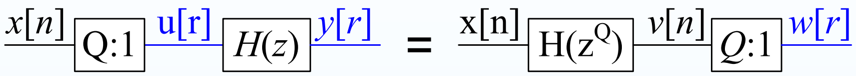

Assume that $h[r]$ is of length $M + 1$ so that $h_Q[n]$ is of length $QM + 1$.

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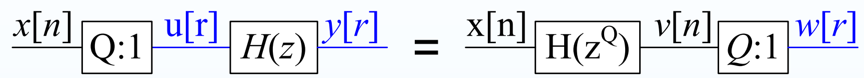

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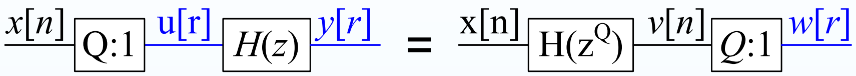

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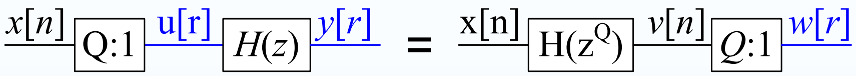

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$$w[r]=v[Qr]=\sum_{s=0}^{QM} h_Q[s]x[Qr-s]$$

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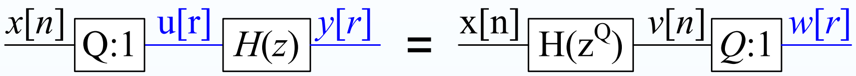

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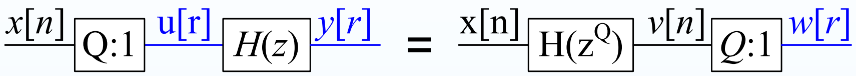

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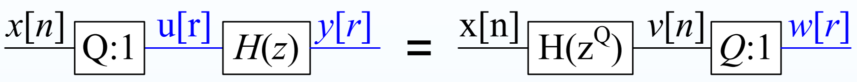

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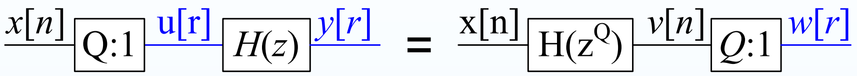

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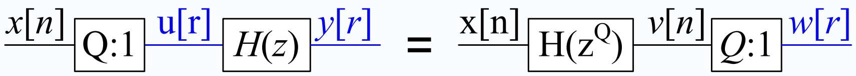

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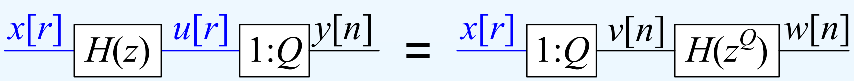

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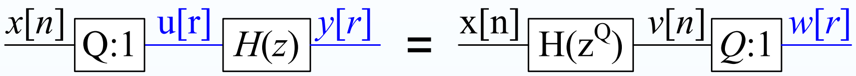

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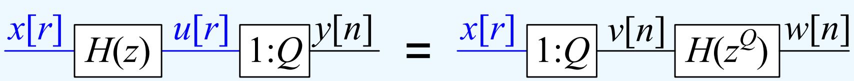

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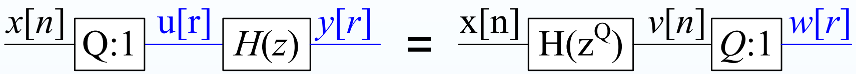

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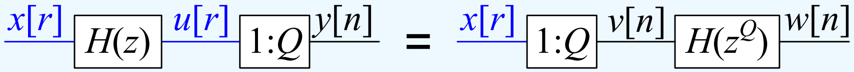

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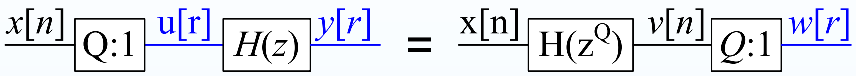

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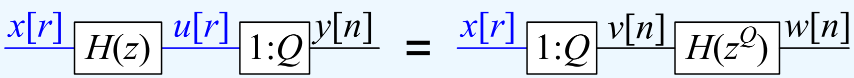

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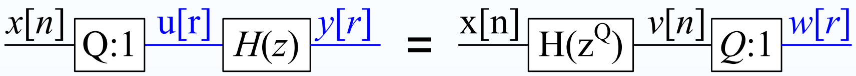

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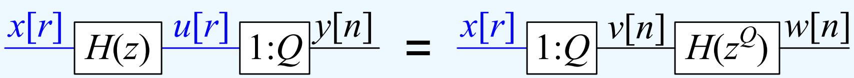

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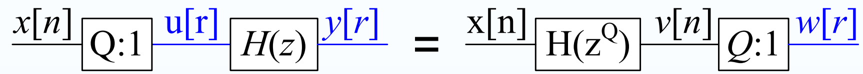

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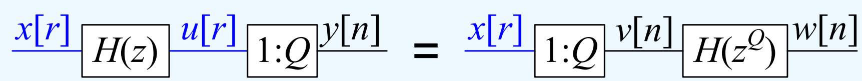

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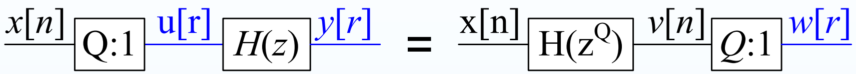

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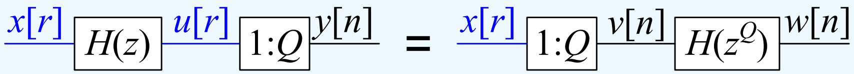

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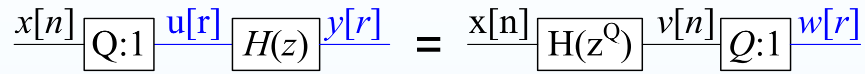

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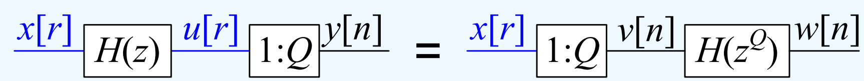

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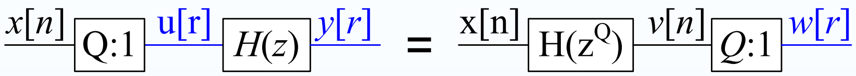

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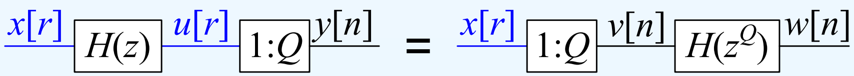

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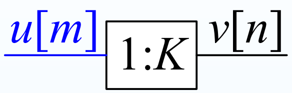
$$= \sum_{m=0}^{M} h[m]x[r-m] = u[r] = y[Qr]$$

☺

Upsampled z-transform

$$V(z) = \sum_n v[n] z^{-n}$$



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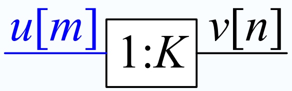

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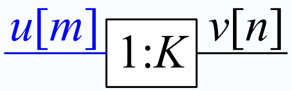

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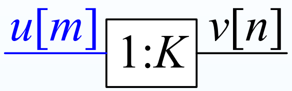

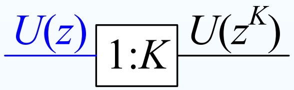

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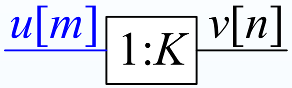

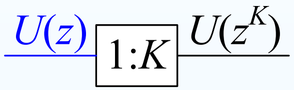

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Upsampled z-transform

$$V(z) = \sum_n v[n]z^{-n} = \sum_{n \text{ s.t. } K|n} u[\tfrac{n}{K}]z^{-n}$$
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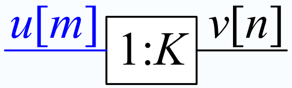

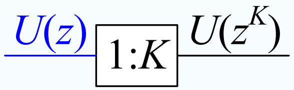

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Spectrum is horizontally shrunk and replicated K times.

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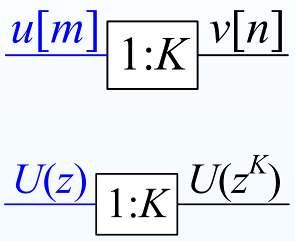

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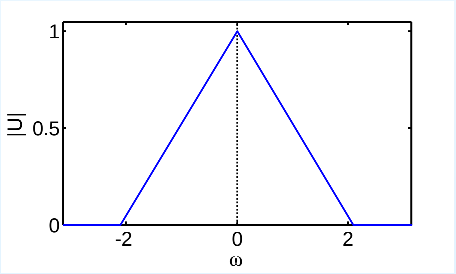
Example:



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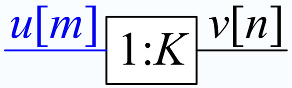

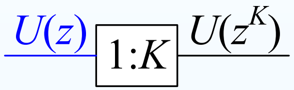

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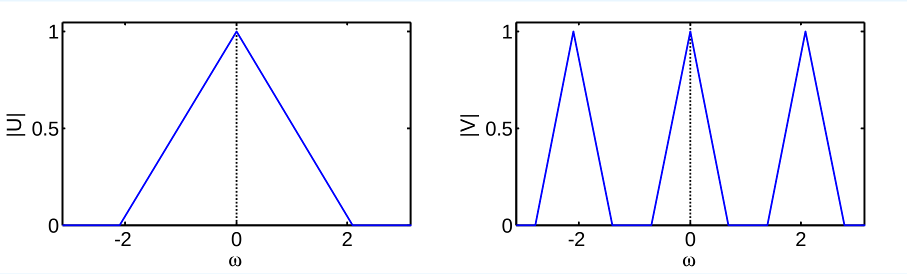

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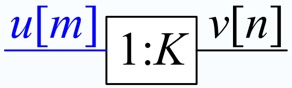

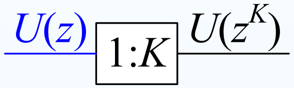

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

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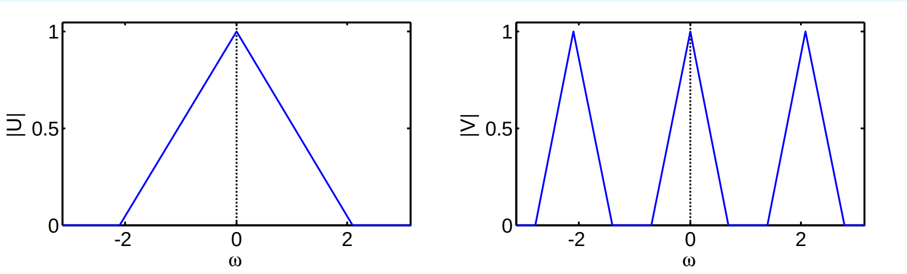

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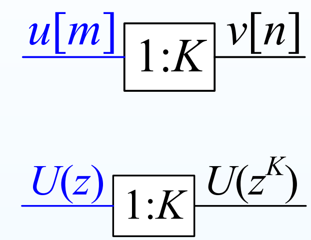

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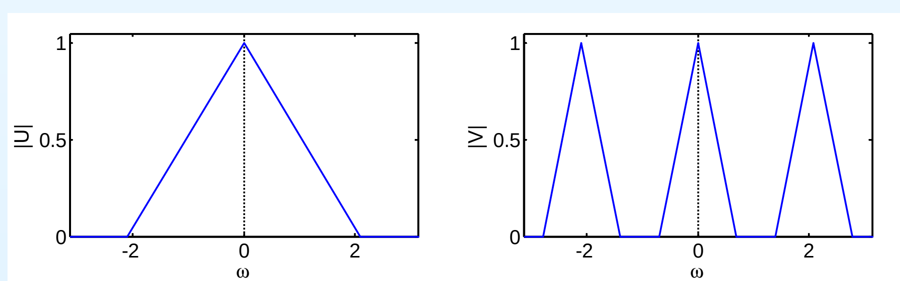
Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi}\int \left|U(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi}\int \left|V(e^{j\omega})\right|^2 d\omega$



Upsampled z-transform

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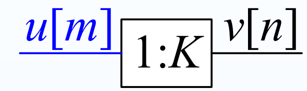

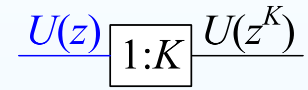

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Spectrum is horizontally shrunk and replicated K times.

Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

Upsampling normally followed by a LP filter to remove images.

Example:

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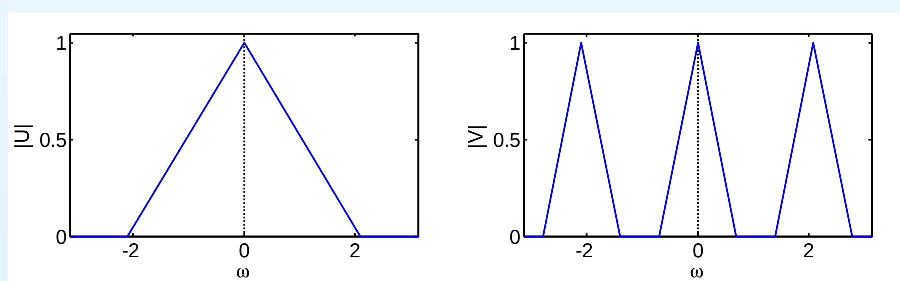
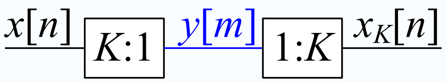


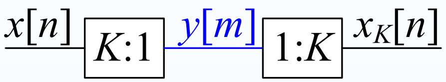
Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n]$



Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



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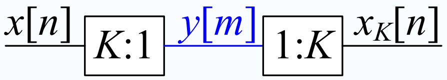

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases}$

Downsampled z-transform

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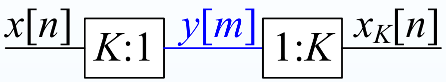

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Downsampled z-transform

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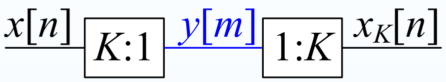

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$X_K(z) = \sum_n x_K[n]z^{-n}$

Downsampled z-transform

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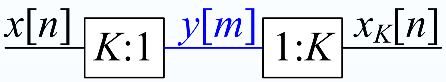

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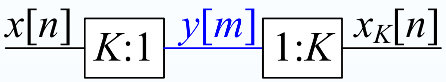
Downsampled z-transform



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Downsampled z-transform

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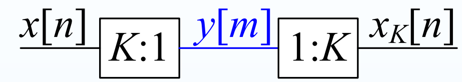

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Downsampled z-transform

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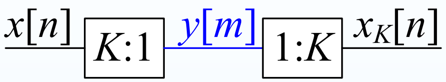

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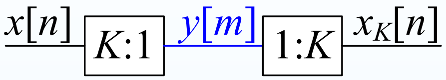
From previous slide:

$$X_K(z) = Y(z^K)$$

Downsampled z-transform



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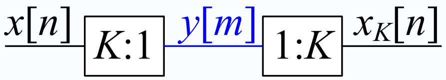
From previous slide:

$$X_K(z) = Y(z^K)$$
$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}})$$

Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



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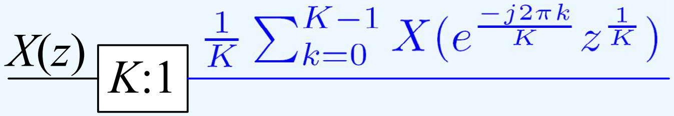
From previous slide:

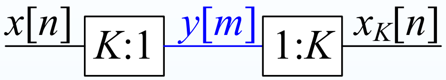


$$X_K(z) = Y(z^K)$$

$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



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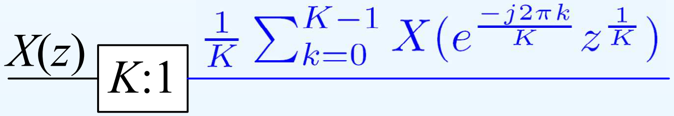
From previous slide:

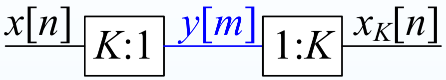


$$X_K(z) = Y(z^K)$$
$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$$

Frequency Spectrum:

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

Downsampled z-transform



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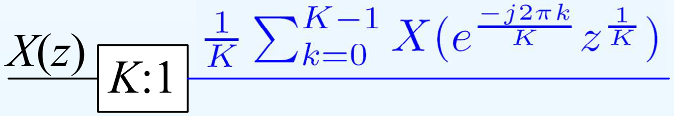

$$X_K(z) = Y(z^K)$$

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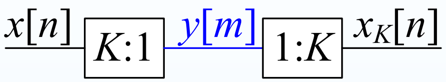
Frequency Spectrum:

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

$$= \frac{1}{K}\left(X(e^{\frac{j\omega}{K}}) + X(e^{\frac{j\omega}{K}-\frac{2\pi}{K}}) + X(e^{\frac{j\omega}{K}-\frac{4\pi}{K}}) + \cdots\right)$$

Downsampled z-transform

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



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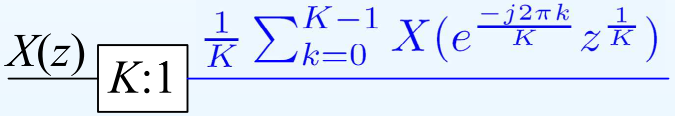
From previous slide:



$$X_K(z) = Y(z^K)$$
$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

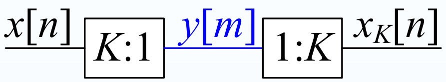
Frequency Spectrum:

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$
$$= \frac{1}{K}\left(X(e^{\frac{j\omega}{K}}) + X(e^{\frac{j\omega}{K}-\frac{2\pi}{K}}) + X(e^{\frac{j\omega}{K}-\frac{4\pi}{K}}) + \cdots\right)$$

Average of K aliased versions, each expanded in ω by a factor of K.

Downsampled z-transform

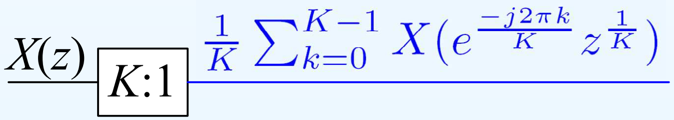
Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$X_K(z) = \sum_n x_K[n]z^{-n} = \frac{1}{K}\sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n]z^{-n}$$
$$= \frac{1}{K}\sum_{k=0}^{K-1}\sum_n x[n]\left(e^{\frac{-j2\pi k}{K}}z\right)^{-n} = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z)$$

From previous slide:



$$X_K(z) = Y(z^K)$$
$$\Rightarrow Y(z) = X_K(z^{\frac{1}{K}}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}}z^{\frac{1}{K}})$$

Frequency Spectrum:

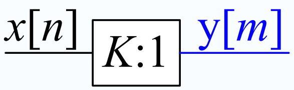
$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$
$$= \frac{1}{K}\left(X(e^{\frac{j\omega}{K}}) + X(e^{\frac{j\omega}{K}-\frac{2\pi}{K}}) + X(e^{\frac{j\omega}{K}-\frac{4\pi}{K}}) + \cdots\right)$$

Average of K aliased versions, each expanded in ω by a factor of K.

Downsampling is normally preceded by a LP filter to prevent aliasing.

Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$



Downsampled Spectrum

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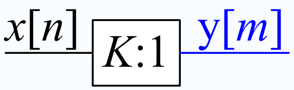
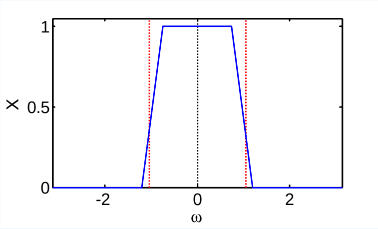


Example 1:

$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$



Downsampled Spectrum

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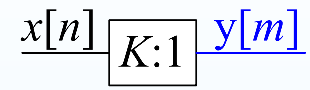
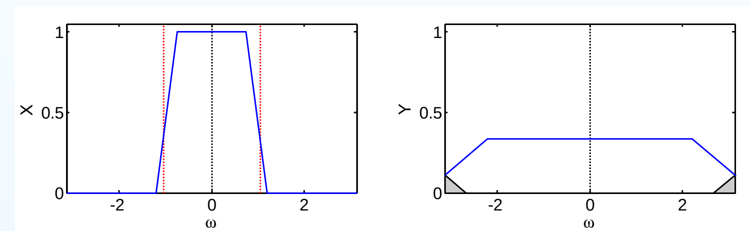


Example 1:

$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$

Shaded region shows aliasing



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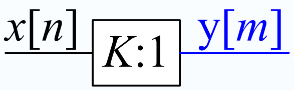

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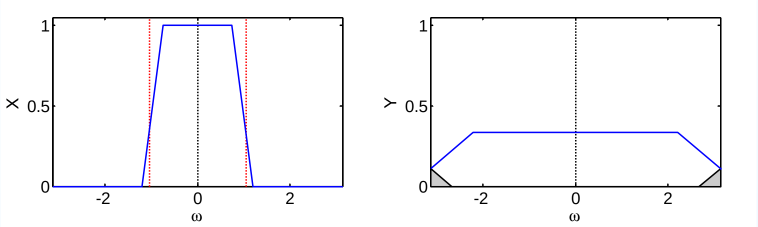
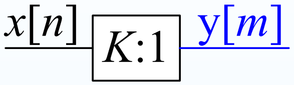


Energy decreases: $\frac{1}{2\pi}\int \left|Y(e^{j\omega})\right|^2 d\omega \approx \frac{1}{K} \times \frac{1}{2\pi}\int \left|X(e^{j\omega})\right|^2 d\omega$

Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega - 2\pi k)}{K}})$$



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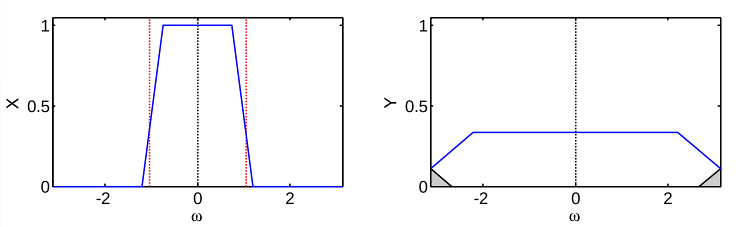

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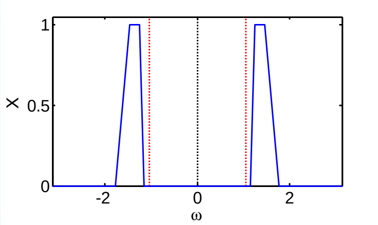
Example 2:

$K = 3$

Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$



Downsampled Spectrum

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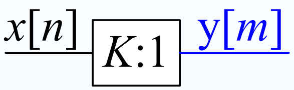

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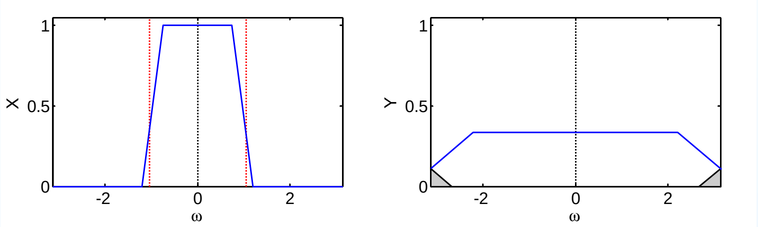

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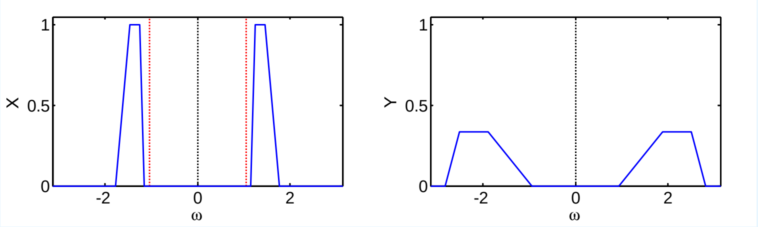
Example 2:

$K = 3$

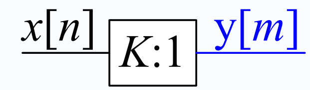
Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$



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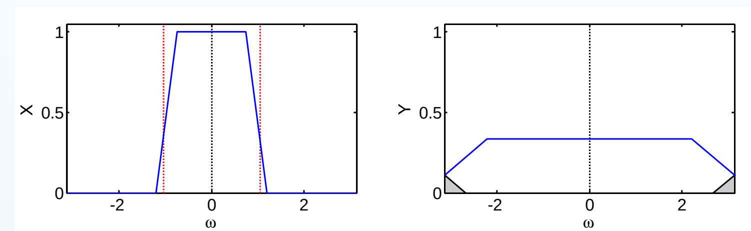

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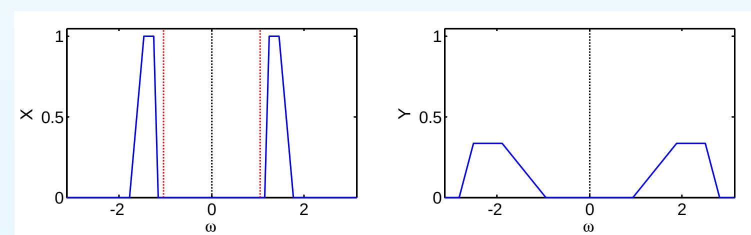
Energy all in $\frac{\pi}{K} \leq |\omega| < 2\frac{\pi}{K}$

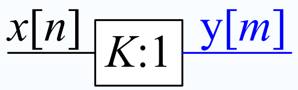
No aliasing: ☺



No aliasing: If all energy is in $r\frac{\pi}{K} \leq |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$



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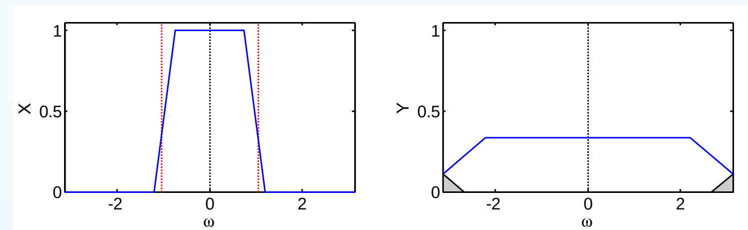
$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$

Shaded region shows aliasing



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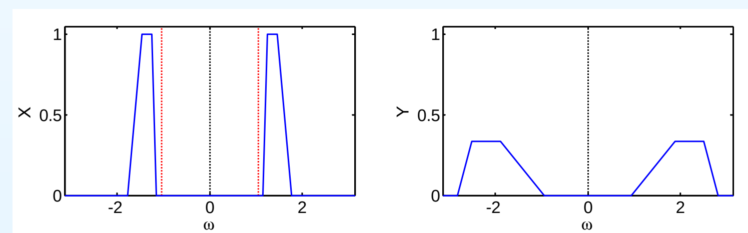
Example 2:

$K = 3$

Energy all in $\frac{\pi}{K} \leq |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



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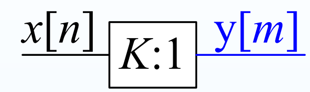
Normal case ($r = 0$): If all energy in $0 \leq |\omega| \leq \frac{\pi}{K}$

Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$



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$K = 3$

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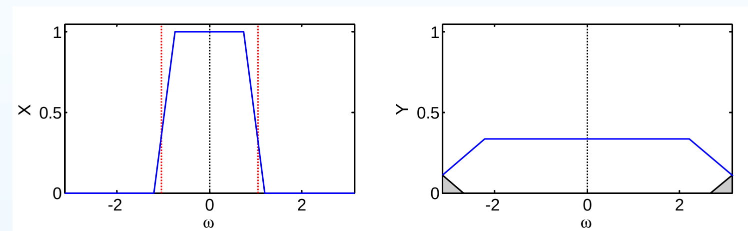

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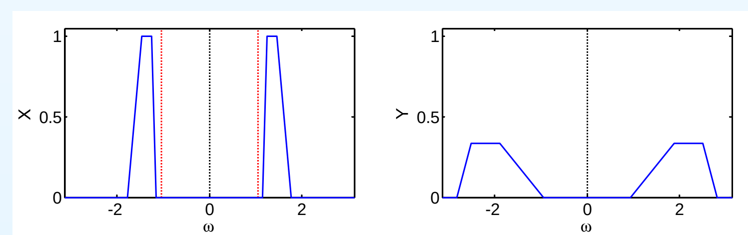
Example 2:

$K = 3$

Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



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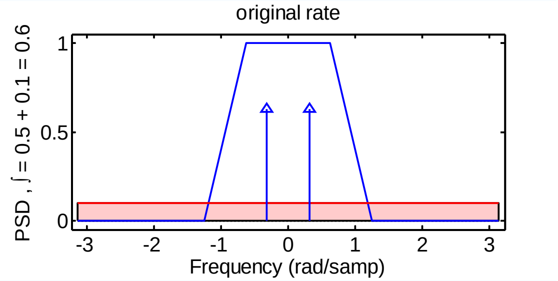
Normal case ($r = 0$): If all energy in $0 \le |\omega| \le \frac{\pi}{K}$

Downsampling: Total energy multiplied by $\approx \frac{1}{K}$ ($= \frac{1}{K}$ if no aliasing)

Average power $\approx$ unchanged ($=$ energy/sample)

Power Spectral Density

Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

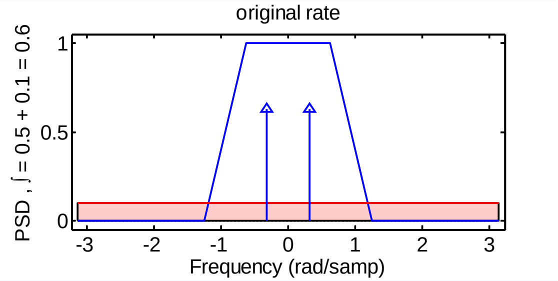


Power Spectral Density

+

Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD



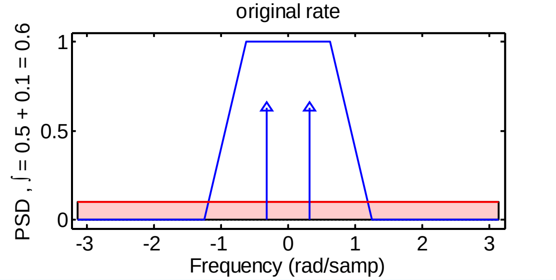
Power Spectral Density

+

Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{PSD}(\omega) d\omega$$



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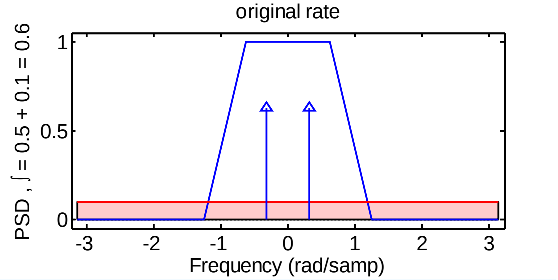
Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{PSD}(\omega) d\omega \;= 0.6$$



Power Spectral Density

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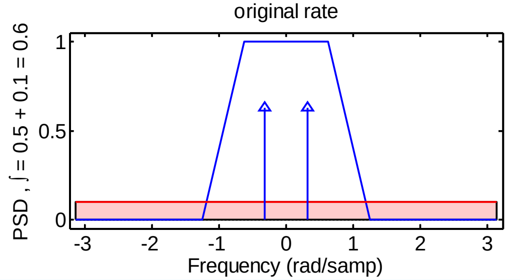
Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{PSD}(\omega)d\omega \ = 0.6$$

Component powers:

Signal = 0.3, Tone = 0.2, Noise = 0.1



Power Spectral Density +

Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

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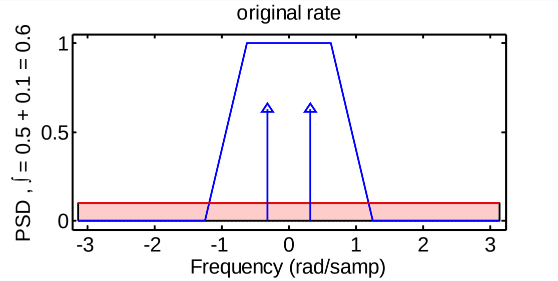
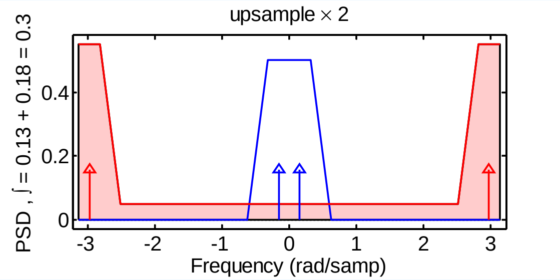


Upsampling:

Same energy per second

$\Rightarrow$ Power is $\div K$



Power Spectral Density

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Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

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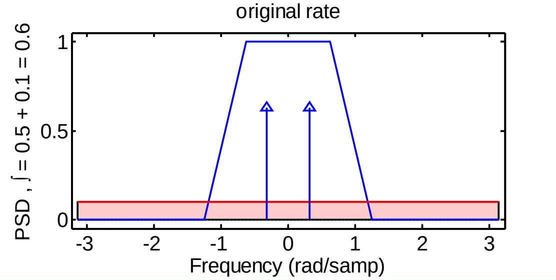

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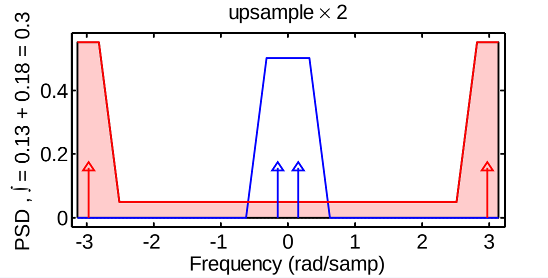

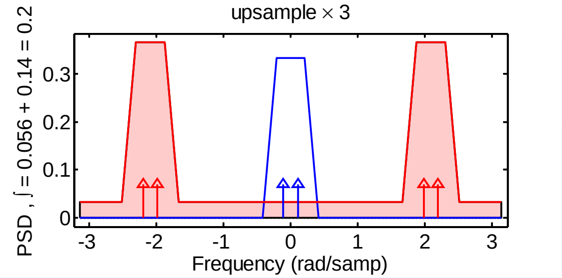

Power Spectral Density

Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

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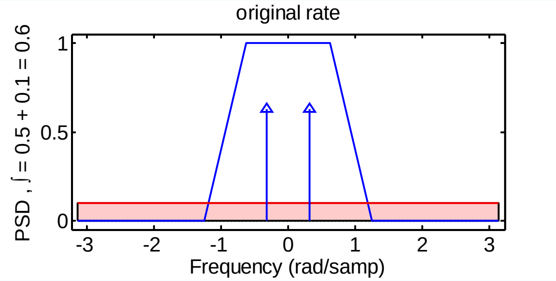

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Same energy per second

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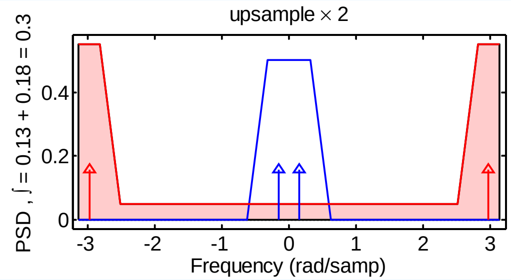
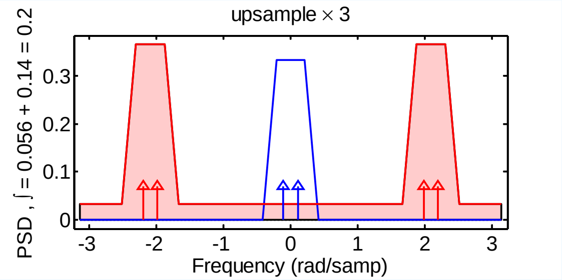
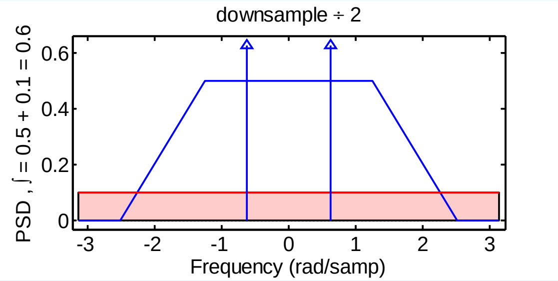




Downsampling:

Average power is unchanged.



Power Spectral Density +

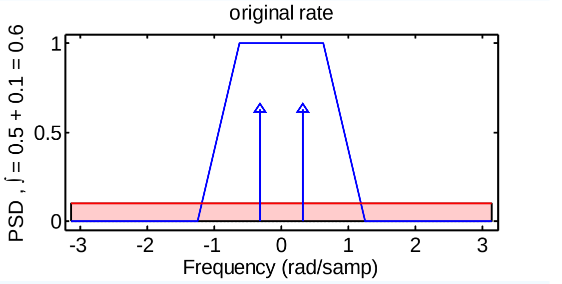
Example: Signal in $\omega \in \pm 0.4\pi$ + Tone @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{PSD}(\omega)d\omega \;= 0.6$$

Component powers:

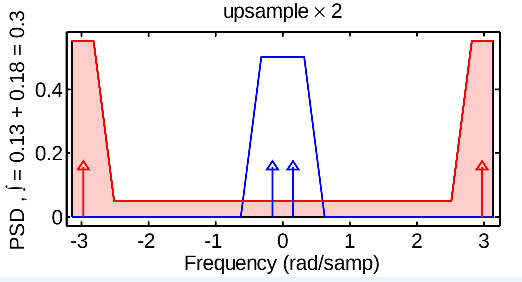
Signal = 0.3, Tone = 0.2, Noise = 0.1

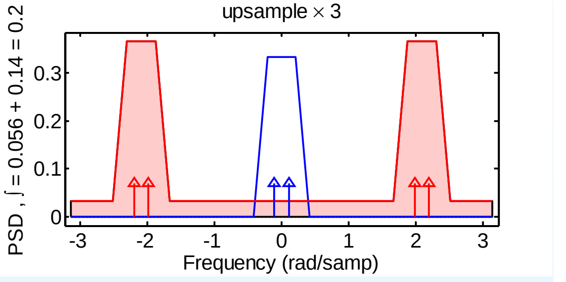


Upsampling:

Same energy per second

$\Rightarrow$ Power is $\div K$

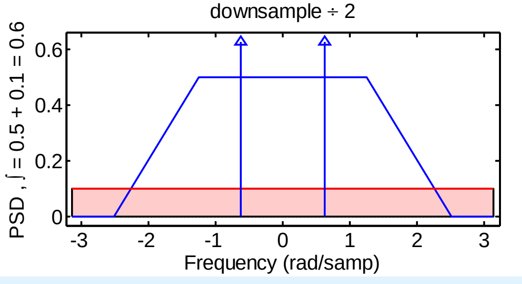


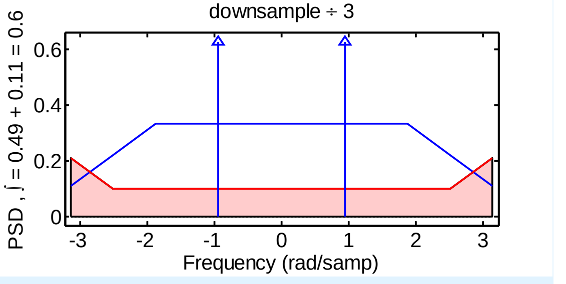


Downsampling:

Average power is unchanged.

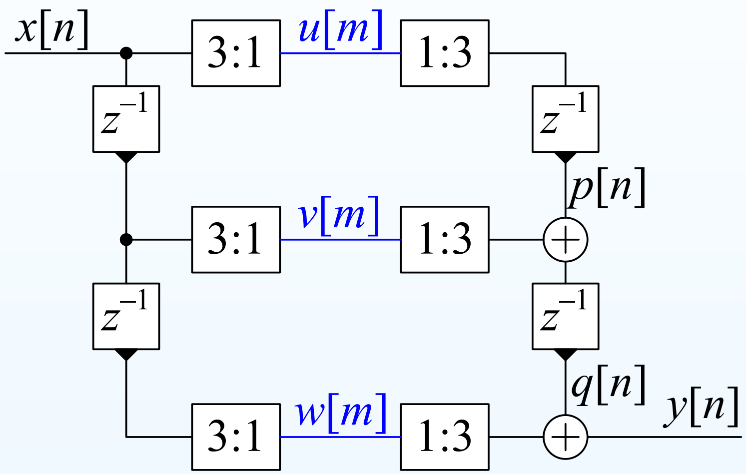
$\exists$ aliasing in the $\div 3$ case.





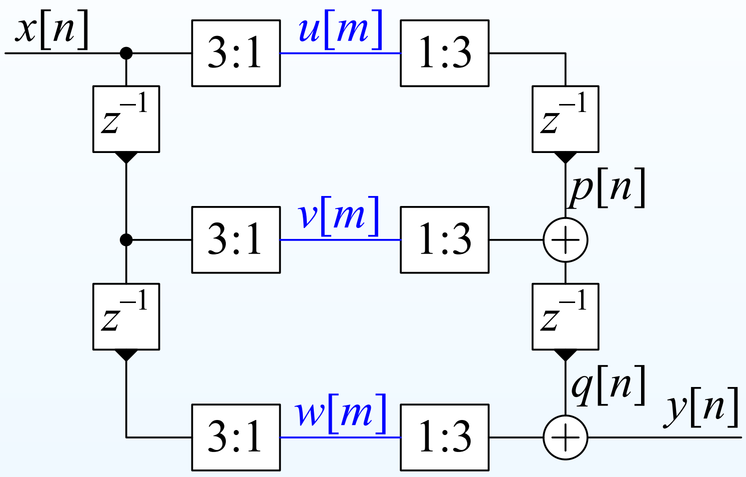
Perfect Reconstruction

$x[n]$ `cdefghijklmn`



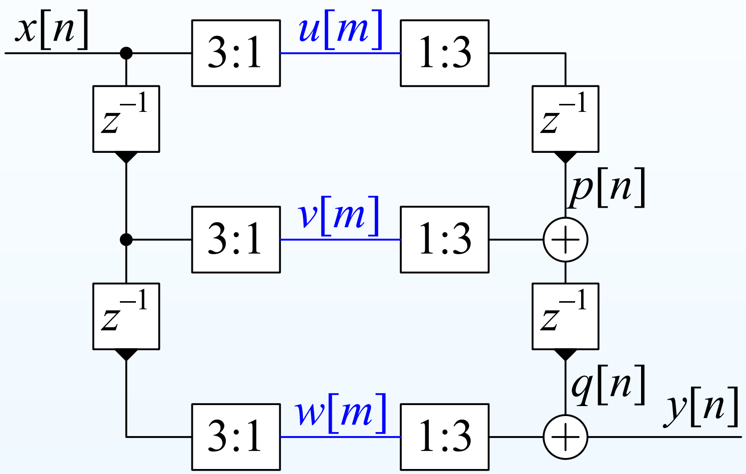
Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`



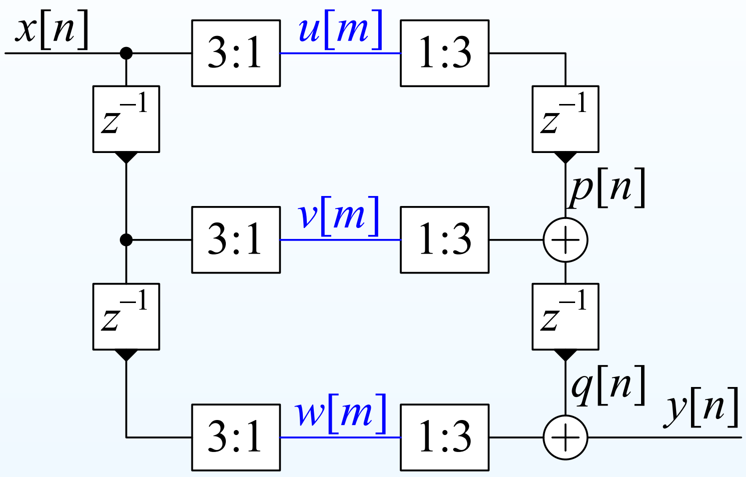
Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`



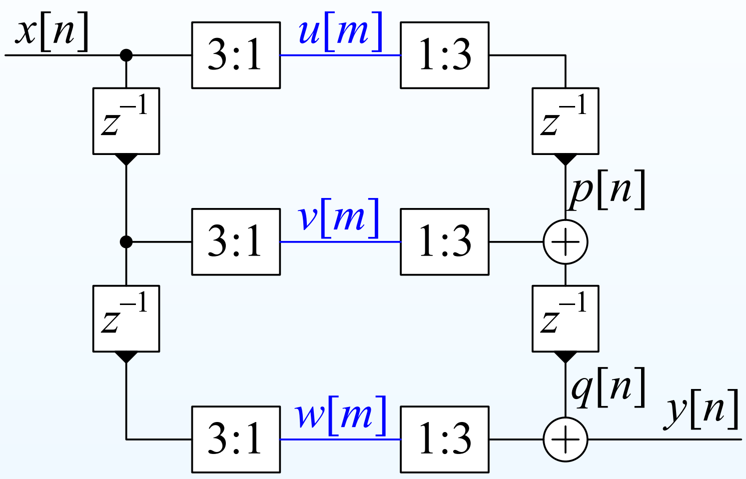
Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`



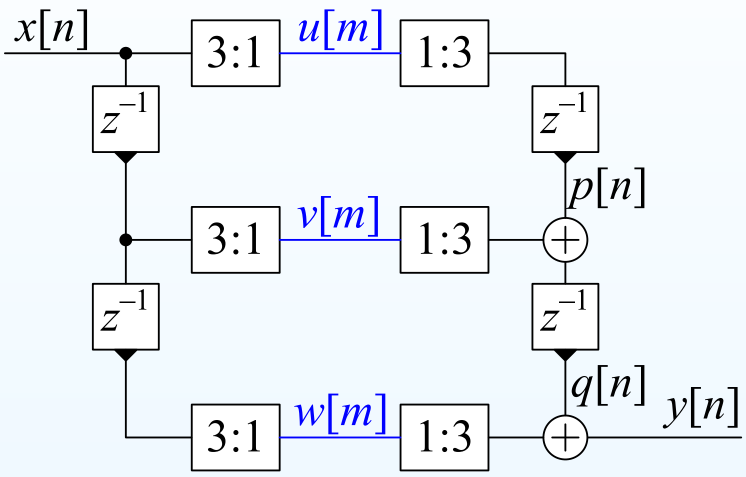
Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`



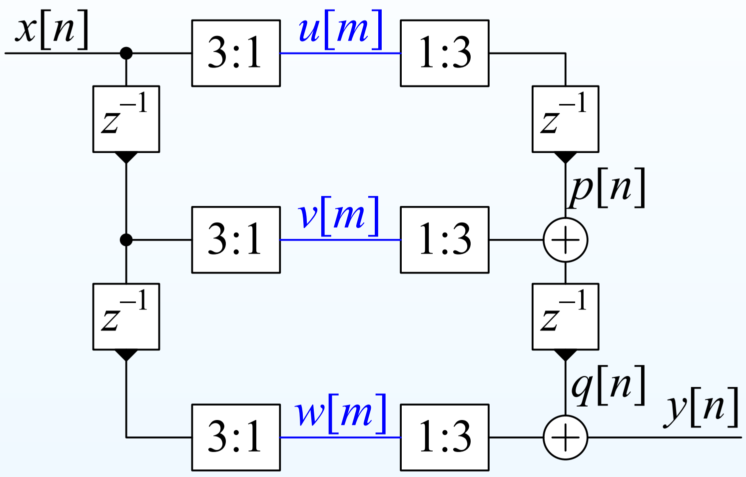
Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`



Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`



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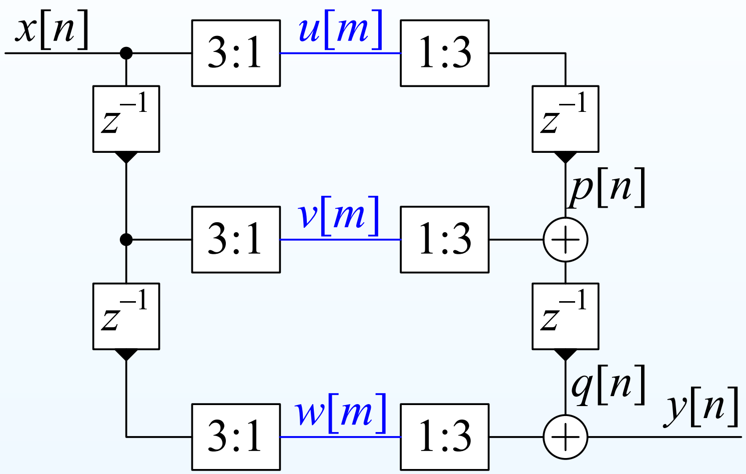
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

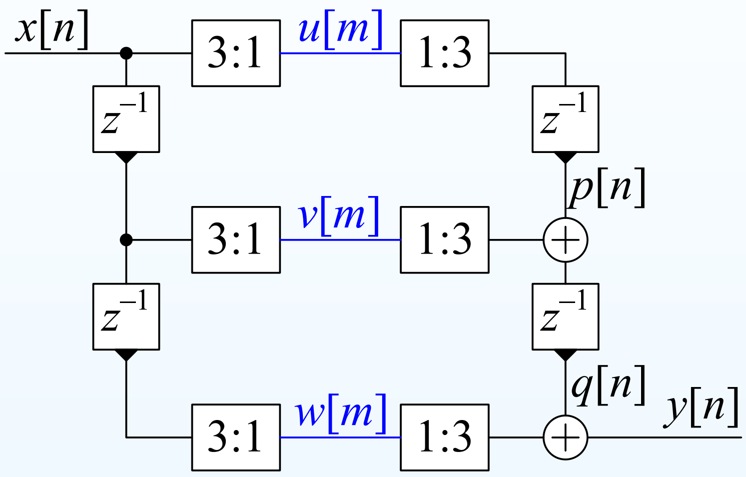


Input sequence $x[n]$ is split into three streams at $\frac{1}{3}$ the sample rate:

$$u[m] = x[3m],\ v[m] = x[3m-1],\ w[m] = x[3m-2]$$

Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`



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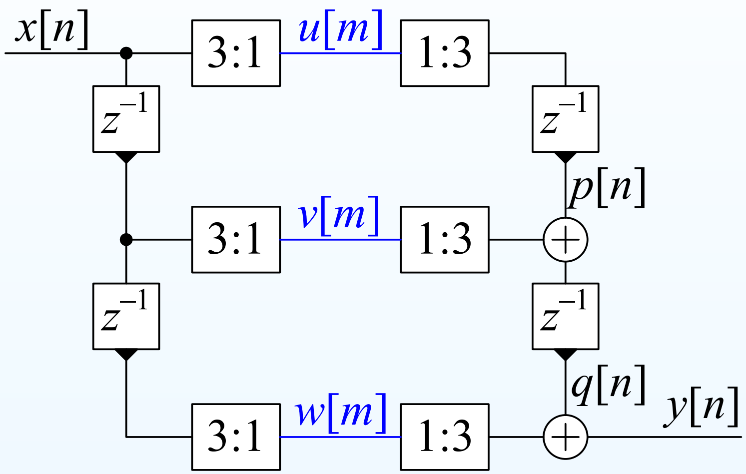
$$u[m] = x[3m],\ v[m] = x[3m-1],\ w[m] = x[3m-2]$$

Following upsampling, the streams are aligned by the delays and then added to give:

$$y[n] = x[n-2]$$

Perfect Reconstruction

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`
$v[m]$	`b  e  h  k`
$q[n]$	`-bc-ef-hi-kl`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`



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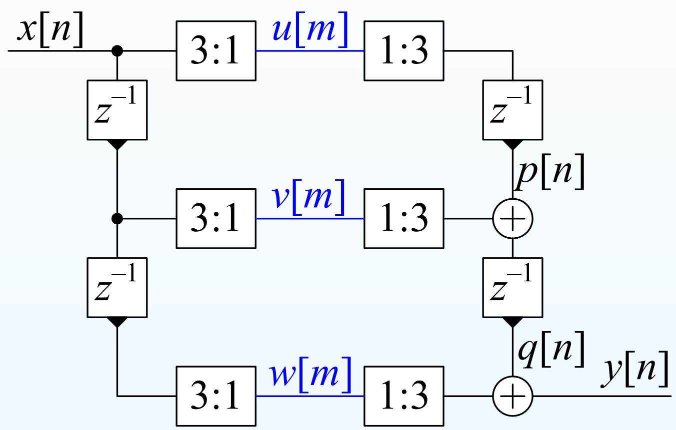
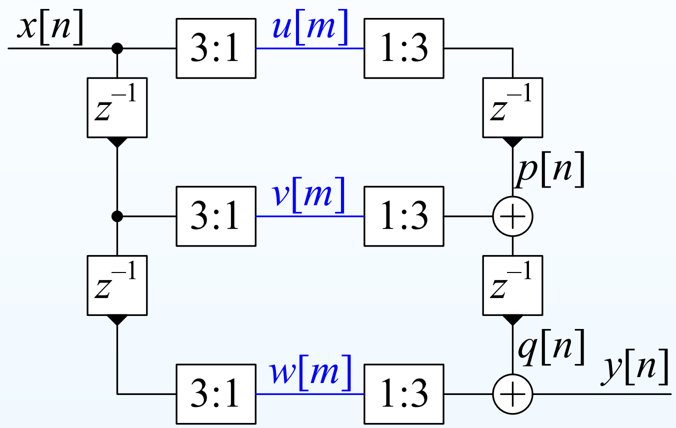
Perfect Reconstruction: output is a delayed scaled replica of the input

Commutators



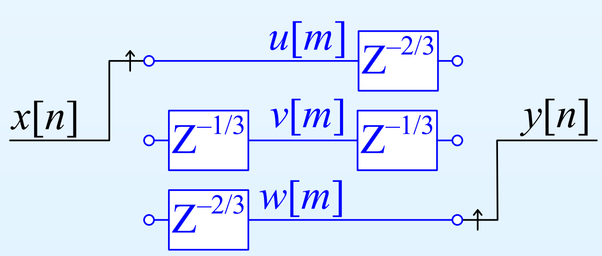
$x[n]$	cdefghijklmn
$u[m]$	c f i l
$v[m]$	b e h k
$w[m]$	a d g j
$y[n]$	abcdefghijkl

Commutators

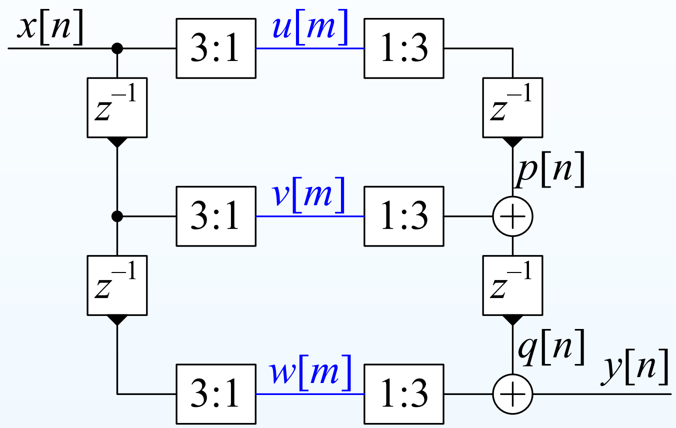


$x[n]$	`cdefghijklmn`
$u[m]$	`c f i l`
$v[m]$	`b e h k`
$w[m]$	`a d g j`
$y[n]$	`abcdefghijkl`

The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.

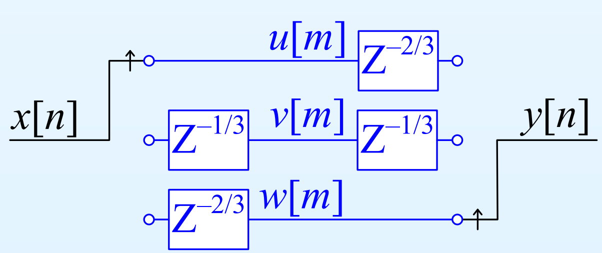


Commutators

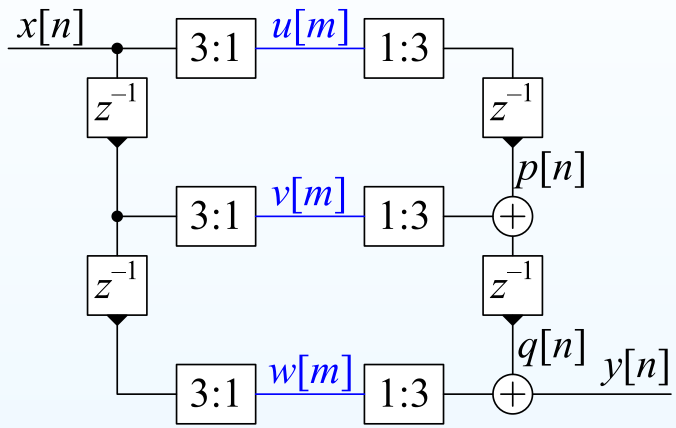


$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.
Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.



Commutators

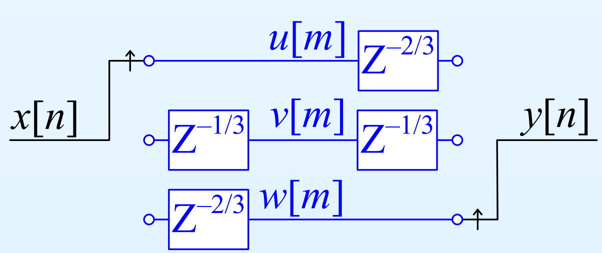


$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

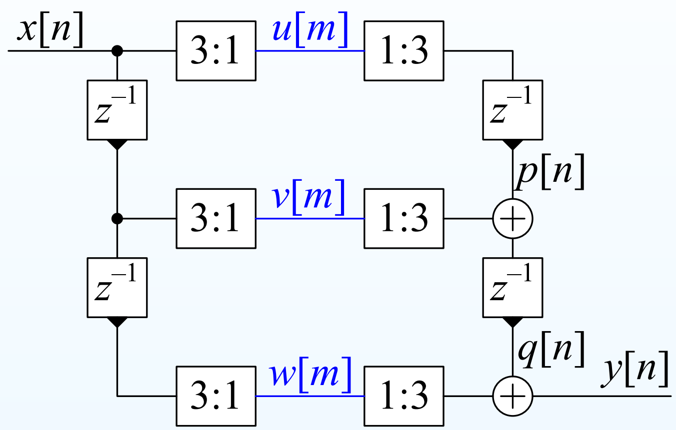
The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.

Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.

The output commutator takes values from the streams in sequence.



Commutators



$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$v[m+\frac{1}{3}]$	`e  h  k  l`
$w[m+\frac{2}{3}]$	`d  g  j  m`
$y[n]$	`abcdefghijkl`

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Fractional delays, $z^{-\frac{1}{3}}$ and $z^{-\frac{2}{3}}$ are needed to synchronize the streams.
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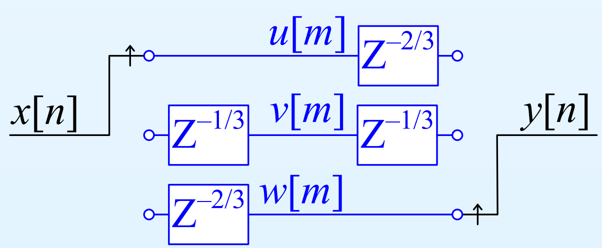
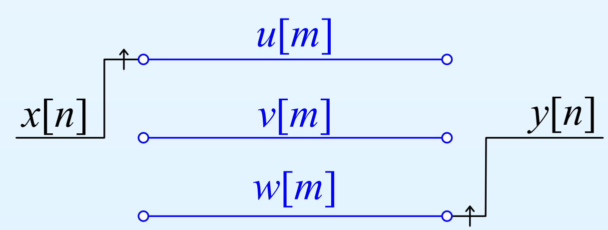
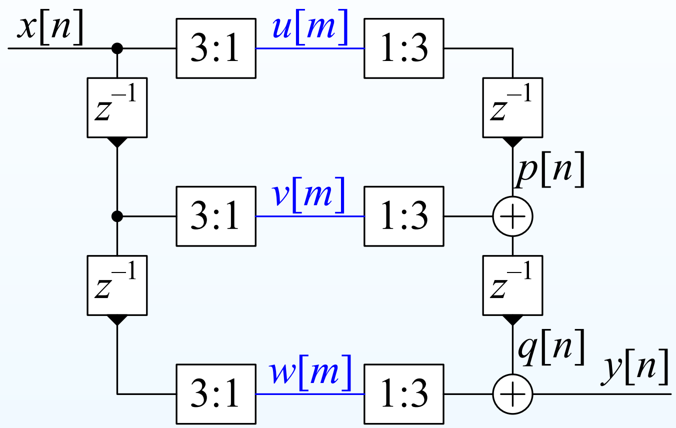
For clarity, we omit the fractional delays and regard each terminal, $\circ$, as holding its value until needed.





Commutators



$x[n]$	c	d	e	f	g	h	i	j	k	l	m	n
$u[m]$	c			f			i			l		
$v[m]$	b			e			h			k		
$w[m]$	a			d			g			j		
$v[m+\frac{1}{3}]$			e			h			k			l
$w[m+\frac{2}{3}]$		d			g			j			m	
$y[n]$	a	b	c	d	e	f	g	h	i	j	k	l

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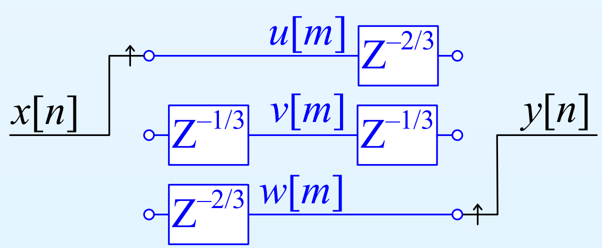
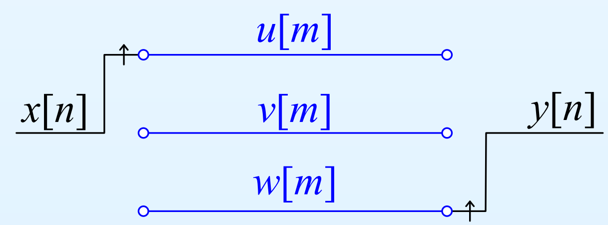
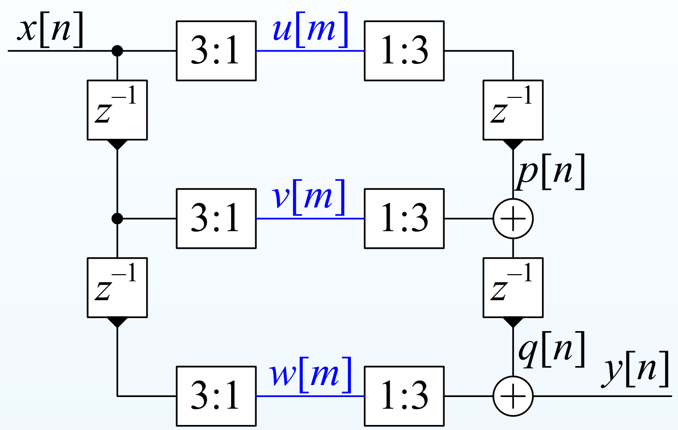
For clarity, we omit the fractional delays and regard each terminal, $\circ$, as holding its value until needed. Initial commutator position has zero delay.





Commutators



$x[n]$	`cdefghijklmn`
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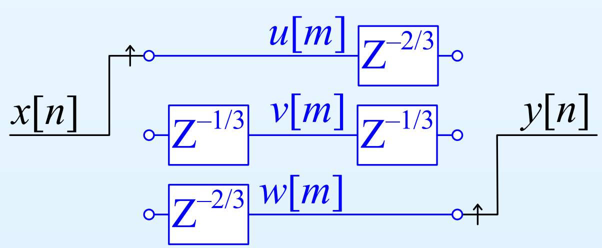

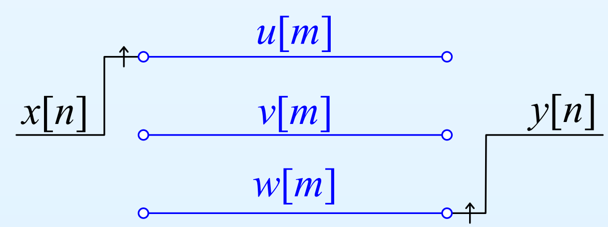

The commutator direction is against the direction of the z^{-1} delays.

Summary

- Multirate Building Blocks
  - Upsample: $X(z) \stackrel{1:K}{\rightarrow} X(z^K)$
    Invertible, Inserts $K-1$ zeros between samples
    Shrinks and replicates spectrum
    Follow by LP filter to remove images

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    Follow by LP filter to remove images
  - Downsample: $X(z) \stackrel{K:1}{\rightarrow} \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$
    Destroys information and energy, keeps every K^{th} sample
    Expands and aliasses the spectrum
    Spectrum is the average of K aliased expanded versions
    Precede by LP filter to prevent aliases

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For further details see Mitra: 13.

MATLAB routines

resample	change sampling rate