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Generating Functions of Class-Numbers

by

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1. *Introduction.*

In the notation of Kronecker [1]) and Humbert [2]), let $F(n)$ and $G(n)$ denote the number of uneven classes and the whole number of classes of binary quadratic forms of determinant $-n$, with the usual conventions that classes equivalent to

$$a(x^2+y^2)\,, \quad a(2x^2+2xy+2y^2)$$

are to count as $\frac{1}{2}$ and $\frac{1}{3}$ respectively, and that

$$F(0)=0\,, \quad G(0)=-\tfrac{1}{12}.$$

Also let [3])

$$G(n)-F(n)=F_1(n)\,, \quad F(n)-F_1(n)=E(n)\,,$$
$$F(n)+3F_1(n)=J(n)\,, \quad F(n)-3F_1(n)=I(n)\,.$$

The power series in which the general coefficients are such class-numbers admit of a transformation theory analogous to the theory of Jacobi's imaginary transformations of the theta-functions. The formulae, however, are not of such a simple character as the Jacobian formulae because they involve certain infinite integrals; alternatively, the formulae may be regarded as supplying a means of expressing the integrals as the sum of two convergent series. The first of these two points of view is probably the more important because it is a simple matter to obtain asymptotic expansions by means of which the infinite integrals may easily be calculated.

Two transformations of generating functions of class-numbers have been discovered by Mordell [4]); the method by which

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1) L. Kronecker [Journal für Math. **57** (1860), 248—255].

2) G. Humbert [Journal de Math. (6) **3** (1907), 337—449].

3) Evidently $F_1(n)$ is the number of even classes.

4) L. J. Mordell [Quarterly Journal of Math. **48** (1920), 334; Messenger of Math. **49** (1920), 65—72].

he proved them depended on the use of properties of solutions of certain functional equations.

In this paper I propose to give a direct and systematic discussion of the transformation theory based on a quite straightforward application of the methods of contour integration. The difference between the integral function used by Mordell as the solution of his functional equations and the modified forms of Hermite's expansions which I use is rather remarkable.

The notation which I shall use for theta-functions is the classical second notation of Jacobi (modified, as usual, by the substitution of ϑ_4 for Jacobi's ϑ), so that [5])

$$\vartheta_3(x,q)=\sum_{n=-\infty}^{\infty} q^{n^2}e^{2nix}, \qquad \vartheta_4(x,q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2nix},$$

$$\vartheta_1(x,q)=\sum_{n=-\infty}^{\infty}(-i)^{2n+1}q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}, \qquad \vartheta_2(x,q)=\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}e^{(2n+1)ix};$$

the parameter q will be omitted from these functional symbols whenever its absence cannot cause confusion; the argument will also be omitted when it is zero, so that

$$\vartheta_2(0,q)=\vartheta_2, \quad \vartheta_3(0,q)=\vartheta_3, \quad \vartheta_4(0,q)=\vartheta_4,$$
$$\left(\frac{d\vartheta_1(x,q)}{dx}\right)_{x=0}=\vartheta_1'=\vartheta_2\vartheta_3\vartheta_4.$$

I shall also write

$$q=e^{\pi i\tau}, \quad q_1=e^{-\pi i/\tau},$$
$$-\pi i\tau=\alpha, \quad \pi i/\tau=\beta,$$

so that

$$\alpha\beta=\pi^2.$$

In order that we may have $|q|<1$, it is, of course, essential that the real parts of α and β should be positive, and, in particular, that α and β themselves should be positive when they are real.

The notation which I shall use for generating functions of class-numbers is the notation introduced by Humbert, namely [6])

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[5]) I prefer not to follow the practice (initiated by H. J. S. Smith) of using a double-suffix notation for the ordinary theta-functions in work on the theory of numbers.

[6]) The use of a dash in two senses, to indicate derivates of theta-functions and to indicate a change of sign of q in generating functions, should not cause confusion.

$$(1.01)\quad \mathscr{A}(q)=\sum_{n=0}^{\infty} q^{n+\frac{3}{4}}\,F(4n+3)\,,\qquad \mathscr{A}'(q)=\mathscr{A}(qe^{\pi i})\,,$$

$$(1.02)\quad \mathscr{B}(q)=\sum_{n=0}^{\infty} q^{n}\,F(4n)\,,\qquad \mathscr{B}'(q)=\mathscr{B}(-q)\,,$$

$$(1.03)\quad \mathscr{C}(q)=\sum_{n=0}^{\infty} q^{n}\,I(n)\,,\qquad \mathscr{C}'(q)=\mathscr{C}(-q)\,.$$

In addition I shall use a notation introduced by Petr [7]), with some slight amplifications, and therefore I shall write

$$(1.04)\quad \mathscr{T}(q)=2\sum_{n=0}^{\infty} q^{n+7/8}\,F(8n+7)\,,\qquad \mathscr{T}'(q)=\mathscr{T}(qe^{\pi i})\,,$$

$$(1.05)\quad \mathscr{U}(q)=4\sum_{n=0}^{\infty} q^{n} J(n)\,,\qquad \mathscr{U}'(q)=\mathscr{U}(-q)\,.$$

I recall the well known relations

$$(1.06)\quad F(4n+1)=G(4n+1)\,,$$

$$(1.07)\quad F(4n+2)=G(4n+2)\,,$$

$$(1.08)\quad 4F(8n+3)=3G(8n+3)\,,$$

$$(1.09)\quad 2F(8n+7)=G(8n+7)\,,$$

$$(1.10)\quad F(4n)=2F(n)\,,$$

$$(1.11)\quad G(4n)=F(4n)+G(n)\,,$$

$$(1.12)\quad \sum_{n=0}^{\infty} q^{n+\frac{1}{4}}\,F(4n+1)=\tfrac{1}{4}\vartheta_3^2\vartheta_2\,,$$

$$(1.13)\quad \sum_{n=0}^{\infty} q^{n+\frac{1}{2}}\,F(4n+2)=\tfrac{1}{4}\vartheta_3\vartheta_2^2\,,$$

$$(1.14)\quad \sum_{n=0}^{\infty} q^{2n+\frac{3}{4}}\,F(8n+3)=\tfrac{1}{8}\vartheta_2^3\,,$$

$$(1.15)\quad \mathscr{B}(q)+\mathscr{C}(q)=\tfrac{1}{4}\vartheta_3^3.$$

These formulae are due to Kronecker [8]) and Hermite [9]).

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[7]) K. Petr [Rozpravy České Akademie Císaře Františka Josefa **9** (1900), No. 38]. Petr writes simply U for what I here call $U(-q^2)$.

[8]) L. Kronecker [Journal für Math. **57** (1860), 248—255]; see also H. J. S. Smith [Report British Assoc. **1865**, 348].

[9]) C. Hermite [Journal de Math. (2) **7** (1862), 25—44].

The results to be obtained in the present paper are formulae connecting $\mathscr{A}(q)$, $\mathscr{A}'(q)$, $\mathscr{B}(q)$, $\mathscr{B}'(q)$ with the corresponding functions of q_1. I shall also obtain a formula connecting the two functions

$$\mathscr{T}'(q), \quad \mathscr{T}'(q_1);$$

this formula seems to be completely new, and it might be difficult to prove it by Mordell's methods.

In the contour integrals which I shall use, it is to be understood in all cases that the paths of integration are straight lines; it is also to be understood that c is a positive number so small that the integrands of the various integrals have no poles in the strip of the plane of breadth $2c$ which is bisected by the real axis with the exception of those poles which lie on the real axis itself.

2. *Expansions of quotients of theta-functions.*

Before proceeding, I shall state certain expansions of a type which are easily derived from relations between various expansions associated with Humbert's generating functions. The twelve expansions in question form three sets, each containing four expansions, such that the members of a set are obtainable from each other by altering the variable by a half-period. The expansions are as follows:

$$(2.01)\quad \vartheta_2 \frac{\vartheta_3(x)\vartheta_4(x)}{\vartheta_1(x)} = -2i \sum_{n=-\infty}^{\infty} \frac{q^{n^2}e^{2nix},}{q^{-n}e^{-ix}-q^n e^{ix}},$$

$$(2.02)\quad \vartheta_2 \frac{\vartheta_3(x)\vartheta_4(x)}{\vartheta_2(x)} = 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2}e^{2nix}}{q^{-n}e^{-ix}+q^n e^{ix}},$$

$$(2.03)\quad \vartheta_2 \frac{\vartheta_1(x)\vartheta_2(x)}{\vartheta_3(x)} = -2i \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}}{q^{-n-\frac{1}{2}}e^{-ix}+q^{n+\frac{1}{2}}e^{ix}},$$

$$(2.04)\quad \vartheta_2 \frac{\vartheta_1(x)\vartheta_2(x)}{\vartheta_4(x)} = -2i \sum_{n=-\infty}^{\infty} \frac{q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}}{q^{-n-\frac{1}{2}}e^{-ix}-q^{n+\frac{1}{2}}e^{ix}},$$

$$(2.05)\quad \vartheta_3 \frac{\vartheta_2(x)\vartheta_4(x)}{\vartheta_1(x)} = -\, i \sum_{n=-\infty}^{\infty} q^{n^2}e^{2nix}\frac{1+q^{2n}e^{2ix}}{1-q^{2n}e^{2ix}},$$

$$(2.06)\quad \vartheta_3 \frac{\vartheta_1(x)\vartheta_3(x)}{\vartheta_2(x)} = i \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}e^{2nix}\frac{1-q^{2n}e^{2ix}}{1+q^{2n}e^{2ix}},$$

$$(2.07)\quad \vartheta_3 \frac{\vartheta_2(x)\vartheta_4(x)}{\vartheta_3(x)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}\frac{1-q^{2n+1}e^{2ix}}{1+q^{2n+1}e^{2ix}},$$

$$(2.08)\quad \vartheta_3\,\frac{\vartheta_1(x)\vartheta_3(x)}{\vartheta_4(x)}=-i\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}\,\frac{1+q^{2n+1}e^{2ix}}{1-q^{2n+1}e^{2ix}},$$

$$(2.09)\quad \vartheta_4\,\frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_1(x)}=-i\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}e^{2nix}\,\frac{1+q^{2n}e^{2ix}}{1-q^{2n}e^{2ix}},$$

$$(2.10)\quad \vartheta_4\,\frac{\vartheta_1(x)\vartheta_4(x)}{\vartheta_2(x)}=\ \ i\sum_{n=-\infty}^{\infty} q^{n^2}e^{2nix}\,\frac{1-q^{2n}e^{2ix}}{1+q^{2n}e^{2ix}},$$

$$(2.11)\quad \vartheta_4\,\frac{\vartheta_1(x)\vartheta_4(x)}{\vartheta_3(x)}=-i\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}\,\frac{1-q^{2n+1}e^{2ix}}{1+q^{2n+1}e^{2ix}},$$

$$(2.12)\quad \vartheta_4\,\frac{\vartheta_2(x)\vartheta_3(x)}{\vartheta_4(x)}=\sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)ix}\,\frac{1+q^{2n+1}e^{2ix}}{1-q^{2n+1}e^{2ix}}.$$

I have not seen these expansions anywhere in print, though, by expanding the series on the right as double series and then rearranging them as Fourier series, it is easy to reduce the expansions to certain expansions given by Hermite [10]).

It is also a simple matter to establish the expansions in the following manner. It may be verified that (for the first of the twelve) the expression

$$\frac{-2i\vartheta_1(x)}{\vartheta_2\vartheta_3(x)\vartheta_4(x)}\sum_{n=-\infty}^{\infty}\frac{q^{n^2}e^{2ix}}{q^{-n}e^{-ix}-q^{n}e^{ix}}$$

is a doubly periodic function of x (with periods π and $\pi\tau$) which has no poles at the zeros of any of the functions

$$\vartheta_3(x)\,,\quad \vartheta_4(x)\,,\quad 1-q^{2n}e^{2ix}\,.$$

Hence, by Liouville's theorem, the expression is a constant; and the value of this constant is

$$\lim_{x\to 0}\frac{-2i\vartheta_1(x)}{\vartheta_2\vartheta_3(x)\vartheta_4(x)}\sum_{n=-\infty}^{\infty}\frac{q^{n^2}e^{2nix}}{q^{-n}e^{-ix}-q^{n}e^{ix}}=\frac{-2i}{\vartheta_2\vartheta_3\vartheta_4}\lim_{x\to 0}\frac{\vartheta_1(x)}{e^{-ix}-e^{ix}}$$

$$=\frac{\vartheta_1'}{\vartheta_2\vartheta_3\vartheta_4}=1.$$

This establishes the first result, and the others can be proved in a similar manner.

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[10]) Oeuvres de Charles Hermite 2 (1908), 244.

3. *The modified form of Humbert's expansions.*

I now state certain formulae involving the generating functions; these formulae are derived from Humbert's [11]) expansions by the same process as that by which the formulae of § 2 are derived from Hermite's expansions. The formulae are

$$(3.01)\quad -\vartheta_2\,\frac{\vartheta_3'(x)\vartheta_4(x)}{\vartheta_2(x)}=4\cdot\mathscr{A}(q)\,\vartheta_1(x)-4i\sum_{n=-\infty}^{\infty}\frac{(-1)^n n q^{n^2}e^{2nix}}{q^{-n}e^{-ix}+q^n e^{ix}},$$

$$(3.02)\quad \vartheta_2\,\frac{\vartheta_4'(x)\vartheta_3(x)}{\vartheta_1(x)}=4\cdot\mathscr{A}(q)\,\vartheta_2(x)+4\sum_{n=-\infty}^{\infty}\frac{n q^{n^2}e^{2nix}}{q^{-n}e^{-ix}-q^n e^{ix}},$$

$$(3.03)\quad \vartheta_2\,\frac{\vartheta_2'(x)\vartheta_2(x)}{\vartheta_4(x)}=4\cdot\mathscr{A}(q)\,\vartheta_3(x)+4\sum_{n=-\infty}^{\infty}\frac{(n-\frac{1}{2})q^{(n-\frac{1}{2})^2}e^{(2n-1)ix}}{q^{\frac{1}{2}-n}e^{-ix}-q^{n-\frac{1}{2}}e^{ix}},$$

$$(3.04)\quad -\vartheta_2\,\frac{\vartheta_2'(x)\vartheta_1(x)}{\vartheta_3(x)}=4\cdot\mathscr{A}(q)\,\vartheta_4(x)+4\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n-\frac{1}{2})^2}e^{(2n-1)ix}}{q^{\frac{1}{2}-n}e^{-ix}+q^{n-\frac{1}{2}}e^{ix}},$$

$$(3.05)\quad -\vartheta_3\,\frac{\vartheta_2'(x)\vartheta_4(x)}{\vartheta_3(x)}=4\cdot\mathscr{B}(q)\,\vartheta_1(x)+$$

$$+i\sum_{n=-\infty}^{\infty}(-1)^n(2n-1)q^{(n-\frac{1}{2})^2}e^{(2n-1)ix}\,\frac{1-q^{2n-1}e^{2ix}}{1+q^{2n-1}e^{2ix}},$$

$$(3.06)\quad \vartheta_3\,\frac{\vartheta_1'(x)\vartheta_3(x)}{\vartheta_4(x)}=4\cdot\mathscr{B}(q)\,\vartheta_2(x)+$$

$$+\sum_{n=-\infty}^{\infty}(2n-1)q^{(n-\frac{1}{2})^2}e^{(2n-1)ix}\,\frac{1+q^{2n-1}e^{2ix}}{1-q^{2n-1}e^{2ix}},$$

$$(3.07)\quad \vartheta_3\,\frac{\vartheta_4'(x)\vartheta_2(x)}{\vartheta_1(x)}=4\cdot\mathscr{B}(q)\,\vartheta_3(x)+\sum_{n=-\infty}^{\infty}2nq^{n^2}e^{2nix}\,\frac{1+q^{2n}e^{2ix}}{1-q^{2n}e^{2ix}},$$

$$(3.08)\quad \vartheta_3\,\frac{\vartheta_3'(x)\vartheta_1(x)}{\vartheta_2(x)}=4\cdot\mathscr{B}(q)\,\vartheta_4(x)+\sum_{n=-\infty}^{\infty}(-1)^n 2nq^{n^2}e^{2nix}\,\frac{1-q^{2n}e^{2ix}}{1+q^{2n}e^{2ix}}.$$

The first four formulae are derivable from one another by changing the variable by a half-period and (in the case of complex half-periods) using one of the formulae of § 2. The same remark applies to the last four formulae, some of which have been stated by Mordell [12]) with the corresponding formulae for $\mathscr{C}(q)$.

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[11]) G. Humbert [Journal de Math. (6) **3** (1907), 349—350].

[12]) L. J. Mordell [Messenger of Math. **45** (1916), 75—80].

4. *The transformation of* $\mathscr{B}(q)$.

We now attack one of the main problems. In formula (3.08) put $x=0$; then

$$4\mathscr{B}(q)\,\vartheta_4(0,q)=-\sum_{n=-\infty}^{\infty}(-1)^n\,2nq^{n^2}\,\frac{1-q^{2n}}{1+q^{2n}}$$

$$=-\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{2\pi z}{\sin\pi z}e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz,$$

by Cauchy's theorem on residues, since the only poles of the integrand between the two lines forming the contour are at the points $z=n$ $(n=-\infty,\dots,-2,-1,1,2,\dots,\infty)$ and the residue at $z=n$ is

$$(-1)^n 2ne^{-\alpha n^2}\frac{e^{\alpha n}-e^{-\alpha n}}{e^{\alpha n}+e^{-\alpha n}}.$$

Now consider

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{2\pi z}{\sin\pi z}e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz.$$

On the path of integration we may write

$$\frac{1}{\sin\pi z}=-2i\sum_{n=0}^{\infty}e^{(2n+1)\pi iz},$$

so that

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{2\pi z}{\sin\pi z}e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz$$

$$=-\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}4\pi iz\,e^{(2n+1)\pi iz-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz$$

$$=-\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}f_n(z)dz,$$

say. We calculate these integrals in the following manner. The poles of $f_n(z)$ are simple poles at the points

$$z_m=(m+\tfrac{1}{2})\pi i/\alpha \qquad (m=-\infty,\dots,-1,0,1,\dots,\infty)$$

and the residue at z_m is

$$-\frac{4\pi^2(m+\frac{1}{2})}{\alpha^2}\exp\left\{\frac{(m+\frac{1}{2})(m-2n-\frac{1}{2})\pi^2}{\alpha}\right\}=\lambda_{n,m},$$

say.

Now, P denoting the 'principal value' of an integral, it follows from Cauchy's theorem that

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic} - \mathrm{P}\int_{-\infty+z_n}^{\infty+z_n}\right\} f_n(z)dz = \lambda_{n,0}+\lambda_{n,1}+\dots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n},$$

and therefore

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} f_n(z)dz = \lambda_{n,0}+\lambda_{n,1}+\dots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n} + \frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty} f_n(z+t)dt.$$

Next, by rearranging repeated series, we see that

$$\begin{aligned}\tfrac{1}{2}\lambda_{0,0}&+\sum_{n=1}^{\infty}\{\lambda_{n,0}+\lambda_{n,1}+\dots+\lambda_{n,n-1}+\tfrac{1}{2}\lambda_{n,n}\}\\ &=\sum_{m=0}^{\infty}\{\tfrac{1}{2}\lambda_{m,m}+\lambda_{m+1,m}+\lambda_{m+2,m}+\dots\}\\ &=\tfrac{1}{2}\sum_{m=0}^{\infty}\lambda_{m,m}\frac{1+e^{-(2m+1)\pi^2/\alpha}}{1-e^{-(2m+1)\pi^2/\alpha}}\\ &=-\frac{2\pi^2}{\alpha^2}\sum_{m=0}^{\infty}(m+\tfrac{1}{2})\,q_1^{(m+\frac{1}{2})^2}\frac{1+q_1^{2m+1}}{1-q_1^{2m+1}}.\end{aligned}$$

Further we have

$$\begin{aligned}\mathrm{P}&\int_{-\infty}^{\infty} f_n(z_n+t)\,dt\\ &=\mathrm{P}\int_{-\infty}^{\infty}4\pi i\left\{\frac{(n+\frac{1}{2})\pi i}{\alpha}+t\right\}e^{-\alpha t^2-(n+\frac{1}{2})^2\pi^2/\alpha}\frac{e^{\alpha t}+e^{-\alpha t}}{e^{\alpha t}-e^{-\alpha t}}dt\\ &=4\pi i\,q_1^{(n+\frac{1}{2})^2}\int_{-\infty}^{\infty}te^{-\alpha t^2}\coth\alpha t\,dt,\end{aligned}$$

the symbol P being omitted in the last line because there is no longer a pole on the path of integration. It will be observed that the result of this analysis is to obtain a path of integration which passes through the stationary point, not of $f_n(z)$, but of its important factor $e^{(2n+1)\pi i z-\alpha z^2}$.

The analysis is, in fact, a rudimentary form of the „method of steepest descents".

It therefore follows that

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{2\pi z}{\sin\pi z}e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz$$

$$=\frac{2\pi^2}{\alpha^2}\sum_{m=0}^{\infty}\left\{(m+\tfrac{1}{2})q_1^{(m+\frac{1}{2})^2}\frac{1+q_1^{2m+1}}{1-q_1^{2m+1}}\right\}-2\left\{\sum_{n=0}^{\infty}q_1^{(n+\frac{1}{2})^2}\right\}\int_{-\infty}^{\infty}te^{-\alpha t^2}\coth\alpha t\,dt.$$

The other integral [13] $-\frac{1}{2\pi i}\int_{-\infty-ic}^{\infty-ic}$ may be treated in a similar manner, by changing the sign of i throughout the previous work. Consequently

$$4\,\mathscr{B}(q)\,\vartheta_4(0,q)=\frac{2\pi^2}{\alpha^2}\sum_{m=-\infty}^{\infty}\left\{(m+\tfrac{1}{2})\,q_1^{(m+\frac{1}{2})^2}\frac{1+q_1^{2m+1}}{1-q_1^{2m+1}}\right\}$$
$$-2\vartheta_2(0,q_1)\int_{-\infty}^{\infty}te^{-\alpha t^2}\coth\alpha t\,dt\ ,$$

that is to say, by (3.06),

$$4\,\mathscr{B}(q)\,\vartheta_4(0,q)=\frac{\pi^2}{\alpha^2}\left[\frac{\vartheta_3(0,q_1)\vartheta_1'(0,q_1)\vartheta_3(0,q_1)}{\vartheta_4(0,q_1)}-4\,\mathscr{B}(q_1)\vartheta_2(0,q_1)\right]$$
$$-2\vartheta_2(0,q_1)\int_{-\infty}^{\infty}te^{-\alpha t^2}\coth\alpha t\,dt.$$

By using Jacobi's imaginary transformation for $\vartheta_4(0,q)$, namely $\vartheta_4(0,q)\sqrt{(-i\tau)}=\vartheta_2(0,q_1)$, and writing $\pi u/\alpha$ for t, we see at once that

$$(4.1)\quad \int_0^{\infty}ue^{-\beta u^2}\coth\pi u\,du=\tfrac{1}{4}\vartheta_3^3(0,q_1)-\mathscr{B}(q_1)-(-i\tau)^{3/2}\,\mathscr{B}(q).$$

This is one form of the required transformation of $\mathscr{B}(q)$.

Interchange α and β; we deduce that

$$(4.2)\quad \int_0^{\infty}ue^{-\alpha u^2}\coth\pi u\,du=\tfrac{1}{4}\vartheta_3^3(0,q)-\mathscr{B}(q)-(-i\tau)^{-3/2}\,\mathscr{B}(q_1).$$

By combining the last two formulae and using Jacobi's imaginary transformation of $\vartheta_3(0,q)$, we see immediately that

$$(4.3)\quad \int_0^{\infty}ue^{-\beta u^2}\coth\pi u\,du=(-i\tau)^{3/2}\int_0^{\infty}ue^{-\alpha u^2}\coth\pi u\,du\ ;$$

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[13]) Two integrals so related will, for brevity, be called ,conjugate'.

a direct proof of this formula has been constructed by Ramanujan [14]) with the help of the theory of reciprocal functions.

I defer the consideration of the consequences of these results to § 7.

5. *The transformation of* $\mathscr{B}'(q)$.

We can treat $\mathscr{B}'(q)$ in much the same way as $\mathscr{B}(q)$. In formula (3.08) put $x = 0$ and change the sign of q, so that

$$4\mathscr{B}'(q)\vartheta_3(0,q) = -\sum_{n=-\infty}^{\infty} 2nq^{n^2}\frac{1-q^{2n}}{1+q^{2n}}$$
$$= -\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic}\right\} 2\pi z \cot \pi z \cdot e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}\,dz.$$

Now consider

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} 2\pi z \cot \pi z \cdot e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}\,dz.$$

On the path of integration we may write

$$\cot \pi z = -i\left(1+2\sum_{n=1}^{\infty} e^{2n\pi i z}\right),$$

so that

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} 2\pi z \cot \pi z \cdot e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}\,dz$$
$$= -\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} 2\pi i z\left(1+2\sum_{n=1}^{\infty} e^{2n\pi i z}\right)e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}\,dz$$
$$= -\frac{1}{2\pi i}\left[\tfrac{1}{2}\int_{-\infty+ic}^{\infty+ic} \Phi_0(z)dz + \sum_{n=1}^{\infty}\int_{-\infty+ic}^{\infty+ic} \Phi_n(z)dz\right],$$

say. We calculate these integrals by expressing $\int_{-\infty+ic}^{\infty+ic} \Phi_n(z)dz$ in terms of $\int_{-\infty+\zeta_n}^{\infty+\zeta_n} \Phi_n(z)dz$, where

$$\zeta_n = n\pi i/\alpha. \qquad (n = 0, 1, 2, \ldots)$$

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14) S. Ramanujan [Quarterly Journal of Math. 46 (1916), 253].

If, as before, we write $z_m=(m+\frac{1}{2})\pi i/\alpha$, the residue of $\Phi_n(z)$ at z_m is

$$-\frac{4\pi^2(m+\frac{1}{2})}{\alpha}\exp\left\{\frac{(m+\frac{1}{2})(m-2n+\frac{1}{2})\pi^2}{\alpha}\right\}=\mu_{n,\,m},$$

say, and we have

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\Phi_n(z)dz=\sum_{m=0}^{n-1}\mu_{n,\,m}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Phi_n(\zeta_n+\iota)dt,$$

the empty sum in the case $n=0$ being interpreted as zero.

Next, by rearranging repeated series, we see that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\mu_{n,\,m}=\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}\mu_{n,m}$$

$$=\sum_{m=0}^{\infty}\frac{\mu_{m+1,\,m}}{1-e^{-(2m+1)\pi^2/\alpha}}$$

$$=-\frac{4\pi^2}{\alpha^2}\sum_{m=0}^{\infty}(m+\tfrac{1}{2})\frac{q_1^{(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}.$$

Further we have

$$\int_{-\infty}^{\infty}\Phi_n(\zeta_n+t)dt=\int_{-\infty}^{\infty}4\pi i\left\{\frac{n\pi i}{\alpha}+t\right\}e^{-\alpha t^2-n^2\pi^2/\alpha}\frac{e^{\alpha t}-e^{-\alpha t}}{e^{\alpha t}+e^{-\alpha t}}dt$$

$$=4\pi i\,q_1^{n^2}\int_{-\infty}^{\infty}te^{-\alpha t^2}\tanh\alpha t\,dt.$$

It therefore follows that

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}2\pi z\cot\pi z\,.\,e^{-\alpha z^2}\frac{e^{\alpha z}-e^{-\alpha z}}{e^{\alpha z}+e^{-\alpha z}}dz$$

$$=\frac{2\pi^2}{\alpha^2}\sum_{m=-\infty}^{\infty}\left\{(m+\tfrac{1}{2})\frac{q_1^{(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}\right\}$$

$$-\left\{1+2\sum_{n=1}^{\infty}q_1^{n^2}\right\}\int_{-\infty}^{\infty}te^{-\alpha t^2}\tanh\alpha t\,dt.$$

The conjugate integral may be treated in a similar manner, by changing the sign of i throughout the previous work. Consequently

$$4\,\mathscr{B}'(q)\vartheta_3(0,q)=\frac{4\pi^2}{\alpha^2}\sum_{m=-\infty}^{\infty}\left\{(m+\tfrac{1}{2})\frac{q_1^{(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}\right\}$$
$$-2\vartheta_3(0,q_1)\int_{-\infty}^{\infty}te^{-\alpha t^2}\tanh\alpha t\,dt\,,$$

that is to say, by (3.03),

$$4\,\mathscr{B}'(q)\vartheta_3(0,q)=\frac{\pi^2}{\alpha^2}\left[\frac{\vartheta_2(0,q_1)\vartheta_1'(0,q_1)\vartheta_2(0,q_1)}{\vartheta_4(0,q_1)}-4\vartheta_3(0,q_1)\,\mathscr{A}(q_1)\right]$$
$$-2\vartheta_3(0,q_1)\int_{-\infty}^{\infty}te^{-\alpha t^2}\tanh\alpha t\,dt.$$

By using Jacobi's imaginary transformation of $\vartheta_3(0,\,q)$ and writing $\pi u/\alpha$ for t, we see at once that

$$\int_0^{\infty}ue^{-\beta u^2}\tanh\pi u\,du=\tfrac{1}{4}\vartheta_2^3(0,q_1)-\mathscr{A}(q_1)-(-i\tau)^{3/2}\,\mathscr{B}'(q),$$

and therefore, disposing of the theta-function by (1.14),

$$(5.1)\qquad \int_0^{\infty}ue^{-\beta u^2}\tanh\pi u\,du=e^{-3/4\pi i}\,\mathscr{A}'(q_1)-(-i\tau)^{3/2}\,\mathscr{B}'(q)$$

An alternative method of disposing of the theta-function is to write

$$\int_0^{\infty}ue^{-\beta u^2}\tanh\pi u\,du=(-i\tau)^{3/2}\{\tfrac{1}{4}\vartheta_4^2(0,q)-\mathscr{B}'(q)\}-\mathscr{A}(q_1),$$

so that, by (1.15), we have

$$(5.2)\qquad \int_0^{\infty}ue^{-\beta u^2}\tanh\pi u\,du=(-i\tau)^{3/2}\,\mathscr{C}'(q)-\mathscr{A}(q_1).$$

I remark that the integral on the left admits of a transformation, though, unlike the corresponding integral of § 4, the transformation is not symmetrical. To obtain it, take the well known formula

$$(5.3)\qquad \tanh\pi u=2\int_0^{\infty}\frac{\sin 2\pi tu}{\sinh\pi t}\,dt,$$

which gives

$$\int_0^\infty ue^{-\beta u^2}\tanh \pi u\,du = 2\int_0^\infty\int_0^\infty ue^{-\beta u^2}\frac{\sin 2\pi tu}{\sinh \pi t}\,dt\,du$$

$$= 2\int_0^\infty\int_0^\infty ue^{-\beta u^2}\frac{\sin 2\pi tu}{\sinh \pi t}\,du\,dt$$

$$= \left(\frac{\pi}{\beta}\right)^{3/2}\int_0^\infty \frac{te^{-\alpha t^2}}{\sinh \pi t}\,dt.$$

The required transformation is consequently

$$\int_0^\infty ue^{-\beta u^2}\tanh \pi u\,du = (-i\tau)^{3/2}\int_0^\infty \frac{ue^{-\alpha u^2}}{\sinh \pi u}\,du. \tag{5.4}$$

6. *The transformation of* $\mathscr{T}'(q)$.

The last fundamental transformation which I shall consider is that of the function defined by (1.04), namely

$$e^{-7/8\pi i}\,\mathscr{T}'(q) = 2\sum_{n=0}^{\infty}(-1)^n q^{n+7/8}\,\mathrm{F}(8n+7).$$

To discuss this function take Humbert's formula [15])

$$\vartheta_2\vartheta_3\vartheta_4\sum_{n=0}^{\infty} q^{2n+7/4}\,\mathrm{F}(8n+7) = 2\sum_{n=0}^{\infty}(-1)^{n+1}\,n^2\frac{q^{n^2+n}}{1+q^{2n}}$$

$$= -\sum_{n=-\infty}^{\infty} 4n^2\frac{q^{4n^2}}{q^{-2n}+q^{2n}} + \sum_{n=-\infty}^{\infty}(2n+1)^2\frac{q^{(2n+1)^2}}{q^{-2n-1}+q^{2n+1}},$$

and replace q by $i\sqrt{q}$; we get

$$e^{-1/8\pi i}\,\vartheta_1'(0, i\sqrt{q})\sum_{n=0}^{\infty}(-1)^n q^{n+7/8}\,\mathrm{F}(8n+7)$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n 4n^2\frac{q^{2n^2}}{q^{-n}+q^{n}} + \sum_{n=-\infty}^{\infty}(-1)^n(2n+1)^2\frac{q^{2(n+\frac{1}{2})^2}}{q^{-n-\frac{1}{2}}-q^{n+\frac{1}{2}}},$$

and hence, in the usual manner,

$$e^{-1/8\pi i}\,\vartheta_1'(0, i\sqrt{q})\sum_{n=0}^{\infty}(-1)^n q^{n+7/8}\,\mathrm{F}(8n+7)$$

$$= \frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic}\right\}\frac{4\pi z^2}{\sin \pi z}\;\frac{e^{-2\alpha z^2}}{e^{\alpha z}+e^{-\alpha z}}\,dz$$

$$-\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic} + \int_{\infty+ic}^{-\infty+ic}\right\}\frac{4\pi z^2}{\cos \pi z}\;\frac{e^{-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}\,dz.$$

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[15]) G. Humbert [Journal de Math. (6) 3 (1907), 368].

The four integrals on the right have to be discussed separately; first we have

$$-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{4\pi z^2}{\sin \pi z}\;\frac{e^{-\alpha z^2}}{e^{\alpha z}+e^{-\alpha z}}\,dz$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}8\pi i z^2 e^{-2\alpha z^2}\,\frac{e^{(4n+1)\pi i z}+e^{(4n+3)\pi i z}}{e^{\alpha z}+e^{-\alpha z}}\,dz$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}g_n(z)dz\;,$$

say. The residue of $g_n(z)$ at [16] z_m is

$$-(-1)^m\frac{4\pi^3(m+\tfrac{1}{2})^2}{\alpha^3}[\exp \pi i z_m+\exp 3\pi i z_m]\exp(4n\pi i z_m-2\alpha z_m^2)$$

$$=\varrho_{n,\,m}\,,$$

say; also

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}g_n(z)dz=\varrho_{n,\,0}+\varrho_{n,\,1}+\;\cdots\;+\varrho_{n,\,n-1}+\tfrac{1}{2}\varrho_{n,\,n}$$

$$+\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}g_n(z_n+t)dt\,,$$

and hence, by rearranging repeated series, we see that

$$\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic}g_n(z)dz=\sum_{m=1}^{\infty}(\tfrac{1}{2}\varrho_{m,\,m}+\varrho_{m+1,\,m}+\varrho_{m+2,\,m}+\cdots)$$

$$+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\mathrm{P}\int_{-\infty}^{\infty}g_n(z_n+t)dt$$

$$=\tfrac{1}{2}\sum_{m=0}^{\infty}\varrho_{m,\,m}\frac{1+e^{-(4m+2)\pi^2/\alpha}}{1-e^{-(4m+2)\pi^2/\alpha}}+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\mathrm{P}\int_{-\infty}^{\infty}g_n(z_n+t)dt\,.$$

Now

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[16]) As hitherto, we write

$$z_m=(m+\tfrac{1}{2})\pi i/\alpha,\;\zeta_m=m\pi i/\alpha.$$

$$\tfrac{1}{2}\sum_{m=0}^{\infty}\varrho_{m,m}\frac{1+e^{-(4m+2)\pi^2/\alpha}}{1-e^{-(4m+2)\pi^2/\alpha}}=-\frac{2\pi^3}{\alpha^3}\sum_{m=0}^{\infty}(-1)^m(m+\tfrac{1}{2})^2q_1^{2(m+\frac{1}{2})^2}\frac{q_1^{-2m-1}+q_1^{2m+1}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}$$

$$=-\frac{4\pi^3}{\alpha^3}\sum_{m=\infty}^{\infty}(-1)^m(m+\tfrac{1}{2})^2\frac{q_1^{2(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}$$

$$-\frac{2\pi^3}{\alpha^3}\sum_{m=0}^{\infty}(-1)^m(m+\tfrac{1}{2})^2q_1^{2(m+\frac{1}{2})^2}(q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}})$$

$$=-\frac{2\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(m+\tfrac{1}{2})^2\frac{q_1^{2(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}-\frac{2\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(m+\tfrac{1}{2})^2q_1^{m(2m+1)}.$$

Also

$$\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}g_n(z_n+t)dt$$

$$=4(-1)^nq_1^{2(n+\frac{1}{2})^2}\,\mathrm{P}\int_{-\infty}^{\infty}\left(\frac{(n+\frac{1}{2})\pi i}{\alpha}+t\right)^2\frac{q_1^{-n-\frac{1}{2}}e^{-\pi it}+q_1^{n+\frac{1}{2}}e^{\pi it}}{i(e^{\alpha t}-e^{-\alpha t})}e^{-2\alpha t^2}dt$$

$$=4(-1)^nq_1^{2(n+\frac{1}{2})^2}(q_1^{-n-\frac{1}{2}}+q_1^{n+\frac{1}{2}})\int_{-\infty}^{\infty}\frac{(2n+1)\pi t}{\alpha}\,\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$-4(-1)^nq_1^{2(n+\frac{1}{2})^2}(q_1^{-n-\frac{1}{2}}-q_1^{n+\frac{1}{2}})\int_{-\infty}^{\infty}\left(t^2-\frac{(n+\frac{1}{2})^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt.$$

Collecting our results and treating the conjugate integral in a similar manner, we find that, for the first pair of integrals,

$$\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{4\pi z^2}{\sin\pi z}\,\frac{e^{-2\alpha z^2}}{e^{\alpha z}+e^{-\alpha z}}dz$$

$$=-\frac{4\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(m+\tfrac{1}{2})^2\frac{q_1^{2(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}-\frac{4\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(m+\tfrac{1}{2})^2\,q_1^{m(2m+1)}$$

$$+8\sum_{n=0}^{\infty}(-1)^nq_1^{2(n+\frac{1}{2})^2}(q_1^{-n-\frac{1}{2}}+q_1^{n+\frac{1}{2}})\int_{-\infty}^{\infty}\frac{(2n+1)\pi t}{\alpha}\,\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$-8\sum_{n=0}^{\infty}(-1)^nq_1^{2(n+\frac{1}{2})^2}(q_1^{-n-\frac{1}{2}}-q_1^{n+\frac{1}{2}})\int_{-\infty}^{\infty}\left(t^2-\frac{(n+\frac{1}{2})^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$=-\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(2m+1)^2\frac{q_1^{2(m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}-\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(2m+1)^2q_1^{m(2m+1)}$$

$$+8\sum_{n=-\infty}^{\infty}(-1)^n q_1^{n(2n+1)}\int_{-\infty}^{\infty}\frac{(2n+1)\pi t}{\alpha}\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}\,dt$$

$$-8\sum_{n=-\infty}^{\infty}(-1)^n q_1^{n(2n+1)}\int_{-\infty}^{\infty}\left(t^2-\frac{(n+\frac{1}{2})^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}\,dt.$$

The transformation of the first pair of integrals is now complete and we turn to the other pair. We have

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{4\pi z^2}{\cos\pi z}\frac{e^{-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}\,dz$$

$$=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}8\pi z^2\left[\frac{e^{\pi i z}}{e^{\alpha z}-e^{-\alpha z}}-\sum_{n=1}^{\infty}\frac{e^{(4n-1)\pi i z}-e^{(4n+1)\pi i z}}{e^{\alpha z}-e^{-\alpha z}}\right]e^{-2\alpha z^2}dz$$

$$=\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{8\pi z^2 e^{\pi i z-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}\,dz-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{-\infty+ic}^{\infty+ic}h_n(z)\,dz,$$

say. Leaving the first integral on the right for the moment, we observe that the residue of $h_n(z)$ at ζ_m is

$$-(-1)^m\frac{4\pi^3 m^2}{\alpha^3}[\exp(-\pi i\zeta_m)-\exp\pi i\zeta_m]\exp(4n\pi i\zeta_m-2\alpha\zeta^2{}_m)=\sigma_{n,m},$$

say; further

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}h_n(z)\,dz=\sigma_{n,1}+\sigma_{n,2}+\ldots+\sigma_{n,n-1}+\tfrac{1}{2}\sigma_{n,n}$$

$$+\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}h_n(\zeta_n+t)\,dt,$$

and hence, by rearranging repeated series, we see that

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{-\infty+ic}^{\infty+ic}h_n(z)\,dz=\sum_{m=1}^{\infty}(\tfrac{1}{2}\sigma_{m,m}+\sigma_{m+1,m}+\sigma_{m+2,m}+\ldots)$$

$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty}\mathrm{P}\int_{-\infty}^{\infty}h_n(\zeta_n+t)\,dt$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}\sigma_{m,m}\frac{1+e^{-4m\pi^2/\alpha}}{1-e^{-4m\pi^2/\alpha}}+\frac{1}{2\pi i}\sum_{n=1}^{\infty}\mathrm{P}\int_{-\infty}^{\infty}h_n(\zeta_n+t)\,dt.$$

Now

$$\tfrac{1}{2}\sum_{m=1}^{\infty}\sigma_{m,\,m}\frac{1+e^{-4m\pi^2/\alpha}}{1-e^{-4m\pi^2/\alpha}}=-\frac{2\pi^3}{\alpha^3}\sum_{m=1}^{\infty}(-1)^m m^2 q_1^{2m^2}\frac{q_1^{-2m}+q_1^{2m}}{q_1^{-m}+q_1^{m}}$$

$$=\frac{4\pi^3}{\alpha^3}\sum_{m=1}^{\infty}(-1)^m m^2\frac{q_1^{2m^2}}{q_1^{-m}+q_1^{m}}-\frac{2\pi^3}{\alpha^3}\sum_{m=1}^{\infty}(-1)^m m^2 q_1^{2m^2}(q_1^{-m}+q_1^{m})$$

$$=\frac{2\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m m^2\frac{q_1^{2m^2}}{q_1^{-m}+q_1^{m}}-\frac{2\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m m^2 q_1^{m(2m+1)}\,.$$

Also

$$\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}h_n(\zeta_n+t)dt$$

$$=4(-1)^n q_1^{2n^2}\mathrm{P}\int_{-\infty}^{\infty}\left(\frac{n\pi i}{\alpha}+t\right)^2\frac{q_1^{-n}e^{-\pi i t}-q_1^{n}e^{\pi i t}}{i(e^{\alpha t}-e^{-\alpha t})}e^{-2\alpha t^2}dt$$

$$=4(-1)^n q_1^{2n^2}(q_1^{-n}-q_1^{n})\int_{-\infty}^{\infty}\frac{2n\pi t}{\alpha}\,\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$-4(-1)^n q_1^{2n^2}(q_1^{-n}+q_1^{n})\int_{-\infty}^{\infty}\left(t^2-\frac{n^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt\;,$$

and lastly

$$\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{8\pi z^2 e^{\pi i z-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}dz=4\int_{-\infty}^{\infty}t^2\,\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt\,.$$

Collecting our results and treating the conjugate integral in a similar manner, we find that, for the last pair of integrals,

$$-\frac{1}{2\pi i}\left\{\int_{-\infty-ic}^{\infty-ic}+\int_{\infty+ic}^{-\infty+ic}\right\}\frac{4\pi z^2}{\cos\pi z}\,\frac{e^{-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}dz$$

$$=-\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m 4m^2\,\frac{q_1^{2m^2}}{q_1^{-m}+q_1^{m}}+\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m 4m^2 q_1^{m(2m+1)}$$

$$+4\sum_{n=-\infty}^{\infty}(-1)^n q_1^{2n^2}(q_1^{-n}+q_1^{n})\int_{-\infty}^{\infty}\left(t^2-\frac{n^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$-4\sum_{n=-\infty}^{\infty}(-1)^n q_1^{2n^2}(q_1^{-n}-q_1^{n})\int_{-\infty}^{\infty}\frac{2n\pi t}{\alpha}\,\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$=-\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m 4m^2\frac{q_1^{2m^2}}{q_1^{-m}+q_1^m}+\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m 4m^2 q_1^{m(2m+1)}$$

$$+8\sum_{n=-\infty}^{\infty}(-1)^n q_1^{n(2n+1)}\int_{-\infty}^{\infty}\left(t^2-\frac{n^2\pi^2}{\alpha^2}\right)\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$+8\sum_{n=-\infty}^{\infty}(-1)^n q_1^{n(2n+1)}\int_{-\infty}^{\infty}\frac{2n\pi t}{\alpha}\;\frac{e^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt.$$

Now combine the expressions obtained for the two pairs of integrals; we get

$$e^{-1/_8\pi i}\vartheta_1'\left(0,\, i\sqrt{q}\right)\sum_{n=0}^{\infty}(-1)^n q^{n+7/_8}\mathrm{F}(8n+7)$$

$$=-\frac{\pi^3}{\alpha^3}\left[\sum_{m=-\infty}^{\infty}(-1)^m 4m^2\frac{q_1^{2m^2}}{q_1^{-m}+q_1^m}+\sum_{m=-\infty}^{\infty}(-1)^m(2m+1)^2\frac{q_1^{(2m+\frac{1}{2})^2}}{q_1^{-m-\frac{1}{2}}-q_1^{m+\frac{1}{2}}}\right]$$

$$-\frac{\pi^3}{\alpha^3}\sum_{m=-\infty}^{\infty}(-1)^m(4m+1)q_1^{m(2m+1)}$$

$$+\frac{2\pi^2}{\alpha^2}\sum_{n=-\infty}^{\infty}(-1)^n(4n+1)q_1^{n(2n+1)}\int_{-\infty}^{\infty}\frac{e^{-2\alpha t}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$+\frac{8\pi}{\alpha}\sum_{n=-\infty}^{\infty}(-1)^n(4n+1)q_1^{n(2n+1)}\int_{-\infty}^{\infty}\frac{te^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$=-\frac{\pi^3}{\alpha^3}e^{-1/_8\pi i}\vartheta_1'\left(0,\, i\sqrt{q_1}\right)\sum_{n=0}^{\infty}(-1)^n q_1^{n+7/_8}\mathrm{F}(8n+7)$$

$$+e^{-1/_8\pi i}q_1^{-1/_8}\vartheta_1'\left(0,\, i\sqrt{q_1}\right)\left[-\frac{\pi^3}{2\alpha^3}+\frac{\pi^2}{\alpha^2}\int_{-\infty}^{\infty}\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt\right.$$

$$\left.+\frac{4\pi}{\alpha}\int_{-\infty}^{\infty}\frac{te^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt\right].$$

The next step is to dispose of the derivates of the theta-functions; this can be effected by expressing them in terms of the moduli and quarter-periods of Jacobian elliptic functions, with the help of the formula

$$\vartheta_1'(0,\, q)=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})^3\,;$$

we thus get

$$e^{-{}^1/_8\pi i}\vartheta_1'\left(0,\, i\sqrt{q}\right)=2q^{{}^1/_8}\prod_{n=1}^{\infty}\{(1+q^{2n-1})(1-q^{2n})\}^2$$

$$=\left(\frac{2\mathrm{K}}{\pi}\right)^{{}^3/_2}(4kk')^{{}^1/_4},$$

so that

$$\frac{e^{-{}^1/_8\pi i}\,\vartheta_1'\left(0,\, i\sqrt{q_1}\right)}{e^{-{}^1/_8\pi i}\,\vartheta_1'\left(0,\, i\sqrt{q}\right)}=\left(\frac{\mathrm{K}'}{\mathrm{K}}\right)^{{}^3/_2}=(-i\tau)^{{}^3/_2}.$$

Hence we have

$$\sum_{n=0}^{\infty}(-1)^n q^{n+{}^7/_8}\mathrm{F}(8n+7)+(-i\tau)^{-{}^3/_2}\sum_{n=0}^{\infty}(-1)^n q_1^{n+{}^7/_8}\mathrm{F}(8n+7)$$

$$=\frac{(-i\tau)^{{}^3/_2}}{q_1^{{}^1/_8}}\left[-\frac{\pi^3}{2\alpha^3}+\frac{\pi^2}{\alpha^2}\int_{-\infty}^{\infty}\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt+\frac{4\pi}{\alpha}\int_{-\infty}^{\infty}\frac{te^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt\right].$$

The next step is to simplify the integrals on the right. Since the residue at the origin of

$$\frac{(4\alpha z-\pi i)\,e^{\pi iz-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}$$

is $-\dfrac{\pi i}{2\alpha}$, we have

$$\int_{-\infty+ic}^{\infty+ic}\frac{(4\alpha z-\pi i)\,e^{\pi iz-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}dz$$

$$=-\frac{\pi^2}{2\alpha}+\mathrm{P}\int_{-\infty}^{\infty}\frac{(4\alpha t-\pi i)\,e^{\pi it-2\alpha t^2}}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$=-\frac{\pi^2}{2\alpha}+4\alpha\int_{-\infty}^{\infty}\frac{te^{2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt+\pi\int_{-\infty}^{\infty}\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt.$$

In this result it is permissible to take $4\alpha c=\pi$, and then we get

$$-\frac{\pi^3}{2\alpha^3}+\frac{\pi^2}{\alpha^2}\int_{-\infty}^{\infty}\frac{e^{-2\alpha t^2}\sin\pi t}{e^{\alpha t}-e^{-\alpha t}}dt+\frac{4\pi}{\alpha}\int_{-\infty}^{\infty}\frac{te^{-2\alpha t^2}\cos\pi t}{e^{\alpha t}-e^{-\alpha t}}dt$$

$$=\frac{\pi}{\alpha^2}\int_{-\infty+{}^1/_4\pi i/\alpha}^{\infty+{}^1/_4\pi i/\alpha}\frac{(4\alpha z-\pi i)e^{\pi iz-2\alpha z^2}}{e^{\alpha z}-e^{-\alpha z}}dz$$

$$=\frac{4\pi q_1^{{}^1/_8}}{\alpha}\int_{-\infty}^{\infty}\frac{ue^{-2\alpha u^2}}{e^{\alpha u}e^{{}^1/_4\pi i}-e^{-\alpha u}e^{-{}^1/_4\pi i}}du$$

$$=\frac{2\sqrt{2}\pi q_1^{1/8}}{\alpha}\int_{-\infty}^{\infty}\frac{\sinh\alpha u\,.\,e^{-2\alpha u^2}}{\sinh^2\alpha u+\cosh^2\alpha u}\,du$$

$$=2\sqrt{2}q_1^{1/8}\left(\frac{\pi}{\alpha}\right)^3\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi t}{\cosh 2\pi t}\,dt.$$

The required transformation is therefore

$$(6.1)\quad \sum_{n=0}^{\infty}(-1)^n q^{n+7/8}\mathrm{F}(8n+7)+\frac{1}{(-i\tau)^{3/2}}\sum_{n=0}^{\infty}(-1)^n q_1^{n+7/8}\mathrm{F}(8n+7)$$
$$=\frac{2\sqrt{2}}{(-i\tau)^{3/2}}\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi t}{\cosh 2\pi t}\,dt,$$

that is to say

$$(6.2)\quad e^{-7/8\pi i}\,\mathscr{T}'(q)+\frac{1}{(-i\tau)^{3/2}}e^{-7/8\pi i}\,\mathscr{T}'(q_1)=\frac{4\sqrt{2}}{(-i\tau)^{3/2}}\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi t}{\cosh 2\pi t}\,dt.$$

By interchanging α and β, we infer that

$$(6.3)\qquad \int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi t}{\cosh 2\pi t}\,dt=(-i\tau)^{3/2}\int_{-\infty}^{\infty}ue^{-2\alpha u^2}\frac{\sinh\pi u}{\cosh 2\pi u}\,du.$$

To prove this last result directly, observe that [17])

$$(6.4)\qquad \frac{\sinh\pi t}{\cosh 2\pi t}=\sqrt{2}\int_{-\infty}^{\infty}\frac{\sinh\pi u\sin 4\pi t u}{\cosh 2\pi u}\,du,$$

and therefore

$$\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi t}{\cosh 2\pi t}\,dt=\sqrt{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi u\sin 4\pi t u}{\cosh 2\pi u}\,du\,dt$$
$$=\sqrt{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}te^{-2\beta t^2}\frac{\sinh\pi u\sin 4\pi t u}{\cosh 2\pi u}\,dt\,du$$
$$=\left(\frac{\pi}{\beta}\right)^{3/2}\int_{-\infty}^{\infty}ue^{-2\alpha u^2}\frac{\sinh\pi u}{\cosh 2\pi u}\,du,$$

which immediately gives the result in question.

7. *Sets of transformations of generating functions.*

From the results of §§ 4—6 it is possible to construct a large number of formulae, each formula containing three terms, two

[17]) Cf. J. Harkness and F. Morley [Introduction to the theory of analytic functions, (1898), 226].

of which are generating functions proceeding in powers of q and q_1 respectively, and the third term being an infinite integral; it is always possible to secure that the formulae contain no theta-functions by means of devices such as were used in obtaining (5.1) and (5.2).

The most interesting formulae consist of three self-contained sets of six formulae; each set of six formulae involves twelve generating functions in all, namely three series proceeding in powers of q and nine series obtained from these three by replacing q successively by $-q$, q_1 and $-q_1$.

To save space, I shall state only three out of each set of six formulae; in each set the three omitted are the reciprocals of those which are stated, and they are immediately obtainable by interchanging q with q_1.

The first set of six formulae consists of relations which connect the six functions

$$\mathscr{A}(q),\quad \mathscr{B}(q),\quad \mathscr{C}(q),\quad \mathscr{A}'(q),\quad \mathscr{B}'(q),\quad \mathscr{C}'(q)$$

with their reciprocals; these formulae are immediate consequences of the results of §§ 4 and 5. The formulae are as follows:

(7.01) $$\mathscr{A}(q) - (-i\tau)^{-3/2}\,\mathscr{C}'(q_1) = -\int_0^\infty te^{-\alpha t^2}\tanh \pi t\,dt,$$
$$= -(-i\tau)^{-3/2}\int_0^\infty \frac{te^{-\beta t^2}}{\sinh \pi t}dt,$$

(7.02) $$e^{-3/4\pi i}\,\mathscr{A}'(q) - (-i\tau)^{-3/2}\,.\,\mathscr{B}'(q_1) = \int_0^\infty te^{-\alpha t^2}\tanh \pi t\,dt$$
$$= (-i\tau)^{-3/2}\int_0^\infty \frac{te^{-\beta t^2}}{\sinh \pi t}dt,$$

(7.03) $$\mathscr{B}(q) - (-i\tau)^{-3/2}\,\mathscr{C}(q_1) = -\int_0^\infty te^{-\alpha t^2}\coth \pi t\,dt$$
$$= -(-i\tau)^{-3/2}\int_0^\infty te^{-\beta t^2}\coth \pi t\,dt,$$

and the three reciprocal formulae.

I shall merely state the second and third sets of formulae, and shall leave the proofs of them to the reader. In each case a proof can be constructed very easily (with the help of formulae of the first set) by expressing the first function on the left in each formula in terms of either $\mathscr{B}(q)$ and $\mathscr{C}(q)$ or of $\mathscr{B}'(q)$ and $\mathscr{C}'(q)$, whichever is applicable to the function in question.

The second set of six formulae consist of relations which connect the six functions

$$2\sum_{n=0}^{\infty} q^n F_1(n), \qquad \sum_{n=0}^{\infty} q^n G(n), \qquad \sum_{n=0}^{\infty} q^{n+3/4} F_1(4n+3),$$

$$2\sum_{n=0}^{\infty} (-1)^n q^n F_1(n), \qquad \sum_{n=0}^{\infty} (-1)^n q^n G(n), \qquad \sum_{n=0}^{\infty} (-1)^n q^{n+3/4} F_1(4n+3)$$

with the reciprocal functions obtained from them by writing q_1 in place of q. The formulae are as follows:

$$(7.04) \qquad 2\sum_{n=0}^{\infty} q^n F_1(n) + (-i\tau)^{-3/2} \sum_{n=0}^{\infty} q_1{}^n G(n) = -\int_0^{\infty} te^{-\alpha t^2} \coth \pi t \, dt,$$

$$(7.05) \qquad 2\sum_{n=0}^{\infty} (-1)^n q^n F_1(n) + (-i\tau)^{-3/2} \sum_{n=0}^{\infty} q_1{}^{n+3/4} F_1(4n+3) = -\int_0^{\infty} \frac{te^{-\alpha t^2}}{\sinh \pi t} \, dt,$$

$$(7.06) \qquad \sum_{n=0}^{\infty} (-1)^n q^n G(n) - (-i\tau)^{-3/2} \sum_{n=0}^{\infty} (-1)^n q_1{}^{n+3/4} F_1(4n+3) = -\int_0^{\infty} \frac{te^{-\alpha t^2}}{\sinh \pi t} \, dt,$$

and the three reciprocal formulae.

The third set of formulae consists of relations which connect the six functions

$$\sum_{n=0}^{\infty} q^n G(4n), \qquad \sum_{n=0}^{\infty} q^n \{5F_1(n) - F(n)\}, \qquad \sum_{n=0}^{\infty} q^{n+3/4} G(4n+3),$$

$$\sum_{n=0}^{\infty} (-1)^n q^n G(4n), \qquad \sum_{n=0}^{\infty} (-1)^n q^n \{5F_1(n) - F(n)\}, \qquad \sum_{n=0}^{\infty} (-1)^n q^{n+3/4} G(4n+3)$$

with the reciprocal functions obtained from them by writing q_1 in place of q. The formulae are as follows:

$$(7.07)\quad \sum_{n=0}^{\infty} q^n G(4n) + (-i\tau)^{-3/2} \sum_{n=0}^{\infty} q_1{}^n \{5F_1(n) - F(n)\}$$
$$= -2\int_0^{\infty} te^{-\alpha t^2} \coth \pi t \, dt,$$

$$(7.08)\quad \sum_{n=0}^{\infty} (-1)^n q^n G(4n) - (-i\tau)^{-3/2} \sum_{n=0}^{\infty} (-1)^n q_1{}^{n+3/4}\, G(4n+3)$$
$$= -2\int_0^{\infty} \frac{te^{-\alpha t^2}}{\sinh \pi t}\, dt,$$

$$(7.09)\quad \sum_{n=0}^{\infty} (-1)^n q^n \{5F_1(n) - F(n)\} + (-i\tau)^{-3/2} \sum_{n=0}^{\infty} q_1^{n+3/4}\, G(4n+3)$$
$$= -2\int_0^{\infty} \frac{te^{-\alpha t^2}}{\sinh \pi t}\, dt,$$

and the three reciprocal formulae.

I remark that, so far as I know, no investigations of arithmetical properties of the class-number

$$5F_1(n) - F(n)$$

appear to have been published.

In addition to these sets, a few isolated formulae exist. The formula involving the function $\mathscr{U}(q)$, defined by (1.05) and its reciprocal is self-reciprocal. This formula is

$$(7.10)\quad \mathscr{U}(q) + (-i\tau)^{-3/2}\, \mathscr{U}(q_1) = -8\int_0^{\infty} te^{-\alpha t^2} \coth \pi t dt.$$

The corresponding formula involving

$$\sum_{n=0}^{\infty} q^n E(n)$$

is also self-reciprocal; but this formula, namely

$$(7.11)\quad \sum_{n=0}^{\infty} q^n E(n) = \tfrac{1}{12}\vartheta_2^3(0,\ q) = (-i\tau)^{-3/2} \sum_{n=0}^{\infty} q_1{}^n E(n),$$

is obviously trivial.

The formulae which involve

$$\mathscr{U}'(q) \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n q^n E(n)$$

respectively are

$$(7.12)\qquad \mathscr{U}'(q) - 4(-i\tau)^{-3/2}\,\mathscr{T}(q_1^2) = -8\int_0^\infty \frac{te^{-\alpha t^2}}{\sinh \pi t}\,dt,$$

$$(7.13)\quad \sum_{n=0}^{\infty}(-1)^n q^n \mathrm{E}(n) = \tfrac{1}{12}\vartheta_4^3(0,q) = \tfrac{2}{3}(-i\tau)^{-3/2}\sum_{n=0}^{\infty} q_1^{2n+3/4}\mathrm{F}(8n+3),$$

the latter being trivial.

Finally, by the result of § 6, there is the self-reciprocal formula

$$(7.14)\qquad e^{-7/8\pi i}\,\mathscr{T}'(q) + (-i\tau)^{-3/2}e^{-7/8\pi i}\,\mathscr{T}'(q_1)$$
$$= 8\sqrt{2}\int_0^\infty te^{-2\alpha t^2}\frac{\sinh \pi t}{\cosh 2\pi t}\,dt$$
$$= \frac{8\sqrt{2}}{(-i\tau)^{3/2}}\int_0^\infty te^{-2\beta t^2}\frac{\sinh \pi t}{\cosh 2\pi t}\,dt.$$

8. *A relation between three infinite integrals.*

By eliminating generating functions from three suitably chosen members of one of the sets of formulae given in § 7, it is possible to construct an interesting relation which connects three infinite integrals. Thus, in the formulae (7.01), 7.02), (7.02) respectively, take

$$\tau = \frac{iy}{1+iy}, \quad \tau = \frac{i}{y-i}, \quad \tau = iy,$$

where the real part of y is positive.

When the phases of the three complex numbers

$$\frac{y}{1+iy}, \quad \frac{1}{y-i}, \quad y$$

are taken to lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, we see without difficulty that

$$\frac{y}{y-i} = \frac{ye^{+\frac{1}{2}\pi i}}{1+iy}.$$

Hence the three formulae become

$$\left(\frac{y}{1+iy}\right)^{3/2} \mathscr{A}\,(e^{-\pi y/(1+iy)}) - \mathscr{C}\,(e^{-\pi/y}) = -\int_0^\infty \frac{te^{-\pi t^2/y}e^{-\pi i t^2}}{\sinh \pi t}\,dt\,,$$

$$(1+iy)^{-3/2} \mathscr{A}\,(e^{-\pi y/(1+iy)}) - \mathscr{B}\,(e^{-\pi y}) = \int_0^\infty \frac{te^{-\pi y t^2}e^{\pi i t^2}}{\sinh \pi t}\,dt\,,$$

$$\mathscr{B}\,(e^{-\pi y}) - y^{-3/2}\,\mathscr{C}\,(e^{-\pi/y}) = -\int_0^\infty te^{-\pi y t^2}\coth \pi t\,dt\,.$$

By eliminating the generating functions from these formulae, we immediately obtain the relation required, namely

$$(8.1)\qquad \int_0^\infty te^{-\pi y t^2}\coth \pi t\,dt = \int_0^\infty \frac{te^{-\pi y t^2}e^{\pi i t^2}}{\sinh \pi t}\,dt + y^{-3/2}\int_0^\infty \frac{te^{-\pi t^2/y}e^{-\pi i t^2}}{\sinh \pi t}\,dt\,.$$

It is not very difficult to construct a direct proof of this formula; we shall first prove a subsidiary formula, namely [18])

$$(8.2)\qquad \int_0^\infty \frac{e^{-\pi i x^2}\sin 2\pi x t}{\sinh \pi x}\,dx = \frac{\cosh \pi t - e^{\pi i t^2}}{2\sinh \pi t}\,.$$

where t is real.

Consider the integral

$$\int \frac{e^{-\pi i z^2}}{\cosh \pi z \cosh \pi(z-t)}\,dz$$

taken along two lines parallel to the real axis and passing through the points $-\frac{1}{2}i$, $\frac{1}{2}i$ respectively. The upper line is to have indentations below it at $\frac{1}{2}i$ and $\frac{1}{2}i+t$, and the lower line is to have indentations above it at $-\frac{1}{2}i$ and $-\frac{1}{2}i+t$, so that there are no poles of the integrand in the strip between the indented lines. It therefore follows from Cauchy's theorem that

$$\frac{1}{\pi i}\left\{\mathrm{P}\int_{-\infty-\frac{1}{2}i}^{\infty-\frac{1}{2}i} + \mathrm{P}\int_{\infty+\frac{1}{2}i}^{-\infty+\frac{1}{2}i}\right\}\frac{e^{-\pi i z^2}}{\cosh \pi z\cosh \pi(z-t)}\,dz$$

is equal to the sum of the residues of the integrand at the simple poles $\frac{1}{2}i$, $\frac{1}{2}i+t$, $-\frac{1}{2}i$, $-\frac{1}{2}i+t$, that is to say it is equal to

$$2e^{\frac{1}{4}\pi i}\,\frac{1-e^{-\pi i t^2}\cosh \pi t}{\pi \sinh \pi t}\,.$$

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[18]) For an indirect and much more elaborate proof of (8.2), see S. Ramanujan [Messenger of Math. **44** (1915), 81—82].

If we now write $x-\frac{1}{2}i$ and $x+\frac{1}{2}i$ for z on the two parts of the path of integration, we find that

$$-\frac{1}{\pi i}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{-\pi i(x-\frac{1}{2}i)^2}-e^{-\pi i(x+\frac{1}{2}i)^2}}{\sinh\pi x\sinh\pi(x-t)}\,dx=2e^{\frac{1}{4}\pi i}\,\frac{1-e^{-\pi i t^2}\cosh\pi t}{\pi\sinh\pi t},$$

that is to say

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{-\pi i x^2}}{\sinh\pi(x-t)}\,dx=i\,\frac{1-e^{-\pi i t^2}\cosh\pi t}{\sinh\pi t}.$$

Hence, replacing x by $x+t$, we have

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{-\pi i x^2}e^{-2\pi i x t}}{\sinh\pi x}\,dx=i\,\frac{e^{\pi i t^2}-\cosh\pi t}{\sinh\pi t},$$

and (8.2) is an immediate consequence of this result.

We are now in a position to give a direct proof of (8.1); by using (8.2), we have

$$\int_{-\infty}^{\infty}te^{-\pi y t^2}\coth\pi t\,dt-\int_0^{\infty}\frac{te^{-\pi y t}e^{\pi i t^2}}{\sinh\pi t}\,dt$$

$$=\int_0^{\infty}te^{-\pi y t^2}\,\frac{\cosh\pi t-e^{\pi i t^2}}{\sinh\pi t}\,dt$$

$$=2\int_0^{\infty}\int_0^{\infty}te^{-\pi y t^2}\,\frac{e^{-\pi i x^2}\sin 2\pi x t}{\sinh\pi x}\,dx\,dt$$

$$=2\int_0^{\infty}\int_0^{\infty}te^{-\pi y t^2}\,\frac{e^{-\pi i x^2}\sin 2\pi x t}{\sinh\pi x}\,dt\,dx$$

$$=y^{-3/2}\int_0^{\infty}\frac{xe^{-\pi x^2/y}\,e^{-\pi i x^2}}{\sinh\pi x}\,dx,$$

and the truth of (8.1) is now evident.

9. *Asymptotic expansions of infinite integrals.*

In conclusion I put on record the asymptotic expansions of the various integrals which occur in § 7. Two sets of expansions will be given, one suitable for small values of α, the other for large values of α. It will be supposed that α is not necessarily real, but the real part of α must, of course, be positive to ensure that $|q|<1$.

All the asymptotic expansions possess the property that, for complex values of α, the absolute value of the error due to stopping at any term never exceeds the absolute value of the first term neglected; if, in addition, α is real (and therefore positive), the error is of the same sign as the first term neglected.

We first prove this general property of the expansions as follows: by Taylor's theorem, with Darboux's form of the remainder for complex values of α and with Lagrange's form of the remainder in the special case when α is positive, on the hypothesis that t is always positive we have

$$e^{-\alpha t^2}=\sum_{n=0}^{N-1}\frac{(-1)^n\alpha^n t^{2n}}{n!}+\theta\frac{(-1)^N\alpha^N t^{2N}}{N!},$$

where $|\theta|\leqq 1$, $0\leqq\theta\leqq 1$ in the general and the special cases respectively.

Now, when $f(t)$ is positive and $|\theta|\leqq 1$, we have

$$\left|\int_0^\infty \theta\,\frac{(-1)^N\alpha^N t^{2N}}{N!}f(t)dt\right| \leqq \left|\int_0^\infty \frac{(-1)^N\alpha^N t^{2N}}{N!}f(t)dt\right|,$$

while, when $f(t)$ and α are both positive and $0\leqq\theta\leqq 1$, we see that the expressions

$$\int_0^\infty \theta\,\frac{(-1)^N\alpha^N t^{2N}}{N!}f(t)dt,\quad \int_0^\infty \frac{(-1)^N\alpha^N t^{2N}}{N!}f(t)dt$$

evidently have the same sign. In each of the cases to be considered, we shall be dealing with an integral in which the function typified by $f(t)$ is positive; this makes obvious the asserted properties of the asymptotic expansions.

We now discuss the set of asymptotic expansions suitable for small values of α.

The first integral to be considered is

$$\int_0^\infty te^{-\alpha t^2}\coth\pi t\,dt=\frac{1}{2\alpha}+2\int_0^\infty e^{-\alpha t^2}\frac{t}{e^{2\pi t}-1}dt$$

$$\sim\frac{1}{2\alpha}+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\alpha^{n-1}}{(n-1)!}\int_0^\infty\frac{t^{2n-1}}{e^{2\pi t}-1}dt$$

$$\sim\frac{1}{2}\left[\frac{1}{\alpha}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{n!}\alpha^{n-1}\right],$$

by using Binet's integral for the Bernoullian numbers B_n, which have the following values:

$$B_1=\tfrac{1}{6},\quad B_2=\tfrac{1}{30},\quad B_3=\tfrac{1}{42},\ldots.$$

Thus we have

$$(9.1)\qquad \int_0^\infty te^{-\alpha t^2}\coth \pi t\,dt=\frac{1}{2}\left[\frac{1}{\alpha}+\sum_{n=1}^\infty \frac{(-1)^{n-1}B_n}{n!}\alpha^{n-1}\right].$$

The second integral to be considered is

$$\int_0^\infty te^{-\alpha t^2}\tanh \pi t\,dt=\frac{1}{2\alpha}-2\int_0^\infty e^{-\alpha t^2}\frac{t}{e^{2\pi t}+1}dt$$

$$\sim \frac{1}{2\alpha}-2\sum_{n=1}^\infty \frac{(-1)^{n-1}\alpha^{n-1}}{(n-1)!}\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}+1}dt.$$

Since

$$\frac{1}{e^{2\pi t}+1}=\frac{1}{e^{2\pi t}-1}-\frac{2}{e^{4\pi t}-1},$$

we have at once

$$(9.2)\quad \int_0^\infty te^{-\alpha t^2}\tanh \pi t\,dt \sim \frac{1}{2}\left[\frac{1}{\alpha}+\sum_{n=1}^\infty \frac{(-1)^{n}B_n}{n!}\left(1-\frac{1}{2^{2n-1}}\right)\alpha^{n-1}\right].$$

The third integral to be considered is

$$\int_0^\infty \frac{te^{-\alpha t^2}}{\sinh \pi t}dt \sim \sum_{n=1}^\infty \frac{(-1)^{n-1}\alpha^{n-1}}{(n-1)!}\int_0^\infty \frac{t^{2n-1}}{\sinh \pi t}dt.$$

Since

$$\frac{1}{\sinh \pi t}=\frac{2}{e^{\pi t}-1}-\frac{2}{e^{2\pi t}-1},$$

we have at once

$$(9.3)\qquad \int_0^\infty \frac{te^{-\alpha t^2}}{\sinh \pi t}dt \sim \frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^{n-1}B_n}{n!}(2^{2n}-1)\alpha^{n-1}.$$

The last integral to be considered, namely

$$8\sqrt{2}\int_0^\infty te^{-2\alpha t^2}\frac{\sinh \pi t}{\cosh 2\pi t}dt,$$

is rather more troublesome; we start with the expansion

$$\frac{\sinh x}{\cosh 2x}=\frac{Q_1 x}{1!}-\frac{Q_2 x^3}{3!}+\frac{Q_3 x^5}{5!}-\ldots,$$

in which the coefficients have the values

$$Q_1 = 1,\ Q_2 = 11,\ Q_3 = 361,\ Q_4 = 24611, \dots ;$$

I mention in passing that the values of the coefficients up to Q_{20} have been computed by Glaisher [19]) who has obtained a number of properties of the general coefficient Q_n. We can now evaluate the integral

$$\int_0^\infty t^{2n+1} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt$$

by taking the formula (6.4), namely

$$\frac{\sinh \pi u}{\cosh 2\pi u} = 2\sqrt{2} \int_0^\infty \frac{\sinh \pi t \sin 4\pi t u}{\cosh 2\pi t}\, dt$$

(where u is supposed real), differentiating $2n+1$ times with respect to u under the integral sign (a procedure which is easily justified), and then putting $u=0$. This process gives

$$\pi^{2n+1} Q_{n+1} = 2\sqrt{2}(4\pi)^{2n+1} \int_0^\infty t^{2n+1} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt\,,$$

so that

$$8\sqrt{2} \int_0^\infty t^{2n+1} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt = \frac{Q_{n+1}}{2^{4n}}\,.$$

From the formula

$$8\sqrt{2} \int_0^\infty t e^{-2\alpha t^2} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt \sim \sum_{n=0}^{\infty} \frac{(-2\alpha)^n}{n!}\, 8\sqrt{2} \int_0^\infty t^{2n+1} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt$$

it now follows immediately that

$$(9.4)\quad 8\sqrt{2} \int_0^\infty t e^{-2\alpha t^2} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n Q_{n+1}}{2^{3n}\, n!}\, \alpha^n.$$

It is worth mentioning that the expansion (9.4) is not useful for purposes of computation unless $|\alpha|$ is very small, since the expansion starts with the terms

$$1 - \frac{11\alpha}{8} + \frac{361\alpha^2}{128} - \frac{24611\alpha^3}{3072} + \dots\,.$$

In this respect (9.4) differs from (9.1) which is of a certain amount of use in the extreme case $\alpha = \pi$.

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[19]) J. W. L. Glaisher [Quarterly Journal of Math. **45** (1914), 202].

We next discuss the asymptotic expansions suitable for large values of $|\alpha|$; these expansions are easily determined by the transformation formulae, since $|\beta|$ is small when $|\alpha|$ is large. Thus we have

$$\int_0^\infty te^{-\alpha t^2} \coth \pi t \, dt = \left(\frac{\pi}{\alpha}\right)^{3/2} \int_0^\infty te^{-\beta t^2} \coth \pi t \, dt$$

$$\sim \frac{1}{2\pi}\left(\frac{\pi}{\alpha}\right)^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{n!} \beta^n\right].$$

and consequently

$$(9.5) \qquad \int_0^\infty te^{-\alpha t^2} \coth \pi t \, dt \sim \frac{1}{2\pi}\left(\frac{\pi}{\alpha}\right)^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n \pi^{2n}}{n! \alpha^n}\right].$$

Similarly

$$(9.6) \qquad \int_0^\infty te^{-\alpha t^2} \tanh \pi t \, dt \sim \frac{1}{2\pi}\left(\frac{\pi}{a}\right)^{1/2} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n \pi^{2n}}{n! \alpha^n} (2^{2n} - 1)\right],$$

$$(9.7) \qquad \int_0^\infty \frac{e^{-\alpha t^2}}{\sinh \pi t} \, dt \sim \frac{1}{2\pi}\left(\frac{\pi}{\alpha}\right)^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n B_n \pi^{2n}}{n! \alpha^n} \left(1 - \frac{1}{2^{2n-1}}\right)\right];$$

$$(9.8) \quad 8\sqrt{2} \int_0^\infty te^{-2\alpha t^2} \frac{\sinh \pi t}{\cosh 2\pi t} \, dt \sim \left(\frac{\pi}{\alpha}\right)^{3/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n Q_{n+1} \pi^{2n}}{2^{3n} n! \alpha^n}\right].$$

The sets of asymptotic expansions are now complete.

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